GPO PRICE $ _____

CFSTI PRICE(S) $ _____

Hard copy (HC) 1.00

Microfiche (MF) .50

ff 653 July 65

FORCED SOUND PROPAGATION INDUCED BY A FRACTIONALLY ACCOMMODATING PISTON

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N66-33439

Introduction and Formulation

This paper examines the steady state sound disturbances induced in a semi-infinite gas by an oscillating plane piston which reflects a fraction, σ, of the incoming particles specularly and the remaining fraction, $1-\sigma$, diffusely and isothermally. In the frame of the moving boundary the diffusely reflected particles come off with a Maxwellian distribution at the background gas temperature. The specular case ($\sigma = 1$) was examined by the present author[1] and also separately by H. Weitzner[2] for the 1964 symposium of this series. The present work generalizes these earlier treatments.

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Our analysis is aided by the introduction of artificial symmetry as shown in Fig. 1. Two pistons at $x = 0^{\pm}$ are allowed to oscillate simultaneously in opposite directions so as to initiate symmetric disturbances in a gas filling the full space. It is assumed that disturbances are governed by the linearized, isothermal BGK equation

$$\frac{\partial g_1}{\partial t} + u\frac{\partial g_1}{\partial x} = -\nu\left(g_1 - n_1(x)g_0 + n_0 u_1(x)\frac{\partial g_0}{\partial u}\right) \tag{1}$$

where $g_1(t,x,u)$ is the first order perturbation about a uniform, one dimensional Maxwellian, and

*Research initiated at the Institut für Plasmaphysik, Munich, Germany. Work continued and completed at M.I.T. under NASA Grant NSG 496.

FACILITY FORM 602

N66 33439 (ACCESSION NUMBER)

(THRU)

21 (PAGES)

1 (CODE)

CR-77047 (NASA CR OR TMX OR AD NUMBER)

12 (CATEGORY)

$$g_0(u) = (2\pi a_0^2)^{-1/2} \exp\left(-u^2/2a_0^2\right)$$

The first order perturbation density and velocity are $n_1(x)$ and $u_1(x,t)$. Temperature perturbations are suppressed to zero.

The boundaries at $x = 0^\pm$ oscillate with velocities $v(t) = \epsilon v \exp(-i\omega t)$. This exponential time dependence is assumed throughout. All disturbances are assumed to decay sufficiently rapidly with $|x|$ so that half range Fourier transforms may be applied in the respective half-planes. Thus,

$$g_1(k,0^+) = \int_0^\infty g_1(u,v)e^{-ikx}dx, \quad \int_0^\infty \frac{\partial g_1}{\partial x} e^{-ikx}dx = ikg_1 - g_1(u,0^+) \tag{2}$$

and $g_1(u,0^-)$ is obtained from a similar integral in $x<0^-$. The presence of $\exp(-ikx)$ in $g_1(k)$ differs from the usual convention and implies that $g_1(k)$ is non-analytic in $\mathrm{Im}(k)>0$, and fully analytic in $\mathrm{Im}(k)<0$. The perturbation density and velocity are similarly transformed.

Applying these transforms to Eq.(1) and factoring the time dependence, we obtain

$$(-i\omega + iku)g_{1\pm} = -v\left(g_{1\pm} - n_{1\pm}g_0 + n_0 u_{1\pm}\frac{\partial g_0}{\partial u}\right) \pm g_1(u,0^\pm) \tag{3}$$

where the + and − refer to Fourier transforms in the right and left half-planes, respectively. We express n_0u_1 in terms of n_1 with the aid of the zeroth moment of Eq.(3). Following the Ref. 1 procedure, we solve for $g_{1\pm}(u)$, integrate to produce n_1, and rearrange our results to

$$n_1(k) = \frac{Q(0^+)}{D(k)}, \quad D(k) = 1 - L(k)$$

$$Q(0^+) = \pm\int_0^\infty \frac{u\,g_1(u,0^+)\,du}{(-i\omega + iku)} \pm \int_{-\infty}^0 \frac{u\,g_1(u,0^+)\,du}{(-i\omega + iku)} + Q_v \tag{4}$$

$$Q_v = \frac{iv}{k} n_0 u_1(0^+) \int_{-\infty}^{+\infty} \frac{\frac{dg_0}{du} du}{(-i\omega_0 + iku)}, \quad \varepsilon(k) = v \int_{-\infty}^{+\infty} \frac{\left(g_0 - \frac{\omega}{k}\frac{dg_0}{du}\right) du}{(-i\omega_0 + iku)} \tag{5}$$

where $\omega_0 = \omega + iv$ and from symmetry $u_1(0^-) = -u_1(0^+)$.

In Ref. 1, we gave separate expressions for the reflected distribution at the right piston for the limiting cases of specular and diffuse-isothermal reflection. Expanding these expressions to first order in v, subtracting away $n_0 g_0(u)$ and adding the specular and diffuse results in proper proportion gives

$$g_1(u,0^+) = \begin{cases} (1-\sigma)\frac{n_0 v}{2c_0} g_0 - (1+\sigma) n_0 v \frac{dg_0}{du} - (1-\sigma) n_0 s^-(0^+) g_0 + \sigma g_1(-u,0^+) & u > 0 \\ g_1(u,0^+) & u < 0 \end{cases} \tag{6}$$

where $c_0 = \int_0^\infty u g_0(u) du$ and $n_0 s^-(0^+) = \int_{-\infty}^0 u g_1(u,0^+) du$.

The substitution of this result into Eq.(4) and the application of analogous methods for $x = 0^-$ yields

$$Q(0^+) = m(k,\sigma_0) - (1-\sigma) S_-(0) I_+(k) + \sigma S_+(k) + S_-(k) \tag{7a}$$

$$Q(\sigma) = n(k,\sigma_0) + (1-\sigma) S_-(0) I_-(k) - \sigma S_-(k) - S_+(k) \tag{7b}$$

where we have defined

$$m(\sigma_0) = Q_v + (1-\sigma_0)\frac{n_0 v}{2c_0}\int_0^\infty \frac{u g_0 du}{(-i\omega_0 + iku)} + (1+\sigma_0)\frac{n_0 v}{a_0^2}\int_0^\infty \frac{u^2 g_0 du}{(-i\omega_0 + iku)}$$

$$n(\sigma_0) = Q_v - (1-\sigma_0)\frac{n_0 v}{2c_0}\int_{-\infty}^0 \frac{u g_0 du}{(-i\omega_0 + iku)} + (1+\sigma_0)\frac{n_0 v}{a_0^2}\int_{-\infty}^0 \frac{u^2 g_0 du}{(-i\omega_0 + iku)}$$

$$I_+(k) = -\frac{i\omega_0}{c_0}\int_0^\infty \frac{u g_0 du}{(-i\omega_0 + iku)}, \quad I_-(k) = -\frac{i\omega_0}{c_0}\int_{-\infty}^0 \frac{u g_0 du}{(-i\omega_0 + iku)} \tag{8}$$

$$S_+(k) = \int_0^\infty \frac{u g_1(u,0^+) du}{(-i\omega_0 + iku)}, \quad S_-(k) = \int_{-\infty}^0 \frac{u g_1(u,0^+) du}{(-i\omega_0 + iku)}$$

In establishing Eqs.(7) and (8) we use $\partial g_0/\partial u = -(u/a_0^2)g_0$. The functions $S_\pm(k)$ have been introduced with the aid of $g_1(u,0^+) = g_1(-u,0^-)$, which follows from symmetry. Note that

$$S_-(0) = -S_+(0) = \frac{1}{(-i\omega_0)}\int_{-\infty}^{0} u\, g_1(u,0^+)\, du = \frac{n_0 S^-(0^+)}{(-i\omega_0)}$$

Q_v is known explicitly, since $n_0 u(0^+) = \int_{-\infty}^{\infty} u g_1(u,0^+)du = n_0 v$ from Eq.(6). We have subscripted σ_0 in the $\mathfrak{m}(\sigma_0)$ and $\mathfrak{n}(\sigma_0)$ definitions, since σ_0 in these functions refers to the way the zero order distribution is reflected.

Together Eqs.(4) and (7) produce a pair of coupled Weiner-Hopf equations

$$n_+(k) = \frac{\mathfrak{m}(k,\sigma_0) - (1-\sigma)S_-(0)\, I_+(k) + \sigma S_+(k) + S_-(k)}{D(k)} \qquad (9a)$$

$$n_-(k) = \frac{\mathfrak{n}(k,\sigma_0) + (1-\sigma)S_-(0)\, I_-(k) - \sigma S_-(k) - S_+(k)}{D(k)} \qquad (9b)$$

which must be solved for the unknown density transforms $n_\pm(k)$ and source transforms $S_\pm(k)$. A subsequent inversion of these transforms yields $n_\pm(x)$ and the incident distributions at the pistons, $g_1(u,0^\pm)$.

The analytic properties of the various functions in Eq.(9) should be known to facilitate solution. From Eq.(8) one concludes that $I_+(k)$ and $S_+(k)$ are analytic in the whole k plane except along a ray from $k = 0$ to $k = \alpha\infty$, $\alpha = \omega_0/|\omega_0|$ in $\mathrm{Im}(k)>0$. Analogous cuts for $I_-(k)$ and $S_-(k)$ reside in $\mathrm{Im}(k)<0$. The remaining functions $\mathfrak{m}$, $\mathfrak{n}$ and D are non-analytic on the line dividing the k plane from $k = -\alpha\infty$ to $k = +\alpha\infty$.

Across its cut each function suffers a jump, e.g.

$$\Delta S_- = S_{-2} - S_{-1} = \frac{\pi n_0 \omega_0}{k^2}\, g_1\!\left(\frac{\omega_0}{k}, 0^+\right), \quad \mathrm{Im}(k) < 0 \qquad (10)$$

where the subscripts "$_1$" and "$_2$" refer to k values just to the right and left of the cut, respectively. In the corresponding regions $D(k)$ assumes the distinct analytic forms

$D_1(k)$ and $D_2(k)$. Each form may be analytically continued throughout the full k plane, and each is found to be analytic everywhere except for an essential singularity at $k = 0$. As k passes to zero in the right half-plane or within the 90° sectors centered about the branch cut, $D_1(k\to 0)\to D_1(0) = (\omega/\omega_0)^2$. Similarly, for paths left of the cut or within the 90° sectors $D_2(k\to 0)\to D_2(0) = D_1(0)$. ~~Landau[4] has shown that the dispersion relation for forced plasma oscillations exhibits comparable features.~~

It can be shown that $D_1(k)$ possesses, at most, a single zero to the right of the D(k) cut and an infinity of zeros to the left near $k = 0$. Similarly, $D_2(k)$ can have one zero to the left of the cut, and has infinitely many zeros to the right near the origin. As in Ref. 1, the functions $D_{1,2}(k)$ can be related to the plasma dispersion function. Utilizing this function's asymptotic representation, we find that for $v/\omega \gg 1$ the $D_{1,2}(k)$ zeros are at $k = \pm k_1 = \pm\omega/s_0(1+2\omega/v+O(\omega^2/v^2))$, which correspond to the isothermal sound modes found in continuum treatments of this problem. The loci of these zeros for arbitrary v/ω must be determined numerically. From Refs. 1 and 2, however, we may conclude that the qualitative trajectories follow the dashed Fig. 2 curves. For some minimum v/ω the zeros cross the cut, but previous studies[2,6] attach no physical significance to this event.

Lastly, we observe that the introduction of a finite upper bound on molecular speeds, say $|u|<c$, $c \gg s_0$ moves the branch points for all the "plus" and "minus" functions from the origin to $k = \pm\omega_0/c$, respectively. This opens up a Re(k) axis strip of width $2v/c$ along which all these functions, including D(k), are analytic.

Special Solutions

A. Specular Reflection ($\sigma = \sigma_0 = 1$)

The sum of the Eqs. (9a) and (9b) for specular reflection is

$$n(k) = n_+(k)+n_-(k) = \frac{\mathcal{N}(k,\sigma)+\mathcal{N}(k,\sigma)}{D(k)} = \frac{\mathcal{S}(k,\sigma)}{D(k)} \tag{11}$$

Eq.(11) is equivalent to Eq.(9) of Ref. 1 with temperature perturbations set to zero. Like D(k) the function $\mathcal{D}(k,1)$ possesses a branch cut from $k = -\alpha\omega$ to $+\alpha\omega$. To the right and left of this cut it has the forms $\mathcal{D}_1(k)$ and $\mathcal{D}_2(k)$, respectively.

Eq.(11) has the Fourier inversion

$$n_1(x) = \frac{1}{2\pi}\left\{\int_{-\infty}^{0}\frac{\mathcal{D}_2(k)}{D_2(k)}e^{ikx}dk + \int_{0}^{\infty}\frac{\mathcal{D}_1(k)}{D_1(k)}e^{ikx}dk\right\} \tag{12}$$

For $x>0^+$ this may be evaluated over the Fig. 2 contour. The integral at infinity makes no contribution. From the pole at k_1 and the branch cut we obtain

$$n_1(x>0^+) = \frac{i\gamma\mathcal{D}_1(k_1)e^{ik_1x}}{\partial D_1/\partial k\,|_{k_1}} + \frac{1}{2\pi}\int_0^{i\infty}\left[\frac{\mathcal{D}_1}{D_1} - \frac{\mathcal{D}_2}{D_2}\right]e^{ikx}dk \tag{13}$$

The coefficient γ is unity if k_1 resides to the right of the cut and zero otherwise. For arbitrary v/ω Eq.(13) must be evaluated numerically, as in Ref. 1. But for $v/\omega>>1$ or $|\omega_0/a_0|x>>1$ asymptotic descriptions can be found.

The integrals

$$f_+(k) = \frac{-i}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)dt}{t-k} \equiv [f(t)]_+, \quad f_-(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)dt}{t-k} \equiv [f(t)]_- \tag{14}$$

will separate[3] f(k) into parts which are non-analytic in only the upper or lower half-plane provided that f(k) is analytic on a real axis strip, and that $f(k+i\infty)\sim k^{-p}$, $p>0$ for k in this strip. The contour for $[\ ]_+$ passes along the real t axis and above the pole at $t = k$, k real. The $[\ ]_-$ contour passes below this pole. Then, $f(k) = [f(t)]_+ + [f(t)]_-$.

Application of Eq.(14) to Eq.(11) yields

$$n_+ = \left[\frac{\delta(t,\sigma_0)}{D(t)}\right]_+ = \frac{\delta(k_1)}{\partial D/\partial k_1 (k-k_1)} - \frac{1}{2\pi i}\int_0^{\infty}\left[\frac{\delta_1}{D_1} - \frac{\delta_2}{D_2}\right]\frac{dt}{(t-k)} \tag{15}$$

where $[\ ]_+$ has been closed over a Fig. 2 type contour. Use of $[\ ]_-$ gives n_-. Multiplication of Eq.(9) by $D(k)$ and use of $[\ ]_+$ for arbitrary σ gives

$$S_+(k) = -\left[D n_- - \mathcal{M}(\sigma_0)\right]_+ , \quad S_-(k) = \left[D n_+ - \mathcal{M}(\sigma_0)\right]_- \tag{16}$$

and also $\Delta S_- = \Delta D n_+ - \Delta \mathcal{M}$, $\mathrm{Im}(k)<0$. From this last result and Eq.(10)

$$g_1(u,\sigma^+) = \left\{\frac{\pi}{u}\left(1 + \frac{\omega_0 u^2}{\omega_0 a_0^2}\right) n_+(k = \omega_0/u) + \frac{2\pi u}{\omega_0 a_0^2} n_+ v\right\} g_0(u), \quad u<0 \tag{17}$$

The transform solution is completed by using n_+ for specular reflection in Eq.(16). Inversion of Eq.(15) for $x>0$ duplicates the specular density disturbance predicted by Eq.(13).

B. Diffuse Reflection ($\sigma = \sigma_0 = 0$)

In this case the expressions in Eq.(9) decouple and Eq.(16) becomes ($\sigma_1 = 0$)

$$n_+(k) = \frac{\mathcal{M}(k,\sigma_0) - S_-(0)\, I_+(k) + S_-(k)}{D(k)} \tag{18}$$

which may be solved by the Wiener-Hopf technique. In the real axis strip of width $2\pi/c$ $D(k+\tau_0)=1$. Therefore from the standard factorization formulas

$$D^{\pm}(k) = \exp\left\{\left[\ln D(t)\right]_{\pm}\right\} \tag{19}$$

The $[\ ]_{\pm}$ operators are defined in Eq.(14). The function

$D^+(k)$ can be zero and/or non-analytic in only $\mathrm{Im}(k)>0$. All zeros and non-analytic character of $D^-(k)$ are in $\mathrm{Im}(k)>0$. Clearly, $D(k) = D^+(k)D^-(k)$.

Write Eq.(18) as

$$D^+(k)\, n_+(k) = \frac{\mathcal{M}(k,\sigma_0) - S_-(0)\, I_+(k)}{D^-(k)} + \frac{S_-(k)}{D^-(k)} \tag{20}$$

From Eq.(8) $\mathcal{M}(k,\sigma_0)$, $I_+(k)$ and $S_-(k)$ all decay as $1/k$ for $k\to i\infty$. By assumption $n_+(k)$ has similar asymptotic behavior. Application of $[\ ]_+$ to Eq.(20), therefore yields

$$n_+(k) = \frac{1}{D^+(k)}\left[\frac{\mathcal{M}(t,\sigma_0) - S_-(0)\, I_+(t)}{D^-(t)}\right]_+ = \frac{D^-(k)\, C_+(k)}{D(k)} \tag{21a}$$

$$S_-(k) = -D^-(k)\left[\frac{\mathcal{M}(t,\sigma_0) - S_-(0)\, I_+(t)}{D^-(t)}\right]_- = -D^-(k)\, C_-(k) \tag{21b}$$

Eqs.(21a), (21b) define $C_\pm(k)$. Putting $k = 0$ in Eq.(21b), we complete the transform solution with

$$S_-(0) = \frac{\left[\dfrac{\mathcal{M}(t,\sigma_0)}{D^-(t)}\right]_-\Bigg|_{t=0}}{\left[\dfrac{I_+(t)}{D^-(t)}\right]_+\Bigg|_{t=0}} \quad \text{where} \quad \left[\frac{I_+(t)}{D^-(t)}\right]_-\Bigg|_0 = \frac{1}{D^-(0)} - \left[\frac{I_+(t)}{D^-(t)}\right]_+\Bigg|_0 . \tag{22}$$

The exact density disturbance for diffuse reflection is then given by Fourier inversion of Eq.(21a) over the Fig. 2 contour

$$n_1(x>0^+) = \frac{i\gamma D^-(k_1)\, C_+(k_1)\, e^{ik_1x}}{\partial D_1/\partial k\big|_{k_1}} + \frac{1}{2\pi}\int_0^{i\infty} D^-\left[\frac{C_{+1}}{D_1} - \frac{C_{+2}}{D_2}\right] e^{ikx}\, dk \tag{23}$$

An involved numerical calculation is needed to compute $n_1(x)$ exactly for arbitrary v/α. However, asymptotic descriptions of the solution are possible, as we shall see. Note that the use of n_+ from Eq.(21a) in Eq.(17) gives the incident diffuse distribution.

C. Asymptotic Treatment (arbitrary σ)

Consider the sum $n_+(k)+\sigma n_-(k) \equiv N(k)/D(k)$. Since $n_-(k)$ makes no $x>0^+$ contribution, inversion over the Fig. 2 contour must give

$$n_+(x>0^+) = \frac{iVM(k)e^{ikx}}{\partial D_1/\partial k}\Big|_{k_s} + \frac{1}{2\pi}\int_0^{\infty}\left[\frac{M_1}{D_1} - \frac{M_2}{D_2}\right]e^{ikx}dk \tag{24}$$

From Eq. (7) we see that $M(k) = Q(k,0^+)+\sigma Q(k,0^-)$ is formally independent of $S_+(k)$. Thus, the integrand in Eq. (24) can be rearranged to

$$\left[\frac{M_1(k)D - D_1(k)\left(\Delta^0 n + \sigma\Delta^0 n - (1-\sigma)S_-(0)\Delta I_+\right)}{D_1(k)\,D_2(k)}\right]e^{ikx} \tag{25}$$

The jumps ΔD, ΔI_+ etc. are readily computed from the Eq. (5) and (6) definitions. Each contains $g_0(\omega_0/k)$, so that the overall integrand has the exponential factor

$$\exp\left\{\frac{-\omega_0^2}{2a_0^2k^2} + ikx\right\}$$

In Ref. 1, we showed that the significant saddle point for this factor is at $k_s = [\omega_0^2/(a_0^2x)]^{1/3}e^{i\theta}$, $\theta = \pi/2-2/3$ arctan (ω/ν) with θ measured from the positive Re(k) axis. Thus, following Ref. 1, our integral can be approximated by the method of steepest descent. The exponential factors in the integrand are evaluated at k_s and brought outside the integral sign. Since as $x\to\infty$, $k_s\to 0$, we compute $D_1(k_s)$ and $M_1(k_s)$ at $k = 0$ for the leading term in our asymptotic estimate. But, $M_1(0,\sigma) = \sinh(0)+\cosh(0)$ which is fully known. Further, we find that $S_-(0)\Delta I(k_s)$ may be dropped in comparison to $M_1(0)\Delta D(k_s)$ as $k_s\to 0$. The leading asymptotic branch cut contribution, $F_c(x)$, is, therefore, independent of S_+ and is

$$B_c(x) = \frac{(1+\sigma)}{\sqrt{3}}\frac{n_0V}{a_0}R^2\tau^{1/3}\exp\left\{-\tfrac{3}{2}\tau^{2/3}e^{i\phi}\right\}e^{i\phi/2}$$

$$\tau = \frac{\omega|R|x}{a_0}, \quad R = 1+i\nu/\omega, \quad \phi = -\tfrac{2}{3}\arctan(\omega/\nu) \tag{26}$$

An examination of $M(k,\sigma)$ which we omit shows that for $\nu/\omega \gg 1$ and arbitrary σ, $M(k_1) = 2\,{}^{\circ}m(0,\sigma_0)$. Evaluating $\partial D/\partial k$ asymptotically, as in Ref. 1, and computing ${}^{\circ}m(0,\sigma_0)$ from Eq.(8), we obtain for $|\omega_0/a_0|x \gg 1$ the asymptotic, continuum result

$$n_1(x>0^+) = \frac{n_0 \gamma}{a_0} e^{ik_1 x} + B_c(x) \quad , \; k_1 = \frac{\omega}{a_0}\left(1 + \frac{2i\omega}{\nu}\right) \tag{27}$$

The General Solution

Eq.(9) can be rearranged to the vector form

$$\begin{pmatrix} n_+ \\ S_+ \end{pmatrix} = \bar{\bar{G}}(k) \begin{pmatrix} n_- \\ S_- \end{pmatrix} + \begin{pmatrix} \dfrac{m + \sigma n - (1-\sigma) S_-(0)(I_+ - \sigma I_-)}{D(k)} \\ n + (1-\sigma) S_-(0) I_- \end{pmatrix}$$

where

$$\bar{\bar{G}}(k) = \begin{pmatrix} -\sigma & \dfrac{(1-\sigma^2)}{D(k)} \\ -D(k) & -\sigma \end{pmatrix} \tag{28}$$

The general theory of simultaneous Wiener-Hopf equations[3,7] tells us that, if $\bar{\bar{G}}(k)$ can be factored into two non-singular matrices $\bar{\bar{G}}(k) = \bar{\bar{G}}^+(k)\bar{\bar{G}}^-(k)$, with $\bar{\bar{G}}^+(k)$ and $\bar{\bar{G}}^-(k)$ non-analytic in $\mathrm{Im}(k)>0$ and $\mathrm{Im}(k)<0$, respectively, then Eq.(28) can be solved by methods analogous to those yielding our earlier diffuse solution. Now for $\sigma = 0$ and $\sigma = 1$ acceptable factorizations are

$$\bar{\bar{G}}(k,0) = \begin{pmatrix} 0 & 1/D^+ \\ -D^+ & 0 \end{pmatrix} \begin{pmatrix} D^- & 0 \\ 0 & 1/D^- \end{pmatrix} = \bar{\bar{G}}^+(k,0)\,\bar{\bar{G}}^-(k,0)$$

$$\bar{\bar{G}}(k,1) = \begin{pmatrix} -1 & 0 \\ -1+\varepsilon_+ & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varepsilon_- & 1 \end{pmatrix} = \bar{\bar{G}}^+(k,1)\,\bar{\bar{G}}^-(k,1) \tag{29}$$

with $D(k) = 1-\varepsilon(k)$ and $\varepsilon_\pm = [\varepsilon(t)]_\pm$. Use of these diffuse and specular matrices yields our earlier results. For

arbitrary σ, however, factorization by inspection, linear vector space transformations, and matrix exponentiation all seem to fail. The usual alternative is to reduce Eq.(28) to a set of coupled Fredholm equations, which can then be solved by iteration. Cercignani[8] solves the Krammers problem in this way. We prefer, however, to develop iterative schemes based directly on the Eq.(9) transform equations. This corresponds to the iterative factorization of $\bar{G}(\zeta)$, as suggested in Muskhelishvili's latest edition[9].

A. Specular Iteration

Assume the expansions $n_\pm = \Sigma\alpha^q n_\pm^{(q)}$, $S_\pm = \Sigma\alpha^q S_\pm^{(q)}$ and $g_j(u,0\pm) = \Sigma\alpha^q g_j^{(q)}(u,0)$ with $\alpha = 1-\sigma$. Substitute these series in Eq.(9). Since $m(k,\sigma_0)$ and $\mathcal{N}(k,\sigma_0)$ are known exactly, they are left unexpanded. To zero order in α we extract and add the equations for $n_+^{(0)}$ and $n_-^{(0)}$. There results

$$n^{(0)} = \frac{\mathcal{N}(k,\sigma_0) + \mathcal{N}(k,\sigma_0)}{D(k)} = \frac{\mathcal{S}^{(0)}(k,\sigma_0)}{D(k)} \tag{30}$$

But, with $n = n^{(0)}$ this is identical to Eq.(11) except that σ_0 is no longer necessarily unity. Thus, Eqs.(11) through (17) give $n_\pm^{(0)}$, $S_\pm^{(0)}$, and $g_j^{(0)}(u,0^\pm)$ when σ_0 is properly chosen.

Physically, this $q = 0$ solution describes the sound disturbances induced by a piston which reflects $g_{j\,\mathrm{inc}}(u)$ with the correct fractional accommodation, while the returned perturbation, $g_j^{(0)}(u\lessgtr 0,0)$, is specularly reflected.

For $q \geq 1$ writing

$$n^{(q)}(k) = \left\{\frac{-S_-^{(q-1)}I_+ - S_+^{(q-1)}(0)I_- - S_+^{(q-1)} + S_-^{(q-1)}}{D(k)}\right\} = \frac{\mathcal{S}^{(q)}(k,\sigma_0)}{D(k)} \tag{31a}$$

$$S_\pm^{(q)} = \mp\left[D\,n_\pm^{(q)}\right]_\pm,\quad g_j^{(q)}(u\lessgtr 0,0) = \pm\frac{\sigma_0}{2}\left(1+\frac{\omega_j^2}{\omega_0^2}\right)n_\pm^{(q)}\left(\frac{\omega_j}{u}\right) \tag{31b}$$

The qth iterate describes the additional disturbance, when a fraction α of the incident distribution $g_1^{(q-1)}(u\leq 0, 0^+)$ is converted to Maxwellian form at the pistons and reemitted. From this new emission follows the specularly reflected return distribution, $g_1^{(q)}(u\leq 0, 0^+)$.

In the free molecular regime the returned distributions are $O(\nu/\omega)$ smaller than that those originally emitted, as may be verified with Eq.(31). Thus, for $\nu/\omega \ll 1$ and $\alpha = 0 \to 1$ our iterative procedure will rapidly converge. In the continuum regime the incident distributions differ from Maxwellian form only to order ω/ν. But, at each iterative stage only this difference is reemitted, so that rapid convergence is also expected for $\omega/\nu \ll 1$.

Finally, for a small enough to insure uniform convergence for real k, we invert the transform series term by term over the Fig. 2 contour, obtaining the exact formal solution.

$$n_1(x>0^+) = \sum_{q=0}^{\infty} \alpha^q \left\{ \frac{i\gamma \Delta^{(q)}(k_1)}{\partial D_1/\partial k_1}\Big|_{k_1} e^{ik_1 x} + \frac{1}{2\pi}\int_0^{\infty} \left[\frac{\Delta_1^{(q)}}{D_1} - \frac{\Delta_2^{(q)}}{D_2}\right] e^{ikx}\, dk \right\} \tag{32}$$

B. Diffuse Iteration

Let $n_\pm = \Sigma\sigma^j n_\pm^{(j)}$, $S_\pm = \Sigma\sigma^j S_\pm^{(j)}$ and $g_1(u, 0^+) = \Sigma\sigma^j g_1^{(j)}$. We substitute these series into Eq.(9). The $j = 0$ equation is simply Eq.(18) with $n_+ \to n_+^{(0)}$ and $S_- \to S_-^{(0)}$. The corresponding solutions are given by Eq.(21) and (22). For $j \geq 1$ we find

$$n_+^{(j)}(k) = \frac{D^-(k)\, C_+^{(j)}(k)}{D(k)}, \quad S_-^{(j)}(k) = -D^-(k)\, C_-^{(j)}(k) \tag{33}$$

$$C_\pm^{(j)} = \left[\frac{\left(S_-^{(j-1)}(0) - S_-^{(j)}(0)\right) I_+ + S_+^{(j-1)}}{D^-(k)}\right]_{+0}, \quad \left(S_-^{(j-1)}(0) - S_-^{(j)}(0)\right) = -\frac{\left[\frac{S_+^{(j-1)}}{D^-}\right]_{+0}}{\left[\frac{I_+}{D^-}\right]_{+0}}$$

In this case the $j = 0$ result gives the disturbances when $u_0 \delta_0(\omega)$ is fractionally accommodated and $g_1^{(0)}(u_0<0, \omega)$ is diffusely reflected. Subsequent iterates give the additional disturbances, if a fraction σ of the incident distribution to each order is actually specularly reflected. Again, the "weak" incident distributions when $v/m \ll 1$, and the small deviations of these distributions from Maxwellian form when $v/m \gg 1$ assures rapid convergence in these regimes. We conclude that for σ suitably small, an alternate representation for the solution is

$$M_1(x>0^+) = \sum_{j=0}^{\infty} \sigma^j \left\{ \frac{i\gamma D(k) C_1^{(j)}(k)}{\partial D_1/\partial k} e^{ikx} + \frac{1}{2\pi} \int_0^{\infty} D \left[\frac{C_{1,1}^{(j)}}{D_1} - \frac{C_{1,2}^{(j)}}{D_2} \right] e^{ikx} \, dk \right\} \tag{34}$$

Conclusion

Forced sound propagation with fractional accommodation has been examined with the aid of half range Fourier transforms applied to the isothermal BGK equation. The resultant system of coupled Wiener-Hopf equations has been solved exactly for specular and diffuse reflection, while for intermediate $\sigma = 1-\alpha$, solutions are obtained by iteration. The general solution consists of a possible exponential mode plus a free-stressing mode of simple asymptotic form. The relative magnitudes of these modes depends only weakly on σ in the regimes examined.

If the energy-conserving BGK model had been employed, then, in addition to the unknowns n_1 and S_2, it would be necessary to determine two perturbation temperature transforms, $T_2(k)$ and two heat flux transforms $Q_1(k)$, with $Q_1(0) = -Q_2(0) = (-1/2m_0)\int_0^{\infty} u(u^2-v^2+w^2) f_1(u,v,w) d^3v$. An additional exponential diffusion mode would be present in $x \geq 0^+$, and the continuum sound speed would assume its adiabatic value γ^2. At least for a nearly unity, the improved free stressing estimate would be stronger by a factor $(R+2)^2/9$, as seen by comparison with Ref. 1. The general character of sound predicted by the new model should agree, however, with that found for the isothermal case.

Acknowledgement

The author expresses appreciation to Mrs. Linda Kline for her aid in preparing the manuscript.

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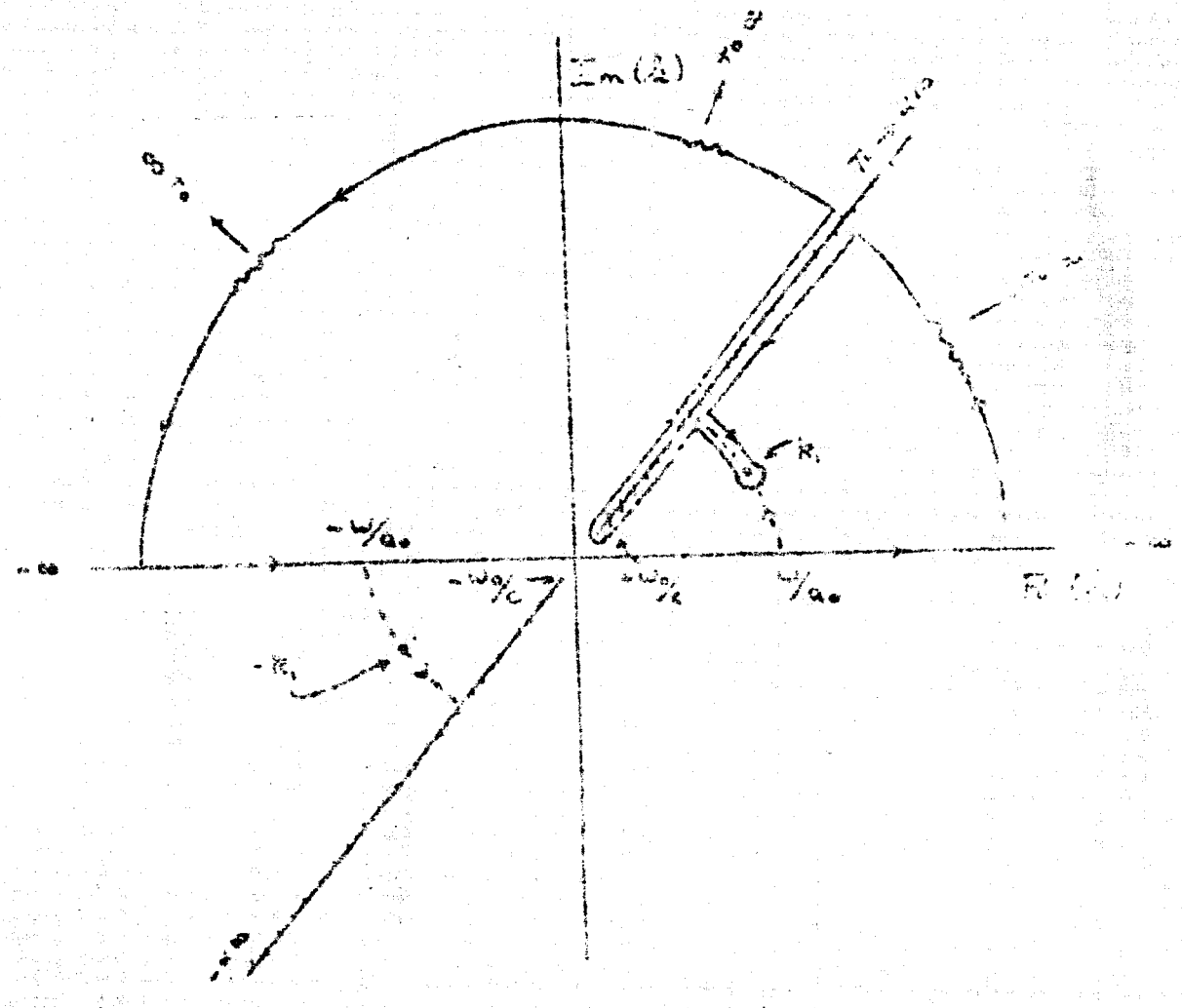

Fig 2

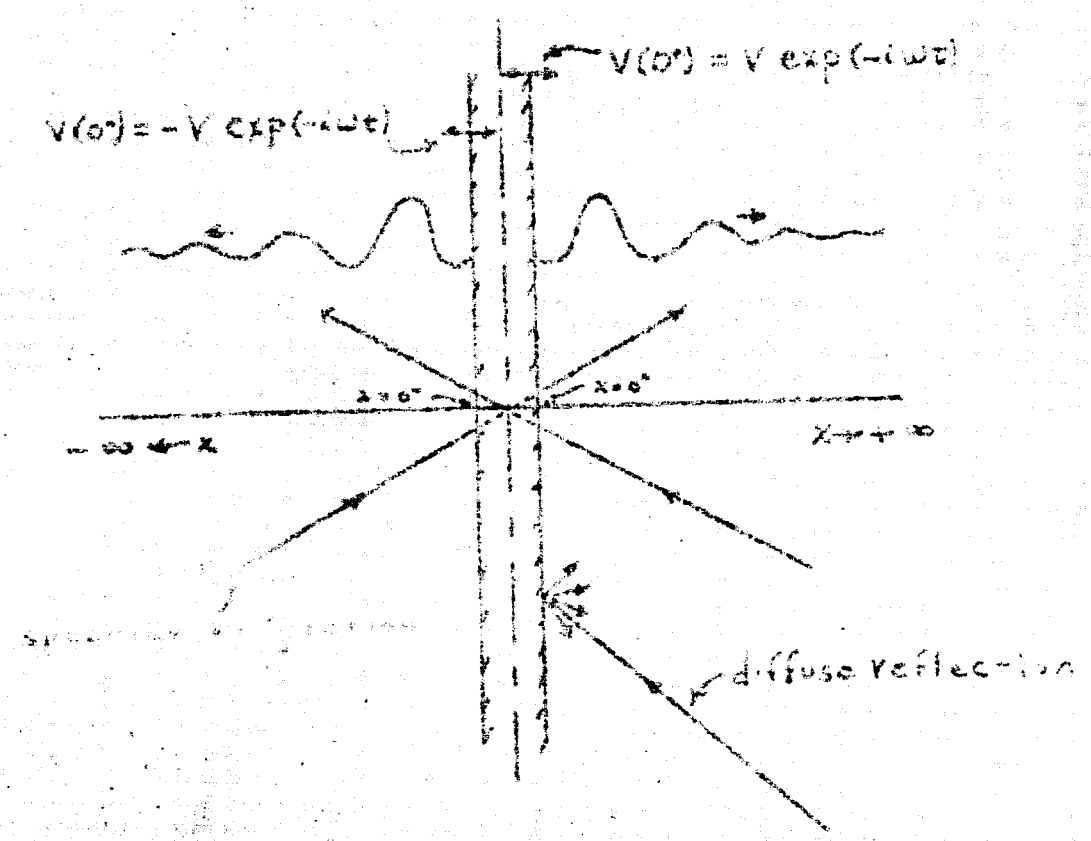

Fig 1

RAREFIED GAS DYNAMICS

Fig. 1 Symmetric sound disturbances initiated at $x = 3^{\pm}$.

Fig. 2 Integral contour for transform inversion.