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NOTE ON THE FLUIDDYNAMIC JUMP CONDITIONS AT A MOVING INTERFACE

By Robert C. Coston

NASA
Langley Research Center

SUMMARY

A new derivation of the fluiddynamic jump conditions at a moving interface is presented. Comparison with an earlier derivation shows the new method to be more general and more direct.

TEXT

There are a number of approaches to deriving the fluiddynamic jump conditions at a moving and deforming surface of discontinuity. Thomas (ref. 1, pp. 180-185) starts from the conservation equations essentially in the form

$$\frac{d}{dt}\int_V \rho \, dV = 0$$

$$\frac{d}{dt}\int_V \rho \vec{v} \, dV = \oint_S \overleftrightarrow{T} \cdot \hat{\nu} \, dS + \int_V \rho \vec{X} \, dV$$

$$\frac{d}{dt}\int_V \rho E \, dV = \int_V \overleftrightarrow{T} : \overleftrightarrow{D} \, dV - \oint_S \vec{q} \cdot \hat{\nu} \, dS \qquad (1)$$

with

E	internal energy density
$\overleftrightarrow{T}$	mechanical stress tensor
$\overleftrightarrow{D}$	deformation tensor $\left(D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})\right)$
$\vec{q}$	heat flux vector

and evaluates the total time derivatives directly for a volume V which momentarily intersects a discontinuity surface Σ in the limit as the perpendicular dimension of V vanishes. In general the limiting values of the total time derivatives are non-zero, since the regions of integration, V and S, in equations (1) move with the fluid. In order to evaluate them, Thomas has presented a modified form of

the transport theorem

$$\frac{d}{dt}\int_V f(\vec{x},t)\,dV = \int_V \frac{\partial f}{\partial t}\,dV + \oint_S f\vec{v}\cdot\hat{\nu}\,dS - \int_\Sigma [f]\,N\,dS \qquad (2)$$

where N is the speed of displacement of surface Σ and where both $f(\vec{x},t)$ and $v(\vec{x},t)$ may be discontinuous across Σ.

As written, equation (2) is, perhaps, misleading, since it tends to imply that

$$\int_V \frac{\partial f}{\partial t}\,dV = \int_{V_1} \frac{\partial f}{\partial t}\,dV + \int_{V_2} \frac{\partial f}{\partial t}\,dV$$

where V_1 and V_2 are the two parts of volume V separated by surface Σ. But in general $\frac{\partial f}{\partial t}$ is infinite on Σ due to the discontinuity in $f(\vec{x},t)$ and to the motion of Σ; hence a more explicit form of equation (2) is preferable:

$$\frac{d}{dt}\int_V f(\vec{x},t)\,dV = \int_{V_1} \frac{\partial f}{\partial t}\,dV + \int_{V_2} \frac{\partial f}{\partial t}\,dV + \oint_S f\vec{v}\cdot\hat{\nu}\,dS - \int_\Sigma [f]\,N\,dS \qquad (3)$$

A comparison of this equation with the transport theorem

$$\frac{d}{dt}\int_V f(\vec{x},t)\,dV = \int_V \frac{\partial f}{\partial t}\,dV + \oint_S f\vec{v}\cdot\hat{\nu}\,dS \qquad (4)$$

shows at once that the contribution from the singularity in $\frac{\partial f}{\partial t}$ on Σ has been deleted from the volume integral and appended in the integral over Σ.

Equation (3) is, however, not as general as the transport theorem (4), since its derivation assumed that

$$\frac{d}{dt}\int_V f(\vec{x},t)\,dV = \frac{d}{dt}\int_{V_1} f(\vec{x},t)\,dV + \frac{d}{dt}\int_{V_2} f(\vec{x},t)\,dV$$

which rules out the possibility of $f(\vec{x},t)$ itself being infinite on surface Σ. Such fields - though of minor importance at fluid shock fronts - are essential at surfaces of electromagnetic discontinuity, where surface charge density and surface current density frequently occur. It would appear impossible to generalize equation (3) to include this case for the following reason: We may attempt

to write

$$\frac{d}{dt}\int_V f(\vec{x},t)\,dV = \frac{d}{dt}\int_{V_1} f\,dV + \frac{d}{dt}\int_{V_2} f\,dV + \frac{d}{dt}\int_{\Sigma} F\,dS$$

with $F = \lim_{\Delta n \to 0}\int_{\Delta n} f\,dn$ and Δn a line element normal to and crossing Σ.

But $\frac{d}{dt}\int_{\Sigma} F\,dS$ is undefined when velocity components tangential to Σ are discontinuous, for the rate at which the area of integration on Σ stretches is then double valued.

An alternate approach to the jump conditions which does not encounter this difficulty starts from the recognition that equations (1) are a more restricted form of the conservation laws than

$$\int_V \frac{\partial \rho}{\partial t}\,dV + \oint_S \rho\vec{v}\cdot\hat{\nu}\,dS = 0$$

$$\int_V \frac{\partial}{\partial t}\rho\vec{v}\,dV + \oint_S \rho\vec{v}(\vec{v}\cdot\hat{\nu})\,dS = \oint_S \overleftrightarrow{T}\cdot\hat{\nu}\,dS + \int_V \rho\vec{X}\,dV$$

$$\int_V \frac{\partial}{\partial t}\rho E\,dV + \oint_S \rho E(\vec{v}\cdot\hat{\nu})\,dS = \int_V \overleftrightarrow{T} : \overleftrightarrow{D}\,dV - \oint_S \vec{q}\cdot\hat{\nu}\,dS \qquad (5)$$

The integrations in set (5) are taken with t fixed, of course, and the regions of integration, V and S, may therefore be regarded as moving and deforming in an arbitrary manner, while in set (1) V and S are restricted to move with the fluid. An alternate unrestricted form of the conservation equations is obtained by applying the transport theorem (4) (with $\vec{m}$ substituted for $\vec{v}$) to eliminate the partial time derivatives in equations (5)

$$\frac{d}{dt}\int_V \rho\,dV + \oint_S \rho(\vec{v}-\vec{m})\cdot\hat{\nu}\,dS = 0$$

$$\frac{d}{dt}\int_V \rho\vec{v}\,dV + \oint_S \rho\vec{v}(\vec{v}-\vec{u})\cdot\hat{\nu}\,dS = \oint_S \overleftrightarrow{T}\cdot\hat{\nu}\,dS + \int_V \rho\vec{X}\,dV$$

$$\frac{d}{dt}\int_V \rho E\,dV + \oint_S \rho E(\vec{v}-\vec{u})\cdot\hat{\nu}\,dS = \int_V \overleftrightarrow{T}:\overleftrightarrow{D}\,dV - \oint_S \vec{q}\cdot\hat{\nu}\,dS \qquad (6)$$

Here $\vec{u} = \vec{u}(\vec{x},t)$ designates the arbitrary abstract velocity field which describes the motion of the regions of integration V and S, while $\vec{v}(\vec{x},t)$ describes the motion of the fluid elements. The abstract velocity $\vec{u}$ and the fluid velocity $\vec{v}$ are (in concept) unrelated. Should the regions of integration be chosen to move with the fluid ($\vec{u} = \vec{v}$), the restricted form (1) is obtained.

Equations (6) may now be used to obtain the jump conditions directly by the usual limiting process. Since the abstract velocity field $\vec{u}$ is arbitrary, we may require that

$$\begin{aligned} &\vec{u}(\vec{x},t) \text{ is continuous across surface } \Sigma \\ &\vec{u}\cdot\hat{n} = N \text{ on } \Sigma \end{aligned} \qquad (7)$$

where $\hat{n}$ is the unit normal on Σ and N the speed of displacement of Σ in the sense of $\hat{n}$. It follows that the total time derivatives in set (6) vanish in the limit as the dimension of V normal to Σ vanishes, provided ρ, $\vec{v}$, and E are bounded on Σ; and if $\overleftrightarrow{T}$, $\vec{q}$, and $\vec{X}$ are also bounded, we obtain essentially the same jump conditions found by Thomas

$$\begin{aligned} &\left[\rho\vec{v}\right]\cdot\hat{n} - N\left[\rho\right] = 0 \\ &\left[(\rho\vec{v})\vec{v}\right]\cdot\hat{n} - N\left[\rho\vec{v}\right] = \left[\overleftrightarrow{T}\right]\cdot\hat{n} \\ &\left[\rho E\vec{v}\right]\cdot\hat{n} - N\left[\rho E\right] = \left[\vec{v}\right]\cdot\left\langle\overleftrightarrow{T}\right\rangle\cdot\hat{n} - \left[\vec{q}\right]\cdot\hat{n} \end{aligned} \qquad (8)$$

where $\left[f\right] = f^+ - f^-$ and $\left\langle f\right\rangle = \frac{1}{2}(f^+ + f^-)$ with $\pm$ designation relative to the unit normal $\hat{n}$.

This method is also applicable to a field $f(\vec{x},t)$ which is infinite (though integrable) on the interface, since $\vec{\omega}$ is taken to be continuous across Σ even though tangential components of $\vec{v}$ may be discontinuous. Therefore, subject to appropriate assumptions, we may set

$$\lim_{\Delta n \to 0} \frac{d}{dt} \int_V f \, dV = \frac{d}{dt} \int_\Sigma F \, dS \tag{9}$$

with $$F = \lim_{\Delta n \to 0} \int_{\Delta n} f \, dn$$

The term $\frac{d}{dt} \int_\Sigma F \, dS$ may then be recast in a form suitable for jump conditions by the appropriate kinematic relation for surface integrals, as given by Truesdell and Toupin (ref. 2) pp. 346-347. An example of this procedure applied to the jump condition on electric current, with surface charge and surface current prescribed on the boundary, is given in reference 3.

REFERENCES

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