A DEFINITION OF THE LAPLACE TRANSFORM*

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In this note, it is shown that one can define the Laplace transform of a function φ having suitable growth ($|\varphi(x)| < Me^{\alpha x}$) and continuity properties by means of the symbolic relationship

$$(1) \qquad \int_0^\infty e^{-px}\varphi(x)dx = p^{-1} \lim_{r \downarrow 0} \{e^{\frac{1}{4p}\Delta_2} \cdot \varphi(r^2)\}$$

where Δ_2 denotes the Laplacian operator $D_r^2 + \frac{1}{r} D_r$. In cases when p is large and $\varphi(r^2)$ satisfies the Goursat conditions, a meaning can be attached to the bracketed term by summing the series

$$(2) \qquad \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{(4p)^j} \Delta_2^j \cdot \varphi(r^2) \quad .$$

The following short table gives some elementary examples of such functions $\varphi(x)$. Included are (i) the function $\varphi(x)$, (ii) the corresponding sum (2), and (iii) the limiting expression in the right member of (1).

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$\varphi(x)$	$e^{t\Delta_2}\cdot\varphi(r^2)$	$p^{-1}\lim_{r\downarrow 0} e^{\frac{1}{4p}\Delta_2}\cdot\varphi(r^2)$
x^n	$n!(4t)^n L_n(-r^2/4t)$	$n!/p^{n+1}$
$L_n(x)$	$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(4t)^k L_k(-r^2/4t)$	$\frac{1}{p}(1-\frac{1}{p})^n$
e^{ax}	$(1-4at)^{-1}e^{ar^2/(1-4at)}$	$(p-a)^{-1}$
$J_0(2\sqrt{ax})$	$e^{-4at}J_0(2\sqrt{ar})$	$p^{-1}e^{-a/p}$

In this, $L_n(x)$ denotes the simple Laguerre polynomial of degree n with $L_n(0) = 1$. For the last entry, $\Delta_2^j J_0(2\sqrt{ar}) = (-4a)^j J_0(2\sqrt{ar})$ so that (2) can be summed.

The validity of (1) follows directly from the Poisson integral representation for a solution of the initial-value heat problem

$$(3)\qquad \frac{\partial}{\partial t}u(x_1,x_2,t) = (D_1^2 + D_2^2)u(x_1,x_2,t);\ u(x_1,x_2,0) = \varphi(x_1^2 + x_2^2).$$

Here $D_i \equiv \frac{\partial}{\partial x_i}$. The solution of this problem is defined (for $1-4at > 0$ and $t > 0$) by (see [1], p. 171 for the definition of e^{tD^2})

$$
(4)\quad \begin{cases}
u(x_1,x_2,t) = e^{t(D_1^2+D_2^2)} \cdot \varphi(x_1^2+x_2^2) \\
= \dfrac{1}{4\pi t}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi^2+\eta^2)e^{-[(x_1-\xi)^2+(x_2-\eta)^2]/4t}\,d\xi d\eta \\
= (\pi t)^{-1}e^{-(x_1^2+x_2^2)/4t}\displaystyle\int_0^{\infty}\int_0^{\infty}\varphi(\xi^2+\eta^2)e^{-(\xi^2+\eta^2)/4t}\cosh\left(\frac{x_1\xi}{2t}\right)\cosh\left(\frac{x_2\eta}{2t}\right)d\xi d\eta,
\end{cases}
$$

this last step following from the symmetry of φ in its arguments. Upon letting x_1 and x_2 tend to zero in this, we get

$$
(5)\quad \begin{cases}
u(0,0,t) = (\pi t)^{-1}\displaystyle\int_0^{\infty}\int_0^{\infty}\varphi(\xi^2+\eta^2)e^{-(\xi^2+\eta^2)/4t}\,d\xi d\eta \\
= (\pi t)^{-1}\displaystyle\int_0^{\infty}\int_0^{\pi/2}\varphi(\rho^2)e^{-\rho^2/4t}\,\rho d\rho d\theta \\
= (2t)^{-1}\displaystyle\int_0^{\infty}\varphi(\rho^2)e^{-\rho^2/4t}\,\rho d\rho = (4t)^{-1}\int_0^{\infty}\varphi(x)e^{-x/4t}\,dx
\end{cases}
$$

The result (1) follows from this by choosing $t = \frac{1}{4p}$, $r^2 = x_1^2+x_2^2$, and noting that $D_1^2+D_2^2 = D_r^2 + \frac{1}{r}D_r$. The validity of the above limiting operation follows from the fact that $|\cosh a - 1| \le a^2 e^{|a|}$, the growth properties of φ, and the inequality

$$
|u(x_1,x_2,t) - u(0,0,t)| \le |u(x_1,x_2,t) - e^{-x_1^2/4t}u(0,x_2,t)|
$$
$$
+ e^{-x_1^2/4t}|u(0,x_2,t) - e^{-x_2^2/4t}u(0,0,t)| + |e^{-(x_1^2+x_2^2)/4t} - 1|\cdot|u(0,0,t)|.
$$

1. I.I. Hirschman and D.V. Widder, The Convolution Transform, Princeton Univ. Press, 1955.

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