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ON THE SOLUTION OF THE GENERAL INHOMOGENEOUS LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER USING LIE SERIES REPRESENTATION [1)]

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1. Introduction

In the last ten years, LIE series /1/ have proved to be a powerful tool for solving differential equations, thoroughly comparable and sometimes superior to the rest of the methods applied in this field.- In the present work, we are going to present two alternative ways of solving the general inhomogeneous linear differential equation of second order, both of them based on Lie formalism and to be used for numerical calculations on a computer; one of them is essentially an extension of a method for solving the general homogeneous linear differential equation of second order proposed by Groebner /2/, while the other one makes use of recurrence formulas connecting different powers of the Lie operator D.- In order to stress the practical significance of these methods, we are going to present two physical applications, one from rigid body mechanics and connected with satellite stabilization, the other one from electricity implying Mathieu functions.

2. Groebner's Method

The equation in question reads:

$$Y''(t) - f_1(t)\,Y'(t) - f_2(t)\,Y(t) = f_3(t) \qquad (1)$$

where we suppose $f_i(t)$ to be regular in the considered domain.

The following system is equivalent to (1).-

$$Y_0' = t' = 1$$
$$Y = Y_1$$
$$Y_1' = Y_2$$
$$Y_2' = f_3 + f_2 Y_1 + f_1 Y_2$$

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which is formally solved by /2/:

(3)

in our case, D is given by

(4)

The small letters are to indicate that after application of the operator the Y-variables have to be replaced by their initial values. According to Groebner /2/, we split the operator in the following way:

(5)

with

(5a)

and

(5b)

where D_1 will produce the main part and D_2 the correction terms. In view of that, the total solution reads /4/:

(6)

where the symbol $\bar{a}$ added after the bracket indicates the fact that after application of the D-operators y_1, y_2 have to be replaced by $e^{tD_1}y_1$ and $e^{tD_2}y_2$, respectively.

We now turn to an evaluation of the first term at the right site of (6):

D_1 may be written in matrix form:

(7)

where

(8)

and

(9)

Applying the operator to the variable, we obtain:

(10)

where

Repeating this operation, we get:

(10a)

and

(10b)

and, generally:

(10c)

For the homogeneous case, a repeated application of D_1 results in multipying the expression by the coefficient matrix A itself, a role played by the more complicated matrix B, in our case. The main part of the solution is, therefore, given by:

$$[illegible] \tag{11}$$

using [illegible]

Now we have to calculate B^n:

$$[illegible] \tag{12}$$

Assuming the eigenvalues [illegible], we diagonalize B, i.e.:

$$[illegible] \tag{13}$$

by solving the secular equation:

$$[illegible] \tag{14}$$

or, in extenso:

$$[illegible] \tag{14a}$$

so that

$$[illegible] \tag{15}$$

T and T^{-1} are, respectively, given by:

$$[illegible] \tag{16}$$

and

$$T^{-1} = \frac{1}{\tau_2(\lambda_1-\lambda_2)} \begin{pmatrix} \tau_2, & -\lambda_2, & 0 \\ -\tau_2, & \lambda_1, & 0 \\ 0, & -\lambda_1\tau_3+\lambda_2\tau_3, & \lambda_1\tau_2-\lambda_2\tau_2 \end{pmatrix} \tag{17}$$

From (13), we have:

$$B = T\Lambda T^{-1} \tag{13a}$$

and

$$B^n = (T\Lambda T^{-1})(T\Lambda T^{-1})\ldots(T\Lambda T^{-1}) = T\Lambda^n T^{-1} \tag{13b}$$

so that we obtain for (11):

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu \begin{pmatrix} Y_2 \\ Y_1 \\ 1 \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B^{\nu T} \begin{pmatrix} Y_2 \\ Y_1 \\ 1 \end{pmatrix} = \tag{18}$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} (T^{-1})^T \Lambda^\nu T^T \begin{pmatrix} Y_2 \\ Y_1 \\ 1 \end{pmatrix} =$$

$$= (T^{-1})^T \begin{pmatrix} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \lambda_1^\nu & 0 & 0 \\ 0 & \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \lambda_2^\nu & 0 \\ 0 & 0 & \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \lambda_3^\nu \end{pmatrix} T^T \begin{pmatrix} Y_2 \\ Y_1 \\ 1 \end{pmatrix} = \tag{19}$$

$$= (T^{-1})^T \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^T \begin{pmatrix} Y_2 \\ Y_1 \\ 1 \end{pmatrix}$$

Consequently, the total solution reads:

$$\begin{pmatrix} y_2(t) \\ y_1(t) \\ 0 \end{pmatrix} = e^{tD} \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} = (T^{-1})^T \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^T \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} +$$

$$+ \sum_{\alpha=0}^{\omega} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D^\alpha \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} \right]_{\bar{a}} d\tau \tag{20}$$

where the perturbation integral can be evaluated by an iteration method according to /4/ ; its evaluation is promising insofar as it offers several possibilities of adaptation, viz., by choosing the numbers of iterations, the step size, and the break-off of the ω -summation.

3. The Method of Recurrence Formulas:

Applying D_ν times to y_1 and splitting up the operator powers we obtain:

$$D^\nu y_1 = D^{\nu-2}(D^2 y_1) = D^{\nu-2}(f_3 + f_1 y_2 + f_2 y_1) \tag{21}$$

Evidently

$$D^{\nu-2} f_3 = f_3^{(\nu-2)} \tag{22}$$

Using the well-known Leibniz rule for D-operators, we obtain for the second and third terms of (21), respectively:

$$D^{\nu-2}(f_1 y_2) = \sum_{\rho=0}^{\nu-2} \binom{\nu-2}{\rho} D^\rho f_1 D^{\nu-2-\rho} y_2 = \sum_{\rho=0}^{\nu-2} \binom{\nu-2}{\rho} D^\rho f_1 D^{\nu-1-\rho} y_1 \tag{23}$$

and

$$D^{\nu-2}(f_2 y_1) = \sum_{\rho=0}^{\nu-2} \binom{\nu-2}{\rho} D^\rho f_2 D^{\nu-2-\rho} y_1 \tag{24}$$

so that the recurrence formula for D is given by:

[illegible] (25)

With the help of this result, the total solution reads:

[illegible] (26)

a formula analogous to that given in /1/ which it is not so difficult to write a code for. It is, of course, possible to splitt known functions from the total solution, also in this case; one of the ways in which this splitting is possible is evidently equivalent to the method used by Groebner, his part $e^{tD_1}(y_2, y_1, 1)^T$ essentially being the hyperbolic (trigonometric) main term of our method.-In contrast to Groebner's method, no way of estimating the error made by breaking off the computation seems to exist for the recurrence formulas, up to now. Nevertheless, they may prove to be superior to the first way, from a physicist's point of view, owing to their easier coding and the fact that an analytic method of error estimating may be replaced by experience on the machine, for practical purposes.

4. An Example From Rigid Body Mechanics:

The equation of motion of a plane mathematical pendulum of length l and mass m in the case of a suspensory point vibrating in vertical direction according to the law cos t (a, constant, g gravitational acceleration) is given by /5/:

[illegible] (27)

if the elongation α is sufficiently small.

If this case:

$$f_1(t) = -\delta(t) \tag{29}$$

$$f_2(t) = -\frac{1}{L} + \frac{1}{L}\omega^2 \cos\omega t \tag{30}$$

the solution is given by (3):

$$Y(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu Y \tag{3a}$$

Assuming $\lambda_1 \neq \lambda_2 \neq \lambda_3$, we obtain the following eigenvalues of the matrix B:

$$\lambda_{1,2} = \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} + f_2} = \frac{\beta(t)}{2} \pm \sqrt{\frac{\beta^2(t)}{4} - \left(\frac{1}{L} - \frac{1}{L}\omega^2 \cos\omega t\right)} \tag{15a}$$

and

$$\lambda_3 = 0 \tag{15b}$$

so that the T_{ij} (the components of transformation matrix) are given by:

$$T_{11} = -\frac{\beta(t)}{2} + \sqrt{\frac{\beta^2(t)}{4} - \left(\frac{1}{L} - \frac{1}{L}\omega^2 \cos\omega t\right)}$$

$$T_{12} = -\frac{\beta(t)}{2} - \sqrt{\frac{\beta^2(t)}{4} - \left(\frac{1}{L} - \frac{1}{L}\omega^2 \cos\omega t\right)}$$

$$T_{13} = 0$$

$$T_{21} = -\left(\frac{1}{L} - \frac{1}{L}\omega^2 \cos\omega t\right) \tag{31}$$

$$T_{22} = -\left(\frac{1}{L} - \frac{1}{L}\omega^2 \cos\omega t\right)$$

$$T_{23} = 0$$
$$T_{31} = f_3$$
$$T_{32} = f_3$$
$$T_{33} = 1$$

The solution is given by (2o).

Using recurrence formulas, the solution reads:

$$Y(t) = \sum_{\nu=2}^{\infty} \frac{t^{\nu}}{\nu!} \sum_{\rho=0}^{\nu-2} \Big[f_3^{(\nu-2)} - \binom{\nu-2}{\rho} D^{\rho} \delta_{+}(t) \cdot D^{\nu-1-\rho} Y_1 + D^{\rho}\Big[-\frac{1}{L} + \frac{a}{L}\omega^2 \cos\omega t\Big] D^{\nu-2-\rho} Y_1 \Big] + Y_1 + t Y_2$$

5. An Example From Electricity:

Another problem described by the following circuital equation may be solved within our formalism:

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC(t)} = \frac{E(t)}{L}$$

i.e., a circuit containing a driving e.m.f. an inductance L in series with a capacitance C varying with time and a constant resistance R. Assuming $C(t) = C_0(1+\varepsilon \cos 2\omega_1 t) = \frac{C_0}{1-\varepsilon \cos 2\omega_1 t}$ (if ε is small enough) and writing $Q=Y$, $1/LC_0 = \omega_0^2$, $z = \omega_1 t$, $\frac{\omega_1 R}{L} = 2\varkappa$, $\bar{a} = (\omega_0/\omega_1)^2$ and $E(t)/\omega_1^2 L = f_3(z)$, we obtain the following inhomogeneous Mathieu equation:

$$Y'' + 2\varkappa Y' + (\bar{a} - 2q\cos 2z)Y = f_3$$

where $f_1 = -2^2$

$$f_2(z) = 2q\cos 2z - \bar{a}$$

(z corresponds to t in the general treatment)

The formal solution is given by:

$$Y(t) = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!} D^\nu Y$$

The eigenvalues of B are:

$$\lambda_{1,2} = -x \pm \sqrt{x^2 + 2q\cos 2z - \bar{a}}$$

$$\lambda_3 = 0$$

while the elements of the transformation matrix are given by:

$$T_{11} = -x + \sqrt{x^2 + 2q\cos 2z - \bar{a}}$$

$$T_{12} = -x - \sqrt{x^2 + 2q\cos 2z - \bar{a}}$$

$$T_{13} = 0$$

$$T_{21} = 2q\cos 2z - \bar{a}$$

$$T_{22} = 2q\cos 2z - \bar{a}$$

$$T_{23} = 0$$

$$T_{31} = f_3$$

$$T_{32} = f_3$$

$$T_{33} = 1$$

Using these results, the general solution may be written in the way given in the general discussion.

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REFERENCES

/1/ "Die Lie-Reihen und ihre Anwendungen"/The Lie Series and Their Application/ by W. Groebner, Verlag der Wissenschaften, Berlin 1960

/2/ "A Compact Method of Solving the General Homogeneous Linear Differential Equation of Second Order Using Lie Series" by W. Groebner, NASA Scient. Rep. No.3.Research Grant No.NGR-52-046-001

/3/ "On the Solution of the General Homogeneous Linear Differential Equation of Second Order Using Lie Series" by A. Schett and J. Weil, NASA Scient. Rep. No.2, Research Grant No.NGR-52-046-001, to be published in Acta Physica Austriaca.

/4/ "Untersuchungen ueber die numerische Behandlung von Anfangswertproblemen gewoehnlicher Differentialgleichungen mit Hilfe von Liereihen und Anwendungen uaf die Berechnung von Mehrkoerperproblemen"/Investigations on the Numerical Treatment of Initial Value Problems of Ordinary Differential Equations ba Means of Lie Series and Applications to the Calculation of Many Body Problems/by F. Reutter and H. Knapp, Forschungsberichte des Landes Nordrhein-Westfalen 1327/Scientific Report of Nordrhein-Westfalen/

/5/ "Mathieu- und Spheroidfunktionen"/Mathieu-and Speroid Functions/ by Meixner and Schaefke, Springer Verlag, 1954

Zusammenfassung

In der vorliegenden Arbeit wird die lineare, inhomogene, gewoehnliche Differentialgleichung 2.Ordnung mit beliebigen regulaeren Koeffizientenfunktionen mittels Liereihen geloest. Es werden zwei Wege beschritten, von denen einer auf Rekursionsformeln und der andere auf die iterative Auswertung eines "Stoerintegrals" fuehrt; ersterer ist programmiertechnisch einfacher, laeßt aber keine exakte Fehlerabschaetzung zu, waehrend fuer das Stoerintegral der Fehler angegeben werden kann.