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STATISTICAL MECHANICS OF CHARGED PARTICLES IN THE PRESENCE OF MAGNETIC IRREGULARITIES*

by

M. Fibich

Belfer School of Science, Yeshiva University

New York, New York

and

G. Sandri

Aeronautical Research Associates of Princeton, Inc.

50 Washington Road, Princeton, New Jersey

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ABSTRACT

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We introduce a model to describe the interaction of cosmic rays (either of solar or of galactic origin) with the solar wind. The model consists in replacing the magnetic irregularities in the wind by highly localized scattering centers. The statistical mechanical theory corresponding to this model is traced in detail from Liouville's theorem through a kinetic equation to the conventional diffusion equation. The statistical nature of the fields within the magnetic irregularities is described in terms of an appropriate Fourier analysis.

Author

TABLE OF CONTENTS

SECTION		PAGE
	Abstract	1
I	Introduction	1
II	Formulation of the Problem	3
III	Multiple Time Scale Expansion	6
IV	The H-theorem	14
V	The Diffusion Regime	17
VI	Scattering Cross Section	21
VII	Conclusions	28
	References	29

I. INTRODUCTION

To study the interaction of cosmic rays (solar and galactic) with the solar wind, we assume that it is justified to replace the effect of the magnetic field irregularities distributed throughout interstellar space by fixed scattering centers. Furthermore, we assume that the interaction among the cosmic rays can be neglected. We formulate, in detail, the statistical mechanics that corresponds to this model in Section II.

In Section III we address ourselves to the question of finding in what sense and to what degree of approximation it is possible to use a diffusion equation to represent this system. By means of the multiple time scale technique (Ref 1), we obtain the kinetic equation for the distribution function of the cosmic rays. We prove that the lowest order result, in powers of the dilution of the scattering centers, is the kinetic equation previously employed for studying this problem (Ref 2). The dilution parameter is appropriately small in the solar environment, i.e., the product of the mean density of magnetic kinks and the kink volume is approximately 1.3×10^{-2} . In Section IV we prove the H-theorem for our kinetic equation. In Section V we obtain solutions of the kinetic equations that correspond, for large times, to the diffusion equation (Ref 3). In these calculations, we make the assumption that the gradient of the distribution function is small, and we do not include (at this point) any large scale magnetic field (Ref 5). The assumption about the smallness of the gradient, although justified when particles have already diffused on an astronomical scale (say a fraction of 1 a.u.), does not hold well for shorter times. We will discuss separately the problem of large gradients and the derivation of the density dependence of the cosmic ray gas transport properties from our theory. This problem has attracted considerable attention recently (Ref 4). In Section VI, by expanding the scattering operator with which we constructed the kinetic equation, we calculate the scattering cross section of a cosmic ray by a magnetic kink for small angle deflections.

The magnetic kink is described by a Fourier series with a narrow band concentrated at low frequencies. Therefore, our formulation of the problem is particularly well suited for correlation of the magnetic field data with the diffusion data.

II. FORMULATION OF THE PROBLEM

We consider a system of N-charged particles (cosmic rays) and an irregular magnetic field which is highly localized around M stationary points in space. We assume that the interaction of a particle with the localized field can be derived from a potential function Φ. Furthermore, we neglect the interaction among the charged particles themselves. Thus, the Hamiltonian of the system is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=1}^{N} \sum_{j=1}^{M} \Phi_{ij} \left(\vec{x}_i - \vec{X}_j\right) \tag{2.1}$$

where $\{\vec{x}_i, \vec{p}_i\}$ are the coordinates and momenta of the N-particles, and $\{\vec{X}_i\}$ are the coordinates of the M localized fields (magnetic kinks). We have not included in Eq 2.1 the wall potential that confines specularly the particles to a finite volume V since we shall be concerned with the limit of an infinite system and the wall potential becomes inoperative in this limit.

Let us define a joint distribution function

$$D_{NM}\left(\{\vec{x}_i, \vec{p}_i\}, \{\vec{X}_j\}\right)$$

that gives the probability density for the N particles to be at the points $\{\vec{x}_i, \vec{p}_i\}$ in phase space and the fields to be localized around the points $\{\vec{X}_i\}$. The function D_{NM} is normalized according to

$$1 = \int \prod_{i=1}^{M} \frac{d\vec{X}_i}{V} \int \prod_{j=1}^{N} \frac{d\vec{x}_j d\vec{p}_j}{V} D_{NM} \tag{2.2}$$

The system described satisfies well defined Hamiltonian equations of motion. The joint distribution function D_{NM} obeys, therefore, the Liouville equation

$$\frac{\partial D_{NM}}{\partial t} + [D_{NM}, H] = 0 \tag{2.3}$$

where $[D_{NM},H]$ is the Poisson bracket of the distribution function with the Hamiltonian of Eq 2.1. We choose D_{NM} to be symmetric under permutations of $\{x_i, p_i\}$ among themselves and of the $\{X_i\}$ among themselves.

The reduced distribution functions are introduced

$$f_m(\vec{x}_1, \vec{p}_1; \vec{X}_1, \cdots \vec{X}_{m-1}) = \int \prod_{i=m}^{M} \frac{d\vec{X}_i}{V} \int \prod_{i=2}^{N} \frac{d\vec{x}_i}{V} d\vec{p}_i \, D_{NM}(\vec{x}_1, \vec{x}_2 \cdots \vec{x}_N, \vec{p}_1, \vec{p}_2 \cdots \vec{p}_N; \vec{X} \cdots \vec{X}_M) \tag{2.4}$$

The equation that f_m obeys follows from Eq 2.3

$$\frac{\partial f_m}{\partial t} + H_m f_m = \frac{M-(m-1)}{V} L_m f_{m+1} \tag{2.5}$$

where H_m and L_m are the operators

$$H_m = \vec{v} \cdot \nabla - \frac{1}{m} \sum_{j=1}^{m-1} \nabla \Phi (\vec{x} - \vec{X}_j) \cdot \frac{\partial}{\partial \vec{v}} \tag{2.6}$$

$$L_m f_{m+1}(\vec{x}, \vec{p}; \vec{X}_1, \cdots \vec{X}_m) = \frac{1}{m} \int d\vec{X}_m \nabla \Phi(\vec{x} - \vec{X}_m) \cdot \frac{\partial}{\partial \vec{v}} f_{m+1}(\vec{x}, \vec{p}; \vec{X}_1 \cdots \vec{X}_m) \tag{2.7}$$

We now take the limit of an infinite system by letting $M \to \infty$, $V \to \infty$ maintaining the mean density of the field points, $n_s \equiv M/V$, constant.

The explicit equations for the first two distribution functions become, then

$$\left(\frac{\partial}{dt} + \vec{v}\cdot\nabla\right) f_1(\vec{x},\vec{v},t) = \frac{n_s}{m}\int d\vec{X}\, \nabla\Phi(\vec{x}-\vec{X})\cdot\frac{\partial}{d\vec{v}} f_2(\vec{x},\vec{v},\vec{X},t) \tag{2.8}$$

$$\left(\frac{\partial}{dt} + \vec{v}\cdot\nabla - \frac{1}{m}\nabla\Phi(\vec{x}-\vec{X})\cdot\frac{\partial}{d\vec{v}}\right) f_2(\vec{x},\vec{v},\vec{X},t) = \frac{n_s}{m}\int d\vec{X}'\, \nabla\Phi(\vec{x}-\vec{X}')\cdot\frac{\partial}{d\vec{x}} f_3(\vec{x},\vec{v},\vec{X},\vec{X}',t) \tag{2.9}$$

The function f_1 describes the probability density for a single cosmic ray while the function f_2 describes the joint probability density for one cosmic ray and one scattering center.

The fields described by Φ are assumed to be so highly localized that we can assign a radius r_o to the "size" of the field inhomogeneities. The operator on the right hand side of Eq 2.5, and similarly in Eqs 2.8 and 2.9, is then of order $n_s r_o^3$ compared to the left hand side of these equations. The quantity $n_s r_o^3$ is the cube of the ratio of the average size of an irregularity to the mean distance between irregularities. This ratio is quite small in the solar cavity ($\sim 1.3 \times 10^{-2}$) and thus defines a convenient parameter of smallness ϵ for a power series expansion.

III. MULTIPLE TIME SCALE EXPANSION

As was mentioned earlier, we restrict our attention to the distribution function late in the kinetic stage. The term ∇f_1 in Eq 2.8, therefore, is of order λ^{-1} where the mean free path is

$$\lambda \sim \frac{1}{n_s \sigma}$$

and $\sigma \sim \pi r_o^2$. Therefore, $r_o/\lambda \approx \epsilon$ which means then that ∇f_1 in Eq 2.8 is of order ϵ . On the other hand, ∇f_2 in Eq 2.9 should not be considered of order ϵ since f_2 varies sharply in the region $|\bar{x} - \bar{X}| \sim r_o$; i.e., during a collision.

The various time scales are defined, following the method of Ref 1, by

$$\tau_0 = t, \quad \tau_1 = \epsilon t, \quad \tau_2 = \epsilon^2 t, \quad \tau_n = \epsilon^n t$$

and the expansions of f_1 and f_2 expanded in the powers of ϵ are given by

$$f_1 = f_1^{(0)} + \epsilon f_1^{(1)} + O(\epsilon^2)$$

$$f_2 = f_2^{(0)} + f_2^{(1)} + O(\epsilon^2)$$

The variables τ_n are to be treated as independent variables during the calculation and are to be related to t (via $\tau_n = \epsilon^n t$) at the end of the calculation.

By equating powers of ϵ in Eq 2.8, we readily obtain

$$\frac{\partial f_1^{(0)}}{\partial \tau_0} = 0 \tag{3.1}$$

$$\frac{\partial f_1^{(1)}}{\partial \tau_0} + \left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\nabla\right) f_1^{(0)} = \frac{1}{m}\int d\vec{\underline{X}}\, \nabla\Phi(\vec{x}-\vec{\underline{X}})\cdot\frac{\partial}{\partial \vec{v}} f_2^{(0)} \qquad (3.2)$$

$$\frac{\partial f_1^{(2)}}{\partial \tau_0} + \left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\partial\right) f_1^{(1)} + \frac{\partial f_1^{(0)}}{\partial \tau_2} = \frac{1}{m}\int d\vec{\underline{X}}\, \nabla\Phi(\vec{x}-\vec{\underline{X}})\cdot\frac{\partial}{\partial \vec{v}} f_2^{(1)} \qquad (3.3)$$

and similarly from Eq 2.9

$$\frac{\partial f_2^{(0)}}{\partial \tau_0} + \left(\vec{v}\cdot\nabla - \frac{1}{m}\nabla\Phi\cdot\frac{\partial}{\partial \vec{v}}\right) f_2^{(0)} = 0 \qquad (3.4)$$

$$\frac{\partial f_2^{(1)}}{\partial \tau_0} + \left(\vec{v}\cdot\nabla - \frac{1}{m}\nabla\Phi\cdot\frac{\partial}{\partial \vec{v}}\right) f_2^{(1)} + \frac{\partial f_2^{(0)}}{\partial \tau_1} = \frac{1}{m}\int d\vec{\underline{X}}'\, \nabla\Phi(\vec{x}-\vec{\underline{X}}')\cdot\frac{\partial}{\partial \vec{v}} f_3^{(0)} \qquad (3.5)$$

For $f_3^{(0)}$ we have the equation

$$\frac{\partial f_3^{(0)}}{d\tau_0} + \left(\vec{v}\cdot\nabla - \frac{1}{m}\nabla\Phi(\vec{x}-\vec{\underline{X}})\cdot\frac{\partial}{\partial \vec{v}} - \frac{1}{m}\nabla\Phi(\vec{x}-\vec{\underline{X}}')\cdot\frac{\partial}{\partial \vec{v}}\right) f_3^{(0)} = 0 \qquad (3.6)$$

Eq 3.1 shows that $f_1^{(0)}$ does not vary on the fast time scale. To solve Eq 3.2 we need $f_2^{(0)}$. This two-body function is obtained from Eq 3.4

$$f_2^{(0)}(\vec{x},\vec{v},\vec{\underline{X}},\tau_0) = C_2(\tau_0)\, f_2^{(0)}(\vec{x},\vec{v},\vec{\underline{X}},0) \qquad (3.7)$$

where we have introduced the collision operator C_2 by

$$C_2(\tau_0) \equiv \exp[-H_2 \tau_0]$$

From Eq 2.6 the operator H_2 is

$$H_2 = \vec{v} \cdot \nabla - \frac{1}{m} \nabla \Phi(\vec{x} - \vec{\underline{X}}) \cdot \frac{\partial}{d\vec{v}} \tag{3.8}$$

We shall assume that as $\tau_0 \to \infty$ the collision operator $C_2(\tau_0)$ reaches an asymptotic limit $C_2(\infty)$, that is, we assume that bound states do not exist. In this limit we then have

$$\frac{\partial f_2^{(0)}}{\partial \tau_0} \xrightarrow[\tau_0 \to \infty]{} 0$$

and therefore, from Eq 3.4

$$H_2 f_2^{(0)} \xrightarrow[\tau_0 \to \infty]{} 0$$

We can therefore conclude, using Eq 3.8, that

$$\frac{1}{m} \nabla \Phi \cdot \frac{\partial}{d\vec{v}} \left[C_2(\infty) f_2^{(0)}(\vec{x}, \vec{v}, \vec{\underline{X}}, 0) \right] = \vec{v} \cdot \nabla \left[C_2(\infty) f_2^{(0)}(\vec{x}, \vec{v}, \vec{\underline{X}}, 0) \right] \tag{3.9}$$

Note that since $f_2^{(0)}$ is a function of $\vec{x} - \vec{X}$,

$$\nabla_{\vec{X}} f_2^{(0)} = \nabla_{\vec{x} - \vec{\underline{X}}} f_2^{(0)}$$

Using this result in Eq 3.2, we obtain in the asymptotic limit

$$\frac{\partial f_1^{(1)}}{\partial \tau_0} + \left(\frac{\partial}{\partial \tau_1} + \vec{v} \cdot \nabla \right) f_1^{(0)} \xrightarrow[\tau_0 \to \infty]{} \int d\vec{\underline{X}}\, \vec{v} \cdot \nabla \left[C_2(\infty) f_2^{(0)}(\vec{x}, \vec{v}, \vec{\underline{X}}, 0) \right] \tag{3.10}$$

By virtue of Eq 3.1, all terms, but the first, in Eq 3.10 are independent of τ_0. If we now demand that the expansion of f_1 in τ_1 be uniformly valid, we must set

$$\frac{\partial f_1^{(1)}}{\partial \tau_0} = 0 \tag{3.11}$$

Carrying out the integral on the right hand side of Eq 3.10 along the direction of $\vec{v}$, we are left with an integral over the area perpendicular to $\vec{v}$, $dS_\perp$,

$$\left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\vec{\nabla}\right) f_1^{(0)}(\vec{x},\vec{v},\tau_1) = \frac{1}{m}\int dS_\perp |\vec{v}| \lim_{\ell\to\infty}\left\{C_2(\infty) f_2^{(0)}(\vec{x},\vec{v},\vec{X}_0)\right\} \tag{3.12}$$

We now assume that the function f_2 $(\vec{x},\vec{v},\vec{X})$ becomes a product of $f_1(\vec{x},\vec{v})$ and the distribution of the irregularities $F_1(\vec{X})$ when the distance between the cosmic ray and the magnetic irregularity, $|\vec{x} - \vec{X}|$, is very large compared with the kink size. What appears then in Eq 3.12 is the difference between the product distributions evaluated before and after a "collision". A collision clearly changes $\vec{v}$ into $\vec{v}'$ (with $|\vec{v}'| = |\vec{v}|$ for elastic collisions). Hence,

$$\left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\vec{\nabla}\right) f_1^{(0)} = \frac{1}{m}\int dS_\perp |\vec{v}| F_1(\vec{X})\left\{f_1^{(0)}(\vec{x},\vec{v}',\tau_1) - f_1^{(0)}(\vec{x},\vec{v},\tau_1)\right\} \tag{3.13}$$

If we know the kinematics of a collision, we may transform the integral over $dS_\perp$ into an integral over the element $d\theta$ of the scattering angle. The differential scattering cross section, $\sigma(\theta)$, is given by

$$\sigma(\theta) = \frac{dS_\perp}{d\theta}$$

Assuming for simplicity that F_1 is constant in space, Eq 3.13 then becomes

$$\left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\vec{\nabla}\right) f_1^{(0)}(\vec{x},\vec{v},\tau_1) = \int d\theta\, \sigma(\theta)\, |\vec{v}| \left[f_1^{(0)}(\vec{x},\vec{v}',\tau_1) - f_1^{(0)}(\vec{x},\vec{v},\tau_1) \right] \quad (3.14)$$

Thus, we have proven that Eq 3.14, which has been used previously to investigate the diffusion problem (Ref 2), is the lowest order term in the kinetic equation derived using the Hamiltonian given in Eq 2.1.

The kinetic equation, correct to the next order in ϵ, is obtained from Eq 3.3. Equation 3.3 requires $f_2^{(1)}$ which is determined from Eq 3.5. After solving Eq 3.6 for $f_3^{(0)}$ from Eq 3.6 we find

$$f_3^{(0)}(\vec{x},\vec{v},\vec{\bar{X}},\vec{\bar{X}}',\tau_0) = e^{-H_3\tau_0} f_3^{(0)}(\vec{x},\vec{v},\vec{\bar{X}},\vec{\bar{X}}',0) \quad (3.15)$$

where, with recorse to Eq 2.6

$$H_3 = \vec{v}\cdot\nabla - \frac{1}{m}\left[\nabla\bar{\Phi}(\vec{x}-\vec{\bar{X}}) + \nabla\bar{\Phi}(\vec{x}-\vec{\bar{X}}')\right]\cdot\frac{\partial}{\partial\vec{v}} \quad (3.16)$$

We assume that the operator $C_3(\tau_0) \equiv \exp[-H_3\tau_0]$ has an asymptotic value $C_3(\infty)$, namely

$$\lim_{\tau_0\to\infty} f_3^{(0)}(\tau_0) = C_3(\infty) f_3^{(0)}(0) \quad (3.17)$$

We then can write

$$H_3\left[C_3(\infty) f_3^{(0)}(0)\right] = 0$$

and using Eq 3.16, we find

$$\frac{1}{m}\nabla\Phi(\vec{x}-\vec{X}')\frac{\partial}{d\vec{v}}\left[C_3(\infty)f_3^{(0)}(0)\right]=\left(\vec{v}\cdot\nabla-\frac{1}{m}\nabla\Phi(\vec{x}-\vec{X})\cdot\frac{\partial}{\partial\vec{v}}\right)\left[C_3(\infty)f_3^{0}(0)\right] \quad (3.18)$$

Since $f_3^{(0)}$ is a function of $\vec{x}-\vec{X}=\vec{\xi}$ and $\vec{x}-\vec{X}'=\vec{\zeta}'$, $\vec{\nabla}\cdot\nabla$ in Eq 3.18 may be replaced by $\vec{\nabla}\cdot(\nabla_{\vec{\zeta}}+\nabla_{\vec{\zeta}'})$. The right hand side of Eq 3.5 thus becomes

$$\frac{1}{m}\int d\vec{X}'\nabla\Phi(\vec{x}-\vec{X}')\cdot\frac{\partial}{\partial\vec{v}}f_3^{(0)}\xrightarrow[\tau_0\to\infty]{}\int d\vec{\xi}'\left\{\vec{v}\cdot(\nabla_{\vec{\xi}}+\nabla_{\vec{\xi}'})-\frac{1}{m}\nabla\Phi(\vec{\xi})\cdot\frac{\partial}{d\vec{v}}\right\}\cdot$$
$$\cdot\left[C_3(\infty)f_3^{(0)}(0)\right]$$
$$=\left(\vec{v}\cdot\nabla-\frac{1}{m}\nabla\Phi(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}}\right)\int d\vec{\xi}'\left[C_3(\infty)f_3^{(0)}(\vec{\xi},\vec{\xi}',\tau_0=0)\right]$$
$$+\int d\vec{\xi}'\vec{v}\cdot\nabla_{\vec{\xi}'}\left[C_3(\infty)f_3^{(0)}(0)\right] \quad (3.19)$$

We note the decomposition

$$H_3=\left(\vec{v}\cdot\nabla_{\vec{\xi}}-\frac{1}{m}\nabla\Phi(\vec{\xi})\cdot\nabla_v\right)+\left(\vec{v}\cdot\nabla_{\vec{\xi}'}-\frac{1}{m}\nabla\Phi(\vec{\xi}')\cdot\nabla_{v'}\right)=H_2(\vec{\xi})+H_2(\vec{\xi}')$$

and the commutation rule

$$\left[H_2(\vec{\xi}),H_2(\vec{\xi}')\right]=\frac{1}{m}\left[\nabla\phi(\vec{\xi})\cdot\nabla_{\vec{\xi}'}-\nabla\phi(\vec{\xi}')\cdot\nabla_{\vec{\xi}}\right]$$

The contribution from the first integral on the right hand side of Eq 3.19 vanishes since

$$\int d\vec{\xi}' C_3(\infty) f_3^{(0)}(0) = V \lim_{\tau_0 \to \infty} f_2^{(0)}(\tau_0)$$

and $H_2 f_2^{(0)} = 0$ in the asymptotic limit. The second integral on the right hand side of Eq 3.19 in a manner similar to that used to obtain Eq 3.14, becomes

$$C_2(\infty) \int d\theta\, \sigma_2(\theta) |\vec{v}| \left[f_2^{(0)}(\vec{X}, \vec{\bar{X}}, \vec{v}', 0) - f_2^{(0)}(\vec{X}, \vec{\bar{X}}, \vec{v}, 0) \right]$$

Eq 3.5, with $f_3^{(0)}$ evaluated in the asymptotic limit then becomes

$$\frac{\partial f_2^{(1)}}{d\tau_0} + H_2 f_2^{(1)} + \frac{\partial f_2^{(0)}}{d\tau_1} = \int d\theta\, \sigma_2 |\vec{v}| \left[f_2^{(0)}(\vec{v}', 0) - f_2^{(0)}(\vec{v}, 0) \right] = I_2^{(0)}(0) \quad (3.20)$$

which has the solution

$$f_2^{(1)}(\tau_0) = e^{-H_2 \tau_0} f_2^{(1)}(0) + e^{-H_2 \tau_0} \int_0^{\tau_0} d\tau_0' e^{H_2 \tau_0'} \left\{ I_2^{(0)}(0) - \frac{\partial f_2^{(0)}(\tau_0', \tau_1)}{d\tau_1} \right\}$$

$$= e^{-H_2 \tau_0} f_2^{(1)}(0) + \int_0^{\tau_0} d\tau_0' \left\{ I_2^{(0)}(\tau_0') - \frac{\partial}{\partial \tau_1} f_2^{(0)}(\tau_0, \tau_1) \right\} \quad (3.21)$$

where we have made use of Eq 3.7. In the asymptotic limit the integrand becomes independent of τ_0' and the integral gives rise to a secular term. Therefore the integrand must vanish and we have

$$\frac{\partial f_2^{(0)}}{\partial \tau_1} = n_3 \int d\theta\, \sigma_2(\theta) |\vec{v}| \left[f_2^{(0)}(\tau_0, \tau_1, \vec{v}') - f_2^{(0)}(\tau_0, \tau_1, \vec{v}) \right] \quad (3.22)$$

and

$$\lim_{\tau_0 \to \infty} f_2^{(1)}(\tau_0) = C_2(\infty) f_2^{(1)}(0) \tag{3.23}$$

When we use this last result in Eq 3.3 and eliminate the secular contribution by means of the requirement that

$$\frac{\partial f_1^{(2)}}{\partial \tau_0} = 0 \tag{3.24}$$

we obtain

$$\left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\nabla\right) f_1^{(1)} + \frac{\partial f_1^{(0)}}{\partial \tau_2} = n_s \int d\theta\, \sigma(\theta) |\vec{v}| \left[f_1^{(1)}(\vec{v}') - f_1^{(1)}(\vec{v})\right] \tag{3.25}$$

The term $\partial f_1^{(0)}/\partial \tau_2$ can be set equal to zero since no secularities occur in Eq 3.25. We note that the precise form of Eq 3.22 is not a necessary intermediate step in obtaining Eq 3.25.

IV. THE H-THEOREM

We prove here the entropy principle (H-theorem) in the form appropriate to the lowest order kinetic equation. The angular averaging performed below is tailored for the discussion of motion in the presence of fixed scattering centers. We can rewrite Eqs 3.14 and 3.15 in the form

$$\left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\nabla\right) f_1^{(0)} = \nu\left(\bar{f}_1^{(0)} - f_1^{(0)}\right) \tag{4.1}$$

$$\left(\frac{\partial}{\partial \tau_1} + \vec{v}\cdot\nabla\right) f_1^{(1)} = \nu\left(\bar{f}_1^{(1)} - f_1^{(1)}\right) \tag{4.2}$$

where

$$\nu \equiv \int d\theta\, \sigma(\theta)\, |\vec{v}| \tag{4.3}$$

and $\bar{f}_1^{(j)}$ is the angular average defined by

$$\bar{f}_1^{(j)} \equiv \frac{1}{\nu}\int d\theta\, \sigma(\theta)\, |\vec{v}|\, f_1^{(j)}(\vec{v}\,') \tag{4.4}$$

Let us note that Eq 4.1 has an H-theorem. The entropy function in lowest order, $H^{(0)}$ is defined by

$$H^{(0)} = \frac{1}{\nu}\int d^3x \int dv\, v^2 \int d\theta\, \sigma(\theta)\, |\vec{v}|\, f_1^{(0)} \ln f_1^{(0)} \tag{4.5}$$

The theorem states that

$$\partial H^{(0)}/\partial \tau_1 \leq 0$$

for any distribution function $f_1^{(0)}$.

Proof: We use the notation

$$\nu^{-1}\int d\nu\, \nu^2 \int d\theta |\vec{v}| \sigma(\theta) = \int d\sigma(\vec{v},\theta)$$

We then have

$$\frac{\partial H^{(0)}}{\partial \tau_1} = \int d^3x \int d\sigma(\vec{v},\theta)\left(1+\ln f_1^{(0)}\right)\frac{\partial f_1^{(0)}}{\partial \tau_1} = \int d^3x \int d\sigma(\vec{v},\theta)\cdot$$

$$\cdot\left(1+\ln f_1^{(0)}\right)\left[\nu\left(\bar{f}_1^{(0)} - f_1^{(0)}\right) - \vec{v}\cdot\nabla f_1^{(0)}\right] \tag{4.6}$$

Now we assume f_1^0 vanishes on the boundaries of the volume of integration. Therefore,

$$\int d^3x \left(1+\ln f_1^{(0)}\right)\vec{v}\cdot\nabla f_1^{(0)} = -\int d^3x f_1^{(0)} \vec{v}\cdot\nabla \ln f_1^{(0)} = -\int d^3x\, \vec{v}\cdot\nabla f_1^{(0)} = 0 \tag{4.7}$$

The remainder of Eq 4.6 can be written as

$$\frac{\partial H^{(0)}}{\partial \tau_1} = \nu \int d^3x \int d\sigma(\vec{v},\theta) \ln\left(\frac{f_1^{(0)}}{\bar{f}_1^{(0)}}\right)\left(\bar{f}_1^{(0)} - f_1^{(0)}\right) \tag{4.8}$$

where we used the fact that

$$\overline{\bar{f}_1^{\,0} - f_1^{\,0}} = 0$$

and similarly that

$$\left(f_1^{(0)} - f_1^{(0)}\right) \ln \bar{f}^{(0)} = 0$$

Clearly, from Eq 4.8, we note that

$$\frac{\partial H^0}{\partial \tau_1} \leq 0 \qquad \text{Q.E.D.}$$

V. THE DIFFUSION REGIME

It is known that the kinetic equation, Eq 4.1, leads in the limit $\tau_1 \gg \nu^{-1}$, to a diffusion equation. We shall show here that for sufficiently large times, the "normal" solutions of the kinetic equation have diffusive behavior.

Let us Fourier-Laplace transform Eq 4.1 using the notation

$$\begin{Bmatrix} g^{(0)}(\vec{k},\vec{v},p) \\ \bar{g}^{(0)}(\vec{k},\vec{v},p) \end{Bmatrix} = \mathcal{L}\mathcal{F}\begin{Bmatrix} f^{(0)}(\vec{x},\vec{v},t) \\ \bar{f}^{(0)}(\vec{x},\vec{v},t) \end{Bmatrix} = \int_0^\infty d\tilde{t}\, e^{-p\tilde{t}} \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \begin{Bmatrix} f^{(0)}(\vec{x},\vec{v},t) \\ \bar{f}^{(0)}(\vec{x},\vec{v},t) \end{Bmatrix} \tag{5.1}$$

$$F(\vec{k},\vec{v}) = \mathcal{F}\left\{f^{(0)}(\vec{x},\vec{v},0)\right\} \tag{5.2}$$

We readily obtain

$$(p + i\vec{k}\cdot\vec{v} + \gamma)\, g^{(0)} = \gamma \bar{g}^{0} + F \tag{5.3}$$

Dividing Eq 5.3 by $p + i\vec{k}\cdot\vec{v} + \nu$ and performing the angular average by means of the definition Eq 4.4, we find

$$\bar{g}^{0}(\vec{k},\vec{v},p) = \overline{\frac{F(\vec{k},\vec{v})}{p+\gamma+i\vec{k}\cdot\vec{v}}} \Big/ \left(1 - \overline{\frac{\gamma}{p+\gamma+i\vec{k}\cdot\vec{v}}}\right) \tag{5.4}$$

The long-time behavior of the velocity distribution is governed by the solutions of $D(\bar{k},p) = 0$ where the function $D(\bar{k},p)$ is given by

$$D(\vec{k},p) = 1 - \overline{\frac{\nu}{p+\nu+i\vec{k}\cdot\vec{v}}} \tag{5.5}$$

We have, in fact

$$\overline{g^{(0)}}(\vec{k},\vec{v},\tau) = \sum_i R_i(\vec{k})e^{ip_i(\vec{k})\tau} \tag{5.6}$$

where $p_i(\bar{k})$ are the solutions of $D(\bar{k},p) = 0$ and $R_i(\bar{k})$ are the residues of $\bar{g}^0(\bar{k},\bar{v},p_i(\bar{k}))$. We shall approximate the solutions of $D(k,p) = 0$ by expanding in powers of $\bar{k}\cdot\bar{v}/\nu$

$$D(\vec{k},p) = 1 - \frac{\nu}{p+\nu} + i\nu\overline{\frac{\vec{k}\cdot\vec{v}}{(p+\nu)^2}} + \frac{\nu}{2}\overline{\frac{(\vec{k}\cdot\vec{v})^2}{(p+\nu)^3}} + \dots = 0 \tag{5.7}$$

For most cases of symmetric cross section $\overline{\vec{k}\cdot\vec{v}} = 0$, so to lowest order we keep only

$$\overline{(\vec{k}\cdot\vec{v})^2} = k^2u^2$$

which defines the "average" velocity u^2 . There are three solutions $p_i(k)$ to Eq 5.7. The first two are

$$p_{1,2}(k) = -\nu \pm ku/\sqrt{2}$$

and the third is

$$p_3(k) \approx \nu (ku)^2/4\nu$$

The first two give rise to terms proportional to $\exp[-\nu\tau_1]$ in $\overline{g^{(o)}}(\bar{k},\bar{v},\tau_1)$, which damp out rapidly. We concentrate then on $p_3(k)$ and

$$\overline{g^o}(\bar{k},\bar{v},\tau_1) = \nu \frac{F(\bar{k},\bar{v})}{p_3(k)+\nu+i\bar{k}\cdot\bar{v}} \exp\left[\frac{(ku)^2}{(4\nu)^2}\nu\tau_1\right] \tag{5.8}$$

specializing to an initial condition

$$f^o(\vec{x},\vec{v},0) = \phi^o(\vec{v})\,\delta(\vec{x}-\vec{x}_o) \tag{5.9}$$

which, because of Eq 5.2, corresponds to

$$F(\bar{k},\bar{v}) = \phi^o(\bar{v})\, e^{-i\bar{k}\cdot\vec{x}_o} \tag{5.10}$$

We obtain, from Eq 5.8, t_o lowest order in ku/ν

$$\overline{g^o}(ku\tau_1) = \nu\phi^o(\bar{v}) \exp\left[\left(\frac{ku}{4\nu}\right)^2 \nu\tau_1\right] e^{-i\bar{k}\cdot\vec{x}_o} \tag{5.11}$$

After an inverse Fourier transform, Eq 5.11 reduces to

$$\bar{f}^{o}(\bar{X}, \bar{V}, \tau_1) = \frac{\overline{\phi^{o}(\bar{V})}}{(4\pi D\tau_1)^{3/2}} \exp\left[-\frac{(\bar{X} - \bar{X}_o)^2}{4D\tau_1}\right] \tag{5.12}$$

where we have introduced the diffusion coefficient D given by

$$D = \frac{u^2}{4\nu} \tag{5.13}$$

We, thus, see that for large times our kinetic equation yields automatically the results of the usual diffusion treatment of the problem (Ref 3).

VI. SCATTERING CROSS SECTION

Up to this point we assumed the cross section for the scattering of a particle by a magnetic irregularity to be known. We want now to see how we can possibly calculate such a cross section.

We have assumed that the field of a typical irregularity is confined to a region in space, approximated by a sphere or radius r_o, and that $n_s r_o^3 \ll 1$, where n_s is the average density of centers of irregularities.

It is necessary to have further knowledge about the structure of the magnetic irregularity in order to calculate the cross section. Obviously, we cannot make wild guesses as to the explicit form of the magnetic field of an irregularity which is known to be very complex. It will be sufficient, however, as we shall see, to make two reasonable assumptions about the model of an irregularity.

1. If we pick any straight line cutting through the (irregularity) sphere, any component of the magnetic field fluctuates very rapidly along the line. The integral of any component of the magnetic field along this line practically vanishes.

2. The fluctuating magnetic field can be described by a Fourier series of large wave numbers k (small wave length) but of relatively narrow band, i.e., $k/k_m \ll 1$, where k is the width of the band and k_m is the mean value of the wave numbers.

We shall employ the collision operator $C_2(t)$ expressed in Eqs 3.7 and 3.8 to calculate the change in the velocity of a particle when it encounters an irregularity. Let us rewrite the expression for $C_2(t)$ with $\nabla\phi$ of Eq 3.8 replaced by

$$\frac{e}{mc}\vec{\nabla}\times\vec{B}\left(\vec{\xi}\right) = \vec{\nabla}\times\vec{\omega}\left(\vec{\xi}\right)$$

where $\overline{B}$ is the magnetic field and is a function of $\overline{\xi} = \overline{x} - \overline{X}$.

$$C_2(t) = \exp\left[-\left(\vec{v}\cdot\vec{\nabla}_{\xi} + \vec{v}\times\vec{\omega}(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}}\right)t\right] \tag{6.1}$$

Any function $g(\overline{v})$ changes in time according to

$$g(\vec{v}(t)) = C_2(t)\,g(\vec{v}(0)) \tag{6.2}$$

The case that we are interested in here is $g(\overline{v}) = \overline{v}$. Generally it is very difficult to calculate $C_2(t)$ or its asymptotic value, in particular as is the case here, when $\overline{\omega}(\overline{\xi})$ may have appreciable gradients. We shall be able, however, to expand $C_2(t)$ in powers of $\overline{v} \times \overline{\omega}$ and retain terms only up to second order. The reason why this expansion gives a reasonably good approximation is due to assumption (1), since, as we shall see, only line integrals of $\overline{\omega}$ appear in the expansion. We thus expand $C_2(t)$ as follows

$$C_2(t) = e^{-t\vec{v}\cdot\vec{\nabla}}\left\{1 - \int_0^t dt'\, e^{t'\vec{v}\cdot\nabla}\,\vec{v}\times\vec{\omega}(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}}\, e^{-t'\vec{v}\cdot\vec{\nabla}} + \right.$$

$$\left. + \int_0^t dt' \int_0^{t'} dt''\, e^{t'\vec{v}\cdot\vec{\nabla}}\,\vec{v}\times\vec{\omega}(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}}\, e^{-(t'-t'')\vec{v}\cdot\vec{\nabla}}\,\vec{v}\times\vec{\omega}(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}}\, e^{-t''\vec{v}\cdot\vec{\nabla}} + \cdots \right\} \tag{6.3}$$

We can thus express $v(t)$ by means of the expansion

$$\vec{v}(t) = \vec{v} + \Delta\vec{v}^{(1)} + \Delta\vec{v}^{(2)} \tag{6.4}$$

where $\overline{v} \equiv \overline{v}(0)$. The first order velocity change is given by

$$\overrightarrow{\Delta V}^{(1)} = -\left\{ e^{-t\vec{v}\cdot\nabla} \int_0^t dt' e^{t'\vec{v}\cdot\nabla} \vec{v}\times\vec{\omega}(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}} e^{t'\vec{v}\cdot\nabla} \right\}\vec{v} \qquad (6.5)$$

Similarly, the second order velocity change is given by

$$\overrightarrow{\Delta V}^{(2)} = \left\{ e^{-t\vec{v}\cdot\nabla} \int_0^t dt' \int_0^{t'} dt'' e^{t'\vec{v}\cdot\nabla} \vec{v}\times\vec{\omega}(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}} e^{-(t'-t'')\vec{v}\cdot\nabla} \vec{v}\times\vec{\omega}(\vec{\xi})\cdot\frac{\partial}{\partial\vec{v}} e^{-t''\vec{v}\cdot\nabla} \right\}\vec{v} \qquad (6.6)$$

Let us first calculate $\overrightarrow{\Delta v}^{(1)}$ remembering that

$$e^{-t\vec{v}\cdot\nabla} f(\vec{\xi}) = f(\vec{\xi}-\vec{v}t)$$

We find

$$\overrightarrow{\Delta V}^{(1)} = -\vec{v}\times\int_0^t d\tau\, \vec{\omega}(\vec{\xi}-\vec{v}\tau) \qquad (6.7)$$

The last integral, according to assumption (1), vanishes except for at most a contribution from one-half wave length of each of the Fourier components of $\vec{\omega}$. But, since this contribution may be either positive or negative, the average

$$\langle \overrightarrow{\Delta V}^{(1)} \rangle = 0 .$$

The second order term, $\overrightarrow{\Delta v}^{(2)}$, may be written in the form

$$\Delta\vec{V}^2 = \int_0^t dt' \int_0^{t'} dt'' \Big\{ \vec{v}\times\vec{\omega}(\vec{\xi}-\vec{v}t+\vec{v}t')\times\vec{\omega}(\vec{\xi}-\vec{v}t+\vec{v}t'') -$$

$$-(t'-t'')\left[\vec{v}\times\vec{\omega}(\vec{\xi}-\vec{v}t+\vec{v}t')\right]\cdot\nabla\,\vec{v}\times\vec{\omega}(\vec{\xi}-\vec{v}t+\vec{v}t'')\Big\} \tag{6.8}$$

Since integrals like those in Eq 6.7 vanish, we should retain in Eq 6.8 only terms in which a given component of the vector appears twice. We may then write for the ith component

$$\Delta V_i^{(2)} = v_i \int_0^t d\tau \int_\tau^t d\tau'\, \omega_i(\vec{\xi}-\vec{v}\tau)\,\omega_i(\vec{\xi}-\vec{v}\tau') -$$

$$-\int_0^t d\tau \int_\tau^t d\tau'(\tau'-\tau)\Big\{\omega_j(\vec{\xi}-\vec{v}\tau)\Big(v_k^2\frac{\partial}{\partial\xi_i} - v_i v_k\frac{\partial}{\partial\xi_k}\Big)\omega_j(\vec{\xi}-\vec{v}\tau') +$$

$$+\omega_k(\vec{\xi}-\vec{v}\tau)\Big(v_j^2\frac{\partial}{\partial\xi_i} - v_i v_j\frac{\partial}{\partial\xi_j}\Big)\omega_k(\vec{\xi}-\vec{v}\tau')\Big\} \tag{6.9}$$

where i,j,k is the cyclic order of the coordinates. If we are now interested in the change in velocity in the direction perpendicular to the initial velocity, we may choose $v_i = 0$, $v_j = 0$ $v_k = v$ and we find

$$\Delta V_i^{(2)} = -v^2 \int_0^t d\tau \int_\tau^t d\tau'(\tau'-\tau)\,\omega_j(\vec{\xi}-\vec{v}\tau)\frac{\partial}{\partial\xi_i}\omega_j(\vec{\xi}-\vec{v}\tau') \tag{6.10}$$

The integral over τ' contributes at most along one-half a wave length. Let the Fourier decomposition of $\omega_j(\vec{\xi})$ be

$$\omega_j(\vec{\xi}) = \sum_{\vec{k}} \gamma_j(\vec{k}) e^{i\vec{k}\cdot\vec{\xi}} \tag{6.11}$$

then

$$\int_{\tau}^{t} d\tau'(\tau'-\tau)\omega_j(\vec{\xi}-\vec{v}\tau') = \sum_{\vec{k}} \gamma_j(\vec{k}) e^{i\vec{k}\cdot(\vec{\xi}-\vec{v}\tau)} \, Re \int_0^{\pi/\vec{k}\cdot\vec{v}} d\alpha\, \alpha\, e^{i\vec{k}\cdot\vec{v}\alpha}$$

$$= \sum_{\vec{k}} \gamma_j(\vec{k}) e^{i\vec{k}\cdot(\vec{\xi}-\vec{v}\tau)} \frac{2}{(\vec{k}\cdot\vec{v})^2} \approx \frac{2}{(k_m v)^2} \omega_j(\vec{\xi}-\vec{v}\tau) \tag{6.12}$$

where $\lambda_m = 2\pi/k_m$ is the mean wave length of the fluctuating field. Eq 6.10 then becomes

$$\Delta v_i^{(2)} \cong -\left(\frac{\lambda_m}{2\pi}\right)^2 \int_0^t dt \frac{\partial}{\partial \xi_i} \omega_j^2(\vec{\xi}-\vec{v}\tau) \tag{6.13}$$

Let us assume that $\omega_j^2(\vec{\xi})$ is a slowly varying function within the sphere $|\vec{\xi}| \leq r_o$ and has the same form for all components

$$\begin{aligned} \omega_j^2(\vec{\xi}) &= \omega^2(\vec{\xi}) \qquad |\vec{\xi}| \leq r_o \\ &= 0 \qquad\qquad |\vec{\xi}| > r_o \end{aligned} \tag{6.14}$$

Since measurements of the interstellar magnetic field are not sufficient to determine the function $\omega^2(\vec{\xi})$, we replace the integrand in Eq 6.13 by its average value. The limits of the integral over τ in Eq 6.13 are the roots of the equation

$$|\vec{\xi} - \vec{v}\tau| = \sqrt{\xi^2 - 2\vec{\xi}\cdot\vec{v}\tau + v^2\tau^2} = r_0 \tag{6.15}$$

so that Eq 6.13 reduces to

$$\Delta v_i^{(2)} = -2\left(\frac{\lambda_m}{2\pi}\right)^2 \frac{\nabla_i \omega_0^2}{v} \sqrt{\left(\frac{\vec{\xi}\cdot\vec{v}}{v}\right)^2 - \xi^2 + r_0^2} \tag{6.16}$$

A similar expression is valid for the other perpendicular direction to the initial velocity, the jth component. The "impact parameter" b is defined by

$$b = \sqrt{\xi^2 - \left(\frac{\vec{\xi}\cdot\vec{v}}{v}\right)^2} \tag{6.17}$$

The absolute value of the change in velocity perpendicular to the initial direction $\Delta v_\perp$ yields the following expression for the tangent of the scattering angle θ

$$\tan\theta = \frac{|\Delta v_\perp|}{v} = \frac{2\sqrt{2}}{3}\left(\frac{\lambda_m}{2\pi}\right)^2 \frac{|\nabla\omega_0|^2}{v^2}\sqrt{r_0^2 - b^2} \tag{6.18}$$

Finally, the scattering cross section is

$$\sigma(\theta) = -\frac{b\,db}{\sin\theta\, d\theta} = \frac{9}{8}\left(\frac{2\pi}{\lambda_m}\right)^4 \left(\frac{v^2}{|\nabla\omega_0^2|}\right)^2 \text{ as } \theta \tag{6.19}$$

where otherwise

$$\tan\theta_{max.} = \gamma r_0, \quad \gamma = \frac{\sqrt{8}}{3}\left(\lambda \;/2\pi\right)^2\left(\nabla\omega_0^2/v^2\right) \tag{6.20}$$

The cross section of Eq 6.19 is large but it is confined to a narrow cone. We find, in fact, for the total cross section the geometrical value

$$\sigma_T = \int d\Omega\,\sigma(\theta) = \frac{\pi}{\gamma^2}\tan^2\theta_{max} = \pi r_0^2 \tag{6.21}$$

In the collision integral of Eq 4.1 the forward scattering does not contribute. The pertinent cross section σ_s is then

$$\sigma_S = \sigma(\theta) - \sigma(0) = \frac{1}{\gamma^2}\left[\frac{1+\tan^2\theta}{\cos\theta} - 1\right] \tag{6.22}$$

The corresponding total cross section is

$$\sigma_{ST} = \left(\pi r_0^2\right)\left(\frac{3}{4}\gamma^2 r_0^2\right) = \pi r_0^2\left(\frac{3}{4}\right)\left(\frac{\lambda_m}{R}\right)^4 \tag{6.23}$$

We thus see a depression of the geometric result by the factor $(3/4)(\lambda_m/R)^4$.

VII. CONCLUSIONS

The interaction of cosmic rays with the magnetic kinks carried by the solar wind has been formulated using statistical mechanical methods. The major simplifying assumptions are: (1) infinitely massive kinks and (2) vanishing average magnetic field. We have derived rigorously the kinetic equation appropriate for this system and shown that its normal solutions have the desired diffusive behavior for large times. Our theory focuses on the regime that corresponds to a dilute system of magnetic irregularities. Dr. Boldt (Ref 5) has pointed out that since there is recent evidence for magnetic ripples (a system of dense, overlapping, kinks) on the magnetic sector boundaries the dense kink regime, i.e., uniform turbulence, is of considerable interest.

We believe that a major merit of our theory is the ease with which the observational information concerning magnetic field measurements can be incorporated in the analysis of cosmic ray diffusion in the solar wind.

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