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# ON THE DUALITY BETWEEN THE PROBLEMS OF STRETCHING AND OF BENDING OF PLATES

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# ON THE DUALITY BETWEEN THE PROBLEMS OF STRETCHING AND OF BENDING OF PLATES

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## 1. Introduction

The analogy between the problems of stretching and of bending of plates has been known for more than half a century. Accordingto Mindlin (1) the first application of this analogy was made by Wieghardt in a paper published in 1908 and dealt with a plate of simply connected shape subjected to boundary loading. Other contributions are due to Westergaard and Mindlin (1) who considered the problems of plane strain and of plate bending for a multipliconnected domain. The latter generalized the treatment to include surface and boundary loads, dislocations and thermal effects. The use of the analogy seems to be limited, however, to the consideration of the deflection of the plate in the bending problem and of Airy's stress function in the stretching problem. In fact a more general analogy exists between the two problems and is a particular case of the static geometric analogy in shell theory (2). It takes the form of a complete duality between the basic equations of the two problems whereby one set of equations is transformed into the other set by interchanging according to a certain correspondence the dependent variables of the two problems. The purpose of this paper is to present this duality in its totality including in the stretching problem displacements, strains, and in-plane changes of curvature and in the bending problem stress functions, stress couples and transverse shears. In both problems, the displacement and the stress function methods of solution, displacement and force boundary conditions and simply and multipliconnected plates are considered.

The problem of plane strain is not treated directly but can be obtained from the stretching problem by changing E into $\frac{E}{1-\nu^2}$ and $\nu$ into $\frac{\nu}{1-\nu}$ where E is Young's modulus and $\nu$ Poisson's ratio (3). This correspondence between the problems of plane strain and of stretching of plates, or generalized plane stress, does not agree with that given in

reference (1) through eqs. 2.7. The latter seems to be only an approximation. The duality for anisotropic plates is also outlined.

The presentation of the stretching-bending analogy will be made by specializing to the case of the plate general equations developed for a shell in orthogonal curvilinear coordinates (4). Cartesian coordinates will be used but when needed formulas in curvilinear coordinates will be specialized to the case of a curved boundary.

## 2. Basic Equations

### a) Equilibrium Equations. Stress-Stress Function Relations.

The differential equations of equilibrium of a plate may be written in the form

$$N_{11,1} + N_{21,2} + p_1 = 0 \qquad 1\text{-}1$$

$$N_{12,1} + N_{22,2} + p_2 = 0 \qquad 1\text{-}2$$

$$N_{12} = N_{21} \qquad 1\text{-}3$$

$$M_{11,1} + M_{21,2} - Q_1 = 0 \qquad 1\text{-}4$$

$$M_{12,1} + M_{22,2} - Q_2 = 0 \qquad 1\text{-}5$$

$$Q_{1,1} + Q_{2,2} + p_3 = 0 \qquad 1\text{-}6$$

where $N_{ij}$ are the in-plane stress resultants, $Q_i$ the transverse shears, $M_{ij}$ the stress couples, $p_i$ the components of the applied load vector $\bar{p}$ per unit area of the middle plane of the plate, Figs. 1, 2, and where a comma followed by an index i indicates differentiation with regard to the cartesian coordinate $x_i$.

Eqs. 1 are supplemented by the additional relation

$$M_{12} = M_{21} \qquad 1\text{-}7$$

Denoting by $N^*_{ij}$ the general solution of the equilibrium equations of the stretching problem for zero surface load we can write

$$N^*_{11} = \Psi_{,22} \qquad 2\text{-}1$$

$$N^*_{22} = \psi_{,11} \qquad 2\text{-}2$$

$$N^*_{12} = N^*_{21} = -\psi_{,12} \qquad 2\text{-}3$$

where $\psi$ is Airy's stress function. If surface load is acting, a particular solution of the equilibrium equations is obtained by means of the integrals

$$J_1 = \int_{x_o(x_2)}^{x_1} p_1(t, x_2)dt \qquad 3\text{-}1$$

$$J_2 = \int_{y_o(x_1)}^{x_2} p_2(x_1, t)dt \qquad 3\text{-}2$$

the general solution for the stress resultants takes then the form

$$N_{11} = N^*_{11} - J_1 \qquad 4\text{-}1$$

$$N_{22} = N^*_{22} - J_2 \qquad 4\text{-}2$$

$$N_{12} = N^*_{12} \qquad 4\text{-}3$$

If the load components $p_1$ and $p_2$ derive from a potential function V according to the relations

$$p_1 = -V_{,1} \qquad 5\text{-}1$$

$$p_2 = -V_{,2} \qquad 5\text{-}2$$

It is possible to write

$$J_1 = J_2 = -V \qquad 6$$

The general solution of the equilibrium equations of the bending problem may be written in terms of a particular solution and of 2 stress functions

$G_1$ and $G_2$ in the form

$$M_{11} = G_{2,2} - D(K_2 + \nu K_1) \qquad 7\text{-}1$$

$$M_{22} = G_{1,1} - D(K_1 + \nu K_2) \qquad 7\text{-}2$$

$$M_{12} = M_{21} = -\frac{1}{2}(G_{1,2} + G_{2,1}) \qquad 7\text{-}3$$

$$Q_1 = F_{3,2} - D(K_2 + \nu K_1)_{,1} \qquad 7\text{-}4$$

$$Q_2 = -F_{3,1} - D(K_1 + \nu K_2)_{,2} \qquad 7\text{-}5$$

where D is a constant that will be identified with the flexural rigidity of the plate,

$$F_3 = \frac{1}{2}(G_{2,1} - G_{1,2}) \qquad 7\text{-}6$$

and $K_1$ and $K_2$ are two functions satisfying the differential equation

$$(K_2 + \nu K_1)_{,11} + (K_1 + \nu K_2)_{,22} = \frac{p_3}{D} \qquad 8$$

and are otherwise arbitrary. The effective transverse shears acting on edges parallel to the $x_1$ and $x_2$ axes, respectively, are

$$Q_{1e} = Q_1 + M_{12,2} = -G_{1,22} - D(K_2 + \nu K_1)_{,1} \qquad 9\text{-}1$$

$$Q_{2e} = Q_2 + M_{21,1} = -G_{2,11} - D(K_1 + \nu K_2)_{,2} \qquad 9\text{-}2$$

b) Strain-Displacement and Compatibility Relations.

Let $u_1$, $u_2$ and w be the components of the displacement vector $\bar{u}$ of a point in the middle plane of the plate, $\omega_3$ be the rotation about the normal to the plate, $\varepsilon_{11}$, $\varepsilon_{22}$ and ( $\varepsilon_{12} + \varepsilon_{21}$) be the in-plane extensional and shear strains and $x_{11}$, $x_{22}$ and $x_{12} = x_{21}$ be the flexural curvatures and twists. In addition to these strains it will prove convenient to define additional strain quantities $x_{13}$ and $x_{23}$ associated with the stretching problem. The strain-displacement relations may be written in the form (3)

$$\varepsilon_{11} = u_{1,1} \qquad 10\text{-}1$$

$$\varepsilon_{22} = u_{2,2} \qquad 10\text{-}2$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(u_{1,2} + u_{2,1}) \qquad 10\text{-}3$$

$$x_{13} = \omega_{3,1} \qquad 10\text{-}4$$

$$x_{23} = \omega_{3,2} \qquad 10\text{-}5$$

where

$$\omega_3 = \frac{1}{2}(u_{2,1} - u_{1,2}) \qquad 10\text{-}6$$

$$x_{11} = -w_{,11} \qquad 10\text{-}7$$

$$x_{22} = -w_{,22} \qquad 10\text{-}8$$

$$x_{12} = x_{21} = -w_{,12} \qquad 10\text{-}9$$

If we denote by $x_{13e}$ and $x_{23e}$ changes of in-plane curvature of lines originally parallel to the $x_1$ and $x_2$ axes respectively, we can write

$$x_{13e} = x_{13} + \varepsilon_{12,1} = u_{2,11} \qquad 10\text{-}10$$

$$x_{23e} = x_{23} - \varepsilon_{21,2} = -u_{1,22} \qquad 10\text{-}11$$

The in-plane strains satisfy identically in the displacements the relations

$$\varepsilon_{22,1} - \varepsilon_{12,2} - x_{23} = 0 \qquad 11\text{-}1$$

$$\varepsilon_{21,1} - \varepsilon_{11,2} - x_{13} = 0 \qquad 11\text{-}2$$

$$x_{23,1} - x_{13,2} = 0 \qquad 11\text{-}3$$

If $x_{13}$ and $x_{23}$ are eliminated from eqs. 11 there results the usual compatibility equation of the stretching problem.

$$\varepsilon_{22,11} - (\varepsilon_{12} + \varepsilon_{21})_{,12} + \varepsilon_{11,22} = 0 \qquad 12$$

The compatibility equations of the bending problem are

$$x_{22,1} - x_{12,2} = 0 \qquad 13\text{-}1$$

$$x_{21,1} - x_{11,2} = 0 \qquad 13\text{-}2$$

c) Stress-Strain Relations.

We consider a linearly elastic homogeneous and isotropic

plate of thickness h with Young's modulus E, Poisson's ratio $\nu$ and coefficient of thermal expansion $\alpha$. The stress-strain relations may be written in the form

$$\varepsilon_{11} - \varepsilon_o = \frac{1}{Eh}(N_{11} - \nu N_{22}) \qquad 14\text{-}1$$

$$\varepsilon_{22} - \varepsilon_o = \frac{1}{Eh}(N_{22} - \nu N_{11}) \qquad 14\text{-}2$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1+\nu}{Eh} N_{12} \qquad 14\text{-}3$$

$$M_{11} - M_o = D(x_{11} + \nu x_{22}) \qquad 14\text{-}4$$

$$M_{22} - M_o = D(x_{22} + \nu x_{11}) \qquad 14\text{-}5$$

$$M_{12} = M_{21} = D(1-\nu)x_{12} \qquad 14\text{-}6$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad 15$$

and where $\varepsilon_o$ is an initial strain and $M_o$ an initial stress couple due to an increase t of temperature measured from some initial state. Letting

$$t_o = \frac{1}{h}\int_{-h/2}^{h/2} t\,dx_3 \qquad 16\text{-}1$$

$$\delta t = \frac{12}{h^2}\int_{-h/2}^{h/2} t x_3\,dx_3 \qquad 16\text{-}2$$

and

$$x_o = \frac{\alpha\,\delta t}{h} \qquad 16\text{-}3$$

we can write

$$\varepsilon_o = \alpha t_o \qquad 16\text{-}4$$

$$M_o = -D(1 + \nu)x_o \qquad 16\text{-}5$$

3. Stretching-Bending Duality

The basic equations of the plate separate into two uncoupled systems corresponding to the stretching and bending problems, respectively.

There is a duality between the two systems of equations that is a particular case of the static-geometric analogy of shell theory where it is established, however, for zero surface load. The analogy may be generalized, in more than one way, to include the case of non zero surface load. For this purpose the superscript * will denote quantities associated with the homogeneous solution of the equilibrium equations, i.e. the general solution of these equations for zero distributed load. We define $\varepsilon^*_{11}$ and $\varepsilon^*_{22}$ through the relations

$$\varepsilon^*_{11} - \varepsilon_o = \frac{1}{Eh}(N^*_{11} - \nu N^*_{22}) \qquad 17\text{-}1$$

$$\varepsilon^*_{22} - \varepsilon_o = \frac{1}{Eh}(N^*_{22} - \nu N^*_{11}) \qquad 17\text{-}2$$

$x^*_{13}$ and $x^*_{23}$ through relations similar to eqs. 11-1, 2

$$\varepsilon^*_{22,1} - \varepsilon_{12,2} - x^*_{23} = 0 \qquad 18\text{-}1$$

$$\varepsilon_{21,1} - \varepsilon^*_{11,2} - x^*_{13} = 0 \qquad 18\text{-}2$$

and $x^*_{13e}$ and $x^*_{23e}$ through relations similar to eqs. 10-10, 11

$$x^*_{23e} = x^*_{23} - \varepsilon_{21,2} \qquad 19\text{-}1$$

$$x^*_{13e} = x^*_{13} + \varepsilon_{12,1} \qquad 19\text{-}2$$

It is then possible to write the equations of the stretching problem in a way such that they are transformed into the equations of the bending problem and vice versa if corresponding or dual dependent variables of

the two problems are interchanged, if $\nu$ is replaced by $-\nu$ and $\frac{1}{Eh}$ is replaced by -D. The duality is summarized below.

| Stretching | | | Bending |
|---|---|---|---|
| Equilibrium | | | Compatibility |
| $N^*_{11,1} + N_{21,2} = 0$ | $N^*_{11}$ | $-x_{22}$ | $-x_{22,1} + x_{12,2} = 0$ |
| $N_{12,1} + N^*_{22,2} = 0$ | $N_{12}$ | $x_{21}$ | $x_{21,1} - x_{11,2} = 0$ |
| | $N_{21}$ | $x_{12}$ | |
| Compatibility | | | Equilibrium |
| $\varepsilon^*_{22,1} - \varepsilon_{12,2} - x^*_{23} = 0$ | $\varepsilon^*_{22}$ | $M_{11}$ | $M_{11,1} + M_{21,2} - Q_1 = 0$ |
| $-\varepsilon_{21,1} + \varepsilon^*_{11,2} + x^*_{13} = 0$ | $\varepsilon_{12}$ | $-M_{21}$ | $M_{12,1} + M_{22,2} - Q_2 = 0$ |
| $x^*_{23,1} - x^*_{13,2} =$ | $\varepsilon_{21}$ | $-M_{12}$ | $Q_{1,1} + Q_{2,2} = -p_3 =$ |
| $\frac{1}{Eh}\left[(J_2 - \nu J_1)_{,11} +\right.$ | $\varepsilon^*_{11}$ | $M_{22}$ | $-D\left[(K_2 + \nu K_1)_{,11} + (K_1 + \nu K_2)_{,22}\right] =$ |
| $\left.(J_1 - \nu J_2)_{,22}\right] = \varepsilon^*_{22,11} -$ | $x^*_{23}$ | $Q_1$ | $M_{11,11} + 2M_{12,12} + M_{22,22}$ |
| $2\varepsilon_{12,12} + \varepsilon^*_{11,22}$ | $x^*_{13}$ | $-Q_2$ | |
| Stress-Strain Relations | $J_1$ | $K_1$ | Stress-Strain Relations |
| $\varepsilon^*_{11} - \varepsilon_o = \frac{1}{Eh}(N^*_{11} - \nu N^*_{22})$ | $J_2$ | $K_2$ | $M_{22} - M_o = D(x_{22} + \nu x_{11})$ |
| $\varepsilon^*_{22} - \varepsilon_o = \frac{1}{Eh}(N^*_{22} - \nu N^*_{11})$ | $\nu$ | $-\nu$ | $M_{11} - M_o = D(x_{11} + \nu x_{22})$ |

| | | | |
|---|---|---|---|
| $\varepsilon_{12} = \varepsilon_{21} = \frac{1+\nu}{Eh} N_{12}$ | $(Eh)^{-1}$ | $-D$ | $M_{12} = M_{21} = D(1-\nu)x_{12}$ |
| | $\varepsilon_o$ | $M_o$ | |
| $\varepsilon_o = -\frac{1-\nu}{Eh} N_o$ | $N_o$ | $-x_o$ | $M_o = -D(1+\nu)x_o$ |
| <u>Strain-Displacement Relations</u> | | | <u>Stress-Stress Function Relations</u> |
| $\varepsilon^*_{11} = u_{1,1} + \frac{1}{Eh}(J_1 - \nu J_2)$ | $u_1$ | $G_1$ | $M_{22} = G_{1,1} - D(K_1 + \nu K_2)$ |
| $\varepsilon^*_{22} = u_{2,2} + \frac{1}{Eh}(J_2 - \nu J_1)$ | $u_2$ | $G_2$ | $M_{11} = G_{2,2} - D(K_2 + \nu K_1)$ |
| $\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(u_{1,2} + u_{2,1})$ | $\omega_3$ | $F_3$ | $M_{12} = M_{21} = -\frac{1}{2}(G_{1,2} + G_{2,1})$ |
| $x^*_{23} = \omega_{3,2} + \frac{1}{Eh}(J_2 - \nu J_1)_{,1}$ | | | $Q_1 = F_{3,2} - D(K_2 + \nu K_1)_{,1}$ |
| $x^*_{13} = \omega_{3,1} - \frac{1}{Eh}(J_1 - \nu J_2)_{,2}$ | | | $Q_2 = -F_{3,1} - D(K_1 + \nu K_2)_{,2}$ |
| $\omega_3 = \frac{1}{2}(u_{2,1} - u_{1,2})$ | | | $F_3 = \frac{1}{2}(G_{2,1} - G_{1,2})$ |
| $x^*_{23e} = x^*_{23} - \varepsilon_{21,2} =$ | $x^*_{23e}$ | $Q_{1e}$ | $Q_{1e} = Q_1 + M_{12,2} =$ |
| $-u_{1,22} + \frac{1}{Eh}(J_2 - \nu J_1)_{,1}$ | $x^*_{13e}$ | $-Q_{2e}$ | $-G_{1,22} - D(K_2 + \nu K_1)_{,1}$ |
| $x^*_{13e} = x^*_{13} + \varepsilon_{12,1} =$ | | | $Q_{2e} = Q_2 + M_{21,1} =$ |
| $u_{2,11} - \frac{1}{Eh}(J_1 - \nu J_2)_{,2}$ | | | $-G_{2,11} - D(K_1 + \nu K_2)_{,2}$ |

| Stress-Stress Function Relations | | Curvature-Displacement Relations |
|---|---|---|
| $N^*_{11} = \Psi_{,22}$ | $\Psi$ \| $w$ | $x_{22} = - w_{,22}$ |
| $N^*_{22} = \Psi_{,11}$ | | $x_{11} = - w_{,11}$ |
| $N_{12} = N_{21} = - \Psi_{,12}$ | | $x_{12} = x_{21} = - w_{,12}$ |

The duality indicated above remains valid if the asterisk is associated instead with the dependent variables of the bending problem. In this case we have

$$M^*_{11} = G_{2,2} = M_{11} + D(K_2 + \nu K_1) \qquad 20\text{-}1$$

$$M^*_{22} = G_{1,1} = M_{22} + D(K_1 + \nu K_2) \qquad 20\text{-}2$$

$$Q^*_1 = F_{3,2} = Q_1 + D(K_2 + \nu K_1)_{,1} \qquad 20\text{-}3$$

$$Q^*_2 = - F_{3,1} = Q_2 + D(K_1 + \nu K_2)_{,2} \qquad 20\text{-}4$$

$x^*_{11}$ and $x^*_{22}$ are defined through the relations

$$M^*_{11} - M_o = D(x^*_{11} + \nu x^*_{22}) \qquad 21\text{-}1$$

$$M^*_{22} - M_o = D(x^*_{22} + \nu x^*_{11}) \qquad 21\text{-}2$$

which yield

$$x^*_{11} = x_{11} + K_2 \qquad 22\text{-}1$$

$$x^*_{22} = x_{22} + K_1 \qquad 22\text{-}2$$

It may be seen then that $N_{11}$ is dual of $-x^*_{22}$, $N_{22}$ of $-x^*_{11}$, $x_{23}$ of $Q^*_1$, etc.

4. Duality Between Methods of Solution.

The duality between the methods of solution is a direct consequence of the duality between the basic equations. Thus the reduction of the stretching problem to a differential equation for $\psi$ through use of the compatibility equations is the dual of the reduction of the bending problem to a differential equation for w through use of the equilibrium equations. These differential equations are

$$\Delta\Delta\psi = -Eh\,\Delta\varepsilon_o + (J_2 - \nu J_1)_{,11} + (J_1 - \nu J_2)_{,22} \qquad 23$$

and

$$\Delta\Delta w = \frac{\Delta M_o}{D} + (K_2 + \nu K_1)_{,11} + (K_1 + \nu K_2)_{,22} \qquad 24$$

$$= \frac{\Delta M_o}{D} + \frac{p_3}{D}$$

where $\Delta$ is Laplace's operator

$$\Delta(\ ) = (\ )_{,11} + (\ )_{,22} \qquad 25$$

It is noted that if a bending problem dual of a given stretching problem is sought, the determination of the load $p_3$ is made by means of $K_1$ and $K_2$ which are dual of $J_1$ and $J_2$, respectively. These are determined through eqs. 3 and may involve arbitrary functions of $x_2$ and $x_1$, respectively, as indicated by the limits of integration $x_o(x_2)$ and $y_o(x_1)$. If $p_1$ and $p_2$ derive from a potential V according to eqs. 5 and 6 the duality gives

$$\frac{p_3}{D} \Leftrightarrow -(1+\nu)\Delta V$$

where the symbol <=> is used to indicate an equality obtained through the dual correspondence. In such a case a particular solution $\Delta\psi_p$ for $\Delta\psi$ and the dual solution for $\Delta w$ may be written as

$$\Delta\psi_p = - Eh\,\varepsilon_o - (1 - \nu)V$$

$$\Delta w_p \Leftrightarrow \frac{\varepsilon_o}{D} - (1 + \nu)V$$

It is noticed that the effect of $\varepsilon_o$ is similar to that of a fictitious loading corresponding to the potential V.

If a stretching problem dual of a given bending problem is sought the load components $p_1$ and $p_2$ are obtained through the duality relations

$$p_1 \Leftrightarrow K_{1,1} \qquad 26\text{-}1$$

$$p_2 \Leftrightarrow K_{2,2} \qquad 26\text{-}2$$

$K_1$ and $K_2$ are related to $p_3$ through eq. 8 and may be determined in an infinity of ways. For example one may write

$$K_2 + \nu K_1 = \frac{c}{D}\int dx_1 \int p_3\, dx_1$$

$$K_1 + \nu K_2 = \frac{1-c}{D}\int dx_2 \int p_3 dx_2$$

where c is an arbitrary constant. This type of solution may be interpreted as a particular solution of the plate bending equilibrium equations obtained by considering the plate as made of independent strips parallel to the coordinate axes with the load $p_3$ divided between the two families of parallel strips in the proportions c and 1-c. The type of end supports of these strips is arbitrary.

Another way of determining $K_1$ and $K_2$ consists in taking

$$K_1 = K_2 = K$$

and determining a particular solution of the differential equation

$$\Delta K = \frac{p_3}{D(1+\nu)}$$

The load components $p_1$ and $p_2$ in the dual problem derive then from a potential V dual of -K.

$$V \Leftrightarrow -K \qquad 27$$

A particular solution for $\Delta w$ is then

$$\Delta w_p = \frac{M_o}{D} + (1+\nu)K$$

and the dual solution for $\Delta\psi$ is

$$\Delta\psi_p \Leftrightarrow -Eh\, M_o + (1-\nu)K$$

The solution of the stretching problem by the displacement method which expresses the equilibrium equations in terms of the displacement components, has as dual the solution of the bending problem by the stress function method which expresses the compatibility relations in terms of the stress functions. These equations may be written in the form

$$\Delta u_1 + \frac{1+\nu}{1-\nu}(u_{1,1} + u_{2,2})_{,1} + \frac{2(1+\nu)}{Eh} p_1 - \frac{2(1+\nu)}{1-\nu} \varepsilon_{o,1} = 0 \qquad 28\text{-}1$$

$$\Delta u_2 + \frac{1+\nu}{1-\nu}(u_{1,1} + u_{2,2})_{,2} + \frac{2(1+\nu)}{Eh} p_2 - \frac{2(1+\nu)}{1-\nu} \varepsilon_{o,2} = 0 \qquad 28\text{-}2$$

and

$$\Delta G_1 + \frac{1-\nu}{1+\nu}(G_{1,1} + G_{2,2})_{,1} - 2D(1-\nu)K_{1,1} - \frac{2(1-\nu)}{1+\nu} M_{o,1} = 0 \qquad 28\text{-}3$$

$$\Delta G_2 + \frac{1-\nu}{1+\nu}(G_{1,1} + G_{2,2})_{,2} - 2D(1-\nu)K_{2,2} - \frac{2(1-\nu)}{1+\nu} M_{o,2} = 0 \qquad 28\text{-}4$$

The above equations may be rewritten equivalently in the form

$$\left(\frac{u_{1,1} + u_{2,2}}{1-\nu}\right)_{,1} - \left(\frac{u_{2,1} - u_{1,2}}{2}\right)_{,2} + \frac{1+\nu}{Eh} p_1 - \frac{1+\nu}{1-\nu} \varepsilon_{o,1} = 0 \qquad 29\text{-}1$$

$$\left(\frac{u_{1,1} + u_{2,2}}{1-\nu}\right)_{,2} + \left(\frac{u_{2,1} - u_{1,2}}{2}\right)_{,1} + \frac{1+\nu}{Eh} p_2 - \frac{1+\nu}{1-\nu} \varepsilon_{o,2} = 0 \qquad 29\text{-}2$$

and

$$\left(\frac{G_{1,1} + G_{2,2}}{1+\nu}\right)_{,1} - \left(\frac{G_{2,1} - G_{1,2}}{2}\right)_{,2} - D(1-\nu)K_{1,1} - \frac{1-\nu}{1+\nu} M_{o,1} = 0 \qquad 29\text{-}3$$

$$\left(\frac{G_{1,1} + G_{2,2}}{1+\nu}\right)_{,2} + \left(\frac{G_{2,1} - G_{1,2}}{2}\right)_{,1} - D(1-\nu)K_{2,2} - \frac{1-\nu}{1+\nu} M_{o,2} = 0 \qquad 29\text{-}4$$

It is known that the general solution of the stretching problem in the absence of surface load and thermal strain may be expressed in terms of analytic functions of the complex variable

$$z = x_1 + ix_2 \qquad 30$$

If the load components and the initial strain are deleted from eqs. 28-1, 2 these reduce to Cauchy-Riemann conditions allowing to write

$$\frac{u_{1,1} + u_{2,2}}{1-\nu} + i\frac{u_{2,1} - u_{1,2}}{2} = f'(z) \qquad 31$$

The superscript * will not be used here for simplicity of notation.

$f'(z)$ in eq. 31 is an arbitrary analytic function of z. It is taken for convenience as the derivative of an arbitrary analytic function $f(z)$. If we let subscripts r and i denote real and imaginary parts, respectively, of a complex number we can write

$$f_{r,1} = f_{i,2} \qquad 32\text{-}1$$

$$f_{r,2} = -f_{i,1} \qquad 32\text{-}2$$

and from eq. 31

$$u_{1,1} + u_{2,2} = (1-\nu)f_{r,1} = (1-\nu)f_{i,2} \qquad 33\text{-}1$$

$$u_{2,1} - u_{1,2} = 2f_{i,1} = -2f_{r,2} \qquad 33\text{-}2$$

Again eqs. 33 reduce to Cauchy Riemann conditions when the right hand sides are put to zero. The particular solution needed to write the general

solution of eqs. 33 is easily found on noting that $f_{r,1}$ and $f_{i,2}$ are harmonic functions. The result may be written in the form

$$u_1 + iu_2 = \frac{3-\nu}{4} f - \frac{1+\nu}{4} z\bar{f}' + \bar{g} \qquad 34$$

where f and g are arbitrary analytic functions of z and a bar, such as in $\bar{g}$, indicates the complex number conjugate of the number without the bar, i. e.

$$\bar{g} = g_r - ig_i \qquad 35\text{-}1$$

$$\bar{f}' = f_{r,1} - if_{i,1} = f_{i,2} + if_{r,2} \qquad 35\text{-}2$$

Similarly, for the homogeneous problem of the bending of plates we can write the general solution for the stress functions in the form

$$G_1 + iG_2 = \frac{3+\nu}{4} F - \frac{1-\nu}{4} z\bar{F}' + \bar{G} \qquad 36$$

where F and G are arbitrary analytic functions. Eq. 34 allows expressing the strains and the stress resultants of the stretching problem in terms of f and g. Eq. 31 yields directly

$$\frac{\varepsilon_{11} + \varepsilon_{22}}{1-\nu} + i\,\omega_3 = f' \qquad 37\text{-}1$$

and using eq. 34 and the strain displacement relations, obtain

$$\varepsilon_{11} - \varepsilon_{22} + i(\varepsilon_{12} + \varepsilon_{21}) = -\frac{1+\nu}{2} z\bar{f}'' + 2\bar{g}' \qquad 37\text{-}2$$

In order to express the stress function $\psi$ in terms of the analytic functions, eqs. 37 with use of the stress strain relations and the stress-stress function relations yield

$$\frac{\varepsilon_{11} + \varepsilon_{22}}{1-\nu} = \frac{1}{Eh}\Delta\psi = f_{r,1} \qquad 38\text{-}1$$

$$\frac{\varepsilon_{11} - \varepsilon_{22}}{1+\nu} = \frac{1}{Eh}(\psi_{,22} - \psi_{,11}) = -\frac{1}{2}(x_1 f_{r,11} + x_2 f_{i,11}) + \frac{2}{1+\nu} g_{r,1} \qquad 38\text{-}2$$

$$\frac{\varepsilon_{12} + \varepsilon_{21}}{2(1+\nu)} = -\frac{1}{Eh}\psi_{,12} = -\frac{1}{4}(-x_1 f_{i,11} + x_2 f_{r,11}) - \frac{1}{1+\nu} g_{i,1} \qquad 38\text{-}3$$

The 3 equations for $\psi$ are compatible. The general solution of eq. 38-1 may be written in the form

$$\psi = \frac{Eh}{4}\left(x_1 f_r + x_2 f_i - \frac{4}{1+\nu} h_r\right) \qquad 39$$

where $h_r$ is an arbitrary harmonic function. Requiring $\psi$ as expressed in eq. 39 to be also a solution of eqs. 38-2, 3 yields the condition

$$h' = g + a \qquad 40$$

where h is the analytic function having $h_r$ as real part and a is an arbitrary complex constant. $\psi$ may also be written in the form

$$\psi = \frac{Eh}{4}\left(\bar{z} f - \frac{4}{1+\nu} h\right)_r \qquad 41$$

If f and h are considered as the two arbitrary analytic functions, g contains through eq. 40 the arbitrary constant a which contributes only a rigid body translation.

In the bending problem the equations dual of eqs. 37 and 41 are

$$\frac{M_{11} + M_{22}}{1+\nu} + iF_3 = F' \qquad 42\text{-}1$$

$$M_{22} - M_{11} - i(M_{12} + M_{21}) = -\frac{1-\nu}{2} \quad z\bar{F}'' + 2\bar{G}' \qquad 42\text{-}2$$

$$w = -\frac{1}{4D}\left(\bar{z} F - \frac{4}{1-\nu} H\right)_r \qquad 42\text{-}3$$

where

$$H' = G + A \qquad 43$$

We note also that

$$\Delta w = -\frac{F_{r,1}}{D} \qquad 44$$

The transverse shears are obtained through the relations

$$Q_1 = F_{3,2} = F_{i,12} = F_{r,11} \qquad 45\text{-}1$$

$$Q_2 = -F_{3,1} = -F_{i,11} = F_{r,12} \qquad 45\text{-}2$$

5. Dependent Variables at the Boundary.

Consider a generally multipliconnected domain bounded by closed curves such as shown in Fig. 3 and let $\varphi$ denote the angle, measured positively counterclockwise, between the $x_1$ axis and the normal $\bar{n}$ to a boundary curve oriented outward. n will denote a rectilinear coordinate along the normal $\bar{n}$ and s will denote the arclength along the boundary curve oriented as shown in Figure 3.

Considering the stretching problem, the stress resultants at the boundary obey the tensorial transformation formulas

$$N_{nn} = N_{11}\cos^2\varphi + 2N_{12}\sin\varphi\cos\varphi + N_{22}\sin^2\varphi \qquad 46\text{-}1$$

$$N_{ss} = N_{11}\sin^2\varphi - 2N_{12}\sin\varphi\cos\varphi + N_{22}\cos^2\varphi \qquad 46\text{-}2$$

$$N_{ns} = N_{12}(\cos^2\varphi - \sin^2\varphi) + (N_{22} - N_{11})\sin\varphi\cos\varphi \qquad 46\text{-}3$$

The stress-stress function relations in orthogonal curvilinear coordinates (4) when applied to the directions n and s at the boundary take the form

$$N^*_{nn} = \Psi_{,ss} + \varphi_{,s}\Psi_{,n} \qquad 47\text{-}1$$

$$N^*_{ss} = \Psi_{,nn} \qquad 47\text{-}2$$

$$N^*_{ns} = -\Psi_{,ns} + \varphi_{,s}\Psi_{,s} \qquad 47\text{-}3$$

where it is recalled that the asterisk refers to the homogeneous solution of the equilibrium equations. We note that

$$\psi_{,sn} \neq \psi_{,ns} \tag{48}$$

and $\psi_{,sn}$ must be interpreted as

$$\psi_{,sn} = \psi_{,ns} - \varphi_{,s}\,\psi_{,s} \tag{49}$$

$N^*_{nn}$, $N^*_{ss}$, and $N^*_{ns}$ are related to $N^*_{11}$, $N^*_{22}$ and $N^*_{12} = N_{12}$ through the same transformation formulas as eqs. 46, whence using eqs. 4

$$N_{nn} = N^*_{nn} - J_{nn} \tag{50-1}$$

$$N_{ss} = N^*_{ss} - J_{ss} \tag{50-2}$$

$$N_{ns} = N^*_{ns} - J_{ns} \tag{50-3}$$

where

$$J_{nn} = J_1 \cos^2\varphi + J_2 \sin^2\varphi \tag{51-1}$$

$$J_{ss} = J_1 \sin^2\varphi + J_2 \cos^2\varphi \tag{51-2}$$

$$J_{ns} = (J_2 - J_1) \sin\varphi \cos\varphi \tag{51-3}$$

$J_1$ and $J_2$ may be interpreted as the principal components of a tensor having the $x_1$ and $x_2$ axes as principal axes. The normal and tangential components of displacement $u_n$ and $u_s$ satisfy the vector transformation formulas

$$u_n = u_1 \cos\varphi + u_2 \sin\varphi \tag{52-1}$$

$$u_s = -u_1 \sin\varphi + u_2 \cos\varphi \tag{52-2}$$

The strains $\varepsilon_{nn}$, $\varepsilon_{ss}$ and $\varepsilon_{ns} = \varepsilon_{sn}$ satisfy the same transformation formulas

as the stress resultants. In terms of $u_n$ and $u_s$ they take the form (4)

$$\varepsilon_{nn} = u_{n,n} \qquad 53\text{-}1$$

$$\varepsilon_{ss} = u_{s,s} + \varphi_{,s} u_n \qquad 53\text{-}2$$

$$\varepsilon_{ns} = \varepsilon_{sn} = \frac{1}{2}(u_{s,n} + u_{n,s} - \varphi_{,s} u_s) \qquad 53\text{-}3$$

The strains $x_{n3}$ and $x_{s3}$ are obtained from $x_{13}$ and $x_{23}$ through vector transformation formulas as eqs. 52 and are related to $\omega_3$ through the relations

$$x_{n3} = \omega_{3,n} \qquad 54\text{-}1$$

$$x_{s3} = \omega_{3,s} \qquad 54\text{-}2$$

where

$$\omega_3 = \frac{1}{2}(u_{s,n} - u_{n,s} + \varphi_{,s} u_s) \qquad 55$$

$x_{n3}$ and $x_{s3}$ are related to $\varepsilon_{ss}$, $\varepsilon_{nn}$ and $\varepsilon_{ns}$ through the compatibility equations (3).

$$x_{n3} = -\varepsilon_{nn,s} + \varepsilon_{sn,n} - \varphi_{,s}(\varepsilon_{sn} + \varepsilon_{ns}) \qquad 56\text{-}1$$

$$x_{s3} = \varepsilon_{ss,n} - \varepsilon_{ns,s} + \varphi_{,s}(\varepsilon_{ss} - \varepsilon_{nn}) \qquad 56\text{-}2$$

The change of in-plane curvature of the boundary curve, except for the effect of the extensional strain $\varepsilon_{ss}$, is

$$x_{s3e} = x_{s3} - \varepsilon_{sn,s} = (\omega_3 - \varepsilon_{sn})_{,s} = -(u_{n,s} - \varphi_{,s} u_s)_{,s} \qquad 57$$

As done in cartesian coordinates, $\varepsilon^*_{nn}$, $\varepsilon^*_{ss}$ and $\varepsilon^*_{sn}$ are defined by relating them through the stress strain relations to the homogeneous solution of the equilibrium equations

$$\varepsilon^*_{nn} = \varepsilon_{nn} + \frac{1}{Eh}(J_{nn} - \nu J_{ss}) \qquad 58\text{-}1$$

$$\varepsilon^*_{ss} = \varepsilon_{ss} + \frac{1}{Eh}(J_{ss} - \nu J_{nn}) \qquad 58\text{-}2$$

$$\varepsilon^*_{ns} + \varepsilon^*_{sn} = \varepsilon_{ns} + \varepsilon_{sn} + \frac{2(1+\nu)}{Eh} J_{ns} \qquad 58\text{-}3$$

$x^*_{n3}$ and $x^*_{s3}$ are related by definition to $\varepsilon^*_{nn}$, $\varepsilon^*_{ss}$ and $\varepsilon^*_{ns} = \varepsilon^*_{sn}$ through the compatibility equations 56 and are also related to $x^*_{13}$ and $x^*_{23}$ through vector transformation formulas. Obtain

$$x^*_{n3} = x_{n3} - \frac{1}{Eh}\left[(J_1 - \nu J_2)_{,2} \cos\varphi - (J_2 - \nu J_1)_{,1} \sin\varphi\right] \qquad 59\text{-}1$$

$$x^*_{s3} = x_{s3} + \frac{1}{Eh}\left[(J_1 - \nu J_2)_{,2} \sin\varphi + (J_2 - \nu J_1)_{,1} \cos\varphi\right] \qquad 59\text{-}2$$

Finally

$$x^*_{s3e} = x^*_{s3} - \varepsilon^*_{sn,s}$$

$$= x_{s3e} + \frac{1}{Eh}\left[(J_1 - \nu J_2)_{,2} \sin\varphi + (J_2 - \nu J_1)_{,1} \cos\varphi\right] - \frac{1+\nu}{Eh} J_{ns,s} \qquad 60$$

Similar relations for the bending problem may be written using the duality with the stretching problem. The same correspondence as indicated in cartesian coordinates applies provided subscripts 1 and 2 are replaced by n and s, respectively.

Before considering the duality in the boundary conditions, formulas that relate the stress function $\psi$ and its derivatives to statical equivalents of the boundary stress resultants $N^*_{nn}$ and $N^*_{ns}$ will be established. Their dual will be formulas relating w and its derivatives to the changes of curvature $x_{ss}$ and $x_{ns}$.

Also formulas for determining the displacements $u_1$ and $u_2$ from known extensional strain $\varepsilon_{ss}$ and change of in-plane curvature $x_{s3e}$ of a boundary curve will be derived. Their dual will be formulas allowing the determination of the stress functions $G_1$ and $G_2$ from known bending moment and effective transverse shear at the boundary.

6. Determination of $\psi$ from $N^*_{nn}$ and $N^*_{ns}$. Determination of w from $x_{ss}$ and $x_{sn}$.

If $\bar{N}_n$ denotes the stress resultant vector acting at a boundary point, then by statics

$$\bar{N}_n = \bar{N}_1 \cos\varphi + \bar{N}_2 \sin\varphi \qquad 61$$

where $\bar{N}_1$ and $\bar{N}_2$ are the stress resultant vectors acting on cuts perpendicular to the $x_1$ and $x_2$ axes, respectively.

Using the relations

$$\cos\varphi = x_{1,n} = x_{2,s} \qquad 62\text{-}1$$

$$\sin\varphi = x_{2,n} = -x_{1,s} \qquad 62\text{-}2$$

and the stress-stress function relations the cartesian components $N_{n1}$ and $N_{n2}$ of $\bar{N}_n$ are expressed in the form

$$N_{n1} = \psi_{,2s} - J_1 x_{2,s} \qquad 63\text{-}1$$

$$N_{n2} = -\psi_{,1s} + J_2 x_{1,s} \qquad 63\text{-}2$$

The components $V_1$ and $V_2$ of the resultant of the forces acting on a segment of a boundary curve, described in the positive sense between an arbitrary origin of arclength $(x_1^\circ, x_2^\circ)$ and a current point $(x_1, x_2)$ at arclength s from the origin, are

$$V_1 = \int_0^s N_{n1}\, ds = \psi_{,2} - \psi^\circ_{,2} - \int_0^s J_1\, dx_2 \qquad 64\text{-}1$$

$$V_2 = \int_0^s N_{n2}\, ds = -\psi_{,1} + \psi^\circ_{,1} + \int_0^s J_2\, dx_1 \qquad 64\text{-}2$$

where superscript ° refers to the arbitrary origin $(x^\circ_1, x^\circ_2)$ on each boundary curve.

The moment $C_3$ with regard to point $(x_1, x_2)$ of the forces described above has the differential

$$dC_3 = V_1\, dx_2 - V_2\, dx_1$$

$$= d\psi - \psi^\circ_{,1}\, dx_1 - \psi^\circ_{,2}\, dx_2 - \left[\int_0^s J_1\, dx_2\right] dx_2 - \left[\int_0^s J_2\, dx_1\right] dx_1$$

whence

$$C_3 = \psi - \psi^\circ - (x_1 - x^\circ_1)\psi^\circ_{,1} - (x_2 - x^\circ_2)\psi^\circ_{,2} -$$

$$\int_0^s \left\{\left[\int_0^s J_2\, dx_1\right] dx_1 + \left[\int_0^s J_2\, dx_2\right] dx_2\right\} \qquad 64\text{-}3$$

Eqs. 64 may be written more concisely in the form

$$\psi_{,1} = \psi^\circ_{,1} - V^*_2 \qquad 65\text{-}1$$

$$\psi_{,2} = \psi^\circ_{,2} + V^*_1 \qquad 65\text{-}2$$

$$\psi = \psi^\circ + (x_1 - x^\circ_1)\psi^\circ_{,1} + (x_2 - x^\circ_2)\psi^\circ_{,2} + C^*_3 \qquad 65\text{-}3$$

where $V^*_1$, $V^*_2$, and $C^*_3$ have the same meaning as $V_1$, $V_2$, and $C_3$ but are due to the homogeneous solution only. Eqs. 65 show that if the forces acting on a boundary curve that are due to the homogeneous solution only have the resultants $R^*_1$ and $R^*_2$ and the moment $M^*_3$ with regard to a point $(x^\circ_1, x^\circ_2)$ of the boundary curve, then $\psi$, $\psi_{,1}$ and $\psi_{,2}$ are multi-valued and their "jumps" after a complete turn around the boundary curve are

$$\delta(\psi_{,1}) = -R^*_2 \qquad 66\text{-}1$$

$$\delta(\psi_{,2}) = R^*_1 \qquad 66\text{-}2$$

$$\delta\psi = M^*_3 \qquad 66\text{-}3$$

For a simply connected domain not subjected to concentrated forces or torques the right hand sides of eqs. 66 are necessarily zero and $\psi$ and its derivatives are singlevalued. In the dual bending problem, $w_{,1}$, $w_{,2}$ and w are expressed in terms of integrals along a boundary curve of changes of curvature. Here the duals of $N^*_{n1}$ and $N^*_{n2}$ are, respectively,

$$-x_{s2} = -x_{ss}\cos\varphi - x_{sn}\sin\varphi = w_{,2s} \qquad 67\text{-}1$$

and

$$x_{s1} = -x_{ss}\sin\varphi + x_{sn}\cos\varphi = -w_{,1s} \qquad 67\text{-}2$$

If $x_{s1}$ and $x_{s2}$ are known $w_{,1}$, $w_{,2}$ and w may be determined through the relations

$$w_{,1} = w^\circ_{,1} - \int_0^s x_{s1}\,ds \qquad 68\text{-}1$$

$$w_{,2} = w^\circ_{,2} - \int_0^s x_{s2}\,ds \qquad 68\text{-}2$$

$$w = w^\circ + (x_1 - x^\circ_1)\,w^\circ_{,1} + (x_2 - x^\circ_2)w^\circ_{,2} - x_1\int_0^s x_{s1}\,ds - x_2\int_0^s x_{s2}\,ds$$

$$+\int_0^s (x_1 x_{s1} + x_2 x_{s2})\,ds \qquad 68\text{-}3$$

To the multivaluedness of $\psi$ and its derivatives in the stretching problem correspond in the bending problem the dislocations

$$\delta(w_{,1}) = -\oint x_{s1}\, ds \qquad 69\text{-}1$$

$$\delta(w_{,2}) = -\oint x_{s2}\, ds \qquad 69\text{-}2$$

$$\delta w = x_1^\circ(\delta w_{,1}) + x_2^\circ(\delta w_{,2}) + \oint (x_1 x_{s1} + x_2 x_{s2})\, ds \qquad 69\text{-}3$$

In such a case we will say that the changes of curvature $x_{ss}$ and $x_{sn}$ along the boundary curve are not compatible.

7. <u>Determination of $u_1$ and $u_2$ from $\varepsilon_{ss}$ and $x_{s3e}$. Determination of $G_1$ and $G_2$ from $M_{nn}$ and $Q_{ne}$.</u>

If the extensional strain $\varepsilon_{ss}$ and the in-plane change of curvature $x_{s3e}$ are known on a boundary curve, the displacements may be determined by integration of the strain displacement relations

$$u_{s,s} + \varphi_{,s}\, u_n = \varepsilon_{ss} \qquad 70\text{-}1$$

$$(u_{n,s} - \varphi_{,s}\, u_s)_{,s} = -x_{s3e} \qquad 70\text{-}2$$

letting

$$\omega = -u_{n,s} + \varphi_{,s}\, u_s \qquad 71$$

and integrating eq. 70-2 between s = 0 and s along each boundary curve obtain

$$\omega = \omega^\circ + \int_0^s x_{s3e}\, ds \qquad 72$$

$\omega$ is the rotation of the boundary curve measured positively counterclockwise and is distinct from $\omega_3$. Solving eqs. 52 for $u_1$ and $u_2$ in terms of $u_n$ and $u_s$, obtain after differentiation and use of eqs. 62

$$u_{1,s} = -\omega x_{2,s} + \varepsilon_{ss} x_{1,s} \qquad 73\text{-}1$$

$$u_{2,s} = \omega x_{1,s} + \varepsilon_{ss} x_{2,s} \qquad 73\text{-}2$$

Integrating eqs. 73 between s = 0 and s and performing integrations by parts the result may be written in the form

$$u_1 = u_1^\circ - (x_2 - x_2^\circ)\omega^\circ - x_2 \int_0^s x_{s3e}\, ds + \int_0^s (x_2 x_{s3e} + \varepsilon_{ss} x_{1,s})\, ds \qquad 74\text{-}1$$

$$u_2 = u_2^\circ + (x_1 - x_1^\circ)\omega^\circ + x_1 \int_0^s x_{s3e}\, ds - \int_0^s (x_1 x_{s3e} - \varepsilon_{ss} x_{2,s})\, ds \qquad 74\text{-}2$$

In general, arbitrarily given boundary strains $\varepsilon_{ss}$ and $x_{s3e}$ are not compatible, in the sense that they imply dislocations $\delta\omega$, $\delta u_1$ and $\delta u_2$ obtained from eqs. 72 and 74 by performing the integrations around each boundary curve.

$$\delta\omega = \oint x_{s3e}\, ds \qquad 75\text{-}1$$

$$\delta u_1 = -x_2^\circ \delta\omega + \oint (x_2 x_{s3e} + \varepsilon_{ss} x_{1,s})\, ds \qquad 75\text{-}2$$

$$\delta u_2 = x_1^\circ \delta\omega - \oint (x_1 x_{s3e} - \varepsilon_{ss} x_{2,s})\, ds \qquad 75\text{-}3$$

The determination of $G_1$ and $G_2$ from given $M_{nn}$ and $Q_{ne}$ is made through equations dual of eqs. 70

$$G_{s,s} + \varphi_{,s} G_n = M^*_{nn} \qquad 76\text{-}1$$

$$(G_{n,s} - \varphi_{,s} G_s)_{,s} = -Q^*_{ne} \qquad 76\text{-}2$$

where $M^*_{nn}$ and $Q^*_{ne}$ are due to the homogeneous solution of the equilibrium equations and are related to $M_{nn}$ and $Q_{ne}$ through relations dual of eqs. 58-2 and 60, i. e.

$$M^*_{nn} = M_{nn} + D(K_{ss} + \nu K_{nn}) \qquad 77\text{-}1$$

$$Q^*_{ne} = Q_{ne} + D\left[(K_1 + \nu K_2)_{,2} \sin\varphi + (K_2 + \nu K_1)_{,1} \cos\varphi\right] - D(1 - \nu)K_{ns,s} \qquad 77\text{-}2$$

letting

$$F = -G_{n,s} + \varphi_{,s} G_s \qquad 78$$

the equations dual of eqs. 72 and 74 are

$$F = F^\circ + \int_0^s Q^*_{ne}\, ds \qquad 79\text{-}1$$

$$G_1 = G^\circ_1 - (x_2 - x^\circ_2)F^\circ - x_2 \int_0^s Q^*_{ne}\, ds + \int_0^s (x_2 Q^*_{ne} + M^*_{nn} x_{1,s})ds \qquad 79\text{-}2$$

$$G_2 = G^\circ_2 + (x_1 - x^\circ_1)F^\circ + x_1 \int_0^s Q^*_{ne}\, ds - \int_0^s (x_1 Q^*_{ne} - x_{2,s} M^*_{nn})\, ds \qquad 79\text{-}3$$

To given $M^*_{nn}$ and $Q^*_{ne}$ correspond in general multivalued stress functions. If the integrations in eqs. 79 are performed around each boundary curve there results equations dual of eqs. 75,

$$\delta F = \oint Q^*_{ne}\, ds \qquad 80\text{-}1$$

$$\delta G_1 = - x^\circ_2\, \delta F + \oint (x_2 Q^*_{ne} + M^*_{nn} x_{1,s})\, ds \qquad 80\text{-}2$$

$$\delta G_2 = x^\circ_1\, \delta F - \oint (x_1 Q^*_{ne} - M^*_{nn} x_{2,s})\, ds \qquad 80\text{-}3$$

If we let $R^*_3$ denote the resultant and $M^*_1$ and $M^*_2$ the moments with regard to the coordinate axes of the distributed forces $Q^*_{ne}$ and stress couples

$M^*_{nn}$ acting on the boundary curve eqs. 80 state that

$$\delta F = -\delta(G_{n,s}) + \delta(\varphi_{,s} G_s) = R^*_3 \qquad 81\text{-}1$$

$$\delta G_1 = M^*_1 - x^\circ_2 R^*_3 \qquad 81\text{-}2$$

$$\delta G_2 = M^*_2 + x^\circ_1 R^*_3 \qquad 81\text{-}3$$

It may be interesting to note that eqs. 75 may be obtained by application of the theorem of complementary virtual work, also called theorem of virtual forces, to a curved beam in the shape of a boundary curve having the strains $\varepsilon_{ss}$ and $x_{s3e}$. If the beam is cut at the arbitrary origin of arc-length and the two faces of the cut are subjected to unit virtual forces acting in opposite directions and consisting of a bending moment and of two forces parallel to the coordinate axes, then eqs. 75 state the equality between external complementary work i. e. the dislocations and the internal complementary work.

Similarly, eqs. 80 may be obtained by application of the theorem of virtual work to the curved beam subjected to the distributed load $Q^*_{ne}$ and to the distributed torque $M^*_{nn}$. Here $\delta F$, $\delta G_1$ and $\delta G_2$ are a concentrated force and two concentrated moments, respectively, that have to be applied at one face of the cut at point $(x^\circ_1, x^\circ_2)$ in order to keep the beam in equilibrium.

## 8. Duality between Boundary Conditions.

### a) Displacement Boundary Conditions in the Stretching Problem Stress-Function or Force Boundary Conditions in the Bending Problem

A stretching problem where $u_1$ and $u_2$ are to take specified values on the boundary is generally a well defined problem having a unique solution. In the dual bending problem $G_1$ and $G_2$ take assigned values

on the boundary. They determine the bending stress couple $M^*_{nn}$ and the effective transverse shear $Q^*_{ne}$ at the boundary just as the in-plane displacements of the boundary determine its extensional strain $\varepsilon_{ss}$ and its change of curvature $x_{s3e}$. If the stretching problem is solved by determining the displacements the remaining quantities of interest such as strains and stresses may be computed through the strain-displacement relations and the stress strain relations, respectively. In the dual bending problem, knowledge of the stress functions allows the determination of the stress couples and transverse shears through the stress-stress function relations and the curvatures through the stress-strain relations. Whereas in the stretching problem the determination of the stress function $\psi$ is of no particular interest when the displacements and stresses have been determined, the determination of w in the bending problem may be accomplished through eqs. 68 when applied along a path drawn inside the plate. The integration constants $w^\circ$, $w^\circ_{,1}$ and $w^\circ_{,2}$ in eqs. 68 pertain in the present context to some arbitrary point in the plate and contribute a rigid body displacement. The result of the integrations in eqs. 68 may lead however to multivalued functions for w and its derivatives just as in the stretching problem the determination of $\psi$ and its derivatives from known stress resultants may lead to multivalued functions. This occurs if the stress resultants of the homogeneous solution acting on a boundary curve are not self equilibrating and in the bending problem if the changes of curvatures $x_{ss}$ and $x_{sn}$ along a boundary curve are not compatible. When displacement boundary conditions are imposed in the stretching problem of a multiply connected domain there is no way of determining a priori whether the stress resultants of the homogeneous solution will be self equilibrating on each boundary curve. Similarly there is in general no way of determining a priori whether stress-function boundary conditions in the bending problem will lead to multivalued functions for w and its derivatives, i. e., to dislocations. However, stress-function boundary conditions do not have as immediate a physical meaning

as do displacement boundary conditions in the stretching problem. Consider instead that the boundary values of the stress functions $G_1$ and $G_2$ are to be determined from specified boundary values of the bending stress couple $M_{nn}$ and the effective transverse shear $Q_{ne}$. This may be done through eqs. 79 which contain, however, for each boundary curve, three arbitrary constants $G^\circ_1$, $G^\circ_2$, and $F^\circ$. Because of these arbitrary constants a bending problem with given force boundary conditions may be formulated into an infinite number of bending problems with stress function boundary conditions. In examining the significance of this let us examine at the same time the dual stretching problem and consider the case where instead of displacement boundary conditions the extensional strain $\varepsilon_{ss}$ and the change of curvature $x_{s3e}$ take specified values on the boundary curves. If $\varepsilon_{ss}$ and $x_{s3e}$ are known on each boundary curve of a generally multipliconnected domain, the boundary displacements may be determined through eqs. 72 and 74 except for a rigid body displacement associated with each curve and formed of the translation $(u^\circ_1, u^\circ_2)$ and of the rotation $\omega^\circ$. This is independent of whether $\varepsilon_{ss}$ and $x_{s3e}$ imply dislocations. That to each particular choice of these rigid body displacements corresponds a physically different problem with the same boundary values of $\varepsilon_{ss}$ and $x_{s3e}$ may be seen by visualizing the boundary curves connected to free rigid bodies, keeping one of the bodies fixed, and subjecting each of the remaining ones to a certain rigid body displacement by applying an arbitrary force vector $(R^*_1, R^*_2)$ in the plane of the plate and an arbitrary torque $M^*_3$. Since displacement boundary conditions determine uniquely the problem there is a unique correspondence between the set of rigid body displacements and the set of forces $R^*_1$, $R^*_2$ and $M^*_3$. To each set of rigid body displacements $(u^\circ_1, u^\circ_2, \omega^\circ)$ corresponds a stretching problem whose stress function $\psi$ is multivalued according to eqs. 66. Thus in the dual bending problem to each set of arbitrary constants $G^\circ_1$, $G^\circ_2$, and $F^\circ$ correspond a bending problem with dislocations. There follows that in order to be able to solve uniquely the bending problem with force boundary conditions by means of stress functions

conditions expressing given dislocations $\delta w$, $\delta(w_{,1})$ and $\delta(w_{,2})$, if any, must be appended to the boundary conditions. In order to do this $x_{s1}$ and $x_{s2}$ on the right sides of eqs. 69 are first expressed in terms of $x_{nn}$, $x_{ss}$, and $x_{ns}$ through eqs. 67 then in terms of the stress functions by means of the stress-strain and stress-stress function relations. This yields

$$x_{s2} = \frac{1}{D(1-\nu^2)}(G_{1,1}+G_{2,2})x_{2,s} - \frac{1}{D(1-\nu)}(G_{2,s} - F_3 x_{1,s}) - (K_1 + \frac{M_o}{D(1+\nu)})x_{2,s} \qquad 82\text{-}1$$

$$x_{s1} = \frac{1}{D(1-\nu^2)}(G_{1,1}+G_{2,2})x_{1,s} - \frac{1}{D(1-\nu)}(G_{1,s} + F_3 x_{2,s}) - (K_2 + \frac{M_o}{D(1+\nu)})x_{1,s} \qquad 82\text{-}2$$

Eqs. 69 become

$$\delta(w_{,1}) = -\frac{1}{D(1-\nu^2)}\oint(G_{1,1}+G_{2,2})dx_1 + \frac{1}{D(1-\nu)}\oint F_3\,dx_2 + \oint(K_2 + \frac{M_o}{D(1+\nu)})\,dx_1 + \frac{\delta G_1}{D(1-\nu)} \qquad 83\text{-}1$$

$$\delta(w_{,2}) = -\frac{1}{D(1-\nu^2)}\oint(G_{1,1}+G_{2,2})\,dx_2 - \frac{1}{D(1-\nu)}\oint F_3 dx_1 + \oint(K_1 + \frac{M_o}{D(1+\nu)})dx_2 + \frac{\delta G_2}{D(1-\nu)} \qquad 83\text{-}2$$

$$\delta w = x_1^\circ\delta(w_{,1}) + x_2^\circ\delta(w_{,2}) + \frac{1}{2D(1-\nu^2)}\oint(G_{1,1}+G_{2,2})dr^2 - \frac{2}{D(1-\nu)}\oint F_3\,dA -$$

$$\oint (K_2 x_1 \, dx_1 + K_1 x_2 \, dx_2 + \frac{M_o}{2D(1+\nu)} dr^2) + \frac{1}{D(1-\nu)} \oint (G_1 \, dx_1 + G_2 \, dx_2) -$$

$$\frac{1}{D(1-\nu)} (x^\circ_1 \delta G_1 + x^\circ_2 \delta G_2) \qquad 83\text{-}4$$

where

$$r^2 = x_1^2 + x_2^2 \qquad 84\text{-}1$$

$$dA = \frac{1}{2}(x_1 \, dx_2 - x_2 \, dx_1) \qquad 84\text{-}2$$

and

$$F_3 = \frac{1}{2}(G_{2,1} - G_{1,2}) \qquad 7\text{-}6$$

The equations dual of eqs. 83 express $\delta(\psi_{,1})$, $\delta(\psi_{,2})$ and $\delta\psi$ i.e. $-R^*_2$, $R^*_1$, and $M^*_3$, respectively, in terms of the displacements,

$$-R^*_2 = \frac{Eh}{1-\nu^2} \oint (u_{1,1} + u_{2,2}) dx_1 - \frac{Eh}{1+\nu} \oint \omega_3 \, dx_2 + \oint (J_2 - \frac{Eh \, \varepsilon_o}{1-\nu}) \, dx_1$$

$$- \frac{Eh \, \delta u_1}{1+\nu} \qquad 85\text{-}1$$

$$R^*_1 = \frac{Eh}{1-\nu^2} \oint (u_{1,1} + u_{2,2}) dx_2 + \frac{Eh}{1+\nu} \oint \omega_3 \, dx_1 + \oint (J_1 - \frac{Eh \, \varepsilon_o}{1-\nu}) \, dx_2$$

$$- \frac{Eh \, \delta u_2}{1+\nu} \qquad 85\text{-}2$$

$$M^*_3 = - x^\circ_1 R^*_2 + x^\circ_2 R^*_1 - \frac{Eh}{2(1-\nu^2)} \oint (u_{1,1} + u_{2,2}) dr^2 + \frac{2Eh}{1+\nu} \oint \omega_3 \, dA$$

$$- \oint (J_2 x_1 dx_1 + J_1 x_2 dx_2 - \frac{Eh \, \varepsilon_o}{2(1-\nu)} \, dr^2) - \frac{Eh}{1+\nu} \oint (u_1 dx_1 + u_2 dx_2) +$$

$$\frac{Eh}{1+\nu} (x^\circ_1 \delta u_1 + x^\circ_2 \delta u_2) \qquad 85\text{-}3$$

where

$$\omega_3 = \frac{1}{2}(u_{2,1} - u_{1,2}) \qquad 10\text{-}6$$

It may be verified that the terms in eqs. 85 containing $J_1$ and $J_2$ combine with $R^*_1$, $R^*_2$ and $M^*_3$ to form $R_1$, $R_2$ and $M_3$, respectively. Thus boundary conditions expressing specified values of $R_1$, $R_2$ and $M_3$ are obtained from eqs. 85 by deleting the asterisk and the terms containing $J_1$ and $J_2$. If it is recalled that $M_3$ is the moment of the boundary forces with regard to the point $(x^\circ_1, x^\circ_2)$, then the term $(M_3 + x^\circ_1 R_2 - x^\circ_2 R_1)$ in eq. 85-3 is the moment with regard to the origin of coordinates.

If the assigned values of $M^*_{nn}$ and $Q^*_{ne}$ on a boundary curve have the resultant $R^*_3$ and the moments $M^*_1$ and $M^*_2$ with regard to the coordinate axes then F, $G_1$ and $G_2$ must be sought as multivalued functions experiencing around that boundary curve the "jumps" required by eqs. 81. In terms of $G_n$ and $G_s$ we can write

$$\delta G_n = \delta G_1 \cos\varphi + \delta G_2 \sin\varphi \qquad 86\text{-}1$$

$$\delta G_s = -\delta G_1 \sin\varphi + \delta G_2 \cos\varphi \qquad 86\text{-}2$$

and using eqs. 81 obtain

$$\delta G_n = M^*_n - \frac{1}{2} R^*_3 (r^2)^\circ_{,s} \qquad 87\text{-}1$$

$$\delta G_s = M^*_s + \frac{1}{2} R^*_3 (r^2)^\circ_{,n} \qquad 87\text{-}2$$

$$\delta(G_{n,s}) = \varphi^\circ_{,s}\, \delta G_s - R^*_3 \qquad 87\text{-}3$$

where $M^*_n$ and $M^*_s$ are the normal and tangential components at point $(x^\circ_1, x^\circ_2)$ of the moment vector $(M^*_1, M^*_2)$. It was assumed above that $\varphi_{,s}$, i.e. the curvature of the boundary curve, is continuous at $(x^\circ_1, x^\circ_2)$.

If a concentrated transverse shear Q or a concentrated bending moment M are acting at a point of the boundary, then, as shown by eqs. 79, F, $G_1$, and $G_2$ have discontinuities at that point equal to Q, $-M \sin\varphi$ and $M\cos\varphi$, respectively.

Consider now the case where the bending problem with force boundary conditions and the stretching problem with displacement boundary conditions are to be solved by means of the differential equations for w and $\psi$, respectively. If there are dislocations in the first problem and non-vanishing $R^*_1$, $R^*_2$, and $M^*_3$ in the second problem, multivalued solutions must be sought for w and $\psi$. The boundary conditions of the bending problem are expressed in terms of w by first expressing the effective transverse shear

$$Q_{ne} = Q_n + M_{ns,s} \qquad 88$$

in terms of the stress couples using the equilibrium equation (3)

$$Q_n = M_{nn,n} + M_{sn,s} + \varphi_{,s}(M_{nn} - M_{ss}) \qquad 89$$

then

$$Q_{ne} = M_{nn,n} + (M_{sn} + M_{ns})_{,s} + \varphi_{,s}(M_{nn} - M_{ss}) \qquad 90$$

The right hand side of eq. 90 is espressed by means of the stress strain relations in the form

$$Q_{ne} = M_{o,n} + D(x_{nn} + \nu x_{ss})_{,n} + 2D(1-\nu)x_{ns,s} + D(1-\nu)\varphi_{,s}(x_{nn} - x_{ss}) \qquad 91$$

Through use of the compatibility equation (3)

$$x_{ss,n} - x_{ns,s} + \varphi_{,s}(x_{ss} - x_{nn}) = 0 \qquad 92$$

eq. 91 reduces to

$$Q_{ne} = M_{o,n} + D(x_{nn} + x_{ss})_{,n} + D(1-\nu)x_{ns,s} \qquad 93\text{-}1$$

For $M_{nn}$ we have

$$M_{nn} = M_o + D(x_{nn} + \nu x_{ss}) \qquad 93\text{-}2$$

Eqs. 93 become boundary conditions for w upon replacing the curvatures by their expressions in terms of w. The result may be written in the form

$$Q_{ne} = M_{o,n} - D(\Delta w)_{,n} + D(1-\nu)(-w_{,ns} + \varphi_{,s} w_{,s})_{,s} \qquad 94\text{-}1$$

$$M_{nn} = M_o - D\left[\Delta w - (1-\nu)(w_{,ss} + \varphi_{,s} w_{,n})\right] \qquad 94\text{-}2$$

It may occur that $Q_{ne}$ and $M_{nn}$ are specified only through statical equivalents. Such a case is encountered for example in a multipliconnected plate one of whose boundary curves is attached to a rigid body that is subjected to a vertical force $R_3$ and to bending moments $M_1$ and $M_2$ with regard to the coordinate axes. Then in addition to the boundary conditions specifying zero curvatures of the boundary curve we have the conditions

$$\oint Q_{ne}\, ds = R_3 \qquad 95\text{-}1$$

$$\oint (x_2 Q_{ne} + M_{nn} x_{1,s}) ds = M_1 \qquad 95\text{-}2$$

$$\oint (-x_1 Q_{ne} + M_{nn} x_{2,s}) ds = M_2 \qquad 95\text{-}3$$

Eqs. 95 are expressed in terms of w through use of eqs. 94. In doing this it is noted that

$$x_{ns} = -w_{,ns} + \varphi_{,s} w_{,s} \qquad 96\text{-}1$$

is singlevalued and in the line integrals of eqs. 95-2, 3 the terms containing $x_{ns}$ and

$$x_{ss} = -w_{,ss} - \varphi_{,s} w_{,n} \qquad 96\text{-}2$$

are integrated by parts. This yields after taking account of eqs. 67

$$\oint (x_2 x_{sn,s} - x_{ss} x_{1,s}) ds = -\oint (x_{sn} x_{2,s} + x_{ss} x_{1,s}) ds = \oint w_{,1s}\, ds \qquad 97\text{-}1$$

$$\oint (-x_1 x_{sn,s} - x_{ss} x_{2,s}) ds = \oint (x_{sn} x_{1,s} - x_{ss} x_{2,s})\, ds = \oint w_{,2s} \qquad 97\text{-}2$$

Eqs. 95 take then the form

$$D\oint (\Delta w)_{,n}\, ds = \oint M_{o,n}\, ds - R_3 \qquad 98\text{-}1$$

$$D\oint (x_2(\Delta w)_{,n} + x_{1,s}\Delta w)ds = \oint (x_2 M_{o,n} + x_{1,s} M_o)ds + D(1-\nu)\delta w_{,1} - M_1 \qquad 98\text{-}2$$

$$D\oint (x_1(\Delta w)_{,n} - x_{2,s}\Delta w)ds = \oint (x_1 M_{o,n} - x_{2,s} M_o)ds - D(1-\nu)\delta w_{,2} + M_2 \qquad 98\text{-}3$$

By analogy with the preceding, the displacement boundary conditions of a stretching problem are expressed in terms of the stress function $\psi$ by first determining from the given boundary displacements the extensional strain $\varepsilon_{ss}$ and the in-plane change of curvature $x_{s3e}$ of the boundary curves, using eqs. 53-2 and 57. Then $\varepsilon^*_{ss}$ and $x^*_{s3e}$ are determined using eqs. 58-2 and 60. The procedure for expressing $\varepsilon^*_{ss}$ and $x^*_{s3e}$ in terms of $\psi$ is dual of the procedure for expressing $M_{nn}$ and $Q_{ne}$ in terms of w. The dual of eqs. 94 are

$$x^*_{s3e} = \varepsilon_{o,n} + \frac{1}{Eh}(\Delta\psi)_{,n} - \frac{1+\nu}{Eh}(-\psi_{,ns} + \varphi_{,s}\psi_{,s})_{,s} \qquad 99\text{-}1$$

$$\varepsilon^*_{ss} = \varepsilon_o + \frac{1}{Eh}\left[\psi_{,nn} - \nu(\psi_{,ss} + \varphi_{,s}\psi_{,n})\right] \qquad 99\text{-}2$$

The boundary conditions 99 are not sufficient in general to solve the problem since as was pointed out earlier arbitrary rigid body displacements may be superimposed on the displacements of each boundary curve without altering the values of $x^*_{s3e}$ and $\varepsilon^*_{ss}$. These rigid body displacements are associated uniquely with the system of forces $R^*_1$, $R^*_2$, $M^*_3$ mentioned earlier and cause $\psi$ and its first derivatives to be multivalued and to have around each boundary curve "jumps" $\delta(\psi_{,1})$, $\delta(\psi_{,2})$, and $\delta\psi$ related to $R^*_1$, $R^*_2$ and $M^*_3$ through eqs. 66. Thus the solution of the differential equation for $\psi$ must be sought such that $\psi$ and its first derivatives be multivalued functions. The 3 constants $\delta(\psi_{,1})$, $\delta(\psi_{,2})$, and $\delta\psi$ associated with each boundary curve may be determined after having solved for the displacements by equating at one point of each boundary

curve the displacements $u_1$ and $u_2$ and the rotation $\omega$ to their assigned values. In the terminology of statically indeterminate structures $R^*_1$, $R^*_2$ and $M^*_3$ are redundants. They may be determined without explicitly solving for the displacements through the theorem of virtual forces. In the application of this theorem virtual forces are generated through arbitrary variations of $R^*_1$, $R^*_2$ and $M^*_3$ according to eqs. 66, and the actual strains of the plate are expressed in terms of $\psi$ by means of the stress-strain and stress-stress function relations.

b) Stress Boundary Conditions in The Stretching Problem. Curvature or Displacement Boundary Conditions in the Bending Problem.

Stress boundary conditions in a stretching problem where $N_{nn}$ and $N_{ns}$ take assigned values on the boundary curves may be expressed in terms of the stress function $\psi$ using the stress-stress function relations. After a particular solution of the equilibrium equations is determined the boundary conditions may be formulated so that $N^*_{nn}$ and $N^*_{ns}$ take assigned values on the boundary curves. $\psi$ and its first derivatives may be computed in terms of known boundary stress resultants through eqs. 65 which are reproduced below.

$$\psi_{,1} = \psi^\circ_{,1} - V^*_2 \qquad 100\text{-}1$$

$$\psi_{,2} = \psi^\circ_{,2} + V^*_1 \qquad 100\text{-}2$$

$$\psi = \psi^\circ + (x_1 - x^\circ_1)\,\psi^\circ_{,1} + (x_2 - x^\circ_2)\,\psi^\circ_{,2} + C^*_3 \qquad 100\text{-}3$$

It is recalled that superscript ° refers on each boundary curve to an arbitrary origin $(x^\circ_1, x^\circ_2)$ of arclength s. $V^*_1$, $V^*_2$ and $C^*_3$ are the resultants and moment, respectively, with regard to a current boundary point $(x_1, x_2)$ of the stress resultants $N^*_{n1}$ and $N^*_{n2}$ acting on the arc described in the positive sense between $(x^\circ_1, x^\circ_2)$ and $(x_1, x_2)$. The normal derivative of $\psi$ is found from $\psi_{,1}$ and $\psi_{,2}$ in the form

$$\psi_{,n} = \psi_{,1} \cos\varphi + \psi_{,2} \sin\varphi$$

or

$$\psi_{,n} = \psi^{\circ}_{,1} \cos\varphi + \psi^{\circ}_{,2} \sin\varphi - V^*_s \qquad 101$$

where

$$V^*_s = - V^*_1 \sin\varphi + V^*_2 \cos\varphi \qquad 102$$

$V^*_s$ is the tangential component at point $(x_1, x_2)$ of the force vector $(V^*_1, V^*_2)$. Eqs. 100-3 and 101 show that stress boundary conditions determine $\psi$ and $\psi_{,n}$ on the boundary except for terms involving 3 arbitrary constants $\psi^{\circ}_{,1}$ $\psi^{\circ}_2$, and $\psi^{\circ}$ on each boundary curve. The determination of these constants will be examined subsequently. If the stress resultants due to the homogeneous solution have on a boundary curve the resultants $R^*_1$, $R^*_2$ and $M^*_3$ then $\psi_{,1}$, $\psi_{,2}$ and $\psi$ are multivalued and their jumps when making a complete turn around this boundary curve are given through eqs. 66, i.e.

$$\delta(\psi_{,1}) = - R^*_2 \qquad 103\text{-}1$$

$$\delta(\psi_{,2}) = R^*_1 \qquad 103\text{-}2$$

$$\delta\psi = M^*_3 \qquad 103\text{-}3$$

For a simply connected domain without concentrated forces, equilibrium requires the right hand sides of eqs. 103 to be zero and $\psi$ and its derivatives are signlevalued. If a concentrated force is acting within the domain of the plate $\psi$ and its derivatives are multivalued since the boundary stress resultants are in equilibrium with the applied concentrated force. If concentrated forces and a moment are acting at a point of the boundary, they are jump discontinuities of $V^*_1$, $V^*_2$, and $C^*_3$ when crossing that point. Then $\psi_1$, $\psi_{,2}$ and $\psi$ go through corresponding discontinuities as required

by eqs. 100. The boundary conditions of the bending problem that are dual of the stress boundary conditions of the stretching problem take the form of assigned values for the curvature $x_{ss}$ and twist $x_{sn}$ on each boundary curve. The transformation of such boundary conditions into boundary conditions for w and its normal derivative $w_{,n}$ is dual of the formulation of the stress boundary conditions of the stretching problem in terms of $\psi$ and $\psi_{,n}$. The appropriate formulas are eqs. 68.

Whereas in a stretching problem stress boundary conditions are first formulated in terms of stress resultants and may be subsequently expressed in terms of the stress function, in a bending problem displacement boundary conditions are usually first formulated in terms of w and are not transformed into curvature boundary conditions. For a multipliconnected plate without dislocations boundary values assigned to w and $w_{,n}$ are singlevalued. In the dual stretching problem the stress resultants due to the homogeneous solution are self-equilibrating on each boundary curve and $\psi$ and $\psi_{,n}$ are singlevalued.

In the bending problem, with or without dislocations, the specified boundary values of w and $w_{,n}$ are usually sufficient to solve completely the problem. If, however, the boundary conditions were to be formulated, in terms of $x_{ss}$ and $x_{ns}$ using eqs. 96 they would not be sufficient to solve the problem in the case of a multipliconnected domain. The reason for this is that an arbitrary rigid body displacement defined by $w^\circ_{,1}$, $w^\circ_{,2}$, and $w^\circ$ could be superimposed on each boundary curve without changing the values of $x_{ss}$ and $x_{ns}$. If one of the curves bounding the plate is held fixed, the rigid body displacements applied to the remaining boundary curves will cause additional boundary forces and will therefore alter the states of stress and deformation. Thus an infinite number of problems satisfy the same assigned values for $x_{ss}$ and $x_{ns}$. All of these problems may be obtained from one of them if one of the boundary curves is held fixed and if all the remaining boundary curves are connected to free rigid

bodies subjected each to an arbitrary force perpendicular to the plate and an arbitrary moment vector parallel to the plate.

Considering now the dual stretching problem it is apparent that stress boundary conditions are not sufficient to solve uniquely the problem in the case of a multipliconnected domain. The arbitrary constants $\psi^\circ_{,1}$, $\psi^\circ_{,2}$ and $\psi^\circ$ that appear in the boundary conditions, eqs. 100-3 and 101, are dual of the constants $w^\circ_{,1}$, $w^\circ_{,2}$ and $w^\circ$ defining the rigid body displacements of the boundary curves in the bending problem. If the constants $\psi^\circ_{,1}$, $\psi^\circ_{,2}$ and $\psi^\circ$ were selected in a certain manner, they would imply multivalued displacements $u_1$ and $u_2$, i. e., dislocations; just as in the bending problem, the rigid body displacements cause, in general, non-vanishing resultant force and resultant moment on the boundary curves and require multivalued stress functions $G_1$ and $G_2$. In order to obtain a unique solution of a stretching problem with stress boundary conditions by means of the stress function $\psi$, conditions expressing assigned dislocations, if any, must therefore be appended to the boundary conditions. These conditions are eqs. 75 which were established by determining the displacements of a boundary curve from its extensional strain $\varepsilon_{ss}$ and its in-plane change of curvature $x_{s3e}$. In the present context the left hand sides of eqs. 75 are imposed dislocations and $\varepsilon_{ss}$ and $x_{s3e}$ on the right hand sides have to be expressed in terms of $\psi$. The procedure is dual of the one followed in expressing in terms of w the resultant force and moments acting on a boundary curve, eqs. 98. In writing the relations dual of eqs. 98 we note the duality relations

$$\delta(w_{,1}) \Leftrightarrow \delta(\psi_{,1}) = -R^*_2 \qquad 104\text{-}1$$

$$\delta(w_{,2}) \Leftrightarrow \delta(\psi_{,2}) = R^*_1 \qquad 104\text{-}2$$

and

$$R^*_3 = \delta F \Leftrightarrow \delta\omega \qquad 105\text{-}1$$

$$M^*_1 = \delta G_1 + x^\circ_2 R^*_3 \Leftrightarrow \delta u_1 + x^\circ_2 \delta\omega \qquad 105\text{-}2$$

$$M^*_2 = \delta G_2 - x^\circ_1 R^*_3 \Leftrightarrow \delta u_2 - x^\circ_1 \delta\omega \qquad 105\text{-}3$$

The relations between $R_3$, $M_1$, $M_2$, and $R^*_3$, $M^*_1$, $M^*_2$ are obtained in the form

$$R_3 = R^*_3 - D \oint \left[(K_2 + \nu K_1)_{,1} dx_2 - (K_1 + \nu K_2)_{,2} dx_1\right] \qquad 106\text{-}1$$

$$M_1 = M^*_1 - D\oint x_2 \left[(K_2 + \nu K_1)_{,1} dx_2 - (K_1 + \nu K_2)_{,2} dx_1\right] - D\oint (K_1 + K_2) dx_1$$

$$- D(1-\nu) \oint (K_{ns} dx_2 - K_{nn} dx_1) \qquad 106\text{-}2$$

$$M_2 = M^*_2 + D\oint x_1 \left[(K_2 + \nu K_1)_{,1} dx_2 - (K_1 + \nu K_2)_{,2} dx_1\right] - D\oint (K_1 + K_2) dx_2$$

$$+ D(1-\nu) \oint (K_{ns} dx_1 + K_{nn} dx_2) \qquad 106\text{-}3$$

The dual of eqs. 98 take the form

$$\oint (\Delta\psi)_{,n} ds = Eh\,\delta\omega - Eh \oint \varepsilon_{o,n} ds + \oint \left[(J_2 - \nu J_1)_{,1} dx_2 - (J_1 - \nu J_2)_{,2} dx_1\right] \qquad 107\text{-}1$$

$$\oint (x_2(\Delta\psi)_{,n} + x_{1,s}\Delta\psi) ds = Eh(\delta u_1 + x^\circ_2 \delta\omega) - (1+\nu) R_2$$

$$- Eh\oint (x_2 \varepsilon_{o,n} + x_{1,s} \varepsilon_o) ds + \oint x_2 \left[(J_2 - \nu J_1)_{,1} dx_2 - (J_1 - \nu J_2)_{,2} dx_1\right]$$

$$+ \oint (J_1 + J_2) dx_1 \qquad 107\text{-}2$$

$$\oint (x_1(\Delta\psi)_{,n} - x_{2,s}\Delta\psi) ds = - Eh(\delta u_2 - x^\circ_1 \delta\omega) - (1+\nu) R_1$$

$$- Eh\oint (x_1 \varepsilon_{o,n} - x_{2,s} \varepsilon_o) ds + \oint x_1 \left[(J_2 - \nu J_1)_{,1} dx_2 - (J_1 - \nu J_2)_{,2} dx_1\right]$$

$$- \oint (J_1 + J_2) dx_2 \qquad 107\text{-}3$$

Consider now the case where the stretching problem with stress boundary conditions and the bending problem with displacement boundary conditions are solved by means of the displacement and stress-functions methods, respectively. The stress boundary conditions must be expressed in terms of the displacements $u_1$ and $u_2$ and the displacements boundary conditions in terms of the stress functions $G_1$ and $G_2$. This may be done in the first case through use of the stress-strain-displacements relations and in the second case through the curvature-stress-stress function relations. The two procedures are dual. If there are dislocations in the stretching problem, they are dealt with directly by seeking a multivalent solution for the displacements. In the bending problem the boundary conditions obtained by the procedure discussed above are not sufficient to obtain a unique solution in the case of a multipliconnected domain. This is due, as explained earlier, to the passage from w and $w_{,n}$ to curvatures $x_{ss}$ and $x_{sn}$. The stress functions $G_1$ and $G_2$ must be sought as multivalued functions experiencing around each boundary curve the "jumps" defined by eqs. 81, i.e.

$$\delta G_1 = M^*_1 - x^\circ_2 R^*_3 \qquad 108\text{-}1$$

$$\delta G_2 = M^*_2 + x^\circ_1 R^*_3 \qquad 108\text{-}2$$

$$\delta F = -\delta(G_{n,s}) + \delta(\varphi_{,s} G_s) = R^*_3 \qquad 108\text{-}3$$

The resultant force $R^*_3$ and the moments $M^*_1$ and $M^*_2$ associated with each boundary curve may be determined, after solving for the deflection w by equating at one point of each boundary curve w, $w_{,1}$ and $w_{,2}$ to the imposed boundary values. In the terminology of statically indeterminate structures $R^*_3$, $M^*_1$ and $M^*_2$ are redundants. They may be determined without solving for the displacements by application of the theorem of virtual forces. These are generated by arbitrary variations of $R^*_3$, $M^*_1$, and $M^*_2$.

The procedure is similar to (but not dual of) that described earlier for the determination of $R^*_1$, $R^*_2$ and $M^*_3$ in the stretching problem.

c) Mixed Boundary Conditions.

The case to be considered now is one in which, in either of the two problems, displacement boundary conditions are specified on part of the boundary and stress boundary conditions are specified on the remaining part. It is apparent that a given problem and its dual have both mixed boundary conditions. Further, the part of the boundary with stress or displacement boundary conditions becomes in the dual problem the part of the boundary with displacement or stress boundary conditions, respectively.

Consider first the stretching problem of a simply connected domain bounded by a closed curve C and let $C_d$ and $C_s$ be the parts of C where displacement and stress boundary conditions are specified, respectively. If $C_d$ and $C_s$ are formed each of a simple arc as shown in Fig. 4 the formulation of the dual bending problem presents no difficulty: on $C_d$ the dual of $u_1$ and $u_2$ are $G_1$ and $G_2$. They determine the bending stress couple $M^*_{nn}$ and the effective transverse shear $Q^*_{ne}$. On $C_s$ the known stress resultants allow determining $\psi$ and $\psi_{,n}$ through eqs. 65, thus their dual w and $w_{,n}$. The dual of the arbitrary constants $\psi^\circ$, $\psi^\circ_{,1}$ and $\psi^\circ_{,2}$ occuring in eqs. 65 are $w^\circ$, $w^\circ_{,1}$ and $w^\circ_{,2}$. They represent a rigid body displacement and may be disposed of arbitrarily. If $C_d$ and $C_s$ are formed each of more than one arc as shown in Fig. 5 there are as many sets of three constants $\psi^\circ$, $\psi^\circ_{,1}$ and $\psi^\circ_{,2}$ as there are arcs in $C_s$. One of these sets may be chosen arbitrarily. The remaining sets are unknown because the stress resultants on an arc of $C_d$ separating two arcs of $C_s$ are unknown. There follows that in the dual bending problem the displacements w and $w_{,n}$ cannot be completely specified on $C_s$. They involve on each arc of $C_s$ 3 unknown constants $w^\circ$, $w^\circ_{,1}$ and $w^\circ_{,2}$ representing a rigid body displacement of that arc. One of these rigid body

displacements may be chosen arbitrarily. In order to define uniquely the dual bending problem, i.e. in order to determine the rigid body displacements mentioned above, it is necessary, in addition to satisfying the boundary conditions in terms of $M_{nn}$ and $Q_{ne}$, that the assigned values of $G_1$, $G_2$ and $F = - G_{n,s} + \varphi_{,s} G_s$ satisfy at the points common to the arcs of $C_d$ and $C_s$, eqs. 79. The reason for this is that the same bending stress couple and the same effective transverse shear would have been obtained on $C_d$ had the original stretching problem been modified by superimposing on the displacements $u_1$ and $u_2$ of each arc of $C_d$ an arbitrary rigid body displacement. Such rigid body displacements would cause on each arc of $C_d$ an additional resultant force in the plane of the plate and an additional resultant moment vector perpendicular to the plate and would therefore affect the stress function at each point common to 2 arcs of $C_d$ and $C_s$ by 3 constants $\psi^\circ$, $\psi^\circ_{,1}$ and $\psi^\circ_{,2}$. The dual of these constants define the rigid body displacements of the arcs of $C_s$ in the dual bending problem. The recourse to eqs. 79 may also be viewed in a different way. The rigid body displacements of the dual bending problem affect $M_{nn}$ and $Q_{ne}$ on the arcs of $C_s$, and these in turn determine $G_1$ and $G_2$ according to eqs. 79. Consider now the bending problem of a simply connected domain bounded by a closed curve C and let w and $w_{,n}$ be specified on $C_d$ and $M^*_{nn}$ and $Q^*_{ne}$ be specified on $C_s$. The dual of w and $w_{,n}$ are $\psi$ and $\psi_{,n}$. They determine the stress resultants on $C_d$ in the dual stretching problem. The dual of $M^*_{nn}$ and $Q^*_{ne}$ are $\varepsilon_{ss}$ and $x_{s3e}$. These allow determining $u_1$ and $u_2$ on $C_s$ using eqs. 72 and 74, except for 3 unknown constants $u^\circ_1$, $u^\circ_2$, $\omega^\circ$ associated with each arc of $C_s$ and representing a rigid body displacement of that arc. In order to define uniquely the dual stretching problem in the case where $C_d$ and $C_s$ are formed each of more than one arc it is necessary that the assigned values of $\psi$, $\psi_{,1}$ and $\psi_{,2}$ satisfy eqs. 65 at the points common to the arcs of $C_d$ and $C_s$.

There remains to consider problems in which the two boundary conditions are one of the force type and one of the displacement type. For example the dual of a bending problem where w and $M_{nn}$ are specified on the boundary (C) is a stretching problem where $\psi$ and $\varepsilon_{ss}$ are specified. To $w_{,n}$ and $Q_{ne}$ correspond $\psi_{,n}$ and $x_{s3e}$. The physical significance of $\psi$ and $\psi_{,n}$ in terms of boundary stress resultants may be found in eqs. 100 and 101. Similarly, the dual of stretching problems where $u_n$ and $N_{ns}$ or $u_s$ and $N_{nn}$ are specified on (C) are bending problems where $G_n$ and $x_{ns}$ or $G_s$ and $x_{ss}$, respectively, are specified. $G_n$ and $G_s$ are related to resultants of the boundary effective shear and stress couple through eqs. 79.

The case of a multipliconnected domain with mixed boundary conditions should not raise any new significant questions beyond those discussed earlier.

9. Application of St. Venant's Principle.

Consider a stretching problem with stress boundary conditions on an arc AB of a boundary curve. Let $V^*_1$, $V^*_2$ and $C^*_3$ be the resultants and the moment with regard to B, respectively, of the stress resultants acting on AB and let AB be oriented positively from A to B. From eqs. 65 we can write

$$\psi^B_{,1} - \psi^A_{,1} = - V^*_2 \qquad 109\text{-}1$$

$$\psi^B_{,2} - \psi^A_{,2} = V^*_1 \qquad 109\text{-}2$$

$$\psi^B - \psi^A - (x^B_1 - x^A_1)\,\psi^A_{,1} - (x^B_2 - x^A_2)\,\psi^A_{,2} = C^*_3 \qquad 109\text{-}3$$

If the distribution of the stress resultants on arc AB is modified keeping $V^*_1$, $V^*_2$ and $C^*_3$ unchanged the left hand sides of eqs. 109 remain unchanged. By St. Venant's principle this modification of the boundary conditions affects significantly the solution only in a certain neighborhood of arc AB.

There follows that in the dual bending problem a modification of w and $w_{,n}$ on arc AB of the boundary keeping the terminal values at A and B unchanged affects significantly the solution only in the same neighborhood of arc AB. It is interesting to note that such modification of the boundary values of w and $w_{,n}$ causes new distribution of the stress couple and effective transverse shear on AB that is not necessarily statically equivalent to the original distribution. When this is the case the additional redistribution of the boundary forces that is required to maintain equilibrium must be limited by St. Venant's principle to the neighborhood of arc AB.

Consider now a bending problem with force boundary conditions on an arc AB of a boundary curve. If these forces are replaced by a statically equivalent system of forces the solution of the bending problem is modified significantly only in a certain neighborhood of arc AB. In the dual stretching problem the extensional strain $\varepsilon_{ss}$ and the in-plane change of curvature $x_{s3e}$ are modified on arc AB keeping unchanged the displacements and rotation at A and B. Such modification of the boundary conditions affects significantly the solution only in the same neighborhood of AB.

It appears from the preceding that St. Venant's principle which is stated usually in terms of statically equivalent systems of boundary forces may be extended to include what might be called equivalent systems of boundary strains.

## 10. Duality for Anisotropic Plates.

The duality between the problems of bending and of stretching of plates developed for a homogeneous and isotropic material will now be generalized for a homogeneous material having one plane of elastic symmetry that is parallel to the middle plane of the plate. With this restriction on the elastic properties of the plate, in-plane stresses are not connected with transverse shear strains and transverse shear stresses are not connected with in-plane strains. The three dimensional stress strain relations are written in the form (5)

$$e_{11} = \frac{1}{E_{11}} \left( \sigma_{11} - \nu_{21} \sigma_{22} - \nu_{31} \sigma_{33} \right) + \frac{\mu_1}{G_{12}} \sigma_{12} \qquad 110\text{-}1$$

$$e_{22} = \frac{1}{E_{22}} \left( - \nu_{12} \sigma_{11} + \sigma_{22} - \nu_{32} \sigma_{33} \right) + \frac{\mu_2}{G_{12}} \sigma_{12} \qquad 110\text{-}2$$

$$e_{33} = \frac{1}{E_{33}} \left( - \nu_{13} \sigma_{11} - \nu_{23} \sigma_{22} + \sigma_{33} \right) + \frac{\mu_3}{G_{12}} \sigma_{12} \qquad 110\text{-}3$$

$$2e_{12} = \frac{1}{G_{12}} \left( \mu_1 \sigma_{11} + \mu_2 \sigma_{22} + \mu_3 \sigma_{33} + \sigma_{12} \right) \qquad 110\text{-}4$$

$$2e_{23} = \frac{1}{G_{23}} \left( \sigma_{23} + \eta_{12} \sigma_{13} \right) \qquad 110\text{-}5$$

$$2e_{31} = \frac{1}{G_{31}} \left( \eta_{21} \sigma_{23} + \sigma_{31} \right) \qquad 110\text{-}6$$

where

$$\frac{\nu_{21}}{E_{11}} = \frac{\nu_{12}}{E_{22}} \qquad 111\text{-}1$$

$$\frac{\nu_{32}}{E_{22}} = \frac{\nu_{23}}{E_{33}} \qquad 111\text{-}2$$

$$\frac{\nu_{13}}{E_{33}} = \frac{\nu_{31}}{E_{11}} \qquad 111\text{-}3$$

$$\frac{\eta_{12}}{G_{23}} = \frac{\eta_{21}}{G_{31}} \qquad 111\text{-}4$$

Eqs. 110 contain thirteen independent elastic constants. The stress strain relations for the stretching and bending problems are established by neglecting $\sigma_{33}$ in eqs. 111 and by adopting Kirchhoff's hypothesis concerning the

deformation of normals to the plate. The following relations are thus obtained

$$\varepsilon_{11} = \frac{1}{E_{11}h}(N_{11} - \nu_{21} N_{22}) + \frac{\mu_1}{G_{12}h} N_{12} \qquad 112\text{-}1$$

$$\varepsilon_{22} = \frac{1}{E_{22}h}(-\nu_{12} N_{11} + N_{22}) + \frac{\mu_2}{G_{12}h} N_{12} \qquad 112\text{-}2$$

$$2\varepsilon_{12} = \frac{1}{G_{12}h}(\mu_1 N_{11} + \mu_2 N_{22} + N_{12}) \qquad 112\text{-}3$$

$$x_{11} = \frac{12}{E_{11}h^3}(M_{11} - \nu_{21} M_{22}) + \frac{12\mu_1}{G_{12}h^3} M_{12} \qquad 113\text{-}1$$

$$x_{22} = \frac{12}{E_{22}h^3}(-\nu_{12} M_{11} + M_{22}) + \frac{12\mu_2}{G_{12}h^3} M_{12} \qquad 113\text{-}2$$

$$2x_{12} = \frac{12}{G_{12}h^3}(\mu_1 M_{11} + \mu_2 M_{22} + M_{12}) \qquad 113\text{-}3$$

Eqs. 112 and 113 contain 6 independent elastic constants.

It is more convenient for the purpose of establishing the duality to write eqs. 112 in the form

$$\varepsilon_{11} = a_{11}N_{11} + a_{12}N_{22} + a_{13}N_{12} \qquad 114\text{-}1$$

$$\varepsilon_{22} = a_{21}N_{11} + a_{22}N_{22} + a_{23}N_{12} \qquad 114\text{-}2$$

$$\varepsilon_{12} = a_{31}N_{11} + a_{32}N_{22} + a_{33}N_{12} \qquad 114\text{-}3$$

and to solve eqs. 113 for the stress couples in the form

$$M_{11} = b_{11} x_{11} + b_{12} x_{22} + b_{13} x_{12} \qquad 115\text{-}1$$

$$M_{22} = b_{21} x_{11} + b_{22} x_{22} + b_{23} x_{12} \qquad 115\text{-}2$$

$$M_{12} = b_{31} x_{11} + b_{32} x_{22} + b_{33} x_{12} \qquad 115\text{-}3$$

From eqs. 112 and 113 we can write

$$a_{12} = a_{21} \qquad 116\text{-}1$$

$$a_{13} = 2a_{31} \qquad 116\text{-}2$$

$$a_{23} = 2a_{32} \qquad 116\text{-}3$$

$$b_{12} = b_{21} \qquad 116\text{-}4$$

$$b_{13} = 2b_{31} \qquad 116\text{-}5$$

$$b_{23} = 2b_{32} \qquad 116\text{-}6$$

The matrices of elements $a_{ij}$ and $b_{ij}$ are related through the relation

$$[b_{ij}] = \frac{h^2}{12} [a_{ij}]^{-1} \qquad 117$$

The 6 independent elastic constants may be taken as $a_{11}$, $a_{22}$, $a_{33}$, $a_{12}$, $a_{13}$ and $a_{23}$ or as $b_{11}$, $b_{22}$, $b_{33}$, $b_{21}$, $b_{13}$, and $b_{23}$. Eqs. 114 are transformed into eqs. 115 if in addition to the duality between strains and stress couples and between stress resultants and curvatures the following duality is adopted

$$a_{11} \Leftrightarrow - b_{22} \qquad 118\text{-}1$$

$$a_{22} \Leftrightarrow -b_{11} \qquad 118\text{-}2$$

$$a_{33} \Leftrightarrow -b_{33} \qquad 118\text{-}3$$

$$a_{12} \Leftrightarrow -b_{21} \qquad 118\text{-}4$$

$$a_{13} \Leftrightarrow b_{23} \qquad 118\text{-}5$$

$$a_{23} \Leftrightarrow b_{13} \qquad 118\text{-}6$$

Consider now the case where the material has three orthogonal planes of elastic symmetry coinciding with the coordinate planes. For such orthotropic plates we have

$$\mu_1 = \mu_2 = 0 \qquad 119$$

Eqs. 112 reduce to

$$\varepsilon_{11} = \frac{1}{E_{11}h}(N_{11} - \nu_{21} N_{22}) \qquad 120\text{-}1$$

$$\varepsilon_{22} = \frac{1}{E_{22}h}(N_{22} - \nu_{12} N_{11}) \qquad 120\text{-}2$$

$$\varepsilon_{12} = \frac{N_{12}}{2Gh} \qquad 120\text{-}3$$

where we have let $G = G_{12}$.

Eqs. 113 solved for the stress couples take the form

$$M_{11} = \frac{E_{11}h^3}{12(1-\nu_{21}\nu_{12})}(x_{11} + \nu_{12} x_{22}) \qquad 121\text{-}1$$

$$M_{22} = \frac{E_{22}h^3}{12(1-\nu_{21}\nu_{12})}(x_{22} + \nu_{21} x_{11}) \qquad 121\text{-}2$$

$$M_{12} = \frac{Gh^3}{6} x_{12} \qquad 121\text{-}3$$

The correspondence that must be adopted between the elastic constants of the stretching and bending problems in order that eqs. 120 and 121 be dual is indicated below.

$$\frac{1}{E_{11}h} \Leftrightarrow -\frac{E_{22}h^3}{12(1 - \nu_{21}\nu_{12})} \qquad 122\text{-}1$$

$$\frac{1}{E_{22}h} \Leftrightarrow -\frac{E_{11}h^3}{12(1 - \nu_{21}\nu_{12})} \qquad 122\text{-}2$$

$$\frac{1}{Gh} \Leftrightarrow -\frac{Gh^3}{3} \qquad 122\text{-}3$$

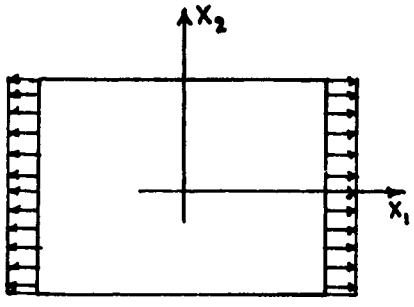
$$\nu_{21}, \nu_{12} \Leftrightarrow -\nu_{21}, -\nu_{12} \qquad 122\text{-}4$$

## 11. Examples of Dual Problems

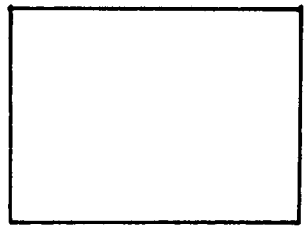
The examples given below are formed of pairs of dual problems. The figures and the accompanying explanations are meant to define the problems qualitatively and may assume knowledge of their solution. They do not represent a quantitative application of the duality. As a guide to the examples we recall some dual properties for homogeneous problems. A direct stress resultant in a direction is dual of the curvature in the perpendicular direction. The in-plane change of curvature of a curve is dual of the effective transverse shear along the curve. A free edge in the stretching problem is clamped or moves as a rigid body in the bending problem. An edge that moves as a rigid body in the stretching problem becomes free in the bending problem.

If the forces acting on a closed boundary curve of a multiply connected domain are not statically equivalent to zero there are dislocations in the dual problem emanating from that boundary curve.

1.

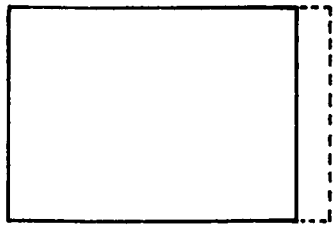



Constant tension in $x_1$ direction

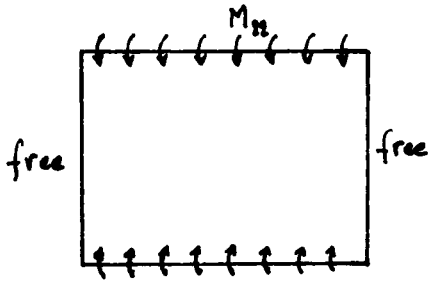


Constant curvature in $x_2$ direction

2.

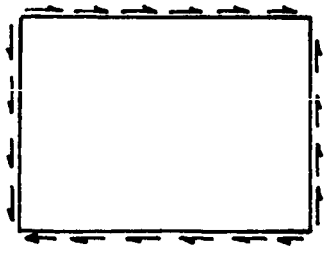


Constant extension in $x_1$ direction

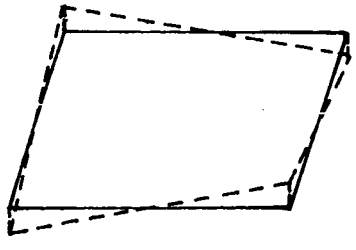



Bending by constant $M_{22}$

3.

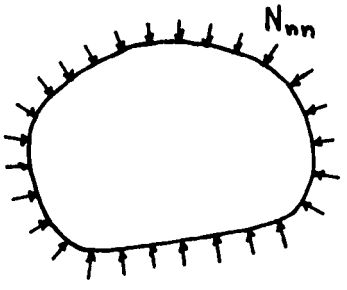


Constant shear

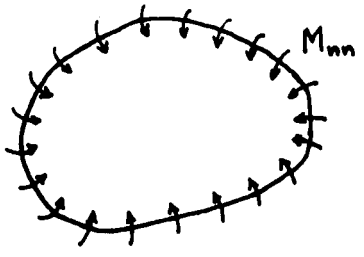


Constant twist
Anticlastic bending

4.

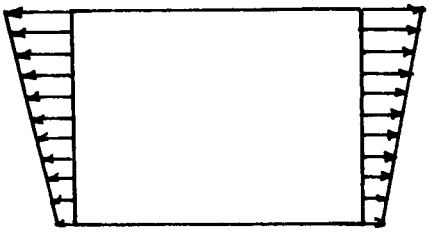


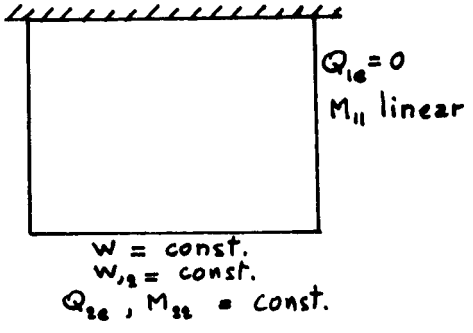

Constant pressure




Pure bending

5.



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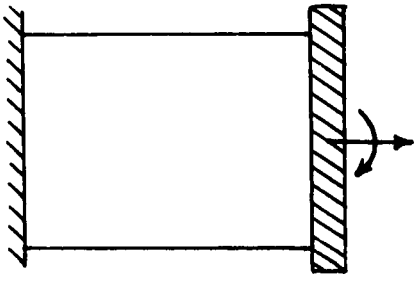

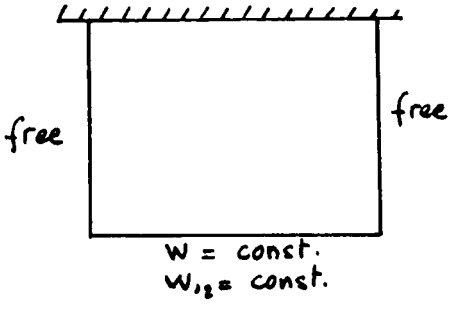


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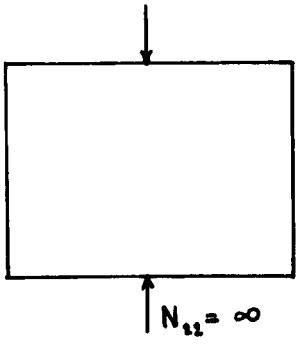


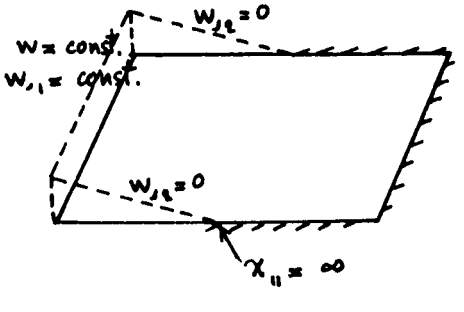

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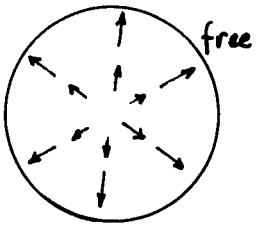
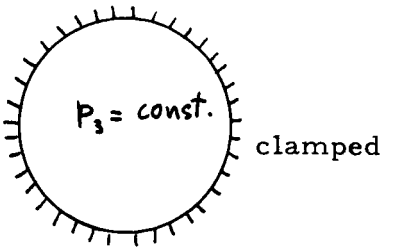



Rotating disc

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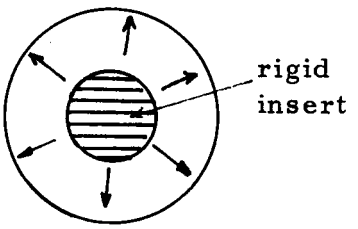


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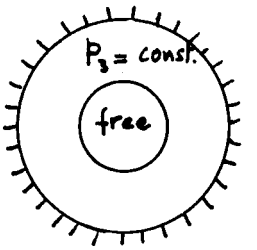


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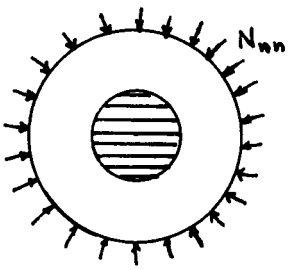


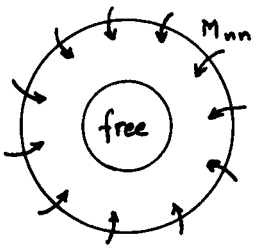


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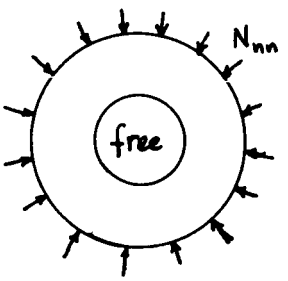


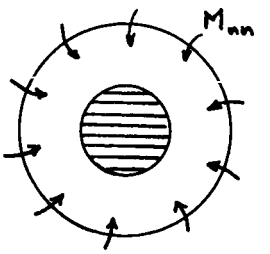

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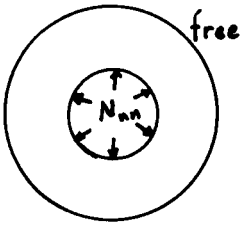
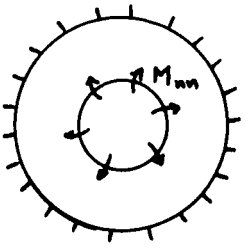
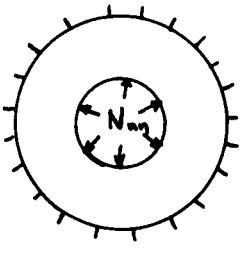
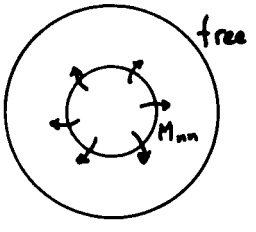

free
$N_{nn}$

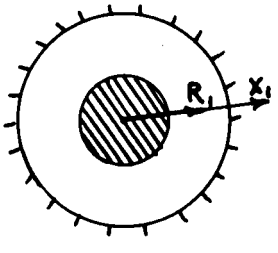

$M_{nn}$

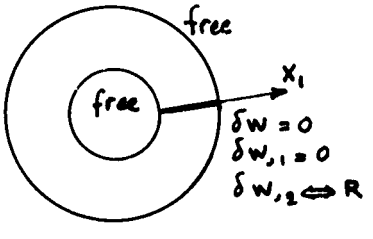
13.


$N_{nn}$

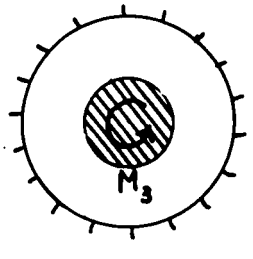

free
$M_{nn}$

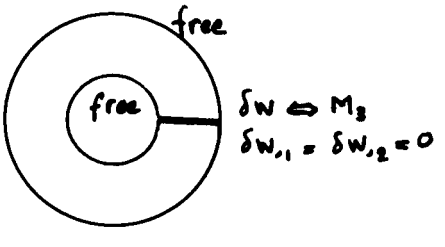
14.


$R_1$
$X_1$

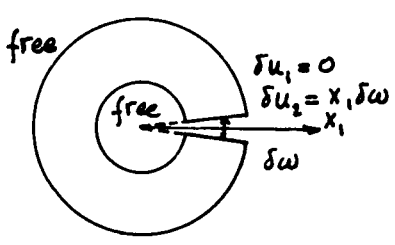

free
free
$X_1$
$\delta w = 0$
$\delta w_{,1} = 0$
$\delta w_{,2} \Leftrightarrow R$

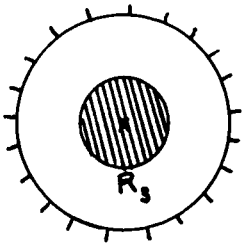
15.


$M_3$

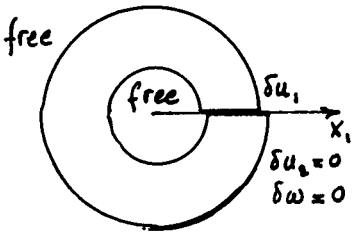

free
free
$\delta w \Leftrightarrow M_3$
$\delta w_{,1} = \delta w_{,2} = 0$

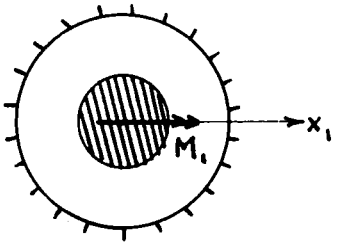
16.


free
free
$\delta u_1 = 0$
$\delta u_2 = x_1 \delta\omega$
$x_1$
$\delta\omega$


$R_3$

17.


free
free
$\delta u_1$
$x_1$
$\delta u_2 = 0$
$\delta\omega = 0$


$M_1$
$x_1$

## 12. Summary and Conclusion

The duality between the dependent variables of the problems of stretching and of bending of plates allows transforming the basic equations of one problem into the basic equations of the other problem. In general terms the dual correspondence holds between displacements and stress functions, in-plane stress resultants and flexural curvatures and twist, strains and stress coupled, in-plane changes of curvature and effective transverse shears.

Solution of the stretching problem by the displacement method is dual of the solution of the bending problem by the stress function method and the use of Airy's stress function in the former problem is dual of the use of the deflection in the latter.

Displacement boundary conditions define uniquely a stretching problem. They may require the stress function and its first derivatives to be multivalued in the case of a multiply connected domain. In the dual bending problem boundary conditions taking the form of specified values for the stress functions may imply dislocations.

Boundary conditions specifying the bending stress couple and the effective transverse shear are dual of stretching boundary conditions specifying the extensional strain and the in-plane change of curvature. Such boundary conditions do not define a unique problem in the case of a multiply-connected domain. For a unique problem conditions specifying dislocations are necessary in the bending problem and conditions specifying rigid body displacements of the boundary curves are necessary in the stretching problem.

Displacement boundary conditions define uniquely a bending problem but may imply multivalued stress functions in the case of a multiply connected domain. In the dual stretching problem boundary conditions specifying boundary values of Airy's stress function and of its normal derivative may imply dislocations.

Boundary conditions specifying the in-plane stress resultants are dual of bending boundary conditions specifying the flexural curvature and twist

along the boundary. Such boundary conditions do not define a unique problem in the case of a multiply connected domain. For a unique problem conditions specifying dislocations are necessary in the stretching problem and conditions specifying rigid body displacements of the boundary curves are necessary in the bending problem.

To St. Venant's principle as applied to statically equivalent systems of boundary forces corresponds a dual principle that may be applied to equivalent systems of boundary deformations.

The duality may be generalized to anisotropic plates having their middle plane as a plane of elastic symmetry and in particular to orthotropic plates.

A property of the duality that may be of interest for experimental applications is the correspondence between mechanical and geometrical quantities. The former are generally "abstract" quantities that may be difficult or even impossible to measure directly while the latter by their geometrical character are always accessible to direct measurement. For example, in-plane strains are easier to measure than stress couples and displacements are directly measurable while stress functions are not.

The reduction of the bending and stretching problems to a single mathematical problem has not apparently been used in a systematic way. In addition to its theoretical interest the dual formulation of the stretching and of the bending problem should be of value in the development of automatic and dual approximate methods of analysis.

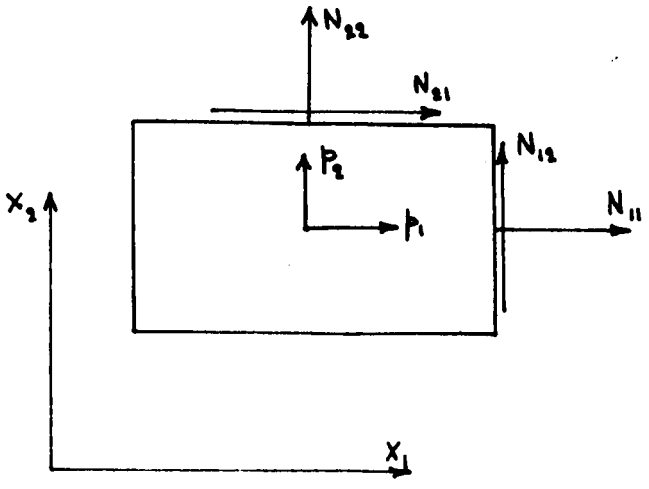

$N_{22}$
$N_{21}$
$N_{12}$
$N_{11}$
$P_2$
$P_1$
$X_2$
$X_1$

Fig. 1

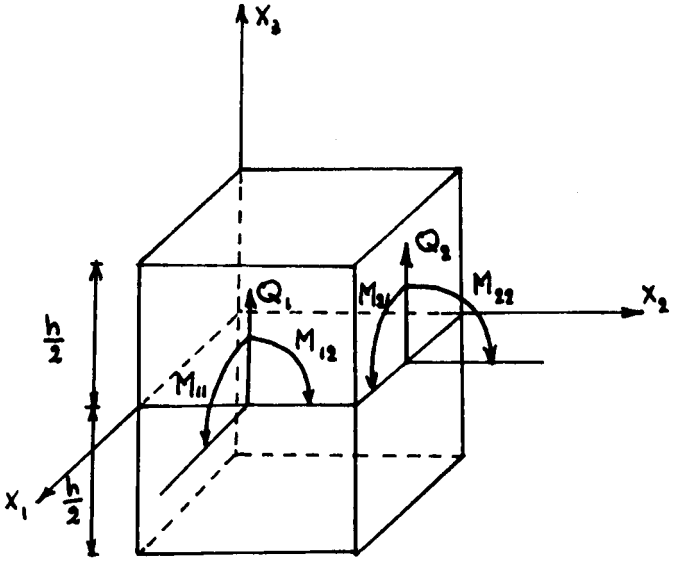

$X_3$
$Q_2$
$Q_1$
$M_{21}$
$M_{22}$
$X_2$
$M_{12}$
$M_{11}$
$\frac{h}{2}$
$X_1$
$\frac{h}{2}$

Fig. 2

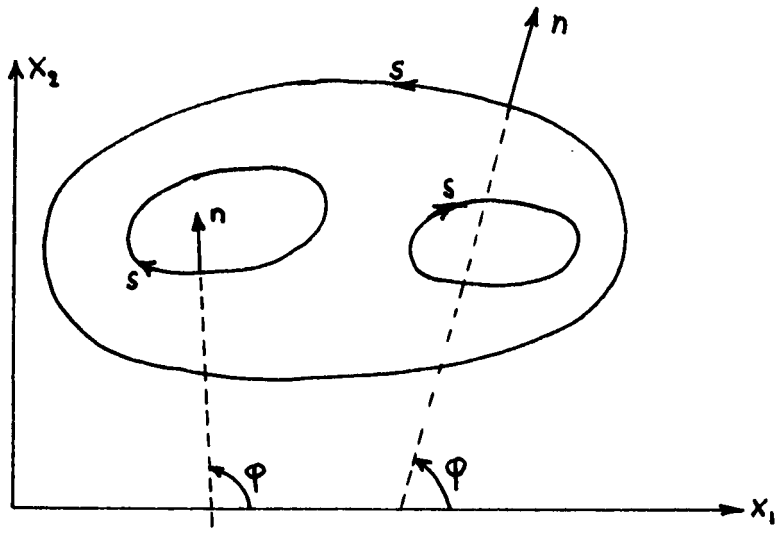



Fig. 3

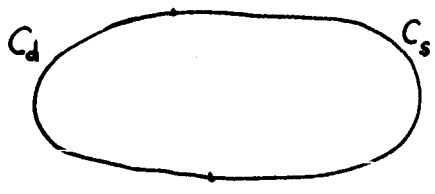



Fig. 4

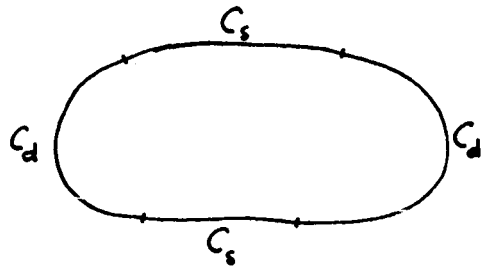



Fig. 5

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