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A PROPERTY OF THE L_2-NORM OF A CONVOLUTION

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INTRODUCTION. It is known that the convolution of two members, f and g, of $L_2(-\infty, +\infty)$ can be a null function without either f or g being a null function. But, if one defines f_ν by setting $f_\nu(x) = e^{i\nu x}f(x)$ for all x, f_ν and g will have a convolution that is not a null function for a suitable choice of ν. There is apparently no information available on how the L_2-norm of the latter convolution depends on ν.

A partial answer to this problem will be provided in the present paper. There will be derived a lower bound on the supremum in ν of the L_2-norm of the convolution of f_ν and g. The lower bound will be expressed in terms of a notion of ϵ-approximate support which is an $L_1(-\infty, +\infty)$ analog of the concept of support of a continuous function on a locally compact space. The inequality will be shown to be sharp in the sense that one can construct an f and a g for which the lower bound is approached arbitrarily closely.

DEFINITIONS AND NOTATION. Because of the need for uniqueness and because of the nature of the L_1-norm, an appropriate analog for $L_1(-\infty, +\infty)$ of the notion of support is the following.

DEFINITION. <u>The ϵ-approximate support of a member f of $L_1(-\infty, +\infty)$ is defined to be the closed interval $I_{\epsilon,f}$ such that</u>

a) <u>$I_{\epsilon,f}$ is symmetric about the smallest real number x_0 for which</u>

$$\int_{-\infty}^{x_0} |f(x)|\,dx = \left(\tfrac{1}{2}\right)||f||_1$$

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b) $\int_{I_{\epsilon,f}} |f(x)|\,dx = (1-\epsilon)||f||_1$,

$||f||_1$ <u>being the L_1-norm of f.</u> The existence and uniqueness of x_0 and $I_{\epsilon,f}$ are clear from the absolute continuity of the indefinite integral of $|f|$.

For any Lebesgue-measurable set E the measure of E is denoted by m(E) and the characteristic function is denoted by (E). Given any two measurable functions on the real numbers, f and g, such that for almost all x, $f(y)g(x-y)$ is in $L_1(-\infty, +\infty)$ one denotes by $f * g$ the function for which $(f*g)(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)dy$ a.e. Given any f in $L_1(-\infty, +\infty) \cap L_2(-\infty, +\infty)$ one defines the Fourier transform of f, denoted by $\hat{f}$, by requiring that $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-i\alpha x)f(x)dx$ for all real α. Thus, the definition of $\hat{f}$ for an arbitrary f in $L_2(-\infty, +\infty)$ is determined.

RESULTS. One lemma is required for proof of the principal result. It appears below.

LEMMA. <u>Given any two non-negative, non-null functions h and k in $L_1(-\infty, +\infty)$ such that $||h||_1 = ||k||_1 = 1$, then</u>

(1) $$\sup_{-\infty < x < +\infty} (h*k)(x) \geq \sup_{0 < \epsilon < 1} (1-\epsilon)^2 \left[m(I_{\epsilon,h}) + m(I_{\epsilon,k})\right]^{-1}$$

PROOF. For any real ϵ such that $0 < \epsilon < 1$, we define h_ϵ and k_ϵ, non-negative and non-null members of $L_1(-\infty, +\infty)$, by the equations below.

(2) $h_\epsilon(z) = {}_{I_{\epsilon,h}}(z)\, h(z)$, all z

(3) $k_\epsilon(z) = {}_{I_{\epsilon,k}}(z)\, k(z)$, all z

Next, one can use (2) and (3) to write:

(4) $m(I_{\varepsilon,h}) \cdot m(I_{\varepsilon,k})$

$$= m\left\{\left[x \mid I_{\varepsilon,h} \cap (x - I_{\varepsilon,k}) \neq \emptyset\right]\right\}$$

$$= m\left\{\left[x \mid \quad y \quad y \in I_{\varepsilon,h},\ x-y \in I_{\varepsilon,k}\right]\right\}$$

$$= m\left\{\left[x \mid (h_\varepsilon * k_\varepsilon)(x) \neq 0\right]\right\}$$

Then since $k_\varepsilon(x)h_\varepsilon(y)$ belongs to $L_1\left[(-\infty, +\infty)X(-\infty, +\infty)\right]$, one can combine (4) with the Fubini theorem for multiple integrals and well-known properties of the transformation T defined by $T(x,y) = (x-y,y)$ to write the following sequence of equalities.

(5) $$\left[m(I_{\varepsilon,h}) \cdot m(I_{\varepsilon,k})\right]\left[\sup_{-\infty < x < +\infty} (h*k)(x)\right]$$

$$\geq m\left(\left\{x \mid (h_\varepsilon * k_\varepsilon)(x) \neq 0\right\}\right)\left[\sup_{-\infty < x < +\infty} (h*k)(x)\right]$$

$$\geq m\left\{\left[x \mid (h_\varepsilon * k_\varepsilon)(x) \neq 0\right]\right\}\left[\sup_{-\infty < x < +\infty} (h_\varepsilon * k_\varepsilon)(x)\right]$$

$$\geq \int_{-\infty}^{+\infty} (h_\varepsilon * k_\varepsilon)(x)\, dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[h_\varepsilon(y)\, k_\varepsilon(x-y)\right] dxdy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[h_\varepsilon(y)\, k_\varepsilon(x)\right] dxdy$$

$$= \left(\int_{-\infty}^{+\infty} h_\varepsilon(y)dy\right)\left(\int_{-\infty}^{+\infty} k_\varepsilon(x)dx\right)$$

$$= (1-\varepsilon)^2$$

The conclusion of this lemma follows directly from (5).

This lemma permits one to prove the following theorem.

THEOREM. <u>Let f and g be any two members of</u> $L_2(-\infty, +\infty)$ <u>such that</u> $||f||_2 = ||g||_2 = 1$. <u>Let</u> $f_\xi(x) = e^{i\xi x} f(x)$ <u>for all</u> ξ. <u>Let</u>

$F(x) = |\hat{f}(x)|^2$ and $G(x) = |\hat{g}(x)|^2$ for all x. Then

$$(6) \quad \sup_{-\infty < \nu < +\infty} ||f_\nu * g||_2 \geq \sup_{0 < \epsilon < 1} (1-\epsilon)\left\{2\pi[m(I_{\epsilon,F}) + m(I_{\epsilon,G})]^{-1}\right\}^{\frac{1}{2}}$$

The inequality (6) is sharp in the sense that for every positive number η there are choices of f and g for which the right side of the inequality is finite and for which the ratio of the expression on the left-hand side of (6) to the expression on the right-hand side exceeds 1 by less than η.

PROOF. The lemma and the Plancherel theorem combined yield (6).

To prove the rest of the theorem, let η be a fixed but arbitrary positive number. Then two members, p and q, of $L_2(-\infty, +\infty)$ will be defined and shown to have the asserted properties relative to η. These functions will be defined in terms of their Fourier transforms.

$$(7) \quad \hat{p}(\omega) = \begin{cases} (\pi\Delta)^{-\frac{1}{2}}(1 + i\omega)^{-1}, & |\omega| \leq \tan\frac{\pi}{2}\Delta \\ 0, & |\omega| > \tan\frac{\pi}{2}\Delta \end{cases}$$

$$(8) \quad \hat{q}(\omega) = \begin{cases} \pi^{\frac{1}{2}}(1 + i\Delta\omega)^{-1}, & |\omega| \leq \Delta^{-1}\tan\frac{\pi}{2}\Delta \\ 0, & |\omega| > \Delta^{-1}\tan\frac{\pi}{2}\Delta \end{cases}$$

Here Δ is assumed to be positive and less than 1. Then, with the aid of the definitions of $\hat{p}$ and $\hat{q}$ and the Plancherel theorem, it can be seen that one has:

$$(9) \quad \sup_{-\infty < \nu < +\infty} ||p_\nu * q||_2 = \sup_{-\infty < \nu < +\infty} \left[2\pi\int_{-\infty}^{+\infty} |\hat{p}(\omega - \nu)|^2 \, |\hat{q}(\omega)|^2 \, d\omega\right]^{\frac{1}{2}}$$

$$= \left[2\pi \int_{-\infty}^{+\infty} |\hat{p}(\omega)|^2 \, |\hat{q}(\omega)|^2 \, d\omega\right]^{\frac{1}{2}}$$

And the latter integral has the following evaluation:

$$(10)\quad \int_{-\infty}^{+\infty} |\hat{p}(\omega)|^2 \, |\hat{q}(\omega)|^2 \, d\omega$$

$$= \left(2\Delta^{-1} \tan \tfrac{\pi}{2}\Delta + 2 \tan \tfrac{\pi}{2}\Delta\right)^{-1} \left(\frac{\tan \frac{\pi}{2}\Delta}{\frac{\pi}{2}\Delta}\right) \left(\frac{1 - \frac{2}{\pi} \tan^{-1}(\Delta \tan \frac{\pi}{2}\Delta)}{1 - \Delta}\right)$$

However, one can see:

$$(11)\quad m(I_{0,P}) = 2 \tan \tfrac{\pi}{2}\Delta$$

$$(12)\quad m(I_{0,Q}) = 2\Delta^{-1} \tan \tfrac{\pi}{2}\Delta$$

where P and Q are determined by setting $P(\omega) = |\hat{p}(\omega)|^2$ and $Q(\omega) = |\hat{q}(\omega)|^2$ for all ω. Thus, there results:

$$(13)\quad \left(2\Delta^{-1} \tan \tfrac{\pi}{2}\Delta + \tan \tfrac{\pi}{2}\Delta\right)^{-\frac{1}{2}} \leq \sup_{0 < \epsilon < 1} (1-\epsilon) \left\{2\pi\left[m(I_{\epsilon,P}) + m(I_{\epsilon,Q})\right]^{-1}\right\}^{\frac{1}{2}}$$

Hence, combining (6), (9), (10), and (13), one can conclude that when Δ is small enough for

$$\left[\frac{\tan \frac{\pi}{2}\Delta}{\frac{\pi}{2}\Delta} \cdot \frac{1 - \frac{2}{\pi} \tan^{-1}(\Delta \tan \frac{\pi}{2}\Delta)}{1 - \Delta}\right]^{\frac{1}{2}} - 1$$

to be less than η, then the same is true of

$$\left(\sup_{-\infty < \nu < +\infty} \|p_\nu * q\|_2\right) \Big/ \left\{\sup_{0 < \epsilon < 1} (1-\epsilon) \sqrt{2\pi} \left[m(I_{\epsilon,P}) + m(I_{\epsilon,Q})\right]^{-\frac{1}{2}}\right\} - 1$$

It is, of course, clear from (9) and (10) that $\sup_{-\infty < \nu < +\infty} \|p_\nu * q\|_2$ is finite.

Thus, the second part of the theorem has been approved.

COROLLARY. <u>Let the notation of the theorem hold. Further, let f and g be restrictions to $(-\infty, +\infty)$ of entire functions of exponential type such that the types of f and g are E_1 and E_2, respectively. Then</u>

$$\sup_{-\infty < \nu < +\infty} \|t_\nu * q\|_2 \geq \tfrac{\pi}{2} (E_1 + E_2)^{-\frac{1}{2}}$$

PROOF. As indicated by Theorem 21 of [1] the transforms of f and g vanish outside $[-E_1, E_1]$ and $[-E_2, E_2]$ respectively. Thus,

(14) $m(I_{0,F}) \leq 4E_1$

(15) $m(I_{0,G}) \leq 4E_2$

Since the indefinite integrals of $|f|$ and $|g|$ are absolutely continuous, (14) and (15) permit the following inequality.

$$(16) \quad \sup_{0 < \epsilon < 1} (1-\epsilon) \left\{ 2\pi \left[m(I_{\epsilon,F}) + m(I_{\epsilon,G}) \right]^{-1} \right\}^{\frac{1}{2}}$$

$$\geq \left\{ 2\pi \left[4E_1 + 4E_2 \right]^{-1} \right\}^{\frac{1}{2}}$$

The assertion of the corollary follows from (16) and the theorem.

REFERENCES

1. Paley, R.E.A.C. and Wiener, N., Fourier Transforms in the Complex Domain, American Mathematical Society, Providence, R.I., 1934.