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THEORY OF SHELLS AND PLATES

Proceedings of the 4th All-Union Conference on Shells and Plates held at Erevan 24-31 October 1962

Edited by S. M. Durgar'yan

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AKADEMIYA NAUK ARMYANSKOI SSR. INSTITUT MATEMATIKI I MEKHANIKI

Academy of Sciences of the Armenian SSR. Institute of Mathematics and Mechanics

THEORY OF SHELLS AND PLATES

(Teoriya obolochek i plastin)

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Izdatel'stvo Akademii Nauk Armyanskoi SSR
Erevan 1964

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FOREWORD

According to resolutions of the Scientific Council of the Academy of Sciences of the USSR on the topic of "Scientific Bases of Strength and Plasticity" and of the Presidium of the Academy of Sciences of the Armenian SSR, the fourth regular All-Union Conference on the Theory of Shells and Plates took place in Yerevan* between 24 and 31 October 1962. The Conference was organized by the Institute of Mathematics and Mechanics of the Academy of Sciences of the Armenian SSR, by the Yerevan State University, and the Yerevan Polytechnic Institute im. K. Marx.

The work of the conference was managed by an organizing committee with the following membership: R. A. Adadurov, N. A. Alumyae, S. A. Ambartsumyan (chairman), N. Kh. Arutyunyan, V. V. Bolotin, A. N. Volkov (scientific secretary), A. S. Vol'mir, I. I. Vorovich, A. L. Gol'denveizer (deputy chairman), S. M. Durgar'yan (scientific secretary), A. V. Karmishin (deputy chairman), N. A. Kil'chevskii, S. G. Lekhnitskii, A. I. Lur'e, Kh. M. Mushtari, V. V. Novozhilov, V. M. Panferov, A. V. Pogorelov, G. N. Savin, O. M. Sapondzhyan, A. A. Umanskii, and T. T. Khachaturyan. D. S. Gevondyan served as the technical secretary.

The conference conducted its business in plenary sessions and in subcommittees for:

1) general problems of the theory of shells and plates;
2) stability and nonlinear problems of the theory of shells and plates;
3) dynamics of shells and plates;
4) thermo-elasticity, creep, and plasticity of shells and plates;
5) structural mechanics of shells and plates.

The conference was attended by 437 delegates from 29 cities of the Soviet Union.

The papers were arranged in [Cyrillic] alphabetical order according to the name of the author; papers read at plenary sessions of the conference are here arranged separately [first part of the collection].

Discussions on a number of papers, which took place at the conference, are not included in this collection.

At the end of this collection there appears a list of papers which were included in the program of the conference, but were not presented by the authors for publication.

The organizing committee of the conference supervised the editing of the manuscripts, which was performed by B. L. Abramyan, N. O. Gulkanyan, S. M. Durgar'yan, R. S. Minasyan, D. V. Peshtmaldzhyan, and K. S. Chobanyan, staff members of the Institute of Mathematics and Mechanics of the Academy of Sciences of the Armenian SSR.

* [Also known as Erevan.]

PAPERS READ AT PLENARY SESSIONS

APPLICATION OF METHODS OF THE THEORY OF PROBABILITY IN THE THEORY OF PLATES AND SHELLS

V. V. Bolotin

(Moscow)

INTRODUCTION

A great deal of interest has been expressed during the last few years in the application of methods of the theory of probability and mathematical statistics to the theory of plates and shells. Until recently, only the desirability of applying these methods to problems of the theory of plates and shells was expressed, as, for example, by A. S. Vol'mir /17/ and V. I. Feodos'ev /60/, in Langhaar's survey of the theory of stability of elastic systems /38/, and also in T'sin Hu-sen's and Khu Khai-chan's review /64/ of the book by Kh. M. Mushtari and K. Z. Galimov /47/. At present, however, active investigations are proceeding in several directions. These trends are in part reviewed in /6, 12, 27, 79, 130/.

The application of probability methods to the theory of plates and shells is based on the fact that the behavior of thin-walled structures subjected to static or dynamic loads depends on a number of random factors. These factors can be divided into two groups by their origin. One group includes geometric and physical parameters of the design proper, i. e., random deviations from the ideal geometric shape, diversity of methods for obtaining boundary conditions, spread of elastic and strength characteristics of the materials, etc. The second group includes random factors depending on the load and other "external" service and experimental conditions. Henceforth we shall distinguish between "internal" and "external" parameters, although this division is, of course, arbitrary.

These factors are encountered not only in the analysis of thin-walled, but, to a varying extent also of any other structure. Nevertheless, the applications of probability (also known as statistical) methods are being developed most intensively in the theory of plates and shells. This may be due to two causes. Firstly, thin-walled structures are particularly sensitive to small changes in initial shape, small variations in the boundary conditions, etc. Minor causes may frequently have major effects. An example is the well-known scatter of the experimental critical forces in shells, which is due to small deviations from the idealized state of the test specimens. Secondly, it is precisely the application of statistical methods to thin-walled structures which is of major interest today. As example we can cite the elements of aircraft skins, subjected to random loads due to atmospheric turbulence, pulsations in the turbulent boundary layer, sound waves from the engines, meteoritic dust, etc.

With reference to the above we should mention the existence of a more general division of mechanics which we shall call "Statistical Dynamics of a Solid Body Undergoing Deformation." This in its turn is part of the "Statistical Dynamics," which is a field of applied mathematics devoted to the application of probability methods to mechanical, electrical, radioengineering, cybernetic, and similar systems. We do not require new methods to investigate the probability behavior of mechanical systems. As a rule, such methods can be obtained from other branches of statistical mechanics; they must only be adapted to the problems dealing with the mechanics of a body undergoing deformation, in particular, to the theory of plates and shells.

Which problems require the application of probability methods in the theory of plates and shells? All problems can be divided into two groups. The first group includes problems concerning the reliability of thin-walled structures subjected to static and dynamic loads. These problems can be formulated in different manners. In some cases the probability of a dangerous state attained at the end of the planned period of operation, or the average or most probable service life has to be determined. In other cases, the laws governing the distribution of parameters determining the state of strain (e.g., residual strains accumulating toward the end of the service life) have to be found. Laws governing the distribution of "external" or "internal" parameters (e.g., tolerances in the manufacture of components, or the level of fluctuating loads), which ensure standard reliability of the structure, may have to be derived. These examples are not exhaustive. The formulation of the problem depends on the intended service and operating conditions of the structure, i.e., whether it will be used only once, or is designed for long service, whether loading is a prolonged steady-rate random process or a short unsteady process, whether the structure is expected to fail as a result of an intensive action of short duration, through accumulation of plastic deformations or development of fatigue cracks, etc. Formulating the problem itself is important, but this belongs to the theory of reliability rather than to the subject of the present conference, and is therefore not within the scope of this paper.

The second group of problems relates to the analysis and interpretation of results of testing plates and shells. The most typical example is stability tests of thin shells. It is widely known that, due to the extensive scatter exhibited by the experimental results obtained for the critical forces, no complete comparison of experimental data with the results of the nonlinear theory can be carried out. In order to make this comparison possible it is necessary to know the statistical distribution of the initial irregularities, fluctuations in obtaining the boundary conditions, and other factors affecting the magnitude of the critical loads. We recall two typical statements of problems in this field. The basic problem consists in finding the relationships governing the distribution of the critical forces on the basis of known distributions of random parameters which characterize the shell and the conditions of the experiment. The opposite problem — of deducing the laws governing the distribution of "internal" parameters for shells undergoing tests on the basis of the empirical distribution of the critical loads — may also arise. Obviously, these problems are not directly related to the reliability problem and may arise also when the latter does not exist.

In dealing with the question of methods for solving statistical problems of the theory of plates and shells we must note that the selection of a method is primarily determined by the probability nature and properties of those parameters upon which the behavior of the structure depends. Some parameters are random quantities, others are random functions of time or position (not infrequently of both), or, as it is said, they represent a random process. The properties of the structure are, as a rule, characterized by random quantities; the loads must generally be treated as random processes. This division is, of course, quite arbitrary. It is known that a certain class of random functions can be represented as a sum of a finite or infinite number of appropriately selected deterministic (nonrandom) functions which depend on random parameters. If the number of these parameters is finite, we can make a transition from operations on random functions to operations on the corresponding random parameters. On the other hand, if the number is infinite but countable, then it may be that the behavior of the random functions can, with a good approximation, be described by a moderate number of parameters. Thus, the initial irregularities of a shell are, strictly speaking, random functions of the position. By expanding these functions in series in terms of certain deterministic functions we can describe the irregularities by using random numbers, i.e., the coefficients of these series. Another example is impulsive loads of short duration. The shape of random impulses is usually quite simple and as such can be described by deterministic functions of time and of several random parameters. Conversely, random loads such as the pressure due to pulsations in the boundary layer and sound waves from engines in operation, wave action on vessels and [shore] protecting structures, etc., can be adequately described only within the framework of the theory of random processes.

Probability methods used in the theory of plates and shells can be divided into three groups: quasisteady methods, correlation and related methods and, finally, methods based on the use of kinetic equations. Each method has its advantages and disadvantages and its fields of application. These fields may overlap; however, for each concrete problem there is always a preferable method.

Quasisteady methods are based on the known equations of the theory of probability. These methods can be used successfully with those problems in which the random factors can be described by a finite (not excessive) number of random quantities. In order to use this method we must have the solutions of the corresponding deterministic problems in the entire range of variations of the random parameters; the subsequent operations reduce to transformation of the probability distributions for these parameters. The applicability of quasisteady methods is not limited to problems in which the loads vary slowly ("quasisteadily"). If the random dynamic loads can be represented in the form of deterministic functions of time, depending on a finite number of random quantities, then the quasisteady methods can also be effectively applied.

Correlation methods are based on the utilization of the relationships between correlation (or moment) functions of "input" parameters (e.g., loads) and "output" parameters (deflections, internal forces and stresses). These relationships can be expressed by differential and integral-differential equations, as well as in elementary cases by finite relationships. The spectral method, method of canonical expansions, etc.,

can be regarded as modifications of correlation methods. These methods, whose theoretical foundations were developed by A. N. Kolmogorov, A. Ya. Khinchin, and N. Wiener, have come into extensive use in modern physics and technology. In particular, the correlation methods have become the operating tools of the theory of turbulence, the theory of information, statistical dynamics of automatic control systems, etc. The field of application of these methods are problems in which the external load is a correlated random process. These methods are most effective when the load is a Gaussian process and the system is linear. However, nonlinear systems can also be solved by using methods of nonlinear mechanics.

The third group of methods is based on the consideration of differential and integral-differential equations which describe the evolution with time of distribution functions of random parameters. The Boltzmann equation in the kinetic theory of gases and the Smoluchowski-Einstein equation in theory of Brownian motion are typical examples of such equations. Methods based on the use of kinetic equations have been worked out in detail in particular for processes without heredity effects (Markov processes). Hence these methods should be applied to problems in which the output quantities can be treated as components of a Markov process. The main advantage of these methods is the possibility of obtaining the distribution functions directly. However, this advantage cannot always be utilized, since solving the kinetic equations is difficult. The above applies in particular to nonsteady-state solutions.

Each of these three groups of methods is applicable to a certain range of problems in the theory of plates and shells. The quasisteady methods are used primarily for interpretation and analysis of stability tests of shells. The correlation methods are used primarily to solve problems of small oscillations of plates and shells under the action of loads representing a steady-state random process. Finally, the kinetic equations are used to describe the behavior of shells at such large random "impacts" that "popping" [collapse accompanied by a loud bang] becomes possible. Systematization of work by the methods used is thus basically identical with systematization by groups of problems in the theory of plates and shells.

Below we give a survey of studies belonging to each of the aforementioned groups. In conclusion we shall give an overall evaluation of the present state, and describe possible trends of future studies.

1. QUASISTEADY METHODS

The term quasisteady methods will be used here to denote methods which are based entirely on equations of the elementary theory of probability without resorting to the theory of random processes. The use of quasisteady methods presupposes preliminary solution of the corresponding deterministic (static or dynamic) problems.

It should be noted that the term "quasisteady methods" is in this case used arbitrarily and does not precisely convey the content and field of application of these methods. In our opinion the name "semideterministic methods" would be more appropriate, thus reflecting the division of the

problem into two stages, the first of which is purely deterministic. In this paper, however, the term "quasisteady methods" will be used, which will cause less misunderstanding than terms such as "statical method", "methods of inertialess transformations" and others used in the literature.

Quasisteady methods are extensively used in statistical dynamics. Thus, the problem well-known in radioengineering and in the theory of automatic controls of a random signal passing through a detector or inertialess relay, reduces to the corresponding transformation of the probability density for the input signal. Consequently, it can be stated that the problem is solved by the quasisteady method.

If we restrict ourselves to mechanical systems, we can refer to works by N. S. Streletskii, A. R. Rzhanitsyn, and their followers, who have extensively used quasisteady methods for statistical substantiation of safety factors in engineering structures. A summary of these works can be found in /9/. Closest to the theme of this paper is the paper by A. R. Rzhanitsyn /52/, in which he considers the problem of combined buckling and bending of beams in the presence of random initial deflections and random eccentricities. By specifying distribution functions for the initial deflections and eccentricities as well as for the axial force and the yield point of the material, and by using the deterministic relationship between the first three parameters and the edge stresses in the beam, A. R. Rzhanitsyn seeks the probability of a dangerous state, and gives a statistical interpretation of the known "buckling coefficient" on the basis of his calculations.

Quasisteady methods were introduced into the theory of plates and shells by this author in /4/. A certain generalization and expansion of the field of application is contained in /9/. Applications of quasisteady methods to various problems of shell stability are given by A. S. Vol'mir /19/, V. M. Goncharenko /23/, and B. P. Makarov /43-45/.

The application of quasisteady methods to the theory of plates and shells presupposes fulfillment of the following conditions:

a. Factors influencing the behavior of the structure can be described by a finite number of random parameters $u_1, u_2, \cdots, u_m, q_1, q_2, \cdots, q_r$ with a known joint probability density $p(u_1, u_2, \cdots, u_m; q_1, q_2, \cdots, q_r)$. Here u_α denotes the "internal" parameters which characterize the shape and the mechanical properties of the structure, while q_α denotes the parameters characterizing the load and other "external" service or experimental conditions.

b. Factors characterizing the behavior of the structure (displacements, strains, forces, moments, and stresses), can also be described by a finite number of random parameters $v_1, v_2, \cdots, v_n$.

c. The aforementioned two groups of parameters are connected by the known deterministic relationships

$$v_\alpha = V_\alpha(u_1, u_2, \cdots, u_m, q_1, q_2, \cdots, q_r), \quad (\alpha = 1, 2, \cdots, n), \tag{1.1}$$

which are assumed to be uniquely determined and piecewise continuous. To obtain (1.1) it is generally required to solve the Cauchy problem for a structure loaded by forces varying with time. If the initial conditions are random, they should be included among the parameters $u_1, u_2, \cdots, u_m$.

With these assumptions, the joint probability density $p(v_1, v_2, \cdots, v_n)$ is found from known formulas for the probability densities of functions of

random quantities. In particular, if $n \leqslant m$ and if equations (1.1) permit the solution

$$u_\alpha = U_\alpha (v_1, v_2, \cdots, v_n, u_{n+1}, \cdots, u_m, q_1, q_2, \cdots, q_r), \tag{1.2}$$

where U_α are uniquely determined continuous functions of $v_1, v_2, \cdots, v_n$, then the solution is given by /4/

$$p(v_1, v_2, \cdots, v_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(U_1 \cdots U_n, u_{n+1}, \cdots, u_m, q_1, \cdots q_r) \times$$

$$\times \left| \frac{\partial (U_1, U_2, \cdots, U_n)}{\partial (v_1, v_2, \cdots, v_n)} \right| du_{n+1} \cdots du_m dq_1 \cdots dq_r, \tag{1.3}$$

where

$$\frac{\partial (U_1, U_2, \cdots U_n)}{\partial (v_1, v_2, \cdots v_n)} = \begin{vmatrix} \frac{\partial U_1}{\partial v_1} \frac{\partial U_2}{\partial v_1} \cdots \frac{\partial U_n}{\partial v_1} \\ \frac{\partial U_1}{\partial v_2} \frac{\partial U_2}{\partial v_2} \cdots \frac{\partial U_n}{\partial v_2} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial U_1}{\partial v_n} \frac{\partial U_2}{\partial v_n} \cdots \frac{\partial U_n}{\partial v_n} \end{vmatrix} \tag{1.4}$$

is the Jacobian of U_α with respect to v_α.

We can generalize (1.3) to include the case when the functions $U_1, U_2, \cdots, U_n$ are not uniquely determined (when the integration interval is divided into a number of subintervals), the case when these functions are piecewise continuous (when the probability density for $v_1, v_2, \cdots v_n$ contains delta-shaped overshoots), etc. Details will not be discussed here.

When $m < n < m + r$, (1.3) can be rewritten. If $n > m + r$, part of the parameters $v_1, v_2, \cdots, v_n$ will, obviously, be mutually dependent. We may then assume that $n = m + r$ and seek the joint probability density for $v_1, v_2, \cdots, v_{m+r}$. Instead of (1.2) we then obtain

$$u_\alpha = U_\alpha (v_1, v_2, \cdots, v_{m+r}), \qquad (\alpha = 1, 2, \ldots, m),$$

$$q_\alpha = Q_\alpha (v_1, v_2, \cdots, v_{m+r}), \qquad (\alpha = 1, 2, \ldots, r).$$

The joint probability density $p(v_1, v_2, \cdots, v_{m+r})$ is

$$p(v_1, v_2, \cdots, v_{m+r}) = p(U_1, \cdots, U_m, Q_1 \cdots, Q_r) \left| \frac{\partial (U_1, \ldots, U_m, Q_1, \ldots Q_r)}{\partial (v_1, v_2, \ldots, v_{m+1})} \right|. \tag{1.5}$$

Let us now find $P(*)$, the probability of the limiting state. The term "limiting state" as used here denotes not only "popping" but also the appearance of large elastic or residual deformations, etc. Analysis shows that the surface corresponding to "pop" conditions in the $v_1, v_2, \cdots, v_n$ space forms only part of the boundary surface $\Psi(v_1, v_2, \cdots, v_n) = 0$, which separates the region of limiting states from the region of normal service.

Let us assume that points in the $v_1, v_2, \cdots v_n$ space for which $\Psi(v_1, v_2, \cdots, v_n) > 0$ belong to the region of limiting states. Then the probability of a limiting state is

$$P(*) = \int \cdots \int_{\Psi(v_1, v_2, \cdots, v_n) > 0} p(v_1, v_2, \cdots, v_n)\, dv_1 dv_2 \cdots dv_n. \tag{1.6}$$

This probability can also be found in a different manner /9/. If region $\Psi(v_1, v_2, \cdots, v_n) > 0$ has a corresponding region $\Psi(u_1, u_2, \cdots, u_m, q_1, q_2, \cdots, q_r) > 0$, in the $u_1, u_2 \cdots, u_m, q_1, q_2, \cdots, q_r$ space, then

$$P(*) = \int \cdots \int_{\Psi(u_1, u_2, \cdots, u_m, q_1, q_2, \cdots, q_r) > 0} p(u_1, u_2, \cdots, u_m, q_1, q_2, \cdots, q_r)\, du_1, \cdots, du_m, dq_1, \cdots dq_r. \tag{1.7}$$

A different problem arises in the analysis and interpretation of data. The "internal" parameters of the system $u_1, u_2, \cdots, u_m$ are the independent random quantities; it is required to find the joint probability density for the limiting values of $q_1, q_2, \cdots, q_r$, which will be denoted by $q_1^*, q_2^*, \cdots, q_r^*$. We shall assume that the deterministic relationships between $q_1^*, q_2^*, \cdots, q_r^*$ and $u_1, u_2, \cdots, u_m$, (the "internal" parameters of the structure) as well as the joint probability density $p(u_1, u_2, \cdots, u_m)$ are known. Assume further that $r < m$ and, in addition, let there exist uniquely determined and continuous relationships

$$u_\alpha = U_\alpha(q_1^*, q_2^*, \cdots, q_r^*, u_{r+1}, \cdots, u_m), \quad (\alpha = 1, 2, \ldots, r). \tag{1.8}$$

Then the joint probability density $p(q_1^*, q_2^*, \cdots, q_r^*)$ can be found from expressions similar to (1.3)

$$p(q_1^*, q_2^*, \cdots, q_r^*) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(U_1, U_2, \cdots, U_r, u_{r+1}, \cdots, u_m) \left| \frac{\partial(U_1, U_2, \cdots, U_r}{\partial(q_1^*, q_2^*, \cdots, q_r^*} \right| du_{r+1} \cdots du_m. \tag{1.9}$$

The limitations imposed upon (1.9) can be easily removed; the case when the relationship is not uniquely determined is discussed in /9/.

It is also possible to state the inverse problem: to find, from a given probability distribution for the limiting loads, the probability distribution for the "internal" parameters of a shell. The solution of this problem is again given by expressions similar to (1.3), (1.5), and (1.9). Assume for example that $r = m$, and let $q_\alpha^* = Q_\alpha^*(u_1, u_2, \cdots, u_m)$ be uniquely determined and continuous. Then

$$p(u_1, u_2, \cdots, u_m) = p(Q_1^*, Q_2^*, \cdots, Q_m^*) \left| \frac{\partial(Q_1^*, Q_2^*, \cdots, Q_m^*)}{\partial(u_1, u_2, \cdots, u_m)} \right|. \tag{1.10}$$

The probability distribution $p(q_1^*, q_2^*, \cdots, q_r^*)$ for the limiting forces can be obtained from (1.7). We assume that $q_1, q_2, \cdots, q_r$ are fixed, being equal to $q_1^*, q_2^*, \cdots, q_r^*$, respectively. Then

$$p(u_1, u_2, \cdots, u_m, q_1, q_2, \cdots, q_r) = p(u_1, u_2, \cdots, u_m)\, \delta(q_1 - q_1^*)\, \delta(q_2 - q_2^*) \cdots \delta(q_r - q_r^*). \tag{1.11}$$

Substituting (1.11) into the right-hand side of (1.7) we obtain the conditional probability of the limiting state

$$P(*\mid q_1^*, q_2^*, \cdots, q_r^*) = \int\cdots\int p(u_1, u_2, \cdots, u_m)\, du_1 du_2 \cdots du_m, \tag{1.12}$$

$$\Psi_1(u_1, u_2, \cdots, u_m \mid q_1^*, q_2^*, \cdots, q_r^*) > 0.$$

It is easy to see that (1.12) gives the joint distribution functions for $q_1^*, q_2^*, \cdots, q_r^*$. From this

$$p(q_1^*, q_2^*, \cdots, q_r^*) = \frac{\partial^r P(*\mid q_1^*, q_2^*, \cdots, q_r^*)}{\partial q_1^* \partial q_2^* \cdots \partial q_r^*}. \tag{1.13}$$

This method of obtaining the probability density for critical forces was first described (for $r = 1$) by V. M. Goncharenko /23/.

We now present some particular problems which are solved by quasi-steady methods. Several studies have been devoted to the following problem: a plate or shell is loaded by external forces which are specified to within one common multiplier — the parameter q. This parameter increases slowly (quasisteadily) and monotonically, from zero to some finite value. It is required to find the distribution of the strain parameters (usually in generalized coordinates which characterize the normal deflection), which is attained toward the end of the loading process. In order that the problem be solvable, condition (b) must be satisfied. It is easy to show that this will be true in this problem if the system is conservative and if not more than two stable forms of equilibrium exist when the load parameter is fixed. In this case the equilibrium mode which is achieved can be easily determined from quasisteady considerations if the history of the system is known. The majority of solved deterministic problems of the nonlinear theory of plates and shells /17, 47/ satisfies condition (b). If relationships (1.1) cannot be obtained from quasisteady considerations, then the Cauchy problem must be solved.

Paper /4/ considers the probability distribution of the total deflection of an elastic shallow cylindrical panel with movable edges, compressed by axial forces whose intensity is q. It is assumed that the initial deflection parameter is a random quantity, while the load parameter is deterministic. The solution of the corresponding deterministic problem, given by A. S. Vol'mir /17/ was applied. The probability of "popping" for the same problem was calculated in the same paper on the assumption that the initial deflections have a symmetrical normal distribution. Figure 1 shows this probability as a function of the load. It is in connection with this problem that the effect of negative initial deflections on the distribution of the "popping" probability was first stressed. Similar curves for the problem of the stability of a closed circular cylindrical shell were obtained by V. M. Goncharenko /23/. One family of these curves is shown in Figure 2.

The distribution of critical forces was most extensively investigated. In the elementary case, when the load is specified to within one parameter q and the irregularities to within one parameter u, (1.9) takes on the form /4/

$$p(q_*) = \frac{p[U(q_*)]\left|\dfrac{\partial U(q_*)}{\partial q_*}\right|}{\displaystyle\int_{u_1}^{u_u} p(u)\, du}. \tag{1.14}$$

The renormalizing multiplier in the denominator takes into account the circumstance that "popping" may take place only when $u_l < u < u_u$.

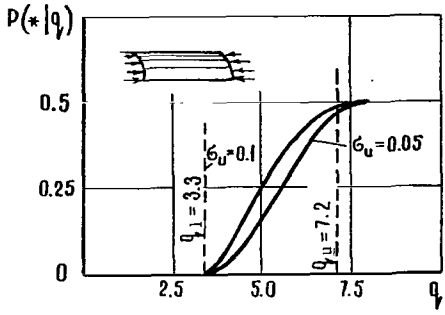

FIGURE 1.

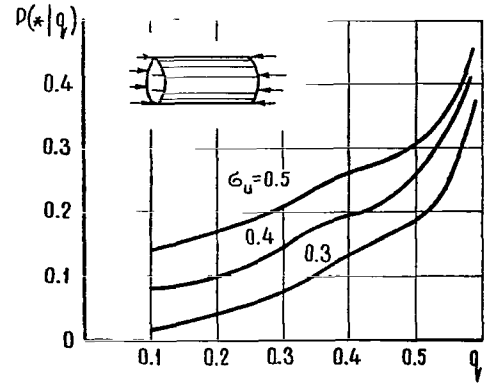

FIGURE 2.

Figure 3 shows the probability density $p(q_*)$ for a shallow cylindrical panel /4/. The table gives the mathematical expectation $\bar{q}_*$ calculated for a symmetrical normal distribution of u for the mean-square values $\sigma_n = 0.1$ and $\sigma_n = 0.25$. The theoretical values were in close agreement with those obtained experimentally.

TABLE 1

σ_u	$\bar{q}_*$	$\frac{\bar{q}_*}{q_u}$ %	$\frac{\bar{q}_*}{q_l}$ %
0.1	4.84	67	146
0.25	4.50	62	136

B. P. Makarov /44/ carried out a thorough investigation of the theoretical laws governing the distribution of critical forces in the following problems: axial compression of a closed circular cylindrical shell; compression of a shell by a transverse normal pressure, including compression on all sides; torsion of a circular cylindrical shell; compression of a cylindrical panel by a normal pressure. Extensive use was made of known solutions of the corresponding deterministic problems, due to A. S. Vol'mir /18/, Kh. M. Mushtari and M. S. Kornishin /34, 47/, N. I. Krivosheev /35/, and also Donnell and Wan /83/, Nash /114, 115/, and others.

We shall now discuss several interesting results obtained by B. P. Makarov. Figure 4 shows the probability distribution $p(q_*)$ for a cylindrical shell undergoing compression on all sides, when various assumptions are made about the distribution of the initial deflections (1 — uniform distribution for $0 \leqslant u \leqslant u_1$, 2, 3, 4 — normal distributions for the standard [deviation]

$\sigma_u = 0.3$ and different values of the average $\bar{u}$). It can be seen that the distribution of the initial deflections has a pronounced effect on the distribution of the critical forces. Similar curves for a cylindrical shell undergoing axial compression are given in Figure 5. It was assumed that the distribution of the initial deflection parameter

$$u = \frac{w_0}{R} n^2 m^{3/2}$$

(w_0 is the initial deflection, n is the number of waves around circumferences, m is the ratio of half-wave lengths in circumferential and axial directions) was normal. It is interesting that some curves have two extrema: one near the lower critical value, the second slightly displaced toward the upper critical value. A similar result was obtained later by V. M. Goncharenko /23/.

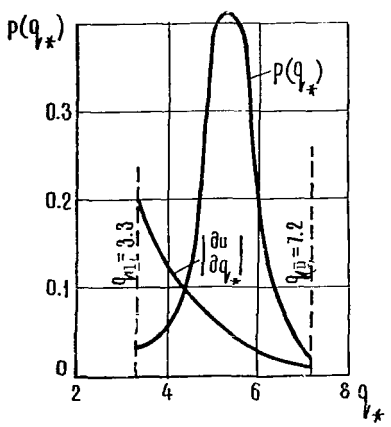

FIGURE 3.

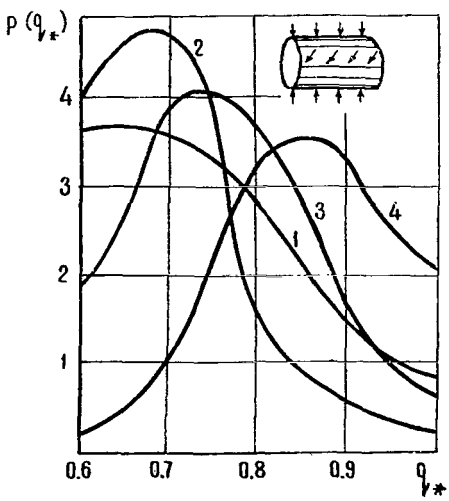

FIGURE 4.

B. P. Makarov /45/ has investigated on a cylindrical panel subject to axial compression, the combined effect of the scatter of initial deflections and support conditions. It was found that for some distributions of these values the probability density of critical forces has two extrema.

Of great interest is the application of quasisteady methods to the analysis of data obtained in stability tests of shells. Using empirical formulas, proposed by Donnell and Wan /83/ and Nash /114/, for the dimensionless initial deflection of a cylindrical shell, A. S. Vol'mir /19/ analyzed the influence of the ratio R/h on the distribution of critical forces. He showed that, from the statistical point of view, we should expect a successive decrease in the average critical pressure $\bar{q}_*$ as the ratio increases (Figure 6).

B. P. Makarov /43/ gives a statistical analysis of the results of 222 tests performed on cylindrical shells subjected to axial compression, contained in the paper by Harris, Suer, Suene, and Benjamin /92/, who

only plotted curves corresponding to a 10 % and 1 % probability of "popping." The experimental data are given in Figure 7, while Figure 8 presents histograms for different ranges of R/h. For small values of R/h the histograms have two maxima, which is in agreement with theoretical conclusion obtained previously in /44/. However, it is still possible that the existence of two maxima is due to the use of two groups of data with different statistical properties (the results analyzed were obtained in several experiments).

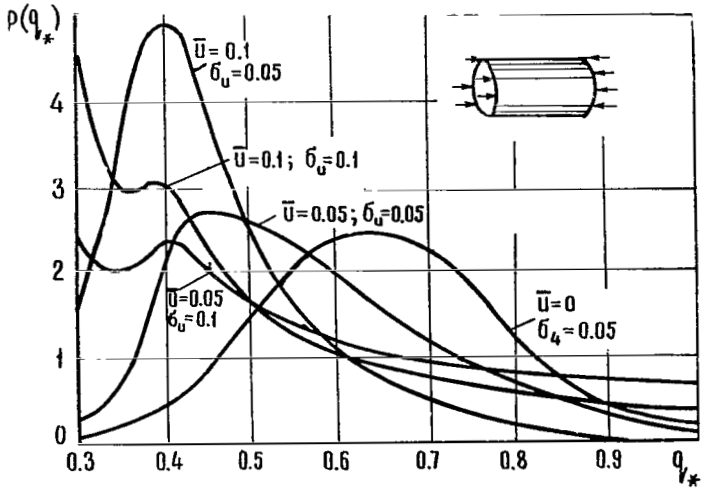

FIGURE 5.

An important problem is to find the laws governing the distribution of initial deflections when the distribution of critical forces is known. Figure 9 shows a histogram for a certain interval of R/h, obtained by B. P. Makarov, It includes for comparison a probability-density curve corresponding to a normal distribution of u for a mean $\bar{u} = 0.1$ and a standard [deviation] $\sigma_u = 0.06$.

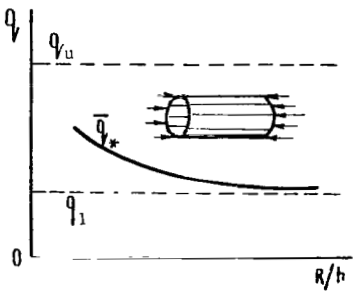

FIGURE 6.

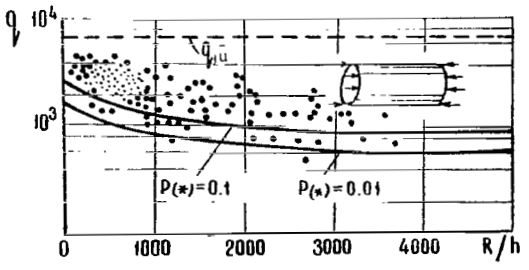

FIGURE 7.

Up to now we have considered problems of quasisteady monotonic loading. There exist, however, problems of the stability of plates and shells which cannot be treated by quasisteady methods no matter how slowly the load is applied. Let, for example, a shell subjected to the action of fixed external forces have more than two stable forms of equilibrium. When the value of q_* is attained from below, either stable state may be obtained. A choice between them cannot be made on the basis of quasisteady considerations. This indeterminacy can be removed, and condition (b) satisfied by considering, for example, loading at a finite (although low) rate and taking into account the initial conditions. Once the deterministic problem is solved, the quasisteady method can be applied. Difficulties involved in the selection of solutions arise and are overcome in the first, purely deterministic, stage of investigation.

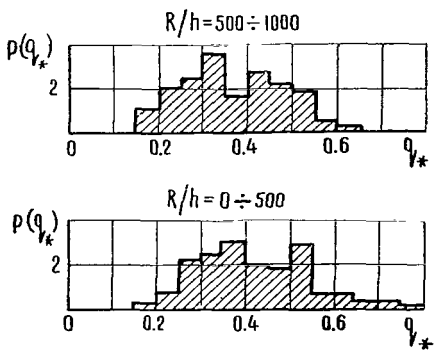

FIGURE 8.

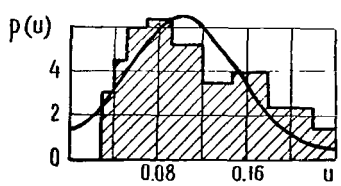

FIGURE 9.

A method for solving certain dynamic problems of the stability of shells is given in /9/. As an example we now consider the problem of dynamic "popping" of a shallow cylindrical panel subjected to a normal impulsive load $q=0$ $(t<0)$ and $q=q_0e^{-ct}$ $(t\geqslant 0)$. The corresponding deterministic problem was solved by a group of authors /6/. Figure 10 shows the region of instability in the πc, q_0 plane. Let c and q_0 be random quantities. Then the probability of "popping" can be determined from (1.7). Some results, obtained for a uniform distribution of c and q_0, are presented in Figure 11. Another example is the problem of dynamic "popping" of a cylindrical panel subjected to axial compression by a load which increases at a random though constant rate from zero to some value $q>q_u$. Figure 12 presents results obtained in solving the deterministic problem /7/. Having obtained the deterministic dependence of the maximum deflections v_{max} occurring immediately after "popping" on the rate c, and using equations similar to (1.3) and (1.5), we can easily determine the probability density for $p(v_{max})$. Results obtained for a normal distribution are given in Figure 13.

According to Bartlett /2/ the loadings considered above represent degenerate random processes whose occurrence is expressed in terms of deterministic functions which depend on a finite (practically, quite small) number of random parameters. Many loads acting on structures (for

example, loads due to atmospheric turbulence, wave pressure, etc.) cannot be satisfactorily described in this manner. In this case, methods of the theory of random processes have to be applied. Depending upon whether these methods use certain averaged characteristics of the process, or directly the joint probability densities for some quantities, we can distinguish two groups, the first of which will be discussed now.

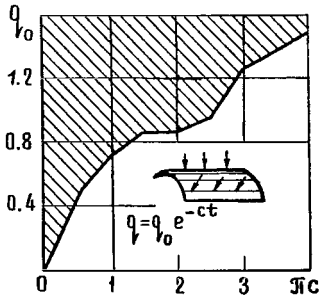

FIGURE 10.

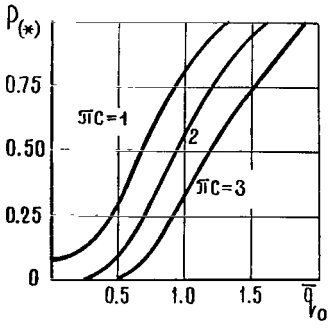

FIGURE 11.

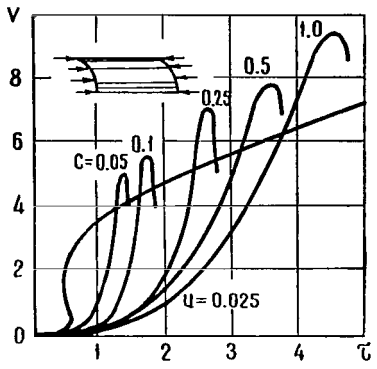

FIGURE 12.

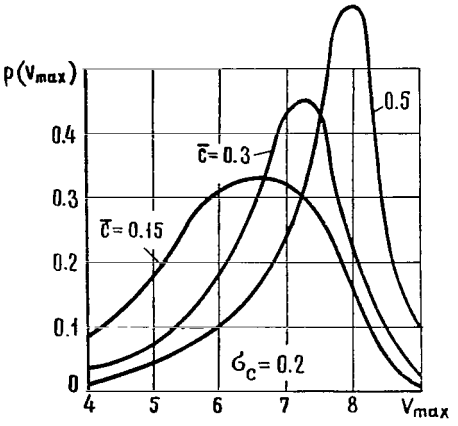

FIGURE 13.

2. CORRELATION METHODS

Correlation methods are based on the relationships between certain characteristics of input and output processes, i.e., between correlation

functions or between functions related to the latter. Let $q_1(t), q_2(t), \cdots, q_n(t)$ be certain random functions of the time t (for example, generalized forces for a plate or shell). We shall use a bar to denote averaging over the ensemble of functions. We shall also assume that the average of the functions $\overline{q_\alpha(t)} = 0$ (this requirement is easily satisfied by appropriate transformations). We now consider the following process

$$K_{q_\alpha q_\beta}(t_1, t_2) = \overline{q_\alpha(t_1)\, q_\beta(t_2)},$$

$$K_{q_\alpha q_\beta q_\gamma}(t_1, t_2, t_3) = \overline{q_\alpha(t_1)\, q_\beta(t_2)\, q_\gamma(t_3)} \tag{2.1}$$

$$(\alpha, \beta, \gamma, \cdots = 1, 2, \cdots n)$$

etc., where $K_{q_\alpha q_\beta}(t_1, t_2)$, $K_{q_\alpha q_\beta q_\gamma}(t_1, t_2, t_3)$, etc., are the second-, third-, etc., order correlation functions. Complete description of the process requires the knowledge of the full system of correlation functions. An appreciable part of the information on the random process is contained in the average $\overline{q_\alpha(t)}$ and in the second-order correlation functions $K_{q_\alpha q_\beta}(t_1, t_2)$, which are henceforth simply called correlation functions. In a number of problems (for example, in those problems where it is known a priori that the generalized coordinates have a normal distribution) it suffices to know the correlation functions in order to obtain the joint probability densities and solve the reliability problems.

Correlation functions for generalized forces can be found if the corresponding functions for the local load density are known. Let, for example, the normal load $q(x_1, x_2, t)$ be a random function of position and time. Expanding the normal deflection in a series of the eigenfunctions $\varphi_\alpha(x_1, x_2)$

$$w(x_1, x_2, t) = \sum_\alpha v_\alpha(t)\, \varphi_\alpha(x_1, x_2). \tag{2.2}$$

we obtain

$$K_{q_\alpha q_\beta}(t_1, t_2) = \iiiint \overline{q(x_1, x_2, t)\, q(\xi_1, \xi_2, t)}\, \varphi_\alpha(x_1, x_2)\, \varphi_\beta(\xi_1, \xi_2)\, ds\, d\sigma. \tag{2.3}$$

Here ds and $d\sigma$ are elements of the area of the middle plane, the integration being performed over its entire area.

The description of a random process becomes simplified when it is a steady-state process, that is, if all its statistical characteristics remain unchanged with time. The concept of steady-state random processes is a very convenient abstraction for describing real processes. The statistical characteristics of atmospheric turbulence, of the noise of engines in operation, of sea waves, etc., can be regarded as constant within a sufficiently wide range of observations. The average values for steady-state random process are constant, and the correlation functions (2.1) depend only on the differences $t_1 - t_2$, $t_1 - t_3$, etc. In particular, the second-order correlation functions are

$$K_{q_\alpha q_\beta}(\tau) = \overline{q_\alpha(t)\, q_\beta(t+\tau)}. \tag{2.4}$$

Almost all the steady-state cases of random loads which are of interest also have the property of ergodicity. This means that a sufficiently

long loading process yields practically all the information about its steady-state properties. For ergodic random processes averaging over the ensemble can be replaced by averaging with respect to time. In particular,

$$K_{q_\alpha q_\beta}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} q_\alpha(t)\, q_\beta(t+\tau)\, dt. \tag{2.5}$$

We also introduce the Fourier representation for the correlation function

$$K_{q_\alpha q_\beta}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \Phi_{q_\alpha q_\beta}(\omega)\, e^{i\omega\tau}\, d\tau. \tag{2.6}$$

Functions $\Phi_{q_\alpha q_\beta}(\omega)$ will be called joint spectral densities. If the process contains discrete components, then the spectral densities will have properties similar to delta-functions.

Correlation functions and the corresponding spectral densities are convenient not only because they permit us to describe the most important properties of the random process in the simplest manner. They are also relatively easily measurable characteristics of the process. Special instruments (correlators, spectral analyzers, etc.) have been developed which make it possible to obtain these characteristics directly during measuring or from recordings.

Let us now consider several of the more important kinds of random loads. Pressure pulsations due to atmospheric turbulence have been studied both experimentally and theoretically /16, 48/. These pulsations can, to a good approximation, be regarded as a random process, constant with time and homogeneous in space.

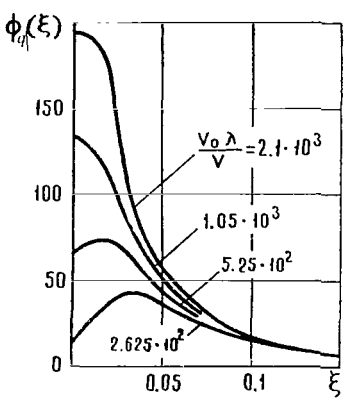

FIGURE 14.

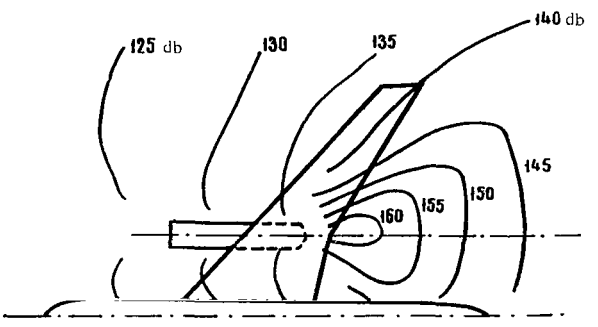

FIGURE 15.

Figure 14 shows the spectral pressure density $\Phi_q(\xi)$ for different values of the Reynolds number $v_0\lambda/\nu$ (v_0 is the average flow velocity, λ is the turbulence

scale, ν is absolute viscosity). The independent variable is the reduced frequency $\xi = \omega\lambda/v_0$. The most frequently used expression for the spectral density has the form /58/

$$\Phi_q(\xi) = \overline{q^2}\frac{\lambda}{\pi v_0}\frac{1+3\xi^2}{(1+\xi^2)^2}.$$

Pulsations in a turbulent boundary layer in the vicinity of an aircraft skin have also been studied by a number of authors. Tack and Lambert /133/ suggest the following expression for the correlation function for pressure

$$K_q(\xi, \eta, \tau) = \overline{q^2}\frac{\sin\dfrac{\xi - v_0\tau}{\lambda_x}}{\dfrac{\xi - v_0\tau}{\lambda_x}}\exp\left(-\frac{\eta}{\lambda_y} - \frac{\tau}{\tau_0}\right).$$

Here λ_x and λ_y are turbulence scales along and across flow, τ_0 is relaxation time, ξ and η are distances between two points along and across flow, respectively.

Correlation functions and spectral densities for sea waves were studied by V. V. Ekimov /32/ and others; of the foreign authors we mention Bartsch /70/.

During the past years an intensive investigation was begun of pressure pulsations caused by jet engine noise /61, 77, 91, 123, 125/. These pressure fields have a pronounced nonhomogeneity. This can be seen, for example, from Figure 15 which shows lines of equal noise in decibels. The spectrum of pressure pulsations is very wide, extending practically over the entire sound range (Figure 16).

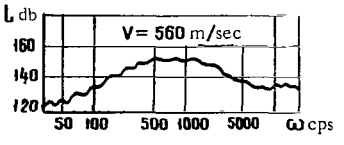

FIGURE 16.

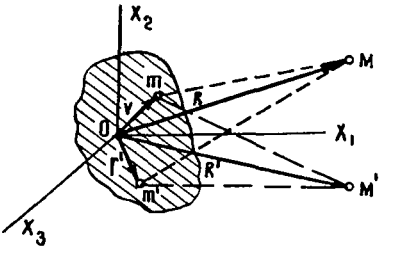

FIGURE 17.

Obtaining the correlation functions of the sound pressure is difficult both theoretically and experimentally. The most convenient characteristic is the "space correlation spectrum," i.e., the Fourier transformation with respect to time of correlation functions of pressure

$$K_q(R, R'; \omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\overline{q(R, t)\, q(R', t+\tau)}\, e^{-i\omega\tau}\, d\tau \qquad (2.7)$$

(R and R' are the radius vectors of points M, M' in Figure 17). The theory due to Lighthill /101/ makes it possible to determine the space

correlation spectrum theoretically. The perturbation of the density ρ of the acoustic medium is given by the nonhomogeneous wave equation

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \Delta \rho = \varkappa, \tag{2.8}$$

where c is the velocity of sound in the medium and the "density of sources" $\varkappa$ is expressed in terms of the Reynolds stress components $\tau_{\alpha\beta}$ as

$$\varkappa = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \frac{\partial^2 \tau_{\alpha\beta}}{\partial x_\alpha \partial x_\beta}.$$

Having solved (2.8) in terms of the retarded potential and using (2.7), we obtain

$$K_q(R, R'; \omega) = \frac{1}{16\pi^2} \iint \frac{K_\alpha(r, r'; \omega)\, e^{\frac{i\omega}{c}(|R-r| - |R'-r'|)}}{|R-r|\,|R'-r'|}\, dr\, dr'. \tag{2.9}$$

Here r and r' are the radius vectors of points M, M' of the emitting region; it is assumed that the control surface does not intersect this region (Figure 17).

Equation (2.9) becomes appreciably simplified only in the case of a so-called "distant" acoustic field, in which the dimensions of the emitting region may be neglected as compared to their distances from the points at which the pressure is measured: $|R-r| \approx |R|, |R'-r'| \approx |R'|$. In particular, for radial correlation we find

$$K_q(R, R'; \omega) \simeq \frac{\Psi(\omega)\, e^{\frac{i\omega}{c}(R-R')}}{RR'}, \tag{2.10}$$

where $\Psi(\omega)$ is a function characterizing the spectral characteristics of the source. In particular, it follows from (2.10) that sound pressure will be a temporary white noise even when the source power is delta-correlated in time.

As can be seen from data presented above, spectral and correlation characteristics are very convenient for the description of real loads acting on aircraft, vessels, and other structures. The main purpose of correlation methods consists in finding the characteristics of output processes when the correlation functions (or other characteristics related to them) of the input processes are known.

If stochastic differential equations relating the input and output parameters are given, differential equations interrelating the correlation functions are easily obtained. This is achieved by term-by-term multiplication of the stochastic equations by the unknown functions and their products and by averaging over the ensembles. In the general case an infinite sequence of differential equations containing higher-order correlation functions is obtained, as are also mixed correlation functions of input and output quantities. By introducing appropriate statistical hypotheses, investigation of the infinite system need only be carried out to second- or third-order correlations. This method was used by Maxwell

for solving Boltzmann's equation in gas kinetics; in physical kinetics this method is called the method of moments. This method is also used in the theory of turbulence for obtaining differential equations relating the correlations of the stress tensor from the Navier-Stokes equation.

The method of moments is particularly simple when applied to linear systems. However, here the method of transient pulse functions and the spectral method, which are extensively discussed in textbooks and in engineering literature /40, 46, 49, 53/ are more effective. Let $q_\alpha(t)$ be the inputs and $v_\alpha(t)$ the outputs of a linear system ($\alpha = 1, 2, \cdots m$) and $H_\alpha(t)$ be the reaction per unit pulse applied at time $t=0$. The solution of the stochastic equations which satisfy zero initial conditions at $t = t_0$, is written in the form

$$v_\alpha(t) = \int_{t_0}^{t} H_\alpha(t-\tau)\, q_\alpha(\tau)\, d\tau \qquad (\alpha = 1, 2, \cdots, m).$$

From this we obtain

$$K_{v_\alpha v_\beta}(t_1, t_2) = \int_{t_0}^{t}\int_{t_0}^{t} H_\alpha(t_1-\tau_1)\, H_\beta^*(t_2-\tau_2)\, K_{q_\alpha q_\beta}(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 \tag{2.11}$$

and similar expressions for correlation functions of higher order (the asterisk denotes a complex conjugate quantity). For steady-state random processes there exists a corresponding expression relating the spectral densities of input and output:

$$\Phi_{v_\alpha v_\beta}(\omega) = \frac{\Phi_{q_\alpha q_\beta}(\omega)}{F_\alpha(i\omega)\, F_\beta^*(i\omega)}. \tag{2.12}$$

where $F_\alpha(i\omega)$ ($\alpha = 1, 2, \cdots$) are transfer functions of the linear system. Equation (2.12) and the functions in it have an explicit physical meaning and are very convenient for calculations; on them are based many engineering applications in the theory of automatic controls, in the theory of communications, etc.

Let us assume that the motion of a mechanical system is described by

$$\rho_\alpha \ddot{v}_\alpha + 2\varepsilon_\alpha \dot{v}_\alpha + \rho_\alpha \omega_\alpha^2 v_\alpha = q_\alpha(t) \qquad (\alpha = 1, 2, \cdots, m), \tag{2.13}$$

where ε_α are the damping coefficients, ρ_α are the inertia coefficients, and ω_α are the partial frequencies of natural oscillations; the dots denote differentiation with respect to time. Equation (2.12) then becomes

$$\Phi_{v_\alpha v_\beta}(\omega) = \frac{\Phi_{q_\alpha q_\beta}(\omega)}{\rho_\alpha \rho_\beta \left(\omega_\alpha^2 + \frac{2i\varepsilon_\alpha \omega}{\rho_\alpha} - \omega^2\right)\left(\omega_\beta^2 - \frac{2i\varepsilon_\beta \omega}{\rho_\beta} - \omega^2\right)}. \tag{2.14}$$

If the system is nonlinear, then a solution can, in principle, be obtained by integrating the moments equations. Difficulties usually arise when doing this (e.g., in the theory of homogeneous turbulence /16/), but these can be overcome by appropriately linearizing the equations. Many linearization

theories have been developed /49, 53, 55/ which are largely similar to the methods of nonlinear mechanics. Most extensively used is the method of statistical linearization (I. E. Kazakov, Booton) which is similar to the method of harmonic linearization or harmonic balance in nonlinear mechanics. We may also mention the method of direct linearization (V. S. Pugachev), the method of slowly changing amplitude and phase (Rice, V. I. Bunimovich), etc.

Determination of correlation and spectral characteristics of output quantities is not an end in itself. The aim is to find the joint probability distribution for the output quantities (frequently together with the first and second derivatives) and to estimate the reliability of the structure on this basis. The complete system of correlation functions contains exhaustive information on the process. In practice, however, only first- and second-order functions are considered. If the output process has a normal distribution (e. g., in a linear system with a normal-distribution input), then knowledge of second-order average and correlation functions is sufficient to obtain the distribution functions, and may also suffice in certain other cases when the method by which the random output quantities are formed is known from other considerations.

After the distribution of input quantities is found we have to apply the reliability theory. The limiting state of the structure ("failure" in the reliability theory) can take place as a result of brittle breakdown or of reaching the elastic limit, as a result of large elastic or plastic deformations (for example, the "popping" of shells) as a result of accumulation of residual damage and development of fatigue cracks, mechanical wear, etc.

A number of reliability problems in structures loaded by random forces is solved by methods of the theory of overshoots of random functions. This theory, originated by Rice /51/, was developed primarily for problems in radioengineering and automatic controls. An important contribution to the development of the theory of overshoots was made by the disciples of S. P. Strelkov (V. I. Tikhonov, R. L. Stratonovich, and others). A comprehensive survey of this field was recently published by V. I. Tikhonov /56/. We have at present at our disposal effective methods of finding the average number of intersections of a given level per unit time, the distribution of extrema and zeros of a random function, the distribution of absolute maxima over a specified time interval, etc. If, for example, the limiting state of a structure is defined as that in which a specified stress (strain) is exceeded, the problem reduces to finding the probability of overshoot of a random function during the specified time interval, equal to the period of reliable service. "Popping" of a shell can also be interpreted as a random overshoot beyond the boundaries of a certain region of the phase or configurational space. Solving the problem of accumulation of residual damage requires knowledge of the distribution of the extrema of the random function, or, more precisely, the distribution of the random function over "cycles." While for processes in which the stresses vary over a narrow range only the distribution of the maxima must be known. It was shown in /15/ that the solution of wide-range processes requires a knowledge of the joint probability densities for several successive extrema. The accumulation of plastic deformations and creep strain must be considered as an irreversible (more precisely, partially

irreversible) process, caused by another (input) process. In our opinion this problem is very important, but it would seem that its solution should be sought by the methods of kinetic equations. These methods should also be used for the more complete (nonphenomenological) theory of accumulation of residual damage and mechanical wear.

We shall now give a systematic review of work done on the application of correlation and related methods to the theory of plates and shells. For solving problems of the strength of structures these methods were apparently first applied by Lippman /39/, Fang /58/, and Miles /112/ at the beginning of the fifties. These authors considered these problems primarily in conjunction with designing aircraft structures to withstand gusts and [tail] buffeting. Miles first considered random vibrations of structures in conjunction with the problem of damage accumulation according to the Mayner-Bakharev concept and gave a partial solution of this problem. These papers do not explicitly deal with plates and shells. This also applies to later works by Press and Houbolt /120/, Ribner /122/, Thomson and Barton /57/. However, it was alreay pointed out that there is no substantial difference between the applications of statistical methods to beams, beam systems, plates, shells, etc. The difference disappears almost immediately after the vibrational problem is expressed in terms of eigenfunctions and ordinary differential equations are obtained which have the same form for any elastic system.

Vibrations of round and rectangular plates supported along the entire contour, subjected to random loads, were first considered theoretically by Eringen /86/. Figure 18 shows the correlation function of the deflection at the center of a round plate due to a concentrated random force applied at the center.

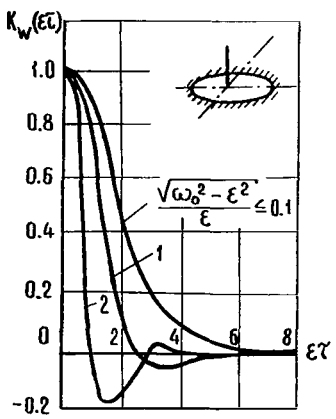

FIGURE 18.

The general theory of vibrations of elastic systems loaded by random forces was presented by this author on the basis of correlation methods /9/. In particular, the role of the correlation between the various degrees of freedom of an elastic system was investigated. It was shown that mutual correlation can be neglected if the damping coefficients for the natural vibrations of the elastic system are sufficiently small in comparison with the differences between the partial frequencies and if the spectral density of the input does not have excessively sharp maxima.

It is known that the modes of natural vibrations can be determined precisely only in a few cases. Usual approximative methods give reliable results only for the lower modes of vibrations. For this reason it seems reasonable to use asymptotic representations for the eigenfunctions and natural frequencies, as in a method developed by this author /8, 10/. In paper /11/ this idea was applied to solve the problem of vibrations of a clamped plate. Complete spatial correlation was assumed, while the time correlation was given in the form of the spectral-density graph. The mean-square value

of normal stresses near the clamped edge was calculated. This was developed further by M. F. Dimentberg /28, 29/, and will be discussed later.

If the spectrum of excitation frequencies is sufficiently wide, a large number of degrees of freedom are excited immediately and the use of correlation methods becomes excessively difficult. However, a substantial simplification is possible due to the plurality of the excited modes of vibration. Noting that (Figure 19)

$$k_1 a_1 = m_1 \pi + O(1), \quad k_2 a_2 = m_2 \pi + O(1), \quad (m_1, m_2 = 1, 2, \cdots),$$

are true for the wave numbers k_1 and k_2 of a plate or of a shallow shell whose sides are a_1 and a_2, we can find an approximate formula for the average number of natural frequencies per unit frequency range /13/

$$\frac{dN}{d\omega} \approx \frac{a_1 a_2}{2\pi^2} \left(\frac{\rho h}{D}\right)^{1/2} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{1 - \frac{\omega_0^2}{\omega^2} (\varkappa \cos^2\theta + \sin^2\theta)^2}}, \tag{2.15}$$

where

$$\chi = \frac{R_1}{R_2}, \quad \omega_0 = \frac{1}{R_1} \left(\frac{E}{\rho}\right)^{1/2}.$$

and R_1 and R_2 are the principal radii of curvature of the middle surface. When $\omega_0 = 0$, (2.15) becomes Courant's equation for the density of the natural frequencies of a plate. The integral in (2.15) is expressed by elliptic integrals. It was found that the distance between neighboring frequencies amounts to several cycles per second for thin plates and shells and to several tens of cycles per second for comparatively thick plates and shells. The width of the the spectrum of excitation frequencies can, as we have seen, amount to hundreds and even thousand of cycles per second.

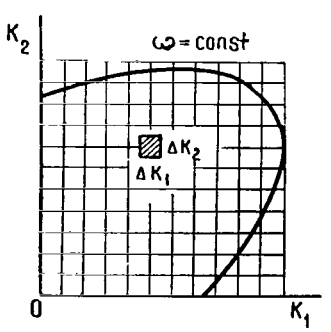

FIGURE 19.

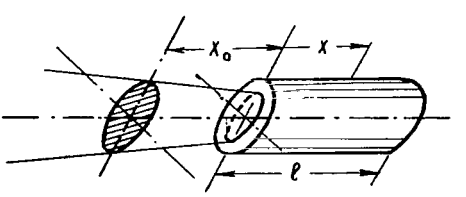

FIGURE 20.

This author has shown /14/ that under certain sufficiently general conditions, it is possible to give integral estimates for the correlation functions and the mean-squares and spectral densities. More precisely, summation over the lattice points of a "quantum" network is replaced by

integration over the entire region of wave numbers $k_1 \geqslant 0, k_2 \geqslant 0$ and division by the area of one "quantum" cell $\Delta k_1 \Delta k_2$ (Figure 19). As an example we now give the expression for the mean-square stresses which arise in a sufficiently large plate due to a steady-state random pressure which is delta-correlated in space. For points sufficiently distant from the contour we have

$$\overline{\sigma^2} \approx \frac{9\pi}{2}(3 + 2\mu + 3\mu^2)\frac{a_1 a_2}{h^2}\left(\frac{D}{\rho h}\right)^{1/2}\int_0^\infty \frac{\Psi(r)\,dr}{\psi(r)\,r},$$

while the maximum stresses at the clamped edge will be

$$\overline{\sigma^2} \approx \frac{288\pi(\sqrt{2}-1)}{\sqrt{2}}\frac{a_1 a_2}{h^2}\left(\frac{D}{\rho h}\right)^{1/2}\int_0^\infty \frac{\Psi(r)\,dr}{\psi(r)\,r},$$

where $\Psi(r)$ is a function characterizing the spectral density of the load, $\psi(r)$ is the relative energy dissipation, and $r^2 = k_1^2 + k_2^2$.

The method of integral estimates makes it possible to obtain in closed form the asymptotic solution for a number of problems and to investigate the effect of such properties of external load as damping, boundary conditions, etc., on the statistical characteristics of the input quantities.

In recent years considerable interest has arisen in problems concerning the vibration of elements of aircraft skins due to random pressures caused by engine noise, pulsations in the boundary layer, etc. These problems have two aspects; one is purely acoustical, being related to problems of sound-proofing, diffraction of the noise on the vibrating surface, etc. The second aspect is related to the strength and reliability of the aircraft structures. It is expressed by the term "acoustic fatigue" (fatigue failure of structures due to random forces whose characteristic frequencies lie in the sonic range).

Work on this subject was first published five or six years ago. We mention the paper by Lyon /105/ on the response of a string to random noise fields. The state of the science in 1958 is reflected in /78/, published under Crandall's editorship, and in /61, 79/. Later, Dyer /84/ considered the vibrations of a plate, freely supported along its contour, due to a distant noise field.

M. F. Dimentberg /28/ applied the asymptotic expansion of eigenfunctions to solve the problem of stresses arising in a clamped plate in a distant noise field. He first investigated the problem of how the "congruence conditions" vary for clamped plates, taking into account the correlation at multiple frequencies. Thereafter /29/ he investigated under the same assumptions, the vibrations of cylindrical panels clamped along their straight edges and showed that the mean-square stresses are appreciably lower than in a flat panel. This is in agreement with results obtained experimentally. Yu. A. Fedorov /59/ has considered the vibrations of a closed cylindrical shell in a pressure field generated by axisymmetrically arranged sources (Figure 20). The longitudinal correlation was introduced by (2.10); the angular correlation was determined by methods of geometric acoustics. These results pertain to a certain intermediate case between "distant" and "close" noise fields, since the space nonhomogeneity of the pressure was partly taken into account. The correction which had to be

made in comparison with the theory of a "distant" field can be seen in Figure 21.

Recently, Lin /102/ investigated the vibrations of regularly shaped continuous plates, reinforced by ribs, in a distant noise field. Tack and Lambert /133/ considered vibrations caused by pulsations in the turbulent boundary layer. Acoustic aspects of interaction between the plate and a random pressure field were investigated by L. M. Lyamshev /41, 42/.

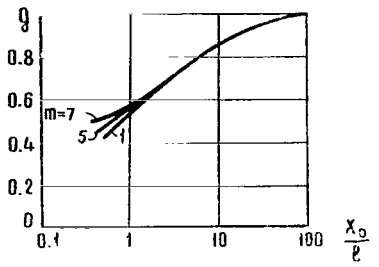

FIGURE 21.

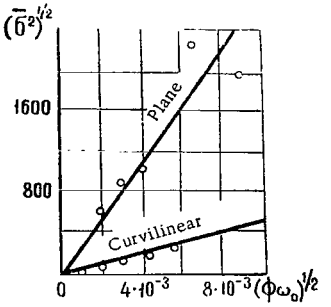

FIGURE 22.

Experimental investigations of vibrations and fatigue failure in sound-pressure fields were apparently first carried out by Lassiter, Hess, and Hubbard /99/. The results of these experiments are presented in Figures 22 and 23. Figure 23 shows the dependence of the time until failure on the noise level. As the noise level increases from 140 to 166 decibels (that is, as the average sound pressure increases approximately by a factor of 20) the service life decreases from 500 minutes to 30-40 seconds. Recently, Clarkson and Ford /77/ published the results of measurements of spectral densities and correlation functions of stresses arising in the skin of the "Caravelle" aircraft. Also, Smith and Lambert /132/ investigated experimentally the vibrations of a plate, caused by pulsations in the boundary layer. Certain other results are given in /61/.

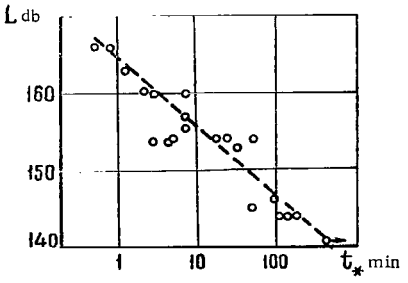

FIGURE 23.

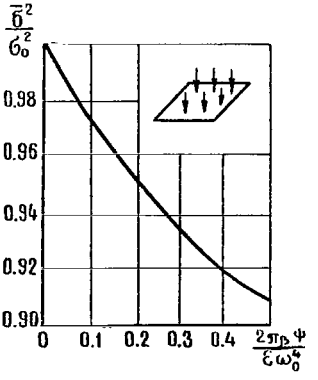

FIGURE 24.

The effect of nonlinear factors on random vibrations of plates was noted in a number of papers. Thus, according to /99/, typically nonlinear effects, i.e., "tightening" and "breakoff" of vibrations, were noticed already at comparatively low stresses (about 350 kg/cm^2).

As was previously pointed out, an extensive literature is devoted to the analysis of nonlinear problems of statistical dynamics by correlation methods. Works by Lyon /106-110/ and Caughey /73, 76/ are closest to the theme of this survey. In particular, Lyon /107/ first estimated the effect of nonlinear factors on the behavior of a taut string in a distant noise field. Recently, Lin /103/ performed corresponding calculations for a rectangular plate. The effect of nonlinearity on the mean-square stresses is shown in Figure 24.

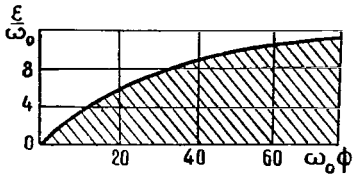

FIGURE 25.

Several works have been published on parametric vibrations of elastic systems excited by random forces. Eringen and Samuels /88/ considered the stability of a beam compressed by fluctuating forces, using the concept of "mean-square" stability. Slightly later Samuels /124/ investigated the stability of a closed circular cylindrical shell compressed by a fluctuating normal load. In Figure 25 the instability region in the parametric plane is shown hatched. Related problems of statistical dynamics were considered still earlier by R. L. Stratonovich and Yu. M. Romanovskii /54/.

3. METHODS OF KINETIC EQUATIONS

Let the motion of a mechanical system be described by the differential equations

$$x_\alpha = f_\alpha(x_1, x_2, \cdots, x_s, q_1, q_2, \cdots, q_r) \quad (\alpha = 1, 2, \cdots, s), \tag{3.1}$$

where $x_\alpha(t)$ are dynamic variables and $q_\alpha(t)$ are generalized forces. If the generalized forces and the initial conditions are determined, then (3.1) defines uniquely the behavior of the system.

Henceforth we assume that the generalized forces are random functions of time. We shall call (3.1), as customary, stochastic equations. The problem arises of finding the joint probability density for the dynamic variables $p(x_1, x_2, \cdots, x_s | t)$ when the initial distribution $p(x_1, x_2, \cdots, x_s | 0)$ is given. The evolution with time of this probability density is described by some differential or integral-differential equation

$$\dot{p}(x_1, x_2, \cdots, x_s | t) = L[p(x_1, x_2, \cdots, x_s | t)], \tag{3.2}$$

which we shall call kinetic.

Kinetic equations were introduced into science in the middle of the last century. Boltzmann's kinetic equation is the earliest, although not the

simplest example. Later, Rayleigh, Fokker, Planck, Smoluchowski, and Einstein applied kinetic equations to the theory of Brownian motion. Nowadays, these equations are used effectively in many fields, such as physics /65/, theoretical biology /2/, radioengineering /55/, theory of automatic controls /50/, etc.

The successful application of kinetic equations is primarily due to the development of the theory of Markov processes. A simple Markov process is a process without aftereffect, in which the probability distribution at time t_1 depends only on the distribution at instant $t_2 < t_1$ but not on the previous history of the system.

As an elementary example we now consider a simple Markov process with one variable $x(t)$. We use the notation $x = x(t)$ and $x_\tau = x(t+\tau)$. It can be shown that if the limits of the expressions

$$\varkappa_k(t) = \lim_{\tau\to 0} \frac{\overline{(x_\tau - x)^k}}{\tau} \tag{3.3}$$

exist, the probability density $p(x)$ is given by the kinetic equation

$$\dot{p}(x) = \sum_{k=1}^{\infty} \frac{1}{k!}\left(-\frac{\partial}{\partial x}\right)^k [\varkappa_k(x)\, p(x)]. \tag{3.4}$$

The functions $\varkappa_k(x)$ are called the intensities of the Markov process. Of them $\varkappa_1(x)$ characterizes the mean process flow, $\varkappa_2(x)$ characterizes the dispersion, etc. The intensity coefficients are related to correlation functions for the derivative $x(t)$ by

$$K_{\dot{x}}(t_1, t_2, \cdots, t_k) = \varkappa_k(t)\, \delta(t_1 - t_2) \cdots \delta(t_1 - t_k) + \cdots \tag{3.5}$$

(dots denote terms characterizing weaker effects).

If for $\varkappa_k(x)$ all the $k > 2$ are zero, the Markov process is called continuous and (3.4) becomes

$$\dot{p}(x) = -\frac{\partial}{\partial x}[\varkappa_1(x)\, p(x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\varkappa_2(x)\, p(x)] \tag{3.6}$$

and is called the one-dimensional Fokker-Planck equation. It describes, for example, the random wandering of a point along the x-axis. Equation (3.6) can be generalized to include multidimensional continuous Markov processes

$$\dot{p}(x_1, x_2, \cdots, x_s) = -\sum_{\alpha=1}^{s} \frac{\partial}{\partial x_\alpha}[\varkappa_\alpha(x_1, x_2, \cdots, x_s)\, p_1(x_1, x_2, \cdots, x_s)] +$$
$$+ \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}[\varkappa_{\alpha\beta}(x_1, x_2, \cdots, x_s)\, p(x_1, x_2, \cdots, x_s)], \tag{3.7}$$

where

$$\varkappa_\alpha = \lim_{\tau\to 0} \frac{\overline{x_{\alpha\tau} - x_\alpha}}{\tau}, \qquad \varkappa_{\alpha\beta} = \lim_{\tau\to 0} \frac{\overline{(x_{\alpha\tau} - x_\alpha)(x_{\beta\tau} - x_\beta)}}{\tau} \tag{3.8}$$

Equation (3.7) and its conjugate equation are usually called Fokker-Planck-Kolmogorov equations.

The solution of the problem reduces to the integration of an equation of the parabolic type for specified initial conditions (the probability distribution at $t=0$) and boundary conditions which require that the function $p(x_1, x_2, \cdots, x_s)$ comes attenuated at large values of x_α. It is also possible to formulate limiting problems, related to finding the probability of attaining the limits, the probability of overshoots, etc. Thus, finding the average time required to reach the boundary S of a closed region V from a state which at time $t=0$ is defined by $x_1^0, x_2^0, \cdots, x_s^0$ $(x_\alpha^0 \subset V)$, reduces to the integration of L. S. Pontryagin's equation

$$\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} \varkappa_{\alpha\beta}\frac{\partial^2 T}{\partial x_\alpha^0 \partial x_\beta^0} + \sum_{\alpha=1}^{s} \varkappa_\alpha \frac{\partial^2 T}{\partial x_\alpha^{02}} = -1 \tag{3.9}$$

with the boundary condition that T vanish on the boundary S of the region. We denote the solution of the limiting problem by $T(x_1^0, x_2^0, \cdots, x_s^0)$. If the initial state has the distribution $p(x_1^0, x_2^0, \cdots, x_s^0)$, in region V, then the average time $\tilde{T}$ required to reach the boundary will be

$$\tilde{T} = \int \cdots \int_V T(x_1^0, x_2^0, \cdots, x_s^0)\, p(x_1^0, x_2^0, \cdots, x_s^0)\, dx_1^0\, dx_2^0 \cdots dx_s^0. \tag{3.10}$$

What physical properties must a system possess in order that we may describe its behavior by equations such as (3.7); or, what are the stochastic equations (3.1) to which the kinetic equations (3.7) correspond? These questions are not at all unnecessary, since Markov processes are mathematical abstractions which cannot always replace real (smoothed-out) processes (Figure 26). If the variation of the generalized coordinates is a Markov process, the variation of the generalized velocities will be a delta-correlated process ("white" noise). This can, in particular, be seen from (3.5). Consequently, variations in generalized coordinates of a system possessing finite inertia cannot be represented as a Markov process.

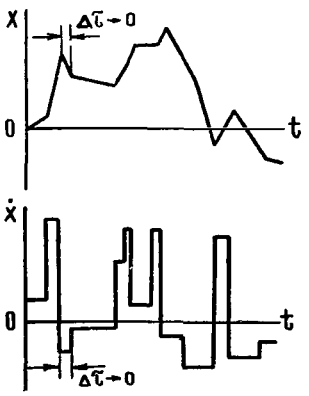

FIGURE 26.

If the system has a finite intertia and the generalized forces are "white" noises, a joint evolution of generalized coordinates and generalized velocities will correspond to the Markov process. In fact, in the equations

$$\rho_\alpha v_\alpha + 2\varepsilon_\alpha v_\alpha + f_\alpha(v_1, v_2, \cdots, v_n) = q_\alpha(t), \tag{3.11}$$
$$(\alpha = 1, 2, \cdots, n),$$

the right-hand sides of which satisfy the conditions

$$\overline{q_\alpha(t)} = 0, \quad \overline{q_\alpha(t)\, q_\beta(t+\tau)} = c_{\alpha\beta}\, \delta(\tau), \tag{3.12}$$

all higher correlation functions are zero. Introducing $s = 2n$ dynamic variables $x_\alpha = v_\alpha (\alpha \leqslant n)$, $x_\alpha = \dot{v}_\alpha (n < \alpha \leqslant 2n)$, we obtain from (3.8)

$$\varkappa_\alpha = \begin{cases} \dot{v}_\alpha & (\alpha \leqslant n), \\ -\dfrac{1}{\rho_\alpha}(2\varepsilon_\alpha \dot{v}_\alpha + f_\alpha) & (n < \alpha \leqslant 2n), \end{cases}$$

$$\varkappa_{\alpha\beta} = \begin{cases} \dfrac{c_{\alpha\beta}}{\rho_\alpha \rho_\beta} & \begin{pmatrix} n < \alpha \leqslant 2n \\ n < \beta \leqslant 2n \end{pmatrix}, \\ 0 \text{ in the remaining cases.} \end{cases}$$

Equation (3.7) takes on the form

$$\dot{p}(v_1, v_2, \cdots, v_n, \dot{v}_1, \dot{v}_2, \cdots, \dot{v}_n) = -\sum_{\alpha=1}^{n} \dot{v}_\alpha \frac{\partial p}{\partial v_\alpha} + \sum_{\alpha=1}^{n} \frac{2\varepsilon_\alpha}{\rho_\alpha}\left(p + \dot{v}_\alpha \frac{\partial p}{\partial \dot{v}}\right) +$$

$$+ \sum_{\alpha=1}^{n} \frac{1}{\rho_\alpha} \frac{\partial}{\partial \dot{v}_\alpha}(f_\alpha p) + \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \frac{c_{\alpha\beta}}{\rho_\alpha \rho_\beta} \frac{\partial^2 p}{\partial \dot{v}_\alpha \partial \dot{v}_\beta}. \tag{3.13}$$

In general it can be shown /55/ that to a dynamic system

$$\dot{x}_\alpha = g_\alpha(x_1, x_2, \cdots, x_s) + \sum_{\beta=1}^{s} h_{\alpha\beta}(x_1, x_2, \cdots, x_s)\, \xi_\beta(t), \tag{3.14}$$

where $\xi_\beta(t)$ are independent "white" noises of unit intensity

$$\overline{\xi_\alpha(t)\, \xi_\beta(t+\tau)} = \delta(\tau)\, \delta_{\alpha\beta},$$

there corresponds a stochastic equation (3.7) when

$$\varkappa_\alpha(x_1, x_2, \cdots, x_s) = g_\alpha + \frac{1}{2} \sum_{\alpha=1}^{s} \sum_{\beta=1}^{s} \frac{\partial h_{\alpha\gamma}}{\partial x_\beta} h_{\beta\gamma},$$

$$\varkappa_{\alpha\beta}(x_1, x_2, \cdots, x_s) = \sum_{\gamma=1}^{s} h_{\alpha\gamma} h_{\beta\gamma}. \tag{3.15}$$

"White" noise can serve as a satisfactory approximation in certain cases, particularly if damping is sufficiently small, since the spectral densities then change only insignificantly within the limits of the passband of the system and can be regarded as constant. However, "white" noise is still an excessive abstraction, since it corresponds to processes possessing infinite energy.

If the loads are steady-state random functions with fractional rational spectral densities, the theory of Markov processes can also be applied. In fact, a steady-state random process with a fractionally rational spectral density can always be treated as the result of the passing of "white" noise

through a linear system with constant parameters. Supplementing the system by equations which describe this passing and setting up the Fokker-Planck-Kolmogorov equation for the expanded system, we can find an effective solution for a number of problems. We can point to a number of examples from the theory of automatic controls /62, 63/.

Thus, the range of application of the theory of Markov processes is very wide. Problems only arise in obtaining the solutions. Steady-state solutions corresponding to a time-independent probability distribution can be found easily. The simplest of them are the Maxwell-Boltzmann and the Gibbs distributions. We recall here the restrictions which must be imposed on a dynamic system in order that its steady-state behavior be governed by the above distributions.

1. In (3.11) let $n = 1$ and $f = \frac{\partial V}{\partial v}$, where V is the potential energy of the system. Then, to the fluctuation equation

$$\rho \ddot{v} + 2\varepsilon \dot{v} + \frac{\partial V}{\partial v} = q(t)$$

there corresponds the kinetic equation (3.13)

$$p(v, \dot{v}) = -\dot{v}\frac{\partial p}{\partial v} + \frac{2\varepsilon}{\rho}\left(p + \dot{v}\frac{\partial p}{\partial \dot{v}}\right) + \frac{1}{\rho}\frac{\partial}{\partial \dot{v}}\left(\frac{\partial V}{\partial v}p\right) + \frac{c}{2\rho^2}\frac{\partial^2 p}{\partial \dot{v}^2}, \tag{3.16}$$

the steady-state solution of which is

$$p(v, \dot{v}) = C\exp\left[-\frac{A\varepsilon}{c}\left(\frac{\rho\dot{v}^2}{2} + V\right)\right], \tag{3.17}$$

i.e., a Maxwell-Boltzmann distribution.

2. Let $n > 1$, $f_\alpha = \frac{\partial V}{\partial v_\alpha}$, $\varepsilon_\alpha = \varepsilon =$ const and, in addition

$$c_{\alpha\beta} = c\delta_{\alpha\beta}, \tag{3.18}$$

where $c =$ const [and $\delta_{\alpha\beta}$ is the Kronecker delta]. Equation (3.18) states that $c_{\alpha\beta}$ is an isotropic tensor. Equation (3.13) then becomes

$$\dot{p} = -\sum_{\alpha=1}^{n} \dot{v}_\alpha \frac{\partial p}{\partial v_\alpha} + 2\varepsilon \sum_{\alpha=1}^{n} \frac{1}{\rho_\alpha}\left(p + \dot{v}_\alpha \frac{\partial p}{\partial \dot{v}_\alpha}\right) +$$

$$+ \sum_{\alpha=1}^{n} \frac{1}{\rho_\alpha}\frac{\partial}{\partial \dot{v}_\alpha}\left(\frac{\partial V}{\partial v_\alpha}p\right) + \frac{c}{2}\sum_{\alpha=1}^{n}\frac{1}{\rho_\alpha^2}\frac{\partial^2 p}{\partial \dot{v}_\alpha^2}. \tag{3.19}$$

This equation has the steady-state solution

$$p = C\exp\left[-\frac{4\varepsilon}{c}\left(\sum_{\alpha=1}^{n}\frac{\rho_\alpha \dot{v}_\alpha^2}{2} + V\right)\right], \tag{3.20}$$

which again is a Maxwell-Boltzmann distribution.

These two cases exhaust the problems which are described by the stochastic equations (3.11) and which lead to a Maxwell-Boltzmann

distribution. This was noted by several authors, in particular Ornstein /117/, who discussed the "Brownian Motion Systems where Several Temperatures Obtain" (see also the book by Chandrasekhar /65/).

We note that the distribution in generalized coordinates is a Gibbs distribution

$$p = C_1 \exp\left(-\frac{4\varepsilon}{c} V\right). \tag{3.21}$$

The kinetic equation, whose steady-state solution is the above distribution, can be also obtained from the following considerations. We assume that the system is inertialess. Physically this corresponds to the case when inertia forces can be neglected in comparison with frictional forces. An example is the motion of a Brownian particle in a very viscous fluid (at very low Reynolds numbers). In the limiting case under consideration we can treat the evolution of the system as a Markov process in a configuration space. The stochastic equations then become

$$2\varepsilon v_\alpha + \frac{\partial V}{\partial v_\alpha} = q_\alpha(t), \quad (\alpha = 1, 2, \cdots, n);$$

the corresponding Fokker-Planck-Kolmogorov equation takes the form

$$p = \sum_{\alpha=1}^{n} \frac{1}{2\varepsilon} \frac{\partial}{\partial v_\alpha}\left(\frac{\partial V}{\partial v_\alpha} p\right) + \frac{c}{8\varepsilon^2} \sum_{\alpha=1}^{n} \frac{\partial^2 p}{\partial v_\alpha^2}. \tag{3.22}$$

It is easy to show that distribution (3.21) satisfies (3.22).

Let us now clarify the meaning of the stated limitations for the problems of statistical dynamics of plates and shells. While the assumption of the existence of a potential V is not restrictive, the assumption that the damping coefficients are equal is contradicted by generally known experimental data. On the other hand, condition (3.18) imposes very rigid restrictions on the properties of the load. As an example, we consider a plate of constant thickness. Let $\varphi_\alpha(x_1, x_2)$ be the orthonormal modes of the natural vibrations. The generalized forces are determined from (2.3). Making use of the second of equations (3.12) we find

$$c_{\alpha\beta} = \int_{-\infty}^{\infty} \left[\iiiint q(x_1, x_2, t)\, q(\overline{(\xi_1, \xi_2, t+\tau)\varphi_\alpha}(x_1, x_2)\, \varphi_\beta(\xi_1, \xi_2)\, ds\, d\sigma\right] d\tau. \tag{3.23}$$

It is easy to show that $c_{\alpha\beta} = c\delta_{\alpha\beta}$, when

$$q(x_1, x_2, t)\, q(\overline{\xi_1, \xi_2, t+\tau}) = c\delta(x_1 - \xi_1)\, \delta(x_2 - \xi_2)\, \delta(\tau). \tag{3.24}$$

This equation states that the load must be delta-correlated not only in time, but also in space. Hence, the intensity of loading must be the same everywhere. This is typical of problems in the theory of Brownian motion. An example is the collisions of molecules with a thin plate placed in a liquid of uniform temperature. Obviously, purely molecular pressure fluctuations are of no interest in engineering problems. Fluctuating loads acting on real structures are gusts due to atmospheric turbulence, pulsations in the boundary layer, sound pressure due to engine noise,

pressure of sea waves, etc. These loads are characterized by a space-correlation scale which is comparable with the size of the structure or, at least, comparable with the characteristic length of the elastic disturbances of the middle surface. Thus, the correlation scale in a distant acoustic field is, as can be seen from (2.10), of the order of a sound-wave length. At a frequency of $\omega = 100$ cps, $\lambda \approx 3.3$ m; at $\omega = 1000$ cps, $\lambda \approx 0.33$ m.

Assume, for example

$$\overline{q(x_1, x_2, t)\, q(\xi_1, \xi_2, t+\tau)} = c_1 \delta(\tau)$$

(load fully correlated in space). Then, from (3.23)

$$c_{\alpha\beta} = c_1 \iint \varphi_\alpha(x_1, x_2)\, dx_1\, dx_2 \iint \varphi_\beta(x_1, x_2)\, dx_1\, dx_2.$$

For a rectangular plate supported in a horizontal plane, we have

$$c_{\alpha\beta} = \frac{\text{const}}{m_\alpha n_\alpha m_\beta n_\beta},$$

where m_α and n_α are positive odd integers, which correspond to the vibrational modes $\varphi_\alpha(x_1, x_2)$. If one of the numbers m_α, n_α, m_β, n_β is odd, then $c_{\alpha\beta} \equiv 0$.

A systematic review of studies of Markov processes must include references to the fundamental works by Smoluchowski and Einstein on the theory of Brownian motion (amongst the "classics" of Brownian motion, Langevin represented, so to speak, the "correlation" school). These works can be found in /66/. We mention also the later works by Uhlenbeck, Ornstein, Yu. A. Krutkov, Wang, Kramers, and others /36, 98, 117, 135, 136/. Several studies were also published after the war /96, 113, 131/. Equations such as (3.13) can, for particular cases, be found in /117/; see also The Handbook of Engineering Mechanics, edited by Flugge /90/.

The study by A. A. Andronov, A. A. Vitt, and L. S. Pontryagin /1/, published in 1933, is of fundamental importance. The theory of continuous Markov processes, in the form in which it had shortly before been developed by A. N. Kolmogorov, is applied in this work to the study of mechanical systems in a quite general form. The authors considered the behavior of a mechanical system having many degrees of freedom, subjected to the simultaneous action of deterministic forces and random Brownian-type impacts, and solved several problems on steady-state probability distributions and on the transition time from one state into another. This book was the first to give equation (3.9) for the mathematical expectation of the transition time, which was later called the Pontryagin equation. Study /1/ was also the first to deal with the relationship between stability and probability distribution; it was shown in it that stable varieties correspond to maxima of probability density. As examples the authors considered first- and second-order systems, including also freely vibrating systems.

The theory of Markov processes was first applied to the theory of shells by I. I. Vorovich /20/ in 1957. He suggested that the forces acting on a shell consist both of slowly varying and of fluctuating forces (Brownian-

type impacts), and determined the probability distribution from an equation such as (3.22), using subsequently its steady-state solution (3.21), i.e., the Gibbs distribution. The probability of "popping" is defined by I. I. Vorovich as the probability of a system outside some region of the configuration space in the steady state. The effect of random parameters of the shell is taken into account by the total-probability expression. In a recently published note /21/ I. I. Vorovich again uses the Gibbs distribution (3.21), attempting to obtain relationships of the quasisteady method by the transition $c \to 0$. These studies are described in I. I. Vorovich's review published in this collection.

A series of studies on the application of the theory of Markov processes to the statistical dynamics of shells was made by V. M. Goncharenko. In an earlier paper /22/, V. M. Goncharenko remained within the framework of this work, only replacing the Gibbs distribution (3.21) by a microcanonical distribution ($c \to 0$). In /24, 25/ V. M. Goncharenko already considers the evolution of a system as a Markov process in the phase space, without assuming the tensor $c_{\alpha\beta}$ to be isotropic (although he does not state the conditions imposed on the load in conjunction with the isotropy assumption). Finally, in /26/ he defines the problem of determining the probability of "popping" as a problem of determining the probability of the first crossing of the potential barrier.

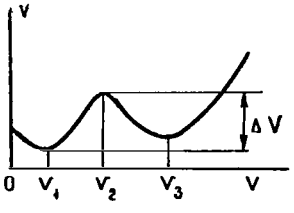

FIGURE 27.

We now consider paper /26/ in more detail. It describes a system with one degree of freedom whose probability distribution is given by (3.17). Using results due to Kramers /98/, the author gives the following approximation for the probability of "popping" at the time Δt

$$P(*/\Delta t) \approx \frac{\Delta t}{2\pi\rho} \sqrt{-\frac{\left(\frac{\partial^2 V}{\partial v^2}\right)_{v=v_1}}{\left(\frac{\partial^2 V}{\partial v^2}\right)_{v=v_2}}} \left[\sqrt{\varepsilon^2 - \rho\left(\frac{\partial^2 V}{\partial v^2}\right)_{v=v_1}} - \varepsilon\right] \times$$
$$\times \exp\left(-\frac{2\varepsilon}{c}\Delta V\right). \qquad (3.25)$$

Here v_1 is the generalized coordinate which corresponds to the undisturbed (stable) equilibrium position, v_2 corresponds to the unstable position; $\Delta V = V(v_2) - V(v_1)$ is the height of the potential barrier (Figure 27). It is assumed that at $t = 0$ the system is situated in the vicinity of point v_1 and that $\Delta t \ll \tau_p$ (τ_p is the relaxation time). Later, V. M. Goncharenko compared the results obtained from (3.25) with those obtained by assuming that the variation in the generalized coordinate is a Markov process. The probability ratio

$$k = \frac{P_0(*/\Delta t)}{P(*/\Delta t)}$$

(the numerator defines the probability according to the second assumption) for one particular case is presented in Table 2. Here δ denotes the damping constant of the natural vibrations.

TABLE 2

δ		0.002	0.004	0.006	0.008	0.02	0.04	0.06
k	$\bar{q} = 6.7$	2000	980	660	480	200	98	65
	$\bar{q} = 9.7$	1900	950	630	470	140	95	63
	$\bar{q} = 13.0$	1800	900	600	450	180	89	61

As can be seen from the table, the assumption that the variation in the generalized coordinate is a Markov process results in a quite appreciable error in determining the probabilities. It is true that the results pertain to the case of $\Delta t \ll \tau_p$. It is frequently thought that such large errors will not result if for $\Delta t \gg \tau_p$ the variation of the generalized coordinate is assumed to be a Markov process. This may be true with respect to the average time required to reach the potential barrier, the average number of series of such crossings, etc. However, for large values of Δt these characteristics are of no interest from the viewpoint of reliability. In fact, when Δt is large, the structure fails due to accumulation of residual deformations caused by repeated cyclical loads, and also due to fatigue. From the viewpoint of these phenomena the interesting characteristics are the average number of crossings of a given [potential] level, the distribution of extremum values, etc.

In order to find the average number of crossings of a given [potential] level per unit time the joint probability density for the generalized coordinates and velocities must be known. For a Markov process, the average number of crossings of a given level per unit time is infinite. If the level is crossed at any given instant, then an unlimited number of crossings are possible in an infinitesimal neighborhood of this instant. A Markov process in configuration space is thus an excessive idealization; we must therefore consider a Markov process in phase space. This, in its turn, becomes too simplified for finding the distribution of extrema. In fact, in order to find the distribution of extrema, we must know the joint probability density for the generalized coordinates, velocities, and accelerations. For loads such as "white" noise this probability density cannot be determined; this means that we must consider loads of finite energy.

Of interest from this viewpoint is the work by M. F. Dimentberg /30/. Assuming that the load is an exponentially correlated process, and applying the theory of Markov processes, the author found the steady-state distribution of coordinates, velocities, and accelerations, whence he was able to obtain the steady-state distribution of extrema. He also made a limit transition to "white" noise, assuming that damping is sufficiently small. It was found that the result depends largely on the manner in which this transition is made. This proves again that a more careful way of approximating loads by "white" noises is needed. A further development is given by M. F. Dimentberg in /31/, included in this collection.

We mention among foreign works dealing with this problem those of Lyon /106-107/, and also of Smith and his collaborators /127-129/. They are closely related to the problem of acoustic fatigue, the theory of Markov

processes being used to evaluate the effect of membrane stresses on the strength of vibrating plates.

We also mention the extensive literature on the application of the theory of Markov processes to the related fields of statistical dynamics. Some ideas on this rapidly developing trend are given in /46, 55, 65, 126/.

4. COMPARISON OF METHODS, TRENDS IN FUTURE INVESTIGATIONS

Having familiarized ourselves with the three groups of methods used in the statistical dynamics of plates and shells and with the results which these methods yield, we can finally compare them. Quasisteady methods are the principal and natural means for solving those problems in which the input parameters are random quantities and (or) deterministic functions of time which depend on a finite number of random quantities. The fewer are the random quantities which characterize the system and load, the simpler becomes the solution given by quasisteady methods. With increasing number of random parameters, it may become advisable to use the methods of the theory of random processes.

The field of application of correlation methods and the methods of kinetic equations are those problems in which the input parameters are nondegenerate random processes. Correlation methods are very effective in solving linear problems, particularly in the case of normal distribution processes. The methods of nonlinear mechanics can be used to solve problems with small nonlinearity. The methods of kinetical equations are particularly effective in problems concerning the theory of Brownian motion, since with their aid exact probability distributions can be obtained in a number of cases. Unfortunately, these distributions (Maxwell-Boltzmann, Gibbs, etc.) are only of limited usefulness in statistical dynamics of plates and shells.

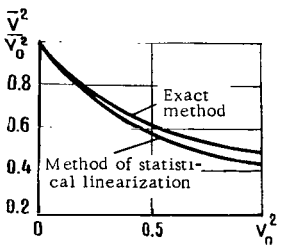

FIGURE 28.

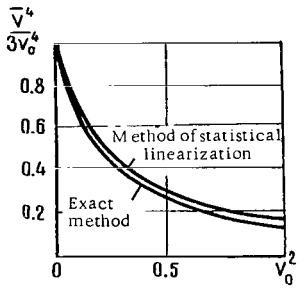

FIGURE 29.

Results obtained by correlation methods and by the methods of kinetical equations were compared repeately.

The comparison for the dynamic system

$$\ddot{v} + 2\varepsilon\dot{v} + \omega^2(1+g)v = \xi(t), \tag{4.1}$$

where $\xi(t)$ is the white noise, $g(v)$ is an odd continuous function of the genralized coordinate v, was made by Lyon /107/ and Crandall /80/. Lyon considered the function $g=\beta v^3$, while Crandall considered the general case. They applied, on one hand, the method of statistical linearization (assuming that the output is a Gaussian process); on the other hand, they set up the Fokker-Planck equation and obtained its solution using an equation such as (3.17). Figures 28 and 29, taken from /108/, give a comparison of the second and fourth moments of the generalized coordinate, found by these two methods. It is seen that the method of statistical linearization gives good agreement with the exact results. A similar result can be obtained for the data given by E. M. Hazen /63/ who considered the passing of the process $q(t)=m+\xi(t)$ with the spectral density

$$\Phi(\omega)=\frac{\Phi_0}{2\pi}\left|\frac{1}{1+i\omega\tau_0}\right|^2$$

through a system with a piecewise-linear characteristic. The results are compared in Table 3.

TABLE 3

Φ_0	m	$\bar{v}$		$\overline{(v-\bar{v})^2}$	
		Exact method	Method of statistical linearization	Exact method	Method of statistical linearization
0.5	0.5	0.2500	0 2500	0.0208	0.0208
	1.5	0.7485	0.7480	0.0195	0.0193
	2.0	0.9522	0.9520	0.00635	0.0066
1	0.5	0.2500	0.2500	0.08268	0.0833
	1.5	0.7284	0.7360	0.0591	0.0670
	2.0	0.9047	0.9070	0.02539	0.0267
5	0.5	0.1503	0.1540	0.6666	0.6770
	1.5	0.4316	0.4430	0.5443	0.5530
	2.0	0.5563	0.5670	0.4594	0.4600

It should be noted that the above comparisons pertained to nonlinear systems which have a single equilibrium position. For a system having two or more stable equilibrium positions (for example, a shell loaded by forces whose magnitudes are between the lower and upper critical values), the accuracy of the method of statistical linearization (at least in the version used above) will no longer be satisfactory.

It would still seem that in many fields of statistical dynamics of nonlinear systems, correlation methods and methods based on kinetic equations can be fully competitive. For example, the method of statistical linearization can be generalized so that it will yield satisfactory solutions even when the nonlinearity is large. On the other hand, wide prospects are opened up by applying the theory of Markov processes to systems with piecewise-linear characteristics /50, 63/. The range of problems which can be thus solved can be appreciably expanded if the approximate

solutions of the Fokker-Planck-Kolomogorov equations are obtained by direct variational methods or by the methods of nonlinear mechanics. Thus, it is possible that the coefficients of Fokker-Planck-Kolomogorov equations can be approximated by quadratic formulas. It is easy to see that this is related to the method of statistical linearization in the correlation theory. Difficulties due to the variety of excited vibrational modes can, in our opinion, also be overcome.

I would like to point out still another aspect of the problem. It is a widely held opinion that the theory of Markov processes is more complicated than, let us say, the correlation theory. I cannot agree with this. It was shown above that the formalism of the theory of Markov processes is generally not more complex than the correlation theory. The latter is only better known to engineers due to its many applications in statistical dynamics. Successful attempts have been made /55/ to present certain fields of statistical dynamics by using both methods equally.

We shall now discuss the formulation of problems arising in the future development of the statistical dynamics of shells and plates. It is frequently said that the most important problem is the collection of statistical data pertaining to the parameters of the structure and to the external loads. In fact, without these data all theoretical considerations are unrealistic. However, here we face a vicious circle. Without statistical data one cannot use the theoretical solutions; at the same time, without knowing the theory it is impossible to select the statistical information correctly and purposefully. For this reason the main problem is in our opinion the extensive popularization of statistical methods. It is most important that the methods, results, and future potential of the statistical dynamics be familiar not only to specialists in the field of the theory of shells and plates, but also to experimenting design engineers, etc. Only under this condition will it be possible to collect statistical information properly.

The following problem is the development of theoretical investigations in the statistical dynamics of plates and shells. This, however, may be dangerous in one respect. There exists a vast literature on the application of the theory of probability and mathematical statistics to various fields of technology and physics. Many solved problems from other parts of statistical dynamics can be easily rephrased in terms of the theory of plates and shells. In the first development stages of the theory, works based on simple analogies with previously solved problems are useful since they aid in the popularization of statistical methods. However, in the future a concentrated effort should be made to solve certain difficulties in the statistical dynamics of plates and shells. We indicate the following possible directions of future study:

1. Taking into account the variety of excited vibrational modes (particularly in nonlinear problems).

Most problems of statistical dynamics considered before pertained to systems with one or several degrees of freedom. Expanding in terms of eigenfunctions and cutting off the series at some term, we reduce the plate or the shell, which is a system with an infinite number of degrees of freedom, to a system with a finite number of degrees of freedom. This number can be sufficiently large, particularly if the load has a wide time spectrum.

2. Taking into account the laws of space-time correlation for loads.

As was shown in this survey, the relationships governing the distribution of real loads can be quite complicated. Analysis of the effect of these laws on the behavior of plates and shells does not, in general, present fundamental difficulties, especially for linear systems. Nevertheless, this problem has practically not yet been studied.

3. Elaboration of problems of statistical dynamics related to the reliability theory.

As was pointed out, the final result of a study should be an estimate of the reliability and service life of the structure. Here we enter a field which does not properly belong to statistical dynamics, although it is allied to it. However, a study in the field of statistical dynamics should yield results which can be further discussed from the point of view of reliability. Most problems cannot be solved by our present knowledge of theory of overshoots of random functions. Thus, up to now we do not know how to determine the joint probability densities of extrema following one another in wide-band random processes.

4. The application of probability methods directly to distributed systems.

In all the works mentioned, systems with distributed parameters were replaced by models with concentrated parameters. It would be interesting to apply directly the methods of the theory of probability and mathematical statistics to distributed systems. The problem is solved comparatively easily for linear systems if, for example, Green's function is known. Certain applications can be found in /89/. For dynamical problems an original approach was developed by Campet de Feriet /33/; it is, however, still unclear to what extent this approach will be effective.

In addition to purely theoretical goals we are also confronted by the very important problem of adapting the methods of the theory of probability and mathematical statistics to practical engineering calculations. Work is being performed in this field; its success will depend entirely on the successul combination of the efforts of theoreticians, designers, and experimental engineers.

The practical value of studies on the application of probability methods to the mechanics of a solid body undergoing deformation is beyond doubt. Its significance was pointed out in the report of the academic secretary of the Division of Technical Sciences of the Academy of Sciences of the USSR, Academician A. A. Blagonravov, at a Congress of the Division on 21 November 1961 /3/: "In various branches of technology, the study of the vibrations and strength of a structure subjected to random forces has assumed importance. I refer to vibrations of aircraft structures passing through atmospheric turbulences and, in general, through media with random nonhomogeneities, vibrations of structures due to sound pressure (the so-called problem of "acoustic fatigue"), the strength of ships under wave action, the strength of structures subjected to seismic disturbances, etc. The study of these problems requires investigation of the statistical characteristics of external influences and of the behavior of elastic and inelastic bodies subjected to random effects, and also the development of methods for estimating the service life and reliability of engineering structures on the basis of comparatively short and uncomplicated pilot tests." Future investigations should be devoted to solving these problems.

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CERTAIN PROBLEMS OF THE USE OF STATISTICAL METHODS IN THE THEORY OF STABILITY OF PLATES AND SHELLS

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I. During the past few years interest has been shown on the use of statistical methods in problems of stability of elastic systems. A considerable number of works has appeared and different points of view have been expressed. Certain results are expected from these methods. In this paper an attempt is made to survey the knowledge accumulated in this field and to consider certain prospects. We note beforehand that we shall not consider problems of the statistical theory of strength, the theory of accumulation of fatigue effects, etc., which are solved by studying the effect of random factors on an elastic system.

Since statistical methods are comparatively new to most workers in the field of the theory of elasticity, the author thought it advisable to include in this survey a general description of statistical methods used in many problems of radiophysics and the theory of controls, which mathematically are similar to problems of the stability of shells.

Statistical methods were first applied by M. Maier /1/ and N. F. Khotsialov /2/ to problems of structural mechanics in order to select safety factors. These methods were then developed extensively by N. S. Streletskii /3-7/ and A. R. Rzhanitsyn /8-13/, who used them as their working tools. We note a number of foreign works on the same subject /14-17/.

The general statement of the problem is as follows: Let quantities $x_1, x_2, \cdots, x_n$ symbolize the principal data of the problem, i. e., the loads, the geometry of the elastic system, and its mechanical properties. We shall assume that they are random quantities having a known distribution $\Psi(x_1, x_2, \cdots, x_n)$. Assume that the criterion for satisfactory service of the installation is

$$\varphi(x_1, \cdots, x_n) \geqslant 0. \tag{1}$$

The set of points (1) of the space $x_1, \cdots, x_n$ is denoted by Ω. Obviously, in this case the quantity J

$$J = \int_{\Omega} \Psi(x)\, dx\Big|, \tag{2}$$

may serve as a measure of the reliability of the installation.

The major difficulty here is finding Ψ by extensive experimentation, selecting the function φ and working out methods for evaluating the integral (2).

All the works mentioned are devoted to solving these problems in one form or another.

The reasons why statistical methods are nowadays applied to problems of shell stability are, in addition to general considerations pertaining to the structural mechanics of beam systems and to the strength of machine parts, etc., as follows:

A thin shell may have several equilibrium shapes under the same load. In such cases, even if we could completely overcome the mathematical difficulties involved in solving boundary-value problems of the nonlinear theory of shells, we could not without using probability methods decide which equilibrium shape would most likely be obtained experimentally. In addition, this is true also with respect to idealized schemes, where no deviations occur in the shape of the middle surface of the shell, and where the support and loading conditions are ideal. Here we have a substantial difference from the pattern characteristic for beams. It becomes nec essary to introduce the so-called upper and lower critical loads. The upper critical load is the upper limiting value of the load at which, according to Lyapunov, the undisturbed state is stable. It can be shown /18, 19/ that this upper bound is found by Euler's linearization method.

The lower critical load is the upper bound of the load at which the only equilibrium shape desirable in service is obtained.

For example, a cylindrical shell under axial compression has two stable equilibrium shapes in the range between the upper and lower critical loads. From the equations proper we cannot determine the probability of obtaining experimentally one shape or the other. Solution of this problem requires the use of statistical methods.

In addition, initial imperfections in the shape of the middle surface have a considerable influence on the magnitude of the critical load of a shell. This effect was investigated thoroughly (for more details see /20, 21, 22/). However, the shape and magnitude of initial imperfections are definitely random factors. V. I. Feodos'ev /23/ and A. S. Vol'mir /20/ suggested the use of statistical methods in strength calculations of shells. This question was also raised by T'sin Hu-Sen and Khu Khai-chan /24/.

We shall first list all the problems whose solution can be made easier by the use of statistical methods:

1. Specification of permissible shell loads, taking fully into account the service conditions of the shell and imperfections in construction.

2. Specification of tolerances for the main parameters of the shell. We refer here primarily to the precision required for the middle surface of the shell.

3. Developing an efficient system for analyzing the experimental results and for recommending experimental setups.

We shall now attempt to establish certain requirements which, in our opinion, must be remembered when developing the statistical theory of stability of elastic systems.

1. The statistical theory of stability of elastic systems should as far as possible take into account all random factors affecting the shell. Only under this condition will the mutual compensation of the harmful effects of random excitations on the stability of the elastic system be exposed, and the optimum utilization of statistical methods be achieved.

2. The statistical theory of stability of elastic systems should enable us to differentiate between the various stable equilibrium shapes possible

for a system at a given load. In other words, if an elastic system has several stable equilibrium shapes, each of these should have a different probability of being obtained experimentally. This requirement can only be satisfied by taking into account the effect of random factors on the displacements of the shell. Hence, in the statistical theory of shell stability we cannot restrict ourselves to the methods used in problems of structural mechanics, but must find ways to apply the theory of random processes.

3. The theory should result in computational algorithms which are feasible at present.

We should note the difficulties which arise in the development of the statistical theory of stability of elastic systems. These are primarily the very meager experimental data on the statistical properties of factors responsible for the random character of the deformations in a system. Despite the large number of experiments performed on shells, we cannot obtain from them the required statistical information. Thus, for example, in only a few cases was it possible to determine the shape of the middle surface of a shell with the required accuracy and obtain useful results by statistical analysis. There are no statistical data on the dispersion of the shell thickness, elastic properties, etc. The situation is even worse as regards statistical data on the external forces acting on the shell under actual service conditions.

The second difficulty is due to the substantial nonlinearity of the phenomenon. The effect of random excitations on a nonlinear system has not as yet been sufficiently investigated. There are many instances in which the existing methods do not provide adequate solutions.

The third difficulty consists in the fact that the statistical theory of stability of elastic systems deals with random processes which have not been sufficiently investigated mathematically. As a rule, these are not Markov processes and additional work is required before the available mathematical methods can be used.

Strictly speaking, we should consider here distributions in infinite-dimensional spaces, since elastic systems have an infinite number of degrees of freedom.

In addition to this there exists the subjective difficulty that until recently researchers in the theory of shells showed little interest in the theory of probability, while researchers in the theory of probability were not interested in problems of the theory of shells.

The main features of the theory should in our opinion include the following:

1) development of methods for describing statistically the random factors which determine the random character of the deformation of the shell, and for setting up experiments for their determination;

2) development of methods for describing statistically the random factors which describe the deformation of the shell and for setting up experiments for their determination;

3) relationships between the statistical characteristics of the random factors which describe the deformation of the shell, and the statistical characteristics of the factors which determine the random character of the deformation of the shell.

The solution of these problems will serve as a basis for the consideration of purely applied problems of stability of elastic systems.

II. We shall now analyze the data available on the above-mentioned features of the theory. We begin with the factors which determine the random character of the deformation of the shell. We divided them into the following groups which can, with sufficient accuracy, be considered independent (at least, in the initial stage of developing the theory):

1) scatter of the mechanical and geometric parameters of the shell;
2) scatter of the support conditions;
3) scatter of the external loads.

Very little statistical data are available on these random factors.

However, for the first group such data can be readily obtained without greatly complicating the setting-up of experiments on the stability of shells. The parameters of each shell should only be carefully measured and its mechanical characteristics determined before the experiment. Profile-measuring apparatus can be extensively used for geometric measurements. It is desirable to investigate this problem insofar as it relates the scatter of the geometric and mechanical characteristics of shells to the process by which they were made.

The probability properties of this group can be specified by a distribution law $\theta(a_1, a_2, \cdots, a_n)$ for a specified number of parameters which describe the scatter of elements of this group. A difficulty may arise in introducing the distribution since some of these parameters are functional. In this case they should be approximated over a substantial part of their distribution by a set with a finite number of numerical parameters. A parameter such as the initial deflection $w_0(x, y)$ of a shell is typical in this respect. Assuming that this parameter is a random function, we select an entire system of functions φ_k such that for the most substantial set of realizations of w_0 we obtain sufficiently exact relationships $w_0(x, y) = \sum_{k=1}^{m} a_k \varphi_k(x, y)$. Obviously, the statistical properties of w_0 can now be described by the distribution of a_k. The statistical properties of the shell thickness and other functional parameters can be described similarly. This method for describing the properties of this group of factors was used in /101/, and is used in all subsequent works.

It is also possible to describe the properties of this group of factors in a different manner. We can, for example, specify functions $w_0(P_k)$ at m points P_k; an approximate statistical description can then be given by the distribution of $\theta(w_0(P_k))$.

The selection of the method for describing this group of parameters depends, primarily, on the method used for solving the problem as a whole.

Determination of the statistical characteristics of external loads is more complicated. They depend essentially on the service conditions of the shell and should be specified in each case separately.

It is advisable to separate the service conditions into computational cases, each of which has its own scattering characteristics which should be obtained from the relevant experiments.

However, specific data are already available on the statistical properties of some types of loading. These include the effects of turbulence apparatus and of the sound pressure due to jet engines, which are of considerable importance in shells for aircraft structures.

Many studies have been devoted to atmospheric turbulence. Here we shall only consider those characteristics of turbulent flow which are

important for the problems disccused. General theoretical information on isotropic homogeneous turbulence can be found in works by A. M. Obukhov and A. M. Yaglom /25/, A. M. Obukhov /26/, and also in books by G. Batchelor /27/, Landau and Lifshits /28/, and in the recently published book by Bai-Shi-I.

According to this theory, the velocity vector is assumed to be a homogeneous steady-state process with respect to spatial variables, having characteristics independent of direction. We denote the vector of velocity pulsation by $\bar{w}$

$$\bar{w} = \bar{v} - \text{м. о. } \bar{v} \tag{3}$$

and introduce the correlation tensor

$$R_{ij}(\bar{r}) = \text{м. о. } (\bar{x}) \cdot w_i(\bar{x} + \bar{r}). \tag{4}$$

For an incompressible fluid it can be that (/26-28/)

$$R_{ij} = F(r) r_i r_j + G(r) \delta_{ij}, \tag{5}$$

where $F(r)$ and $G(r)$ are interrelated by

$$4F + rF' + \frac{1}{r} G' = 0. \tag{6}$$

The most important statistical properties of a homogeneous isotropic velocity field in an incompressible fluid are thus described by a single function whose determination is the greatest difficulty of the theory. Since we shall consider experimental data on the statistical properties of pulsations, the following relationship will replace (5):

$$R_{ij} = \frac{R_t(r) - R_n(r)}{r^2} r_i r_j + R_n \delta_{ij}, \tag{7}$$

where R_t and R_n are the longitudinal and transverse correlation functions of the velocity. Equation (7) is convenient because it contains correlation quantities which can be easily determined by experiment. Karman and Howarth /29/ proposed the following relationships:

$$R_t(r) = ae^{-r/L}; \quad R_n(r) = a\left(1 - \frac{r}{L}\right)e^{-r/L}, \tag{8}$$

where L, the so-called turbulence scale, and a, the dispersion of one of the velocity components, are determined experimentally.

Experimental values of L and a are given in a number of works, i. a., H. W. Liepman /30/, Y. C. Fung /31/, M. Kramer /32/, H. Press, M. T. Meadows, and J. Hadlock /33/, F. W. Diederich /34/, P. McCready /35/, G. Clementson /36/, and M. Strasberg /37/. Some of these works deal with the effect of a turbulent atmosphere on elastic objects, but they also present important data on the character of turbulence. Extensive experimental material is also contained in the National Report of the USSR on the "Meteorologiya" Association for 1957 /38/.

Monographs by M. I. Yudin /39/, T'sin Hu-sen /40/, M. Pelegrin /41/, R. L. Bisplinghoff, H. Ashley and R. L. Halsman /42/ and Ya. Ts. Ryn /43/ also contain some data on atmospheric turbulence as applied to the study of flutter and buffeting and also in connection with the effect of turbulence on certain control systems. We draw attention to the monograph by M. I. Yudin, who already in 1946 (apparently, first) used statistical methods to estimate the effect of atmospheric turbulence on aircraft. Some experimental data on atmospheric turbulence are given by N. Z. Pinus /44/ and Yu. V. Kurinova and S. P. Khachatryan /45/. Among later works, the study by Yu. P. Dobrolenskii /46/ is of considerable interest. In it, a and L are determined by new, improved methods. We mention also Yu. M. Romanovskii's dissertation /47/, the first chapter of which contains data on atmospheric turbulence.

The correlation functions (8) have the following spectral densities

$$S_l(\Omega) = \frac{2aL}{\pi}\frac{1}{1+\Omega^2L^2}; \quad S_n(\Omega) = \frac{aL}{\pi}\frac{1+3\Omega^2L^2}{(1+\Omega^2L^2)^2}. \tag{8'}$$

Relationships (8) and (8') can be used to investigate the effect of atmospheric turbulence on elastic systems.

Another kind of random effect on elastic systems — sound pressure — is at present being extensively studied both experimentally and theoretically. A survey of the literature on this problem was given by L. Goodman and J. Rattayya /48/. The noise of an aircraft originates from many sources. A description and analysis of the relative intensity of these sources was given by Hubbart /49/. It is nowadays accepted that the principal source of noise is the jet engine. A survey of studies of jet-engine noise was performed by Richards /50/. We shall only consider those works in which the noise effect is considered from the viewpoint of probability. It has been established by several experiments that the spectrum of jet-engine noise is continuous and that the spectral density is almost constant in that frequency range which is of importance in elastic systems. We refer to the works by Wolfe /51/, Kennard /52/, and Callaghan, Howes, and Coles /53/.

It seems now that this is an established fact. It is extremely important, since it considerably simplifies from the mathematical aspect the problem of the effect of engine noise on elastic systems. The uniform distribution of the spectral density enables us to consider engine noise as a so-called "white noise," so that the Kolmogorov-Fokker-Planck equation can be used and in many cases a complete solution obtained (for more details see below).

The "white noise" idealization of engine noise was used by Miles /54, 55/. Hubbard and Hess /56/ considered an aircraft-skin panel subjected to the noise of a jet engine also idealized as white noise. The same was done by Hess, Fralich, and Hubbard /57/ in the study of fatigue strength, and in many other works devoted to this problem. All these works contain experimental data on the intensity of jet-engine noise.

Data are thus available on the statistical properties of two of such important classes of loading as atmospheric turbulence and engine noise.

No experimental data are known to the author on scatter of the manner of shell supports; we therefore only note that this group of factors can be described statistically in the same way as the first group. (This also applies to the shell deformations.)

III. We shall now consider the possibilities of establishing relationships between the st. ch. (statistical characteristics) of the three above-mentioned groups and those of the shell deformations.

It follows from the preceding that the shell should be regarded as an elastic nonlinear system acted upon by random forces. Such problems were considered in physics and astronomy, various assumptions being made about the nature of the random forces, and have now become important in the theory of controls and in radiophysics. The mathematical formulation of these problems in the theory of shells has much in common with the formulation of similar problems in theoretical physics, the theory of controls, and radiophysics. We shall therefore discuss briefly the methods used in solving these problems in the fields mentioned.

M. Smoluchowski /58-60/ investigating the constrained motion of Brownian particles was the first to consider the effect of random forces on an elastic system. In deriving the fundamental equation defining the distribution he assumed that the particle inertia is small, so that his results are valid only when the system has been observed for some time.

Later, Fokker /61/ and Planck /62/ found a differential equation, independent of Smoluchowski's assumption, describing the distribution of Brownian particles acted upon by arbitrary forces

$$\frac{\partial f(t, x, \tau, y)}{\partial \tau} = -\frac{\partial}{\partial y}[a(\tau, y) f(t, x, \tau, y)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[b(\tau y) f(t, x, \tau, y)]. \quad (9)$$

Here $f(t, x, \tau, y)$ is the probability that the random coordinate will have the value y at time τ, if at time t it had the value x.

Independently of Fokker and Planck, A. N. Kolomogorov /63/ arrived at (9), providing a rigorous mathematical derivation which made clear the conditions under which this equation is applicable. In addition, he obtained a second important equation, which must be satisfied by f with respect to the arguments t and x. We shall henceforth call (9) the KFP equation.

It was shown by A. A. Andronov, L. S. Pontryagin, and A. A. Vitt /64/ that the KFP equation can be used also in the qualitative theory of differential equations, in which case many aspects of this theory take on entirely new meanings.

A. Ya. Khinchin's work /65/, published in 1934, initiated the correlation theory of random processes, and also developed the correlation and spectral theory of steady-state processes. A complete presentation of this theory was given by A. M. Yaglom /66/. All these works formed the basis for the large-scale application of the methods of the theory of random processes in different fields of mechanics, physics, astronomy, and technology. Many examples of their use in mechanics, physics, and astronomy were given by S. Chandrasekhar /67/.

A number of interesting problems were considered by Yu. A. Krutkov, who investigated random effects on many linear and nonlinear systems /68-73/. In particular, I should mention /74/, in which oscillations of a string and a rod subjected to Brownian forces are considered. An extensive bibliography on the subject of Brownian motion is given in /75/.

The theory of steady-state processes is most extensively used in automatic controls, both correlation and spectral theories being applied

to the same degree. This is illustrated in the works of V. S. Pugachev /76/, and E. P. Popov and I. P. Pal'tov /77/. Guidance on the use of these methods in the theory of controls is contained in, e.g., /41, 78, 79/. Solutions of nonlinear problems in the theory of controls are extensively based on methods of statistical linearization, developed by I. E. Kazakov /80-82/ and Booton /83, 84/ and described in detail in /76, 77/.

The method of statistical linearization was employed by A. A. Pervozvanskii /85, 86/ in the theory of controls in the study of free oscillations. Nonlinear problems can also be solved by the method of expansions in terms of small parameters /87, 88/. The KFP method was used to a much lesser extent in the theory of controls. Here only a few works are known to the author, for example, by Bol'shakov /89/, P. S. Landa /90/, P. S. Landa and S. P. Strelkov /91/, Crandall /92/, and Barrett /93/.

As opposed to this, the KFP equation is used extensively in radiophysics. More will be said on this later on. Of course, the choice of the method depends mainly on the problem considered and on the statistical characteristics of the external forces.

During the past few years several attempts were made to generalize the KFP equation to include cases when the random effects are not white noise. The first attempt was apparently made by V. S. Pugachev /94, 95/. J. L. Doob /96/ showed that the KFP equation can be established when the random forces are a linear integral-differential operator of white noise. P. I. Kuznetsov, R. L. Stratonovich, and V. I. Tikhonov /97/ and R. L. Stratonovich /98/ used another method to investigate systems with continuous random processes. We shall show another way of taking into account the continuous part of random forces.

Because of the mathematical similarity of the problems, all the methods mentioned can also be used in the statistical theory of stability of plates and shells. The method of statistical linearization in our opinion offers fewer prospects, since it is apparently suited only to weakly linear systems. This can be seen from the fact that by using this method it is impossible to obtain a polyconic distribution at the output if the input is a Gaussian process. However, this occurrence is quite possible in a nonlinear system, and in the statistical theory of stability of shells this is used as a basis for solving the problem of discrimination (requirement 2).

This shortcoming of the method of statistical linearization might be overcome, if it becomes possible to include this method as a first approximation into some scheme which in the limit yields the exact solution. The only work on this sujbect known to the author is that by G. I. Pyatnitskii /99/.

Certain circumstances must be taken into account when using the above methods. Thus, in the case of shells, allowance must be made, not only for random external forces but also for the scatter of the shell parameters. Analogous effects are less important in radiophysics and in the theory of controls. Furthermore, the excitation of devices used in radiophysics has frequently the character of white noise, while the effects on a shell will mostly not have this character.

IV. We shall now consider the attempts made to analyze statistically problems of the stability of shells. Chronologically, the first paper on this subject was read by this author in June 1957 at the All-Union Conference on the Theory and Applications of Thin Shells at Tartu /100/, being subsequently published in 1959 /101/. It contains the above classification of random factors acting on a shell.

An approximative description of the distribution of functional parameters by a finite number of numerical parameters will be given here. The Bubnov-Galerkin method is used for interrelating them. This has already been discussed in detail. Finally, to take fully into account the entire ensemble of random parameters acting on the shell, the use of the total-probability theorem is suggested.

The vector R, of the external forces acting on the shell, is represented in the form of three components

$$\overline{R}(p, t) = \overline{R}_1, (p, t) + \overline{R}_2 (p, t) + \overline{R}_3 (p, t). \tag{10}$$

Here, $\overline{R}_2 (p, t)$ is a fluctuation term inducing an acceleration of the type of Brownian-motion acceleration, $R_3 (p, t)$ is a continuous random process; we assume that with sufficient accuracy,

$$R_3 (p, t) = \sum_{k=1}^{n} \sum_{l=1}^{n_1} a_{kl} \overline{X}_k (p) \Psi_l (t), \tag{11}$$

where the $\Psi_l (t)$ are fixed functions of time and $\overline{X}_k (p)$ are vector functions which form a basis in the "energy space" /102/.

We shall now consider in more detail the meaning of separating the random component of the load into the two parts R_2 and R_3. Any random load $f(t)$ encountered in practice can generally be represented as some series of nonrandom functions

$$f(t) = \sum_{k=0}^{\infty} a_k \varphi_k (t). \tag{12}$$

The statistical description of $f(t)$ will be specified if the distribution of the infinite system of parameters a_k is given.

The convergence of (12) is defined in a particular manner, and we cannot go into details of this problem. We only note that the use in (12) of a finite number of terms n to describe $f(t)$ is permissible only when the distribution of the coefficients $a_{n+1}, a_{n+2}, \cdots$ is, for a sufficiently large n, concentrated in the vicinity of zero, this concentration increasing with n, and if according to the law governing this distribution, high values of these coefficients have very small probabilities.

However, it may also happen that the probability of large values of a_n does not change, or changes very little, with increasing n. In this case, (12) becomes practically useless, since we then have a diverging series.

We shall then attempt to represent $f(t)$ in the form

$$f(t) = f_1 (t) + f_2 (t);$$

$$f_1 (t) = \sum_{k=0}^{\infty} a_{1k} \varphi_k (t); \quad f_2 (t) = \sum_{k=0}^{\infty} a_{2k} \varphi_k (t), \tag{13}$$

where the distribution of a_{1k} is independent of k ("white noise"), and the distribution of a_{2k} becomes concentrated around zero as k increases. In this case a finite number of terms in the expansion of $f_2 (t)$ will give a satisfactory approximation.

This separation is particularly advantageous if the random process $f(t)$ acts on a system described by ordinary equations. In this case the

distribution of the solutions in respect of f_1 will be governed by KFP equations and can be found comparatively simply in many cases. This fact will be used later. It should, however, be remembered that we shall be interested in a finite number of terms in (12), namely in those terms to which our system responds. It is natural that $f_1(t)$ will contain those a_k which have identical distributions if $0<k\ll N$, i. e., any functions in which harmonics (in time), important in our system, have an identical scatter intensity can be regarded as "white noise."

Separation of the external load into $\overline{R}_2$ and $\overline{R}_3$ thus removes from the external load the part R_2, which is difficult to approximate but for which we shall find a solution in closed form; the effect of the remaining part R_3 is now easier to take into account, since it is better approximated by (11). We arrived at this separation also by considering the correlation functions of the load. However, we shall not discuss this in more detail here.

The main idea behind this separation is made clear by the following analogy. Consider the deflections of a hinged beam subjected to a load including concentrated forces. If we solve this problem by the Fourier method, we obtain poorly converging series, especially for the stresses. It is therefore advisable to separate the load into two parts. The first part will contain the concentrated forces; for it a solution will be found in closed form. The remaining continuous part of the load is expanded in rapidly-converging Fourier series; the corresponding solution is obtained as similar rapidly-converging series.

Let us now write the equations of motion for the shell in the following symbolic form:

$$\rho\frac{\partial^2\overline{w}}{\partial t^2}+2\gamma\frac{\partial\overline{w}}{\partial t}+A(\overline{w})=\overline{R}_1(p,t)+\overline{R}_2(p,t)+\overline{R}_3(p,t), \tag{14}$$

where A denotes the nonlinear part of the elastic forces acting on the shell.

We shall now use the Bubnov-Galerkin method for the approximate solution of the problem, setting

$$\overline{w}=\sum_{k=1}^{n}\overline{X}_k(p)\,q_k. \tag{15}$$

In this case, (14) can be approximated by the system

$$\ddot{q}_k+2hq_k+A_k(q_1,\cdots,q_n)=R_{k1}(t)+R_{k2}(t)+R_{k3}(t), \tag{16}$$

where

$$R_{ki}(t)=\int_{\Omega}\overline{R}_i(p,t)\,\overline{X}_k(p)\,dp. \tag{17}$$

It follows from (17) that R_{k2} is "white noise" and R_{k3} is a continuous random process; according to (2) we have

$$R_{k3}(t)=\sum_{l=1}^{n_1}a_{kl}\Psi_l(t). \tag{18}$$

We now assume that R_{k2} does not depend statistically on the remaining random factors. We further assume that the parameters a_k and a_{kl} assume fixed values, and obtain the distribution of q_k for this case. If only sufficiently large time intervals are considered, then the required conditional distribution can be found, for example, by Smoluchowski's equation /103/

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} [f(A - R_{k1} - R_{k3})] \frac{1}{2h} + \frac{\delta}{4h^2} \nabla_q^2 f. \tag{19}$$

By using this equation we assume that the constituents of $R_{k2}(t)$(the "white noise") are independent of one another and have the same intensity. This assumption limits, on one hand, the range of problems which can be thus solved, but, on the other, it enables the study of a problem to be brought to a conclusion without numerical integration of (19). In the general case, the Laplacian in the right-hand side of (19) is replaced by an arbitrary elliptic operator. This generally complicates the solution of definite problems, so that in practice the corresponding boundary problem has to be solved numerically. Having done this for (19), we find the conditional distribution of $f(q_k, t, a_k, a_{kl})$.

To find the unconditional distribution f^0 we use the total-probability theorem

$$f^0(q_k, t) = \int f(q_k, t, a_k, a_{kl})\, da_k\, da_{kl}, \tag{20}$$

with the aid of which all the random factors acting on the shell can be taken into account.

This approach to the statistical theory of stability of shells makes allowance for all the requirements stated above.

The computational algorithms which arise in the process reduce in the worst case to the numerical solution of a heat-transfer-type equation over the entire space of variation of the coordinates q_k, followed by numerical quadrature. The use of computers makes all these operations feasible.

V. V. Bolotin /104/ suggested another approach to the use of statistical methods in stability problems of elastic systems. This is extensively dealt with in his survey submitted to this conference /105/, and will not be discussed here.

V. M. Goncharenko /106-111/ continued investigation of the behavior of shells, subjected to random loads, on the basis of a concept given in /100, 101/. However, in some parts this concept has been developed and supplemented. Thus, Smoluchowski's equation has been replaced by the KFP equation. In addition, it is assumed that $R_{k2}(t)$ is a "white noise" with a given matrix of spectral planes. For this reason the equation for f assumes the form

$$\frac{\partial f}{\partial t} + \sum_{k=1}^{n} \frac{\partial f}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} (R_{k1} + R_{k3} - A_k) \frac{\partial f}{\partial \dot{q}_k} =$$

$$= 2h \sum_{k=1}^{n} \frac{\partial}{\partial \dot{q}_k} (f\dot{q}_k) + \sum b_{ij} \frac{\partial^2 f}{\partial \dot{q}_i \partial \dot{q}_j}. \tag{21}$$

Of course, replacing (19) by (21) greatly complicates the mathematical analysis of the problem, since (21) contains twice as many independent spatial variables. The phenomenon can now be studied in the entire range $0 \leqslant t < \infty$, and not only for sufficiently large values of t, as was the case with (19). However, the mathematical complexity of the problem has increased to such an extent that the solution has to be obtained by approximative qualitative methods.

In particular, V. M. Goncharenko considered the probability P that a shell overcomes the potential barrier in a time t which is considerably less than the relaxation time of the system /109*, 111/. This study was based on the works by Christiansen and Kramers /112, 113/. The author considered in detail a shallow cylindrical panel hinged to fixed ribs and subjected to a constant pressure.

A suggestion deserving attention that Doob's method /96/ be used for the approximate solution of the problem if the external forces are not "white noise," is contained in /110/. V. M. Goncharenko considered in detail the response of a shell to random excitations of the form

$$\overline{R}_3(p, t) = \sum_{k=1}^{N} \overline{a}_k e^{i(\omega_k t + \varphi_k)}, \tag{22}$$

where a_k and φ_k are random constants with known distributions (random excitation with narrow spectrum). He also investigated the response of a shell to a random longitudinal force such as (22). He also discovered a very interesting nonstatistical effect consisting in an increase of the lower critical load under the action of rapidly oscillating longitudinal foces /111/.

The effect of the scatter in the shape of the middle-surface on the upper critical load of a shell is discussed in /107/. A closed circular cylindrical shell subjected to a longitudinal deterministic load is considered in detail. It is assumed that the main part of the distribution of the initial deflections f_0, is concentrated in the region $f_0 > 0$ (the initial deflections are mostly directed toward the center). The author uses the solution of the corresponding deterministic problem (21) and obtains the law governing the distribution of the critical load if the distribution of f_0 is known. In particular, it is shown that if f_0 has a normal distribution, the differential distribution of the critical loads has two maxima.

B. P. Makarov /114/ considered a problem which, to a certain extent, is the inverse of the above. Knowing the scatter of the upper critical forces, he tried to find the scatter in the shape of the middle surface. Calculations were performed for a cylindrical shell, based on the solution of a deterministic problem given in /115/, and on experiments described in /116/.

V. Until now we have considered problems of the stability of shells which are due to statistical loading, i. e., when the shell is in one of its equilibrium states, when not subjected to the random load. Of course, statistical methods are not less interesting in the dynamic problems of stability of plates and shells. Some considerations referring to this problem are given below.

* This work contains an error. The author claims that the generalized coordinates in /100, 101/ are considered as a Markov process. As can be seen from the preceding, this is not so.

Without attempting to make a very general statement three dynamic problems should be noted, which in our opinion are important and where the use of statistical methods is of considerable interest:

1) induced vibrations of plates and shells due to periodic forces;
2) parametric resonance;
3) free oscillations of plates and shells (flutter).

The basic assumptions made in /100, 101/ remain valid also in this case: all the random factors should be separated into three groups. It is also advisable to describe statistically the functional parameters by specifying the distribution of a finite number of random quantities. All the random factors can be accounted for together by using the total-probability theorem. The relationship between the statistical characteristics of the deformations of the shell and the initial random factors is also best found by the Bubnov-Galerkin method. We then consider the probability of obtaining one of the periodic regimes in a nonlinear system subjected to random forces. This problem was to a large extent studied before in connection with problems of radioengineering and the theory of controls. For this reason we shall only describe the methods involved briefly.

The most suitable way for solving these problems is based on the use of averaging methods. The first works in this direction were apparently those by L. I. Bershtein /117-121/, based on ideas of A. A. Andronov; the effect of fluctuations on the operation of a vacuum-tube generator was investigated; the latter is, as known, a self-oscillating system. Using averaging methods the author obtained differential equations for the envelopes, averaging in a certain manner also the random external forces.

After this, the statistical properties of the abbreviated system were studied on the basis of a method due to A. A. Andronov, A. A. Vitt, and L. S. Pontryagin /64/. Self-oscillating systems of another type, excited by random factors, were similarly investigated by M. E. Zhabotinskii /122/. P. I. Kuznetsov, R. L. Stratonovich, and V. I. Tikhonov also considered random effects on a vacuum-tube generator. The generalized KFP equation was used in this study /97/. V. I. Tikhonov and I. N. Amiantov /123/ studied fluctuations in self-oscillating systems under the action of slow excitations, using the averaging method.

As far as this author knows, R. L. Stratonovich and Yu. M. Romanovskii were the first to consider in /124/ the problem of the parametric effect of a random force on a nonlinear system. They used the averaging method and based their subsequent analysis on the KFP equation, establishing important conditions for the statistical stability of such systems. These authors later considered, in a similar manner, the case where the system, in addition to a random force, is also subjected to the parametric action of a regular harmonic force /125/.

I. B. Chelpanov in /126/ derived some criteria for the stability under parametric resonance of the solutions of the linear equation. S. M. Rytov /127/ used the fitting method for analyzing fluctuations in self-oscillating systems; this method was further developed by R. Kh. Sadekov /128, 129/. R. L. Stratonovich and P. S. Landa /130/ have calculated the probability of excitation and break-off of the self-oscillations in a generator with rigid excitation, using the KFP equation.

The ideas appearing in these works were later developed in /131/, where the probability of transition from one equilibrium state to another was considered for a system of general form.

The method of averaging can be used in the statistical theory of vibrations not only in conjunction with the KFP equation, but also together with subsequent correlational analysis /98/. In addition, self-oscillations can be statistically investigated by means of statistical linearization. This has already been discussed /85, 86/.

Hence, we can expect that the averaging method, with subsequent application of either the KFP equation (or its generalizations /94, 95, 97/) or the method of correlation functions, will be used extensively also in the statistical theory of stability of elastic systems.

The following should also be noted. The use of curtail (averaged) statistical equations greatly extends the range of problems where external random effects can be idealized as "white noise." As a result of this the mathematical analysis of many problems becomes much simpler. More details on this were given by R. L. Stratonovich /98/ and V. I. Tikhonov /132/. The lack of a mathematical basis for this approach appears to this author to be only temporary. A rigorous analysis of the averaging method was given by I. I. Gikhman /133/ for some different effects of random forces.

We shall now illustrate this approach by considering the problem of forced nonlinear vibrations of shells. Let a shallow shell be subjected to an external periodic force, which, for simplicity, will be assumed to be normal to the middle surface. We shall, as before, assume that the random factors determining the scatter of the geometric and elastic properties of the shell and its support conditions, are described by a finite number of random parameters $a_1, a_2, \cdots, a_p$, having a known distribution $\Omega(a)$.

The external force acting on the shell is

$$Z(t) = \sum_{k=0}^{\infty} [P_k(t) \sin krt + Q_k(t) \cos krt] + \xi(t), \tag{23}$$

where P_k and Q_k are slowly-varying functions of time, while $\xi(t)$ is a component of the external forces, which induces Brownian-motion type accelerations, or simply "white noise."

If we write down the equations of motion of a shallow shell subjected to a load (23), and analyze them by the Bubnov-Galerkin method with respect to the spatial coordinates, we obtain

$$\ddot{q}_i + r^2 q_i = A_i(q, \dot{q}, a) + \sum_{k=0}^{\infty} [P_{ik} \sin krt + Q_{ik} \cos krt] + \xi_i, \tag{24}$$

where the A_i are the elastic and frictional forces in the shell, while P_{ik} Q_{ik}, and ξ_i correspond respectively to P_k, Q_k, and ξ in (23).

We shall now describe in more detail the slowly varying random functions P_i and Q_i.

We assume that

$$P_{ik} = \sum_{\sigma=1}^{N} p_{ik\sigma} \Psi_k(t); \qquad Q_{ik} = \sum_{\sigma=1}^{N} q_{ik\sigma} \Psi_k(t). \tag{25}$$

where the $\Psi_k(t)$ are nonrandom functions of time, while $p_{ik\sigma}$ and $q_{ik\sigma}$ are random quantities whose distribution $\Omega_2(p_{ik\sigma}, q_{ik\sigma})$ is known. In the general case we can assume that the joint distribution $\Omega(a, p_{ik\sigma}, q_{ik\sigma})$ is known.

Finally, we assume that the spectral matrix f_{ik} of process $\xi_i(t)$ is known.

Our problem consists in determining the statistical characteristics q and $\dot{q}$ from (24). It will be solved approximately, assuming that a, p_{ik0}, q_{ik0}, q_0, $\dot{q}_0$, and are statistically independent. We first find the distribution of $w^0(q, \dot{q})$, assuming that a, p_{ik0}, q_{ik0}, q_0, and $\dot{q}_0$ assume fixed values. For this we try a solution of (24) in the form

$$q_i(t) = \sum_{k=0}^{\infty} (a_{ik} \sin krt + b_{ik} \cos krt). \tag{26}$$

Inserting (26) into (24) we obtain

$$\sum_{k=0}^{\infty} \{\ddot{a}_{ik} \sin krt + \ddot{b}_{ik} \cos krt + 2\dot{a}_{ik} \cos krt \cdot kr - 2\dot{b}_{ik} \sin krt \cdot kr\} =$$

$$= A_i + \sum_{k=0}^{\infty} (P_{ik} \sin krt + Q_{ik} \cos krt) + \xi_i(t). \tag{27}$$

Equating the Fourier coefficients of the right- and left-hand sides of (27) for each i over an interval of t whose length is $\frac{2\pi}{r}$, we obtain

$$\sum_{k=0}^{\infty} \int_t^{t+\frac{2\pi}{r}} (\ddot{a}_{ik} \sin krt + \ddot{b}_{ik} \cos krt + 2\dot{a}_{ik} \cos krt \cdot kr - 2\dot{b}_{ik} \sin krt \cdot kr) e^{ikrt} dt =$$

$$= \int_t^{t+\frac{2\pi}{r}} \left[A_i + \sum_{k=0}^{\infty} (P_{ik} \sin krt + Q_{ik} \cos krt) \right] e^{ikrt} dt + \int_t^{t+\frac{2\pi}{r}} \xi_i(t) e^{ikrt} dt. \tag{28}$$

We simplify (28) by assuming that it satisfies the conditions for averaging by the Krylov-Bogolyubov method. For example, we consider a_{ik}, b_{ik}, A_i, P_{ik}, and Q_{ik} to be sufficiently smooth functions of their arguments, and r to be sufficiently large. In this simplification we do not so far change terms containing "white noise," which will be dealt with separately.

We then obtain from (28):

$$\ddot{a}_{ik} - 2kr\dot{b}_{ik} = \frac{r}{\pi} \int_t^{t+\frac{2\pi}{r}} A_i \sin kru\, du + P_{ik}(t) + \frac{r}{\pi} \int_t^{t+\frac{2\pi}{r}} \xi_i(u) \sin kru\, du, \tag{29}$$

$$\ddot{b}_{ik} + 2kra_{ik} = \frac{r}{\pi} \int_t^{t+\frac{2\pi}{r}} A_i \cos kru\, du + Q_{ik}(t) + \frac{r}{\pi} \int_t^{t+\frac{2\pi}{r}} \xi_i(u) \cos kru\, du. \tag{30}$$

We shall now consider the terms in the right-hand sides of (29) and (30) which formerly contained "white noise." These terms will, of course, no longer contain "white noise"; however, in this problem they can be replaced by "white noise" without appreciable error. We introduce the notation

$$\alpha_{ik} = \frac{r}{\pi} \int_t^{t+\frac{2\pi}{r}} \xi_i(u) \sin kru\, du; \quad \beta_{ik} = \frac{r}{\pi} \int_t^{t+\frac{2\pi}{r}} \xi_i(u) \cos rku\, du \tag{31}$$

and consider the correlation functions of the processes α_{ik} and β_{ik}.

$$K_{ik}(t_1, t_2) = \text{м. о. } \alpha_{ik}(t_1)\alpha_{ik}(t_2), \quad \begin{matrix} L_{ik}(t_1, t_2) = \text{м. о. } \alpha_{ik}(t_1)\beta_{ik}(t_2), \\ M_{ik}(t_1, t_2) = \text{м. о. } \beta_{ik}(t_1)\beta_{ik}(t_2). \end{matrix} \tag{32}$$

It is easy to see that all these functions vanish when $|t_1 - t_2| > \frac{2\pi}{r}$; hence, they are concentrated in a band of the t_1, t_2 plane in which they are periodic functions of and t_2. We can calculate their average value in the square of the periods $\frac{2\pi}{r} \times \frac{2\pi}{r}$, and construct a "white noise" whose correlation functions are equal to the mean values of the corresponding correlation functions α_{ik} and β_{ik}. This procedure requires cumbersome calculations; we shall therefore use a simpler, less rigorous method which, however, yields the same results.

Using formally the mean-value theorem in (31), we obtain

$$\alpha_{ik} = 2\xi_i(t)\sin kru; \qquad \beta_{ik} = 2\xi_i(t)\cos kru. \tag{33}$$

We now determine the spectral matrix of α_{ik}, β_{ik}. We have

$$\begin{aligned} K_{ik,jl} &= \text{м. о. } \alpha_{ik}(t_1)\alpha_{jl}(t_2) = 4\pi f_{ij}\sin krt_1 \sin lrt_2\,\delta(t_1 - t_2); \\ L_{ik,jl} &= \text{м. о. } \alpha_{ik}(t_1)\beta_{jl}(t_2) = 4\pi f_{ij}\sin krt_1 \cos lrt_2\,\delta(t_1 - t_2); \\ M_{ik,jl} &= \text{м. о. } \beta_{ik}(t_1)\ \beta_{jl}(t_2) = 4\pi f_{ij}\cos krt_1 \cos lrt_2\,\delta(t_1 - t_2). \end{aligned} \tag{34}$$

We use (34) to find the mean values of K, L, and M in the square of the periods $\frac{2\pi}{r} \times \frac{2\pi}{r}$:

$$\overline{K}_{ik,jl} = \frac{r^2}{4\pi^2}\int_0^{\frac{2\pi}{r}}\int_0^{\frac{2\pi}{r}} 4\pi f_{ij}\sin krt_1 \sin lrt_2\,\delta(t_1 - t_2)\,dt_1\,dt_2 = \begin{cases} 0, & k \neq l, \\ rf_{ij}, & k = l. \end{cases} \tag{35}$$

Similarly,

$$\overline{L}_{ik,jl} = 0, \quad \overline{M}_{ik,jl} = \begin{cases} 0, & k \neq l \\ rf_{ij}, & k = l \neq 0 \\ 2rf_{ij}, & k = l = 0. \end{cases} \tag{36}$$

We now replace α_{ik} and β_{ik} in the right-hand sides of (29) and (30) by "white noise" $\overline{\alpha}_{ik}$, $\overline{\beta}_{ik}$ whose correlation functions are given by (35) and (36).

As a result, (29) and (30) take the form

$$-2krb_{ik} = \frac{r}{\pi}\int_t^{t+\frac{2\pi}{r}} A_i \sin kru\,du + P_{ik}(t) + \overline{\alpha}_{ik}; \tag{37}$$

$$2kr\dot{b}_{ik} = \frac{r}{\pi}\int_t^{t+\frac{2\pi}{r}} A_i \cos kru\,du + Q_{ik}(t) + \overline{\beta}_{ik}; \tag{38}$$

$$\ddot{b}_{i0} = \frac{r}{\pi}\int_t^{t+\frac{2\pi}{r}} A_i\,du + Q_{i0}(t) + \overline{\beta}_{i0}. \tag{39}$$

In (37)-(39) we have discarded terms containing second derivatives, in accordance with the general simplifications introduced with the Krylov-Bogolyubov method. In (39), the terms containing second derivatives were retained, since they are larger in the initial time interval than those in (37) and (38).

Assuming A_i to be a linear function of $\dot{q}$, each pair of equations (37) and (38) can be solved for a_{ik} and b_{ik}. We then obtain

$$\dot{a}_{ik} = \Phi_{ik}(a_{ik}, b_{ik}, P_{ik\sigma}, Q_{ik\sigma}, a) + \lambda_{ik},$$
$$\dot{b}_{ik} = \Psi_{ik}(a_{ik}, b_{ik}, P_{ik\sigma}, Q_{ik\sigma}, a) + \mu_{ik}; \tag{40}$$

$$\ddot{b}_{i0} + 2h\dot{b}_{i0} = \Psi_{i0}(a_{ik}, b_{ik}, P_{ik\sigma}, Q_{ik\sigma}, a), \tag{41}$$

where λ_{ik} and μ_{ik} are linear combinations of $\bar{\alpha}_{ik}$ and β_{ik}, being by virtue of (35) and (36), also "white noise." Equations (40) and (41) describe the statistical properties of a_{ik} and b_{ik} with the error peculiar to the averaging method in the theory of differential equations. This was observed by the author in a number of examples.

The conditional distribution of $w^0(a_{ik}, b_{ik}, \dot{b}_{i0}, P_{ik\sigma}, Q_{ik\sigma}, t)$ can now be found from the KFP equation

$$\frac{\partial w^0}{\partial t} + \sum_{i,k\neq 0}\left[\frac{\partial}{\partial a_{ik}}(w^0\Phi_{ik}) + \frac{\partial}{\partial b_{ik}}(w^0\Psi_{ik})\right] +$$
$$+\sum_{i=1}^{n}\frac{\partial}{\partial \dot{b}_{i0}}(\ddot{b}_{i0}w^0) + \sum_{i=1}^{n}\frac{\partial}{\partial \dot{b}_{i0}}[(-2h\dot{b}_{i0} + \Psi_{i0})w^0] = L_n, \tag{42}$$

where L is an elliptic operator whose coefficients are expressed in terms of $\bar{K}$, $\bar{L}$, and $\bar{M}$.

We now assume that w^0 is known. In this case the unconditional distribution of w is found by applying the total-probability theorem

$$w(a_{ik}, b_{ik}, b_{i0}, t) =$$
$$= \int w^0(a_{ik}, b_{ik}, \dot{b}_{i0}, P_{ik\sigma}, Q_{ik\sigma}, a, t)\,\Omega(P_{ik\sigma}, Q_{ik\sigma}, a)\,dP_{ik\sigma}\,dQ_{ik\sigma}\,da. \tag{43}$$

The principal mathematical difficulty of this method is thus the solution of (42).

Apparently, asymptotic integration of (42) offers some prospects when the "white noise" intensity is low. In some cases such an asymptotic representation is possible. In addition, piecewise linear approximations of the system as developed by E. M. Hazen /134/ are possible.

VI. As example we shall now consider the effect of a random transverse load on a round, rigidly supported plate [of thickness H]. The equations of motion are

$$\rho w_{tt} + D\nabla^4 w = Q\cos pt + \frac{H}{r}(w_r F_r)_r - 2\gamma w_t;$$
$$\nabla^4 F = -\frac{E}{2r}(w_r)_r^2; \qquad w\Big|_\Gamma = \frac{\partial w}{\partial n}\Big|_\Gamma = 0. \tag{44}$$

Solving the problem by the Bubnov-Galerkin method, we set

$$w = x(t)(a^2 - r^2)^2. \tag{45}$$

We then obtain

$$\ddot{x} + \frac{2\gamma}{\rho} x + \frac{320D}{3a^4\rho} x + \frac{10HE}{7\rho a^4} x^3 = \frac{5Q}{3\rho} \sin pt. \tag{46}$$

Introducing the new variables

$$y = \frac{x}{H}; \qquad t = \tau \cdot \left(\frac{3a^4\rho}{320D}\right)^{1/2},$$

we obtain from (46)

$$y'' + 2hy' + y + \alpha y^3 = P \sin rt, \tag{47}$$

where

$$h = \frac{\gamma}{\rho}\left(\frac{3a^4\rho}{320D}\right)^{1/2}, \qquad \alpha = \frac{9(1-\mu^2)}{56}. \tag{48}$$

We assume that

$$P(t) = R(t) + \xi(t), \tag{49}$$

where R is the slowly-varying random part and ξ, the rapidly-varying part of the force P, which we shall replace by "white noise."

Writing

$$y = a \sin r\tau + b \cos r\tau; \quad A^2 = a^2 + b^2, \tag{50}$$

and applying the averaging procedure, we obtain

$$a = \alpha_1 + \frac{r}{2\pi}\int_t^{t+\frac{2\pi}{r}} (R+\xi)\sin^2 rs\, ds; \quad b = \alpha_2 + \frac{r}{2\pi}\int_t^{t+\frac{2\pi}{r}} (R+\xi)\sin rs \cos rs\, ds; \tag{51}$$

$$\alpha_1 = \frac{Ph}{2(h^2+r^2)} - \frac{ah+br}{2(h^2+r^2)}\left(\frac{3\alpha A^2}{4} + 1 - r^2\right) + \frac{hr(bh-ar)}{r^2+h^2};$$

$$\alpha_2 = -\frac{Pr}{2(h^2+r^2)} - \frac{bh-ar}{2(h^2+r^2)}\left(\frac{3\alpha A^2}{4} + 1 - r^2\right) - \frac{hr(ah+br)}{r^2+h^2}. \tag{52}$$

In averaging the random terms in the right-hand sides of (51) and (52) we assume that

$$\frac{2\pi}{r} \ll \tau^{\xi}_{\text{cor}} \leqslant \tau, \tag{53}$$

where τ^{ξ}_{cor} is the correlation time of the random function $\xi(t)$ and τ is the

time required for the processes to pass through the system. We obtain after averaging

$$\dot{a} = \alpha_1(a, b) + \frac{1}{2}(R + \xi),$$
$$b = \alpha_2(a, b). \tag{54}$$

We now assume that the random function $P(t)$ has been realized. In this case the conditional distribution of w^0 is found from

$$\frac{\partial w^0}{\partial t} = \frac{\partial}{\partial a}\left[w^0\left(\alpha_1 + \frac{P}{2}\right)\right] + \frac{\partial}{\partial b}(w^0\alpha_2) + q\frac{\partial^2 w^0}{\partial a^2}, \tag{55}$$

where q is determined by the dispersion $\frac{1}{2}\xi$ of the "white noise."

If w^0 is determined from (55), then the unconditional distribution of w is found from the total-probability theorem, as shown above.

There is a case in which this scheme can be carried to the end. We assume that P is a random constant with a distribution $\Omega(P)$.

The steady-state probability distribution for $t = \infty$ can be found from the equations

$$\dot{a} = \alpha_1(a, b) + \frac{1}{2}(R + \xi); \quad \alpha_2(a, b) = 0. \tag{56}$$

We now express b in the second equation (56) in terms of a and substitute the result in the first equation:

$$\dot{a} = \alpha_1[a, b(a)] + \frac{1}{2}(R + \xi). \tag{57}$$

The steady-state probability distribution for a is then obtained from

$$\frac{\partial}{\partial a}\left\{\langle \alpha_1[a, b(a)] + \frac{P}{2}\rangle w^0\right\} + q\cdot\frac{\partial^2 w^0}{\partial a^2} = 0, \tag{58}$$

which yields

$$w_0 = A_1 e^{-\frac{\Phi(a)}{q}}, \quad \Phi(a) = \int_b^a \alpha_1[s, b(s)]\,ds + \frac{Pa}{2}, \tag{59}$$

where A_1 is found from the normalization condition. The unconditional distribution is again found from the total-probability theorem. This expression is, to a certain extent, analogous to the Gibbs distribution. The role of the potential energy of the system is here played by the function $\Phi(a)$, which can be called the "potential energy" of the amplitude of steady-state vibrations.

It seems to us that the method of averaging, together with the KFP equation and the total probability theorem, can serve as basis for developing a statistical theory of flutter.

Similar work has been done by Yu. M. Romanovskii and S. P. Strelkov /135/. They investigated the response of an aircraft wing in a turbulent atmosphere. The wing was assumed to be an elastic system performing flexural-torsional vibrations. The mean-square characteristics of the state of stress were obtained as functions of the velocity. Yu. M. Romanovskii /136/ also considered parametric excitation of an aircraft wing in turbulent flow.

These methods should enable us to establish a statistical theory of parametric resonance in elastic systems. This would be important from the practical point of view.

VII. Use of statistical methods throws new light on the methods and content of the theory of stability of elastic systems, in which increasing interest has been shown recently. For example, the survey by A. S. Vol'mir /137/, read at this conference, is devoted to these problems.

The theory of stability of elastic systems contains, in our opinion, the following:

1) qualitative methods for determining the number of equilibrium shapes (or the number of motion regimes of a specific type) of a system;

2) methods to estimate quantitatively the probability of each of these shapes (or regimes).

Statistical methods provide fundamental solutions of these two problems. In principle there is no difference between problems of equilibrium stability and stability of oscillations, conservative and nonconservative systems, etc.

It is true that in using statistical methods we encounter certain mathematical difficulties. This is, however, true also when solving problems of stability of elastic systems by other methods.

If only as much effort had been devoted to the development of statistical methods in the theory of stability of elastic systems as has been devoted to that of all other methods, these difficulties would have been overcome long ago.

VIII. In concluding this survey I shall attempt to formulate problems in the theory of stability of elastic systems, whose solution is at present of interest.

1. Statistical characteristics of the first group of random factors (scatter in the shape of the middle surface of the shell, its thickness, elastic properties, etc.). By means of large-scale experimentation data should be obtained on the relationships between the probability properties of this group of factors and the features of the production process.

2. Statistical characteristics of the second group of random factors (support conditions). For this it is desirable to obtain the scatter characteristics of the restraints to which shells are subjected in actual structures.

3. Statistical characteristics of the loads acting on the shell. For this it is desirable to separate the service conditions of the shell into computational cases, for each of which instructions should be given on the selection of numerical values of the appropriate statistical characteristics.

The importance of investigating experimentally the statistical effects of turbulent flow and sound waves must be emphasized. These problems are generally known, but we mention them again due to their exceptional importance to the problems considered.

4. Further development of methods for determining the relationships between the characteristics of the initial data and those of the deformations of the shell. Approximative methods have to be found for solving the Kolmogorov-Fokker-Planck equation and its generalization, discussed before.

The asymptotic method of integration appears to offer some prospects. Direct methods and methods for numerical solution should also be considered, as should methods of piecewise linearization.

5. Continuing the analysis of various particular cases of making allowance for random effects on shells of different shapes and different support conditions, varying the assumptions as regards the random factors.

6. Continued study of dynamical problems, primarily on the basis of the averaging method.

7. Development of the statistical theory of flutter.

8. Development of the statistical theory of parametric resonance in plates and shells.

9. Some purely mathematical problems related to the use of statistical methods in problems of stability of elastic systems.

Most important is the problem of developing a general theory of stability of linear systems with periodic coefficients subjected to steady-state processes of statistical nature. In the case of nonlinear systems it is of interest to establish a method of linearization in problems of statistical stability.

Very important is the investigation of averaging methods for solving problems of statistical stability, continuing the work done by I. I. Gikhman /133/.

It is also of interest to investigate the convergence of statistical approximations obtained by the Bubnov-Galerkin method. It is important to provide estimates for the build-up time of processes in nonlinear systems and to study the influence of various factors on this time.

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INTEGRAL-DIFFERENTIAL AND INTEGRAL EQUATIONS OF STATICS AND DYNAMICS OF THIN ELASTIC SHELLS

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This communication is a natural extension of a report read in L'vov in 1961 at the Conference on the Theory of Plates and Shells /21/, which dealt with methods for reducing three-dimensional problems of the theory of elasticity to two-dimensional problems of the theory of shells, without resorting to additional kinematic hypotheses.

Equations of the theory of shells with boundary conditions obtained by nonclassical methods lead to complicated boundary-value problems. On the other hand, the boundary-value problems of the classical theory of shells are also quite complex. These difficulties are, apparently, a constant challenge to find new, efficient methods to solve boundary-value problems of the theory of shells.

Assuming the history of the development of methods for solving boundary-value problems in the theory of shells to be known, I wish to consider methods based on replacing the differential equations in these problems of the theory of shells by, in a certain sense equivalent systems of integral and integral-differential equations. This approach to the solution of statical and dynamical problems in the theory of shells is quite recent, although the first works along these lines on the theory of shells appeared already in 1939-1940 /14, 15/.*

This survey contains a classification of studies in this field of the applied theory of elasticity and a brief description of the methods used. The studies were classified according to the starting premises used to establish the integral and integral-differential equations of the theory of shells. The survey ends with a note on the prospects for the future development of this method and of the fields for its application. The standard symbols of tensor analysis and the notations of the theory of shells are used.

There exist two basically different approaches to the formulation of boundary-value problems of the theory of shells, based on integral-differential and integral equations. The first consists in directly obtaining integral-differential equations of the theory of shells from the reciprocity theorem of the three-dimensional theory of elasticity without resorting to the Kirchhoff-Love hypotheses. The second is based in principal on the classical equations of the theory of shells, obtained by simplifying the Kirchhoff-Love hypotheses. Mixed methods are also sometimes encountered.

* See also Vlasov, V. Z. Obshchaya teoriya obolochek (General Theory of Shells).—Gostekhizdat. 1949.

The appendix to this survey contains a bibliography, which is, however, not exhaustive.

1. Equations derived from the reciprocity theorem of the three-dimensional theory of elasticity.

These equations were obtained by this author in /14/. Without discussing details, we note the main stages in establishing the integro-differential equations of the statics and dynamics of shells. We first consider the equations of statics. Henceforth, a system of displacements, stresses, strains, and external forces pertaining to the boundary-value problem studied will be called principal system. The system of displacements, internal and external forces, and strains introduced according to the broadened method of Somil'yan will be called auxiliary system.

Let the auxiliary system consist of forces, directed along the tangent to the coordinate line x^i in the positive direction and distributed with a linear density $q_{(i)}(\alpha)$ along the normal to the undeformed middle surface of the shell, of displacements, strains, and stresses brought about by these forces, and also of surface forces in equilibrium with the load $q_{(i)}(\alpha)$. The auxiliary system is based on the solution of the problem of a concentrated force acting in an unbounded elastic medium. From the reciprocity theorem we obtain

$$\int_{-h}^{+h} q_{(i)}(\alpha)\, u_i(M, \alpha)\, d\alpha + \iint_{(S)} S^j_{(i)}\, u_j\, ds = \iint_{(S)} X^j v_{(i)j}\, ds + \iiint_{(V)} F^j v_{(i)j}\, dV \qquad (1)$$

$(i, j = 1, 2, 3;$ no summation over $i)$.

Here the point $M(x^j)$ is the foot of the normal to the middle surface over which the load $q_{(i)}(\alpha)$ is distributed; $S^j_{(i)}$ are components of the auxiliary surface forces; X^j and F^j are components of the surface and body forces of the principal system; u_i are covariant components of the principal displacement vector; and $v_{(i)j}$ are components of the auxiliary displacements. The integrals $\iint_{(S)}$ are taken over the boundary and contour surfaces of the shell, while integrals $\iiint_{(V)}$ are taken over its volume.

By means of (1) we can now apply a new method for reducing three-dimensional problems of the theory of elasticity to two-dimensional problems of the theory of shells. However, in contrast to other methods, the boundary-value problem of the theory of shells is here reduced to solving a system of integral-differential equations. This approach to the method is evident from (1). For example, we can find the coefficients of the expansion of the components u_i in terms of any system of functions $\varphi_j(\alpha)$, orthogonal in the interval $(-h, +h)$. Other applications of (1) are also possible. We shall consider one which is closest to the methods of reduction considered in /21/. We assume that the displacements $u_i(M, \alpha)$ can be approximated by the polynomial

$$u_i(M, \alpha) \cong u_i^{(0)} + \alpha u_i^{(1)} + \cdots + \alpha^N u_i^{(N)}, \qquad (2)$$

when the coefficients $[u_i^{(k)}]$ are functions of the point $M(x^j)$ on the middle

surface. It is always possible to find a system of piecewise continuous functions $q^k_{(i)}(\alpha)$ which satisfy the condition

$$\int_{-h}^{+h} q^k_{(i)}(\alpha)\,\alpha^j\,d\alpha = \delta^{kj}, \tag{3}$$

where δ^{kj} is the Kronecker delta. Substituting these functions in (1), we obtain

$$u_i^{(k)} \cong \iint_{(S)} X^j v^{(k)}_{(i)j}\,ds + \iiint_{(V)} F^j v^{(k)}_{(i)j}\,dV - \iint_{(S)} S^{(k)j}_{(i)} u_j\,dS \tag{4}$$

$$(i, j = 1, 2; 3; \quad k = 0, 1, 2, \ldots, N).$$

This is a new method of reducing three-dimensional statical problems of an elastic body to two-dimensional problems of the theory of shells. Grouping together in (4) the terms $\Phi_i^{(k)}(x^j)$ known from the conditions of the problem, we obtain*

$$u_i^{(k)}(x^j) = \Phi_i^{(k)}(x^j) + \iint_{(II)} X^r v^{(k)}_{(i)r}\,dS - \iint_{(I)} S^{(k)r}_{(i)} u_r\,dS - \iint_{(\pm)} S^{(k)r}_{(i)} u_r\,dS \tag{5}$$

$$(i, r = 1, 2, 3; \quad j = 1, 2; \quad k = 0, 1, 2, \ldots, N).$$

Here $\iint_{(I)}$ is an integral taken over that part of the contour surface for which the components of the stress vector are specified, and $\iint_{(II)}$ is an integral taken over that part of the contour surface for which the displacement components of the principal system are specified. Integrals $\iint_{(\pm)}$ contain the covariant components oı the unknown displacements at the boundary surface of the shell. To determine approximately the displacements of the shell we must find the components of the unknown displacements at the boundary surfaces of the shell and for part (I) of the contour surface, as well as the components of the force system for part (II) of the contour surface.

Using the approximation (2) and considering the components of the tensors on the middle surface of the shell, we obtain from (5)

$$u_i^{(k)}(x^j) = \Phi_i^{(k)}(x^j) + \int_{(II)} X^{(p)r} v^{(k)}_{(ip)r}\,ds - \int_{(I)} S^{(k)r}_{(ip)} u_r^{(p)}\,ds - \iint_{(S)} K^{(k)r}_{(ip)} u_r^{(p)}\,dS \tag{6}$$

$$(i, r = 1, 2, 3; \quad j = 1\ \ 2; \quad k, p = 0, 1, 2, \ldots, N).$$

Integral $\iint_{(S)}$ is taken over the middle surface, while $\int_{(I)}$ and $\int_{(II)}$ are

* Henceforth, the sign $\cong$ will be omitted.

contour integrals. In (6) we used the notation

$$v_{(ip)r}^{(k)} = \int_{-h}^{+h} z^p v_{(i)r}^{(k)} dz; \qquad S_{ip}^{(k)r} = \int_{-h}^{+h} z^p S_{(i)}^{(k)r} dz;$$

$$K_{(ip)}^{(k)r} = h^p [[1 - (k_1 + k_2) h + k_1 k_2 h^2] R_{(i)}^{(k)r} + (-1)^p [1 + (k_1 + k_2) h + k_1 k_2 h^2] T_{(i)}^{(k)r}], \quad (7)$$

where $R_{(i)}^{(k)r}$ and $T_{(i)}^{(k)r}$ are the components of the vectors $S_{(i)}^{(k)r}$ at the boundary surfaces $z = \pm h$, appropriately translated, according to Levi-Civita, to the middle surface, and k_i are the principal curvatures of the middle surface of the shell. Equations (7) are integral analogs of the differential equations considered in /21/. However, there is an essential difference between the latter and (6), which do not contain derivatives of the vector components of the forces acting on the shell. Consequently, these equations remain applicable also when concentrated forces act on the boundary surfaces of the shell.

Equations of the dynamics of shells. To obtain equations of motion from the equilibrium equations we have to include the intertia forces $-\rho \frac{\partial^2 u^r}{\partial t^2}$ in the volume forces in (1). Expressing the inertia forces in terms of covariant components and performing the same transformations as before, we obtain

$$u_i^{(k)}(x^j, t) = \Phi_i^{(k)}(x^j, t) + \int_{(II)} X^{(p)r} v_{(ip)r}^{(k)} dS - \int_{(I)} S_{(ip)}^{(k)r} u_r^{(p)} dS -$$

$$- \iint_{(S)} K_{(ip)}^{(k)r} u_r^{(p)} dS - \iint_{(S)} V_{(ip)}^{(k)r} \frac{\partial^2 u_r^{(p)}}{\partial t^2} dS; \quad (8)$$

$$(i, r = 1, 2, 3; \; j = 1, 2; \; k, p = 0, 1, 2, \ldots, N),$$

where

$$V_{(ip)}^{(k)r} = \int_{-h}^{+h} \rho z^p v_{(i)}^{(k)r} [1 - (k_1 + k_2) z + k_1 k_2 z^2] dz. \quad (9)$$

Equations (8) form a system of integral-differential equations of the dynamics of shells in the unknown functions $u_i^{(k)}(x^j, t)$. In the case of a steady-state process, we set

$$u_r^{(p)}(x^j, t) = u_{r0}^{(p)}(x^j) + \sum_{(\omega)} u_{r\omega}^{(p)}(x^j) \cos(\omega t + \varepsilon_\omega); \quad (10a)$$

$$\Phi_i^{(k)}(x^j, t) = \Phi_{i0}^{(k)}(x^j) + \sum_{(\omega)} \Phi_{i\omega}^{(k)}(x^j) \cos(\omega t + \varepsilon_\omega); \quad (10b)$$

$$(i, r = 1, 2, 3; \; k, p = 1, 2, \ldots, N).$$

We assume that the frequencies are discrete. Substituting (10a) and (10b) in (8), we obtain

$$u_{i0}^{(k)}(x^j) = \Phi_{i0}^{(k)}(x^j) + \int_{(II)} X_0^{(p)r} v_{(ip)r}^{(k)} ds - \int_{(I)} S_{(ip)}^{(k)r} u_{r0}^{(p)} ds - \iint_{(S)} K_{(ip)}^{(k)r} u_{r0}^{(p)} dS; \quad (11a)$$

$$u_{i\omega}^{(k)}(x^j) = \Phi_{i\omega}^{(k)}(x^j) + \int_{(II)} X_\omega^{(p)r} v_{(ip)r}^{(k)} ds - \int_{(I)} S_{(ip)}^{(k)r} u_{r\omega}^{(p)} ds - \iint_{(S)} [K_{(ip)}^{(k)r} - \omega^2 V_{(ip)}^{(k)r}] u_{r\omega}^{(p)} dS; \quad (11b)$$

$$(i, r = 1, 2, 3; \; j = 1, 2; \; k, p = 0, 1, 2, \ldots, N).$$

Equations (11a) are identical with (6), while equations (11b) contain the parameter ω. When the homogeneous system has a nontrivial solution, system (11b) has no solutions. These resonance cases will be considered separately.

In the case of nonsteady vibrational processes we apply the Laplace-Carson transformation with respect to the variable t in (8). This leads to the following system of integral-differential equations in the transforms of the unknown functions:

$$\dot{u}_i^{(k)}(x^j, p) = \dot{\Phi}_i^{(k)}(x^j, p) + p^2 \iint_{(S)} V_{(iq)}^{(k)r} u_r^{(q)}(\xi^j, 0)\, dS +$$
$$+ p \iint_{(S)} V_{(iq)}^{(k)r} u_r^{(q)}(\xi^j, 0)\, dS + \int_{(\Pi)} \dot{X}^{(q)r} v_{(iq)r}^{(k)}\, ds - \int_{(l)} S_{(iq)}^{(k)r} \dot{u}_r^{(q)} ds -$$
$$- \iint_{(S)} [K_{(iq)}^{(k)r} + p^2 V_{(iq)}^{(k)r}]\, \dot{u}_r^{(q)}\, dS; \qquad (12)$$
$$(i, r = 1, 2, 3;\ j = 1, 2;\ k, q = 0, 1, \ldots, N).$$

The first three terms in the right-hand sides of (12) contain the initial conditions which must be satisfied by the unknown functions. After the transforms of $u_i^{(k)}$ have been found from (12), their inverse transforms are generally obtained by the Riemann-Mellin formula.

Methods of solution. The general method is well known. Performing quadratures and replacing the derivatives of the unknown functions by ratios of finite differences, the solution of the system of integral-differential equations is approximated by the solution of a system of linear algebraic equations (resolvents). Comparing this procedure with that of approximating systems of differential equations in the theory of shells by systems of algebraic equations, using the relaxation method, we find that the algebraic equations obtained by our method are more exact than those obtained by the relaxation method. This is due to the smoothing effect of integration. On the other hand, systems of algebraic equations which are approximations of systems of integral-differential equations are more complex than systems obtained from differential equations by the relaxation method, since the matrices of the latter contain many zero elements. To reduce this shortcoming, we set up equations with kernels having an extremum value near the point $M(x^j)$, which decreases rapidly with increasing distance from this point. According to K. Lantsosh we shall call them focusing kernels, and the auxiliary load corresponding to them will be called the focusing load. To obtain focusing kernels for the equations considered above we can, for example, produce the normal over which the auxiliary load $q_{(i)}(\alpha)$ is distributed beyond the shell and distribute over it a load statically in equilibrium with $q_{(i)}(\alpha)$. This method was applied by this author in /14/. Other methods to obtain focusing kernels are possible, but they must be considered separately.

By means of focusing kernels we can form systems of resolving algebraic equations with dominant terms. This greatly facilitates the numerical solution. Development of these methods is beyond the scope of this paper.

Nonlinear integral-differential equations of dynamics of shells are derived from a theorem replacing the reciprocity theorem in the nonlinear theory of elasticity. Without presenting here this theorem,

which was proved by this author, we shall now consider the nonlinear integral-differential equations of the dynamics of shells obtained from it for the case of geometric nonlinearity. We assume that finite deformations occur in the shell due to the applied load and under specified conditions at its contour surface, due to its flexibility. For this reason the finite deformations determine the principal, but not the auxiliary state of the shell. The auxiliary state is, as before, determined by the displacements and stresses in the linearly-deformed medium subjected to the aforementioned loads.

Using Karman's hypotheses, known from the theory of flexible plates, we obtain after some transformations and elimination of nonlinear terms of higher degree:

$$u_i^{(k)}(x^j, t) = \Phi_i^{(k)}(x^j, t) + \int\limits_{(\Pi)} X^{(p)r} v_{(ip)r}^{(k)} ds - \int\limits_{(l)} S_{(ip)}^{(k)r} u_r^{(p)} ds -$$

$$- \iint\limits_{(S)} K_{(ip)}^{(k)r} u_r^{(p)} dS + \iint\limits_{(S)} H_{(i)}^{(k)qq} u_q^{(1)} u_q^{(1)} dS +$$

$$+ \iint\limits_{(S)} R_{(i)}^{(k)jq} u_j^{(1)} u_q^{(1)} dS - \iint\limits_{(S)} V_{(ip)}^{(k)r} \frac{\partial^2 u_r^{(p)}}{\partial t^2} dS; \qquad (13)$$

$$(i, r = 1, 2, 3. \quad j, q = 1, 2. \quad k, p = 0, 1, 2, \ldots, N).$$

Here

$$H_{(i)}^{(k)qq} = -\frac{\lambda}{2} \int\limits_{-h}^{+h} g^{qq} \nabla_r v_{(i)}^{(k)r} \left[1 - (k_1 + k_2) z + k_1 k_2 z^2\right] dz. \qquad (14a)$$

$$R_{(i)}^{(k)jr} = -\mu \int\limits_{-h}^{+h} g^{sj} \nabla_s v_{(i)}^{(k)r} \left[1 - (k_1 + k_2) z + k_1 k_2 z^2\right] dz; \qquad (14b)$$

λ and μ are Lamé constants.

Equations (13) form a system of nonlinear integral-differential equations of the theory of shells. This system is intermediate between equations (8) of the linear theory and the general system of equations for "strong flexure." Equations (13) are solved approximately, using the method of focusing loads, which should be discussed separately.

Although the integral-differential equations established contain terms including the boundary conditions, the solutions obtained do not always satisfy all the conditions of the boundary value problem. An investigation on this subject performed by this author is contained in a not yet published paper. Numerical algorithms were obtained which enable us to find a solution of the statical or dynamical boundary-value problem of the theory of shells from the integral-differential equations considered above.

2. Equations of the theory of shells with clearly defined displacements of a plate whose middle plane is a "map" of the middle surface of a shell.

It is assumed that the three-dimensional problem of the theory of elasticity has been reduced to the two-dimensional problem of the theory of shells, and that the auxiliary displacements are induced in the plate by a concentrated unit force. The plate is so chosen that a one-to-one correspondence is established between points respectively on its middle

plane and on the middle surface of the shell. We then apply the reciprocity theorem. Here the integral-differential equations of the theory of shells will contain explicitly, as known functions, quantities which describe the states of stress and strain of the plate.

This method was suggested by us in /16, 17/, dealing with the equilibrium and vibrations of cylindrical shells, and was later developed in /18, 20/ to include shells of arbitrary shape. Integral-differential equations were obtained which determine the displacements due to a concentrated unit force acting along the tangent to the coordinate curves x^i at the middle surface. These equations are:

$$u_{(i)\tau}(M, N) = F^2(M)\sqrt{G_{\alpha\alpha}(M)}\, X^k_{(i)}(N)\, v_{(\tau)k}(N, M) -$$
$$- \iint\limits_{(S)} F^2(M)\sqrt{G_{\alpha\alpha}(M)}\, [K^j_{(\alpha)}(Q, M)\, u_{(i)j}(Q, N) +$$
$$+ H^j_{(\alpha)}(Q, M)\, \omega_{(i)j}(Q, N)]\, dS_Q + \oint F^2(M)\sqrt{G_{\alpha\alpha}(M)}\, [T^j_{(i)}(Q, N) \times$$
$$\times v_{(\alpha)j}(Q, M) + M^j_{(i)}(Q, N)\, \psi_{(\alpha)j}(Q, M) - S^j_{(\alpha)}(Q, M)\, u_{(i)j}(Q, N) -$$
$$- L^j_{(\alpha)}(Q, M)\, \omega_{(i)j}(Q, N)]\, ds_Q. \qquad (15)$$

The function $F^2(M)$ defines the metric in the isothermal coordinate system at the middle surface of the shell, the quantities $G_{ii}(M)$ denote the components of the metric tensor in an arbitrary orthogonal system of coordinates at the middle plane of the plate, the quantities $X^k_{(i)}(N)$ denote the contravariant components of the unit force acting on the middle surface of the shell in the direction of the coordinate curves x^i, while $\omega_{(i)j}$ and $\psi_{(\alpha)i}$ are the angles of rotation of the feet of the normals to the middle surface, corresponding respectively to the principal and auxiliary systems of the displacements. The meaning of the other notations is evident from the preceding and from the physical meaning of the quantities contained in the analytical statement of the reciprocity theorem.

This method, developed in /16-18/, was used by N. I. Remizova in /24-30/ in solving problems of the equilibrium of cylindrical shells; G. I. Tkachuk applied it to the problem of the equilibrium of an axisymmetrical shell /31, 32/, while B. N. Fradlin and S. M. Shakhnovskii applied it in /33-38/ to problems of the equilibrium of shallow shells. Paper /30/ indicates in particular, a method for obtaining integral equations containing forces and moments as unknown functions. Several new problems on the equilibrium of cylindrical shells with cut-outs were solved with the aid of this method by D. V. Vainberg and A. L. Sinyavskii /7, 9/; D. V. Vainberg, V. A. Zaruts'kii, and B. Z. Itenberg considered in /8/ the equilibrium of reinforced shells. The above method, in conjunction with methods of operational calculus, was applied to problems of dynamics of plates and shells by this author /17/, by G. E. Kazantseva /11-13/, and by B. N. Fradlin and S. M. Shakhnovskii /39/. In /18/ it was observed that the problem of the uniqueness of solutions obtained for the integral equations of statics of shells, derived from a system of integral-differential equations through a certain selection of a system of auxiliary displacements, requires separate study.

3. Direct transformation of differential equations of boundary-value problems in the classical theory of shells to systems of integral-differential and integral equations.

We have already discussed methods of setting up integral-differential and integral equations which do not enable us to determine directly that they are "inversions" of differential equations of boundary-value problems in the classical theory of shells. For example, the method considered in the first section of this paper was an independent means to reduce three-dimensional problems of the theory of elasticity to two-dimensional problems of the theory of shells; naturally it led to equations, approximating those of the three-dimensional theory of elasticity, but not literally equivalent to the equations of the classical theory of shells, which are based on the Kirchhoff-Love hypotheses. Here we can talk only about generalized equivalence, since all these equations are derived from the equations of the three-dimensional theory of elasticity by making additional assumptions.

Several works have been published lately in which the boundary-value problems of the theory of shells were "inverted" directly, thus making it possible to replace the classical statement of these problems by the problem of solving fully equivalent systems of integral and integral-differential equations. The inversion is performed by applying the latest results, obtained in the theory of boundary-value problems with respect to the solution of systems of partial differential equations up to and including the fourth order.

This method was used primarily in problems of equilibrium of shallow shells. In several cases solutions were found by quadratures. These results can be found in works by A. A. Berezovskii /1-3/, and S. P. Gavelya and A. M. Kuzenko /10/. It was shown by A. A. Berezovskii and A. F. Shestopal /2/, and also by A. A. Berezovskii /4-5/, that this method can be extended to include solution of nonlinear problems of the theory of shallow shells, in particular, to problems of their stability. N. A. Birger considered in his book /6/ a range of problems in the mechanics of shells, which are reduced to boundary-value problems for systems of ordinary differential equations, replaced by equivalent systems of integral equations. The "inversion" of systems of differential equations is performed by comparatively simple means. N. A. Birger used this method in particular to solve problems of extension and flexure of round plates (disks), and of the symmetrical deformation of axisymmetrical shells.

We note in conclusion that the integral and integral-differential equations obtained by the method considered in the second section of this paper, as well as by inversion method, contain an error, introduced with the Kirchhoff-Love hypotheses, which cannot be eliminated. This remark does not apply to the method considered in the first section.

4. The significance of the method of integral equations. Prospects for the future.

The method of integral and integral-differential equations in the theory of shells makes it possible to develop efficient methods for the numerical solution of the corresponding boundary-value problems by using modern computers. Examples of practical methods of numerical solutions for comparatively simple integral equations can be found in the aforementioned

book by Birger. Obviously, the competitiveness of the method considered cannot be determined by solving simple problems. This method should be used for solving particularly difficult boundary-value problems of the theory of shells, where its advantage over previously used methods is clear. Some problems of this class are mentioned by D. V. Vainberg and A. L. Sinyavskii /9/.

In future, more effort should be devoted to exposing and describing those problems which are solvable by the method considered, but practically unsolvable by another.

As already mentioned, the introduction of focusing kernels assists greatly in the numerical solution of integral-differential equations in the theory of shells, and makes possible their qualitative evaluation.

Establishment of integral-differential and integral equations is an efficient method for obtaining approximate solutions of boundary-value problems in the theory of shells, using various interpolation and iteration processes.

Urgent problems in the above field of the theory of shells consist in extending the method of integral and integral-differential equations to nonlinear and dynamical problems. Some works have already been published on this subject.

Programs for solving these equations by computers should be standardized, and sufficiently accurate and reliable formulas obtained.

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DEVELOPMENT OF THE METHOD OF COMPLEX TRANSFORMATION IN THE LINEAR THEORY OF SHELLS DURING THE LAST FIFTY YEARS

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The general linear theory of isotropic shells was developed to a quite advanced degree already at the end of the last century by Love, Lamb, Besset, and others. However, they mainly considered the vibrations of shells, very little attention being paid to strength calculations, since no urgent need for this existed then.

The present significance of shells as structural elements of high strength and minimum weight arose only in the twentieth century. Among the first to use shells were the designers of reinforced-concrete domes; thus, the first problem of the moment theory of shells to be solved satisfactorily was the axisymmetrical deformation of shells of revolution. In this particular case the equations of the theory of shells form a system of two ordinary second-order differential equations whose integration is difficult because the coefficients are variable. However, if we discard certain small terms containing Poisson's ratio as a multiplier, these equations become symmetric. Introducing an auxiliary complex function, we can then reduce this system to one second-order equation. This fact, discovered by Meissner about 50 years ago /1/, has played a decisive role in developing methods for analyzing symmetrically deformed shells of revolution.

Exact solutions of this second-order equation were obtained for spherical, conical, and toroidal shells. In addition, it was shown that satisfactory approximate solutions can be obtained for axisymmetrical shells of any shape, using the method of asymptotic integration. Pioneering in this field were O. Blumenthal /2/, I. Ya. Shtaerman /3/, and I. Geckeler /4/. Depending on the problem and the shape of the shell, the asymptotic solutions assume the form of either exponential functions or Bessel functions. Thus, for example, when considering the joint between a dome of arbitrary shape and its foundation, we use the exponential solution, while in considering the stresses in this dome in the neighborhood of a concentrated force acting at the vertex, we use the solution obtained in terms of Bessel functions. These functions are also contained in the asymptotic solution for a symmetrically loaded toroidal shell. Since Meissner derived the complex equations for the symmetric deformation of shells of revolution, the above approach to these problems has become fundamental.

Further technological development led to increasing use of shells; this made it necessary to solve problems far more difficult than those

mentioned above. Since the differential equations of the theory of shells form a very cumbersome eighth-order system, it became necessary to simplify it by determining which terms can be neglected, and by transforming the equations. In addition, it was also necessary to consider methods for obtaining approximate solutions for this system. This was done primarily in three ways: The first is based on the fact that in the equations of the theory of shells, the terms containing higher derivatives are small. This enables approximate solutions to be obtained by asymptotic integration. The most complete treatment of this problem, using exponential functions as the rapidly varying functions, can be found in the works of A. L. Gol'denveizer /5/.

The second approach is based on the fact that tangential displacements play a secondary role in the expressions for the changes in curvature and torsion. The idea to simplify the equations of the theory of shells by discarding terms corresponding to these displacements is due to Kh. M. Mushtari /6/. V. Z. Vlasov /7/ continued this simplification by neglecting further terms and reduced the problem of the linear theory of shells to solving a system of two symmetric partial differential equations in two unknowns, namely the normal deflection and the stress function. The error introduced by replacing the general equations of the theory of shells by this simplified system can in general be considerable; however, for many problems it is not large, so that V. Z. Vlasov's equations merit their popularity.

Finally, the third approach, due to this author, consists in generalizing Meissner's complex transformation for the equations of the symmetric deformation of axisymmetric shells, so as to include all problems of the theory of shells /8/.

These three trends should not be contrasted with one another; the first is a method for solving equations of the theory of shells, while the other two are methods for simplification and transformation of these equations, showing varying degrees of generality and exactness. V. Z. Vlasov's equations can be obtained by discarding terms in the equations of the theory of shells, given in complex form. Methods of asymptotic integration can be considered for the equations of the theory of shells, appearing in real or complex form. All three approaches are thus compatible, and we can imagine an investigation in which, for example, V. Z. Vlasov's equations are considered in their complex form, while they are solved by asymptotic integration.

We shall only discuss the method of complex transformation, its development during the last fifty years, and its place in the modern linear theory of shells.

As already mentioned, complex transformation was first applied to the problem of symmetric deformation of axisymmetric shells. Further progress became possible only after compatibility equations had been derived for arbitrary deformations of the middle surface of a shell. These relationships were given by A. L. Gol'denveizer in 1939 /9/; he also showed their complete symmetry with respect to the equilibrium equations of a shell element. This symmetry was termed statical-geometric analogy. The existence of this analogy is a necessary condition for the reducibility of the equations of the theory of shells to complex form. However, this condition alone is insufficient. Complex transformations are only possible

when the relationships between the forces and moments, on one hand, and the strains of the middle surface, on the other, are fully defined. The most consistent form, based on Kirchhoff's hypotheses, of writing the given relationships in the theory of isotropic shells, was difficult to find, this being done only in 1945 by I. Balabukh /10/ and this author /8/. Independently of one another, and by different considerations they arrived at the same form of relationships between stresses and strains in isotropic shells, which differ slightly from those due to A. Love /11/, which he calls the first approximation. It was found that reduction of the equations of the theory of shells to the complex form (without discarding any terms in the equilibrium and compatibility equations) is possible only when using precisely these relationships between forces and strains of the middle surface, and then only when Poisson's ratio is equal to zero and the shell thickness is constant.

Complex transformation was thus initially extended to any deformations of shells of arbitrary shape only for the particular case that Poisson's ratio is zero.

If there had been no further progress, this method would, evidently, be applicable only to this particular case. However, as soon as the equations of the theory of shells were reduced to the complex form in this particular case, it was found that the equations of the complex system contain several terms distinguished one from another only by factors of the order of h/R. Since Kirchhoff's hypotheses are not quite exact, the terms containing these small factors should have been discarded; this would have been equivalent to neglecting certain terms in the equilibrium and compatibility equations of a shell element. However, when the above simplifications are introduced into these equations, complex transformations become possible for any value of Poisson's ratio.

This is the method used for deriving the general equations of the linear theory of shells in complex form; a number of small terms was once and for all discarded during their derivation. These terms must be omitted, regardless of whether a complex transformation is performed or not; taking them into account would require corrections of the order of h/R in the solution, which are unimportant for thin shells.

How consistent the elimination of small terms is when performing complex transformation can be seen from the following example. A. L. Gol'denveizer /5/, p. 230) presents a characteristic equation for a circular cylindrical shell in which, as he states, "all unimportant terms have been eliminated, but not a single important term has been lost." This characteristic equation is identical with the one derived for an arbitrary value of Poisson's ratio from the general equations of the theory of shells, given in complex form.

The consistency of the simplifications introduced when deriving these equations can also be shown by applying them to an arbitrary deformation of a spherical shell and comparing the results with those obtained from the equations of the theory of shells given in real form (for any relationship between forces and moments, on the one hand, and the strains of the middle surface, on the other).

We discussed this at length, because it is sometimes said that complex transformations are accurate only in the particular case that Poisson's ratio is zero, being less exact in the general case. This is, however, not true. Simplifications introduced in order to make complex transformations

possible for any value of Poisson's ratio lead to errors in the coefficients of the differential equations of the theory of shells, which are only of the order h/R. This is also the order of magnitude of errors incurred in solving the equations.

Thus, even when, for some reason we do not wish to perform complex transformation, but prefer to use the equations of the theory of shells in real form, it is advantageous to derive the latter by separating the real from the imaginary parts in the complex equations of the theory of shells. We shall then obtain a real system, from which a large number of small terms, complicating the equations without increasing their exactitude, have been consistently eliminated.

Having considered the role of the complex transformation as a means for eliminating negligible terms from the equations of the theory of shells, we shall now consider its value in solving specific problems. When the equations of the theory of shells are written in complex form, their order is reduced by a factor of two, and they become much less cumbersome. Thus, for example, for axisymmetric shells, the equations in complex form can be reduced to a system of two equations with two unknowns, these being the complex auxiliary functions. The given system has quadratures for symmetric and antisymmetric loads and reduces in these cases to a single second-order differential equation. This result is not new for symmetric loads, being substantially identical with Meissner's result, discussed previously. For antisymmetric loads this result was not known; its discovery is a merit of the complex-transformation method /12/. Of course, this result can be also obtained directly from the equations in real form without complex transformation. However, this would take much longer.

A specific problem of practical importance which has been solved by complex transformation is the analysis of toroidal shells under symmetric and antisymmetric loads. In this case it is impossible to separate the stresses into membrane loads and boundary effects. The standard (exponential) asymptotic method cannot be used here either. All this makes it very difficult to investigate toroidal shells. However, these difficulties have now been mainly overcome (by R. Clark /13/, the present author and E. Zenova /14/, K. F. Chernykh (dissertation, 1955), V. S. Chernina /15/, S. A. Tumarkin /16/, and many others).

Complex transformations yield some results also in the case of more general deformations of axisymmetric shells. In particular, it was shown that an exact solution of the complex equations can be obtained for axisymmetric shells having parallel circles as boundaries /17/. A solution was obtained for an arbitrarily loaded catenary shell bounded by two parallel arcs /77/.

Complex equations for cylindrical shells of arbitrary cross section can be reduced to a single ordinary fourth-order differential equation, which differs by only one term from the complex equation corresponding to those due to V. Z. Vlasov /7/. Consideration of this term is primarily important with long shells.

The application to circular cylindrical shells of the complex equations of the general theory of cylindrical shells results in a biquadratic characteristic equation whose consistency has already been discussed. If, however, we proceed from the equations of the theory of circular cylindrical shells, given in real form, we obtain a fourth-order characteristic

equation of the most general form, whose simplification requires additional consideration. We then obtain finally (as was shown by A. L. Gol'denveizer /5/) the same biquadratic equation which follows immediately from the basic complex equation of the theory of cylindrical shells. This last equation was used by this author and some others in solving specific problems.

For example, R. L. Malkina (/18/ and others) considered the deformation of a cylinder of noncircular cross section. To this conference was presented a paper by Yu. S. Dem'yanovich, devoted to the analysis of a cylindrical shell loaded by a concentrated force. Although this problem has already been solved by V. M. Darevskii (proceeding from the equations of a circular cylindrical shell, given in real form /19/), Yu. Dem'yanovich's paper is nevertheless of methodological interest. It shows how much the use of complex equations simplifies both the intermediate calculations and the final result. Quite simple expressions are obtained for the coefficients of the unary trigonometric series which represent the solution; in addition, it is unnecessary to separate the real and imaginary parts in the symbolic form. This should be done only after the numerical values for the specific problem are substituted in the coefficients of the series.

The results due to R. L. Malkina have similar advantages.

It is frequently claimed that the use of complex equations of the theory of shells presents no advantage in solving specific problems, since the boundary conditions cannot usually be formulated in terms of complex auxiliary functions (complex forces or moments, or complex displacements), but must be expressed by their real or imaginary parts. However, the principal way for solving problems of the theory of shells is still the method of separation of variables, by which equations with ordinary derivatives are obtained. The order of the complex equations is half that of the corresponding real equations; this facilitates their exact or approximate solution.

We return to the symmetric (or antisymmetric) problem for the toroidal shell. Here, both the solution of the homogeneous equations and the finding of particular solutions required nontrivial approaches. The particular solutions could not be obtained from the membrane equations. The asymptotic solution could not be found from exponential functions. It was much easier to understand how these difficulties could be overcome by studying a single complex second-order equation than a system of two fourth-order equations which, in addition, contained a number of small terms (of the order of h/R).

Although, as a rule, real and imaginary parts have to be separated when subjecting the solutions of specific problems to the boundary conditions, the most general solution is, nevertheless, in many cases not only the shortest, but also the easiest to obtain from complex equations.

There exist also types of boundary conditions which permit complex formulation, i. e., which are written in terms of complex auxiliary functions. The solutions of the corresponding problems can, through complex transformations, be reduced to final expressions without having to separate real and imaginary parts at any intermediate stage. An example of such a boundary condition is a free edge on a rigid support which permits sliding along the normal to the edge. In /17/, and also in the aforementioned works by R. L. Malkina and Yu. S. Dem'yanovich, there

are problems which are solved by complex transformations under these boundary conditions. Another important type of problem which does not require separation of real and imaginary parts when subjecting the solution, found in complex form, to the boundary conditions, is that in which the joint between two shells of different shape but equal thickness is considered. In practice it is sometimes necessary to analyze composite shells formed of cylindrical, toroidal, spherical, etc., shells. Examples are box-section bottoms or steam turbine casings. It was shown by K. F. Chernykh /20/ that the boundary conditions can in these cases be formulated in terms of complex auxiliary functions (for "smooth" joints, as well as for joints forming angles). This result was used by V. K. Naumov /21/ to determine the deformations in steam turbine casings. It is used also to analyze structures of ship hulls forming cylinders whose generators consist of several circular arcs.

These boundary conditions do not include all the cases which can be encountered in practice; neither are they the principal types of boundary conditions. However, there also exist and are of significance, types of boundary conditions for which the advantages of complex transformation can be fully exploited. Even if the boundary conditions require separation of real and imaginary parts in the complex auxiliary functions, the method considered has still many advantages which have been already discussed and need not be repeated.

The method of complex transformation, created fifty years ago to solve a particular problem of the theory of shells and generalized 17 years ago to include any problems of this theory, is still being developed. During the last five years important results in this direction were obtained by K. F. Chernykh who, firstly, found that it is possible to set up in complex form the boundary conditions at the joint between two shells (already discussed), secondly, derived in complex form the variational principle of the theory of shells /20/ and, thirdly, studied in detail a version of the complex equations, known as equations in complex displacements /20/. He showed that this version can be appreciably simplified, after which it is in many cases more convenient than the equations in complex forces or moments, predominantly used by this author. Equations in complex displacements were also used by Gol'denveizer in /22/. This version of the complex transformation will evidently be developed further. On its basis K. F. Chernykh gave a solution to Saint Venant's problem for curved pipes with round cross sections /25/. He also showed in one of his latest, still unpublished, works, that equations in complex displacements can be considered as a second approximation of V. Z. Vlasov's equations.

Other studies on complex transformations in the theory of shells, which appeared in the last few years and deserve mention, are those by the Rumanian scientists V. Visarion, C. Stanescu, and L. Librescu (/24, 23/, etc.), in which this method is extended to include some types of anisotropic and laminated shells.

In concluding this survey, I wish to note that the method of complex transformations in the linear theory of shells has (as any other) not only merits but also shortcomings. Firstly, it is not all-encompassing (even within the scope of the linear theory) and cannot be extended in general form to include shells of variable thickness and anisotropic shells. It is inapplicable to dynamic problems of shells and to problems of their stability.

No complex transformation exists in the nonlinear theory of shells (this being true of both geometric and physical nonlinearity). Secondly, the advantages presented by this form of equation are not always so weighty as to give them preference to ordinary (real) equations.

Although unable to solve each and every problem of the theory of shells, the method of complex transformations has a sufficiently wide range of application. Its possibilities should be remembered.

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NONLINEAR PROBLEMS OF STRESS CONCENTRATION NEAR HOLES IN PLATES

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§ 1.

The appearance in the last few years of new synthetic materials, and also the more extensive use of structures consisting of thin plates and shells, make it necessary to formulate the problems of the theory of elasticity more exactly. The problems then become, as a rule, generally nonlinear (both physically and geometrically), or either geometrically or physically nonlinear.

The nonlinearity of many problems of the theory of elasticity is no innovation in this theory. The basis for the nonlinear theory was established together with the science of elasticity. However, the subsequent development of this science was concerned primarily with the linear theory of elasticity /32/, in which many results valuable and useful, both from the theoretical and from the engineering viewpoints were obtained. In particular, powerful and effective methods to solve entire classes of problems were developed in the linear theory. Foremost amongst these are the solutions of the plane problem of the theory of elasticity,* developed by Academician N. I. Muskhelishvili /1/ and his students; these are methods of complex potentials in conjunction with conformal mapping and Cauchy-type integrals, which make it possible to reduce the principal problems of the two-dimensional theory of elasticity** to boundary-value problems of the theory of analytic functions of a complex variable.

These methods were also found to be most effective in the nonlinear plane problem of the theory of elasticity, in particular in the problem of stress concentration near holes. However, due to the mathematical difficulties, the number of works devoted this problem is still extremely small. The list of original works given at the end of this article contains only about twenty titles.

* [The Russian word ploskoi has been translated as plane when appearing together with the words problem, stress, strain and their combinations, while when appearing with the word theory it was translated as two-dimensional. These two terms are synonymous (at least here) and the above use is, in this translator's opinion, in accordance with convention (see for example, titles of Refs. /2-4/).]

** [The Russian word deformatsiya which, again, has two related meanings, that is, deformation and strain, was again translated, accordingly, in keeping with the particular context of the given sentence. As in the case of ploskoi the choice represents the translator's judgment in the matter.]

§ 2.

The first works on the nonlinear plane problem of the theory of elasticity and, in particular, on stress concentration near round holes, were by Adkins, Green, and Shield /2/, Adkins and Green /4/, Adkins, Green, and Nicholas /3/, in which they established the basic system of equations of the nonlinear theory of elasticity for both compressible and incompressible materials, for plane stress as well as for plane strain,* for the most general elasticity relationships

$$\tau^{ij} = \frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_1} g^{ij} + \frac{2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial I_2} B^{ij} + 2\sqrt{I_2} \frac{\partial W}{\partial I_3} G^{ij}, \qquad (1)$$

where τ^{ij} are contravariant components of the stress tensor referred to a system of curvilinear coordinates in the deformed body; $W(I_1, I_2, I_3)$ is the strain-energy density; I_r $(r = 1, 2, 3)$ are the invariants of the strain tensor

$$I_1 = g^{ij} G_{ij}; \quad I_2 = I_3 g_{ij} G^{ij}; \quad I_3 = \frac{G}{g}; \quad B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs};$$
$$|G_{ij}| = G; \quad |g_{ij}| = g;$$

while g^{ik} and G^{ik} are respectively the contravariant components of the metric tensors of the undeformed and deformed state of the elastic body.

Introducing the stress function U which satisfies the equilibrium conditions, the authors of /2-4/ obtained a complete basic system of equations of the plane nonlinear problem of the theory of elasticity, consisting of two equations for the determination of the functions U and D in the case of plane strain, and of [three equations] in the three functions U, D, and $\lambda = \frac{h}{h_0}$ for the generalized plane state of stress, i.e., for the case of a thin plate, where h_0 and h are the half-thicknesses of the plate before and after deformation respectively.

The principal system of equations for the generalized plane state of stress (i.e., for a thin plate), expressed in terms of $(z, \bar{z})$ coordinates, is after deformation

$$\begin{gathered}
\frac{\partial^2 U}{\partial z^2} - 4h_0 \frac{\sqrt{I_3}}{\lambda} \left[\frac{\partial W}{\partial J_1} + (\lambda^2 - 2) \frac{\partial W}{\partial J_2} - (\lambda^2 - 1) \frac{\partial W}{\partial J_3} \right] \frac{\partial \bar{D}}{\partial \bar{z}} \left(\frac{\partial D}{\partial z} - 1 \right) = 0; \\
\frac{\partial^2 U}{\partial z \partial \bar{z}} - 2h_0 \left\{ \frac{\partial W}{\partial J_1} + \left(\frac{\sqrt{I_3}}{\lambda} + \lambda^2 - 2 \right) \frac{\partial W}{\partial J_2} + \right. \\
\left. + (\lambda^2 - 1) \left(\frac{\sqrt{I_3}}{\lambda} - 1 \right) \frac{\partial W}{\partial J_3} \right\} - 4h_0 \frac{\sqrt{I_1}}{\lambda} \left\{ \frac{\partial W}{\partial J_1} + \right. \\
\left. + (\lambda^2 - 2) \frac{\partial W}{\partial J_2} - (\lambda^2 - 1) \frac{\partial W}{\partial J_3} \right\} \frac{\partial D}{\partial \bar{z}} \cdot \frac{\partial \bar{D}}{\partial z} = 0; \qquad (2) \\
\frac{\partial W}{\partial J_1} + 2\left(\frac{\sqrt{J_3}}{\lambda} - 1 \right) \frac{\partial W}{\partial J_3} + \left(\frac{\sqrt{I_3}}{\lambda} - 1 \right)^2 \frac{\partial W}{\partial J_3} + \\
+ 4 \frac{I_3}{\lambda^2} \left(\frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3} \right) \frac{\partial D}{\partial \bar{z}} \cdot \frac{\partial \bar{D}}{\partial z} = 0;
\end{gathered}$$

* [See footnotes on previous page.]

where

$$I_3 = -\frac{\lambda^2}{4a}\ ;\quad a_{\alpha\beta} = g_{\alpha\beta}\ ;\quad g_{33} = \frac{1}{\lambda^2}\ ;$$

$$g = \frac{a}{\lambda^2}\ ;\quad a = |a_{\alpha\beta}|\,.$$

$$J_1 = \lambda^2 + 2\frac{\sqrt{I_3}}{\lambda} + 4\frac{I_3}{\lambda^2}\cdot\frac{\partial D}{\partial \bar z}\cdot\frac{\partial D}{\partial z} - 3\,;$$

$$J_2 = 3 - 2\lambda^2 + \frac{I_3}{\lambda^2} + 2(\lambda^2 - 2)\left[\frac{\sqrt{I_3}}{\lambda} + \frac{I_3}{\lambda^2}\cdot\frac{\partial D}{\partial \bar z}\cdot\frac{\partial \bar D}{\partial z}\right];$$

$$J_3 = (\lambda^2 - 1)\left[\left(1 - \frac{\sqrt{I_3}}{\lambda}\right)^2 - 4\frac{I_3}{\lambda^2}\cdot\frac{\partial D}{\partial \bar z}\cdot\frac{\partial \bar D}{\partial z}\right].$$

The principal system in $(z;\bar z)$ coordinates for plane strain has the same form. These systems can also be written in the coordinates $(\eta, \bar\eta)$ before deformation.

It is easy to see that system (2) is so complicated that only approximate solutions can be obtained.

It was pointed out in /2-4/ that this can be done by the method of the small parameter, which consists in that the unknown functions U, D, and λ, and also all known functions entering in these equations, are represented in the form of expansions

$$\begin{aligned} U &= H_0\,\varepsilon\,[U^{(1)} + \varepsilon\,U^{(2)} + \varepsilon^2\,U^{(3)} + \cdots] \\ D &= \varepsilon\,D^{(1)} + \varepsilon^2 D^{(2)} + \varepsilon^3 D^{(3)} \\ \lambda &= 1 + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \cdots \end{aligned} \tag{3}$$

in powers of the small parameter ε, where $D = z - \eta = u + iv$ is a complex function of displacements, H_0 is a constant, equal to, in the case of plane strain, the shear modulus μ of the material at infinitesimal deformations, and equal to $2h\mu$ in the case of the generalized plane state, i.e., for a thin plate.

Introducing (3) and similar expressions into the principal systems of equations, and equating to zero the coefficients of equal powers of ε, we obtain an infinite system of equations for the unknown quantities $U^{(j)}$, $D^{(j)}$, and $\lambda^{(j)}$ $(j = 1, 2, 3)$, which are the j-th approximation of the required solution.

The first approximation, i.e., the equations for $U^{(1)}$, $D^{(1)}$, and $\lambda = 1$, corresponds to the classical linear theory and leads to the known Kolosov-Muskhelishvili relationships for the first-order complex potentials. The procedure for finding these potentials is known /1/; they have been found for many shapes of holes and types of loading /9/.

The principal system of equations can be expressed in complex coordinates $(z, \bar z)$ after deformation, as well as in the coordinates $(\eta, \bar\eta)$, before deformation. The potentials can therefore be expressed in either system of coordinates.

The equations for functions $\dfrac{\partial^2 U^{(2)}}{\partial \bar z^2}$ and $D^{(2)}$, expressed as second-order potentials in $(z, \bar z)$ coordinates, are /4, 16/

$$\frac{\partial U^{(2)}}{\partial \bar z} = \varphi^{(2)}(z) + z\,\overline{\varphi'^{(2)}(z)} + \overline{\psi^{(2)}(z)} - f(z, \bar z);$$
$$D^{(2)} = k\,\varphi^{(2)}(z) - z\,\overline{\varphi^{(2)}(z)} - \overline{\psi^{(2)}(z)} - f_1(z, \bar z); \tag{4}$$

where $f(z, \bar z)$ and $f_1(z, \bar z)$ are known functions, expressed in terms of the complex potentials $\psi^{(1)}(z)$ and $\varphi^{(1)}(\bar z)$ of the first approximation, and k is a known constant.

The stress tensor components referred to $(z, \bar z)$, the coordinates of points of the body in the deformed state, are

$$\tau^{11} = \tau^{22} = \sigma_{y2} - \sigma_{y1} + i2\tau_{y1y2} = -4\,\frac{\partial^2 U}{\partial \bar z^2};$$
$$\tau^{12} = \sigma_{y1} + \sigma_{y2} = 4\,\frac{\partial^2 U}{\partial z\,\partial \bar z}; \tag{5}$$
$$z = y_1 + iy_2; \quad \bar z = y_1 - iy_2.$$

The complex second-order potentials $\varphi^{(2)}(z)$ and $\psi^{(2)}(z)$ were investigated for the case of a finite or infinite multiply-connected region, and the principal boundary-value problems of the nonlinear plane problem of the theory of elasticity were formulated by G. N. Savin and Yu. I. Koifman /16/.

In contrast with the classical linear theory of elasticity, new ways to formulate boundary-value problems are possible in the nonlinear theory of elasticity in determining second-order potentials.

Thus, the following three approaches are possible for the first principal problem:

1) the external forces are specified at a known contour of the deformed body;

2) the external forces are specified at a known contour of the undeformed body;

3) the boundary of the region is specified for the undeformed body, while the external forces are specified at an unknown contour of the deformed body.

The approach to the second principal problem is similar /16/. The complex second-order potentials $\varphi^{(2)}(z)$ and $\psi^{(2)}(z)$ for the case of an unbounded region, the stresses at infinity being bounded, are

$$\varphi^{(2)}(z) = -\frac{\bar X^{(2)} + i\,\bar Y^{(2)} - E_1}{2\pi(k+1)}\ln z + \frac{N_1 + N_2}{4}z + \varphi_0^{(2)}(z);$$
$$\psi^{(2)}(z) = \frac{k\,(\bar X^{(2)} - i\,\bar Y^{(2)}) + E_2}{2\pi(k+1)}\ln z - \frac{(N_1 - N_2)\,e^{-2i\vartheta}}{2}z + \psi_0^{(2)}(z); \tag{6}$$

where N_1 and N_2 are the principal stresses at infinity, ϑ is the angle between N_1 and Oy_1, while E_1 and E_2 are known constants whose expanded expressions are given in /16/.

It should be noted that in none of the above works is the problem of the convergence of expansions (3) for U, D, and λ even raised.

In deriving the principal system of equations of the two-dimensional nonlinear theory of elasticity, no limitations are imposed on the strain-energy density W.

For incompressible materials (in particular rubber which within wide limits — up to 50% elongation — satisfies the incompressibility condition),

the function W was selected in /2-4, 16/, in the form

$$W = C_1 (I_1 - 3) + C_2 (I_2 - 3); \tag{7}$$

where $C_1 > 0$ and $C_2 > 0$ are the elastic constants of the material, which are determined experimentally. This function $W(z)$ was first proposed by Moony /8/.

No limitations are imposed on W in /2-4, 16/ when deriving general equations of the two-dimensional theory of elasticity and establishing an approximative method for their solution, although in the classical linear theory, the function W is subjected to strictly specified conditions: in order that the ordinary boundary-value problems of the theory of elasticity have a unique solution (Kirchhoff's theorem), the function W must be a positive-definite homogeneous quadratic form. In the nonlinear theory of elasticity, the question of what functions can serve as the strain-energy density of an elastic material, has not yet been solved /19/.

It follows from (1) that to the selected function W there will correspond values of τ^{ij}, i.e., it will have a single elastic body that will correspond to this W.

In the general case, i.e., in the nonlinear theory of elasticity, the function $W(I_1, I_2, I_3)$ cannot be quite arbitrary; this follows from the fact that in the case of infinitesimal deformations, when all bodies obey Hooke's law, this function must be a positive-definite quadratic form.

The papers of T. Doyle and G. Eirksen /19/, and Truesdell /22/, contain a survey of works on this problem; paper /19/ presents a formulation of the conditions necessary for an incompressible material

$$dW > 0;$$
$$\frac{dW}{dI_1} + (1 + \delta_i)^2 \frac{\partial W}{\partial I_2} > 0, \quad (i = 1, 2) \tag{8}$$

where δ_i are the principal strains. In the linear theory the second equation (8) states that the shear modulus μ must be positive.

Adkins and Green /2/ were the first to publish, in 1953, an approximate solution (accurate to a second approximation) of Kirsch's problem, i.e., of the problem of the stress distribution near a round hole for the case of the plane deformation of an incompressible material in a uniaxial state of stress: $\sigma^{(\infty)} = p =$ const at infinity. Slightly later, these authors also considered in /4/ among other problems concerning round holes, the stress concentration near a round hole in a thin, elastic (both geometrically and physically nonlinear) plate subjected to uniform tension p = const at infinity. By a slightly different approach than that used in /2-4/, stress concentration near a round hole was considered by L. A. Tolokonnikov /13/, and I. N. Slezinger and S. D. Barskaya /14/.

L. A. Tolokonnikov /13/ obtained the compatibility conditions for finite plane strain in an incompressible material, expressed in terms of the invariant strain characteristic, taken as the rate of deformation R_i. He also assumed that the physical deformation is described by relationships between R_i and the octahedral tangential stress τ_i.

On this assumption, proceeding from the compatibility and equilibrium equations which are satisfied by introducing the stress function,

L. A. Tolokonnikov obtained the resolving equation in terms of stresses for the problem of finite plane strain in an incompressible material. This equation is also integrated with the aid of the method of the small parameter. He also considered (in complex coordinates of the undeformed body) the stress concentration near a round hole (plane strain) whose contour was initially round, in uniaxial compression (tension), assuming that the law of deformation is

$$\tau_i = \mu \, tgR; \quad R = 2\sqrt{1.5}\, R_i. \tag{9}$$

Here the stress-concentration factor in compression k^{**} is obtained as third approximation:

$$k^{**} = 3\left(1 + 0.2\,\frac{p}{\mu} - \frac{p^2}{\mu^2}\right). \tag{10}$$

For $0 < \frac{p}{\mu} < 0.2$ $(p > 0)$ the stress-concentration factor increases with the load, and decreases for $\frac{p}{\mu} > 0.2$. It is assumed that this is due to the physical nonlinearity of the material, whose effect increases as the deformation proceeds. It appears that the second approximation describes basically the geometric nonlinearity of the problem.

I. N. Slezinger and S. D. Barskaya /14/ obtained for plane strain a resolving system of equations in terms of displacements. They assumed that the problem is linear geometrically but not physically, since they used a law due to N. V. Zvolinskii and P. M. Riz /34, 35/ governing the physical deformations, which states that there exists a linear relationship between the principal stresses, referred to the initial cross-sectional area, and the princiapl elongations.

Using the resolving equation thus found, the authors of /14/ employed the method of the small parameter to obtain an approximate solution of the problem of stress concentration near a round (in the undeformed state) hole in plane strain for the case of uniaxial stresses at inifinity.

Expressions are given (including second-order terms) for the stress components σ_θ and for the displacement components, as well as the expression below and Table 1, containing values of the stress-concentration factor in tension.

$$k = \frac{3 + 6\,(2 - \chi)\,\alpha}{1 + 0.5\,(3 - \chi)\,\alpha}. \tag{11}$$

$$\alpha = \frac{p}{4\mu}; \quad \chi = \frac{\lambda + 3\,\mu}{\lambda + \mu}.$$

where p is the stress at infinity, referred to the initial cross-sectional area, while λ and μ are the Lamé constants.

It follows from Table 1 that for the given elasticity relationships the stress-concentration factor increases in comparison with the linear theory according to which $k^\circ = 3$.

The principal systems of equations of the nonlinear two-dimensional problem of the theory of elasticity, obtained in /2-4/, and the approximate

methods for their solution, considered there for the elementary cases of a round hole or a round washer, show that for other shapes of holes or for noncircular bodies, the problem can only be solved by using the Kolosov-Muskhelishvili complex potentials in conjunction with conformal mapping and Cauchy-type integrals.

TABLE 1

ν^* \ $\frac{p}{\mu}$	0.04	0.08	0.12	0.16	1.20
$\frac{1}{4}$	2.995	2.980	2.955	2.941	2.927
$\frac{1}{2}$	3.030	3.060	3.081	3.115	3.143

* ν is the Poisson ratio.

In this way, Yu. I. Koifman /15/ obtained an approximate (accurate to the second approximation) solution of the problem of stress concentration near an elliptical (in particular, also near a round) hole in the cases of plane strain and of a generalized state of stress, i. e., for a thin plate in a homogeneous state of stress at infinity.

He determined the complex first- and second-order potentials for different conditions at infinity, and found expressions for the stresses σ_ϑ^* and σ_ϑ^{**} at the contours of round and elliptical holes, whose shapes are given both for the deformed and the undeformed body.

Some of these results are presented below:

1. Stress concentration near a round hole

a) Uniaxial tension or compression.

1. Consider the stress concentration at the contour of a hole which is round in the deformed state.

The stresss σ_ϑ^* at the contour of a hole are for uniaxial tension or compression

$$\sigma_\vartheta^* = N\left\{1 - 2\cos 2\vartheta + \frac{\gamma N}{4H_0}\left[(3 - k\delta) - 4\cos 2\vartheta + 4\cos 4\vartheta\right]\right\}, \qquad (12)$$

where ϑ is the polar angle in the deformed body; N is the principal stress at infinity; γ, k, and δ are elastic constants whose values are given in /4, 16/; H_0 is equal to the shear modulus μ for plane strain, and to $2h\mu$ for a thin plate; $2h$ is the thickness of the plate after deformation. The stress-concentration factor is

$$K^* = \frac{(\sigma_\vartheta^*)_{\max}}{N} = 3\left[1 + \frac{\gamma(11 - k\delta)}{12} \cdot \frac{N}{H_0}\right]. \qquad (13)$$

In particular, for incompressible materials we obtain (12) and (13):

for plane strain*

$$\sigma_\vartheta^* = N\left[1 - 2\cos 2\vartheta + \frac{N}{4\mu}\ (1 - 2\cos 2\vartheta + 2\cos 4\vartheta)\right]; \tag{14}$$

$$K^*_{\text{p.s.}} = 3\left(1 + \frac{5}{12}\frac{N}{\mu}\right); \tag{15}$$

for a [thin] plate, i. e., for a generalized two-dimensional state of stress

$$\sigma_\vartheta^* = N\left\{1 - 2\cos 2\vartheta + \frac{5+l}{32(1+l)} \cdot \frac{N}{H_0}\left[\frac{2\,(21+l)}{3\,(5+l)} - 4\cos 2\vartheta + 4\cos 4\vartheta\right]\right\}; \tag{16}$$

$$K^*_{\text{pl}} = 3\left[1 + \frac{81 + 13l}{144\,(1+l)}\frac{N}{H_0}\right]; \tag{17}$$

where $l = \frac{C_2}{C_1}$; C_1 and C_2 are Moony constants.

Table 2 gives stress-concentration factors obtained from (15) and (17) for different values of $\frac{N}{H_0}$. When using (17) it was assumed that $l = \frac{1}{19}$.

TABLE 2

K \ N/H_0	−0.3	−0.2	−0.1	0.1	0.2	0.3	According to linear theory
$K^*_{\text{p.s.}}$	2.625	2.750	2.875	3.125	3.250	3.375	3
K^*_{pl}	2.514	2.676	2.838	3.162	3.344	3.486	3

From (15) and (17), and Table 2, it is evident that introducing nonlinear corrections results in values of K^* which differ considerably from those given by the linear theory.

When comparing this with the results of the linear theory it must be remembered that a final circular contour can be obtained either by extending a plane having an elliptical hole in the direction of the latter's minor axis, or by compressing in the direction of the major axis. It is evident that the greater the tensile force required to obtain a round contour (tension in direction of minor axis), the higher will be the stress concentration at the contour. On the other hand, the greater the compressive force (compression in direction of major axis), the smaller becomes the stress concentration at the contour of the hole.

2. Consider the stress concentration at the contour of a hole which was round before deformation. As a result of deformation this hole has become oval, its shape depending on the conditions at infinity (elongated in the direction of the force when subjected to tension and perpendicular to the direction of the force, when under compression).

The stresses σ_ϑ^{**} along the contour of an initially round hole are after deformation

$$\sigma_\vartheta^{**} = N\left\{1 - 2\cos 2\vartheta + \frac{2\gamma - k - 1}{8}\frac{N}{H_0}\left[\frac{2\gamma\,(3 - k\delta)}{2\gamma - k - 1}4\cos 2\vartheta + 4\cos 4\vartheta\right]\right\}. \tag{18}$$

* Equation (14) was first given in /2/.

where ϑ is the polar angle in the undeformed plane. The remaining notation is the same as in (12).

Using (18) we obtain the stress-concentration factor

$$K^{**} = \frac{(\sigma_\vartheta^{**})_{\max}}{N} = 3\left[1 + \frac{\gamma(11 - k\delta) - 4(k+1)}{12} \cdot \frac{N}{H_0}\right]. \tag{19}$$

For an incompressible material we find from (18) and (19) for plane strain

$$\sigma_\vartheta^{**} = N\left[1 - 2\cos 2\vartheta + \frac{N}{4\mu}(1 + 2\cos 2\vartheta - 2\cos 4\vartheta)\right]; \tag{20}$$

$$K_{\text{p.s.}}^{**} = 3\left[1 - 0.25\frac{N}{\mu}\right]; \tag{21}$$

for a thin plate

$$\sigma_\vartheta^{**} = N\left\{1 - \cos 2\vartheta + \frac{17 + 29l}{96(1+l)} \cdot \frac{N}{H_0}\left[\frac{2(21+l)}{17+29l} + 4\cos 2\vartheta - 4\cos 4\vartheta\right]\right\}; \tag{22}$$

$$K_{\text{pl}}^{**} = 3\left[1 - \frac{47 + 115l}{144(1+l)} \cdot \frac{N}{H_0}\right]. \tag{23}$$

Table 3 gives stress-concentration factors obtained from (21) and (23) for the same values of the elastic constants as used in Table 2.

TABLE 3

K \ N/H_0	−0.3	−0.2	−0.1	0.1	0.2	0.3	According to linear theory
$K_{\text{p.s.}}^{**}$	3.225	3.150	3.075	2.925	2.850	2.775	3
K_{pl}^{**}	3.315	3.210	3.105	2.895	2.790	2.685	3

Comparing the numerical values given in Tables 2 and 3, we notice that they differ. We can assume this to be due to changes in the shape of the hole as a result of deformation. Thus, under 2, when the tension increases, the initially circular hole becomes increasingly flatter in the direction perpendicular to the force (N at inifinity); in this case the stress concentration at the contour point $\vartheta = \frac{\pi}{2}$ will decrease. On the other hand, under compression the round hole becomes flatter in the direction of the force, so that the stress concentration at the contour point $\vartheta = \frac{\pi}{2}$ will increase.

b) All-sided tension or compression.

In all-sided tension or compression the hole retains its initial round shape, only its radius changing. This does not affect the stresses near the hole.

In this case $\sigma_\vartheta^* = \sigma_\vartheta^{**}$; hence the stress-concentration factor is

$$K = K^* = K^{**} = 2\left[1 + 0.5\,\gamma\,\frac{N}{H_0}\right]. \tag{24}$$

Values of K obtained from (24) for an incompressible material, both for plane strain and for a thin plate are given in Table 4 for different values of $\frac{N}{H_0}$.

TABLE 4

K \ N/H_0	−0.3	−0.2	−0.1	0.1	0.2	0.3	According to linear theory
$K_{p.s.}$	1.85	1.90	1.95	2.05	2.10	2.15	2
K_{pl}	1.82	1.88	1.94	2.06	2.12	2.18	2

This table shows the influence of nonlinear corrections in tension and compression.

c) Pure shear.

1. Consider a plane weakened by a round hole in the deformed state and subjected to pure shear. The stresses σ_ϑ along the contour of the hole are

$$\sigma_\vartheta^* = 4N\left[\cos 2\vartheta + \frac{\gamma N}{4H_0}\,(2 - k\delta + 4\cos 4\vartheta)\right]. \tag{25}$$

From (25) we obtain the stress-concentration factor

$$K^* = 4\left[1 + \frac{\gamma\,(6 - k\delta)}{4}\cdot\frac{N}{H_0}\right]. \tag{26}$$

For an incompressible material we find from (25) and (26) in the case of plane strain

$$\sigma_\vartheta^* = 4N\left[\cos 2\vartheta + \frac{N}{8\mu}\,(1 + 4\cos 4\vartheta)\right]; \tag{27}$$

$$K^*_{p.s.} = 4\left[1 + 0.625\,\frac{N}{\mu}\right]; \tag{28}$$

for a thin plate

$$\sigma_\vartheta^* = 4N\left\{\cos 2\vartheta + \frac{5 + l}{32(1 + l)}\cdot\frac{N}{H_0}\left[\frac{27 - l}{3\,(5 + l)} + 4\cos 4\vartheta\right]\right\}; \tag{29}$$

$$K^*_{pl} = 4\left[1 + \frac{87 + 11\,l}{96(1 + l)}\cdot\frac{N}{H_0}\right]. \tag{30}$$

Table 5 gives some values of K^*_{pl} from (30) and $K^*_{p.s.}$ from (28) for different values of $\frac{N}{H_0}$.

TABLE 5

K^* \ N/H_0	0.1	0.2	0.3	According to linear theory
$K^*_{p.s.}$	4.25	4.50	4.75	4
K^*_{pl}	4.35	4.69	5.04	4

This table shows the considerable effect of nonlinear corrections to the linear theory.

2. Consider pure shear of a plane, weakened by a hole which before deformation was a circle of arbitrary radius.

The stresses σ^{**}_ϑ along the contour of the hole are

$$\sigma^{**}_\vartheta = 4N\left\{\cos 2\vartheta + \frac{2\gamma - k - 1}{8}\cdot\frac{N}{H_0}\left[\frac{2\gamma(2-k\delta)}{2\gamma - k - 1} + 4\cos 4\vartheta\right]\right\}. \qquad (31)$$

The stress-concentration factor is

$$K^{**} = 4\left[1 + \frac{\gamma(6-k\delta) - 2(k+1)}{4}\cdot\frac{N}{H_0}\right] \qquad (32)$$

For an incompressible material we obtain

in the case of plane strain

$$\sigma^{**}_\vartheta = 4N\left[\cos 2\vartheta + \frac{N}{8H_0}(1 - 4\cos 4\vartheta)\right]; \qquad (33)$$

$$K^{**}_{p.s.} = 4\left[1 - 0.375\frac{N}{\mu}\right]; \qquad (34)$$

for a thin plate

$$\sigma^{**}_\vartheta = 4N\left\{\cos 2\vartheta + \frac{17+29l}{96(1+l)}\cdot\frac{N}{H_0}\left[\frac{27-l}{17-29l} - 4\cos 4\vartheta\right]\right\}; \qquad (35)$$

$$K^{**}_{pl} = 4\left[1 - \frac{41-117l}{96(1+l)}\cdot\frac{N}{H_0}\right]. \qquad (36)$$

Table 6 gives values of $K^{**}_{p.s.}$ from (34) and K^{**}_{pl} from (36) for some values of $\frac{N}{H_0}$.

Tables 5 and 6 show the effect on the stress-concentration factor of a change in the initial shape of the hole.

An investigation of the stresses near the hole shows that in this case the stress distribution along the hole contour differs appreciably from that obtained under 1. Thus, with increasing load the stress concentration at the contour point $\vartheta = 0$ decreases, $(\sigma^{**}_\vartheta)_{max}$ being obtained at $\vartheta = \frac{\pi}{2}$; however,

here, in contrast to 1, the maximum* stresses are compressive.

TABLE 6

K^{**} \ N/H_0	0.1	0.2	0.3	According to linear theory
$K^{**}_{p.s.}$	4.15	4.30	4.45	4
K^{**}_{pl}	4.19	4.37	4.56	4

2. Stress concentration near an elliptical hole

a) Extension in direction of major axis of elliptical hole.

1. In the deformed state the hole forms an ellipse with a as major and b as minor semiaxis.

The data refer to the case of plane strain of an incompressible material, with $m = \frac{a-b}{a+b} = \frac{1}{3}$.

The stress-concentration factor is

$$K^{*}_{p.s.} = 2\left(1 + 0.225\frac{N}{\mu}\right). \tag{37}$$

It is seen from (37) that taking into account the nonlinearity leads to an increase in the stress-concentration factor in comparison with the linear theory ($K_0 = 2$) when the tension is increased, and to a decrease when the compression is increased.

2. The hole was elliptical before deformation.

In this case the stress-concentration factor is

$$K^{**}_{p.s.} = 2\left(1 - 0.163\frac{N}{\mu}\right). \tag{38}$$

For the same values of $\frac{N}{\mu}$, (38) gives a lower value for $K^{**}_{p.s.}$ than (37). This may be due to the fact that the contour of the hole becomes flatter in tension, the curvature decreasing at the contour point $\vartheta = \frac{\pi}{2}$. In compression, the hole becomes more circular, hence the stress-concentration factor increases.

* In absolute value.

b) Extension in direction of minor axis of elliptical hole $\left(m = \frac{1}{3}\right)$.

1. The stress-concentration factor is

$$K^{*}_{p.s.} = 5\left(1 + 0.463\,\frac{N}{\mu}\right). \tag{39}$$

From the linear theory we obtain $K = 5$. It can be seen from (39) that when $\frac{N}{\mu} = 0.3$, $K^{*}_{p.s.} = 5.693$, and when $\frac{N}{\mu} = -0.3$, $K^{*}_{p.s.} = 4.306$. The deviation from the results of the linear theory is thus considerable.

2. The stress-concentration factor is

$$K^{**}_{p.s.} = 5\left(1 - 0.063\,\frac{N}{\mu}\right). \tag{40}$$

For $\frac{N}{\mu} = \pm 0.3$ we obtain from (40) respectively, $K^{**}_{p.s.} = 4.91$ and $K^{**}_{p.s.} = 5.09$. This differs only insignificantly from the value $K = 5$ given by the linear theory.

c) All-sided extension (compression) of a plane with an elliptical hole.

1. The stress-concentration factor is

$$K^{*}_{p.s.} = 4\left(1 + 0.688\,\frac{N}{\mu}\right). \tag{41}$$

2. The stress-concentration factor is

$$K^{**}_{p.s.} = 4\left(1 - 0.063\,\frac{N}{\mu}\right). \tag{42}$$

Values obtained from (41) and (42) for some values of $\frac{N}{\mu}$ are given in Table 7.

TABLE 7

K \ N/μ	According to linear theory	−0.3	−0.2	−0.1	0.1	0.2	0.3
$K^{*}_{p.s.}$	4	3.18	3.45	3.73	4.28	4.55	4.83
$K^{**}_{p.s.}$	4	4.08	4.05	4.03	3.98	3.95	3.93

It can be seen from Table 7 that the deviation from the value $K = 4$ is highest for problem 1. This should have been expected.

These results lead to certain general conclusions.

1. In the nonlinear theory the stress-concentration factor depends on both type and magnitude of the external load at infinity, as well as on the elastic properties of the material and the type of elastic equilibrium, i.e., whether a state of plane strain or a generalized state of stress (thin plate) exists.

2. When, for an incompressible material, the tension needed to deform the initial contour to a circle increases (in some interval), the stress-concentration factor at the (deformed) contour will increase; for increased compression this factor will decrease.

3. The deviation of the stress-concentration factor from the value given by the linear theory is, in general, considerable. The largest deviation is observed in the case of pure shear.

4. For problem 1, i.e., when the shape of the hole contour is specified in the deformed state of the body, the stress-concentration factor decreases with increasing uniaxial tension, and increases with uniaxial compression. This may be due to the distortion of the initial contour into the specified shape during deformation.

A particularly noticeable change in the stress distribution along the contour occurs in pure shear.

Table 8 gives expressions for the stress-concentration factors in respect to a plane, weakened by an elliptical hole, for all the problems considered above.

TABLE 8

Type of load	Problem	Tension or compression in direction of major axis	Tension or compression in direction of minor axis	All-sided tension or compression
Tension	1	$2\left(1+0.225\frac{N}{\mu}\right)$	$5\left(1+0.463\frac{N}{\mu}\right)$	$4\left(1+0.688\frac{N}{\mu}\right)$
	2	$2\left(1-0.163\frac{N}{\mu}\right)$	$5\left(1-0.063\frac{N}{\mu}\right)$	$4\left(1-0.063\frac{N}{\mu}\right)$
Compression	1	$2\left(1-0.225\frac{N}{\mu}\right)$	$5\left(1-0.463\frac{N}{\mu}\right)$	$4\left(1-0.688\frac{N}{\mu}\right)$
	2	$2\left(1+0.163\frac{N}{\mu}\right)$	$5\left(1+0.063\frac{N}{\mu}\right)$	$4\left(1+0.063\frac{N}{\mu}\right)$
Tension or compression	According to linear theory	2	5	4

3. Effect of reinforcing the holes by elastic rings

This author and Yu.I. Koifman /16/ considered the effect of reinforcing the edge of a circular hole by an elastic annular plate consisting of another,

in general also physically nonlinear material, pressed or soldered into the hole, the state of stress being homogeneous at infinity.

Expressions were given for the stresses σ_r and σ_θ at the surface of contact or soldering (for plane strain of an incompressible material) of an infinite plane with a circular hole, into which a perfectly rigid annular ring (washer) is pressed-in or soldered. It was shown that this reinforcement reduces the stress concentration at the contour of the hole.

Yu. I. Koifman /17/ also solved the problem of reinforcing a circular hole in an infinite plate by a soldered-in ring made from a thin elastic bar of constant cross section, whose elastic-equilibrium conditions are determined from the theory of small deformations of thin curved beams.

It was shown in /17/ that such a linearly elastic reinforcing ring considerably reduces the stress concentration at the contour of the hole.

4. Stress concentrations near free as well as reinforced holes

From the viewpoint of the nonlinear theory of elasticity the most general approach to stress concentrations near holes is presented in /2, 3, 4, 15, 16, 17/. By means of the powerful methods of the two-dimensional linear theory of elasticity and, in particular, of Kolosov-Muskhelishvili complex potentials in conjunction with conformal mapping and Cauchy-type integrals, it is possible to obtain satisfactory solutions (up to the second approximation) of problems of stress concentration near nonreinforced as well as reinforced curvilinear holes.

The holes may be reinforced by a wide platelike or narrow thin elastic rings, as assumed in /16/ and /17/ for a round hole.

The problem of reinforcing holes by rings which either eliminate entirely or reduce to a minimum the stress concentrations near the holes is of great scientific and engineering interest.

§ 3.

The ordinary stress distribution does not hold near holes, where the so-called stress-concentration zones appear. The stresses in these zones may be quite high, in particular near holes with sharp corners. The stresses at these corners can exceed the elastic limit of the material, or, for plastic materials, the yield point.

Even at comparatively low stresses, the stress-strain diagram for many materials deviates from the straight line (Hooke's law); for non-ferrous metals, certain plastics, etc., this deviation is quite appreciable. For the overwhelming majority of materials the stress-strain diagram in uniaxial loading has the shape shown in Figure 1.

Let us assume that a plate, weakened by a small curvilinear hole, is in an uniaxial state of stress, with $\sigma = p =$ const at infinity. The stresses in the plate at a sufficient distance from the hole are equal to p, corresponding to point A in Figure 1, while in the stress-concentration zone

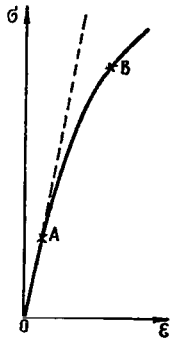

FIGURE 1.

near the hole the stresses will correspond to point B or be even higher. For these points the curve deviates already slightly, but perceptibly from a straight line of Hooke's law.

It is natural to ask how these small deviations from Hooke's law influence the stress-concentration factor and the extent of the stress-concentration zone near a hole, in particular when the hole has corners with small radii of curvature. For this it is necessary to compare, for the same case, the results of the classical linear theory of elasticity with those obtained by taking into account the deviations from Hooke's law.

We thus must know the stress concentration near a curvilinear hole in a physically nonlinear (deviating slightly from the linear) plane field. In other words, we have to solve problems in which the linear relationship between stresses and strains (Hooke's law) in the principal equations of the classical theory of elasticity is replaced by a nonlinear relationship, which for small deformations (e.g., in the case of metals, for strains less than 0.1 %) goes over into Hooke's law.

One method for solving these problems is discussed by this author in /20/, where he assumes the simplest nonlinear relationship between stresses and strains, suggested by G. Cowderer in /21/

$$\begin{aligned} \varepsilon_x &= \frac{\sigma_0}{3K} + \frac{g\,(t_0^2)}{2G}\,(\sigma_x - \sigma_0); \\ \varepsilon_y &= \frac{\sigma_0}{3K} + \frac{g\,(t_0^2)}{2G}\,(\sigma_y - \sigma_0); \\ \psi_{xy} &= \frac{g\,(t_0^2)}{G}\,\tau_{xy}. \end{aligned} \tag{43}$$

Here K and G, are respectively the bulk modulus and the shear modulus of physically nonlinear materials for the case of infinitesimal strains, and

$$\begin{aligned} g\,(t_0^2) &= 1 + g_2\,t_0^2; \\ g_2 &= \frac{9G^2}{4}\left(1 + \frac{G}{3K}\right) a_3; \end{aligned} \tag{44}$$

where a_3 is a constant determined experimentally by axial extension (compression) of a specimen of the material whose tensile (compressive) stress-strain diagram has the shape shown in Figure 1. This diagram can, with sufficient accuracy, be approximated by the relationship

$$\varepsilon = \frac{1}{E}\,[1 + a_3\,\sigma^2]\,\sigma; \tag{45}$$

where ε and σ are respectively the axial strains and stresses in the cross section of the specimen.

If we assume a stress-strain relationship in the form of (43), the solutions of the two-dimensional problem (plane state of stress) of

the theory of elasticity reduces /20, 21/ to finding the stress functions $F(x, y)$ from the equation

$$F_{xxxx} + 2F_{xxyy} + F_{yyyy} + \lambda [(2F_{xx}^2 + F_{yy}^2 + 2F_{xy}^2 - \\ - 2F_{xx}F_{yy})F_{xxxx} + \cdots + 12F_{xy}(F_{xxx}F_{xxy} + \\ + 2F_{xxy}F_{xyy} + F_{xyy}F_{yyy})] = 0; \tag{46}$$

with the appropriate boundary conditions, where

$$\lambda = \frac{Kg_2}{G^2(3K+G)} = \varepsilon\beta^2 \tag{47}$$

is a constant of the material and

$$\varepsilon = \frac{1}{g_2}; \quad \beta^2 = \frac{g_2^2}{G^2} \cdot \frac{K}{3K+G} \tag{48}$$

As shown by experiments, for nonferrous materials (copper and its alloys, etc.) the dimensionless constant g_2 in (44) is of the order of magnitude of the moduli K and G, expressed in $\frac{\text{kg}}{\text{cm}^2}$, i.e., 10^5 to 10^6; hence for these materials $\lambda \left(\left[\lambda\right] \equiv \frac{\text{cm}^4}{\text{kg}^2}\right)$ is of the order of the reciprocals of K and G, i.e., 10^{-5} to 10^{-6}. For example, tensile tests $\sigma \leqslant 1000 \frac{\text{kg}}{\text{cm}^2}$ for pure copper yielded the following results:

$$K = 1.37 \cdot 10^6 \frac{\text{kg}}{\text{cm}^2}; \quad G = 0.46 \cdot 10^6 \frac{\text{kg}}{\text{cm}^2}; \quad g_2 = 0.18 \cdot 10^6; \quad \lambda = 0.255 \cdot 10^{-6} \frac{\text{cm}^4}{\text{kg}^2}. \tag{49}$$

In (46) the parameter λ, given by (47), is small; this suggests that we seek the solution of this equation in the form of the expansion

$$F(x, y) = H_0 [F^{(0)}(x, y) + \varepsilon F^{(1)}(x, y) + \varepsilon^2 F^{(2)}(x, y) + \cdots], \tag{50}$$

where H_0 is a constant:

$$H_0 = \frac{1}{\beta} = \frac{G}{g_2}\sqrt{3 + \frac{G}{K}}, \quad \varepsilon = \frac{1}{g_2}.$$

Substituting this expansion in (46) and equating to zero the coefficients of equal powers of ε, we obtain the infinite system of nonlinear differential equations (51) and (52) in the functions $F^{(0)}(x, y)$ and $F^{(k)}(x, y)$ $(k = 1, 2, 3 \cdots)$

$$\Delta\Delta F^{(0)}(x, y) = 0; \tag{51}$$

$$\Delta\Delta F^{(k)}(x, y) = A_k(F^{(0)}, F^{(1)}, \cdots, F^{(k-1)}); \tag{52}$$

where A_k is a nonlinear operator on the functions and their derivatives of the preceding approximations, i.e., the functions $F^{(0)}$; $F^{(1)}$; $F^{(2)}$; $\cdots$; $F^{(k)}$ and their derivatives. The operator A_1 is given in /21/ and /20/, while the operator A_2 is given by Jindra /23/.

The biharmonic function $F^{(0)}(x, y)$ can be found using all the powerful tools of the two-dimensional problem of the linear theory of elasticity /1/. It is determined by taking into account the forces specified at the contour of the hole and the given conditions at infinity. On the other hand, the functions $F^{(1)}(x, y)$, $F^{(2)}(x, y)\cdots$ are determined from the nonhomogeneous equations (52), for zero forces at the contour of the hole and zero stresses at infinity.

The main difficulty in solving this problem is to find a particular solution of (52). On the other hand, the general solution of the homogeneous equation (52) for the appropriate boundary conditions and conditions at infinity can be easily found by using the powerful tools of the classical (linear) two-dimensional theory of elasticity, developed by the school of Academician N. I. Muskhelishvili /1/.

The problem of the stress distribution near a round hole in a thin plate consisting of a physically nonlinear (as defined by (43)) material, was first solved (with an accuracy up to the second approximation) by Jindra /23/, who gave expressions for the stress components near the hole and, in particular, at its contour.

In this approximation the stress-concentration factor is

$$K = \frac{\sigma_\vartheta}{p} = 3\ (1 - 3.45\ \lambda p^2). \tag{53}$$

The second term in (53) is a correction for the physical nonlinearity of the material. Thus, for copper this correction amounts to about 10% for $p \cong 333.3$, which is large enough to be taken into account.

Later I. A. Tsurpal /24-29/ considered several new problems formulated in the above manner. Thus, in /24/ he discussed the stress concentration near a round hole in a nonlinearly elastic plate subjected to tension on all sides. The solution is given as a third approximation, i.e., the stress function has the form

$$F\ (x, y) = H_0\,[F^{(0)}(x, y) + \varepsilon F^{(1)}(x, y) + \varepsilon^2 F^2\ (x, y)].$$

The stress-concentration factor was found to be

$$K = \left(\frac{\sigma_\vartheta}{p}\right)_{r=R} = 2\ [1 - 1.5\ \lambda p^2 + 10.605\ \lambda^2 p^4]. \tag{54}$$

It is seen from (54) that K primarily depends nonlinearly on the elastic properties of the material and on the external load p.

In this particular problem he investigated the rate of convergence of successive approximations by comparing the values of the stress-concentration factor obtained in the first, second, and third approximation for a copper plate whose elastic constants are given (49). Results of this comparison are given in Table 9.

It can be seen from the table that, at least for axisymmetric states of stress, the second approximation is sufficiently accurate for engineering calculations.

Paper /25/ deals with the stresses in a hollow cylinder subjected to a uniform external and internal pressure (plane strain), taking into account the physical nonlinearity of the pipe material, defined by (45).

Paper /26/ deals with a plate having a round hole, subjected to pure shear at infinity. The stress-concentration factor (in the second approximation) is

$$K = \left(\frac{\sigma_\vartheta}{\tau}\right)_{r=R} = -4\ \sin 2\vartheta + \lambda\tau^2\ (17.38 \sin 2\vartheta - 6.2 \sin 6\vartheta). \tag{55}$$

It follows from (55) that for insignificant deviations from Hooke's law the stress-concentration factor does not remain constant, but depends considerably on the external load and on the elastic properties of the plate material.

TABLE 9

$p \frac{kg}{cm^2}$	Stress-concentration factor according to		
	linear theory	nonlinear theory	
	first approximation	second approximation	third approximation
100	2.000	1.990	1.990
200	2.000	1.988	1.988
300	2.000	1.985	1.985
450	2.000	1.847	1.899

Paper /28/ deals with the determination of the elastic constants K, G, and g_2 for certain materials.

In a paper read at this conference, I. A. Tsurpal considers the contact problem of reinforcing a round hole by an elastic ring made of another material. The formulation of this problem is similar to that solved by Yu. I. Koifman and this author in /16/. In /29/ I. A. Tsurpal considered the effect of the physical nonlinearity on the stresses in a plate with a hole reinforced by either an elastic ring made of another material or by a perfectly rigid ring, or finally, when an elastic washer is soldered into the hole. All these problems are discussed for both all-sided and uniaxial states of stress at infinity.

Investigations /24-27/ of stress concentrations near round holes show that, when using the elastic relationships (43), i.e., at an accuracy of up to and including the second approximation of the physically nonlinear theory, the stress-concentration factor is lower than according to the linear theory.

As in the general case of the nonlinear two-dimensional problem considered in § 2 of the present survey, nothing has been published on the convergence of the successive approximations in the case of the physically nonlinear problems considered in § 3. Neither has anything been published on the stresses near a noncircular hole, using the stress-strain relationship (43). It would also be very interesting to study the effect of the physical nonlinearity of the material on the stress concentration at the corners of a hole.

It is known from /10/ that the method presented in /16/ for solving physically nonlinear problems, using (43), opens a way to solve certain problems of elastoplastic deformation in the stress-concentration zone.

§ 4.

If a flexible elastic plate whose material obeys Hooke's law is weakened by a small hole and subjected to uniform compression along the external

contour of the plate, the latter will be in a plane state of stress, until loss of stability, i.e., buckling occurs. In this case a stress-concentration zone arises near the hole, which can be determined by known methods.

Of considerable theoretical and practical interest is the problem of stress concentration near a hole in an elastic flexible plate when the latter buckles and undergoes postcritical deformations.

Unfortunately, this author knows of no published material on this subject. The only relevant works are those by Ya. F. Kayuk /30/ and /31/, whose report is presented to this conference. Hence I shall dwell only briefly on the results obtained by him. He himself will supply the details.

It is assumed in /30/ and /31/ that the plate material is elastic and obeys Hooke's law. Axisymmetric postcritical deformation of a thin annular plate, weakened by a small circular hole and subjected to uniform compression applied to its external contour, is considered under the following conditions: 1) the external contour is hinged; 2) the external contour is rigidly clamped. The contour of the hole is free from external loads. To estimate the stress concentration near the hole as well as for purposes of comparison, he also considered the postcritical deformation of a similar continuous plate. The solution of this problem was reduced to the integration of Karman's nonlinear equations, using the method of the small parameter. The results are presented in Tables 10 and 11.

Table 10 gives the membrane stresses $T^0_{\theta\theta}$ at the free (inside) contour of the annular plate.

TABLE 10

$\varepsilon = \dfrac{\Delta P}{P_{cr.}}$	0.00	0.10	0.30	0.50	0.70
Hinged $(T^0_{\theta\theta})_{\rho=0.10}$	−2	−2.13	−2.12	−1.92	−1.69
Rigidly clamped $(T^0_{\theta\theta})_{\rho=0.12}$	−2	−2.20	−2.41	−2.67	−2.90

Table 11 gives the moments $M^0_{\theta\theta}$ at the free inside contour of an annular plate and along the corresponding curve on a continuous plate.

TABLE 11

Plate	$M^0_{\theta\theta}$ Hinged $\rho = 0.10$	$M^0_{\theta\theta}$ Rigidly clamped $\rho = 0.12$
Continuous	$5.41\eta + 6.29\eta^3$	$13.66\eta - 6.20\eta^3$
Annular	$14.1\eta + 6.11\eta^3$	$19.67\eta + 3.96\eta^3$

Here, $\eta = \dfrac{\Delta P}{P_{cr} + \Delta P}$.

Tables 10 and 11 show that in the case of a plate with hinged external contour, the concentration of the membrane-stress $T^0_{\theta\theta}$ at the

free contour decreases with increasing postcritical deformations, while the moments $M^0_{\vartheta\vartheta}$ increase in comparison with those at the corresponding (imagined) contour $\rho = 0.10$ of a continuous plate.

In the case of a plate rigidly clamped along its external contour, an increase in the postcritical deformations results in larger membrane stresses $T^0_{\vartheta\vartheta}$ and moments $M^0_{\vartheta\vartheta}$ at the free contour $\rho = 0.12$.

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PAPERS READ AT MEETINGS OF SECTIONS AND SUBSECTIONS

ANALYSIS OF BENDING AND VIBRATIONS OF PLATE SYSTEMS

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The paper deals with practical methods for analyzing plate systems.

Bending of plate systems is analyzed by the methods of the theory of structures, basically by a combination of the principle of superposition with numerical methods. By removing or imposing certain constraints the plate system is divided into a number of independent elements of the same type, in the same way as the primary system is obtained in the theory of structures by the method of forces or the method of displacements. Each typical plate element is analyzed by the relaxation method /1/ for the case of a contour subjected to unit forces or strain factors and to the given load. The solutions are represented in the form of tabulated plate deflections and are fundamental for the subsequent calculations. They are used to set up the equations of deformation continuity (also by numerical methods) at several points of connection.

The equations obtained are similar to the equations of the theory of rod systems, and the known methods of the theory of structures can therefore be used for their solution.

The method proposed reduces considerably the number of equations, as compared with the traditional relaxation method, retaining the principal advantages of the latter, which are its universality and its applicability to systems for which no exact solutions exist (such as a system of plates of variable thickness, or of plates with holes and different contour conditions).

Due to the determination of the conditions at the joints by the more precise method of numerical differentiation /2/ than by the relaxation method, the accuracy of the solution is satisfactory even when only a small number of points is taken inside the contour. This accuracy can later be increased by successive approximations, using the results of the preceding approximation without the need to solve the equations of the relaxation method in explicit form /3/.

1. BENDING OF CONTINUOUS PLATES

Consider rectangular continuous plates hinged on rigid supports. The flexural rigidity of each plate is constant over its span. Applying the method described above, we obtain the equation of deformation continuity of two adjacent spans of the continuous plate, along the line $(n-3)_{\mathrm{I}}-(n-3)_{\mathrm{II}}$ expressed

through the intensity of the end moments

$$\left.\begin{aligned}&\sum_{A1} M_{A1}\lambda_1' \varphi_{A1} + \sum_{B} M_B (\lambda_1' \varphi_{B1} + \lambda_2' \varphi_{B2}) + M_n (\lambda_1' \varphi_{n1} + \lambda_2^1 \varphi_{n2}) + \\ &+ \sum_{A2} M_{A2}\lambda_2'\varphi_{A2} = -\lambda_1' R_{1q} q_1 \lambda_1^2 - \lambda_2' R_{2q} q_2 \lambda_2^2.\end{aligned}\right\} \quad (1)$$

Here $A1$ and $A2$ (Figure 1) represent the support sections with unknown end moments at the nonadjacent supports of plates I and II, and B the section at the common support, except sections n.

$\lambda_j' = \dfrac{\lambda_j}{D_j}\dfrac{D_0}{\lambda_0}$ is the reduced step of the grid for plate j,

$$\varphi_{ij} = 4\delta_{n-1,\,ij} - \delta_{n-2,\,ij} + \frac{4}{27}\delta_{n-3,\,ij},$$

where $i = A$ or B, $j = 1$ or 2;

$$R_{iq} = 4\delta_{n-1,\,q} - \delta_{n-2,\,q} + \frac{4}{27}\delta_{n-3,\,q},$$

$$\delta_{kr} = w_{kr}\frac{D}{\lambda^2}, \qquad \delta_{kq} q\lambda^2 = w_{kq}\frac{D}{\lambda^2};$$

δ_{kr}, δ_{kq} are dimensionless numerical factors in tabulated values of the deflections w_{kr}, w_{kq} of the hinged plate under the action of the end moments /3/ and the given load /4/ respectively.

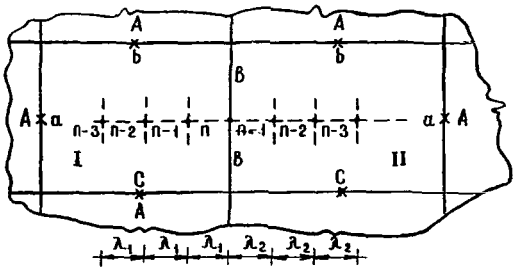

FIGURE 1.

The calculations show that for a continuous plate with square spans it is sufficient to satisfy the connection conditions only at the middle point of the support. In this case equation (1) reduces to the following

$$\left.\begin{aligned}&0.659\, M_{a,\,1}\lambda_1' + M_{b,\,1}\lambda_1' + M_{c,\,1}\lambda_1' + 3.725(\lambda_1' + \lambda_2') M_n + M_{c,2}\,\lambda_2' + \\ &+ M_{b,\,2}\lambda_2' + 0.659 M_{a,\,2}\lambda_2' = -0.2501\,(\lambda_1' R_{n1} + \lambda_2' R_{n2}).\end{aligned}\right\} \quad (2)$$

In the case of a continuous plate in the form of strips with square spans, equation (2) reduces to a three-moment equation /3/.

2. BENDING OF PLATES SUPPORTED ON ELASTIC BEAMS

Consider a square plate loaded by a uniformly distributed load q (Figure 2). Sides AB and CD are hinged on rigid supports, and sides AC and BD on elastic beams of equal rigidity. We shall use as fundamental system a plate with free edges AC and BD, i.e., we remove the elastic beams from the plate.

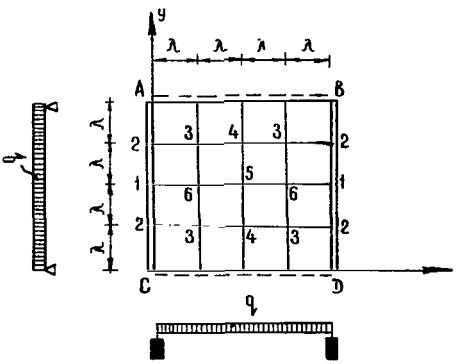

FIGURE 2.

We shall satisfy the connection conditions of the plate and the elastic beams at points 1 and 2 (more accurately along portions of length λ with center at points 1 and 2 respectively). The unknowns here will be the interaction forces between the plate and the elastic beams at these points. To form the equations we make use of the table given in /4/, allowing for the symmetry of deformation.

$$\left.\begin{aligned} &\frac{\lambda^3}{D}(1.0308 + 1.19\gamma)X_1 + \frac{\lambda^3}{D}(1.3148 + 1.79\gamma)X_2 = 3.7037\frac{q\lambda^4}{D}, \\ &\frac{\lambda^3}{D}(0.6574 + 0.895\gamma)X_1 + \frac{\lambda^3}{D}(1.0308 + 1.294\gamma)X_2 = 2.6454\frac{q\lambda^4}{D}. \end{aligned}\right| \qquad (3)$$

Here $\gamma = \frac{2\lambda}{EI}$, EI is the beam rigidity, D is the flexural rigidity of the plate.

The comparison with the exact solution /5/ for the deflections at points 1 and 5 shows a discrepancy of 3.96 and 0.705% for $\gamma = 0.25$, and of 2.86 and 3.39% for the bending moments at the plate center. An accuracy of the same order is obtained for other values of γ as well. It is thus seen that it is sufficient to satisfy the connection conditions at three points of the edge.

The plate is similarly calculated in the case of symmetrical loading by unit forces along portions of length λ with center at points 1 and 2.

Consider now a symmetrical three-span plate with the middle span loaded by a uniformly distributed load (Figure 3, a). The plates are hinged to each other and to the elastic beams (two-sided constraints). The given system can serve as a scheme for calculating ceilings, when the component plates do not form a monolithic floor.

By means of suitable cuts (Figure 3, b) we transform the system considered into a number of separate plates, with edges free or supported on elastic beams. Tabulated values of the deflections are available for each of the plates obtained: tables /4/ for the extreme plates, and the above-obtained data for the middle plate.

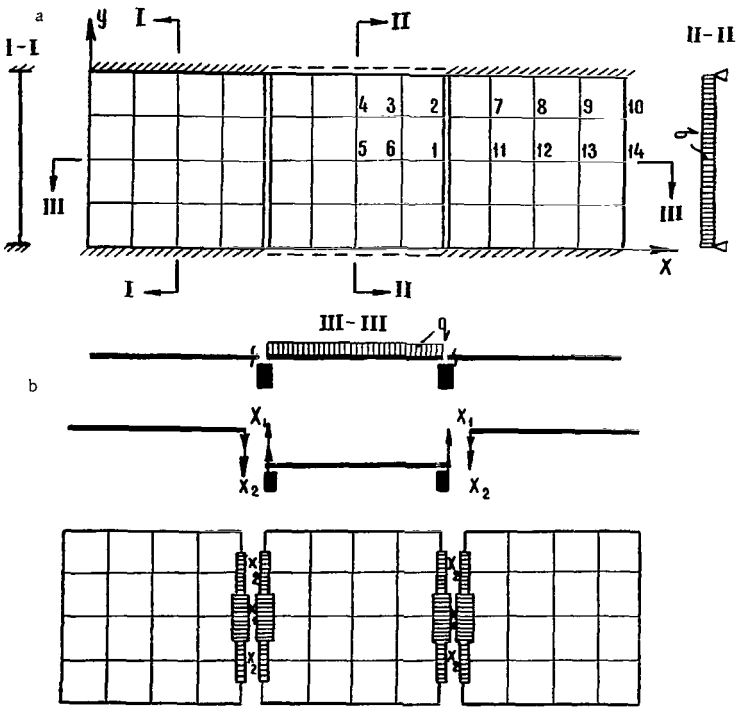

FIGURE 3.

By satisfying the connection conditions at three points of each edge as above we obtain the system of equations

$$\left.\begin{aligned} 0.6261\,X_1 + 0.6701\,X_2 &= 0.8833\,q\lambda; \\ 0.3506\,X_1 + 0.5218\,X_2 &= 0.6510\,q\lambda, \end{aligned}\right\} \qquad (4)$$

whence

$$X_1 = 0.2689\,q\lambda; \quad X_2 = 1.0670\,q\lambda.$$

The values of the deflections of the plate surface, multiplied by $\frac{D}{q\lambda^4}$, are given below:

$$w_1 = 0.4609, \quad w_2 = 0.3447, \quad w_3 = 0.7998, \quad w_4 = 0.9691,$$
$$w_5 = 1.3407, \quad w_6 = 1.0991, \quad w_7 = 0.1267, \quad w_8 = 0.02335,$$
$$w_9 = 0.00299, \quad w_{10} = -0.07368, \quad w_{11} = 0.1955, \quad w_{12} = 0.0599,$$
$$w_{13} = 0.00857, \quad w_{14} = -0.01118.$$

The bending moments at the center of the middle plate are found by the formulas of numerical differentiation:

$$M_x = 0.694\, q\lambda^2, \quad M_y = 0.895\, q\lambda^2.$$

The simultaneous calculation of the three-span system leads to a certain redistribution of the stresses as compared with the calculation of each plate separately. In the example given the allowance for the influence of the adjacent plates on the middle one is equivalent to an increase of the rigidity of the elastic beams by a factor larger than two. At the limit $\frac{1}{\gamma} \to 0$ (very flexible beams) the error in the values of the bending moments at the center of the middle plate increases to 40% for M_x, and to 100% for M_y. The error decreases with the increased of the beam rigidity, and at $\frac{1}{\gamma} > 10$ the hinged plates can be considered as independent from each other, since their supporting beams become practically rigid.

It is thus seen that neglecting the influence of the adjacent plates leads in a certain range to a very considerable error; what is more, this error does not favor an increase of the safety factor, since the decrease of the moments of one direction is linked with the increase of the moments of the other direction.

The analyzing method proposed in this paper can be used for analyzing continuous plates with elastic supports as well. This problem will not be treated here, in view of the absence of tables relative to the action of the moments on the contour. There is, however, no basic difficulty in forming such tables.

3. FREE VIBRATIONS OF PLATE FRAME SYSTEMS WITH RIGID SUPPORTS

We shall consider the free vibrations of plate systems made of plates continuous in one direction, supported on rigid supports.

The differential equation of free vibrations of a thin plate is

$$\nabla^2\nabla^2\bar{w}(\xi, \eta, t) + \frac{\beta^2}{\omega^2}\frac{\partial^2\bar{w}(\xi, \eta, t)}{\partial t^2} = 0, \tag{5}$$

where $\bar{w}(\xi, \eta, t)$ is the deflection of the point with coordinates ξ, η from the equilibrium positions; ξ, η are the dimensionless coordinates: $\xi = \frac{x}{a}$, $\eta = \frac{y}{a}$; a is the length of the rectangular plate along the x axis; ω is the frequency of natural vibrations:

$$\beta^2 = \frac{q\omega^2 a^4}{gD}, \tag{6}$$

where $q = \gamma h$ is the weight of the plate per unit surface.

The solution of (5) will be sought as usual in the form

$$\bar{w} = (A\cos\omega t + B\sin\omega t)\, w(\xi, \eta). \tag{7}$$

The deflection amplitudes will be determined from the equation

$$\nabla^2\nabla^2 w(\xi, \eta) - \beta^2 w(\xi, \eta) = 0, \tag{8}$$

which will be solved by the Levy method for the case of loading of a hinged plate by amplitude end moments $\overline{M}_A(\xi)$ and $\overline{M}_B(\xi)$ distributed according to some law along the parallel supports $\eta = 0$ and $\eta = \mu$:

$$w(\xi, \eta) = \frac{a^2}{D}\sum_{n=1}^{\infty}\frac{\sin n\pi\xi}{2\beta_n}\left\{M_A^n\left[\frac{\operatorname{sh} k_{1n}(\mu-\eta)}{\operatorname{sh} k_{1n}\mu} - \frac{\operatorname{sh} k_{2n}(\mu-\eta)}{\operatorname{sh} k_{2n}\mu}\right] + \right.$$
$$\left. + M_B^n\left[\frac{\operatorname{sh} k_{1n}\eta}{\operatorname{sh} k_{1n}\mu} - \frac{\operatorname{sh} k_{2n}\eta}{\operatorname{sh} k_{2n}\mu}\right]\right\}, \tag{9}$$

where

$$\left.\begin{aligned} k_{1n}^2 &= n^2\pi^2 - \beta, \qquad k_{2n}^2 - k_{1n}^2 = 2\beta_n, \\ k_{2n}^2 &= n^2\pi^2 + \beta, \qquad \mu = \frac{a}{b}; \end{aligned}\right\} \tag{9a}$$

$$M_A^n = 2\int_0^1 \overline{M}_A(\xi)\sin n\pi\xi d\xi; \qquad M_B^n = 2\int_0^1 \overline{M}_B(\xi)\sin n\pi\xi d\xi. \tag{9b}$$

M_A^n, M_B^n are the integral values of the end moments.

The condition of continuity of the deformation on the common supporting edge of two adjacent plates i and $i+1$ is

$$\left.\frac{\partial w_i}{\partial y}\right|_{y=b_i} = \left.\frac{\partial w_{i+1}}{\partial y}\right|_{y=0}. \tag{10}$$

Substituting (9) yields

$$b_i'(M_{ii}^n A_{n,i} + M_{i+1,i}^n B_{n,i}) = b_{i+1}'(M_{i+1,i+1}^n B_{n,i+1} + M_{i+2,i+1}^n A_{n,i+1}), \tag{11}$$

where $b_i' = \frac{b_i}{D_i}\frac{D_0}{b_0}$ is the reduced length of the plate span along the y axis;

$$\left.\begin{aligned} A_{n,i} &= \left(\frac{k_{2n}\mu_i}{\operatorname{sh} k_{2n}\mu_i} - \frac{k_{1n}\mu_i}{\operatorname{sh} k_{1n}\mu_i}\right)\frac{1}{2\beta_n}, \\ B_{n,i} &= (k_{1n}\mu_i \operatorname{cth} k_{1n}\mu_i - k_{2n}\mu_i \operatorname{cth} k_{2n}\mu_i)\frac{1}{2\beta_n} \end{aligned}\right\} \tag{12}$$

are functional coefficients.

Condition (11) connects four integral values of the end moments and can be called the four-moment equation of the theory of vibrations of plate structures. In the particular case when only two plates are connected at each nodal line of the plate structure, equation (11) reduces to a three-moment equation:

$$M_i^n A_{ni} b_i + M_{i+1}^n (B_{n,i} b_i' + B_{n,i+1} b_{i+1}') + M_{i+2}^n A_{n,i+1} b_{i+1} = 0. \tag{13}$$

Equations (11) and (13) coincide in structure with the similar equations of the theory of vibrations of rod systems, and reduce to the latter for $a \to \infty$.

The sequence and method of determining the natural frequencies of the plate system considered remain the same as in the similar rod systems.

The frequency of a given series is determined by equating the determinant of system (11) or (13) to zero at given n. There are an infinite number of series. Each series is characterized by a constant form of the vibrations along the ξ axis.

In the case of a plate continuous in two directions the solution of (8) is to be represented in the form

$$w = \sum_{n=1}^{\infty} F_n(\xi) \sin n\pi\eta + \sum_{m=1}^{\infty} \Phi_m(\eta) \sin m\pi\xi, \tag{14}$$

where

$$F_n(\xi) = A_n \operatorname{sh} k_{1n}\xi + B_n \operatorname{ch} k_{1n}\xi + C_n \operatorname{sh} k_{2n}\xi + D_n \operatorname{ch} k_{2n}\xi,$$

$$\Phi_n(\eta) = \bar{A}_m \operatorname{sh} k_{1m}\eta + \bar{B}_m \operatorname{ch} k_{1m}\eta + \bar{C}_m \operatorname{sh} k_{2m}\eta + \bar{D}_n \operatorname{ch} k_{2m}\eta.$$

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BENDING OF THICK PIPES

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(Yerevan)

1. We consider the stressed state of a thick pipe under the action of a bending load (see figure).

The problem will be solved with the aid of Papkovich's functions /1, 2/.

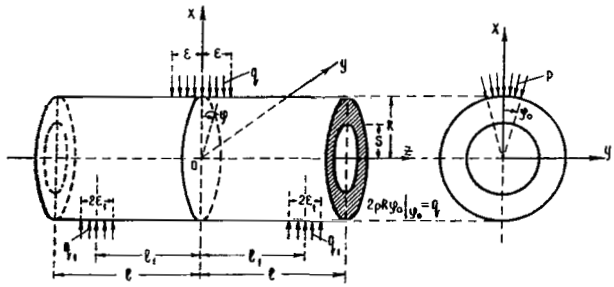

FIGURE.

In cylindrical coordinates r, φ, z the displacement components are expressed through Papkovich's four harmonic functions by the formulas

$$u_r = \Phi_1 \cos\varphi + \Phi_2 \sin\varphi - \frac{1}{4(1-\nu)} \frac{\partial \Psi}{\partial r},$$

$$u_\varphi = -\Phi_1 \sin\varphi + \Phi_2 \cos\varphi - \frac{1}{4(1-\nu)} \frac{1}{r} \frac{\partial \Psi}{\partial \varphi}, \qquad (1.1)$$

$$u_z = \Phi_3 - \frac{1}{4(1-\nu)} \frac{\partial \Psi}{\partial z},$$

where

$$\Psi = r(\Phi_1 \cos\varphi + \Phi_2 \sin\varphi) + z\Phi_3 + \Phi_0, \qquad (1.2)$$

$$\nabla^2 \Phi_i = \frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_i}{\partial r} + \frac{\partial^2 \Phi_i}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 \Phi_i}{\partial \varphi^2} = 0 \quad (i = 0, 1, 2, 3), \qquad (1.3)$$

ν is the Poisson ratio.

Before proceeding to the analysis of the problem formulated, assume that a homogeneous elastic body of revolution is deformed symmetrically relative to plane $x0z$, and let the plane $y = 0$ be the only plane of symmetry (figure). Measuring the angle φ in the xl plane from the x-axis we find then that the displacements u_r and u_z are even functions of φ, and the displacement u_φ is an odd function.

These displacements will be represented in the form of the following series:

$$u_r = \sum_{k=0}^{\infty} u_k(r, z) \cos k\varphi,$$

$$u_\varphi = \sum_{k=1}^{\infty} v_k(r, z) \sin k\varphi, \tag{1.4}$$

$$u_z = \sum_{k=0}^{\infty} w_k(r, z) \cos k\varphi.$$

This same representation is used in Solyanik-Krassa's papers /3, 4/ in connection with several problems of the theory of elasticity.

It is easily seen that in this case $\tau_{r\varphi}$ and $\tau_{\varphi z}$ will be odd functions of φ, and the remaining stress components, even functions.

Note also that in order to ensure such a stressed state in a homogeneous body of revolution the tangential component of the surface forces must be given as an odd function of φ, and the remaining components as even functions. Such a formulation of the problem makes it possible to represent Papkovich's harmonic functions $\Phi_i(r, \varphi, z)$ as expansions by trigonometric functions of φ.

Represent the functions Φ_i in the following form /5/:

$$\Phi_1 = \sum_{k=0}^{\infty} \varphi_{1k}(r, z) \cos(k+1)\varphi,$$

$$\Phi_2 = \sum_{k=0}^{\infty} \varphi_{1k}(r, z) \sin(k+1)\varphi, \tag{1.5}$$

$$\Phi_3 = \sum_{k=0}^{\infty} \varphi_{3k}(r, z) \cos k\varphi,$$

$$\Phi_0 = \sum_{k=0}^{\infty} \varphi_{0k}(r, z) \cos k\varphi.$$

Relationships (1.4) will be satisfied in this case.

The functions φ_{ik} in expressions (1.5) must satisfy the equations

$$\nabla_{k+1}^2 \varphi_{1k} = 0, \quad \nabla_k^2 \varphi_{3k} = 0, \quad \nabla_k^2 \varphi_{0k} = 0. \tag{1.6}$$

where

$$\nabla_\alpha^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{\alpha^2}{r^2}. \tag{1.7}$$

Using formulas (1.5) and (1.4), we obtain the following expression for the expansion coefficients u_k, v_k, and w_k:

$$u_k(r, z) = \varphi_{1k} - \frac{1}{4(1-\nu)} \frac{\partial}{\partial r} (v\varphi_{1k} + z\varphi_{3k} + \varphi_{0k}), \quad (k = 0, 1, \cdots),$$

$$v_k(r, z) = \varphi_{1k} + \frac{1}{4(1-\nu)} \frac{k}{r} (r\varphi_{1k} + z\varphi_{3k} + \varphi_{0k}), \quad (k = 1, 2, \cdots), \tag{1.8}$$

$$w_k(r, z) = \varphi_{3k} - \frac{1}{4(1-\nu)} \frac{\partial}{\partial z} (r\varphi_{1k} + z\varphi_{3k} + \varphi_{0k}), \quad (k = 0, 1, \cdots).$$

Using the expansions for the stress components, we obtain the following expressions for the coefficients of these expansions:

$$\left.\begin{aligned}
\sigma_r^{(k)}(r, z) &= 2G\left[\frac{\partial\varphi_{1k}}{\partial r} - \frac{1}{4(1-\nu)}\frac{\partial^2\omega_k}{\partial r^2} + \frac{\nu}{2(1-\nu)}\Omega_k\right] \\
\sigma_\varphi^{(k)}(r, z) &= 2G\left[\frac{k+1}{r}\varphi_{1k} - \frac{1}{4(1-\nu)}\left(\frac{1}{r}\frac{\partial\omega_k}{\partial r} - \frac{k^2}{r^2}\omega_k\right) + \right. \\
&\qquad \left. + \frac{\nu}{2(1-\nu)}\Omega_k\right] \\
\sigma_z^{(k)}(r, z) &= 2G\left[\frac{\partial\varphi_{3k}}{\partial z} - \frac{1}{4(1-\nu)}\frac{\partial^2\omega_k}{\partial z^2} + \frac{\nu}{2(1-\nu)}\Omega_k\right] \\
\tau_{rz}^{(k)}(r, z) &= G\left[\frac{\partial\varphi_{3k}}{\partial r} + \frac{\partial\varphi_{1k}}{\partial z} - \frac{1}{2(1-\nu)}\frac{\partial^2\omega_k}{\partial r\partial z}\right]
\end{aligned}\right\} \quad (k = 0, 1, \cdots)$$

$$\left.\begin{aligned}
\tau_{r\varphi}^{(k)}(r, z) &= G\left[\frac{\partial\varphi_{1k}}{\partial r} - \frac{k+1}{r}\varphi_{1k} + \frac{k}{2(1-\nu)}\frac{\partial}{\partial r}\left(\frac{\omega_k}{r}\right)\right] \\
\tau_{\varphi z}^{(k)}(r, z) &= G\left[\frac{\partial\varphi_{1k}}{\partial z} - \frac{k}{r}\varphi_{3k} + \frac{k}{2(1-\nu)}\frac{\partial}{\partial z}\left(\frac{\omega_k}{r}\right)\right]
\end{aligned}\right\} \quad (k = 1, 2, \cdots) \tag{1.9}$$

where

$$\omega_k(r, z) = r\varphi_{1k} + z\varphi_{3k} + \varphi_{0k},$$

$$\Omega_k(r, z) = \frac{\partial\varphi_{1k}}{\partial r} + \frac{k+1}{r}\varphi_{1k} + \frac{\partial\varphi_{3k}}{\partial z}. \tag{1.10}$$

The three-dimensional problem of the determination of the functions $\Phi_i(r, \varphi, z)$ has thus been reduced to the two-dimensional problem of determining the functions φ_{ik} (the boundary conditions for the functions φ_{ik} are formed with the aid of the coefficients of the expansions by the coordinate φ of the displacements and stresses given on the body surface).

2. We now proceed to the treatment of the problem of the bending of an infinite pipe by forces applied to its external lateral surface symmetrically relative to the xz plane.

The boundary conditions for this problem are

$$\sigma_r(R, \varphi, z) = f(\varphi, z), \quad \tau_{rz}(R, \varphi, z) = \tau_{\varphi r}(R, \varphi, z) = 0,$$

$$\sigma_r(s, \varphi, z) = \tau_{rz}(s, \varphi, z) = \tau_{\varphi r}(s, \varphi, z) = 0. \tag{2.1}$$

where the function $f(\varphi, z)$ is continuous along separate sections and of bounded variation in the intervals considered.

In particular, the stress $\sigma_r(R, \varphi, z)$ can be represented in the form of a distributed load acting on the portions of the lateral surface of the pipe (figure) in the following way:

$$f(\varphi, z) = \begin{cases} -p & (0 \leqslant \varphi \leqslant \varphi_0;\ 0 \leqslant z \leqslant \varepsilon), \\ -p_1 & (\pi - \varphi_0 \leqslant \varphi \leqslant \pi;\ l_1 - \varepsilon_1 \leqslant z \leqslant l_1 + \varepsilon_1), \\ 0 & \text{(on the remaining part of the lateral surface).} \end{cases} \tag{2.2}$$

We shall also use the following symmetry conditions relative to the xy plane:

$$u_z(r, \varphi, 0) = \tau_{rz}(r, \varphi, 0) = \tau_{\varphi z}(r, \varphi, 0) = 0. \tag{2.3}$$

The functions $\varphi_{ik}(r, z)$ will be represented in the form

$$\begin{aligned} \varphi_{1k}(r, z) &= \int_0^\infty [A_k(\lambda) I_{k+1}(\lambda r) + B_k(\lambda) K_{k+1}(\lambda r)] \cos \lambda z d\lambda, \\ \varphi_{3k}(r, z) &= \int_0^\infty [C_k(\lambda) I_k(\lambda r) + D_k(\lambda) K_k(\lambda r)] \sin \lambda z\, d\lambda, \\ \varphi_{0k}(r, z) &= \int_0^\infty [E_k(\lambda) I_k(\lambda r) + F_k(\lambda) K_k(\lambda r)] \cos \lambda z\, d\lambda, \end{aligned} \tag{2.4}$$

where $I_i(x)$ and $K_i(x)$ are Bessel functions of the first and second kind of an imaginary argument /6/.

By substituting these expressions in the formulas for the stresses and displacements and satisfying the boundary conditions (2.1) and (2.3) we obtain the following system of six equations with six unknowns for the determination of the integration coefficients:

$$\begin{aligned} &A_k a_k^{(1)}(R) + B_k b_k^{(1)}(R) + C_k c_k^{(1)}(R) + D_k d_k^{(1)}(R) + \lambda E_k e_k^{(1)}(R) + \\ &\quad + \lambda F_k f_k^{(1)}(R) = \frac{\psi_k(\lambda, R)}{G\pi\lambda}, \\ &A_k a_k^{(1)}(s) + B_k b_k^{(1)}(s) + C_k c_k^{(1)}(s) + D_k d_k^{(1)}(s) + \lambda E_k e_k^{(1)}(s) + \lambda F_k f_k^{(1)}(s) = 0, \\ &A_k a_k^{(2)}(R) + B_k b_k^{(2)}(R) + C_k c_k^{(2)}(R) + D_k d_k^{(2)}(R) + \lambda E_k e_k^{(2)}(R) + \lambda F_k f_k^{(2)}(R) = 0, \\ &A_k a_k^{(2)}(s) + B_k b_k^{(2)}(s) + C_k c_k^{(2)}(s) + D_k d_k^{(2)}(s) + \lambda E_k e_k^{(2)}(s) + \lambda F_k f_k^{(2)}(s) = 0, \\ &A_k a_k^{(3)}(R) + B_k b_k^{(3)}(R) + C_k c_k^{(3)}(R) + D_k d_k^{(3)}(R) + \lambda E_k e_k^{(3)}(R) + \lambda F_k f_k^{(3)}(R) = 0, \\ &A_k a_k^{(3)}(s) + B_k b_k^{(3)}(s) + C_k c_k^{(3)}(s) + D_k d_k^{(3)}(s) + \lambda E_k e_k^{(3)}(s) + \lambda F_k f_k^{(3)}(s) = 0, \end{aligned} \tag{2.5}$$

where

$$\psi_k(\lambda, R) = \frac{2}{\pi} \int_0^\infty \cos \lambda z\, dz \int_0^\pi f(z, \varphi) \cos k\varphi\, d\varphi, \tag{2.6}$$

$$\begin{aligned} a_k^{(1)}(x) = I_k(\lambda x) - \frac{k+1}{\lambda x} I_{k+1}(\lambda x) - \frac{1}{4(1-\nu)} \Big[\lambda x I_{k+1}(\lambda x) + I_k(\lambda x) + \\ + \frac{k(k+1)}{\lambda x} I_{k+1}(\lambda x) \Big] + \frac{\nu}{2(1-\nu)} I_k(\lambda x), \end{aligned} \tag{2.7}$$

$$
\left.
\begin{aligned}
a_k^{(2)}(x) &= -I_{k+1}(\lambda x) + \frac{1}{2(1-\nu)}\,[\lambda x I_k(\lambda x) - kI_{k+1}(\lambda x)],\\
a_k^{(3)}(x) &= I_{k+2}(\lambda x) + \frac{k}{2(1-\nu)}\left[I_k(\lambda x) - \frac{(k+1)\,I_{k+1}(\lambda x)}{\lambda x}\right],\\
b_k^{(1)}(x) &= -\left\{K_k(\lambda x) + \frac{k+1}{\lambda x}K_{k+1}(\lambda x) - \frac{1}{4(1-\nu)}\times\right.\\
&\times\left.\left[K_k(\lambda x) - \lambda x K_{k+1}(\lambda x) - \frac{k(k+1)}{\lambda x}K_{k+1}(\lambda x)\right] + \frac{\nu}{2(1-\nu)}K_k(\lambda x)\right\},\\
b_k^{(2)}(x) &= -\left\{K_{k+1}(\lambda x) + \frac{1}{2(1-\nu)}\,[\lambda x K_k(\lambda x) + kK_{k+1}(\lambda x)]\right\},\\
b_k^{(3)}(x) &= -\left\{K_{k+2}(\lambda x) + \frac{k}{2(1-\nu)}\left[K_k(\lambda x) + \frac{k+1}{\lambda x}K_{k+1}(\lambda x)\right]\right\};\\
c_k^{(1)}(x) &= \frac{1}{4(1-\nu)}\left[2\nu I_k(\lambda x) - \lambda x I_{k+1}(\lambda x) - I_k(\lambda x) - \frac{k(k+1)}{\lambda x}I_{k+1}(\lambda x)\right],\\
c_k^{(2)}(x) &= \frac{k}{\lambda x}I_k(\lambda x) + I_{k+1}(\lambda x) + \frac{1}{2(1-\nu)}\,[\lambda x I_k(\lambda x) - kI_{k+1}(\lambda x)],\\
c_k^{(3)}(x) &= \frac{k}{2(1-\nu)}\left[I_k(\lambda x) - \frac{k+1}{\lambda x}I_{k+1}(\lambda x)\right],\\
d_k^{(1)}(x) &= \frac{1}{4(1-\nu)}\left[-(1-2\nu)K_k(\lambda x) + \lambda x K_{k+1}(\lambda x) + \right.\\
&\left. + \frac{k(k+1)}{\lambda x}K_{k+1}(\lambda x)\right],\\
d_k^{(2)}(x) &= \frac{k}{\lambda x}K_k(\lambda x) - K_{k+1}(\lambda x) + \frac{1}{2(1-\nu)}\,[\lambda x K_k(\lambda x) + kK_{k+1}(\lambda x)],\\
d_k^{(3)}(x) &= \frac{k}{2(1-\nu)}\left[K_k(\lambda x) + \frac{k+1}{\lambda x}K_{k+1}(\lambda x)\right],\\
e_k^{(1)}(x) &= -\frac{1}{4(1-\nu)}\left[\frac{k(k-1)}{(\lambda x)^2}I_k(\lambda x) + I_k(\lambda x) - \frac{I_{k+1}(\lambda x)}{\lambda x}\right],\\
e_k^{(2)}(x) &= \frac{1}{2(1-\nu)}\left[\frac{k}{\lambda x}I_k(\lambda x) + I_{k+1}(\lambda x)\right],\\
e_k^{(3)}(x) &= \frac{k}{2(1-\nu)}\left[\frac{k-1}{(\lambda x)^2}I_k(\lambda x) + \frac{I_{k+1}(\lambda x)}{\lambda x}\right],\\
f_k^{(1)}(x) &= -\frac{1}{4(1-\nu)}\left[\frac{k(k-1)}{(\lambda x)^2}K_k(\lambda x) + K_k(\lambda x) + \frac{K_{k+1}(\lambda x)}{\lambda x}\right],\\
f_k^{(2)}(x) &= \frac{1}{2(1-\nu)}\left[\frac{k}{\lambda x}K_k(\lambda x) - K_{k+1}(\lambda x)\right],\\
f_k^{(3)}(x) &= \frac{k}{2(1-\nu)}\left[\frac{k-1}{(\lambda x)^2}K_k(\lambda x) - \frac{K_{k+1}(\lambda x)}{\lambda x}\right].
\end{aligned}
\right\}
\quad (2.7)\ \text{(cont'd)}
$$

The solution of system (2.5) and the determination of the integration coefficients end the solution of the problem.

The displacements and stresses are determined from relationships expressed through Fourier series by the coordinate φ and Fourier integrals by the coordinate z.

3. Consider now a finite pipe of length $2l$ subjected to the action of the same bending load as in the above case of an infinite pipe. The boundary conditions will therefore be once more (2.1) and (2.3).

Let the following conditions be fulfilled at the end $z = l$:

$$u_z(r, \varphi, l) = f_1(r, \varphi), \quad \tau_{rz}(r, \varphi, l) = f_2(r, \varphi), \quad \tau_{\varphi z}(r, \varphi, l) = f_3(r, \varphi), \tag{3.1}$$

where f_1 is a bounded and continuous function, and f_2 and f_3 are bounded and continuous by parts within the ranges of variation of the variables r and φ.

We shall represent the functions $\varphi_{ik}(r, z)$ for this problem in the form:

$$\begin{aligned}
\varphi_{1k}(r, z) &= \sum_{m=0}^{\infty} \varphi_{1km}^{(1)}(r) \cos \gamma_m z + \sum_{m=1}^{\infty} \varphi_{1km}^{(2)}(z)\, W_{k+1}(\lambda_m r), \\
\varphi_{0k}(r, z) &= \sum_{m=0}^{\infty} \varphi_{0km}^{(1)}(r) \cos \gamma_m z + \sum_{m=1}^{\infty} \varphi_{0km}^{(2)}(z)\, W_k(\lambda_m r) + \varphi_{0k0}^{(2)}(r, z), \\
\varphi_{3k}(r, z) &= \sum_{m=1}^{\infty} \varphi_{3km}^{(1)}(r) \sin \gamma_m z + \sum_{m=1}^{\infty} \varphi_{3km}^{(2)}(z)\, W_k(\lambda_m r) + \varphi_{3k0}^{(2)}(r, z),
\end{aligned} \tag{3.2}$$

where

$$\gamma_m = \frac{m\pi}{l},$$

$$W_i(\lambda_m r) = \frac{J_i(\lambda_m r)}{J_{k+1}(\lambda_m R)} - \frac{Y_i(\lambda_m r)}{Y_{k+1}(\lambda_m R)}, \tag{3.3}$$

$J_i(x)$ and $Y_i(x)$ are Bessel functions of a real argument of the first and second kinds respectively.

The determination of the integration constants can be reduced to solving linear equations of the type (2.5) in this problem as well.

If conditions (3.1) are replaced by more general conditions, the solution of the problem will reduce to infinite systems of linear equations.

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GENERAL EQUATIONS OF NONSYMMETRICAL SANDWICH SHELLS WITH A LIGHT FILLER

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Consider a thin nonsymmetrical sandwich shell of arbitrary shape loaded with a normal pressure. It is assumed that the filler is uncompressible but light, i. e., that it is subjected to transverse shear only. The normal element of the filler does not bend after the deformation and remains rectilinear. The Kirchhoff-Love hypothesis is true for the bearing layers. The problem will be solved by the variational method.

In the state of equilibrium

$$\delta\Pi = \delta\Pi_b + \delta\Pi_f - \delta Q = 0, \tag{1}$$

where $\delta\Pi$ is the variation of the shell potential; $\delta\Pi_b$ is the variation of the potential energy of the faces; $\delta\Pi_f$ is the variation of the potential energy of the filler; δQ is the variation of the work of the external forces.

The potential of the shell considered will be

$$\begin{aligned}
\delta\Pi = \iint \Big\{ \sum_{i=1,2} \Big[& T_{1i}\Big(\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\delta u_i + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta v_i + \frac{1}{R_1}\delta w\Big) + \\
& + T_{2i}\Big(\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\delta v_i + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta u_i + \frac{1}{R_2}\delta w\Big) + \\
& + S_i\Big(\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\partial v_i - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta v_i + \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\delta u_i - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_i\Big) + \\
& + M_{1i}\Big(\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\delta\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{1}{R_1}\frac{\partial}{A_1\partial\alpha_1}\delta u_i + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \\
& - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{1}{R_2}\delta v_i\Big) + M_{2i}\Big(\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\delta\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{1}{R_2}\frac{\partial}{A_2\partial\alpha_2}\delta v_i + \\
& + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{1}{R_1}\delta u_i\Big) + \\
& + 2H_i\frac{1}{A_1A_2}\Big(\delta\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial}{\partial\alpha_1}\partial w - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial}{\partial\alpha_2}\delta w - \frac{A_1}{R_1}\frac{\partial}{\partial\alpha_2}\delta u_i - \\
& - \frac{1}{R_1}\frac{\partial A_1}{\partial\alpha_2}\delta u_i - \frac{A_2}{R_2}\frac{\partial}{\partial\alpha_1}\delta v_i - \frac{1}{R_2}\frac{\partial A_2}{\partial\alpha_1}\delta v_i\Big)\Big] + \\
& + \frac{c+t_c}{c}\Big[N_1\delta\Big(\alpha + \frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\Big) + N_2\delta\Big(\beta + \frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\Big)\Big] - q\delta w\Big\} A_1A_2 d\alpha_1 d\alpha_2 = 0,
\end{aligned} \tag{2}$$

where T_{1i}, T_{2i}, S_i are forces per unit thickness in the faces; M_{1i}, M_{2i}, H_i are moments in the faces; N_1, N_2 are shearing forces per unit thickness in the filler; A_1, A_2 are Lamé parameters; α_1, α_2 are curvilinear coordinates; $u=\frac{u_1+u_2}{2}$; $v=\frac{v_1+v_2}{2}$ are the half-sums of the tangential displacements; w, u_i, v_i are displacements of the faces; R_1, R_2 are radii of the principal curvatures of the shell; $\alpha=\frac{u_1-u_2}{c+t_c}$, $\beta=\frac{v_1-v_2}{c+t_c}$ are relative differences between the tangential displacements of the lower and upper faces; c is the thickness of the filler layer; $t_c=\frac{t_1+t_2}{2}$ is the reduced thickness of the faces; t_1 is the thickness of the upper face; t_2 is the thickness of the lower face.

The displacements and angles of shear of the filler are related to those of the faces by the following expressions:

$$u_f=u+z\frac{c+t_c}{c}\left(\alpha+\frac{t_c}{c+t_c}\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right)+\frac{t_1-t_2}{4}\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1};$$
$$v_f=v+z\frac{c+t_c}{c}\left(\beta+\frac{t_c}{c+t_c}\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right)+\frac{t_1-t_2}{4}\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}; \tag{3}$$

$$\gamma_1=\frac{c+t_c}{c}\left(\alpha+\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right);\qquad \gamma_2=\frac{c+t_c}{c}\left(\beta+\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right). \tag{4}$$

In the symbols adopted, the tangential displacements of the faces will be

$$u_i=u\pm\frac{c+t_c}{2}\alpha;\qquad v_i=v\pm\frac{c+t_c}{2}\beta. \tag{5}$$

Equation (2) can be represented after integration by parts and regrouping of the terms in the form:

$$\iint\Bigg\{\sum_{i=1,2}\Bigg[\Bigg(\frac{\partial A_2T_{1i}}{\partial\alpha_1}-T_{2i}\frac{\partial A_2}{\partial\alpha_1}+\frac{\partial A_1}{\partial\alpha_2}S_i+\frac{\partial A_1S_i}{\partial\alpha_2}+\frac{1}{R_1}\frac{\partial A_2M_{1i}}{\partial\alpha_1}-$$
$$-\frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}M_{2i}+2\frac{1}{R_1}\frac{\partial A_1H_i}{\partial\alpha_2}+2\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2}H_i\Bigg)\delta u_i+\Bigg(\frac{\partial A_1T_{2i}}{\partial\alpha_2}-T_{1i}\frac{\partial A_1}{\partial\alpha_2}+$$
$$+\frac{\partial A_2}{\partial\alpha_1}S_i+\frac{\partial A_2S_i}{\partial\alpha_1}+\frac{1}{R_2}\frac{\partial A_1M_{2i}}{\partial\alpha_2}-\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2}M_{1i}+2\frac{1}{R_2}\frac{\partial A_2H_i}{\partial\alpha_1}+$$
$$+2\frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}H_i\Bigg)\delta v_i+\Bigg[\frac{T_{1i}}{R_1}+\frac{T_{2i}}{R_2}-\frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\Bigg(\frac{\partial A_2M_{1i}}{\partial\alpha_1}+$$
$$+\frac{\partial A_1H_i}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}H_i-\frac{\partial A_2}{\partial\alpha_1}M_{2i}\Bigg)-\frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\Bigg(\frac{\partial A_1M_{2i}}{\partial\alpha_2}+\frac{\partial A_2H_i}{\partial\alpha_1}+$$
$$+\frac{\partial A_2}{\partial\alpha_1}H_i-\frac{\partial A_1}{\partial\alpha_2}M_{1i}\Bigg)\Bigg]\delta w\Bigg]-q\delta w-$$
$$-\frac{c+t_c}{c}\left[N_1\delta\alpha+N_2\delta\beta+\frac{1}{A_1A_2}\left(\frac{\partial A_2N_1}{\partial\alpha_1}+\frac{\partial A_1N_2}{\partial\alpha_2}\right)\partial w\right]\Bigg\}A_1A_2d\alpha_1d\alpha_2+$$
$$+\Phi=0. \tag{6}$$

The equilibrium equations become particularly simple if they are formed for several sum and difference combinations of the forces and moments of the faces.

Actually, these will be the equilibrium equations of some equivalent shell in which forces and moments act in sum and difference combinations in the sections, as well as shearing stresses in the filler. These combinations can be obtained by substituting in (6) the values of u_t and v_t by formulas (5).

The potential of the equivalent shell differs from the potential of a one-piece shell by the presence of terms characterizing the filler energy. It follows that the system of equations of the sandwich shell will contain two more equations than the system corresponding to the one-piece shell (corresponding to the variation of the parameters α and β) and additional terms in the third equation (corresponding to the variation of the normal displacement due to the shear in the filler).

Using the formulation proposed, we shall transform system (6), by introducing the following symbols:

$$\begin{aligned}
&M_{11}+M_{12}=M_1; \quad m_1=\frac{c+t_c}{2}(T_{11}-T_{12}); \quad \Omega_1=\frac{c+t_c}{2}(M_{11}-M_{12});\\
&M_{21}+M_{22}=M_2; \\
&H_1+H_2=H; \quad m_2=\frac{c+t_c}{2}(T_{21}-T_{22}); \quad \Omega_2=\frac{c+t_c}{2}(M_{21}-M_{22}); \qquad (7)\\
&T_{11}+T_{12}=T_1; \\
&T_{21}+T_{22}=T_2; \quad m_{12}=\frac{c+t_c}{2}(S_1-S_2); \quad \Omega_{12}=\frac{c+t_c}{2}(H_1-H_2).\\
&S_1+S_2=S;
\end{aligned}$$

By the method of variation of the displacements u, v, w, α, β we obtain the following system of equilibrium equations of nonsymmetrical sandwich shells with light filler:

$$\left.\begin{aligned}
&1.\ \frac{\partial A_2T_1}{\partial\alpha_1}-T_2\frac{\partial A_1}{\partial\alpha_1}+\frac{\partial A_1}{\partial\alpha_2}S+\frac{\partial A_1S}{\partial\alpha_2}+\frac{1}{R_1}\left(\frac{\partial A_2M_1}{\partial\alpha_1}-\frac{\partial A_2}{\partial\alpha_1}M_2+\right.\\
&\qquad\left.+2\frac{\partial A_1H}{\partial\alpha_2}+2\frac{R_1}{R_2}\frac{\partial A_1}{\partial\alpha_2}H\right)=0;\\
&2.\ \frac{\partial A_1T_2}{\partial\alpha_2}-T_1\frac{\partial A_2}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}S+\frac{\partial A_2S}{\partial\alpha_1}+\frac{1}{R^2}\left(\frac{\partial A_1M_2}{\partial\alpha_2}-\right.\\
&\qquad\left.-\frac{\partial A_1}{\partial\alpha_2}M_1+2\frac{\partial A_2H}{\partial\alpha_1}+2\frac{R_2}{R_1}\frac{\partial A_2}{\partial\alpha_1}H\right)=0;\\
&3.\ \frac{T_1}{R_1}+\frac{T_2}{R_2}-\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left(\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{\partial A_1H}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}H-\right.\right.\\
&\left.\left.-\frac{\partial A_2}{\partial\alpha_1}M_2\right)+\frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left(\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{\partial A_2H}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}H-\frac{\partial A_1}{\partial\alpha_2}M_1\right)\right]-\\
&\qquad-\frac{c+t_c}{c}\frac{1}{A_1A_2}\left(\frac{\partial N_1A_2}{\partial\alpha_1}+\frac{\partial N_2A_1}{\partial\alpha_2}\right)=q;\\
&4.\ \frac{\partial A_2m_1}{\partial\alpha_1}-m_2\frac{\partial A_2}{\partial\alpha_1}+m_{12}\frac{\partial A_1}{\partial\alpha_2}+\frac{\partial A_1m_2}{\partial\alpha_2}+\\
&\qquad+\frac{1}{R_1}\left(\frac{\partial A_2\Omega_1}{\partial\alpha_1}-\Omega_2\frac{\partial A_2}{\partial\alpha_1}+2\frac{\partial A_2\Omega_{12}}{\partial\alpha_2}+2\Omega_{12}\frac{\partial A_1}{\partial\alpha_2}\right)-\\
&\qquad-A_1A_2\frac{c+t_c}{c}N_1=0;
\end{aligned}\right\}\qquad(8)$$

5

$$\left.\begin{aligned}
&\frac{\partial A_1 m_2}{\partial \alpha_2} - m_1 \frac{\partial A_1}{\partial \alpha_2} + m_{12} \frac{\partial A_2}{\partial \alpha_1} + \frac{\partial A_2 m_1}{\partial \alpha_1} + \\
&+ \frac{1}{R_2}\left(\frac{\partial A_1 \Omega_2}{\partial \alpha_2} - \Omega_1 \frac{\partial A_1}{\partial \alpha_2} + 2\frac{\partial A_2 \Omega_{12}}{\partial \alpha_1} + 2\Omega_{12}\frac{\partial A_2}{\partial \alpha_1}\right) - \\
&- A_1 A_2 \frac{c + t_c}{c} N_2 = 0.
\end{aligned}\right\} \quad \begin{matrix}(8)\\ \text{(cont'd)}\end{matrix}$$

Equations (8) can be transformed into equations in displacements. The formulas connecting the forces with the displacements are

$$\left.\begin{aligned}
T_1 = B\Bigg[&\frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v + \mu\left(\frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u\right) + \\
&+ \frac{w}{R_1 R_2}(R_2 + \mu R_1)\Bigg] + \\
&+ b\left[\frac{1}{A_1}\frac{\partial \alpha}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\beta + \mu\left(\frac{1}{A_2}\frac{\partial \beta}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\alpha\right)\right]; \\
T_2 = B\Bigg[&\frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u + \mu\left(\frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v\right) + \\
&+ \frac{w}{R_1 R_2}(R_1 + \mu R_2)\Bigg] + \\
&+ b\left[\frac{1}{A_2}\frac{\partial \beta}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\alpha + \mu\left(\frac{1}{A_1}\frac{\partial \alpha}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\beta\right)\right];
\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}
M_1 = -D\Bigg\{&\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \right. \\
&\left. - \frac{v}{R_2}\right) + \mu\Bigg[\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \right. \\
&\left.\left. - \frac{u}{R_1}\right)\Bigg]\Bigg\} + d\Bigg[\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\alpha}{R_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\frac{\beta}{R_2} + \\
&+ \mu\left(\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\beta}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\alpha}{R_1}\right)\Bigg]; \\
M_2 = -D\Bigg\{&\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \right. \\
&\left. - \frac{u}{R_1}\right) + \mu\Bigg[\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\left(\frac{\partial w}{A_2 \partial \alpha} - \right. \\
&\left.\left. - \frac{v}{R_2}\right)\Bigg]\Bigg\} + d\Bigg[\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\beta}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\alpha}{R_1} + \\
&+ \mu\left(\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\alpha}{R_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\frac{\beta}{R_2}\right)\Bigg]; \\
S = \frac{1-\mu}{2} B&\left[\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{v}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u}{A_1}\right)\right] + \\
&+ \frac{1-\mu}{2} b\left[\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\beta}{A_2}\right) + \frac{A_1}{A_2}\left(\frac{\alpha}{A_1}\right)\right];
\end{aligned}\right\} \quad (10)$$

$$
\left.\begin{aligned}
H = (1-\mu) D \Bigg[& -\frac{1}{A_1 A_2}\left(\frac{\partial^2 w}{\partial\alpha_1 \partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1} - \right. \\
& \left. - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_2}\right) + \frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u}{\partial\alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2} u\right) + \\
& + \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial v}{\partial\alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1} v\right)\Bigg] + \\
& + (1-\mu) d \Bigg[\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial\alpha}{\partial\alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}\alpha\right) + \\
& + \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial\beta}{\partial\alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}\beta\right)\Bigg];
\end{aligned}\right\} \quad \begin{matrix}(10)\\ (\text{cont'd})\end{matrix}
$$

$$
\left.\begin{aligned}
m_1 = D_0 \Bigg\{ & \Bigg[\frac{1}{A_1}\left(\frac{\partial\alpha}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\beta\right) + \mu\frac{1}{A_2}\left(\frac{\partial\beta}{\partial\alpha_2} + \frac{1}{A_1}\cdot\frac{\partial A_2}{\partial A_1}\alpha\right)\Bigg] + \\
& + \nu\Bigg[\frac{1}{A_1}\frac{\partial u}{\partial\alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2} v + \\
& + \mu\frac{1}{A_2}\left(\frac{\partial v}{\partial\alpha_2} + \frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1} u + \frac{w}{R_2}\right) + \frac{w}{R_1}\Bigg]\Bigg\}; \\
m_2 = D_0 \Bigg\{ & \Bigg[\frac{1}{A_1}\left(\frac{\partial\beta}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_2}\alpha\right) + \mu\frac{1}{A_1}\left(\frac{\partial\alpha}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\beta\right)\Bigg] + \\
& + \nu\Bigg[\frac{1}{A_2}\frac{\partial v}{\partial\alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1} u + \\
& + \mu\frac{1}{A_1}\left(\frac{\partial u}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2} v + \frac{w}{R_1}\right) + \frac{w}{R_2}\Bigg]\Bigg\}; \\
m_{12} = \frac{1-\mu}{2} D_0 & \left[\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\beta}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\alpha}{A_1}\right)\right] + \\
+ \frac{1-\mu}{2} b & \left[\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{v}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u}{A_1}\right)\right];
\end{aligned}\right\} \quad (11)
$$

$$
\left.\begin{aligned}
Q_1 = D \Bigg[& \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\alpha}{R_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{\beta}{R_2} + \\
& + \mu\left(\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\beta}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\alpha}{R_1}\right)\Bigg] - \\
& - d\Bigg\{\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}\right) + \\
& + \mu\Bigg[\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right)\Bigg]\Bigg\}; \\
Q_2 = D_1 \Bigg[& \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\beta}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\alpha}{R_1} + \mu\left(\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\alpha}{R_1}\right) + \right. \\
& \left. + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\beta}{R_2}\right)\Bigg] - \\
& - d\Bigg\{\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right) + \\
& + \mu\Bigg[\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_1} - \frac{v}{R_2}\right)\Bigg]\Bigg\};
\end{aligned}\right\} \quad (12)
$$

$$
\left.\begin{aligned}
Q_{12} = \frac{1-\mu}{2} D_1 \Bigg[& \frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial \alpha}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\alpha\right) + \\
& + \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial \beta}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\beta\right)\Bigg] + \\
+ \frac{1-\mu}{2} d \Bigg[& -\frac{1}{A_1 A_2}\left(\frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial w}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial w}{\partial \alpha_2}\right) + \\
& + \frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u\right) + \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)\Bigg];
\end{aligned}\right\} \quad \begin{matrix}(12)\\ \text{(cont'd)}\end{matrix}
$$

$$
N_1 = G_3 (c + t_c)\left(\alpha + \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}\right), \qquad N_2 = G_3 (c + t_c)\left(\beta + \frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}\right), \qquad (13)
$$

where

$$
\begin{aligned}
& B = \frac{2Et_c}{1-\mu^2}; \quad D_0 = \frac{E(c+t_c)^2}{2(1-\mu^2)}; \quad D = \frac{E(t_1^3 + t_2^3)}{12(1-\mu^2)}; \\
& b = \frac{E(c+t_c)}{2(1-\mu^2)}(t_1 - t_2); \qquad d = \frac{E(c+t_c)}{24(1-\mu^2)}(t_1^3 - t_2^3); \qquad (14) \\
& D_1 = \frac{E(c+t_c)^2}{48(1-\mu^2)}(t_1^3 + t_2^3); \qquad \nu = \frac{t_1 - t_2}{t_c(c+t_c)}.
\end{aligned}
$$

The transformation of the equations in forces into equations in displacements is in the general case a very cumbersome and unexpedient operation. The procedure to be used will be illustrated on the example of a circular cylindrical shell.

In this case:

$$
R_1 = \infty; \quad R_2 = r = \text{const}; \quad A_1 = A_2 = r; \quad \alpha_1 = \xi = \frac{x}{r}; \quad \alpha_2 = \theta = \frac{s}{r}.
$$

The substitution of the values of the forces and moments by (9) to (13) into the equilibrium equations yields

$$
\left.\begin{aligned}
& \left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \theta^2}\right)u + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial \xi \partial \theta} + \mu\frac{\partial w}{\partial \xi} + \\
& + \delta\left[\left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \theta^2}\right)\alpha + \frac{1+\mu}{2}\frac{\partial^2 \beta}{\partial \xi \partial \theta}\right] = 0; \\
& \frac{1+\mu}{2}\frac{\partial^2 u}{\partial \xi \partial \theta} + \left(\frac{\partial^2}{\partial \theta^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \xi^2}\right)v + L(w) + \\
& + \delta\left[\left(\frac{\partial^2}{\partial \theta^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \xi^2}\right)\beta + \frac{1+\mu}{2}\frac{\partial^2 \alpha}{\partial \xi \partial \theta}\right] = 0; \\
& \mu\frac{\partial u}{\partial \xi} + L(v) + \left[1 - \frac{k}{B}\nabla^2 + \frac{t_n'}{12r^2}\nabla^2\nabla^2\right]w + \\
& + \left(\mu\delta - \frac{kr}{B}\right)\frac{\partial \alpha}{\partial \xi} + \delta L_1(\beta) - \frac{kr}{B}\frac{\partial \beta}{\partial \theta} = \frac{r^2}{B}q; \\
& \left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \theta^2}\right)\alpha + \frac{1+\mu}{2}\frac{\partial^2 \beta}{\partial \xi \partial \theta} - \frac{4r^2 k}{(c+t_c)^2 B}\left(\alpha + \frac{\partial w}{r\partial \xi}\right) + \\
& + \nu\left\{\left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \theta^2}\right)u + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial \xi \partial \theta} + \mu\frac{\partial w}{\partial \xi}\right\} = 0.
\end{aligned}\right\} \quad (15)
$$

$$\left.\begin{aligned}&\left(\frac{\partial^2}{\partial\theta^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial\xi^2}\right)\beta+\frac{1-\mu}{2}\frac{\partial^2\alpha}{\partial\xi\partial\theta}+\nu\left\{\left(\frac{\partial^2}{\partial\theta^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial\xi^2}\right)v+\right.\\&\left.+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial\xi\partial\theta}+L_1(w)\right\}-\frac{4r^2k}{(c+t_c)^2B}\left(\beta+\frac{\partial w}{r\partial\theta}\right)=0,\end{aligned}\right\}\quad\begin{matrix}(15)\\(\text{cont'd})\end{matrix}$$

where

$$k=\frac{G_3(c+t_c)^2}{c},\qquad t_n'=t_1^2-t_1t_2+t_2^2.$$

Equations (15) are transformed by introducing the displacement functions

$$\frac{\partial\Phi}{\partial\xi}=u;\quad \frac{\partial\Phi}{\partial\theta}=v;\quad \frac{\partial F}{\partial\xi}=\alpha;\quad \frac{\partial F}{\partial\theta}=\beta \tag{16}$$

into the following three equilibrium equations:

$$\left.\begin{aligned}&1.\quad \nabla^2\nabla^2\Phi+\delta\nabla^2\nabla^2F+\frac{\partial}{\partial\theta}L(w)+\mu\frac{\partial^2 w}{\partial\xi^2}=0;\\&2.\quad \frac{\partial}{\partial\theta}L(\Phi)+\mu\frac{\partial^2\Phi}{\partial\xi^2}+\delta\left[\frac{\partial}{\partial\theta}L_1(F)+\mu\frac{\partial^2F}{\partial\xi^2}\right]-\\&\qquad -\frac{kr}{B}\nabla^2F+\left[1-\frac{k}{B}\nabla^2+\frac{t_n'}{12r^2}\nabla^2\nabla^2\right]w=\frac{r^2}{B}q;\\&3.\quad \nabla^2\nabla^2F-\frac{4r^2k}{B(c+t_c)}\nabla^2F+\nu\nabla^2\nabla^2\Phi+\nu\left[\frac{\partial}{\partial\theta}L_1(w)+\right.\\&\qquad \left.+\mu\frac{\partial^2 w}{\partial\xi^2}\right]-\frac{4rk}{B(c+t_c)^2}w=0,\end{aligned}\right\}\tag{17}$$

where

$$\nabla^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\theta^2};\qquad \delta=\frac{t_1-t_2}{4t_c}(c+t_c);$$

$$L_1=\frac{\partial}{\partial\theta}-\frac{t_1^2+t_1t_2+t_2^2}{12r^2}\left[\frac{\partial^3}{\partial\theta^3}+(2-\mu)\frac{\partial^3}{\partial\xi^2\partial\theta}\right];$$

$$L=\frac{\partial}{\partial\theta}-\frac{t_1^2-t_1t_2+t_2^2}{12r^2}\left[\frac{\partial^3}{\partial\theta^3}+(2-\mu)\frac{\partial^3}{\partial\xi^2\partial\theta}\right].$$

If the bending and chain stresses are commensurable, equations (17) simplify and reduce to

$$\left.\begin{aligned}&1.\quad \nabla^2\nabla^2\Phi+\delta\nabla^2\nabla^2F+\left(\frac{\partial^2}{\partial\theta^2}+\mu\frac{\partial^2}{\partial\xi^2}\right)w=0;\\&2.\quad \left(\frac{\partial^2}{\partial\theta^2}+\mu\frac{\partial^2}{\partial\xi^2}\right)\Phi+\left(1-\frac{k}{B}\nabla^2+\frac{t_n'}{12r^2}\nabla^2\nabla^2\right)w-\\&\qquad -\left[\frac{kr}{B}\nabla^2+\delta\left(\frac{\partial^2}{\partial\theta^2}+\mu\frac{\partial^2}{\partial\xi^2}\right)\right]F=\frac{r^2}{B}q;\end{aligned}\right\}\tag{18}$$

$$3. \qquad \left.\begin{aligned} &\nu\nabla^2\nabla^2\Phi + \left[\frac{rk}{D_0}\nabla^2 + \nu\left(\mu\frac{\partial^2}{\partial\xi^2} + \frac{\partial}{\partial\theta}L_1\right)\right]w + \\ &+\left(\nabla^2\nabla^2 - \frac{r^2k}{D_0}\nabla^2\right)F = 0. \end{aligned}\right\} \qquad \begin{matrix}(18)\\ \text{(cont'd)}\end{matrix}$$

This system is reduced by eliminating the functions Φ and F to the following equation:

$$\left\{\left[\underline{\frac{D}{r^3}\nabla^2\nabla^2\nabla^2\nabla^2} + \frac{2Et_c}{(1-\mu^2)r}\nabla^2\nabla^2 - \frac{2Et_c}{(1-\mu^2)r}\left(\frac{\partial^2}{\partial\theta^2} + \mu\frac{\partial^2}{\partial\xi^2}\right)^2\right]\times\right.$$
$$\times\left[1 - \frac{(1-\delta\nu)D_0}{r^2k}\nabla^2\right] + \frac{1-\delta\nu}{r^3}\nabla^2\nabla^2\nabla^2\nabla^2 -$$
$$\left. -\frac{\nu t_n^{''}D_0}{12r^3}\nabla^2\nabla^2\left[\frac{\partial^4}{\partial\theta^4} + (2-\mu)\frac{\partial^4}{\partial\xi^2\partial\theta^2}\right]\right\}w = r\nabla^2\nabla^2\left[1 - \frac{(1-\delta\nu)D_0}{r^2k}\nabla^2\right]q, \qquad (19)$$

where

$$t_n^{''} = t_1^2 + t_1t_2 + t_2^2.$$

Equation (19) can be used for determining the critical load of a circular cylindrical shell subjected to the action of an all-round hydrostatic pressure.

If the load and deflection are represented by the following expressions:

$$q = -pr\left(\frac{\partial^2 w}{2r^2\partial\xi^2} + \frac{\partial^2 w}{r^2\partial\theta^2} + \frac{w}{r^2}\right), \qquad (20)$$

$$w = A\sin\lambda\xi\sin n\theta, \qquad \lambda = \frac{\pi r}{l}, \qquad (21)$$

the following formula is obtained for the critical load of a nonsymmetrical shell

$$p = \frac{2Et_c}{r(n^2+0.5\lambda^2-1)}\left\{\frac{\lambda^4}{(n^2+\lambda^2)^2} + \underline{\frac{D(n^2+\lambda^2)^2}{2r^2Et_c}} + \right.$$
$$+\frac{D_0(1-\delta\nu)(n^2+\lambda^2)^2}{2Et_cr^2\left[1+\frac{(1-\delta\nu)D_0}{r^2k}(n^2+\lambda^2)\right]} + \frac{2n^2\lambda^2}{(1+\mu)(n^2+\lambda^2)^2} -$$
$$\left. -\frac{\nu t_n^{''}}{24Et_cr^2}\,\frac{n^4+(2-\mu)n^2\lambda^2}{\frac{(1-\delta\nu)}{r^2k}(n^2+\lambda^2)}\right\}. \qquad (22)$$

The underlined terms in (19) and (22) are to be neglected if the flexural rigidity of the faces is neglected.

The following equation is obtained from (15) as a particular case corresponding to the symmetrical deformation of a circular symmetrical cylindrical shell*:

$$\frac{DD_0}{k}w^{\mathrm{VI}} + \left[\frac{D_0}{k}(-T_1-k) - D\right]w^{\mathrm{IV}} + \left(\frac{2D_0Et}{kr^2} - T_1\right)w'' -$$
$$-\frac{2Et}{r^2}w = \left(\frac{\mu}{2}-1\right)\left(1 - \frac{D_0}{k}\frac{d^2}{dx^2}\right)p. \qquad (23)$$

* Equation (23) was obtained by M. Ya. Sharaev.

In the case of hydrostatic pressure, and if the flexural rigidity of the faces is neglected, equation (23) reduces to the well-known equation

$$D_0\left(1-\frac{pr}{2k}\right)w^{\mathrm{IV}}+\left(\frac{pr}{2}-\frac{2D_0Et}{kr^2}\right)w''+\frac{2Et}{r^2}w=p\left(1-\frac{\mu}{2}\right),$$

$$\left(T_1=-\frac{pr}{2}\right).$$

At $k\to\infty$ this equation is transformed into the Timoshenko-Papkovich equation.

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ON THE BENDING OF A CANTILEVER PLATE WEAKENED BY A HOLE OF ARBITRARY SHAPE

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(Saratov)

The paper deals with the problem of the concentration of stresses in a bent thin isotropic plate having a hole of arbitrary shape.

The problem is solved by Muskhelishvili's method /1/, making use of the similitude established by Lekhnitskii /2/. A solution is also given to the first and second fundamental problems, in the same way as done in Burmistrov's paper /3/.

The problem reduces to finding two functions of a complex variable satisfying on the contour of circle γ the condition

$$\Phi(\sigma) + \varkappa\overline{\Phi(\sigma)} - \frac{\sigma^2}{\overline{\omega'(\sigma)}}\left[\overline{\omega(\sigma)}\,\Phi'(\sigma) + \omega'(\sigma)\,\psi(\sigma)\right] = F(\sigma) + iC_1. \qquad (1)$$

Here $\sigma = e^{i\vartheta}$ is a point on γ; $\Phi(\zeta) = \dfrac{\varphi'(\zeta)}{\omega'(\zeta)}$, $\Psi(\xi) = \dfrac{\psi'(\zeta)}{\omega'(\zeta)}$ are defined by

$$\Phi(\zeta) = \Phi_1(\zeta) + \Phi_0(\zeta), \qquad \Phi_0(\zeta) = \sum_{k=1}^{\infty} a_k\zeta^{-k},$$
$$\Psi(\zeta) = \Psi_1(\zeta) + \Psi_0(\zeta), \qquad \Psi_0(\zeta) = \sum_{k=1}^{\infty} a_k'\zeta^{-k}. \qquad (2)$$

The functions $\Phi_1(\zeta)$ and $\Psi_1(\zeta)$ are defined by the stressed state in the continuous plate.

Let a hole be cut at the center of a cantilever plate of width $2b$, and let the hole contour be defined by the equations /3/

$$x = R\left(\cos\vartheta + \sum_{k=1}^{n} S_k\cos k\vartheta\right), \qquad y = R\left(\sin\vartheta - \sum_{k=1}^{n} S_k\sin k\vartheta\right). \qquad (3)$$

Let the plate be bent by a uniformly distributed load of intensity p as indicated in the drawing.

The functions defining the stressed state in a continuous plate are

$$\Phi_1(\zeta) = \frac{2p}{8D}\left[i\omega(\zeta) + b\right], \qquad \Psi_1(\zeta) = -\frac{2p}{8D}\left[i\omega(\zeta) + 2b\right]. \qquad (4)$$

Let the function /3/

$$z = \omega(\zeta) = R\left(\zeta + \sum_{k=1}^{n} S_k\zeta^{-k}\right) \qquad (5)$$

map conformally the exterior of the circle γ into the exterior of an infinite plate with a hole (3).

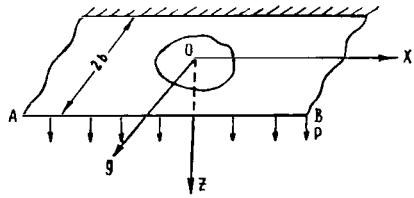

FIGURE.

By integrating (1) as indicated in /3/ we obtain

$$\Phi_0(\zeta) = \frac{1}{\varkappa\left(1-\sum_{k=1}^{n} kS_k\zeta^{-k-1}\right)} \left\{ \frac{\overline{a_1'}}{\zeta} + \sum_{m=2}^{n}\zeta^{-m}\sum_{k=m+1}^{n}(m-1)\,S_k\bar{a}_{k-(m-1)} + \right.$$

$$+ iC_1\sum_{k=1}^{n} kS_k\zeta^{-k-1} + L_2(1+\varkappa)\sum_{k=1}^{n} kS_k\zeta^{-k-1} - L_1 iR\left[\sum_{k=1}^{n}(k+1)\,S_k\zeta^{-k-2} + \right.$$

$$\left. + \sum_{m=2}^{n} S_{m-1}\sum_{k=m}^{n} S_k[k-(m-1)]\zeta^{-k+(m-2)}\right] +$$

$$\left. + \varkappa L_1 iR\left[\sum_{k=2}^{n}(k-1)\,S_k\zeta^{-k} + \sum_{m=1}^{n} S_m\sum_{k=1}^{n} kS_k\zeta^{-k-(m+1)}\right] - L_3 iR\,\frac{1}{\zeta^3} + \frac{L_4}{\zeta^2}\right\}. \qquad (6)$$

The use of the condition of single-valuedness of the displacements /3/ for constant C_1 yields the expression

$$C_1 = \sum_{k=1}^{n} S_k\beta_{k+1} + L_1 R\sum_{k=2}^{n} S_k S_{k-1} + L_3 R S_2. \qquad (7)$$

The coefficients $a_k = \alpha_k + i\beta_k$ in (6) are found from the system

$$\varkappa a_2 - \sum_{k=3}^{n} S_k\bar{a}_{k-1} = S_1[iC_1 + L_2(1+\varkappa)] - L_1 iR\left[\sum_{k=2}^{n} S_k S_{k-1} - \varkappa S_2\right] + L_4$$

$$\varkappa a_3 - 2\sum_{k=4}^{n} S_k\bar{a}_{k-2} = 2S_2[iC_1 + L_2(1+\varkappa)] -$$

$$- 2L_1 iR\left[\sum_{k=3}^{n} S_k S_{k-2} + S_1\right] + \varkappa L_1 iR(2S_3 + S_1^2) - L_3 iR$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\varkappa\left[a_m - \sum_{k=2}^{m-2}(k-1)\,a_{m-k}S_{k-1}\right] - (m-1)\sum_{k=m+1}^{n} S_k\bar{a}_{k-(m-1)} =$$

$$= (m-1)\left\{S_{m-1}[iC_1 + L_2(1+\varkappa)] - L_1 iR\left[S_{m-2} + \sum_{k=m}^{n} S_k S_{k-(m-1)}\right]\right\} +$$

$$+ \varkappa L_1 R\left\{(m-1)\,S_m + \sum_{k=1}^{m-2}[m-(k+1)]\,S_k S_{m-(k+1)}\right\},$$

where $m = 4, 5, \ldots, n+1$.

The bending moment at the hole contour will be given by the following expression:

$$M_\vartheta = [(1-\nu)(1-\varkappa)+4\nu]\,D\left\{L_1R\left(\sin\vartheta - \sum_{k=1}^{n} S_k \sin k\vartheta\right) - E\right\} -$$

$$-\frac{[(1-\nu)(1-\varkappa)+4\nu]\,D}{\varkappa L}\left\{\sum_{m=2}^{n}(m-1)\sum_{k=m+1}^{n} S_k\left[\cos m\vartheta - \right.\right.$$

$$\left.-\sum_{\lambda=1}^{n}\lambda S_\lambda \cos(m-\lambda-1)\vartheta\right]\alpha_{k-(m-1)} -$$

$$-\sum_{m=2}^{n}(m-1)\sum_{k=m+1}^{n} S_k\left[\sin m\vartheta - \sum_{\lambda=1}^{n}\lambda S_\lambda \sin(m-\lambda-1)\vartheta\right]\beta_{k-(m-1)} +$$

$$+C_1\sum_{k=1}^{n} kS_k \sin(k+1)\vartheta + L_2(1+\varkappa)\sum_{k=1}^{n} kS_k\left[\cos(k+1)\vartheta - \right.$$

$$\left.-\sum_{\lambda=1}^{n}\lambda S_\lambda \cos(k-1)\vartheta\right] - L_1R\left\{\sum_{k=1}^{n}(k+1)S_k\left[\sin(k+2)\vartheta - \right.\right.$$

$$\left.-\sum_{\lambda=1}^{n}\lambda S_\lambda \sin(k-\lambda+1)\vartheta\right] + \sum_{m=2}^{n} S_{m-1}\sum_{k=m}^{n}[k-(m-1)]\,S_k \times$$

$$\left.\times\left[\sin(k-m+2)\vartheta - \sum_{\lambda=1}^{n}\lambda S_\lambda \sin(k-m-\lambda+1)\vartheta\right]\right\} +$$

$$+\varkappa L_1R\left\{\sum_{k=2}^{n}(k-1)S_k\left[\sin k\vartheta - \sum_{\lambda=1}^{n}\lambda S_\lambda \sin(k-\lambda-1)\vartheta\right] + \right.$$

$$\left.+\sum_{m=1}^{n}\sin\sum_{k=1}^{n} kS_k\left[\sin(k+m+1)\vartheta - \sum_{\lambda=1}^{n}\lambda S_\lambda \sin(k+m-\lambda)\vartheta\right]\right\} -$$

$$-L_3R\left[\sin 3\vartheta + \sum_{\lambda=1}^{n}\lambda S_\lambda \sin(\lambda-2)\vartheta\right] +$$

$$\left.+L_4\left[\cos 2\vartheta - \sum_{k=1}^{n} kS_k \cos(k-1)\vartheta\right]\right\};$$

where

$$L = 1 + \sum_{k=1}^{n} k^2S_k^2 + 2\left\{\cos\vartheta\sum_{k=2}^{n} k(k-1)S_kS_{k-1} - \right.$$

$$\left.-\sum_{k=1}^{n} kS_k \cos(k+1)\vartheta + \sum_{k=3}^{n}\cos(m-1)\vartheta + \sum_{\lambda=k}^{n}\lambda[\lambda-(k-1)]\,S_\lambda S_{\lambda-(k-1)}\right\};$$

$$L_1 = \frac{2p}{8D}; \qquad L_2 = \frac{2pb}{8D}; \qquad L_3 = -\frac{2p}{8D}; \qquad L_4 = -\frac{4pb}{8D}.$$

The extreme values of the bending moment on the contours of several holes for the two problems are given in the table.

It is interesting to note that the constant C_1 determined by (7) is, in the first problem (empty hole), equal to zero for all the holes having two and more axes of symmetry and different from zero for all holes with a single axis of symmetry. In the second problem (hole filled by an absolutely rigid core) the constant C_1 is equal to zero independently of the hole shape.

TABLE

Hole shape	Problem 1 $\left(\varkappa=-\frac{3+\nu}{1-\nu}\right)$				Problem 2 $(\varkappa=1)$			
	$\vartheta°$	$M_{\vartheta max}$ (in fractions of pR	$\vartheta°$	$M_{\vartheta min}$ (in fractions of pR)	$\vartheta°$	$M_{\vartheta max}$ (in fractions of pR)	$\vartheta°$	$M_{\vartheta min}$ (in fractions of pR)
Circle	200	−4.14	90	−0.47	195	−1.94	—	—
	340	−4.14	—	—	345	−1.94	245	−0.02
Ellipse	0	−5.28	90	−0.36	0	−3.29	240	−0.07
	180	−5.28	—	—	180	−3.29	300	−0.07
Near-square	0	−5.26	75	−0.61	180	−4.22	250	−0.006
	180	−5.26	105	−0.61	360	−4.22	290	−0.006
Near-triangular	0	−6.85	90	−0.34	180	−5.39	295	−0.02

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ON REFINED REISSNER-TYPE THEORIES OF PLATES

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(Tallin)

Numerous studies have lately been devoted to the subject of refined plate theories /1-16/. The first and most well-known of these theories is Reissner's theory /1-3/. Gol'denveizer /9/ has generalized Reissner's theory by replacing the linear distribution of stresses $\sigma^{\alpha 3}$ through the plate thickness by a distribution represented by an arbitrary function $\psi(z)$.

In this note the function $\psi(z)$ is considered as an unknown function, to be determined during the solution of the problem. Using the variational principle, a system of integral-differential equations is derived, consisting of Reissner-type plate equations and an additional equation for determining the function $\psi(z)$.

1. PLATE EQUATIONS

Consider a plate of uniform thickness h loaded by normal forces $\pm \frac{1}{2} P$ along the surfaces $x^3 = z = \pm \frac{1}{2} h$.

The equilibrium equations and the boundary conditions of the plate will be derived from Castigliano's variational principle of the linear theory of elasticity

$$\delta \left[\frac{1}{2} \int_V e_{ik}(\sigma) \sigma^{ik} dV \right] = 0, \qquad (i, k = 1, 2, 3). \tag{1.1}$$

The varied stresses σ^{ik} in (1.1) must satisfy the equilibrium conditions and the statical boundary conditions. These latter have the following form on the plate surfaces $z = \pm \frac{1}{2} h$:

$$\sigma^{\alpha 3} = 0, \qquad \sigma^{33} = \pm \frac{1}{2} P, \quad (\alpha, \beta = 1, 2). \tag{1.2}$$

Following Gol'denveizer /9/, we shall approximate the stresses in the form

$$\sigma^{\alpha\beta} = - M^{\alpha\beta}(x^\gamma) \psi''(z), \quad \sigma^{\alpha 3} = V^\alpha(x^\gamma) \psi'(z),$$
$$\sigma^{33} = P \psi(z), \tag{1.3}$$

where the function $\psi(z)$ is odd and satisfies the boundary conditions

$$\psi\left(\frac{h}{2}\right) = \frac{1}{2}, \quad \psi'\left(\frac{h}{2}\right) = 0, \tag{1.4}$$

while the magnitudes $M^{\alpha\beta}$, V^{α} satisfy the conditions

$$V^{\alpha} - \nabla_{\beta} M^{\alpha\beta} = 0, \tag{1.5}$$

$$\nabla_{\alpha} V^{\alpha} + P = 0. \tag{1.6}$$

The primes denote derivatives with respect to z.

Note that the following equalities are satisfied:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma^{\alpha\beta} dz = M^{\alpha\beta}, \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha 3} dz = V^{\alpha}. \tag{1.7}$$

It follows that $M^{\alpha\beta}$, V^{α} can be considered as moments and shearing stresses of the plate. Equations (1.5) and (1.6) are respectively the equations of plate equilibrium in moments and in shearing stresses.

The substitution of stresses (1.3) in the functional (1.1) and the addition to the integrand of (1.5) and (1.6), multiplied by the Lagrange multipliers $\varphi_{\alpha}(x^{\gamma})$, $w(x^{\gamma})$, yields:

$$\delta \int_V \left\{ \frac{1}{4\mu} \left[\left(a_{\alpha\gamma} a_{\beta\delta} - \frac{\nu}{1+\nu} a_{\alpha\beta} a_{\gamma\delta} \right) M^{\gamma\delta} M^{\alpha\beta} (\psi'')^2 + \right.\right.$$

$$\left. + 2\left(V^{\alpha} V_{\alpha} - \frac{\nu}{1+\nu} P a_{\alpha\beta} M^{\alpha\beta} \right) (\psi')^2 + \frac{1}{1+\nu} P^2 \psi^2 \right] -$$

$$\left. - \varphi_{\alpha} (V^{\alpha} - \nabla_{\beta} M^{\alpha\beta}) + w (\nabla_{\alpha} V^{\alpha} + P) \right\} dV = 0, \tag{1.8}$$

where $a_{\alpha\beta}$ is the metric tensor of the plate.

The variation of (1.8) by the functions $M^{\alpha\beta}$, V^{α}, φ_{α}, w, ψ yields the equilibrium equations (1.5), (1.6) and the following additional equations:

$$A\left[(1+\nu) a_{\alpha\gamma} a_{\beta\delta} - \nu a_{\alpha\beta} a_{\gamma\delta}\right] M^{\gamma\delta} - \frac{1}{2} E (\nabla_{\alpha} \varphi_{\beta} + \nabla_{\beta} \varphi_{\alpha}) - \nu B a_{\alpha\beta} P = 0, \tag{1.9}$$

$$B V_{\alpha} - \mu (\varphi_{\alpha} + \nabla_{\alpha} w) = 0, \tag{1.10}$$

$$P\psi^{\mathrm{IV}} - Q\psi'' + R\psi = 0, \tag{1.11}$$

where

$$A = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\psi'')^2 dz, \quad B = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\psi')^2 dz \tag{1.12}$$

and

$$P = \int_S \left(a_{\alpha\gamma} a_{\beta\vartheta} - \frac{\nu}{1+\nu} a_{\alpha\beta} a_{\gamma\vartheta} \right) M^{\alpha\beta} M^{\gamma\vartheta} ds,$$

$$Q = 2 \int_S \left(V^\alpha V_\alpha - \frac{\nu}{1+\nu} a_{\alpha\beta} M^{\alpha\beta} P \right) ds, \tag{1.13}$$

$$R = \frac{1}{1+\nu} \int_S P^2 ds.$$

The following homogeneous boundary conditions are obtained from (1.8):
1) statical conditions

$$M^{\alpha\beta} n_\alpha n_\beta = 0, \quad M^{\alpha\beta} n_\alpha t_\beta = 0, \quad V^\alpha n_\alpha = 0; \tag{1.14}$$

2) geometrical conditions

$$\varphi^\alpha n_\alpha = 0, \quad \varphi^\alpha t_\alpha = 0, \quad w = 0. \tag{1.15}$$

n_α, t_α are components of the unit normal vector and the unit tangential vector of the contour.

Equations (1.5), (1.6), (1.9) to (1.11) form a system of integral-differential equations which is complete relative to the unknown magnitudes. At constant values of the coefficients A, B equations (1.5), (1.6), (1.9), (1.10) coincide with the well-known Reissner-type plate equations derived by Gol'denveizer /9/.

The following system relative to V_α and w is easily obtained from equations (1.5), (1.6), (1.9), and (1.10):

$$(A - B\nabla^2) V_\alpha + \frac{E}{1-\nu^2} \nabla^2 \nabla_\alpha w + \frac{1}{1-\nu} B \nabla_\alpha P = 0. \tag{1.16}$$

This equation reduces on the strength of (1.6) to

$$\frac{E}{1-\nu^2} \Delta^4 w = AP - \frac{2-\nu}{1-\nu} B \nabla^2 P. \tag{1.17}$$

The solution of the system (1.6), (1.16) can be represented in the form

$$V_\alpha = V_{\alpha 1} + V_{\alpha 2}, \tag{1.18}$$

where $V_{\alpha 1}$ is a particular solution of the system of the form

$$V_{\alpha 1} = -\frac{E}{A(1+\nu^2)} \nabla^2 \nabla_\alpha w - \frac{2-\nu}{1-\nu} \frac{B}{A} \nabla_\alpha P, \tag{1.19}$$

and $V_{\alpha 2}$ is the general solution of the corresponding homogeneous system. If we write

$$V_{\alpha 2} = c_\alpha^\beta \nabla_\beta \Phi \tag{1.20}$$

(c^{β}_{α} is the discriminant tensor), the homogeneous equation reduces to

$$B\nabla^2\Phi - A\Phi = 0. \tag{1.21}$$

The solution of the plate problem has thus been reduced to integrating the system of integral-differential equations (1.11), (1.17), and (1.21).

2. ELIMINATING EQUATION (1.11)

Depending upon the relative magnitude of the parameters

$$r = \frac{1}{2}\frac{Q}{P}, \qquad s = \sqrt{\frac{R}{P}} \tag{2.1}$$

the following solutions are obtained for equation (1.11) with boundary conditions (1.4):

a) if $s > |r|$, then

$$\psi = \frac{1}{\Omega}\,[(\alpha \operatorname{sh}\alpha \sin\beta + \beta \operatorname{ch}\alpha \cos\beta)\operatorname{sh}\alpha Z \cos\beta Z +$$

$$+ (\beta \operatorname{sh}\alpha \sin\beta - \alpha \operatorname{ch}\alpha \cos\beta)\operatorname{ch}\alpha Z \sin\beta Z], \tag{2.2}$$

where

$$\alpha = \frac{h}{2}\sqrt{\frac{s+r}{2}}, \qquad \beta = \frac{h}{2}\sqrt{\frac{s-r}{2}},$$

$$\Omega = \beta \operatorname{sh} 2\alpha - \alpha \sin 2\beta \tag{2.3}$$

and

$$Z = \frac{2z}{h};$$

b) if $s < |r|$, then

$$\psi = \frac{1}{2\Sigma}\,(\vartheta \operatorname{ch}\vartheta \operatorname{sh}\gamma Z - \gamma \operatorname{ch}\gamma \operatorname{sh}\vartheta Z), \tag{2.4}$$

where

$$\gamma = \frac{h}{2}\sqrt{r + \sqrt{r^2 - s^2}}, \qquad \vartheta = \frac{h}{2}\sqrt{r - \sqrt{r^2 - s^2}},$$

$$\Sigma = \vartheta \operatorname{sh}\gamma \operatorname{ch}\vartheta - \gamma \operatorname{sh}\vartheta \operatorname{ch}\gamma; \tag{2.5}$$

c) if $s = |r|$, then

$$\psi = \frac{1}{\Pi}\,[(\operatorname{ch}\zeta + \zeta \operatorname{sh}\zeta)\operatorname{sh}\zeta Z - \zeta \operatorname{ch}\zeta \cdot Z \operatorname{ch}\zeta Z], \tag{2.6}$$

where

$$\zeta = \frac{h}{2}\sqrt{r}, \qquad \Pi = \operatorname{sh} 2\zeta - 2\zeta. \tag{2.7}$$

The coefficients A and B calculated by (1.12) will have the following values for the cases considered:

for case a)

$$A = (\alpha^2 + \beta^2)^2 h^{-3} \Omega^{-2} [2 (\alpha^2 - \beta^2) (1 + \operatorname{ch} 2\alpha \cos 2\beta) + 4\alpha\beta \operatorname{sh} 2\alpha \sin 2\beta + \\ + \alpha^{-1}\beta^{-1} (\operatorname{ch} 2\alpha + \cos 2\beta) (\beta^3 \operatorname{sh} 2\alpha - \alpha^3 \sin 2\beta)],$$

$$B = \frac{1}{4} (\alpha^2 + \beta^2)^2 h^{-1} \Omega^{-2} [-2 (1 + \operatorname{ch} 2\alpha \cos 2\beta) + \\ + \alpha^{-1}\beta^{-1} (\alpha^2 + \beta^2)^{-1} (\beta^3 \operatorname{sh} 2\alpha + \alpha^3 \sin 2\beta) (\operatorname{ch} 2\alpha + \cos 2\beta)]; \qquad (2.8)$$

for case b)

$$A = \gamma^2 \vartheta^2 h^{-3} \Sigma^{-2} [(\gamma^2 + 3\vartheta^2) (\gamma^2 - \vartheta^2)^{-1} \gamma \operatorname{sh} 2\gamma \operatorname{ch}^2 \vartheta - \\ - (3\gamma^2 + \vartheta^2) (\gamma^2 - \vartheta^2)^{-1} \vartheta^{-1} \operatorname{sh} 2\vartheta \operatorname{ch}^2 \gamma - 2\gamma^2 \operatorname{ch}^2 \vartheta - 2\vartheta^2 \operatorname{ch}^2 \gamma],$$

$$B = \frac{1}{4} \gamma^2 \vartheta^2 h^{-1} \Sigma^{-2} [(\gamma^2 + 3\vartheta^2) (\gamma^2 - \vartheta^2)^{-1} \vartheta^{-1} \operatorname{sh} 2\vartheta \operatorname{ch}^2 \gamma - \\ - (3\gamma^2 + \vartheta^2) (\gamma^2 - \vartheta^2)^{-1} \gamma^{-1} \operatorname{ch} 2\gamma \operatorname{ch}^2 \vartheta + 2 \operatorname{ch}^2 \gamma + 2 \operatorname{ch}^2 \vartheta] \qquad (2.9)$$

for case c)

$$A = \frac{1}{3} \zeta^3 h^{-3} \Pi^{-2} [2\zeta (8\zeta^2 - 3) + 3 (8\zeta^2 + 1) \operatorname{sh} 2\zeta - \\ - 2\zeta (4\zeta^2 + 3) \operatorname{ch} 2\zeta + 3\zeta^2 \operatorname{sh} 4\zeta],$$

$$B = \frac{1}{12} \zeta h^{-1} \Pi^{-2} [-(16\zeta^3 + 9\zeta) + 3 \operatorname{sh} 2\zeta + 2\zeta (4\zeta^2 - 3) \operatorname{ch} 2\zeta - 3\zeta \operatorname{ch} 4\zeta + \\ + 3 (\zeta^2 + 1) \operatorname{sh} 4\zeta]. \qquad (2.10)$$

The coefficients A and B obtained are complex functions of the unknown plate parameters. The plate equations (1.5), (1.6), (1.9), (1.10) will therefore be complex integral-differential equations, and it will hardly be possible to use them as such in the solution of nonelementary plate problems. These equations are, however, easily solved by iterations: the problem is first solved by assuming some constant values for A and B; the solution obtained is then used for calculating A and B from the formulas given, and so on. Each iterative step will consist in solving a system of ordinary Reissner-type plate equations. It is of course necessary to determine whether this iterative process will converge.

Note that at the limit $\frac{Q}{P} \to 0$ and $\frac{R}{P} \to 0$ the function ψ reduces to

$$\psi = \frac{3}{4} Z - \frac{1}{4} Z^3 \qquad (2.11)$$

and the corresponding coefficients will be constants:

$$A = \frac{12}{h^3}, \qquad B = \frac{6}{5h}. \qquad (2.12)$$

For these values of the coefficients the plate equations (1.5), (1.6), (1.9), (1.10) coincide with Reissner's plate equations /1-3/.

3. EXAMPLE

Consider an elementary problem which can be solved directly be means of the equations given.

Let a square hinged plate of width $2a$ be loaded by a sinusoidally varying load

$$P = P_0 \sin \xi x \sin \xi y, \tag{3.1}$$

where $\xi = \frac{n\pi}{a}$, and n is an integer.

If the boundary conditions are:

$$\begin{aligned} &w = 0, \ M_x = 0, \ \frac{\partial w}{\partial y} - \frac{2(1+\nu)}{E} B V_y = 0 \quad \text{at} \quad x = \pm a, \\ &w = 0, \ M_y = 0, \ \frac{\partial w}{\partial x} - \frac{2(1+\nu)}{E} B V_x = 0 \quad \text{at} \quad y = \pm a, \end{aligned} \tag{3.2}$$

the system (1.6), (1.16) will have the following solution:

$$\begin{aligned} w &= \frac{P_0(1-\nu^2)}{4\xi^4 E}\left(A + 2\frac{2-\nu}{1-\nu}\xi^2 B\right) \sin \xi x \sin \xi y, \\ V_x &= \frac{P_0}{2\xi} \cos \xi x \sin \xi y, \\ V_y &= \frac{P_0}{2\xi} \sin \xi x \cos \xi y. \end{aligned} \tag{3.3}$$

The corresponding moments will be

$$\begin{aligned} M_x = M_y &= P_0\left(\frac{1+\nu}{4\xi^2} + \frac{\nu}{2}\frac{B}{A}\right) \sin \xi x \sin \xi y, \\ M_{xy} &= P_0\left(\frac{\nu-1}{4\xi^2} + \frac{\nu}{2}\frac{B}{A}\right) \cos \xi x \cos \xi y. \end{aligned} \tag{3.4}$$

Using relationships (1.13), (2.1), (3.3), and (3.4), we obtain that

$$\begin{aligned} r &= \frac{2\xi^2(1-\nu^2-2\nu^2\xi^2 BA^{-1})}{1-\nu^2+4\nu^2\xi^4 B^2 A^{-2}}, \\ s &= \frac{2\xi^2}{\sqrt{1-\nu^2+4\nu^2\xi^4 B^2 A^{-2}}}. \end{aligned} \tag{3.5}$$

It is easily seen here that $s > |r|$. The substitution of (3.5) into (2.3) and of the resulting expressions into (2.8) yields a system of equations for determining the coefficients A and B. The table lists the numerical results of the solution of this system for several values of the coefficient $\frac{a}{hn}$ which characterizes the variation of the surface load $\left(\nu = \frac{1}{3}\right)$.

It is seen from the table that the coefficients A, B are near to the Reissner coefficients down to $\frac{a}{hn} \approx 2$.

TABLE

$\frac{a}{hn}$	Ah^3	Bh	$\frac{B}{A}h^2$
1	15.42	1.097	0.0711
2	12.18	1.173	0.0962
3	12.03	1.187	0.0987
5	12.01	1.191	0.0992
∞	12.00	1.200	0.1000

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TWO PROBLEMS CONCERNING SANDWICH PANELS

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I. CALCULATING SANDWICH PANELS WITH FILLER MADE OF GLASS STAMPINGS OR BARS

Plane and curved sandwich panels with filler made of glass particles punched from one of the faces in a rectangular or triangular grid pattern have been frequently applied lately. These glass particles are welded or soldered by their bases to the other face (Figure 1). It is also possible to punch the glass particles separately, and then fasten them to both faces of the panel (Figure 2, a). The calculating model is in this case the same as that of a panel with filler made of continuous bars (Figure 2, b).

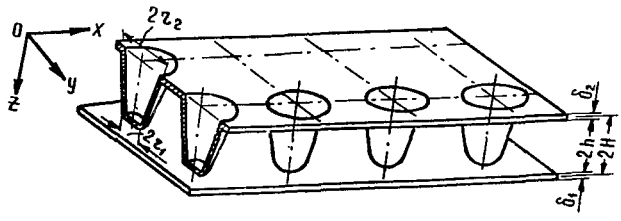

FIGURE 1.

Bending and buckling calculations of such panels are conducted by the formulas obtained for sandwich panels with compact filler and various faces. The rigidity of the punched face is calculated in the same way as the reduced rigidity of a plate weakened by holes and reinforced by soldered tubes (Figure 2, c). The rigidity of the second face is calculated in the same way as the reduced rigidity of a plate reinforced by washers and tubes (Figure 2, d).

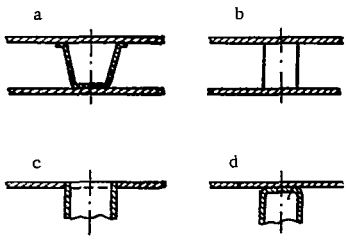

FIGURE 2.

The reduced moduli of normal elasticity of such a filler can be taken as $E_x = E_y = 0$.

1. If the glass particles are arranged in a rectangular grid pattern, the relative shear of the faces in the x direction will cause the sandwich plate to acquire the shape represented by the dotted line of Figure 3. In view of the large

rigidity of the glass particles in the direction of their axis, the shaded part of the faces (Figure 3), bounded by curves of the type

$$x = a \pm r\left(1 - k\sin^2\frac{\pi y}{2b}\right)$$

($r = r_1$ or $r = r_2$; $k\,(0 \leqslant k \leqslant 1)$ is an unknown parameter determined from the condition of minimum of the deformation energy), is not deformed, and only rotates by an angle α.

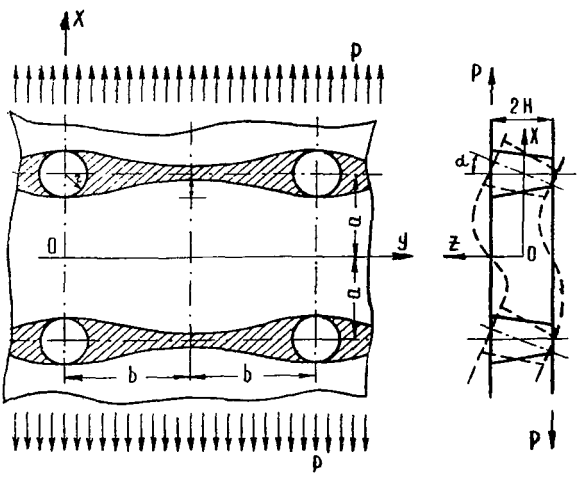

FIGURE 3.

The equation of the elastic surface of the face element, bounded by the lines $x = 0$, $y = 0$, $y = b$ and by the curve $x = f(y) = a - r\left(1 - k\sin^2\frac{\pi y}{2b}\right)$ (this element is repeated continuously along the face), will be represented in the form $w(x, y) = A(y)x + B(y)x^3$ where $A(y)$ and $B(y)$ are to be found from the condition of compatibility of deformations of the elements.

The glass particles will be considered approximately as bars of circular constant cross section jammed between the plate faces. These bars will bend during plate deformations, and the position of the inflection point on them will depend on the ratio between the rigidities of the faces (Figure 5). A part of the glass particles will be considered as cantilever beams jammed between the plate faces.

By equating the energy of deformation of the faces and the glass particles to the energy of deformation of some conventional homogeneous filler at a certain mutual displacement of the faces, and by determining the parameter k from the condition of energy minimum ($k \leqslant 1$), we obtain the following formula for determining G_{xz}:

$$G_{xz} = \frac{1}{4abH^2\left[\dfrac{(1+\beta)^2}{8H(D_1F_1\beta^2 + D_2F_2) + E_cI_c(1-\beta)^2} + \dfrac{1}{3E_cI_c}\right]}, \qquad (1.1)$$

where $\beta = \frac{\alpha_1}{\alpha_2}$ (α_1, α_2 cf. Figure 5) is determined from the condition

$$\frac{\partial G_{xz}}{\partial \beta} = 0, \tag{1.2}$$

which is equivalent to minimizing the energy;

F_1 and F_2 are found from Figure 4 for given a, b, and r; D_1 and D_2 are cylindrical flexural rigidities of the faces; $E_c I_c$ is the rigidity of the glass particles.

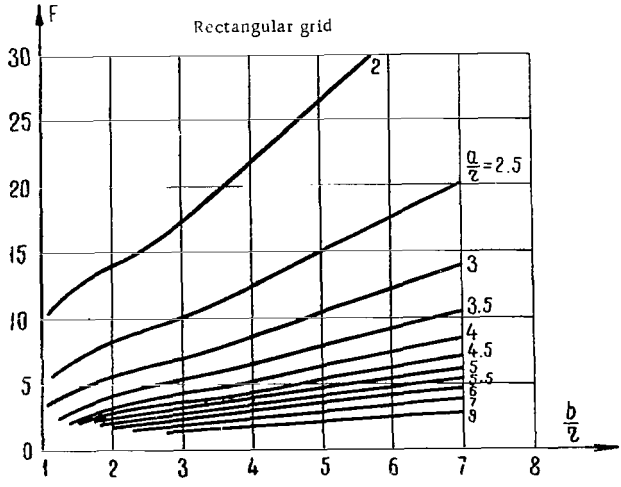

FIGURE 4.

Condition (1.2) yields the following expression for β:

$$\beta = \frac{4HD_2F_2 + E_cI_c}{4HD_1F_1 + E_cI_c}. \tag{1.3}$$

If the faces are identical, $\beta = 1$, and

$$G_{xz} = \frac{1}{2abH\left(\frac{1}{2DF} + \frac{2H}{3E_cI_c}\right)}. \tag{1.4}$$

If the deformation of the glass particles is neglected, this reduces further to

$$G_{xz} = \frac{D_1F_1 + D_2F_2}{2abH}. \tag{1.5}$$

The same formulas (1.1)—(1.5) can be used for determining the reduced modulus G_{yz}; it suffices to intervert the places of m and n when finding F_1 and F_2 from the graph.

If the glass particles are arranged in a triangular grid pattern, the relative shear of the faces in the x direction will cause the sandwich plate to acquire the shape shown in Figure 6. The shaded parts

of the layers, bound by the curves

$$x = f_1(y) = a - r + \frac{kr}{2} + \frac{kr}{2}\sin\frac{\pi y}{b}$$

and

$$x = f_2(y) = -a + r - \frac{kr}{2} + \frac{kr}{2}\sin\frac{\pi y}{b},$$

are not deformed, but only rotate through an angle α.

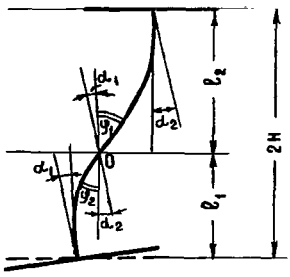

FIGURE 5.

The equation of the elastic surface of the face element, bound by the lines $y = \pm a$ and the curves $x = f_1(y)$ and $x = f_2(y)$, will be represented in the form

$$w(x, y) = A(y)x + B(y)x^3 + C(y)x^5 + M(y),$$

where $A(y)$, $B(y)$, $C(y)$, $M(y)$ are found from the condition of compatibility of deformations of the elements.

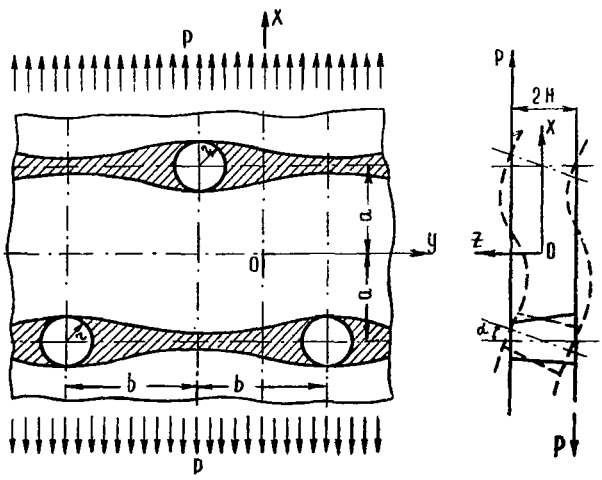

FIGURE 6.

A formula for determining the reduced shear modulus is similarly determined:

$$G_{xz} = \frac{3E_cI_c[4H(D_1F_1\beta^2 + D_2F_2) + E_cI_c(1-\beta)^2]}{4abH^2[4H(D_1F_1\beta^2 + D_2F_2) + E_cI_c(1-\beta)^2 + 3E_cI_c(1+\beta)^2]}, \tag{1.6}$$

where $\beta = \frac{\alpha_1}{\alpha_2}$, determined from the condition

$$\frac{\partial G_{xz}}{\partial \beta} = 0,$$

is given by the expression

$$\beta = \frac{2HD_2F_2 + E_cI_c}{2HD_1F_1 + E_cI_c}. \tag{1.7}$$

At identical faces $\beta = 1$, and

$$G_{xz} = \frac{1}{4abH\left(\frac{1}{2DF} + \frac{H}{3E_cI_c}\right)}, \tag{1.8}$$

and if the bending of the glass particles is neglected

$$G_{xz} = \frac{D_1F_1 + D_2F_2}{4abH}. \tag{1.9}$$

G_{yz} is determined by the same formulas (1.6)—(1.9), with the places of m and n inverted when determining $F_{1,2}$ from Figure 7.

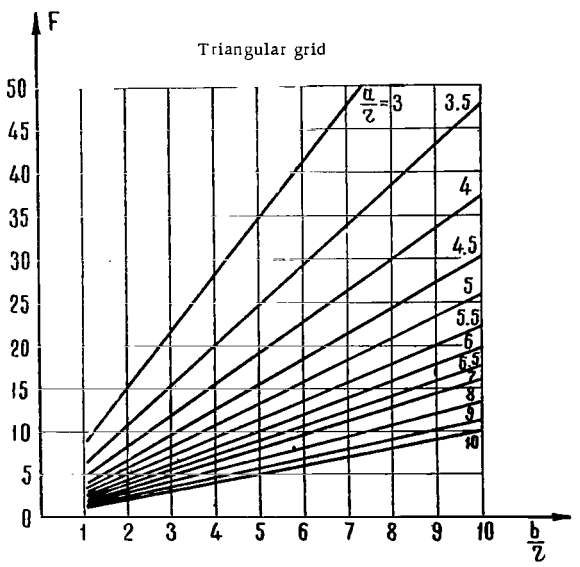

FIGURE 7.

The obtained formulas were tested experimentally.

2. The local stability of the faces under longitudinal compression was studied for the case of the bending represented in Figure 8 (corresponding to the deformation of the plate elements at shear). The critical loads were determined by the energy method.

The following critical values of the compression loads were obtained:

in the case of a rectangular grid pattern

$$T_{xk1} = \frac{(D_1F_1 + D_2F_2)B_1}{ab\sum\limits_{i=1,2} B_i\left[\frac{6n_i}{5\sqrt{(n_i-1)(n_i-1+k_i)}}-1\right]}; \quad T_{xk2} = \frac{B_2}{B_1}T_{xk1}; \tag{1.10}$$

in the case of a triangular grid pattern

$$T_{xk1} = \frac{D_1F_1 + D_2F_2}{2ab}\,\frac{B_1}{\sum\limits_{i=1,2} B_i\left[\frac{16n_i(d_i^2+5k_i^2)}{35(d_i^2-5k_i^2)^{3/2}} + \frac{12n_i}{5d_i} - 1\right]}; \tag{1.11}$$

$$T_{xk2} = \frac{B_2}{B_1}T_{xki}.$$

where $B_i = \frac{E_i\delta_i}{(1-\mu_i^2)}$ is the rigidity of the faces at extension $(i = 1, 2)$.

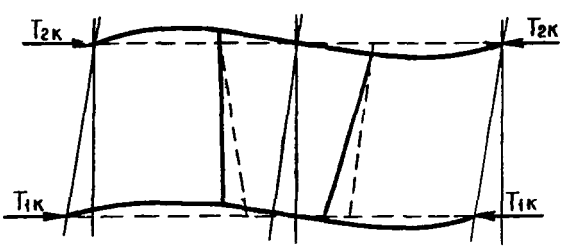

FIGURE 8.

If one of the faces buckles under the load before the other, this leads practically to the bending of the entire plate under a common load

$$2T_{xk} = T_{xki}\left(\frac{B_i}{B_{3-i}}\right) \qquad (i = 1, 2),$$

where T_{xki} is the smaller of the two critical loads.

We shall assume that the values of the parameters k_1 and k_2 coincide with the values which were found from the condition of minimum energy of bending of each layer. The values of k_1 and k_2 are determined from Figures 9 and 10; in the case that they lie above the curve $k_{1,2} = 1$, the value $k_{1,2} = 1$ is to be used in (1.10) and (1.11).

T_{yk} can be calculated by (1.10) and (1.11), by interverting the positions of m and n.

3. The influence of the longitudinal forces $2T_x$ or $2T_y$ applied on the sandwich plate on the modulus of shear of the filler can be allowed for by means of the formulas

$$G_{xz}^* = G_{xz}\left(1 - \frac{2T_k}{2T_{xk}}\right),$$
$$G_{yz}^* = G_{yz}\left(1 - \frac{2T_y}{2T_{yk}}\right), \tag{1.12}$$

where G_{xz}, G_{yz} are values of the moduli without allowing for the compression forces.

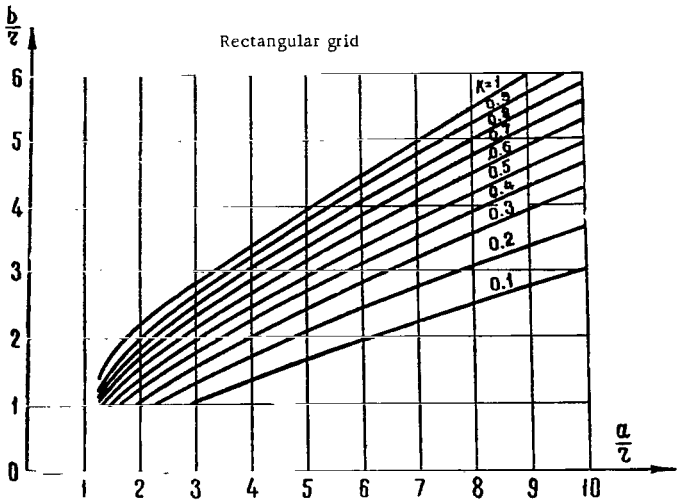

FIGURE 9.

Strictly speaking, the energy of bending of the glass particles ought to be taken into account in the determination of $2T_{xk}$ and $2T_{yk}$; in view, however, of the negligible influence of this energy, we disregard it.

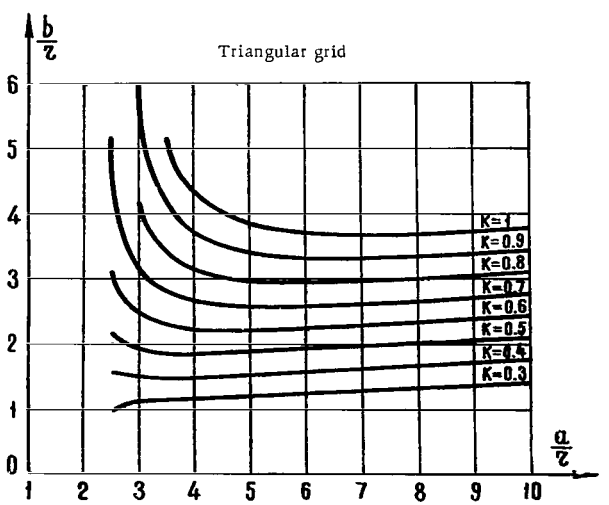

FIGURE 10.

This problem was solved by Aleksandrov, Vol'pert, and Masalovich.

2. OPTIMUM PARAMETERS OF SANDWICH PLATES WITH FILLER MADE OF REINFORCED FOAMLEX AT COMPRESSION

The longitudinal compression of an infinitely wide sandwich plate with identical faces and a filler made of foamlex reinforced by ribs placed in the direction of compression was considered (Figure 11). The faces, foamlex, and reinforcing ribs were glued together. The problem was to select plate parameters corresponding to a minimum plate weight at given load. The conditions to be fulfilled were general stability of the plate, local stability of the faces, and strength of the filler. It was assumed that the conditions of plate strength are fulfilled if the value of the load acting on the plate is equal to 2/3 of the critical load, the stresses in the faces do not exceed the proportional limit of the material, and the reduced stresses in the filler, found by the second or third theory of strength, are allowable.

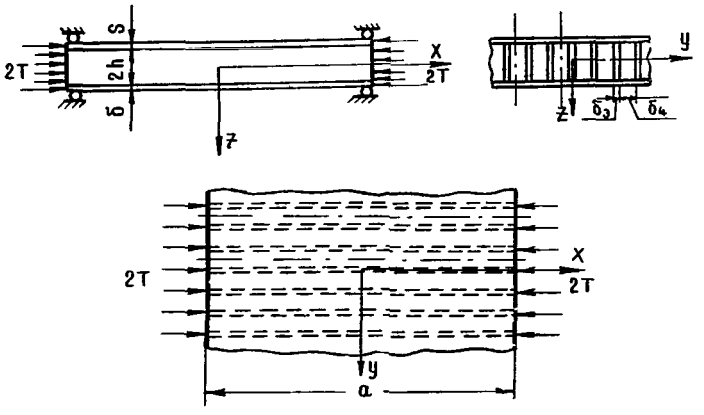

FIGURE 11.

The expression for the critical load of a compressed infinitely wide sandwich plate with identical faces simply supported by two edges (Figure 11) is, allowing for the rigidity of the filler and the flexural rigidity of the faces /1/:

$$2T_k = -\frac{\pi^2 D_{pl}}{a_0^2} m. \qquad (2.1)$$

where

$$D_{pl} = 2\left\{ \bar{B}\left[\left(h+\frac{\delta}{2}\right)^2 + \frac{\delta^2}{12}\right] + \frac{E\cdot h^3}{3(1-\mu^2)} \right\};$$

$$m_t = \frac{\delta^2 \overline{B}}{6D_{\text{pl}}} + \frac{1}{1+k};$$

$$k = \frac{\pi^2 D_{\text{pl}} h}{2a_0^2\left(h + \frac{\delta}{2}\right)^2 G};$$

$\overline{B} = \frac{\overline{E}\delta}{1-\mu^2}$ is the tensile rigidity of the face per unit width; $\overline{E}$, $\overline{\mu}$ are the modulus of elasticity and Poisson's ratio of the face material; E, G, μ are the reduced modulus of elasticity, modulus of shear, and Poisson's ratio of the reinforced-foamlex filler; $a_0 = \frac{a}{\lambda}$ is the reduced plate length; λ is the coefficient of end restraint of the plate edges; $\lambda = 1$ for hinged support, $\lambda = 2$ at fixed support.

If the ultimate strength of the filler at compression and extension is the same, the reduced stresses in the filler can be determined by the theory of maximum shearing stresses. In this case the reduced stresses (σ_p) in the filler of the sandwich plates at longitudinal compression are determined by the expression /2/

$$\sigma_p = \sqrt{(\sigma_{zcM} + \sigma_x)^2 + 4(\tau_{xzkM} + \tau_{xzcM})^2} \leqslant [\sigma]. \qquad (2.2)$$

Here $\sigma_x = \bar{\sigma}_x \cdot \frac{E}{\overline{E}}$ is the stress in the filler due to the compression of the entire plate; $\bar{\sigma}_x$ is the compression stress in the face.

The values of the maximum stresses of pull and shear in the filler σ_{zcM}, τ_{xzcM} caused by the initial symmetrical curvature of the faces are determined by the formulas

$$\sigma_{zcM} = 2G\varphi \frac{2(1-\mu)\left(1+\frac{\varphi}{2}\right) - \frac{\varphi}{2}}{4(1-\mu)-1} \cdot \frac{\bar{\sigma}_x}{\bar{\sigma}_{kc} - \bar{\sigma}_x} \cdot \frac{\omega_c}{\delta},$$

$$\tau_{xzcM} = 2G\varphi \frac{2(1-\mu)\left(1+\frac{\varphi}{2}\right) - 1}{4(1-\mu)-1} \cdot \frac{\bar{\sigma}_x}{\bar{\sigma}_{k.c} - \bar{\sigma}_x} \cdot \frac{\omega_c}{\delta},$$

where $\bar{\sigma}_{kc} = \frac{T}{\delta}$ is the critical value of $\bar{\sigma}_x$.

The value of the critical load in the face at local (symmetrical) buckling ($\bar{\sigma}_{k.c}$) and at a buckling wavelength coinciding with the initial curvature of the face is calculated by the formula

$$\bar{\sigma}_{k.c} = \frac{\varphi^2 \overline{E}}{12(1-\overline{\mu}^2)} + \frac{2G}{\varphi} \cdot \frac{2(1-\mu)\left(1+\frac{\varphi}{2}\right)^2 - \varphi}{4(1-\mu)-1}.$$

The stress $\bar{\sigma}_{k.c}$ will be maximum for the following value of the parameter φ which determines the initial curvature of the faces:

$$\varphi = \sqrt[3]{\frac{96(1-\mu)G(1-\overline{\mu}^2)}{(3-4\mu)\overline{E}}}$$

The value of the maximum shearing stress appearing in the filler as a result of the initial antisymmetrical bending of the plate (τ_{zxkM}) is determined for an infinitely wide plate by the formula

$$\tau_{xzkM} = \frac{2h + \delta}{\dfrac{(2h+\delta)^2}{2\bar{\sigma}_x\delta} - \dfrac{2h}{G} - \dfrac{2a^2}{\pi^2 B}} \cdot \frac{\pi}{a} w_k.$$

If the foamlex has different ultimate strengths at compression and extension, the reduced stresses are determined by the second theory of strength with the aid of the following two expressions:

$$\sigma_p = \frac{(1+\mu)}{2}\left\{\sqrt{(-\sigma_x \pm \sigma_{zcM})^2 + 4(\tau_{xzkM} + \tau_{xzcM})^2} + \right.$$
$$\left. + \frac{(1-\mu)}{(1+\mu)}(-\sigma_x \mp \sigma_{zcM})\right\} \leqslant |\sigma|_p; \qquad (2.3)$$

$$\sigma_p = \frac{(1+\mu)}{2}\left\{-\sqrt{(-\sigma_x \pm \sigma_{zcM})^2 + 4(\tau_{xzkM} + \sigma_{xzcM})^2} + \right.$$
$$\left. + \frac{(1-\mu)}{(1+\mu)}(-\sigma_x \mp \sigma_{zcM})\right\} \leqslant |\sigma|_c,$$

where $|\sigma|_p$ is the allowable stress for the filler at extension; $|\sigma|_c$ is the allowable stress for the filler at compression. The signs ± before the stress σ_{zcM} are selected in such a way as to obtain the maximum reduced stress.

The reduced parameters of a foamlex filler reinforced by ribs (Figure 11) are determined by the following formulas:

$$G = G_{xz} = \frac{G_3\delta_3 + G_4\delta_4}{\delta_3 + \delta_4}; \qquad (2.4)$$

$$E = E_x = E_z = \frac{E_3\delta_3 + E_4\delta_4}{\delta_3 + \delta_4}; \qquad (2.5)$$

$$\mu = \mu_{xz} = \mu_{zx} = \frac{\mu_3\delta_3 G_3 + \mu_4\delta_4 G_4}{\delta_3 G_3 + \delta_4 G_4}; \qquad (2.6)$$

$$\sigma_b = \frac{\sigma_3\delta_3 + \sigma_4\delta_4}{\delta_3 + \delta_4}, \qquad (2.7)$$

where E_3, G_3, μ_3, σ_3 are the modulus of normal elasticity, modulus of shear, Poisson's ratio, and ultimate strength of the material of the reinforcing ribs of the filler at compression (extension); E_4, G_4, μ_4, σ_4 are the modulus of normal elasticity, modulus of shear, Poisson's ratio, and ultimate strength of the foamlex (nonreinforced) at compression (extension).

Formula (2.4) is correct if the reinforcing ribs do not buckle at shear; formulas (2.5), (2.7) are correct if the ribs do not buckle at compression.

Note that compression and shearing stresses appear in the reinforcing ribs of the filler of a plate possessing an initial curvature and compressed in the longitudinal direction.

In order to determine the critical values of these stresses, consider an element made of two reinforcing ribs and three foamlex layers (represented in Figure 11 by strokes and dots). This element behaves as

an infinitely long sandwich plate of width $2h$ simply supported on two edges (the reinforcing elements are considered as simply supported on the faces). Two forms of buckling are considered for such an element: symmetrical and antisymmetrical.

The critical stresses in the ribs for the case of antisymmetrical buckling under compression (without shear) are determined by the formula

$$\sigma_{xk} = \frac{m_t \pi^2 B_3 (\delta_3 + \delta_4)^2}{16 h^2 \delta_3}, \tag{2.8}$$

where

$$m_t = \frac{4}{(1+k)^2} \quad \text{at} \quad k \leqslant 1;$$

$$m_t = \frac{1}{k} \quad \text{at} \quad k > 1; \quad k = \frac{\pi^2 B_3 \delta_4}{16 h^2 G}.$$

The critical stress of symmetrical buckling is determined by the formula

$$\sigma_{xk} = \frac{\xi \cdot 2G}{\delta_2}. \tag{2.9}$$

where ξ is found from Figure 12.

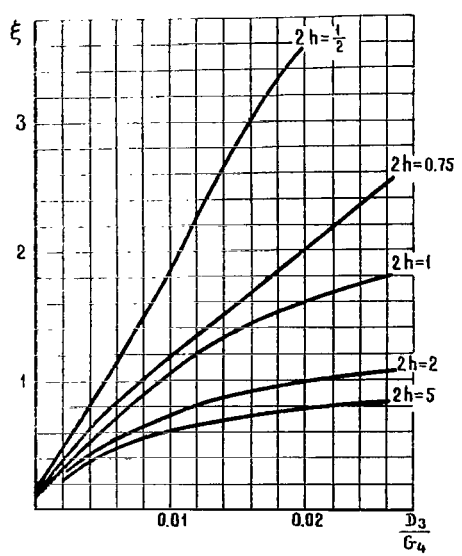

FIGURE 12.

The smaller of the values (2.8) and (2.9) is used in the calculations. Buckling will not occur if $\sigma_{xk} > \bar{\sigma}_x \frac{E_3(1-\bar{\mu}^2)}{\bar{E}}$.

The critical load of antisymmetrical buckling under the simultaneous action of compression and shear on the ribs is determined by the formulas

$$\sigma_{xk} = m_t \frac{\pi^2 B_3 (\delta_4 + \delta_3)^2}{16 h^2 \delta_3}; \quad \tau_k = m_s \frac{\pi^2 B_3 (\delta_4 + \delta_3)^2}{16 h^2 \delta_3}. \tag{2.10}$$

The relationship between m_t and m_s is represented in Figure 13.

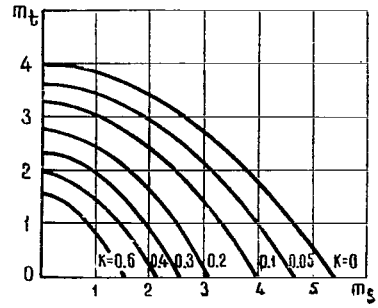

FIGURE 13.

In order to test the stability of the ribs under the simultaneous action of shear and compression and a symmetrical form of bending of the element, we determine the principal compressive stress acting on the rib:

$$\sigma = -\frac{1}{2}\bar{\sigma}_x \frac{E_3 (1-\bar{\mu}^2)}{\bar{E}} - \sqrt{\left(-\frac{1}{2}\bar{\sigma}_x \frac{E_3 (1-\bar{\mu}^2)}{\bar{E}}\right)^2 + \tau_{xzkM}^2 \left(\frac{\delta_3 + \delta_4}{\delta_3 + \frac{G_4}{G_3}\delta_4}\right)^2}.$$

The critical stress on the rib is determined in this case by the formulas

$$T_{xk} = 1.45 \delta_3 \sqrt[3]{\frac{\bar{E} \cdot E_4 G_4}{1-\bar{\mu}^2}} \quad \text{at} \quad \frac{\delta_4}{\delta_3} > 0.635 \sqrt[3]{\frac{\bar{E} \cdot E_4}{(1-\bar{\mu}^2) G_4^2}};$$

$$T_{xk} = 1.16 \delta_3 \sqrt{\frac{E}{1-\bar{\mu}^2} E_4 \frac{\delta_3}{\delta_4}} \quad \text{at} \quad \frac{\delta_4}{\delta_3} < 0.635 \sqrt[3]{\frac{\bar{E} \cdot E_4}{(1-\bar{\mu}^2)}}. \tag{2.11}$$

The stability condition is

$$\sigma \leqslant \bar{\sigma}_{xk}.$$

Note that the value of the reinforcement step δ_4 is restricted by the requirement to ensure the stability of the face element located between two stiffening ribs. This stability is checked for the case of antisymmetrical buckling by (2.8), and for the case of symmetrical buckling by (2.9) and Figure 12, with $2h$ replaced by δ_4, δ_3 by δ, and $2G$ by G.

The weight of unit area of the plate with reinforced filler will be

$$Q = 2\delta\bar{\gamma} + \frac{2h(\delta_4 \gamma_4 + \delta_3 \gamma_3)}{\delta_4 + \delta_3}. \tag{2.12}$$

The problem of selecting the optimum parameters of sandwich plates reduces to solving the system of equations (2.1), (2.3) with the substitution of the reduced elasticity parameters (2.4), (2.5), (2.6), finding the relationship between the elastic parameters of the foamlex and its unit weight, and selecting the plates having the minimum weight at given loading parameter $\frac{2T_k}{a_0}$. The condition $\bar{\sigma}_x \leqslant \bar{\sigma}_{pl}$ ($\bar{\sigma}_{pl}$ is the proportional limit of the face material) was fulfilled in the problem solution.

The problem of selecting the optimum parameters was solved by means of computers.

Curves of the optimum parameters as a function of the loading parameter $\frac{2T_k}{a_0}$, the initial symmetrical curvature of the face w_e, and the antisymmetrical plate bending w_k, are represented in Figure 14. The critical load is determined here from the expression $2T_k = \frac{3}{2}\,2T$ where $2T$ is the service load per unit width of the plate. The minimum weight corresponds to foamlex of minimum unit weight for the given category (for foamlex FK 20, γ = 0.1 g/cm^3, and for foamlex on the base of polystyrene, γ = 0.05 g/cm^3).

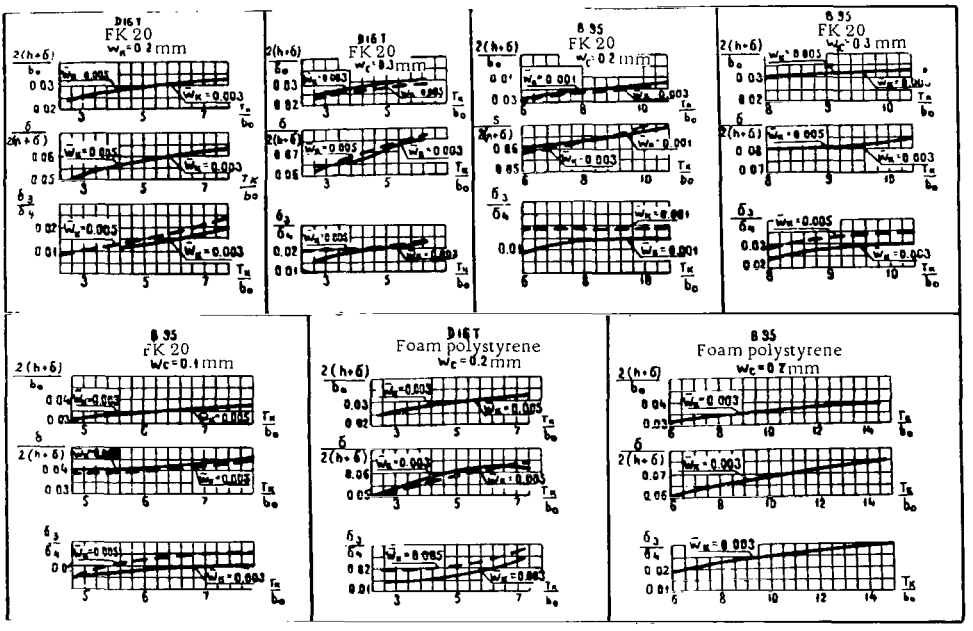

FIGURE 14.

After the optimum parameters had been determined, the strength of the bond of the filler with the faces was tested. The design stresses in these bonds (equal to the stress in the filler [σ]) were in all cases lower than the allowable shearing stresses for glues BF-2 and BF-4 (the ultimate

strength of glues BF-2 and BF-4 at shear is, according to TU MKhP 1367-49, of 150 kg/cm^2 at 20°C).

In order to find the values of δ_3 and δ_4 which ensure the stability of the ribs, formulas (2.8) to (2.10) and Figures 12 and 13 are used (after the ratio $\frac{\delta_3}{\delta_4}$ has been determined from the curves).

This problem was solved by Aleksandrov and Naumova.

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ON THE DEPENDENCE OF THE UPPER CRITICAL PRESSURE OF A CYLINDRICAL SHELL ON THE BOUNDARY CONDITIONS FOR THE TANGENTIAL DISPLACEMENTS

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Usually only the boundary conditions relative to the normal displacements are taken into account when determining the critical loads, while the boundary conditions relative to the tangential displacements are totally neglected. This paper is devoted to an assessment of the influence of the boundary conditions for the tangential displacements on the value of the upper critical pressure of a closed circular cylindrical shell; this assessment is given on the basis of an analysis of the buckling of a schematized semimomentless shell.

Consider a circular cylindrical shell of radius R, length l, and thickness h. The points of its middle surface will be described by the coordinates x and y (in the axial and circumferential directions respectively). The shell will be considered as semimomentless, i.e., we shall consider that the inner force factors in it are expressed through the displacements of the middle surface points u, v, w in the following way:

$$\begin{gathered} T_x = \frac{Eh}{(1-\mu^2)}\left[\frac{\partial u}{\partial x} + \mu\left(\frac{\partial v}{\partial y} - \frac{w}{R}\right)\right]; \\ T_y = \frac{Eh}{(1-\mu^2)}\left[\left(\frac{\partial v}{\partial y} - \frac{w}{R}\right) + \mu\frac{\partial u}{\partial x}\right]; \\ S = \frac{Eh}{2(1+\mu)}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right); \\ M_y = \frac{Eh^3}{12(1-\mu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right); \\ M_x = M_{xy} = 0, \end{gathered} \tag{1}$$

where T_x, T_y, and S are forces per unit length in the middle surface; M_x, M_y, and M_{xy} are the corresponding bending and torsional moments; E and μ are the modulus of elasticity and Poisson's ratio of the shell material.

In other words, we consider an anisotropic shell which lacks flexural and torsional rigidity in the axial direction, and in which the flexural rigidity in the circumferential direction and the tensile and shearing rigidities of the middle surface are equal to the ordinary rigidities of an isotropic shell of thickness h. We shall treat later the question of the error introduced by such a schematization of the shell.

The system of equations describing the buckling of the shell considered under the action of an external transverse pressure P /1/ will have the form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial \varphi\, \partial \xi} + \frac{1-\mu}{2}\frac{\partial^2 u}{\partial \varphi^2} - \mu\frac{\partial w}{\partial \xi} = 0;$$

$$\frac{Eh^3}{(1-\mu^2)}\left[\frac{1+\mu}{2}\frac{\partial^2 u}{\partial \varphi \partial \xi} + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \varphi^2} - \frac{\partial w}{\partial \varphi}\right] + \tag{2}$$

$$+ \frac{Eh^3}{12(1-\mu^2)R^2}\left(\frac{\partial^3 w}{\partial \varphi^3} + \frac{\partial w}{\partial \varphi}\right) = 0;$$

$$\frac{Eh^3}{12(1-\mu^2)R^2}\left[\frac{\partial^4 w}{\partial \varphi^4} + \frac{\partial^2 w}{\partial \varphi^2}\right] - \frac{Eh}{(1-\mu^2)}\left[\left(\frac{\partial v}{\partial \varphi} - w\right) + \mu\frac{\partial u}{\partial \xi}\right] +$$

$$+ PR\left(\frac{\partial^2 w}{\partial \varphi^2} + w\right) = 0,$$

where the following dimensionless coordinates have been used for simplicity:

$$\xi = \frac{x}{R} \text{ and } \varphi = \frac{y}{R}.$$

For this system of equations, which corresponds to a semimomentless shell, we have the following boundary conditions at ξ = const.

Geometrical conditions:

$$1)\ w = 0;\ 2)\ v = 0;\ 3)\ u = 0. \tag{3}$$

Force conditions:

$$1)\ T_x = \left[\frac{\partial u}{\partial \xi} + \mu\left(\frac{\partial v}{\partial \varphi} - w\right)\right] = 0; \tag{4}$$

$$2)\ S = \left(\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \varphi}\right) = 0.$$

Boundary conditions of the type

$$\frac{\partial u}{\partial \xi} = 0;\ M_x = 0;\ Q_x = \frac{\partial M_x}{\partial \xi} + \frac{\partial M_{xy}}{\partial \varphi} = 0 \tag{5}$$

have obviously no meaning for the semimomentless shell considered.

Further on, following /2/ and /3/, we reduce system (2) to the following system of equations:

$$\frac{\partial^4 u}{\partial \xi^4} = -\frac{\partial^3 w}{\partial \xi \partial \varphi^2} + \left[\mu\frac{\partial^3 w}{\partial \xi^3}\right];$$

$$\frac{\partial^4 v}{\partial \xi^4} = \frac{\partial^3 w}{\partial \varphi^3} + \left[(2+\mu)\frac{\partial^3 w}{\partial \xi^2 \partial \varphi}\right]; \tag{6}$$

$$EhR^2\frac{\partial^4 w}{\partial \xi^4} + \frac{Eh^3}{12(1-\mu^2)}\left(\frac{\partial^8 w}{\partial \varphi^8} + 2\frac{\partial^6 w}{\partial \varphi^6} + \frac{\partial^4 w}{\partial \varphi^4}\right) + R^3 P\left(\frac{\partial^6 w}{\partial \varphi^6} + \frac{\partial^4 w}{\partial \varphi^4}\right) = 0.$$

Note that in the first two equations the terms between brackets are small compared with the main terms, and can be neglected; they are, however, important when the main terms mutually cancel /2/ in the expression

$$S = \frac{Eh}{2(1+\mu)}\left(\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \varphi}\right). \tag{7}$$

We shall look for a solution of system (6) in the form

$$u = U \cos n\varphi,$$
$$v = V \sin n\varphi, \qquad (8)$$
$$w = W \cos n\varphi,$$

where U, V, W are functions of the coordinate ξ only.

The third equation of system (6) can be written in this case as follows:

$$\frac{d^4 W}{d\xi^4} - \lambda^4 W = 0, \qquad (9)$$

where

$$\lambda^4 = \frac{R}{Eh}\left[Pn^4(n^2-1) - \frac{Eh^3}{12(1-\mu^2)R^2} n^4 (n^2-1)^2\right]. \qquad (10)$$

From the first two equations of system (6) we find

$$V = \frac{1}{n} W - \left[\frac{2+\mu}{n^3} W''\right];$$
$$U = \frac{1}{n^2} W' + \left[\frac{\mu}{n^4} W'''\right]. \qquad (11)$$

As in equations (6), the terms between brackets are small compared with the main terms, and are important only in the determination of S.

The solution of equation (9), which coincides in form with the ordinary equation of free transverse vibrations of a beam, is obviously of the form

$$W = A_1 \sin\lambda\xi + A_2 \cos\lambda\xi + A_3 \operatorname{sh}\lambda\xi + A_4 \operatorname{ch}\lambda\xi, \qquad (12)$$

where A_i are arbitrary constants.

Boundary conditions (3) and (4) will now be expressed through W, using relationship (11).

1. The condition for ξ = const, $w = 0$ is obviously equivalent to condition $W = 0$.
2. The condition for ξ = const, $v = 0$ leads to the condition $W = 0$.
3. The condition for ξ = const, $u = 0$ leads to the condition $dW/d\xi = W' = 0$.
4. The condition for ξ = const, $T_x = 0$ leads to the condition

$$W'' = 0.$$

5. Finally, the condition for ξ = const, $S = 0$ leads to the condition $W''' = 0$.

Thus, if the boundary conditions for the shell are given, its critical pressure is easily determined from the corresponding characteristic equation for the natural frequencies of a beam. The relationship for the critical pressure is found from expression (10)

$$P_{cr} = \lambda_0^4 \frac{Eh}{Rn^4(n^2-1)} + \frac{Eh^3}{12(1-\mu^2)R^3}(n^2-1), \qquad (13)$$

where λ_0 is the minimum root of the corresponding characteristic equation.

If $n^2 \gg 1$, expression (13) can be simplified, by finding the minimum of P_{cr} by n:

$$P_{cr} = \lambda_0 \frac{\sqrt{6}E}{9(1-\mu^2)^{3/4}} \left(\frac{h}{R}\right)^{\frac{5}{2}}. \tag{14}$$

We shall now consider several specific variants of the boundary conditions.

1. Both ends of the shell are simply supported:

$$\text{at } \xi = 0 \text{ and } \xi = l/R,\ w = T_x = 0;\ \text{therefore}, W = W'' = 0.$$

This is similar to a hinged beam; the minimum root of the corresponding characteristic equation is

$$\lambda_0 = \frac{\pi R}{l}. \tag{15}$$

The expression for the critical pressure at $n^2 \gg 1$ coincides with Papkovich's well-known formula

$$(P_{cr})_{s.s.} = \frac{\pi\sqrt{6}E}{9(1-\mu^2)^{3/4}} \left(\frac{R}{l}\right)\left(\frac{h}{R}\right)^{5/2}. \tag{16}$$

2. Both ends of the shell are hinged:

$$\text{at } \xi = 0 \text{ and } \xi = l/R,\ w = u = 0;\ \text{therefore}, W = W' = 0.$$

This is similar to a beam with built-in ends

$$\lambda_0 = 4.73 \frac{R}{l}. \tag{17}$$

At $n^2 \gg 1$ we obtain that the critical pressure is one and a half times larger than in a beam with simply supported ends:

$$P_{cr} = 1.5\ (P_{cr})_{s.s.}. \tag{18}$$

3. One end is simply supported, the other is hinged:

$$\text{at } \xi = 0,\ w = T_x = 0, \text{ and } \xi = l/R,\ w = u = 0,$$

i.e.,

$$\text{at } \xi = 0,\ W = W'' = 0;\ \text{at } \xi = l/R,\ W = W' = 0.$$

This is similar to a beam with one end built-in and the other supported

$$\lambda_0 = 3.93 \frac{R}{l}. \tag{19}$$

At $n^2 \gg 1$ we have

$$P_{cr} = 1.25\ (P_{cr})_{s.s.}. \tag{20}$$

4. One end is hinged, the other is free:

$$\text{at } \xi=0,\ w=u=0;\ \text{and } \xi=l/R,\ T_x=S=0,$$

i.e.,

$$\text{at } \xi=0,\ W=W'=0,\ \text{and } \xi=l/R,\ W''=W'''=0.$$

This is similar to a beam with one end built-in and the other free

$$\lambda_0=1.87\,\frac{R}{l}\,. \tag{21}$$

At $n^2 \gg 1$

$$P_{cr}=0.6\ (P_{cr})_{s.s.}\ .$$

5. One end is simply supported, and the other cannot move in the axial direction:

$$\text{at } \xi=0,\ w=T_x=0, \qquad \text{at } \xi=l/R;\ u=S=0,$$

i.e.,

$$\text{at } \xi=0;\ W=W''=0; \qquad \text{at } \xi=l/R,\ W'=W'''=0.$$

This is similar to a hinged beam of double length

$$\lambda_0=\frac{\pi R}{2l}\,. \tag{22}$$

At $n^2 \gg 1$

$$P_{cr}=0.5\ (P_{cr})_{s.s.}\ . \tag{23}$$

6. One end is hinged, and the other cannot move in the axial direction:

$$\text{at } \xi=0,\ w=u=0; \qquad \text{at } \xi=l/R,\ u=S=0,$$

i.e.,

$$\text{at } \xi=0;\ W=W'=0; \qquad \text{at } \xi=l/R,\ W'=W'''=0.$$

This is similar to a beam of double length built-in at both ends

$$\lambda_0=4.73\,\frac{R}{2l}\,, \tag{24}$$

and

$$P_{cr}=0.75\ (P_{cr})_{s.s.}. \tag{24'}$$

7. One end is simply supported, the other is free:

$$\text{at } \xi=0,\ w=T_x=0; \qquad \text{at } \xi=l/R,\ T_x=S=0,$$

i.e.,

$$\text{at } \xi=0,\ W=W''=0; \qquad \text{at } \xi=l/R,\ W''=W'''=0.$$

This is similar to a beam with one end hinged and the other free

$$\lambda_0=0. \tag{25}$$

It follows then from (13)

$$P_{cr}=\frac{Eh^3\,(n^2-1)}{12\,(1-\mu^2)R^3}. \tag{26}$$

In other words, the critical pressure coincides in this case with the critical pressure for an infinitely long shell.

Note that system (6) could also have been obtained from the complete system of equations of an isotropic shell, by using the condition /2/

$$\frac{\partial^2 (\ )}{\partial \xi^2} << \frac{\partial^2 (\ )}{\partial \varphi^2}. \tag{27}$$

The use of the semimomentless shell model clarifies the question of the correct boundary conditions. As to the errors introduced by such a schematization, they are the same as the errors introduced by condition (27), i.e., such a model is correct for not too short shells /2, 3/.

We note in conclusion that the allowance for the boundary conditions for the tangential components of the displacements considerably influences the value of the upper critical pressure. This influence is particularly strong for shells with one free end; thus, cases 4) and 7) differ only by the boundary conditions for u, while the expressions for the critical pressures in these cases differ qualitatively (the quantitative difference can be arbitrarily large). This last fact has been corroborated experimentally.

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MOTION OF A RIGID-PLASTIC CIRCULAR CYLINDRICAL SHELL IN A RESISTING MEDIUM

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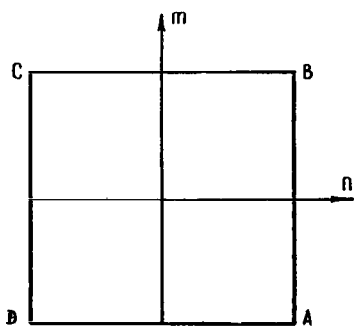
1. INTRODUCTION

The motion of an infinite rigid-plastic beam in a resisting medium was treated in /1/; it was established there that if the medium resistance is proportional to some power of the velocity of displacement, a solution can be found to the problem of the motion of a rigid-plastic body.



Yield square

FIGURE 1.

The motion of a simply supported circular plate in a resisting medium under the action of a uniformly distributed load was treated in /2/ under the assumption that the resistance force is proportional to the first power of the velocity of displacement. The present paper is devoted to the analysis of the motion in a resisting medium of a cylindrical shell rigidly clamped at both ends.

It is assumed that the shell is made of a rigid ideally plastic material behaving according to Tresca's yield condition (Figure 1) and to the corresponding flow rule.

Let the shell be subjected to the action of a uniformly distributed internal impact load of constant intensity p acting during $0 \leqslant t \leqslant t_0$.

The resistance force is assumed to be proportional to the first power of the velocity of displacement, i. e.,

$$q = \alpha \dot{w} . \tag{1.1}$$

It is assumed as in /3/ that $\alpha = \rho \cdot c$, where ρ is the density, and c is the velocity of sound.

The solution of this problem without allowing for the resistance forces is known /3/.

The dynamic behavior of the shell will depend on whether the load will have a "medium" ($p_0 < p \leqslant p_1$) or a "high" ($p > p_1$) value (p_1 is determined in § 3).

In this paper we give a solution for the case of "medium" value of the load. In addition, since the solution for the second phase imposes restrictions on the shell dimensions, we have solved separately the cases of "short" and "long" shells. These solutions are given in §§ 3 and 4, respectively.

2. FUNDAMENTAL EQUATIONS

The equation of motion of the cylindrical shell, allowing for the resistance force, will be

$$\frac{d^2M_x}{dx^2}+\frac{N_\varphi}{a}+p-q-s\frac{d^2w}{dt^2}=0, \tag{2.1}$$

where M_x is the axial bending moment, N_φ, the circumferential force per unit thickness, p, the inner pressure, q, the resistance force, and s, the mass of unit area.

We shall use the dimensionless magnitudes

$$n=\frac{N_\varphi}{4\,kh},\quad m=\frac{M_x}{2kh^2},\quad P=\frac{pa}{4kh},\quad W=\frac{saw}{4\,kh\,t_0^2},$$
$$\gamma=\frac{\alpha\cdot t_0}{s},\quad y=\frac{x}{L},\ \tau=\frac{t}{t_0},\quad C=\frac{L}{\sqrt{ah}}, \tag{2.2}$$

where $2L$ is the shell length, a, its radius, and h, its thickness. In these magnitudes (2.1) is represented in the form

$$m''/2C^2+n+P-\gamma\dot{W}-\ddot{W}=0, \tag{2.3}$$

where the prime designates differentiation by y, and the dot differentiation by the time τ.

Following /3/, we take as yield curve the quandrangle in coordinates $(m,\ n)$ (Figure 1). The general method of solution for plastic-rigid material consists in assuming some "stress profile" for the body, i.e., in assuming some locus of the stress points relative to the yield curve.

It turned out in the given problem that only sides AB, AD, and apices A and B of the yield quadrangle can be used for determining the stresses. The following table can be formed for these sides and apices.

TABLE

Plastic range	Stresses		Vector of strain rates		
	$n\ m$	inequality	$\dot{W}\dot{W}''/2C^2$	equality	inequality
AD	$*-1$	$-1\leqslant n\leqslant+1$	$0\ \lambda$	$\dot{W}=0$	$\dot{W}''\geqslant0$
A	$+1-1$	*	$\mu\ \lambda$	*	$\dot{W}\geqslant0,\ \dot{W}''\geqslant0$
AB	$+1*$	$-1\leqslant m\leqslant+1$	$\mu\ 0$	$\dot{W}''=0$	$\dot{W}\geqslant0$
B	$+1+1$	*	$\mu-v$	*	$\dot{W}>0\ \ \dot{W}''\leqslant0$

3. SHORT SHELL UNDER A MEDIUM LOAD

It can be shown that, just as in the absence of resistance forces /3/, the shell as a whole is in state AB (the cylindrical part of the shell is in state B).

First phase ($0<\tau<1$). Since on side AB $\dot{W}''' = 0$, then

$$\dot{W} = \dot{W}_0(\tau)\, y\,. \tag{3.1}$$

Solving (2.3) allowing for (3.1), and using the boundary conditions $m\ (0) = -1$, $m\ (1) = +1$, we obtain

$$m = C^2y^2 - C^2Py^2 + \frac{1}{3}C^2\gamma\,\dot{W}_0y^3 + \frac{1}{3}C^2\,\ddot{W}_0y^3 + \frac{1}{3}C^2\left[\frac{1}{C^2} + \frac{1}{2}(P-1) - \right.$$
$$\left. - \frac{1}{6}\gamma\,\dot{W}_0 - \frac{1}{6}\,\ddot{W}'\right] y - C. \tag{3.2}$$

W_0, is determined from the boundary condition $m'\ (1) = 0$:

$$W_0 = C_1 + C_2e^{-\gamma\tau} + \frac{3}{C^2\gamma}\left[\frac{C^2}{2}(P-1) - 1\right]. \tag{3.3}$$

The integration constant is determined from the initial conditions

$$W_0 = -\frac{3}{C^2\gamma^2}\left[\frac{C^2}{2}(P-1) - 1\right] + \frac{3}{C^2\gamma^2}\left[\frac{C^2}{2}(P-1) - 1\right]e^{-\gamma\tau} +$$
$$+ \frac{3}{C^2\gamma}\left[\frac{C^2}{2}(P-1) - 1\right]\tau\,. \tag{3.4}$$

The deflection will be, on the strength of (3.1),

$$W = \left\{-\frac{3}{C^2\gamma^2}\left[\frac{C^2}{2}(P-1) - 1\right] + \frac{3}{C^2\gamma^2}\left[\frac{C^2}{2}(P-1) - 1\right]e^{-\gamma\tau} + \right.$$
$$\left. + \frac{3}{C\gamma}\left[\frac{C^2}{2}(P-1) - 1\right]\tau\right\} y. \tag{3.5}$$

Substituting (3.4) in (3.2), we obtain the following expression for the moment:

$$m = \left[\frac{C^2}{2}(P-1) - 1\right] y^3 - C^2(P-1)\,y^2 + \left[\frac{C^2}{2}(P-1) + 3\right] y. \tag{3.6}$$

It is seen from (3.6) that, just as in the case of a plate, the expression for the bending moment is independent of γ. The deflection in the case of motion without resistance can be obtained from (3.5) by passing to the limit $\gamma \to 0$:

$$W = \left(\frac{3}{2C^2}\right)\left[\left(\frac{C^2}{2}\right)(P-1) - 1\right]\tau^2\, y. \tag{3.7}$$

This coincides with solution /4/. Since the expressions for the bending moment coincide with the solution obtained when the medium resistance is

neglected, the value of the medium load will be

$$P_1 = P_0 + \frac{4}{C^2}. \tag{3.8}$$

Second phase ($1 \leqslant \tau \leqslant \tau_1$).

The load is abruptly removed at $\tau = 1$. The motion of the shell is interrupted as a result, but only after the kinetic energy acquired by the shell under the action of P has been dissipated in the plastic deformation.

The stressed state is characterized by the points of segment AB, so that $W = W_0 y$. For $W_0(\tau)$ we obtain, substituting $P = 0$ in (3.8),

$$W_0 = C_3 + C_4 e^{-\gamma\tau} + \frac{3}{C^2\gamma}\left[-\frac{C^2}{2} + 1\right]. \tag{3.9}$$

The integration constants C_3 and C_4 are determined from the condition of continuity of $\frac{\partial W}{\partial \tau}$ and W at $\tau = 1$.

The following expression is obtained for the deflection:

$$W = \left\{\frac{3P}{2\gamma} + \frac{3(C^2+2)}{2C^2\gamma^2} + \frac{3}{C^2\gamma^3}\left[\frac{C^2}{2}(P-1) - 1\right]e^{-\gamma\tau} - \right.$$
$$\left. - \frac{3P}{2\gamma^2}e^{-\gamma(\tau-1)} - \frac{3}{C^2\gamma}\left[\frac{C^2}{2}+1\right]\tau\right\}y. \tag{3.10}$$

This reduces at the limit $\gamma \to 0$ to

$$W = \left[-\left(\frac{3}{2C^2}\right)\left(\frac{C^2}{2}+1\right)\tau^2 + \left(\frac{3P}{2}\right)\tau - \frac{3P}{4}\right]y. \tag{3.11}$$

This expression coincides with the result of /3/ for the second phase.

For the moment we obtain

$$m = -\left[\frac{C^2}{2}+1\right]y^3 + C^2y^2 + \left(3 - \frac{C^2}{2}\right)y - 1. \tag{3.12}$$

Using the condition $\frac{\partial W}{\partial \tau} = 0$, we obtain the time τ_1 after which the shell passes into a state of rest.

$$\tau_1 = \frac{1}{\gamma}\ln\frac{P_0}{2}\left[\frac{1}{2} + \frac{2}{C^2} - \frac{P}{2}(1 - e^{+\gamma})\right]. \tag{3.13}$$

4. LONG SHELL UNDER A MEDIUM LOAD

It was shown in /3/ that the solution obtained in § 3 for the second phase is correct for $0 < C < \sqrt{6}$. If $C^2 > 6$, we obtain $m < -1$ at $y = 0$, which is unadmissible. A different hypothesis for the stream profile is therefore to be assumed at $C^2 > 6$.

It could be assumed that the point A corresponds to the interval $0 \leqslant y \leqslant u$, and the segment AB to the interval $u \leqslant y \leqslant 1$. The corresponding calculations

show however that the yield conditions are not satisfied in this case. It remains to assume that point K corresponds to the interval $0 < y < u$, and side AB to the interval $u < y < 1$. In this case, as seen from the table, the shell remains rigid for $0 < y < u$.

Second phase $(1 < \tau < \tau_1)$.

$$\text{For } 0 < y < u \quad m = -1, \quad n = 0, \quad \dot{W} = 0; \tag{4.1}$$

$$\text{for } u < y < 1 \text{ we have } \dot{W} = \dot{A}\, y + \dot{B}, \; n = -1. \tag{4.2}$$

Solving equation (2.3) with the aid of (4.2), we obtain

$$m = C^2 \left[\frac{\ddot{A} + \gamma \dot{A}}{3} y^3 + (1 + \ddot{B} + \gamma \dot{B}) y^2 \right] + C_1 y + D. \tag{4.3}$$

The boundary conditions for m in the case $u < y < 1$ are

$$m(u) = -1, \; m'(u) = 0, \; m(+1) = +1, \; m'(1) = 0. \tag{4.4}$$

Using these conditions, we determine the unknown integration functions as functions of u:

$$\begin{gathered} \ddot{A} + \gamma \dot{A} = -\left(\frac{12}{C^2}\right)(1-u)^{-2}, \\ \ddot{B} + \gamma \dot{B} = -1 + \left(\frac{6}{C^2}\right)(1+u)(1-u)^{-3}, \\ C = -12\, u\, (1-u)^{-3}, \\ D = 1 - 2\,(1 - 3u)\,(1-u)^{-3}, \end{gathered} \tag{4.5}$$

The condition of continuity of the velocities at $\tau = 1$ yields

$$\dot{B} = -\dot{A}\, y.$$

Using (4.5) we obtain

$$\dot{A} = \frac{6}{C^2 u} \left[\frac{C^2}{6} - (1-u)^{-2} \right]. \tag{4.6}$$

The integration constants are determined from the condition of continuity of W at $\tau = 1$.

We find thus from (4.2) and (3.5)

$$\dot{W}(y, 1) = \dot{A}(1)\, y + \dot{B}(1) = \left\{ -\frac{3}{C^2 \gamma} \left[\frac{C^2}{2}(P-1) - 1 \right] e^{-\gamma} + \right.$$
$$\left. + \frac{3}{C^2 \gamma} \left[\frac{C^2}{2}(P-1) - 1 \right] \right\} y,$$

whence

$$\dot{B}(1) = 0; \; \dot{A}(1) = \frac{3}{C^2 \gamma} \left[\frac{C^2}{2}(P-1) - 1 \right] (1 - e^{-\gamma}).$$

Then

$$u(1) = 0, \; \dot{u}(1) = \frac{\gamma \quad (C^2 - 6)}{3 \left[\frac{C^2}{2}(P-1) - 1 \right] (1 - e^{-\gamma})},$$

whence

$$u = \frac{\gamma\,[C^2 - 6]\;(\tau - 1)}{3\left[\dfrac{C^2}{2}(P-1)\;(1-e^{-\tau})\right]}. \tag{4.7}$$

The solution for the second phase is

$$0 < y < u, \quad m = -1, \; n = 0, \; \dot{W} = 0,$$

$$u < y < 1, \; m = 1 + (1-u)^{-3}\;[-4y^3 + 6\,(1+u)\,y^2 - 12uy - 2\,(1-3u)], \tag{4.8}$$

$$n = -1, \; \dot{W} = \left(\frac{6}{C^2 u}\right)\left[\frac{C^2}{6} - (1-u)^{-2}\right](y-u),$$

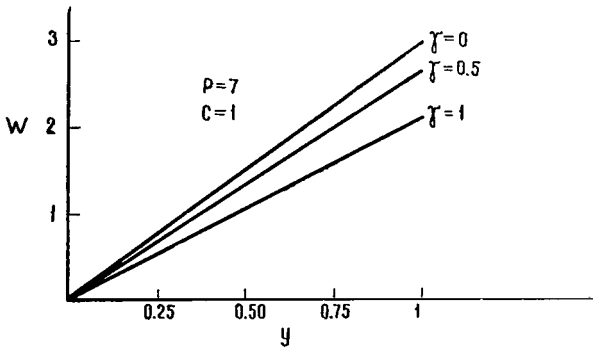
$$u = \frac{2\gamma\,(C^2-6)\;(\tau-1)}{3\left[\dfrac{C^2}{2}(P-1) - 1\right](1-e^{-\tau})},$$

$$\text{for } 1 < \tau < \tau_1 \;\; P_0 \leqslant P \leqslant P_1, \; \sqrt{6} < C.$$

The time after which the shell stops is

$$\tau_1 = \frac{3\left[\dfrac{C^2}{6}(P-1) - 1\right](1-e^{-\tau})}{2\gamma\,C\,(C+\sqrt{6})}. \tag{4.9}$$

Expressions (4.8) represent thus a solution for the second phase $C > \sqrt{6}$. The value of the deflection as a function of γ for $C = 1$, $\tau = 1$, and $P = 7$ is represented in Figure 2.



The deflection as a function of γ

FIGURE 2.

It is seen from expressions (3.5), (3.10), and (4.8) that the influence of the medium resistance on the value of the deflection is very considerable even for a "medium" value of the load.

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DYNAMICS OF A MOMENTLESS SPHERICAL SHELL MADE FROM AN IDEALLY RIGID-PLASTIC MATERIAL

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This paper is devoted to an analysis of the motion of a semispherical momentless shell with built-in edges.

The shell material is assumed to be plastic-rigid without work-hardening, and to behave according to Mises' yield condition and to the flow rule connected with it.

The stress-strain diagram of this material at the simplest stressed state is represented in Figure 1.

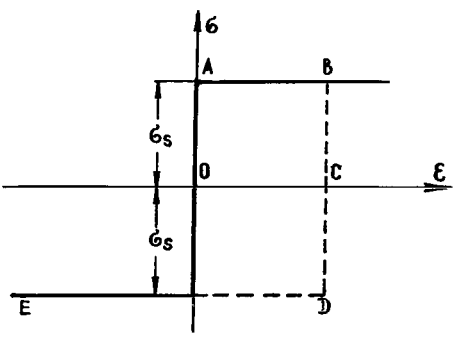

FIGURE 1.

Segment OA corresponds to the initial loading up to the yield point at extension, segment AB to the plastic flow, segments BC and CD to complete unloading with subsequent application of a compressing load up to the yield point at compression, and segment DE to plastic flow under a compressive stress.

An axially symmetrical normal pressure is applied to the shell at the initial time moment . It is assumed that the material of the entire shell, or of a part of the shell (the other part can remain rigid), passes instantaneously into a plastic state.

The equations of motion of a momentless shell, taking large deflections into account, are of the form

$$m\ddot{w} = \frac{1}{r}(N_\varphi + N_\theta) + \frac{1}{r^2}\frac{\partial N_\varphi}{\partial \varphi}\cdot\frac{\partial w}{\partial \varphi} + \frac{N_\varphi}{r^2}\left(\frac{\partial^2 w}{\partial \varphi^2} + \operatorname{ctg}\varphi\frac{\partial w}{\partial \varphi}\right) + Z, \qquad (1)$$

$$m\ddot{v} = \frac{1}{r}\left[\frac{\partial N_\varphi}{\partial \varphi} + \operatorname{ctg}\varphi\,(N_\varphi - N_\theta)\right], \tag{2}$$

where

$$N_\varphi = \sigma_\varphi h; \quad N_\theta = \sigma_\theta h.$$

Here v, w are deflections of the shell along the meridian and along the normal to the surface; m is the mass of unit area of the shell; N_φ and N_θ, and σ_φ and σ_θ are, respectively, the meridional and circumferential membrane forces and stresses (Figure 2).

The strains ε_φ and ε_θ are given by the following expressions:

$$\varepsilon_\varphi = \frac{1}{r}\frac{\partial v}{\partial \varphi} - \frac{w}{r} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \varphi}\right)^2, \tag{3}$$

$$\varepsilon_\theta = \frac{1}{r}(v \operatorname{ctg}\varphi - w). \tag{4}$$

The strain rates will therefore be:

$$\dot{\varepsilon}_\varphi = \frac{1}{r}\frac{\partial \dot{v}}{\partial \varphi} - \frac{\dot{w}}{r} + \frac{1}{r^2}\frac{\partial w}{\partial \varphi}\cdot\frac{\partial \dot{w}}{\partial \varphi}, \tag{5}$$

$$\dot{\varepsilon}_\theta = \frac{1}{r}(\dot{v} \operatorname{ctg}\varphi - \dot{w}). \tag{6}$$

Mises' plasticity condition is /1/ (Figure 3):

$$\sigma_\varphi^2 - \sigma_\varphi\cdot\sigma_\theta + \sigma_\theta^2 = \sigma_s^2. \tag{7}$$

The strain rates are connected with the stress components by the Saint-Venant—Mises equations /1/ (the flow rule)

$$\dot{\varepsilon}_\varphi = \lambda(2\sigma_\varphi - \sigma_\theta), \tag{8}$$

$$\dot{\varepsilon}_\theta = \lambda(2\sigma_\theta - \sigma_\varphi), \tag{9}$$

$$\lambda = \frac{1}{2\sigma_s^2}(\sigma_\varphi\cdot\dot{\varepsilon}_\varphi + \sigma_\theta\dot{\varepsilon}_\theta). \tag{10}$$

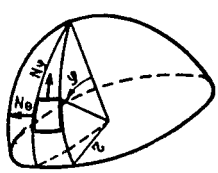

FIGURE 2.

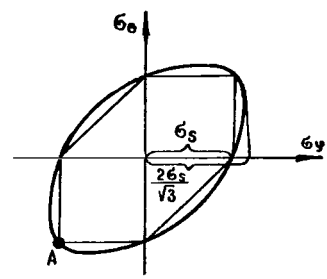

FIGURE 3.

Introducing the symbol

$$\frac{\dot{\varepsilon}_\varphi}{\dot{\varepsilon}_\theta} = \frac{2\sigma_\varphi - \sigma_\theta}{2\sigma_\theta - \sigma_\varphi} = k, \tag{11}$$

and taking formulas (5), (6), (8), (9), and (11) into account, we can replace the flow rule (8), (9), and (10) by the single equation

$$\frac{\partial v}{\partial \varphi} = \frac{\operatorname{ctg}\varphi}{k}\dot{v} + \frac{k-1}{k}\dot{w} - \frac{1}{r}\frac{\partial w}{\partial \varphi}\cdot\frac{\partial \dot{w}}{\partial \varphi}. \tag{12}$$

As to the external pressure Z applied to the shell, it is assumed that it decreases monotonically from the shell pole to the edge, and that it decreases (or at least does not increase) with time.

We shall now prove that the following proposition is true under these assumptions: the parallel of angle φ_0 at which $Z(\varphi_0) = \frac{2N_s}{r}$ splits the shell into a plastic and a rigid part, the region about the pole will be plastic, and the region adjoining the shell edge will be rigid (Figure 4).

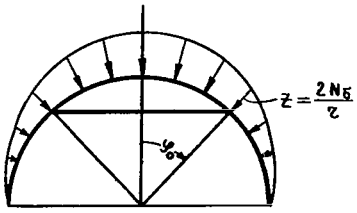

FIGURE 4.

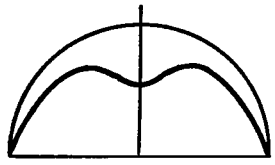

FIGURE 5.

To prove this proposition, write equations (1), (2) for the case of equilibrium

$$N_\varphi + N_\theta + Zr = 0, \tag{13}$$

$$\frac{dN_\varphi}{d\varphi} + \operatorname{ctg}\varphi\,(N_\varphi - N_\theta) = 0. \tag{14}$$

Eliminating N_θ from (13) and (14), we obtain the following equation for N_φ:

$$\frac{dN_\varphi}{d\varphi} + \operatorname{ctg}\varphi\,(2N_\varphi + Zr) = 0. \tag{15}$$

It is known that $Z = \frac{2N_s}{r}$ at $\varphi = \varphi_0$ and decreases further on. It follows then from (15) that

$$\frac{dN_\varphi}{d\varphi} = 0, \quad \text{for} \quad \varphi = \varphi_0, \tag{16}$$

$$\frac{dN_\varphi}{d\varphi} > 0, \quad \text{for} \quad \varphi = \varphi_0. \tag{17}$$

It follows in addition from (13) that

$$\frac{dN_\theta}{d\varphi} = -\frac{\partial Z}{\partial\varphi} r > 0, \quad \text{for} \quad \varphi = \varphi_0. \tag{18}$$

It is easily shown with the aid of (16), (17), (18) that in the case of equilibrium the points representing stresses for angles larger than φ_0 will be situated inside the Mises ellipse (Figure 3), i.e., a part of the shell will remain rigid at $\varphi > \varphi_0$.

By reasoning in the same way it can be shown that in order that the part of the shell adjoining the pole ($\varphi < \varphi_0$) be in equilibrium it is necessary that the points representing the stresses lie outside the Mises ellipse. This cannot happen, since the material is not work-hardened, and therefore this part of the shell passes into a plastic state.

It then follows that if the external pressure Z is everywhere smaller than $\frac{2N_s}{r}$, the entire shell will remain rigid. On the other hand, if at $\varphi = \frac{\pi}{2}$ the pressure is equal to or larger than $\frac{2N_s}{r}$, the entire dome will pass into a plastic state.

Thus, in order to solve the problem it is necessary to solve a system of three equations (1), (2), (12) with three unknowns v, w, N_φ (N_θ is considered as a known function of N_φ (7)) for the region which passed into the plastic state. The following initial and boundary conditions are used for v and w:

$$v = 0; \quad \frac{\partial v}{\partial t} = 0, \quad w = 0; \quad \frac{\partial w}{\partial t} = 0; \quad \text{at} \quad t = 0; \tag{19}$$

$$v = 0; \quad w = 0; \quad \text{at} \quad \varphi = \varphi_0. \tag{20}$$

It also follows from the above considerations that at the initial time moment the stress for the part of the shell which has passed into a plastic state will be represented by point A of Mises' ellipse, i. e.,

$$N_\varphi = -N_s; \quad N_\theta = -N_s; \quad \text{at} \quad t = 0 \tag{21}$$

everywhere for $\varphi \leqslant \varphi_0$.

Note that in this case equation (1) will be of the elliptic type. It follows that when solving the problem formulated for equations of the elliptic type it is necessary to state the Cauchy problem with zero initial conditions.

The method of successive approximations can be used /2/ for solving the system of equations (1), (2), and (12).

We take as first approximation for the stresses

$$N'_\varphi = -N_s; \quad N'_\theta = -N_s.$$

Solving next equation (1), we find the first approximation for w. Substituting this value w_1 in (12) and solving relative to v, we find the first approximation v_1. Substituting v_1 in (2) and solving relative to N_φ (N_θ is expressed through N_φ by Mises' condition), we obtain the second approximation for the stresses. The whole sequence is then repeated.

If the load acts for a very short time interval only, it can turn out that at some subsequent moment the strain rates in the pole become equal to zero, while the velocities of displacement $\dot{v}$ and w in the entire plastic region are negligible. The calculation is to be interrupted at this moment, and it is to be assumed that the shell has stopped after a certain compression.

It can also happen that at some moment the strains in the shell pole will attain such a large value that the material will collapse. In this case too the calculation is to be interrupted, and it is to be considered that the shell has been put out of action as a result of the failure caused by intense pressure.

Finally, at some types of load, in particular in the case of loads having about the pole a sharply displayed increase, it can happen that the strain rates $\dot{\varepsilon}_\varphi$ and $\dot{\varepsilon}_\theta$ become equal to zero although the external load continues to act, or if the external load has already been removed but the rates of deflection of the shell $\dot{v}$ and $\dot{w}$ are still large due to the shell inertia.

In this case (Figure 1) there appears in the shell pole an expanding region where tensile stresses take place, since here a stress of opposite sign is applied to the pole immediately after the unloading.

In this case there will no longer be a simple compression of the shell, and the shell can be put out of action as a result of an abrupt shape alteration (Figure 5).

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STABILITY OF AN ORTHOTROPIC CYLINDRICAL SHELL IN A GAS STREAM

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Consider a thin cylindrical shell made of an odd number $(2k+1)$ of homogeneous anisotropic layers. The layers, arranged symmetrically relative to the coordinate surface of the shell, all have the same thickness and the same physico-mechanical properties. The coordinate surface (α, β) is the middle surface of both the middle layer and the shell as a whole.

It will be assumed that the material of each layer behaves according to the generalized law of Hooke and that at each point there are three planes of elastic symmetry, the principal directions of which coincide with the directions of the orthogonal coordinate lines α, β, γ. It is also assumed that the shell layers remain elastic after the deformation and do not slip relative to each other.

Let a supersonic gas stream flow past the shell at an undisturbed velocity U directed parallel to the shell generator (Figure 1).

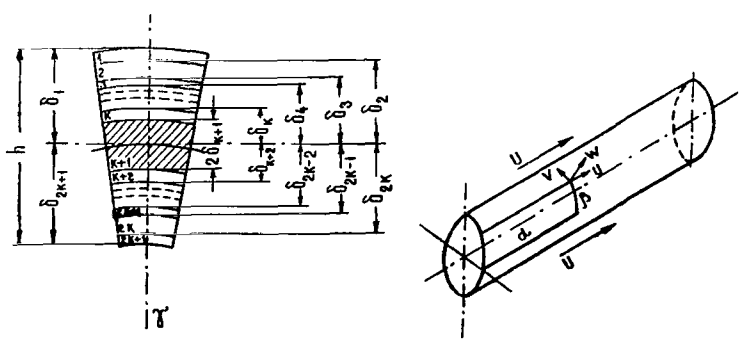

FIGURE 1.

It is assumed that the hypothesis of undeformable normals is true for the whole sandwich shell.

The following starting relationships and equations are obtained for the shell considered on the strength of the assumption made /1/:

a) equations of motion

$$\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = m^* \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = m^* \frac{\partial^2 v}{\partial t^2}, \tag{1.1}$$

$$\frac{\partial^2 M_1}{\partial\alpha^2} + 2\frac{\partial^2 H}{\partial\alpha\partial\beta} + \frac{\partial^2 M_2}{\partial\beta^2} + \frac{1}{R}T_2 + Z = m^*\frac{\partial^2 w}{\partial t^2}, \tag{1.1 (cont'd)}$$

where m^* is the reduced mass, defined by

$$m^* = \frac{2}{g}\left[\gamma_{k+1}\delta_{k+1} + \sum_{i=1}^{k}\gamma_i(\delta_i - \delta_{i+1})\right], \tag{1.2}$$

and γ_i is the unit weight of the i-th layer, δ_i, the distance of the i-th layer from the middle surface of the shell, Z, the external load normally applied to the middle surface, T_1, T_2, M_1, M_2, H, inner forces and moments, and R, the shell radius;

b) elasticity relationships

$$\begin{aligned} T_1 &= c_{11}\varepsilon_1 + c_{12}\varepsilon_2, \\ T_2 &= c_{12}\varepsilon_1 + c_{22}\varepsilon_2, \\ S &= c_{66}\omega; \end{aligned} \tag{1.3}$$

$$\begin{aligned} M_1 &= D_{11}\varkappa_1 + D_{12}\varkappa_2, \\ M_2 &= D_{12}\varkappa_1 + D_{22}\varkappa_2, \\ H &= D_{66}\tau, \end{aligned} \tag{1.4}$$

where the tensile and flexural rigidities c_{ij} and D_{ij} are given by

$$\begin{aligned} c_{ij} &= 2\left[B_{ij}^{(k+1)}\delta_{k+1} + \sum_{n=1}^{k}B_{ij}^{(n)}(\delta_n - \delta_{n+1})\right], \\ D_{ij} &= \frac{2}{3}\left[B_{ij}^{(k+1)}\delta_{k+1}^3 + \sum_{n=1}^{k}B_{ij}^{(n)}(\delta_n^3 - \delta_{n+1}^3)\right], \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} B_{11}^{(n)} &= \frac{E_1^{(n)}}{1-\mu_1^{(n)}\mu_2^{(n)}}, \qquad B_{22}^{(n)} = \frac{E_2^{(n)}}{1-\mu_1^{(n)}\mu_2^{(n)}}, \\ B_{12}^{(n)} &= \frac{\mu_1^{(n)}E_2^{(n)}}{1-\mu_1^{(n)}\mu_2^{(n)}} = \frac{\mu_2^{(n)}E_1^{(n)}}{1-\mu_1^{(n)}\mu_2^{(n)}}, \qquad B_{66}^{(n)} = G_{12}^{(n)}; \end{aligned} \tag{1.6}$$

c) geometrical relationships

$$\begin{aligned} \varepsilon_1 &= \frac{\partial u}{\partial\alpha}, \quad \varepsilon_2 = \frac{\partial v}{\partial\beta} + \frac{w}{R}, \quad \omega = \frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha}, \\ \varkappa_1 &= -\frac{\partial^2 w}{\partial\alpha^2}, \quad \varkappa_2 = -\frac{\partial^2 w}{\partial\beta^2}, \quad \tau = -2\frac{\partial^2 w}{\partial\alpha\partial\beta}, \end{aligned} \tag{1.7}$$

where $u(\alpha, \beta, t)$, $v(\alpha, \beta, t)$, $w(\alpha, \beta, t)$ are tangential and normal displacements of the points of the middle surface of the shell.

The substitution of the values of the inner forces and moments from (1.3) and (1.4) into the equations of motion (1.1) yields, with the aid of (1.7), the following system of differential equations of motion of the shell:

$$c_{11}\frac{\partial^2 u}{\partial\alpha^2} + c_{66}\frac{\partial^2 u}{\partial\beta^2} + (c_{12} + c_{66})\frac{\partial^2 v}{\partial\alpha\partial\beta} + \frac{1}{R}c_{12}\frac{\partial w}{\partial\alpha} = m^*\frac{\partial^2 u}{\partial t^2}; \tag{1.8}$$

$$c_{22}\frac{\partial^2 v}{\partial\beta^2}+c_{66}\frac{\partial^2 v}{\partial\alpha^2}+(c_{12}+c_{66})\frac{\partial u}{\partial\alpha\partial\beta}+\frac{1}{R}c_{22}\frac{\partial w}{\partial\beta}=m^*\frac{\partial^2 v}{\partial t^2};$$

$$D_{11}\frac{\partial^4 w}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2}+D_{22}\frac{\partial^4 w}{\partial\beta^4}+ \tag{1.8 (cont'd)}$$

$$+\frac{1}{R}\left[c_{12}\frac{\partial u}{\partial\alpha}+c_{22}\frac{\partial v}{\partial\beta}\right]+c_{22}\frac{w}{R^2}-Z+m^*\frac{\partial^2 w}{\partial t^2}=0.$$

The problem of the stability of an isotropic, homogeneous, circular cylindrical shell of finite length in a gas stream was treated in /4, 5/. Here we shall consider the same problem for an orthotropic sandwich shell, and shall determine the influence of anisotropy and nonhomogeity on the critical velocity of the flutter.

By replacing Z in (1.8) by the expression /3/

$$Z=-2\varepsilon m^*\frac{\partial w}{\partial t}-\varkappa p_\infty M\frac{\partial w}{\partial\alpha}-\frac{\varkappa p_\infty}{a_\infty}\frac{\partial w}{\partial t}, \tag{1.9}$$

we obtain the equation of stability of a cylindrical shell in a supersonic gas stream; here ε is the damping coefficient, p_∞, pressure of the undisturbed gas stream, a_∞, sound velocity for the undisturbed gas, $\varkappa$, polytropic index, and t, time.

If the frequency of natural transverse vibrations of the shell is small compared with the frequency of natural vibrations of the shell in its own surface, the tangential components of the inertia forces can be neglected. In this case, by introducing the function $\Phi(\alpha,\beta,t)$, connected with u, v, w by the relationships /*/

$$u=\frac{1}{R}\frac{\partial}{\partial\alpha}\left(a_{22}\frac{\partial^2}{\partial\beta^2}-a_{12}\frac{\partial^2}{\partial\alpha^2}\right)\Phi,$$

$$v=-\frac{1}{R}\frac{\partial}{\partial\beta}\left[a_{22}\frac{\partial^2}{\partial\beta^2}-(a_{66}-a_{12})\frac{\partial^2}{\partial\alpha^2}\right]\Phi,$$

$$w=\left[a_{11}\frac{\partial^4}{\partial\alpha^4}+(a_{66}-2a_{12})\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+a_{22}\frac{\partial^4}{\partial\beta^4}\right]\Phi,$$

and allowing for (1.9), we reduce system (1.8) to the following resolving equation:

$$\left[D_{11}\frac{\partial^4}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+D_{22}\frac{\partial^4}{\partial\beta^4}\right]\times$$

$$\times\left[a_{11}\frac{\partial^4}{\partial\alpha^4}+(a_{66}-2a_{12})\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+a_{22}\frac{\partial^4}{\partial\beta^4}\right]\Phi+\frac{1}{R^2}\frac{\partial^4\Phi}{\partial\alpha^4}+$$

$$+\left[a_{11}\frac{\partial^4}{\partial\alpha^4}+(a_{66}-2a_{12})\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+a_{22}\frac{\partial^4}{\partial\beta^4}\right]\times$$

$$\times\left[m^*\frac{\partial^2}{\partial t^2}+\left(2m^*\varepsilon+\frac{\varkappa p_\infty}{a_\infty}\right)\frac{\partial}{\partial t}+\varkappa p_\infty M\frac{\partial}{\partial\alpha}\right]\Phi=0, \tag{1.10}$$

where

$$a_{11}=\frac{c_{11}}{\Omega},\quad a_{12}=\frac{c_{12}}{\Omega},\quad a_{22}=\frac{c_{22}}{\Omega}, \tag{1.11}$$

* [Reference omitted in Russian text.]

$$a_{66} = \frac{1}{c_{66}}, \qquad \Omega = c_{11}c_{22} - c_{12}^2. \tag{1.11 (cont'd)}$$

Assume that the shell is simply supported by the base. The solution of (1.10) can then be represented in the form /5/

$$\Phi(\alpha, \beta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} f_{mn}(t) \sin \frac{m\pi\alpha}{l} \cos \frac{n}{R} \beta, \tag{1.12}$$

where m is the number of half-waves of the bent surface along the generator, n, the number of waves in the circumferential direction, and $f_{mn}(t)$, some functions which remain to be determined.

By substituting (1.12) in (1.10) and applying Bubnov-Galerkin's variational method we obtain the following system of ordinary differential equations for the dimensionless functions $x_{mn} = f_{mn}/h$:

$$\frac{d^2 x_{mn}}{d\tau^2} + 2\gamma_{mn} \frac{dx_{mn}}{d\tau} + \Omega_{mn}^2 x_{mn} + \Psi_{mn} = 0 \quad \begin{pmatrix} m = 1, 2, \cdots \\ n = 0, 1, 2, \cdots \end{pmatrix}, \tag{1.13}$$

$$\tau = \omega_{1n} t, \qquad \gamma_{mn} = \frac{2\varepsilon + \dfrac{\varkappa p_\infty}{m^* a_\infty}}{\omega_{1n}}, \qquad \Omega_{mn}^2 = \frac{\omega_{mn}^2}{\omega_{1n}^2}, \tag{1.14}$$

$$\omega_{mn}^2 = \frac{1}{m^*} \left[D_{11} \left(\frac{m\pi}{l}\right)^4 + 2(D_{12} + 2D_{66}) \left(\frac{m\pi}{l}\right)^2 \frac{n^2}{R^2} + D_{22} \frac{n^4}{R^4} + \right.$$

$$\left. + \frac{1}{R^2} \cdot \frac{\left(\dfrac{m\pi}{l}\right)^4}{a_{11} \left(\dfrac{m\pi}{l}\right)^4 + (a_{66} - 2a_{12}) \left(\dfrac{m\pi}{l}\right)^2 \dfrac{2n^2}{R^2} + a_{22} \dfrac{n^4}{R^4}} \right], \tag{1.15}$$

$$\Psi_{mn} = K\nu \frac{m}{2A_{mn}} \sum_{i=1}^{\infty} i A_{in} \frac{1 - (-1)^{m+i}}{m^2 - i^2} \cdot x_{in},$$

$$A_{mn} = a_{11} \left(\frac{m\pi}{l}\right)^4 + (a_{66} - 2a_{12}) \left(\frac{m\pi}{l}\right)^2 \frac{n^2}{R^2} + a_{22} \frac{n^4}{R^4}, \tag{1.16}$$

$$K = \frac{4\varkappa p_\infty}{m^* h \omega_{1n}^2}, \qquad \nu = \frac{h}{l} M. \tag{1.17}$$

Here ω_{mn} are frequencies of the small natural vibrations of the shell and ν is the reduced velocity parameter.

The solution of system (1.13) can be represented in the form $x_{mn} = y_{mn} e^{\lambda\tau}$. To the critical value of the velocity parameter ν_*, which corresponds to the appearance of flutter, will correspond in this case the passage of the characteristic index λ from the left half-plane of the complex variable into the right half-plane. At $m = 2$ the following formula is obtained for determining the critical value ν_*:

$$\nu_* = \frac{3}{4} \frac{\Omega_{2n}^2 - 1}{K} \sqrt{1 + \frac{2(\Omega_{2n}^2 + 1)\gamma_{2n}^2}{(\Omega_{2n}^2 - 1)^2}}. \tag{1.18}$$

At $m = 4$, in the case of absence of damping, the critical parameter is found from the equation /5/

$$\left(\frac{256}{175}\right)^2 K^4\nu^4 + \left[\frac{144}{49}(1-\lambda^2)(\Omega_{2n}^2-\lambda^2) + \frac{36}{25}(1-\lambda^2)(\Omega_{4n}^2-\lambda^2) + \right.$$

$$\left. + \frac{16}{225}(\Omega_{2n}^2-\lambda^2)(\Omega_{3n}^2-\lambda^2) + \frac{4}{9}(\Omega_{3n}^2-\lambda^2)(\Omega_{4n}^2-\lambda^2)\right] K^2\nu^2 +$$

$$+ (1-\lambda^2)(\Omega_{2n}^2-\lambda^2)(\Omega_{3n}^2-\lambda^2)(\Omega_{4n}^2-\lambda^2) = 0. \qquad (1.19)$$

The solution of (1.19) relative to ν yields the relationship $\nu = f(\lambda)$. To the critical parameter ν_* corresponds the junction of the two adjacent real frequencies λ.

Of greatest interest are those values of the parameter n at which the critical velocity has a minimum value. The values n_* for which ν_* has a minimum value are determined from the equation

$$\frac{6(D_{12}+2D_{66})}{\pi^2} - \frac{1}{R^2l^2}\left[16\cdot\frac{(a_{66}-2a_{12})\left(\frac{2\pi}{l}\right)^2 + 2a_{22}\frac{n^2}{R^2}}{A_{2n}^2} - \right.$$

$$\left. - \frac{(a_{66}-2a_{12})\left(\frac{\pi}{l}\right)^2 + 2a_{22}\frac{n^2}{R^2}}{A_{1n}^2}\right] = 0. \qquad (1.20)$$

It is seen from (1.20) that the critical value of the parameter n_* depends to a considerable extent on the elastic coefficients of the shell material. Note that for a one-layer isotropic shell it depends on the Poisson ratio only /5/.

In the case of a one-layer shell equation (1.20) can be reduced to the form

$$\frac{1}{\pi^2} - \frac{4l^4}{h^2R^2}\cdot\frac{\lambda_{66}(\lambda_{11}\lambda_{22}-\lambda_{12}^2)}{\lambda_{12}+2\lambda_{66}} \times$$

$$\times\left[\frac{16\cdot[2\pi^2(\lambda_{11}\lambda_{22}-\lambda_{12}^2-2\lambda_{12}\lambda_{66})+\lambda_{22}\lambda_{66}\gamma^2n^2]}{[16\pi^4\lambda_{11}\lambda_{66}+4\pi^2(\lambda_{11}\lambda_{22}-\lambda_{12}^2-2\lambda_{12}\lambda_{66})\gamma^2n^2+\lambda_{22}\lambda_{66}\gamma^4n^4]^2}\right.$$

$$\left. - \frac{1}{2}\cdot\frac{\pi^2(\lambda_{11}\lambda_{22}-\lambda_{12}^2-2\lambda_{12}\lambda_{66})+2\lambda_{22}\lambda_{66}\gamma^2n^2}{[\pi^4\lambda_{11}\lambda_{66}+\pi^2(\lambda_{11}\lambda_{22}-\lambda_{12}^2-2\lambda_{12}\lambda_{66})\gamma^2n^2+\lambda_{22}\lambda_{66}\gamma^4n^4]^2}\right] = 0, \qquad (1.21)$$

where

$$\lambda_{ik} = D_{ik}/D_{11};\ \lambda_{11} = 1,\ \lambda_{12} = \mu_2,\ \gamma = l/R.$$

In the case of a one-layer orthotropic shell the critical value of the parameter n is thus determined in a first approximation by equation (1.21), whose coefficients depend on ratios of the type $\frac{D_{ik}}{D_{11}}$.

Figure 2 represents n_* as a function of the parameters λ_{22} and λ_{66} at $l/R = 5$, $h/R = 1/100$. It is seen from these curves that the critical number of waves in the circumferential direction increases with the increase of λ_{22} and decreases with the increase of λ_{66}.

It is seen from Figure 3 that the minimum value of the critical velocity of flutter increases with the increase of λ_{66}.

To determine the influence of the material nonhomogeneity on the critical value of n (and therefore on the critical velocity of flutter), consider a three-layer shell made of isotropic layers. Assume for simplicity that all layers have the same Poisson ratio. We obtain in this case from (1.20),

on the strength of (1.5) and (1.6):

$$\frac{1}{\pi^2} - \gamma^4 \frac{R^2(1-\mu^2)}{\delta_1^2} \frac{1+k(\lambda-1)}{1+k^3(\lambda-1)} \left[\frac{16}{(4\pi^2+n^2\gamma^2)^3} - \frac{1}{(\pi^2+n^2\gamma^2)^3} \right] = 0, \qquad (1.22)$$

where $\lambda = E^{(2)}/E^{(1)}$ is the ratio of the moduli of elasticity of the shell layers, $k = \delta_2/\delta_1$, and μ is the Poisson ratio.

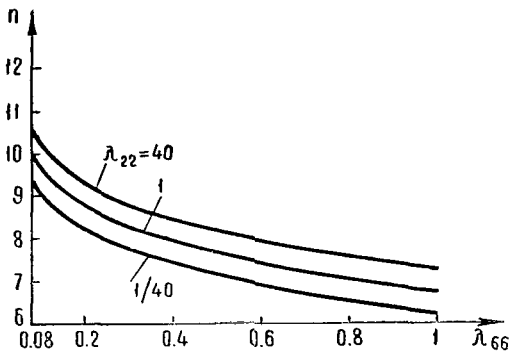

FIGURE 2.

Equation (1.22) is reduced to the equation of an isotropic one-layer shell by substituting $\lambda = 1$.

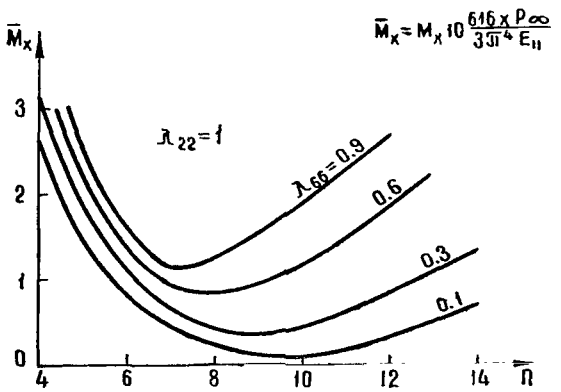

FIGURE 3.

The results of the solution of (1.22) for several values of λ and k are represented in Figure 4 for $l/R = 5$, $h/R = 1/100$. It is seen from these curves that the critical number of waves in the circumferential direction increases with the increase of λ in the entire range of variation of $\lambda (0 < \lambda < \infty)$.

It follows that the value of n_* for an isotropic shell ($\lambda = 1$) will be smaller than the critical number of waves of the corresponding sandwich shell with coefficient λ larger than unity.

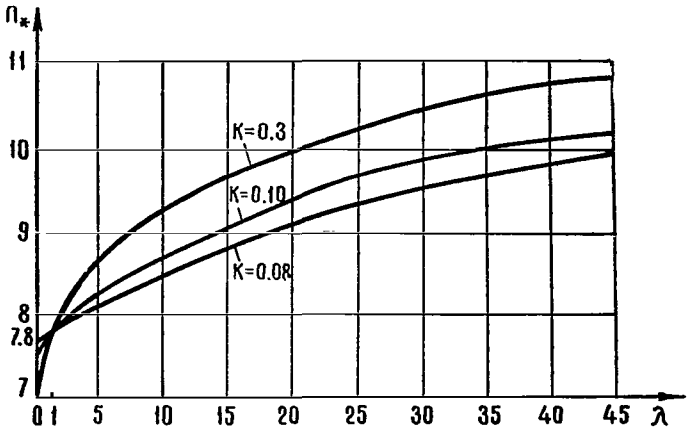

FIGURE 4.

If $\lambda < 1$, the value of n_* for a sandwich shell will be smaller than the critical number of waves of the corresponding isotropic shell.

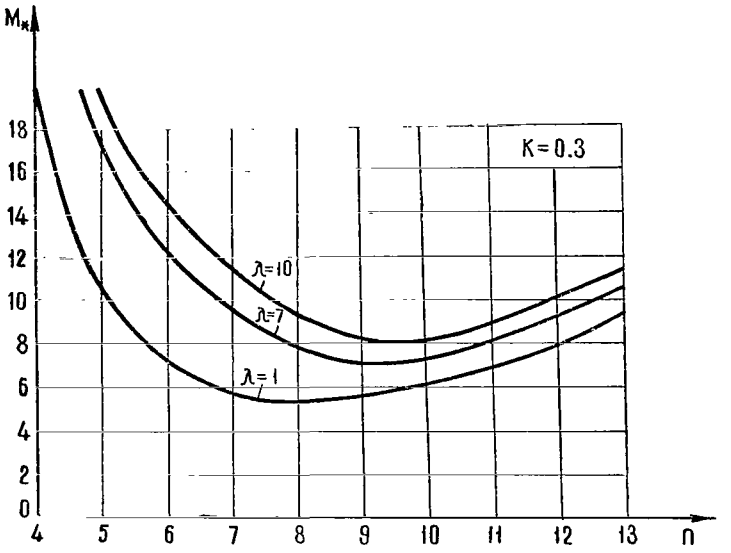

FIGURE 5.

It is seen from the curves represented in Figure 5 that the minimum value of the critical velocity of flutter increases with the increase of λ, and therefore with the increase of the critical number of waves n_*.

The region of shell stability can therefore be extended considerably by varying the geometrical and physical parameters (λ and k).

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STUDY OF THE STABILITY OF ELASTIC RECTANGULAR PLATES WITH A RECTANGULAR HOLE, BUILT-IN BY THE OUTER AND INNER CONTOURS

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(Tashkent)

Consider the following cases of loading with application of forces to the outer contour of the rectangle only, with stresses not exceeding the elastic limit:

1) simultaneous action of compressive forces directed perpendicular to the rectangle sides (Figure 1A);

2) plate compression in the direction of one pair of sides of the rectangle only (Figure 1B);

3) plate compression in the direction of one pair of sides of the rectangle, and extension in the direction of the other pair (Figure 1C).

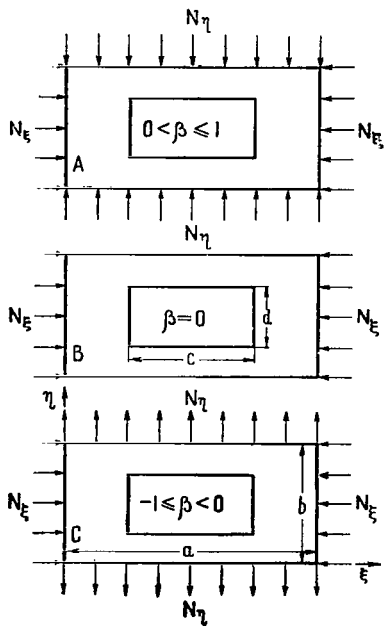

FIGURE 1.

We shall study the influence of the ratio of the loads and the ratio of the sides on the stability coefficient K_{cr} of plates with rectangular hole built-in by both the outer and inner contours. The problem will be solved by the method of finite differences in a dimensionless form.

The equation of stability of rectangular plates in finite differences is /1/

$$2\,[3\,(\lambda^4+4\lambda^2+3)-(\beta\lambda^2+1)\,k]\,\zeta_i-[4(\lambda^2+1)-k](\zeta_k+\zeta_l)-$$
$$-\lambda^2\,[4\,(\lambda^2+1)-\beta k](\zeta_m+\zeta_n)+2\lambda^2\,(\zeta_o+\zeta_p+\zeta_q+\zeta_r)+$$
$$+(\zeta_s+\zeta_t)+\lambda^4\,(\zeta_u+\zeta_v)=0, \qquad (1)$$

where ζ is a dimensionless function of the dimensionless coordinates ξ and η; k is a coefficient determining the unknown critical coefficient of stability K_{cr}; the subscripts of the function ζ represent the grid nodes; α is the ratio of the number of divisions of the grid along the η axis to the number of divisions along the ξ axis; $\alpha=\frac{n_2}{n_1}$; λ_1 is the ratio of the outer $\left(\frac{a}{b}\right)$ and of the inner $\left(\frac{c}{d}\right)$ sides of the rectangle:

$$\lambda_1=\frac{a}{b}=\frac{c}{d}.$$

Designate the ratio of loads on the rectangle by β. Then

$$\beta=\frac{N_\eta}{N_\xi}=\text{const.}$$

It has been established elsewhere by us /1/ that

$$K_{cr}=\frac{n_1^2}{\pi^2\lambda_1^2}k.$$

The stress distribution over the plate will be assumed to be uniform, just as in the case of continuous plates. Were we to assume that β is a function of the coordinates, the problem would have been considerably complicated, and we would have obtained an equation with variable coefficients.

It suffices to consider the unknown points in one quadrant of the plate, since the applied load and the fastening of the plate sides are symmetric relative to the axis of symmetry of the plate. The number of such points for the grid selected is seven (Figure 2). The plate deflections are equal to zero at the nodal points of the grid lying on the inner and outer contours. These points will be designated by "0".

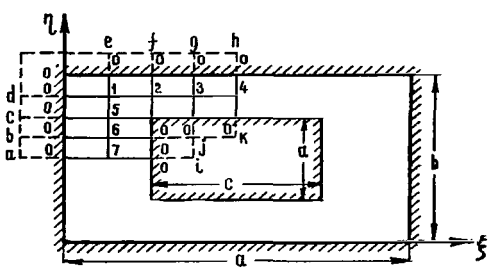

FIGURE 2.

We shall use two approximate expressions for the boundary conditions for the built-in sides of the plate /2, 3/:

$$\zeta_{m+1} = \zeta_{m-1}, \qquad \text{(a)}$$

$$\zeta_{m+1} = 3\zeta_{m-1} - 0.5\zeta_{m-2}. \qquad \text{(b)}$$

Using approximation (a), we obtain the following equalities connecting the functions corresponding to the points lying outside the contour with those corresponding to the points lying inside it:

$$\left.\begin{aligned} &\zeta_a = \zeta_l = \zeta_7,\ \zeta_b = \zeta_{j\xi} = \zeta_6,\ \zeta_c = \zeta_5, \\ &\zeta_{d\xi} = \zeta_{e\eta} = \zeta_1,\ \zeta_f = \zeta_2, \\ &\zeta_g = \zeta_{j\eta} = \zeta_3,\ \zeta_h = \zeta_k = \zeta_4. \end{aligned}\right\} \qquad (2)$$

By writing the stability equation (1) for points 1 through 7 and taking conditions (2) into account we obtain a system of seven equations with seven unknowns. The minimum root $k_{\min}$, and therefore the stability coefficient K_{cr}, are found by equating to zero the determinant of this system:

$$\begin{vmatrix} a_{11} & \cdots & a_{17} \\ \cdot & \cdots & \cdot \\ a_{71} & \cdots & a_{77} \end{vmatrix} = 0, \qquad (3)$$

where

$$a_{11} = 2\left[(3\lambda^4 + 4\lambda^2 + 3) - (\beta\lambda^2 + 1)k\right] + \lambda^4 + 1 = a + \lambda^4 + 1;$$

$$a_{22} = a + \lambda^4;\quad a_{33} = a + 2\lambda^4 + 1,\quad a_{44} = 0.5a + \lambda^4;$$

$$a_{55} = a + 1;\quad a_{66} = a + \lambda^4 + 2;\quad a_{77} = 0.5a + 1;$$

$$a_{12} = a_{21} = a_{23} = a_{32} = a_{34} = a_{43} = -\left[4(\lambda^2 + 1) - k\right];$$

$$a_{15} = a_{51} = a_{56} = a_{65} = a_{67} = a_{76} = -\lambda^2\left[4(\lambda^2 + 1) - \beta k\right];$$

$$a_{25} = a_{52} = 2\lambda^2;\quad a_{16} = a_{61} = a_{57} = a_{75} = \lambda^4;$$

$$a_{13} = a_{31} = a_{24} = a_{42} = 1;$$

$$a_{14} = a_{41} = a_{17} = a_{71} = a_{26} = a_{62} = a_{27} = a_{72} = a_{35} = a_{53} = a_{36} = a_{63} =$$

$$a_{37} = a_{73} = a_{45} = a_{54} = a_{46} = a_{64} = a_{47} = a_{74} = 0.$$

The solution of the determinant equation (3) for given values of the parameters β and $\lambda = \lambda_1$ is given in Table 1, a.

Using now approximation (b), we obtain the following equalities:

$$\left.\begin{aligned} &\zeta_a = \zeta_l = 3\zeta_7,\quad \zeta_b = \zeta_{j\xi} = 3\zeta_6; \\ &\zeta_c = 3\zeta_3,\quad \zeta_d = 3\zeta_1 - 0.5\zeta_2; \\ &\zeta_e = 3\zeta_1 - 0.5\zeta_5,\quad \zeta_f = 3\zeta_2; \\ &\zeta_g = \zeta_{j\eta} = 3\zeta_3,\quad \zeta_n = \zeta_k = 3\zeta_4. \end{aligned}\right\} \qquad (4)$$

As before, we obtain from the stability equation (1) for points 1 through 7 and conditions (4) a system of seven equations with seven unknowns, from which we obtain the equation of stable state

$$\begin{vmatrix} b_{11} & \cdots & b_{17} \\ \cdot & \cdots & \cdot \\ b_{71} & \cdots & b_{77} \end{vmatrix} = 0 \qquad (5)$$

with elements

$$b_{11} = 2\left[(3\lambda^4 + 4\lambda^2 + 3) - (\beta\lambda^2 + 1)k\right] + 3\lambda^4 + 3 = b + 3\lambda^4 + 3;$$

$$b_{22} = b + 3\lambda^4,\quad b_{33} = b + 6\lambda^4 + 1,\quad b_{44} = 0.5b + 3\lambda^4;$$

$$b_{55} = b + 3, \quad b_{66} = b + \lambda^4 + 6, \quad b_{77} = 0.5b + 3;$$

$$b_{21} = b_{23} = b_{32} = b_{34} = b_{43} = -[4(\lambda^2 + 1) - k];$$

$$b_{12} = -[4(\lambda^2 + 1) - k] - 0.5, \quad b_{51} = b_{56} = b_{65} =$$

$$= b_{67} = b_{76} = -\lambda^2[4(\lambda^2 + 1) - \beta k];$$

$$b_{15} = -\lambda^2[4(\lambda^2 + 1) - \beta k] - 0.5\lambda^4;$$

$$b_{25} = b_{52} = 2\lambda^2, \quad b_{16} = b_{61} = b_{57} = b_{75} = \lambda^4;$$

$$b_{13} = b_{31} = b_{24} = b_{42} = 1, \quad b_{14} = b_{41} = b_{17} = b_{71} = b_{26} = b_{62} = b_{27} = b_{72} =$$

$$= b_{35} = b_{53} = b_{36} = b_{63} = b_{37} = b_{73} = b_{45} = b_{54} = b_{46} = b_{64} = b_{47} = b_{74} = 0.$$

From equation (5) we determine $k_{\min}$, i.e., K_{cr}, for the given values of the parameters β, $\lambda = \lambda_1$ (Table 1, b).

TABLE 1

β	$\lambda = \lambda_1 = 1$		$\lambda = \lambda_1 = 2$		$\lambda = \lambda_1 = 3$		$\lambda = \lambda_1 = 4$		By condition
	$k_{\min}$	K_{cr}	$k_{\min}$	K_{cr}	$k_{\min}$	K_{cr}	$k_{\min}$	K_{cr}	
−1	−4.493	−29.137	8.925	14.469	−23.421	−16.875	−29.449	−11.935	
−0.75	4.476	29.024	8.010	12.985	25.065	18.060	−49.989	−17.828	
−0.5	4.386	28.444	7.256	11.764	18.048	13.003	67.120	27.203	
−0.25	4.300	27.885	6.625	10.741	12.535	10.500	35.112	14.230	
0	4.217	27.344	6.089	9.871	11.358	8.183	23.275	9.433	(a)
0.25	4.136	26.819	5.626	9.121	9.527	6.864	17.263	6.997	
0.5	4.057	26.305	5.223	8.468	8.185	5.898	13.669	5.540	
0.75	3.976	25.782	4.868	7.892	7.163	5.161	11.293	4.577	
1	3.820	24.773	4.553	7.382	6.358	4.582	9.611	3.895	
−1	−6.190	−40.142	10.165	16.479	−25.046	−18.046	−31.025	−12.574	
−0.75	6.247	40.509	9.161	14.852	31.899	22.984	−46.734	−18.941	
−0.5	6.114	39.644	8.338	13.517	22.521	16.227	80.170	32.492	
−0.25	5.982	38.789	7.650	12.402	17.176	12.375	41.609	16.863	
0	5.848	37.924	7.068	11.458	13.769	9.921	26.754	10.843	(b)
0.25	5.708	37.012	6.568	10.647	11.431	8.236	19.497	7.902	
0.5	5.546	35.965	6.135	9.945	9.740	7.018	15.268	6.188	
0.75	5.325	34.532	5.756	9.331	8.466	6.100	12.521	5.075	
1	4.914	32.866	5.422	8.790	7.476	5.386	10.601	4.236	

TABLE 2

β	$\lambda = \lambda_1 = 1$	$\lambda = \lambda_1 = 2$	$\lambda = \lambda_1 = 3$	$\lambda = \lambda_1 = 4$	By condition
−1	−29.137 −40.142	14.469 16.479	−16.875 −18.046	−11.935 −12.574	(a) (b)
−0.75	29.024 40.509	12.985 14.852	18.060 22.984	−17.828 −18.941	(a) (b)
−0.5	28.444 39.664	11.764 13.517	13.003 16.227	27.203 32.492	(a) (b)
−0.25	27.885 38.789	10.741 12.402	10.500 12.375	14.230 16.863	(a) (b)
0	27.344 37.924	9.871 11.458	8.183 9.921	9.433 10.843	(a) (b)
0.25	26.819 37.012	9.121 10.647	6.864 8.236	6.997 7.902	(a) (b)
0.5	26.305 35.965	8.468 9.945	5.898 7.0 8	5.540 6.188	(a) (b)
0.75	25.782 34.532	7.8 2 9.331	5.161 6.100	4.577 5.075	(a) (b)
1	24.773 32.866	7.382 8.790	4.582 5.386	3.895 4.236	(a) (b)

Equations (3) and (5) were solved by the method of iterations /4/ on one of the computers of the Institute of Cybernetics of the AN UkrSSR.

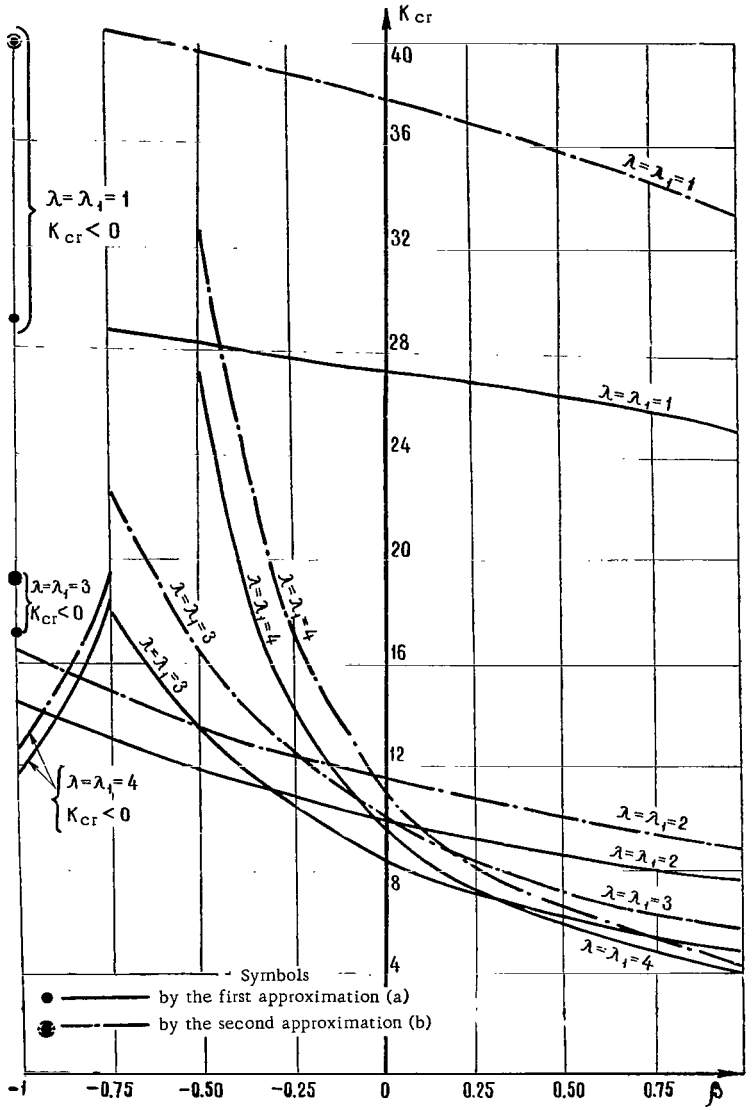

FIGURE 3.

Tables 3 and 4 establish the connection between the maximum and minimum values of the stability coefficients K_{cr} and the parameters β and λ_1 determining the properties of the combined system of loads and the geometrical shape of the plate, and give the boundaries of the range of variation of the parameters in which the plate does not buckle.

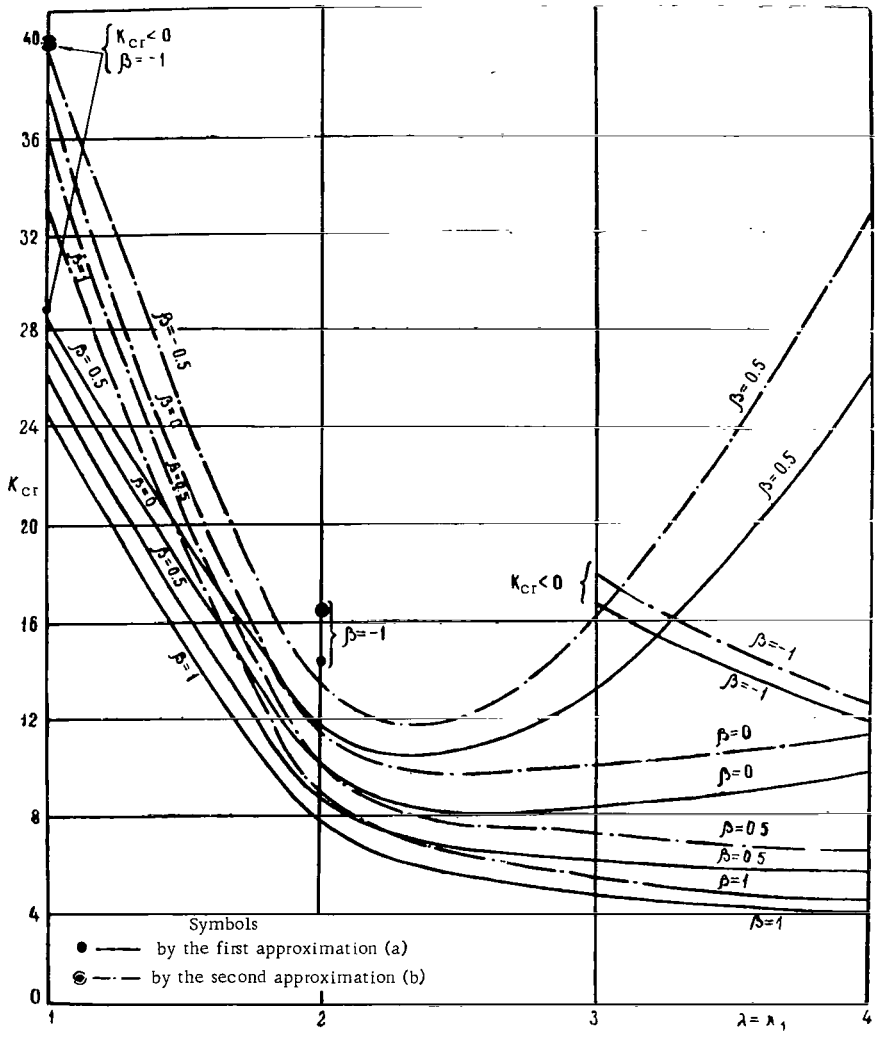

FIGURE 4.

TABLE 3

$-1 \leqslant \beta \leqslant 1$

λ_1	$K_{cr\,max}$	$K_{cr\,min}$	Stability range
1	$\beta = -0.75$	$\beta = 1$	$\beta = -1$
2	$\beta = -1$	$\beta = 1$	—
3	$\beta = -0.75$	$\beta = 1$	$\beta = -1$
4	$\beta = -0.5$	$\beta = 1$	$-1 < \beta \leqslant -0.75$

TABLE 4

$1 \leqslant \lambda_1 \leqslant 4$

β	$K_{cr\ max}$	$K_{cr\ min}$	Stability range
-1	$\lambda_1 = 2$	—	$\lambda_1 = 1,\ 3 \leqslant \lambda_1 \leqslant 4$
-0.75	$\lambda_1 = 1$	$\lambda_1 = 2$	$\lambda_1 = 4$
-0.5	$\lambda_1 = 1$	$\lambda_1 = 2$	—
$-0.25;\ 0;\ 0.25$	$\lambda_1 = 1$	$\lambda_1 = 3$	—
$0.5;\ 0.75;\ 1$	$\lambda_1 = 1$	$\lambda_1 = 4$	—

Note that these values of the stability coefficients were obtained by us as a result of numerical solutions and have not been accompanied by studies of the general properties of K_{cr} as a function of the parameters β and λ_1.

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NONLINEAR INTEGRAL AND INTEGRAL-DIFFERENTIAL EQUATIONS OF SHALLOW SHELLS AND THEIR APPLICATIONS

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(Kiev)

1. The determination of the stressed and strained state of flexible shells and plates is reduced to the solution of nonlinear boundary-value problems

$$\begin{aligned} A\,\boldsymbol{u}(P) &= \boldsymbol{f}_0\,(P) + \boldsymbol{f}_1\,(\boldsymbol{u}(P)), \quad P \in S; \\ \Gamma\,\boldsymbol{u}(P) &= \bar{\varphi}_0\,(P) + \bar{\varphi}_1\,(\boldsymbol{u}(P)), \quad P \in l; \end{aligned} \tag{1}$$

where A is the quadratic elliptic matrix of differential equations; Γ is the rectangular differential matrix of the boundary conditions; $\boldsymbol{u}\,(P)$ is the unknown vector function; $\boldsymbol{f}_0, \boldsymbol{f}_1, \bar{\varphi}_0, \bar{\varphi}_1$ are given vector functions $\boldsymbol{u}\,(P)$ of point P of the region of the middle surface S limited by the contour l.

Using Green's tensor $K\,(P,\,Q)$ for the operator A at homogeneous linear boundary conditions Γ, the boundary-value problem (1) is reduced to an equivalent system of nonlinear integral equations

$$\begin{aligned} \boldsymbol{u}\,(Q) = \boldsymbol{u}_{\mathrm{L}}(Q) - \int\limits_S K\,(P,\,Q)\,\boldsymbol{f}_1\,(\boldsymbol{u}\,(P))\,dS_P + \\ + \oint\limits_l G_P\,K\,(P,\,Q)\,\bar{\varphi}_1\,(\boldsymbol{u}\,(P))\,dl_P, \end{aligned} \tag{2}$$

where G is the differential matrix of the boundary conditions, completing matrix Γ; $\boldsymbol{u}_{\mathrm{L}}\,(Q)$ is the solution of the corresponding linear problem

$$\boldsymbol{u}_{\mathrm{L}}(Q) = -\int\limits_S K\,(P,\,Q)\,\boldsymbol{f}_0\,(P)\,dS_P + \oint\limits_l G_P K\,(P,\,Q)\,\bar{\varphi}_0\,(P)\,dl_P. \tag{3}$$

System (2) represents thus the solution of the nonlinear boundary-value problem (1) as the sum of the linear solution and of a correction for the nonlinearity.

2. Proceeding to the approximate solution of the system of nonlinear integral equations (2), we approximate the Green tensor $K(P,\,Q)$ by the degenerate tensor $\hat{K}\,(P,\,Q)$. Then, as known, the solution of the system of nonlinear integral equations (2) reduces to the solution of a system of nonlinear algebraic equations.

We shall represent $\hat{K}\,(P,\,Q)$ in the form of the sum

$$\hat{K}\,(P,\,Q) = \sum_{k,\,l=1} K^{kl}\,(P,\,Q), \quad k_{lj}^{kl}\,(P,\,Q) = k_j^{kl}\,(Q)\;\omega_{lj}^{kl}(P), \tag{4}$$

where $\omega_{ij}^{kl}(P)$ are elements of the approximating tensor $\Omega^{kl}(P)$, all columns of which are equal* and satisfy the homogeneous linear boundary values of of problem (1): $\Gamma_P\ \Omega^{kl}(P)=0,\ P \in l;\ k_i^{kl}(Q)$ are arbitrary functions of point Q, which will be determined with the aid of the matrix equations of the Bubnov-Galerkin method:

$$\int_S \Omega'^{kl}(P)\, A_P\, \hat{K}(P, Q)\, dS_P = -\int_S \Omega'^{kl}(P)\, \delta(P-Q)\, E\, ds_P. \tag{5}$$

where the prime designates the operation of transposition; E is the unit matrix; $\delta(P-Q)$ is the delta function.

Having realized the operations of multiplication and integration of the matrices, we obtain the equality of two matrices, with identical rows. The equality of the elements of these rows gives the necessary number of linear algebraic equations for determining

$$k_i^{kl}(Q), \quad k,l=1,\ 2,\cdots$$

In the first approximation

$$\int_S \Omega'^{11}(P)\, A_P\, K^{11}(P, Q)\, dS_P = -\int_S \delta(P-Q)\, \Omega'^{11}(P) dS_P. \tag{6}$$

Replacing the tensor $K(P, Q)$ in system (2) by the degenerate tensor $\hat{K}(P, Q)$, we obtain its solution in the form:

$$u(Q) = u_L(Q) + \sum_{k,l=1} u_{kl}(Q), \tag{7}$$

where the vector function $u_{kl}(Q)$ is determined from the system of vectorial nonlinear equations

$$u_{kl}(Q) = -\int_S K^{kl}(P, Q)\, f_1\Big(u_L(P) + \sum_{i,j=1} u_{ij}(P)\Big) dS_P +$$

$$+ \oint_l G_P\, K^{kl}(P, Q)\, \bar{\varphi}_1\Big(u_n(P) + \sum_{i,j=1} u_{ij}(P)\Big) dl_P, \quad k, l=1,\ 2,\ 3,\cdots \tag{8}$$

In a first approximation $k=l=1$

$$u(Q) = u_n(Q) + u_{11}(Q), \tag{9}$$

$$u_{11}(Q) = -\int_S K^{11}(P, Q)\, f_1\Big(u_L(P) + u_{11}(P)\Big) dS_P +$$

$$+ \oint_l G_P\, K^{11}(P, Q)\, \bar{\varphi}_1\Big(u_L(P) + u_{11}(P)\Big) dl_P. \tag{10}$$

3. We shall consider a number of specific cases.

The problem of the large deflections and buckling of a circular simply supported plate under the action of a normal load of intensity $q(r)$, a compressive force p, and a bending moment m, uniformly distributed over

* If A is a diagonal matrix, a diagonal matrix is likewise selected for the approximating matrix $\Omega^{kl}(P)$.

the contour, is reduced to the nonlinear boundary-value problem (1), where /1/

$$u\ \{u,\ \theta\},\qquad A=\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r)E,\qquad \Gamma=\left(\frac{d}{dr}+\mu\frac{1}{r}\right)E,$$

$$f_0\left\{0,\ \frac{12\,(1-\mu^2)}{E\,h^3\,r}\int\limits_0^r q\,(r)\,rdr\right\},\quad \bar{\varphi}_0\ \{-P,\ m\},$$

$$f_1\left\{-\frac{1-\mu}{2r}\theta^2-\theta\frac{d\theta}{dr},\ \frac{12}{h^2}\ \theta\left(\frac{du}{dr}+\mu\frac{u}{r}+\frac{\theta^2}{2}\right)\right\},\ \bar{\varphi}_1\left\{-\frac{\theta^2}{2},\ 0\right\}. \tag{11}$$

Here $0\leqslant r\leqslant a$; E, μ, h, and a are Young's modulus, the Poisson ratio, thickness, and radius of the plate; u, θ are radial displacement and angle of rotation [change of slope] of the middle surface of the plate.

The following expressions are to be substituted in the system of integral-differential equations (2) corresponding to this case:

$$K\ (r,\ x)=\frac{1}{2}\,r\left(\frac{1-\mu}{1+\mu}\,\frac{x}{a^2}+x^{-1}\right)E,\qquad r\leqslant x;\qquad K(r,\ x)=K(x,\ r);$$

$$P=r,\quad Q=x;\quad dS_P=rdr;\quad G=rE,\ \oint\limits_l=\Big|_0^a;$$

$$u_{\mathrm{L}}\left\{-p\,\frac{x}{1+\mu},\ \gamma\omega\ (x)+m\,\frac{x}{1+\mu}\right\};\quad q(r)=q=\mathrm{const}; \tag{12}$$

$$\omega\ (x)=\frac{x^3}{a^3}-\frac{3+\mu}{1+\mu}\,\frac{x}{a};\ \gamma=\frac{3q\,a^3\ (1-\mu^2)}{4\,E\,h^3}$$

Writing

$$\Omega^{11}\ (r)=\omega\ (r)\ E, \tag{13}$$

we obtain from (6)

$$K^{11}\ (r,\ x)=\|\omega\|^{-2}\ \omega(r)\ \omega\,(x)\ E,\qquad \|\omega\|^2=-\int\limits_0^a\omega\ (r)\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\ (r\omega(r))rdr. \tag{14}$$

The substitution of (14) in (10) yields

$$u\ (x)=u_{\mathrm{L}}\ (x)+u_{11}\,(x),\quad u_{11}\ \{u\ \omega(x),\ \theta\,\omega(x)\}, \tag{15}$$

where u is some second-degree polynomial of θ, and θ is a real root of the cubic equation, which in three particular cases of the load on the plate is transformed to

I $\quad p=m=0,\quad q\neq 0,\quad 8.045\ (\theta+\gamma)^3+\dfrac{11}{3}\dfrac{h^2}{a^2}\ \theta=0;$

II $\quad p=q=0,\qquad \beta=-m,$

$$8.045\ \theta^3+13.632\ \beta\ a\ \theta^2+\left(1.779\ \beta^2\,a^2+\frac{11}{3}\,\frac{h^2}{a^2}\right)\theta+1.5\ \beta^3\ a^3=0;$$

III $\quad q=m=0,\qquad p^*=12\dfrac{a^2}{h^2}p,\quad \vartheta=\dfrac{a}{h}\ \theta,$

$$8.045\ \vartheta^2-\frac{41}{48}p^*+\frac{11}{3}=0.$$

4. The problem of large deflections of a simply supported shallow spherical shell under the action of a normal load of intensity $q\ (r)$ and a bending moment m, distributed uniformly over the contour, is reduced to the nonlinear boundary-value problem (1), where /2/

$$\boldsymbol{u}\ \{s,\ \theta\},\qquad A=\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho)E,\qquad \Gamma=\begin{pmatrix}1 & 0\\ 0 & \dfrac{d}{d\rho}+\mu\dfrac{1}{\rho}\end{pmatrix};$$

$$\boldsymbol{f}_0\left\{0,\ \lambda\int\limits_0^\rho q\ (\rho)\ \rho d\rho\right\},\quad \bar{\varphi}_0\{0,m\},\quad \lambda=\frac{[12\ (1-\mu^2)\ ]^{3/2}\ a^4}{Eh^3};\tag{16}$$

$$\boldsymbol{f}_1\left\{k^2\rho\ \theta-\frac{\theta^2}{2},\ \theta s-k^2\rho s\right\},\ \bar{\varphi}_1\ \{0,0\},\ k^2=2\sqrt{12\ (1-\mu^2)}\ \frac{H}{h};$$

H is the rise of the shell; $0\leqslant\rho\leqslant 1$.

In this case we must substitute in the equivalent system of integral equations (2)

$$K\ (\rho,\ x)=\frac{1}{2}\rho\begin{pmatrix}x^{-1}-x & 0\\ 0 & \dfrac{1-\mu}{1+\mu}x+x^{-1}\end{pmatrix},\quad \rho\leqslant x,\quad G=\begin{pmatrix}\rho\dfrac{d}{d\rho} & 0\\ 0 & \rho\end{pmatrix};$$

$$K\ (\rho,\ x)=K\ (x,\ \rho),\ P=\rho,\ Q=x,\ ds_P=d\rho,\ \oint\limits_\gamma=\Big|_0^1;\tag{17}$$

$$\boldsymbol{u}_{\mathrm{L}}\left\{0,\ \alpha\omega_2\ (x)+m\frac{x}{1+\mu}\right\},\quad \alpha=\frac{\lambda q}{16};\ q\ (r)=q=\mathrm{const},\ \omega_2\ (x)=x^3-\frac{3+\mu}{1+\mu}x.$$

Writing

$$\Omega^{11}\ (\rho)=\begin{pmatrix}\omega_1\ (\rho) & 0\\ 0 & \omega_2\ (\rho)\end{pmatrix},\qquad \omega_1\ (\rho)=\rho^3-\rho,\tag{18}$$

we obtain from (6)

$$K^{11}\ (\rho,\ x)=\begin{pmatrix}\|\omega_1\|^{-2}\ \omega_1\ (\rho)\ \omega_1\ (x) & 0\\ 0 & \|\omega_2\|^{-2}\ \omega_2\ (\rho)\ \omega_2\ (x)\end{pmatrix}.\tag{19}$$

The substitution of (19) in (10) yields

$$\boldsymbol{u}\ (x)=\boldsymbol{u}_{\mathrm{L}}\ (x)+\boldsymbol{u}_{11}\ (x),\qquad \boldsymbol{u}_{11}\ \{s\ \omega_1\ (x),\ \theta\omega_2\ (x)\},\tag{20}$$

where

$$\|\omega_1\|^2\ s=a+b\ \theta+c\theta^2,\quad \|\omega_2\|^2\ \theta=ds+e\theta s;$$

$$a=-\frac{27}{160}\alpha^2-\frac{1}{6}k^2\alpha+\frac{1}{8}\alpha m-\frac{3}{128}m^2+\frac{1}{16}k^2m;$$

$$b=-\frac{27}{80}\alpha-\frac{1}{6}k^2+\frac{1}{8}m,\qquad c=-\frac{27}{160};\tag{21}$$

$$d=\frac{1}{6}k^2+\frac{27}{80}\alpha+\frac{1}{8}m,\qquad e=\frac{27}{80}.$$

In the case $k=m=0$ we obtain for the determination of θ the cubic equation

$$(\theta+\alpha)^3+\frac{44\cdot 80^2}{9\cdot 27^2}\theta=0.\tag{22}$$

5. In those cases when it is impossible to form the linear solution (3), we shall establish an approximate linear solution, replacing the Green

tensor $K(P, Q)$ in (3) by the degenerate tensor $\hat{K}(P, Q)$:

$$u_L(Q) = -\int_S \hat{K}(P, Q) f_0(P)\, dS_P + \oint_l G_P \hat{K}(P, Q)\, \bar{\varphi}_0(P) dl_P. \qquad (3'')$$

The approximate solution of boundary-value problem (1) is transformed then to

$$u(Q) = \sum_{k,l=1} u_{kl}(Q), \qquad (7'')$$

where the vector function $u_{kl}(Q)$ is determined from the system of non-linear vectorial equations

$$u_{kl}(Q) = -\int_S K^{kl}(P, Q)\Big\{f_0(P) + f_1\Big(\sum_{i,j=1} u_{ij}(P)\Big)\Big\} dS_P +$$

$$+ \oint_l G_P K^{kl}(P, Q)\Big\{\bar{\varphi}_0(P) + \bar{\varphi}_1\Big(\sum_{i,j=1} u_{ij}(P)\Big)\Big\} dl_P, \quad k,l = 1, 2, 3, \cdots \qquad (8'')$$

In a first approximation $k = l = 1$

$$u(Q) = u_{11}(Q), \qquad (9'')$$

$$u_{11}(Q) = -\int_S K^{11}(P, Q)\Big\{f_0(P) + f_1\Big(u_{11}(P)\Big)\Big\} dS_P +$$

$$+ \oint_l G_P K^{11}(P, Q)\Big\{\bar{\varphi}_0(P) + \bar{\varphi}_1\Big(u_{11}(P)\Big)\Big\} dl_P. \qquad (10'')$$

6. In the case of boundary-value problem (16) at $q = 0$ equation (10'') is reduced to the resolving cubic equation

$$\frac{27^2}{2\cdot 80^2}\theta^3 + \frac{27}{4\cdot 80} k^2 \theta^2 + \left(\frac{22}{9} + \frac{1}{36} k^4\right)\theta + m = 0.$$

The critical value of the slope parameter $k = 2\sqrt{12(1-\mu^2)}\,\frac{H}{h}$ at which the buckling becomes possible is equal to 3.64. The linear theory gives in this case /2/ $3.53 \leqslant k \leqslant 3.59$.

In the case of a simply supported spherical shell quadratic in the plan subjected to a circumferential load, equation (10'') is reduced to the resolving cubic equation /3/

$$\frac{128}{9}\zeta^3 - 4k\zeta^2 + \left[\frac{\pi^4}{3(1-\mu^2)} + \frac{1}{4}K^2\right]\zeta - M = 0.$$

The critical value of the slope parameter $K = 8\,\frac{H}{h}$ is equal in this case to 16.8.

We obtain in both the first and the second cases almost the same value of the ratio $\frac{H}{h}$ (2.03 and 2.01 respectively); this ratio remains therefore practically constant for a spherical shell, independent of its shape in the plan.

7. The problem of the large deflections and buckling of a shallow shell of double curvature under the action of compressive forces T and bending moments M is reduced to the nonlinear boundary-value problem (1), where /4/

$$u\{u,\ v,\ w\},\quad A=\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix},$$

$$A_{11}=\frac{\partial^2}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial y^2},\quad A_{22}=\frac{\partial^2}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial x^2},$$

$$A_{12}=A_{21}=\frac{1+\mu}{2}\frac{\partial^2}{\partial x\partial y},\qquad A_{33}=\frac{h^2}{12}\nabla^2\nabla^2,$$

$$f_0\{0,0,0\},\qquad \bar{\varphi}_0\left\{\frac{1-\mu^2}{Eh}T_1,\ \frac{1-\mu^2}{Eh}T_2,\ 0,\ \frac{12(1-\mu^2)}{Eh^3}M\right\},$$

$$f_1\left\{-k_1\frac{\partial w}{\partial x}-\mu k_2\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\cdot\frac{\partial^2 w}{\partial x^2}-\mu\frac{\partial w}{\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y}-\frac{1-\mu}{2}\left(\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial w}{\partial y}+\right.\right.$$

$$\left.+\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial w}{\partial x}\right),$$

$$-k_2\frac{\partial w}{\partial y}-\mu k_1\frac{\partial w}{\partial y}+\frac{\partial w}{\partial y}\cdot\frac{\partial^2 w}{\partial y^2}-\mu\frac{\partial w}{\partial x}\cdot\frac{\partial^2 w}{\partial x\partial y}-\frac{1-\mu}{2}\left(\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial w}{\partial x}+\right.$$

$$\left.+\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial w}{\partial y}\right),$$

$$\left.N_1\left(\frac{\partial^2 w}{\partial x^2}-k_1\right)+N_2\left(\frac{\partial^2 w}{\partial y^2}-k_2\right)+2N_{12}\frac{\partial^2 w}{\partial x\partial y}\right\},\tag{23}$$

$$\bar{\varphi}_1\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\ 0,\ 0\right\}.$$

If the shell edge is built-in

$$\Gamma=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{\partial}{\partial n} \end{pmatrix},\quad G=\begin{pmatrix} \frac{\partial}{\partial n} & 0 & 0 \\ 0 & \frac{\partial}{\partial n} & 0 \\ 0 & 0 & \frac{\partial}{\partial n}\nabla^2 \\ 0 & 0 & \frac{\partial^2}{\partial n^2} \end{pmatrix}.\tag{24}$$

If the shell edge $x=a$ is hinged

$$\Gamma=\begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{\partial^2}{\partial x^2} \end{pmatrix},\quad G=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial x}\nabla^2 \\ 0 & 0 & \frac{\partial}{\partial x} \end{pmatrix}.\tag{25}$$

If the shell edges $|y=0,\ b$ are built-in, and the other two edges are hinged

$$\Omega^{11}(x,y)=\begin{pmatrix} \cos\frac{\pi}{a}x\sin\frac{\pi}{b}y & \cos\frac{\pi}{a}x\sin\frac{\pi}{b}y & 0 \\ \sin\frac{\pi}{a}x\sin\frac{\pi}{b}y & \sin\frac{\pi}{a}x\sin\frac{\pi}{b}y & 0 \\ 0 & 0 & y^2(y-a)^2\sin\frac{\pi}{a}x \end{pmatrix}. \qquad (26)$$

In the case of a plate $k_1=k_2=0$, and equation (10'') reduce to the cubic equation

$$0.7\frac{Eha^8}{D}W_1^3-\left[1+0.02\frac{a^2T}{D}\right]W_1+0.027\frac{1}{a^2D}M=0,$$

$$D=\frac{Eh^3}{12(1-\mu^2)}.$$

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ON THE SYMMETRICAL DEFORMATION OF AN ORTHOTROPIC TOROIDAL SHELL

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The problem of calculating an isotropic toroidal shell loaded by an axially symmetrical system of forces was treated in /1-7/ and in many other papers.

The present paper is devoted to the study of the symmetrical deformation of an orthotropic toroidal shell. The solution proposed is based on the equations of symmetrical deformation of anisotropic shells /8/. A resolving equation of the problem is obtained, which contains a small parameter (proportional to the shell thickness) at a high-order derivative. This equation is solved by the asymptotic method, and its solution is expressed through tabulated functions /9/. The first asymptotic approximation is studied in detail; simple formulas are then given for determining the maximum stresses and strains in the shell.

1. The calculation of an orthotropic toroidal shell in the case of loading by axisymmetrical forces reduces to determining the complex unknown function V from the differential equation

$$(1+\alpha\sin\theta)\,V''+\alpha\cos\theta V'+il\left(\sin\theta+\frac{i\lambda\alpha^2}{l^3}\frac{\cos^2\theta}{1+\alpha\sin\theta}\right)V=il^3f. \qquad (1.1)$$

Here

$$l^3=\sqrt{12\lambda(1-\nu_1\nu_2)}\,\frac{r_0^2}{R_0h}, \quad \lambda=\frac{E_2}{E_1}, \quad \alpha=\frac{r_0}{R_0}, \quad (i=\sqrt{-1}),$$

where E_1, E_2, ν_1, and ν_2 are the moduli of elasticity and the Poisson ratios in the meridional and circumferential directions of the shell, respectively; the primes represent differentiation by θ. The other designations are shown in Figure 1.

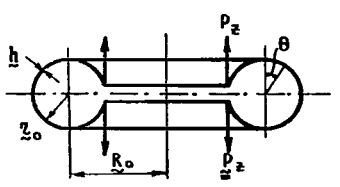

FIGURE 1.

The load function f in the right-hand side of (1.1) is expressed by the formula

$$f=\frac{i\sqrt{12\lambda(1-\nu_1\nu_2)}}{E_2h^2}F_2+\frac{1}{E_1hr_0}\left(\nu_1F_1'+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}F_1\right), \tag{1.2}$$

where

$$\begin{aligned} F_1 &= -\cos\theta\cdot I_1+\sin\theta\left(\frac{P_z}{2\pi}+I_2\right), \\ F_2 &= \sin\theta\cdot I_1+\cos\theta\left(\frac{P_z}{2\pi}+I_2\right). \end{aligned} \tag{1.3}$$

Here

$$I_1=r_0R_0\int_{\theta_0}^{\theta}(q_n\sin\theta+q_1\cos\theta)(1+\alpha\sin\theta)\,d\theta,$$

$$I_2=r_0R_0\int^{\theta}(q_n\cos\theta-q_1\sin\theta)(1+\alpha\sin\theta)\,d\theta,$$

q_n and q_1 are components of the external surface load in the directions of the normal and the tangent to the shell meridian, and P_z is the axial component of the external forces applied to the parallel circle $\theta=\theta_0$.

Knowing the external load acting on the toroidal shell, it is easy to find by means of formulas (1.3) the functions F_1 and F_2, and thus to define completely the function f.

In the particular case that the axial force P_z acts on the shell

$$F_1=\frac{P_z}{2\pi}\sin\theta,\qquad F_2=\frac{P_z}{2\pi}\cos\theta,\qquad F_1'=\frac{P_z}{2\pi}\cos\theta,$$

and formula (1.2) reduces to

$$f=\frac{P_z}{2\pi}\left[\frac{i\sqrt{12\lambda(1-\nu_1\nu_2)}}{E_2h^2}+\frac{1}{E_1hr_0}\left(\nu_1+\frac{\alpha\sin\theta}{1+\alpha\sin\theta}\right)\right]\cos\theta. \tag{1.4}$$

It is easily shown that the second term in the brackets of (1.4) is of order $\frac{h}{r_0}$ compared with unity. It can therefore be neglected compared with the first term, without decreasing the accuracy of the initial equations; then

$$f=iD\cos\theta,\qquad \left(D=\frac{P_z}{2\pi R_0hE_2\alpha^2}\right). \tag{1.5}$$

A similar simplification can be made in formula (1.2). In this case however the function F_1 must not increase during differentiation. Then

$$f=\frac{i\sqrt{12\lambda(1-\nu_1\nu_2)}}{E_2h^2}F_2. \tag{1.6}$$

Having determined the function V from (1.1), the magnitudes characterizing the stressed and strained state of the shell can be found by the following formulas:

normal forces per unit length (stress resultant)

$$T_1 = \frac{\mu \cos\theta \operatorname{Im} V + F_1}{R_0(1+\alpha\sin\theta)}; \qquad T_2 = \frac{\mu}{r_0}\operatorname{Im} V'; \tag{1.7}$$

shearing force per unit length

$$N = \frac{F_2 - \mu\sin\theta \operatorname{Im} V}{R_0(1+\alpha\sin\theta)}; \tag{1.8}$$

bending moments

$$\begin{aligned} M_1 &= \frac{D_1}{r_0}\left(\operatorname{Re} V' + \nu_2 \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\operatorname{Re} V\right), \\ M_2 &= \frac{D_2}{r_0}\left(\nu_1 \operatorname{Re} V' + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\operatorname{Re} V\right); \end{aligned} \tag{1.9}$$

meridional and circumferential strains

$$\begin{aligned} \varepsilon_1 &= \frac{\mu\cos\theta \operatorname{Im} V + F_1}{E_1 h R_0(1+\alpha\sin\theta)} - \frac{\mu\nu_1}{E_2 h r_0}\operatorname{Im} V', \\ \varepsilon_2 &= -\frac{\nu_2(\mu\cos\theta \operatorname{Im} V + F_1)}{E_2 h R_0(1+\alpha\sin\theta)} + \frac{\mu}{E_2 h r_0}\operatorname{Im} V'; \end{aligned} \tag{1.10}$$

angle of rotation [change of slope] of the normal to the shell meridian

$$\vartheta = \operatorname{Re} V; \tag{1.11}$$

axial and circumferential displacements

$$\Delta_z \approx C_0 - r_0 \int_{\theta_0}^{\theta} \operatorname{Re} V \cos\theta \, d\theta, \qquad \Delta_x = R_0(1+\alpha\sin\theta)\,\varepsilon_2; \tag{1.12}$$

maximum stresses in the shell

$$\sigma_i = \pm\frac{6M_i}{h^2} + \frac{T_i}{h}, \qquad (i=1,\ 2). \tag{1.13}$$

In formulas (1.7) to (1.13)

$$\mu = \frac{E_2 h^2}{\sqrt{12\lambda(1-\nu_1\nu_2)}}, \qquad D_i = \frac{E_i h^3}{12(1-\nu_1\nu_2)}, \qquad (i=1,\ 2).$$

Re ... and Im... are the real and imaginary parts, respectively, of the functions V and V'.

2. Equation (1.1) can be represented in the form

$$\varepsilon(pV')' + (q+\varepsilon r)V = f, \tag{2.1}$$

where $\varepsilon = -i/l^3$ is a small parameter (proportional to the shell thickness),

$$p = 1 + \alpha \sin\theta, \qquad q = \sin\theta, \qquad r = -\frac{\lambda\alpha^2\cos^2\theta}{1+\alpha\sin\theta}.$$

Equations of the type (2.1) have been studied in detail in /7/.

The solution of (2.1) is of the form

$$V = v\,[C_1 h_1(t) + C_2 h_2(t)] + V^*, \tag{2.2}$$

where C_1 and C_2 are complex integration constants, determined from the boundary conditions of the specific problem or from the conditions of joining the toroidal shell with shells of other types, $h_1(t)$ and $h_2(t)$ are Airy functions

$$h_i(t) \sim \left(\frac{2}{3}t^{3/2}\right)^{1/3} H_{1/3}^{(i)}\left(\frac{2}{3}t^{3/2}\right), \qquad (i=1,\,2). \tag{2.3}$$

$H_{1/3}^{(i)}$ are first and second Hankel functions of the complex argument $\frac{2}{3}t^{3/2}$,

$$t = is, \qquad s = -lu, \qquad u = \pm\left(\frac{3}{2}\int_0^\theta \sqrt{\frac{|\sin\theta|}{1+\alpha\sin\theta}}\,d\theta\right)^{2/3},$$

$$v = \left[\frac{u}{\sin\theta\,(1+\alpha\sin\theta)}\right]^{1/4}.$$

In practical calculations the following approximate expressions can be used at small values of α:

$$u \approx \theta\left(1 - \frac{1}{5}\alpha\theta\right), \qquad v \approx 1 - \frac{3}{10}\alpha\theta. \tag{2.4}$$

The values of $h_1(t)$, $h_2(t)$ and of their derivatives are given in /9/. At large values of $t = is$ $(s \geqslant 6)$ one can use expressions (2.3) with the first terms of the Hankel functions only /10/.

A particular solution of the nonhomogeneous equation (2.1) is represented by the asymptotic series by the powers of $\frac{1}{\rho}$ $(\rho = -il)$

$$V^* = \rho v \sum_{n=0}^{\infty} \frac{1}{\rho^n} e_n(t) \frac{d^n g}{dt^n}, \qquad (g(u) = v^3 p f), \tag{2.5}$$

where $e_n(t)$ are generalized Airy functions.

The functions $e_n(t)$ have been tabulated for $n = 0,\ 1,\ 2$ /9/.

The first asymptotic approximation for the particular solution will now be considered in more detail. At $n = 0$ we obtain from (2.5), allowing for (1.6),

$$V^* = \frac{l}{\mu}\frac{u}{\sin\theta} e_0(s)\, F_2. \tag{2.6}$$

This reduces at $\theta = 0$ to

$$V^*(0) = \frac{l}{\mu} e_0(0)\, F_2(0).$$

It follows that at the singular points of the torus ($\theta = 0$ and $\theta = \pi$) the particuliar solution is finite (unlike the particular solution obtained on the basis of the membrane shell theory $V_m^* = \frac{iF_2}{\mu \sin\theta}$).

The differentiation of (2.2) by θ yields

$$V' = C_1 [v' h_1(t) - \varkappa(\theta) h_1'(t)] + C_2 [v' h_2(t) - \varkappa(\theta) h_2'(t)] + V^{*\prime}, \qquad (2.7)$$

where

$$v' \approx -\frac{3}{10}\alpha, \qquad \varkappa(\theta) = \frac{l}{pv}.$$

The function $V^{*\prime}$ is found by differentiating (2.6) by θ. Since, in addition,

$$u' = \frac{1}{pv^2}, \qquad \frac{de_0(s)}{d\theta} = -e_0'(s)\, lu',$$

expressions (2.2) and (2.7) solve completely the problem of calculating the orthotropic toroidal shell.

Four boundary conditions (two on each edge of the shell) are necessary in order to determine the four integration constants. Depending upon the edge conditions of the shell, these boundary conditions can be given in forces or in displacements. Expressions (1.7) to (1.12) can be used for formulating the boundary conditions.

Let, for instance, the shell edge $\theta = \theta_0$ be fastened in such a way that the rotation angle ϑ and the circumferential strain ε_2 are equal to zero. No restrictions are imposed on the axial displacement of the shell edge (the force P_z is known). According to (1.11) and (1.10), the conditions $\vartheta = \varepsilon_2 = 0$ lead to the expressions

$$\operatorname{Re} V(\theta_0) = 0,$$

$$\frac{\mu(1 + \alpha \sin\theta_0)}{\alpha v_2} \operatorname{Im} V'(\theta_0) - \mu \cos\theta_0 \operatorname{Im} V(\theta_0) = F_1(\theta_0). \qquad (2.8)$$

Formulas similar to (2.8) can be obtained for other variants of the boundary conditions as well.

In practice, the method of separate fulfillment of the boundary conditions is frequently used. For a toroidal shell the following conditions can serve as criterion of applicability of this method:

$$\theta_k \geqslant \frac{3.8}{l}. \qquad (2.9)$$

Here θ_k is the angular distance between the shell ends considered; the coefficient 3.8 was obtained from the analysis of the variation of the function $h_2(t)$, which increases with t. The numerical value of the coefficient has been determined with the accuracy of 5% normal in technical calculations $\left(\frac{h}{r_0} = \frac{1}{20}\right)$.

3. Consider as an example a toroidal shell truncated along a parallel arc $\theta_0 = -1/2\pi$ loaded on this edge by the axial force P_z. Let the shell edge be free in the sense of radial displacements but incapable of rotating.

Such boundary conditions reduce to the requirements $N = \vartheta = 0$ at $\theta = \pm 1/2\pi$. A particular solution for a toroidal shell loaded by an axial force P_z is

$$V^* = lDu\cot\theta e_0(s). \tag{3.1}$$

The expression for the shearing force will be in the case considered

$$N = \frac{1}{R_0(1+\alpha\sin\theta)}\left(\frac{P_z}{2\pi}\cos\theta - \mu\sin\theta\,\mathrm{Im}\,V\right). \tag{3.2}$$

It follows from (3.2) and (1.11) that at $\theta = \pm {}^1/_2\pi$ $\mathrm{Re}\,V = \mathrm{Im}\,V = 0$. Since $V^*\left(\pm {}^1/_2\pi\right) = 0$, the complex integration constants C_1 and C_2 must be equal in the given case to zero. The stressed state of the orthotropic toroidal shell is thus described in this case by the function V^*.

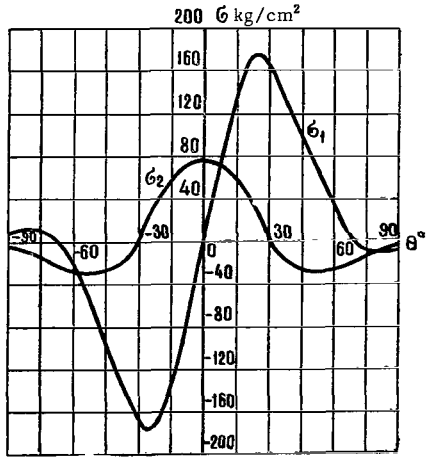

FIGURE 2.

Figure 2 represents the distribution of stresses σ_1 and σ_2 in an orthotropic toroidal shell of geometrical dimensions r_0 = 10 cm, R_0 = 50 cm, h = 0.25 cm. The elastic constants of the shell material have the following values: $E_1 = 1.2\times10^5$ kg/cm²; $E_2 = 0.6\times10^5$ kg/cm²; $\nu_1 = 0.072$; $\nu_2 = 0.036$. The shell is loaded by an axial force P_z = 100 kg.

It is seen from the curves of Figure 2 that the distribution of stresses in the shell is very nonuniform. This nonuniformity is due to the considerable bending moments.

We shall now establish formulas for calculating the maximum stresses and strains in the shell. Let us assume for simplicity that

$$\alpha \approx 0, \quad \sin\theta \approx \theta, \quad \cos\theta \approx 1. \tag{3.3}$$

The approximate expression for the function V^* will then be

$$V^* \approx lDe_0(s) \tag{3.4}$$

(since $u\cot\theta \approx 1$ for small α within a wide range of variation of the angle θ).

After the values (3.3) have been substituted in (1.1), the latter splits into the following two equations:

$$\operatorname{Re} V'' - l^3\theta \operatorname{Im} V = -l^3 D, \quad \operatorname{Im} V'' + l^3\theta \operatorname{Re} V = 0. \tag{3.5}$$

Consider the approximate expression for the meridional bending moment

$$M_1 \approx \frac{D_1}{r_0} \operatorname{Re} V'. \tag{3.6}$$

Obviously, to the maximum value of M_1 there corresponds approximately the condition $\operatorname{Re} V'' = 0$. When this condition is satisfied the first equation of (3.5) reduces to $\theta \operatorname{Im} V = D$. The substitution here of the value of $\operatorname{Im} V$ according to (3.4) yields

$$-s \operatorname{Im} e_0(s) = 1. \tag{3.7}$$

The value of s which satisfies this equation is $s = \pm 1.225$. It follows that the value of the angles θ_m at which maximum meridional bending stresses will appear in the shell is determined by the expression

$$\theta_m \approx \pm \frac{1.225}{l}. \tag{3.8}$$

The formulas for the maximum stresses are in this case (for $\nu_1\nu_2 \approx 0$)

$$\begin{gathered} \sigma_{1u}(\theta_m) \approx \pm 2.99\,\lambda^{-1/4}\left(\frac{r_0R_0}{h^2}\right)^{1/2} p_0, \qquad \left(p_0 = \frac{P_z}{2\pi R_0 h}\right); \\ \sigma_{1p}(\theta_m) \approx \left[1.24\,\lambda^{1/4}\left(\frac{r_0^2}{R_0h}\right)^{1/2} + 0.81\,\lambda^{-1/4}\left(\frac{r_0^2}{R_0h}\right)^{-1/2}\right] p_0. \end{gathered} \tag{3.9}$$

A similar analysis of the second equation of (3.5) leads to the conclusion that the maximum circumferential tensile stress occurs at $\theta \approx 0$. The following values of the maximum stresses are obtained for this value of θ:

$$\begin{gathered} \sigma_{2u}(0) \approx \pm 2.6\,\lambda^{3/4}\left(\frac{r_0^2}{R_0h}\right)^{1/2} p_0, \\ \sigma_{2p}(0) \approx 2.15\lambda^{3/4}\left(\frac{r_0R_0}{h^2}\right)^{1/2} p_0. \end{gathered} \tag{3.10}$$

Expressions (1.12) and (3.4) will be used for deriving the formula for axial divergence of the shell edges for $\theta_0 = -1/2\pi$:

Since /7/

$$\int_{-\infty}^{\infty} \operatorname{Re} e_0(s)\, ds = \pi,$$

we obtain

$$\Delta_z = \sqrt{\frac{12(1-\nu_1\nu_2)}{E_1E_2}}\,\frac{P_z r_0}{h^2}. \tag{3.11}$$

At $\nu_1 = \nu_2 = \nu$, $E_1 = E_2 = E$ it follows from (3.11) that

$$\Delta_z = \sqrt{12(1-\nu^2)}\,\frac{P_z r_0}{Eh^2},$$

which corresponds to the axial divergence of the edges of an isotropic toroidal shell.

Formula (3.11) can be used for the approximate assessment of the deformation of a toroidal shell reinforced along the meridian by stiffening ribs. Substituting in (3.11) $\nu_1 = \nu_2 = \nu$, $E_2 = E$ and $E_1 = E\,[1 + 12(1-\nu^2)\,zI/2\pi R_0 h^3]$ we obtain

$$\Delta_z = \sqrt{\frac{12(1-\nu^2)}{1+zI/I_0}}\,\frac{P_z r_0}{Eh^2}, \tag{3.12}$$

where I is the moment of inertia of a rib relative to the middle surface of the shell; z is the number of ribs; $I_0 = 2\pi R_0 h^3/12(1-\nu^2)$ is the moment of inertia of the shell wall.

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THERMAL STRESSES IN A REINFORCED CYLINDRICAL SHELL

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This paper gives a solution to the problem of determining the temperature stresses and displacements in a cylindrical shell (Figure 1) reinforced by stringers in the case of axisymmetrical heating. The temperature is assumed to be constant through the shell thickness and to vary along its length according to some arbitrary law $t^{sh} = t(\xi)$. The temperature of the stringers can vary in both directions (length and height): $t^{st} = t(\xi, z)$; it is however assumed that its variation with height is linear. The temperature field is assumed to be stationary, the material of the shell and stringers to be elastic and isotropic, and the physical-mechanical characteristics of the material to be temperature-independent. The problem is solved in an exact formulation, as a contact problem. The displacements and stresses in the shell are determined first, and the displacements and stresses in the stringer afterwards. The following conditions must be fulfilled along the line of junction of the shell and stringer:

$$1)\ u^{sh} = u^{st},\quad 2)\ w^{sh} = w^{st},\quad 3)\ v^{sh} = 0,\quad 4)\ \frac{\partial w^{sh}}{\partial \varphi} = 0. \tag{1}$$

The temperature field being axisymmetrical, the shell is deformed symmetrically relative to the places of fastening of the stringers. Consider the part of the shell between adjacent stringers. We shall use as an initial system of equations the complete system of the moment theory of cylindrical shells, with temperature terms added. The system of equations of equilibrium in displacements is

$$\begin{aligned} &L_{11}u + L_{12}v + L_{13}w = R(1+\nu)\alpha\frac{dt}{d\xi},\\ &L_{21}u + L_{22}v + L_{23}w = 0,\\ &L_{31}u + L_{32}v + L_{33}w = -R(1+\nu)\alpha t(\xi), \end{aligned} \tag{2}$$

where $L_{11}, L_{12}, \cdots, L_{33}$ are known differential operators /1/; u, v, w are axial, tangential, and radial displacements respectively; α is the Poisson ratio; ν is a coefficient of linear expansion; R is the shell radius.

The relationships between the forces per unit length and the displacements will be:

$$T_1 = \frac{2Eh}{1-\nu^2}\frac{1}{R}\left(\frac{\partial u}{\partial \xi} + \nu\frac{\partial v}{\partial \varphi} - \nu w\right) - \frac{2Eh}{1-\nu}\alpha t(\xi), \tag{3}$$

$$T_2 = \frac{2Eh}{1-\nu^2}\frac{1}{R}\left(\frac{\partial v}{\partial \varphi} - w + \nu\frac{\partial u}{\partial \xi}\right) - \frac{2Eh}{1-\nu}\alpha t(\xi),$$

$$G_1 = -\frac{2Eh}{1-\nu^2}\frac{h^2}{3R^2}\left(\frac{\partial^2 w}{\partial \xi^2} + \nu\frac{\partial^2 w}{\partial \varphi^2} + \nu\frac{\partial v}{\partial \varphi}\right),$$

$$G_2 = -\frac{2Eh}{1-\nu^2}\frac{h^2}{3R^2}\left(\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial v}{\partial \varphi} + \nu\frac{\partial^2 w}{\partial \xi^2}\right), \qquad (3)\ \text{(cont'd)}$$

$$S_1 = \frac{2Eh}{1+\nu}\frac{1}{R}\left[\frac{1}{2}\left(\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \varphi}\right) + \frac{h^2}{3R^2}\left(\frac{\partial^2 w}{\partial \xi \partial \varphi} + \frac{\partial v}{\partial \xi}\right)\right],$$

$$S_2 = -\frac{Eh}{1+\nu}\frac{1}{R}\left(\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \varphi}\right),$$

$$H_1 = -H_2 = \frac{2Eh}{1+\nu}\frac{h^2}{3R^2}\left(\frac{\partial^2 w}{\partial \xi \partial \varphi} + \frac{\partial v}{\partial \xi}\right).$$

System (2) will be solved by the operatorial method. The solution of the corresponding homogeneous system reduces in this case to the solution of one resolving equation of the eighth order relative to some potential function Φ /1/:

$$\frac{\partial^8 \Phi}{\partial \xi^8} + 4\frac{\partial^8 \Phi}{\partial \xi^6 \partial \varphi^2} + 6\frac{\partial^8 \Phi}{\partial \xi^4 \partial \varphi^4} + 4\frac{\partial^8 \Phi}{\partial \xi^2 \partial \varphi^6} + \frac{\partial^8 \Phi}{\partial \varphi^8} +$$
$$+ (8 - 2\nu^2)\frac{\partial^6 \Phi}{\partial \xi^4 \partial \varphi^2} + 8\frac{\partial^6 \Phi}{\partial \xi^2 \partial \varphi^4} + 2\frac{\partial^6 \Phi}{\partial \varphi^6} + \frac{1-\nu^2}{a^2}\frac{\partial^4 \Phi}{\partial \xi^4} +$$
$$+ 4\frac{\partial^4 \Phi}{\partial \xi^2 \partial \varphi^2} + \frac{\partial^4 \Phi}{\partial \varphi^4} = 0. \qquad (4)$$

In deriving this equation we have neglected the value $a^2 = \frac{h^2}{3R^2}$ in comparison with units ($2h$ is the shell thickness).

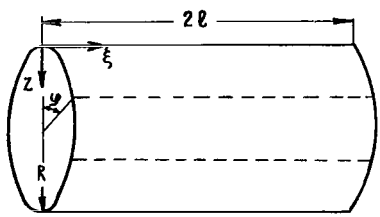

FIGURE 1.

The solution of the nonhomogeneous system reduces to solving a system of the form

$$D\psi_1 = (1+\nu)R\alpha\frac{dt}{d\xi},$$
$$D\psi_2 = -(1+\nu)R\alpha t(\xi), \qquad (5)$$

with the displacements u^*, v^*, w^* determined through the unknown functions of this system by means of the following equations:

$$u^* = D_{11}\psi_1 + D_{13}\psi_2, \qquad (6)$$

$$v^* = D_{21}\psi_1 + D_{23}\psi_2,$$
$$w^* = D_{31}\psi_1 + D_{33}\psi_2. \qquad (6) \text{ (cont'd)}$$

Here D is the determinant of system (2) and $D_{11}, D_{12}, \cdots, D_{33}$ its corresponding minors.

We shall look for a solution of the resolving equation in the form of a trigonometric series by the variable ξ:

$$\Phi(\varphi, \xi) = \sum_{m=0}^{\infty} f_m(\varphi) \cos \lambda_m \xi. \qquad (7)$$

The substitution of (7) in (4) yields for each term of the series an ordinary differential equation with constant coefficients relative to the unknown function $f_m(\varphi)$

$$f_m^{(8)} - (4\lambda_m^2 - 2) f_m^{(6)} + (6\lambda_m^4 - 8\lambda_m^2 + 1) f_m^{(4)} - [4\lambda_m^6 - (8 - 2\nu^2)\lambda_m^4 + 4\lambda_m^2] f_m^{(2)} +$$
$$+ \left(\lambda_m^8 + \frac{1-\nu^2}{a^2}\lambda_m^4\right) f_m = 0. \qquad (8)$$

By solving this equation we obtain a solution of the following type for each term of the expansion:

$$\Phi_m = [e^{a_m\varphi}(C_{1m}\cos b_m\varphi + C_{2m}\sin b_m\varphi) +$$
$$+ e^{-a_m\varphi}(C_{3m}\cos b_m\varphi + C_{4m}\sin b_m\varphi) + e^{a_{1m}\varphi}(C_{5m}\cos b_{1m}\varphi + C_{6m}\sin b_{1m}\varphi) +$$
$$+ e^{-a_{1m}\varphi}(C_{7m}\cos b_{1m}\varphi + C_{8m}\sin b_{1m}\varphi)] \cos \lambda_m \xi, \qquad (9)$$

where $C_{1m}, C_{2m}, \cdots, C_{8m}$ are arbitrary integration constants and a_m, a_{1m}, b_m, b_{1m} are the real part and the coefficient of the imaginary part of the complex roots of the characteristics equation obtained from (8).

The displacements are determined from the relationships

$$u^0 = -\frac{\partial^3\Phi}{\partial\xi\,\partial\varphi^2} + \nu\frac{\partial^3\Phi}{\partial\xi^3} + \frac{h^2}{3R^2}\left[\frac{(1+\nu)(2-\nu)}{1-\nu}\frac{\partial^5\Phi}{\partial\xi^3\,\partial\varphi^2} + \right.$$
$$\left. + \frac{1+\nu}{1-\nu}\frac{\partial^5\Phi}{\partial\xi\,\partial\varphi^4} + 4\nu\frac{\partial^3\Phi}{\partial\xi^3} + \frac{2\nu}{1-\nu}\frac{\partial^3\Phi}{\partial\xi\,\partial\varphi^2}\right],$$
$$v^0 = (2+\nu)\frac{\partial^3\Phi}{\partial\xi^2\,\partial\varphi} + \frac{\partial^3\Phi}{\partial\varphi^3} - \frac{h^2}{3R^2}\left[\frac{2(2-\nu)}{1-\nu}\frac{\partial^5\Phi}{\partial\xi^4\,\partial\varphi} + \right.$$
$$\left. + \frac{4-3\nu+\nu^2}{1-\nu}\frac{\partial^5\Phi}{\partial\xi^2\,\partial\varphi^3} + \frac{\partial^5\Phi}{\partial\varphi^5}\right], \qquad (10)$$
$$w^0 = \frac{\partial^4\Phi}{\partial\xi^4} + 2\frac{\partial^4\Phi}{\partial\xi^2\,\partial\varphi^2} + \frac{\partial^4\Phi}{\partial\varphi^4} +$$
$$+ \frac{h^2}{3R^2}\left[4\frac{\partial^4\Phi}{\partial\xi^4} + \frac{2(2-2\nu+\nu^2)}{1-\nu}\frac{\partial^4\Phi}{\partial\xi^2\,\partial\varphi^2} + \frac{\partial^4\Phi}{\partial\varphi^4}\right].$$

The symmetry condition $w(\varphi) = w(-\varphi)$ leads to the following relationships between the arbitrary constants:

$$C_{1m} = C_{3m}, \quad C_{2m} = -C_{4m}, \quad C_{5m} = C_{7m}, \quad C_{6m} = -C_{8m}.$$

The displacements contain therefore four arbitrary constants.

The temperature along the shell can be represented as a cosine series

$$t^{sh}(\xi) = \sum_{m=0}^{\infty} k_m \cos \lambda_m \xi \tag{11}$$

(k_m are expansion coefficients).

The following expressions will represent then a particular solution of (5):

$$\begin{gathered} u_m^* = k_m \frac{R\alpha}{\lambda_m}\left(1 + \frac{\nu\lambda_m^4 a^2}{\lambda_m^4 a^2 + 1 - \nu^2}\right)\sin \lambda_m \xi, \\ v_m^* = 0, \\ w_m^* = -k_m \frac{R\alpha(1-\nu^2)}{\lambda_m^4 a^2 + 1 - \nu^2}\cos \lambda_m \xi. \end{gathered} \tag{12}$$

Having obtained a complete solution of system (2), we shall find now the expressions for the forces and moments (3). Let the shell be hinged:

$$v = w = 0, \quad T_1 = 0, \quad G_1 = 0 \quad \text{at} \quad \xi = \pm l. \tag{13}$$

These conditions will be satisfied if $\cos \lambda_m \xi = 0$ whence $\lambda_m = \frac{(2n+1)\pi}{2l}$, where n is any positive integer.

The stringers will be considered as beams subjected to extension and bending. The shell transmits to each stringer uniformly distributed forces per unit length $\tilde{q} = 2N_2$. $\tilde{t} = 2S_2$ and moment $\tilde{\mu} = 2H_2$. In addition, the stringer is heated to a temperature $t^{st} = t(\xi, z)$. The equations of equilibrium of the stringer element (Figure 2) will be

$$\begin{gathered} \frac{dp^{st}}{d\xi} d\xi + \tilde{t} R d\xi = 0, \qquad \frac{dQ^{st}}{d\xi} d\xi + \tilde{q} R d\xi = 0, \\ \frac{dM^{st}}{d\xi} d\xi + Q^{st} R d\xi - \tilde{q} R d\xi \frac{R d\xi}{2} - \tilde{t} R d\xi \cdot l - \tilde{\mu} R d\xi + \frac{dM^t}{d\xi} d\xi = 0 \end{gathered} \tag{14}$$

(M^t is the bending moment appearing in the stringer with the variation of the temperature along its height).

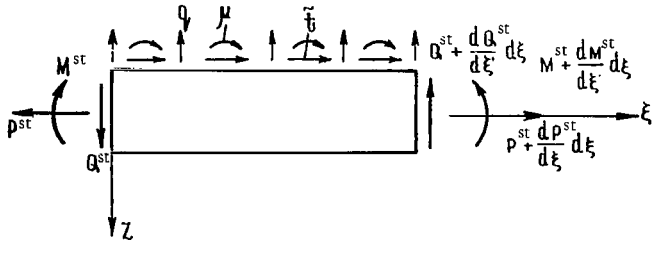

FIGURE 2.

The boundary conditions for the stringer are:

$$\begin{gathered} \text{at} \quad \xi = \pm l \quad w^{st} = 0, \quad p^{st} = 0, \quad M^e = 0; \\ \text{at} \quad \xi = 0 \quad u^{st} = 0. \end{gathered} \tag{15}$$

The condition of symmetry yields the relationship $w^{st}(\xi) = w^{st}(-\xi)$. The

solution of system (14), taking into account the boundary conditions (15) and the equation of bending

$$\frac{d^2 w^{st}}{R^2 d\xi^2} = -\frac{M}{EJ},$$

yields the expressions for the displacements and forces in the stringer.

The arbitrary constants through which the displacements and forces in the shell and stringers are expressed are determined from condition (1).

The cylindrical shells are reinforced in most structures by stringers and frames. The influence of frames on the magnitude and distribution of temperature stresses in the sheathing and stringers was assessed approximately in our studies.

We assume that the frames are absolutely rigid and are disposed close enough to each other two prevent the deflection of the stringers. Thus, without considering the frames separately, we assume that the flexural rigidity of the stringers increases infinitely and the deflection w^{st} is equal to zero, and therefore that the shell deflection along its line of junction with the stringer is equal to zero from the condition of compatibility of deformations. This solution represents a particular case of the above considered solution. Using the expressions obtained for the displacements and stresses in the shell and stringer, and writing in them $J^{st} = \infty$, we determine the arbitrary constants from the relationships existing along the line of junction of the shell and stringer at $\varphi = \varphi^*$:

$$1)\ u^{sh}(\xi, \varphi^*) = u^c(\xi), \qquad 2)\ w^{sh}(\xi, \varphi^*) = 0,$$

$$3)\ v^{sh}(\xi, \varphi^*) = 0, \qquad 4)\ \left.\frac{\partial w^{sh}}{\partial \varphi}\right|_{\varphi=\varphi^*} = 0.$$

The solution considered and the supplement given make it possible to estimate the expected distribution of temperature stresses in a cylindrical shell reinforced by stringers and frames, and also to find the stresses transmitted from the side of the sheathing to the stringer in the presence of a temperature gradient.

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DYNAMIC CONTACT PROBLEM FOR A THICK PLATE IN THE CASE OF AXIAL SYMMETRY

N.M. Borodachev

(Saratov)

1. Consider an infinite thick elastic plate (elastic layer) of thickness h lying on a rigid foundation without friction. Let a circular-cylindrical punch of flat base be placed on the plate (see figure), and let a vertical force $Q + Pe^{i\omega t}$ act on the punch along its axis of symmetry. It is assumed that $Q > P$, and that there is no friction between the punch and the plate.

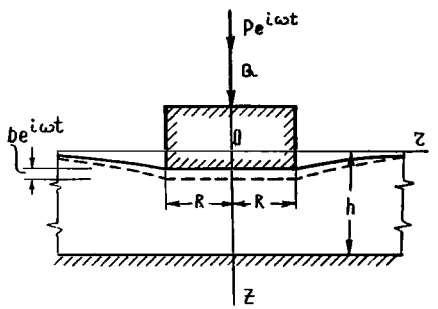

FIGURE.

Problems of this type are met when the foundation on which a machine with a crank movement is set lies on a relatively thin soil layer underlain by a rocky foundation (or more generally by a foundation considerably more compact than the upper layer).

The problem can be solved by superposing the solutions of the following two problems:

a) the problem of the penetration of the punch in the elastic layer under the action of the constant force Q;

b) the problem of the motion of a punch under the action of a dynamic force $Pe^{i\omega t}$.

The first problem is a static contact problem for an elastic layer lying on a rigid foundation. This problem was treated in our papers /1, 2/ relative to the case of axial symmetry. The second problem is treated in the present paper.

2. Before solving the dynamic contact problem we shall establish a general solution of the problem of steady vibrations of an elastic layer of finite thickness lying on a rigid foundation in the case of axial symmetry.

We shall use the cylindrical system of coordinates (r, φ, z), with the z axis perpendicular to the boundaries of the elastic layer. For the case of steady vibrations of frequency ω of an isotropic elastic layer we can write

$$u_r(r, z, t) = u_r^*(r, z)\, e^{i\omega t},$$
$$u_z(r, z, t) = u_z^*(r, z)\, e^{i\omega t}, \tag{2.1}$$

where u_r, u_z are projections of the displacement vector on the r and z coordinate axes, respectively; u_r^*, u_z^*, must satisfy the following equations:

$$(\lambda + 2\mu)\frac{\partial\theta^*}{\partial r} + 2\mu\frac{\partial\eta^*}{\partial z} + \rho\omega^2 u_r^* = 0,$$
$$(\lambda + 2\mu)\frac{\partial\theta^*}{\partial z} - 2\mu\frac{1}{r}\frac{\partial}{\partial r}(r\eta^*) + \rho\omega^2 u_z^* = 0. \tag{2.2}$$

Here λ, μ are Lamé's elastic constants, and ρ is density;

$$\theta^* = \frac{1}{r}\frac{\partial}{\partial r}(r u_r^*) + \frac{\partial u_z^*}{\partial z},$$
$$2\eta^* = \frac{\partial u_r^*}{\partial z} - \frac{\partial u_z^*}{\partial r}. \tag{2.3}$$

The functions θ^*, η^* satisfy in their turn the following equations:

$$\frac{\partial^2\theta^*}{\partial r^2} + \frac{1}{r}\frac{\partial\theta^*}{\partial r} + \frac{\partial^2\theta^*}{\partial z^2} + \varepsilon^2 m^2\theta^* = 0,$$
$$\frac{\partial^2\eta^*}{\partial r^2} + \frac{1}{r}\frac{\partial\eta^*}{\partial r} - \frac{\eta^*}{r^2} + \frac{\partial^2\eta^*}{\partial z^2} + m^2\eta^* = 0, \tag{2.4}$$

where

$$m = \omega\sqrt{\frac{\rho}{\mu}}; \qquad \varepsilon = \sqrt{\frac{\mu}{\lambda + 2\mu}}. \tag{2.5}$$

By applying Hankel's integral transformation by the variable r to equations (2.2)-(2.4), we obtain after the corresponding calculations

$$u_r^*(r, z) = \int_0^\infty \xi\,[\xi(A_1 \operatorname{sh} k_1 z + A_2 \operatorname{ch} k_1 z) - k_2(A_3 \operatorname{ch} k_2 z + A_4 \operatorname{sh} k_2 z)]\, J_1(r\xi)\, d\xi, \tag{2.6}$$

$$u_z^*(r, z) = \int_0^\infty \xi\,[-k_1(A_1 \operatorname{ch} k_1 z + A_2 \operatorname{sh} k_1 z) + \xi(A_3 \operatorname{sh} k_2 z + A_4 \operatorname{ch} k_2 z)]\, J_0(r\xi)\, d\xi. \tag{2.7}$$

Here

$$k_1 = \sqrt{\xi^2 - \varepsilon^2 m^2}; \qquad k_2 = \sqrt{\xi^2 - m^2}; \tag{2.8}$$

$J_n(x)$ is a Bessel function of the first kind; $A_1(\xi)$, $A_2(\xi)$, $A_3(\xi)$, $A_4(\xi)$ are functions determined from the boundary conditions at $z = 0$ and $z = h$. For

large positive values of ξ each of the radicals in (2.8) is taken with positive sign.

In what follows we shall need the case characterized by the following boundary conditions:

the stresses at $z = 0$ are

$$\sigma_z(r, 0, t) = \sigma_z^*(r, 0)\, e^{i\omega t}, \quad \tau_{rz}(r, 0, t) = 0, \tag{2.9}$$

the vertical displacements of the layer and the friction forces are equal to zero at $z = h$:

$$u_z(r, h, t) = 0, \quad \tau_{rz}(r, h, t) = 0. \tag{2.10}$$

For these boundary conditions we obtain

$$A_1 = \frac{2\xi^2 - m^2}{\mu B(\xi)} \Sigma_{z(0)}^*(\xi, 0),$$

$$A_3 = -\frac{2\xi k_1 \operatorname{cth} k_2 h}{\mu B(\xi)} \Sigma_{z(0)}^*(\xi, 0), \tag{2.11}$$

$$A_2 = -A_1 \operatorname{cth} k_1 h, \qquad A_4 = -A_3 \operatorname{th} k_2 h, \tag{2.12}$$

where

$$B(\xi) = (2\xi^2 - m^2)^2 \operatorname{cth} k_1 h - 4\xi^2 k_1 k_2 \operatorname{cth} k_2 h, \tag{2.13}$$

$$\Sigma_{z(0)}^*(\xi, 0) = \int_0^\infty r\sigma_z^*(r, 0)\, J_0(\xi r)\, dr. \tag{2.14}$$

3. The results obtained in § 2 will be applied now to the solution of the dynamic contact problem for a thick plate. Let an infinite thick elastic plate (elastic layer) of thickness h lying without friction on a rigid foundation be subjected to the action of a circular-cylindrical punch with flat base, advancing under the action of an axial force $Pe^{i\omega t}$. It will be assumed that there is no friction between the punch and the plate, and that there is no load outside the punch.

The boundary conditions for the dynamic contact problem considered will be

$$u_z(r, 0, t) = be^{i\omega t} \quad \text{at} \quad 0 < r < R, \tag{3.1}$$

$$\sigma_z(r, 0, t) = 0 \quad \text{at} \quad r > R, \tag{3.2}$$

$$\tau_{rz}(r, 0, t) = 0 \quad \text{at} \quad 0 < r < \infty, \tag{3.3}$$

$$u_z(r, h, t) = 0 \quad \text{at} \quad 0 < r < \infty, \tag{3.4}$$

$$\tau_{rz}(r, h, t) = 0 \quad \text{at} \quad 0 < r < \infty. \tag{3.5}$$

By substituting in (2.7) the expressions for A_1, A_2, A_3, A_4 from (2.11), (2.12) and writing then $z = 0$ we obtain

$$u_z^*(r, 0) = \frac{m^2}{\mu} \int_0^\infty \frac{\xi k_1 \Sigma_{z(0)}^*(\xi, 0)\, J_0(r\xi)}{B(\xi)}\, d\xi. \tag{3.6}$$

This expression satisfies the boundary conditions (3.3)-(3.5). The integral in (3.6) is to be considered as a contour integral. The singularities at points $\xi = \varepsilon m$, $m, \xi_1, \xi_2, \cdots, \xi_n, \cdots, \xi_N$ are to be by-passed along the arcs of the circles located above the singularities. Here ξ_n are the positive roots of the equation

$B(\xi)=0$. By satisfying the boundary conditions (3.1), (3.2) we obtain the dual integral equations

$$\int_0^\infty \frac{m^2\xi k_1 \Sigma^*_{z(0)}(\xi, 0) J_0(r\xi)}{B(\xi)} d\xi = \mu b, \quad 0 < r < R,$$

$$\int_0^\infty \xi \Sigma^*_{z(0)}(\xi, 0) J_0(r\xi)\, d\xi = 0, \quad r > R. \tag{3.7}$$

(Expression (3.6) and the Hankel transformation formula were used in the derivation of these equations.)

Equations (3.7) must be solved relative to $\Sigma^*_{z(0)}(\xi, 0)$, after which the stress $\sigma_z(r, 0, t)$ acting along the contact area is determined from the formula

$$\sigma_z(r, 0, t) = e^{i\omega t} \int_0^\infty \xi \Sigma^*_{z(0)}(\xi, 0) J_0(r\xi)\, d\xi. \tag{3.8}$$

The solution of system (3.7) can be reduced to solving Fredholm's integral equation of the second kind by means of J. Cooke's method /3/. The second equation of (3.7) will be satisfied if we write

$$\Sigma^*_{z(0)}(\xi, 0) = -\frac{4(1-\varepsilon^2)\mu b R}{\pi} \int_0^1 f(x) \cos(R\xi x)\, dx. \tag{3.9}$$

In order that the function $\Sigma^*_{z(0)}(\xi, 0)$ thus defined may satisfy the first equation of (3.7), the function $f(x)$ must be a solution of Fredholm's integral equation of the second kind

$$f(x) - \int_0^1 K(x, y) f(y)\, dy = 1, \tag{3.10}$$

where

$$K(x, y) = \frac{2\alpha}{\pi} \int_0^\infty F(\beta) \cos(\alpha x \beta) \cos(\alpha y \beta)\, d\beta; \tag{3.11}$$

$$F(\beta) = 1 + \frac{2(1-\varepsilon^2)\beta\sqrt{\beta^2-\varepsilon^2}}{(2\beta^2-1)^2 \operatorname{cth}(q\sqrt{\beta^2-\varepsilon^2}) - 4\beta^2\sqrt{\beta^2-\varepsilon^2}\sqrt{\beta^2-1}\operatorname{cth}(q\sqrt{\beta^2-1})}; \tag{3.12}$$

$$\alpha = \omega R\sqrt{\frac{\rho}{\mu}}; \qquad q = \omega h\sqrt{\frac{\rho}{\mu}}. \tag{3.13}$$

The integral in (3.11) is to be considered as a contour integral. The singularities at points $\beta = \varepsilon, 1, \beta_1, \beta_2, \cdots, \beta_n, \cdots, \beta_N$ are to be by-passed along the arcs of small circles lying above the singularities. Here β_n represent the positive roots of the equation

$$(2\beta^2-1)^2 \operatorname{cth}(q\sqrt{\beta^2-\varepsilon^2}) - 4\beta^2\sqrt{\beta^2-\varepsilon^2}\sqrt{\beta^2-1}\operatorname{cth}(q\sqrt{\beta^2-1}) = 0. \tag{3.14}$$

Some problems connected with the determination of the roots of (3.14) have been considered by Marguerre /4/. The integral in (3.11) must be calculated by approximate methods.

Integral equation (3.10) will be solved approximately, by substituting for it a finite system of linear algebraic equations. The integral is replaced by a summation by Simpson's formula:

$$f(x) - \frac{1}{3n}\sum_{m=0}^{n} a_m K(x, x_m) f(x_m) = 1, \tag{3.15}$$

where n is the number (even) of equal parts into which the integration interval (0, 1) is split:

$$a_0 = a_n = 1; \quad a_1 = a_3 = \cdots = a_{n-1} = 4; \quad a_2 = a_4 = \cdots = a_{n-2} = 2.$$

By successively introducing in (3.15) the equalities

$$x = x_0, \ x_1, \ x_2, \cdots, x_n,$$

we obtain the following system of equations:

$$f(x_j) - \frac{1}{3n}\sum_{m=0}^{n} a_m K(x_j, x_m) f(x_m) = 1, \quad (j = 0, 1, 2, \cdots, n). \tag{3.16}$$

By solving this system we obtain the approximate values of the function $f(x)$ at points $x_0, x_1, x_2, \cdots, x_n$. By using then some interpolation formula we can obtain an approximate expression for $f(x)$ in the whole interval (0, 1).

The substitution of the expression for $\Sigma^{*}_{z(0)}(\xi, 0)$ from (3.9) in (3.8) yields

$$\sigma_z(r, 0, t) = -\frac{4(1-\varepsilon^2)\mu b e^{i\omega t}}{\pi}\left[\frac{f(1)}{\sqrt{R^2 - r^2}} - \int_{r/R}^{1} \frac{f'(x)\,dx}{\sqrt{R^2x^2 - r^2}}\right] \quad \text{at } r < R. \tag{3.17}$$

The elastic layer reaction can be found from

$$P_0 e^{i\omega t} = -2\pi \int_0^R \sigma_z(r, 0, t)\, r dr. \tag{3.18}$$

Substituting in (3.18) the expression for $\sigma_z(r, 0, t)$ from (3.17) and integrating by r, we obtain

$$P_0 = 8(1-\varepsilon^2)\mu b R \int_0^1 f(x)\,dx. \tag{3.19}$$

Introduce the designations

$$\gamma = \int_0^1 f(x)\,dx; \quad \gamma = \gamma_1 + i\gamma_2 = \gamma_0 e^{i\delta}; \tag{3.20}$$

$$b = b_1 + ib_2 = b_0 e^{-i\varphi}. \tag{3.21}$$

The elastic layer reaction can then be represented in the form

$$P_0 e^{i\omega t} = 8(1-\varepsilon^2)\mu R \gamma_0 b_0 e^{i(\omega t + \delta - \varphi)}. \tag{3.22}$$

If the external force acting on the punch is equal to $P\cos\omega t$ ($J_m P = 0$), the equation of motion of the punch will be

$$M \cdot \mathrm{Re}\left[\frac{d^2}{dt^2}(be^{i\omega t})\right] + \mathrm{Re}(P_0 e^{i\omega t}) = P\cos\omega t, \tag{3.23}$$

where M is the punch mass. Using (3.21), (3.22), and (3.23), we obtain

$$-M\omega^2 b_0 \cos(\omega t - \varphi) + 8(1-\varepsilon^2)\mu R\gamma_0 b_0 \cos(\omega t + \delta - \varphi) = P\cos\omega t,$$

from which we find

$$\operatorname{tg}\varphi = \frac{\sin\delta}{\cos\delta - c}, \tag{3.24}$$

$$b_0 = \frac{P}{8(1-\varepsilon^2)\mu R\gamma_0\sqrt{1-2c\cdot\cos\delta + c^2}}, \tag{3.25}$$

where

$$c = \frac{M\omega^2}{8(1-\varepsilon^2)\mu R\gamma_0}. \tag{3.26}$$

Therefore, if the force acting on the punch is equal to $P\cos\omega t$, the punch will move according to the law $b_0\cos(\omega t - \varphi)$. The angle of phase shift φ is determined by formula (3.24). Relationship (3.25) establishes the connection between the amplitude of vertical displacements of the punch b_0 and the amplitude of the applied force P. Using (3.17), (3.21), and (3.25), we obtain a formula for determining the normal stresses acting along the area of contact:

$$\sigma_z(r, 0, t) = -\operatorname{Re}\left\{\frac{Pe^{i(\omega t-\varphi)}}{2\pi R\gamma_0\sqrt{1-2c\cdot\cos\delta + c^2}}\left[\frac{f(1)}{\sqrt{R^2-r^2}} - \int_{rR}^{1}\frac{f'(x)\,dx}{\sqrt{R^2x^2-r^2}}\right]\right\} \quad \text{at} \quad r < R. \tag{3.27}$$

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VIBRATIONS OF CYLINDRICAL SHELLS FILLED WITH LIQUID

V.E. Breslavskii

(Khar'kov)

Two basic problems arise when considering the motion of liquid-filled thin shells. The first problem is to study the motion of the system liquid + shell under the action of the given system of forces; Moiseev, Narimanov, Rumyantsev, Sretenskii, Rabinovich, and others have studied it. The second problem is to study the motion of the liquid and its influence on the shell, the pattern of motion of the shell being assumed as known. Pavlenko /1/ has studied the two-dimensional motion of a liquid in a rectangular cavity; Okhotsimskii /2/ has studied the vibrations of a straight circular cylinder as a beam, and the influence of the liquid filling the cylinder; Bublik and Merkulov /3/ have studied the question of dynamic stability of liquid-filled shells. In this paper we study the free vibrations of smooth and ribbed cylindrical shells partially or completely filled with liquid. Both a normal pressure and an axial force can act on the shell. Let the x axis be directed along the cylinder axis and coincide with the vertical. We assume that the cylinder is filled with a completely uncompressible liquid which performs a potential motion during the cylinder vibrations.

We shall consider the vibrations of a shell with simply supported edges (at different boundary conditions the solution will differ negligibly from that obtained for this particular case). The equation of shell vibrations and the boundary conditions will then be satisfied by displacements of the form

$$u = M\cos m\varphi \cos\lambda\xi;\ v = N\sin m\varphi \sin\lambda\xi;\ w = K\cos m\varphi \sin\lambda\xi, \qquad (1)$$

where u, v, w are components along the shell axis, the tangent to the cross-section circle, and the normal, of the displacements of a point of the middle surface; M, N, K are time-dependent magnitudes representing generalized coordinates; φ is an angle defining the position of the point on the cross-sectional circle of the shell or the position of the liquid particle; m is the number of waves in the shell cross section; R is the radius of the middle surface of the shell; $\lambda = \frac{n\pi R}{L}$; n is an integer; L is the shell length; $\xi = \frac{x}{R}$.

It is assumed that the liquid motion depends on the coordinate x but is performed in the plane normal to the shell axis. In this case the equation of flow continuity for a liquid layer will be, in cylindrical coordinates,

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0, \qquad (2)$$

where r is the distance from the given point to the shell axis; $\Phi(r, \varphi\xi)$ is the velocity potential.

The normal components of the velocity of the liquid and shell must coincide at the liquid-shell boundary:

$$\frac{\partial w}{\partial t} = \frac{\partial \Phi}{\partial r}\bigg|_{r=R} \tag{3}$$

Since the shell displacements are periodic relative to φ, the velocity potential can be represented in the form

$$\Phi(r, \varphi, \xi) = \Phi_1(r) \cos m\varphi \sin \lambda\xi, \tag{4}$$

where

$$\Phi_1(r) = \text{const} \cdot r^m.$$

Expression (4) satisfies equation (2) and the boundary condition (3) if

$$\text{const} = \frac{\dot{K}}{mR^{m-1}}.$$

The dot designates differentiation with respect to time. It follows that the expression for the velocity potential will be

$$\Phi(r, \varphi, \xi) = \frac{\dot{K}}{mR^{m-1}} r^m \cos m\varphi \sin \lambda\xi. \tag{5}$$

The natural frequency of the liquid-filled shell will be determined by the Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{Q}}\right) + \frac{\partial \Pi}{\partial Q} = 0, \tag{6}$$

where T is the kinetic energy of the shell and liquid; Π is the potential energy of the shell; t is time.

The kinetic energy of the shell is equal to

$$T_{\text{sh}} = \frac{\mu}{2} \iint_{(s)} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] ds,$$

where s is the area of the middle surface and μ is the mass of the shell per unit area of the middle surface. We obtain after integration

$$T_{\text{sh}} = \frac{\mu\pi R L}{4} (\dot{M}^2 + \dot{N}^2 + \dot{K}^2). \tag{7}$$

To determine the kinetic energy of the liquid we shall make use of the fact that in the case of vortex-free motion of the liquid in a simply connected region the kinetic energy of the entire liquid depends on its motion at the boundaries /4/

$$T_1 = \frac{\rho_1}{2} \iint_{(s)} \left(\Phi \cdot \frac{\partial \Phi}{\partial r}\right)_{r=R} \cdot ds = \frac{\rho_1 \pi L R^2 K^2}{4m}, \tag{8}$$

where ρ_1 is the mass density of the liquid. The total kinetic energy will then be

$$T = T_{sh} + T_1 = \frac{\pi R L}{4}\left[\mu\,(\dot{M}^2 + \dot{N}^2 + \dot{K}^2) + \frac{\rho_1 R K^2}{m}\right]. \tag{9}$$

The potential energy of the shell is determined from the expression /5/

$$\Pi = \frac{E\delta}{2(1-\nu^2)}\iint\limits_{(S)}\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\nu)\left(\varepsilon_1\varepsilon_2 - \frac{\omega_1^2}{4}\right)\right]ds +$$

$$+\frac{E\delta^3}{24(1-\nu^2)}\iint\limits_{(S)}\left[(\varkappa_1+\varkappa_2)^2 - 2(1-\nu)(\varkappa_1\varkappa_2 - \tau^2)\right]ds. \tag{10}$$

where ε_1, ε_2 are relative elongations; ω_1 is shear; $\varkappa_1$ and $\varkappa_2$ are variations of the curvature; τ is torsion; δ is shell thickness; E is Young's modulus; ν is the Poisson ratio.

By replacing in (10) the strains by displacements according to /5/, and by assuming that the vibrations are harmonic

$$M = M' \sin \omega_1 t,$$
$$N = N' \sin \omega_1 t,$$
$$K = K' \sin \omega_1 t,$$

we obtain equation (6) in the form

$$\left.\begin{aligned}
&M'\left(\lambda^2 R + \frac{d}{4R}m^2 - \frac{\mu R}{B}\omega_1^2\right) - N'\frac{1+\nu}{2}\lambda m - K'\nu\lambda = 0,\\
&-M'\frac{1+\nu}{2}\lambda m + N'\left[\frac{m^2}{R} + \frac{d}{4}\lambda^2 R - \frac{\mu R}{B}\omega_1^2 + \frac{\delta^2}{12}\left(\frac{m^2}{R^3} + \frac{d\lambda^2}{R}\right)\right] +\\
&\qquad + K'\left[\frac{m}{R} + \frac{\delta^2}{12}\left(\frac{\lambda^2 m}{R} + \frac{m^3}{R^3} + \frac{d\lambda^2 m}{2R}\right)\right] = 0.\\
&-M'\nu\lambda + N'\left[\frac{m}{R} + \frac{\delta^2}{12}\left(\frac{\lambda^2 m}{R} + \frac{m^3}{R^3} + \frac{d\lambda^2 m}{2R}\right)\right] +\\
&+K'\left[\frac{1}{R} + \frac{\delta^2}{12}\left(\lambda^4 R + \frac{m^4}{R^3} + \frac{2\lambda^2 m^2}{R}\right) - \frac{\mu R}{B}\omega_1^2 - \frac{\rho_1 R^2\omega_1^2}{mB}\right] = 0,
\end{aligned}\right\} \tag{11}$$

where $B = \frac{E\delta}{1-\nu^2}$; $d = 2(1-\nu)$.

Since system (11) must have a nontrivial solution, its determinant will be equal to zero. This yields a third-degree equation relative to ω_1^2, from which three values of the frequencies are obtained for given m and n. To determine the frequencies of the principally normal vibration one can neglect higher powers of $\frac{\mu R}{B}\omega_1^2$ /6/; one obtains then after some transformations

$$\omega_1^2 = -\frac{\omega_0^2}{1 + \frac{Rm}{m^2+1}\cdot\frac{\rho_1}{\mu}}, \tag{12}$$

where $\omega_0^2 = \frac{B}{\mu R^2}\,\frac{(1-\nu^2)\lambda^4 + k(\lambda^2+m^2)^4}{m^2 + (\lambda^2+m^2)^2}$ is the natural frequency of the shell in

vacuum: $k = \frac{\delta^2}{12R^2}$.

If the liquid does not completely fill the shell ($h < L$), we assume that the form of vibrations of the shell is determined by relationships (1). The expression for the velocity potential remains the same as before, and the kinetic energy of the liquid will be

$$T_1 = \frac{\rho_1}{2}\int_0^{2\pi}\int_0^h \left(\Phi\frac{\partial\Phi}{\partial r}\right)_{r=R} R\,d\varphi\,dx = \frac{\pi\rho_1 R^2\dot{K}^2}{4m}\int_0^h \sin^2\frac{n\pi x}{L}\,dx.$$

The kinetic and potential energies of the shell will be the same as in the case of complete filling of the shell. We obtain from the Lagrange equations the frequency determinant, and the following formula for the frequency of the mainly normal vibrations:

$$\omega_1^2 = \frac{\omega_0^2}{1 + 2\frac{Rm}{m^2+1}\frac{\rho_1}{\mu L}\int_0^h \sin^2\frac{n\pi x}{L}\,dx} \tag{13}$$

If a normal pressure acts in addition on the liquid-filled shell, the natural frequency will increase. According to /7/, the natural frequency of an empty shell subjected to the action of a normal pressure ΔP is

$$\omega_e^2 = \omega_0^2 + \frac{B}{\mu R^2}\frac{\psi_1(\lambda^2+m^2)^2(m^2-1)}{m^2+(\lambda^2+m^2)^2}. \tag{14}$$

where

$$\psi_1 = \frac{\Delta P\cdot R(1-\nu^2)}{E\delta}.$$

It follows that the frequency of vibrations of a shell filled with liquid up to a height h and subjected to the action of a normal pressure will be given by the formula

$$\omega_{1,p}^2 = \frac{\omega_e^2}{1 + 2\frac{\rho_1}{\mu}\frac{Rm}{m^2+1}\int_0^h \sin^2\frac{n\pi x}{L}\,dx}.$$

If the shell has flexible ribs and is subjected to both a normal pressure and an axial force, the natural frequency will be given by the formula

$$\omega_{p.r}^2 = \omega_r^2 + \frac{B}{\mu R^2}\frac{\psi_1(\lambda^2+m^2)^2(m^2-1) + \psi_2\lambda^2(\lambda^2+m^2)^2}{m^2+(\lambda^2+m^2)^2}.$$

where $\psi_2 = \frac{N_x(1-\nu^2)}{E\delta}$; N_x is the longitudinal force and ω_r is the natural frequency of the ribbed shell (without load).

The following approximate formula can be used if there are only transverse ribs:

$$\omega_r^2 = \frac{1}{\mu R^2}\left[\frac{\lambda^4 E\delta + \frac{i+Ds}{R^2}(\lambda^2+m^2)^4}{m^2+(\lambda^2+m^2)^2}\right] \tag{15}$$

Here $i=\frac{EI}{l_0}$, I is the moment of inertia of the cross section of the rib with the shell portion attached to it relative to the axis passing through the center of gravity of this element and parallel to the shell axis; l_0 is the distance between ribs; $s=1-\frac{l_1}{l_0}$; l_1 is the length of the connection between the rib and the shell.

If the loaded ribbed shell is filled with liquid, the natural frequency will be determined by the formula

$$\omega^2_{\text{p.r.l}} = \frac{\omega^2_{\text{p.r}}}{1+2\frac{R\cdot m}{m^2+1}\frac{\rho_1}{\mu L}\int\limits_0^h \sin^2\frac{n\pi x}{L}dx}, \tag{16}$$

where h is the liquid level in the shell, and μ is the mass of the ribbed shell per unit middle surface area.

The formulas obtained were tested by means of an empirical determination of the natural frequencies of a cylindrical steel shell filled with water. The shell was placed vertically, and its dimensions were L = 97 cm, R = 12.5 cm, δ = 0.12 cm; E was taken as equal to 2×10^6 kg/cm^2, and ν as equal to 0.3. The scheme of the experimental installation and the method of the experiment are described in /6/ and /7/. The results of the experimental determination of the frequencies of the full f_1^{ex} and empty f_e^{ex} shell are represented in Table 1 together with the corresponding values calculated by (12). The experimental value of the frequencies in cycles/sec was determined from oscillograms.

TABLE 1

m	f_e^{ex}	f_1^{ex}	f_1^{calc}	$\left\lvert\frac{f_1^{ex}-f_1^{calc}}{f_1^{ex}}\cdot100\right\rvert$
2	270	114	110	3.5
3	220	—	—	—
4	300	150	153	2.0
5	430	2 0	248	3.3
6	660	390	385	1.3
7	920	540	552	2.2

It is seen from Table 1 that the values of the frequencies calculated by (12) are quite near to the experimental values. Relationship (12) can therefore be used for calculating the influence of the liquid on the frequency of vibrations. The frequencies were determined in the case of partial filling as well. A considerable discrepancy was established in this case between the empirical and calculated (by (13)) values. The values obtained for $h=\frac{3}{4}L$ are given in Table 2.

An experimental determination of the frequency of vibrations was also realized in the case of action of a normal pressure on a liquid-filled shell. The results obtained at complete filling and an excess pressure ΔP = = 2 kg/cm^2 are represented in Table 3, when f^{calc} is the calculated value of the frequency (cycles/sec), and $f_{l.p}^{ex}$ is the experimental value of the frequency (cycles/sec).

TABLE 2

m	f_e^{ex}	f_1^{ex}	f_1^{calc}	$\left\lvert \frac{f_1^{ex}-f_1^{calc}}{f_1^{ex}} \cdot 100 \right\rvert$
2	270	120	114	5.0
3	220	105	104	0.95
4	300	160	156	2.5
5	460	240	257	7.1
7	920	610	580	4.95

TABLE 3

m	$f_{1.p}^{calc}$	$f_{1.p}^{ex}$	$\left\lvert \frac{f_{1.p}^{calc}-f_{1.p}^{ex}}{f_{1.p}^{ex}} \cdot 100 \right\rvert$
2	122	125	2.5
4	207	200	3.5
5	312	310	0.65
7	635	610	4.1

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PRINCIPLES OF THE STABILITY CALCULATIONS OF THIN SHELLS

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The mast wrecks which have occurred in recent years have raised doubts as to the adequacy of the stability calculations relative to compressed cylindrical shells. Structural elements such as bars and plates are calculated allowing for the possible imperfections and plastic properties of the material, and only the shells are usually considered as perfectly elastic bodies of ideal shape.

An attempt is made in the present report to approximate the influence of initial curvatures and elastic-plastic behavior of the material on the critical load of a circular cylindrical shell.

An initial curvature alters the phenomenon of buckling not only quantitatively, but qualitatively as well. This can be seen from the curves given in Figure 1 of the load parameter $\varkappa$ as a function of the displacement w^* normal to the middle surface, and in particular from the values of the characteristic points where $\varkappa\ (w^*)$ is a single-valued function.

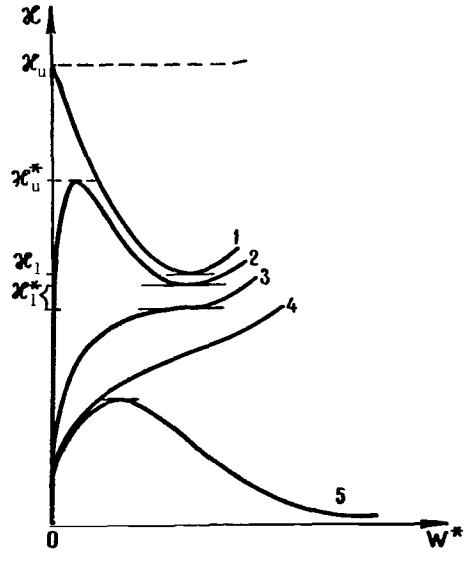

FIGURE 1.

Curve 1 corresponds to an ideal elastic shell (without initial curvature). There are two characteristic points here: the upper critical value of the load parameter $\varkappa_u$, the bifurcation point (the linearized solution is shown in dotted line), and the lower critical value $\varkappa_l$, the minimum of the function $\varkappa(w^*)$. The descending branch of the curve between $\varkappa_u$ and $\varkappa_l$ corresponds to the unstable state of equilibrium, and the ascending branch to the stable state. Curve 2 corresponds to a very slender shell* with an initial curvature so small that the deformation takes place within the elastic limits. The qualitative change is expressed by the appearance of displacements from the very beginning of the loading. There is no bifurcation and the upper critical value of $\varkappa$ coincides with the maximum $\varkappa(w^*)$. Both ascending branches of the curve represent stable states of equilibrium, and the descending branch represents the unstable states. When the energy barrier is overcome in the range $\varkappa_l^* \leqslant \varkappa < \varkappa_u^*$ a discontinuous transition to a new stable equilibrium state becomes possible; this is the case of elastic buckling.

The lower critical value can be defined in a general manner as the minimum stationary value (provided there is at least one more stationary value). This definition is also applicable in the case of a larger curvature (curve 3), when the maximum and minimum coalesce in the inflection point. With the further increase of the initial curvature in a perfectly elastic shell the function $\varkappa(w^*)$ becomes monotonically increasing (curve 4).

If the shell material possesses plastic properties, the relationships represented by curves 3, 4 are not usually realized. A substantial development of the plastic deformations leads to a qualitative change in the buckling (curve 5). After the maximum $\varkappa$ is attained the stable equilibrium is not restored, and the branch of unstable equilibrium states approaches the abscissa asymptotically. This phenomenon, similar to the process of buckling of a compressed-bent bar, will be called buckling of the second kind, or "plastic buckling." In the absence of the second branch of stable states, the elastic buckling is impossible and the critical state coincides with the attainment of the load maximum. Curve 5 is typical for shells of large and medium slenderness at considerable initial curvatures or for shells of low slenderness at any initial curvatures.

The characteristic schemes of shell behavior considered above do not exhaust all possible cases. The rigorous plotting of the relationship in the general case would have necessitated to take into account both the geometrical and the physical nonlinearity of the initial relationships. Such a general problem is not treated here; it is replaced by the following two particular problems:

1) an assessment of the influence of a sufficiently small initial curvature on the lower critical load of the elastic shell;

2) the determination of the critical load in the case of plastic buckling of a shell with initial curvature.

Only the geometrical nonlinearity is allowed for, by definition, in the first problem. The second problem is treated in a geometrically linear formulation, but allowing for the physical nonlinearity (the diagram of the elastic-plastic material is used). The acceptability of the geometrical linearization in the determination of the critical load is proved.

* The classification of the shells according to their slenderness ratio is given at the end.

1. ESTIMATES OF THE LOWER CRITICAL LOAD OF AN ELASTIC SHELL

Unlike the papers of Donnell and Wan /1/, and Loo Tsu-Tao /2/, in which the solution is found by the Ritz method, minimizing by two parameters, we obtain here approximate estimates by forming majorants (minorants) of the solution of the related system of nonlinear equations.

Consider a circular cylindrical shell of radius R and wall thickness t subjected to axial compression and torsion by forces applied to the middle surface at ends $x=0,\ l$. In the case of small initial curvature* $w_0\ (x,\ y)$ the shell equilibrium is described by the nonlinear equations

$$\left.\begin{aligned}(D/t)\ \Delta\Delta w &= \Phi\ (\varphi,\ w + w_0 + y^2/2R),\\ \Delta\Delta\varphi &= -\ E\,\Phi(w,\ w/2 + w_0 + y^2/2R),\end{aligned}\right\} \tag{1.1}$$

where $x,\ y$ are coordinates by the generatrix and the directrix; w is the radial displacement; $\Phi\ (f,\ g) \equiv f_{xx}\, g_{yy} + f_{yy}\, g_{xx} - 2f_{xy}\, g_{xy}$; φ is a stress function; E is the modulus of elasticity; D is the cylindrical rigidity.

The boundary conditions at $x=0,\ l$ are

$$L_i\ [w] = 0, \quad i = 1,\ 2, \tag{1.2}$$

$$(1/2\pi R)\int_0^{2\pi R} \varphi_{yy}\, dy = -\varkappa\sigma, \quad (1/2\,\pi R)\int_0^{2\pi R} \varphi_{xy}\, dy = -\varkappa\tau. \tag{1.3}$$

Here L_i is a linear operator; $\sigma > 0$ and τ are constants of dimension stress.

Consider the system

$$\left.\begin{aligned}(D/t)\ \Delta\Delta\, w &= \Phi\ (\varphi,\ (1+\beta)\ w + y^2/2R),\\ \Delta\Delta\,\varphi &= -\ E\Phi\ (w,\ (1+2\beta)\ w/2 + y^2/2R)\end{aligned}\right\} \tag{1.4}$$

with boundary conditions (1.2), (1.3). Here $\beta > 0$ is a parameter. Let a lower critical load, for which $\varkappa = \varkappa_\beta,\ w = w_\beta,\ \varphi = \varphi_\beta$ exist at sufficiently small values of β. By writing $w_0 = \beta w_\beta$ we can see that the same solution satisfies equations (1.1), although it cannot be asserted that it assumes a stationary value of $\varkappa$. This last can be used only as an assumption suitable at sufficiently low values of β.

Consider the problem of obtaining a two-sided estimate for $\varkappa_\beta$ if the lower critical value $\varkappa_0$ for an ideal shell is known. Substitute in (1.4)

$$w = cw^*, \quad \varphi = k\varphi^*, \quad \varkappa = k\varkappa^*, \quad R = R^* c/k, \tag{1.5}$$

where

$$c = [(1+\beta)\ (1+2\,\beta)]^{-1/2}, \qquad k = (1+\beta)^{-1}$$

We obtain

$$\left.\begin{aligned}(D/t)\ \Delta\Delta\ w^* &= \Phi\ [\varphi^*,\ w^* + (y^2/2R^*)\ (k/c)^2],\\ \Delta\Delta\ \varphi^* &= -\ E\Phi\ (w^*,\ w^*/2 + y^2/2R^*).\end{aligned}\right\} \tag{1.6}$$

The boundary conditions (1.2) and (1.3) are satisfied.

* Meaning that w_0 is small compared with R, the initial curvature can be considerable compared with the wall thickness.

Equations (1.6) differ from the equations for an ideal shell only in the term $R^* (c/k) = R^* (1+\beta)/(1+2\beta)$, which appears in the first of them instead of the larger magnitude R^*. The increase in the radius decreases the critical load, and therefore by taking in (1.6) $c/k = 1$ we obtain a lower estimate.

Let the lower critical value for an ideal shell with a ratio R^*/t be equal to $\varkappa_0 = f(t/R^*)$. By using relationships $\varkappa_\beta > \varkappa_0$ and $R^* = R\,k\,c$ we obtain

$$\varkappa_\beta > k\,f\,(ct/kR).$$

In particular, in the presence of compression only $f(t/R^*) = at/R^*$, whence

$$\psi = \varkappa_\beta/\varkappa_0 > [(1+\beta)(1+2\beta)]^{-1/2}. \tag{1.7}$$

Here $\varkappa_0$ is the lower critical value for an ideal shell with ratio R/t.

In a similar way one can obtain the upper estimate

$$\varkappa_\beta < k\,f(kt/cR)$$

and in the case of axial compression

$$\psi = \varkappa_\beta/\varkappa_0 < (1+2\beta)^{1/2}(1+\beta)^{-1/2}, \tag{1.8}$$

or approximately

$$\psi \approx k = (1+\beta)^{-1}. \tag{1.9}$$

If the values of β are sufficiently small, the total deflective $w_d = (1+\beta)w_\beta$ can be equated approximately to the amplitude of the dent of an ideal shell, equal to $\sim 5t$ at $R/t = 180$ (cf. /3/, p. 330). Let the amplitude of the initial curvature be equal to a_0, then

$$\psi \approx 1 - 0.2a_0/t. \tag{1.10}$$

Formula (1.10) shows that the initial curvature lowers the lower critical load relatively little. It should be remembered, however, that the estimates obtained are true only at low values of β, so that formula (1.10) can be used only at $a_0/t < 1$.

2. PLASTIC BUCKLING (BUCKLING OF THE SECOND KIND)

We consider here buckling at axial compression of elastic-plastic shells of low and medium slenderness; in other words, we have to determine the maximum of the function $\varkappa(w^*)$. The problem is considered in a geometrically linear formulation. The initial curvature is assumed to be axially symmetrical and similar to the form of the buckling of an ideal shell. The approximate values of the critical stress are found under two assumptions: 1) the modulus of elasticity in the circumferential direction remains constant; 2) this modulus decreases with the passage of the meridional stresses beyond the elastic limit.

Let an infinitely long circular cylindrical shell of radius R and wall thickness t (Figure 2) have an initial curvature

$$w_0 = a_0 \sin(\pi x/l),$$

where x is measured along the cylinder axis and l is a length whose value is established below. The axial stress is distributed uniformly along a circle of radius R.

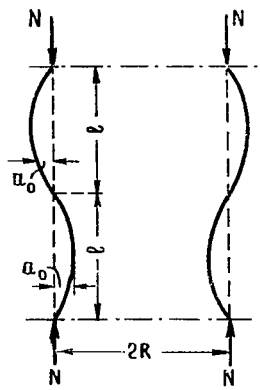

FIGURE 2.

The hypothesis of conservation of normals is used. The relationship between the relative elongation in the meridional direction ε and the corresponding normal stress σ is assumed to be the same as in the case of monoaxial extension:

$$\sigma = F(\varepsilon), \quad \varepsilon = \varepsilon_0 - w'' z, \tag{2.1}$$

where ε_0 is deformation at the origin of the coordinate z, measured along the normal to the deformed middle surface; w is radial displacement measured from the initial (curved) middle surface; the primes stand for differentiation by x.

The deformation is assumed to be everywhere active.

Assume that the elastic properties of the material in the circumferential direction do not vary with the passage of σ beyond the proportional limit. Isolate a strip of width 1 by two meridional sections; we can write for it the equation of equilibrium as for a compressed-bent beam on an elastic foundation of foundation modulus Et/R^2 (E is the modulus of elasticity)

$$-\left(\int_{z_1}^{z_2} \sigma z dz\right)'' + N(w + w_0)'' + Etw/R^2 = 0, \tag{2.2}$$

$$\int_{z_1}^{z_2} \sigma \, dz + N = 0. \tag{2.3}$$

Here N is the absolute value of the compressive force per unit strip width; z_1, z_2 are coordinates of the edge fibers of the section.

By using the method of variation of the equations of equilibrium by w and ε at a stationary value of N /4/ and origin of z fixed in each section we obtain

$$-\left[\int_{z_1}^{z_2} (d\sigma/d\varepsilon)(\delta\varepsilon_0 - z\delta w'') z dz\right]'' + N\delta w'' + Et\delta w/R^2 = 0, \tag{2.4}$$

$$\int_{z_1}^{z_2} (d\sigma/d\varepsilon)\ (\delta\varepsilon_\theta - z\delta w'')\ dz = 0. \tag{2.5}$$

$d\sigma/d\varepsilon$ is a function of x, z, ε_0, w:

$$d\sigma/d\varepsilon = \varphi\ (x, z, \varepsilon_0, w),$$

while $\delta\varepsilon_0$ and δw are functions of x alone. For any fixed x one can select the origin of z so as to satisfy the equation

$$\int_{z_1}^{z_2} \varphi z\, dz = 0. \tag{2.6}$$

Equation (2.4) reduces then to

$$(D_1\ \delta w'')'' + N\delta w'' + Et\delta w/R^2 = 0, \tag{2.7}$$

where

$$D_1 = \int_{z_1}^{z_2} (d\sigma/d\varepsilon)\ z^2\, dz = \int_{z_1}^{z_2} [E_t/(1-\nu^2)]\ z^2\, dz, \tag{2.8}$$

E_t is the tangent modulus, ν is the Poisson ratio; D_1 is obviously the reduced rigidity of the strip section, depending on x and w. Equation (2.6) shows that the origin of z must coincide with the center of gravity of the reduced section.

An approximate solution can be found by using a constant value of D_1 calculated from the given stress-strain diagram in section $x = 0.5\,l$, and by restricting ourselves to satisfying the equilibrium conditions (2.2), (2.3) in this section. The result is one of the following two formulas, depending on whether a yield zone is present on one or on the two sides:

$$\left.\begin{aligned} 0.5\ [1 - (\sigma_0/\sigma_e)^{2/3}]\,(\sigma_T/\sigma_0 - 1) &= a_0/t, \\ (\sigma_T/4\sigma_0)\ [1 - (\sigma_0/\sigma_T)^2 - (\sigma/\sigma_e)^{4/3}] &= a_0/t. \end{aligned}\right\} \tag{2.9}$$

Here $\sigma_0 = N/t$, σ_T is the yield stress, and $\sigma_e = Et/R\ \sqrt{3\,(1-\nu^2)}$. The criterion which establishes which of the two formulas is to be used is the inequality

$$\sigma_0/\sigma_T + (\sigma_0/\sigma_e)^{2/3} \gtrless 1, \tag{2.10}$$

in which the upper sign corresponds to the case of one-sided yield. Equations (2.9) were obtained for $l = l_0 = \pi\sqrt{Rt}\ \sqrt{\sigma_0/\sigma_c}\ [12(1-\nu^2)]^{-1/4}$. If we take $l = l_* > l_0$, the value of a_0/t found from (2.9) must be increased. Designating by a_0^* the true value of the initial amplitude, we obtain

$$a_0^*/t = (a_0/t)\{1 - \sin\,[0.5\,\pi\ (1 - l_0/l_*)]\}^{-1}.$$

The method described gives an upper estimate of the critical stress.

Consider now the second variant. Assume that the modulus decreases in the circumferential direction in such a way that the reaction created by the circumferential stresses remains the same as at the moment of appearance of yield in the meridional direction. The solution is similar to Ježek's solution /5/ for a bar of rectangular section. The results corresponding to $l = \pi\sqrt{Rt}\ [12\ (1-\nu^2)]^{-1/4}$ are given below:

$$\left.\begin{aligned} 0.5\ [1 - (\sigma_0/\sigma_e)^{1/3}]\ (\sigma_T/\sigma_0 - 1) &= a_0/2t, \\ (\sigma_T/4\sigma_0)\ \ [1 - (\sigma_0/\sigma_T)^2 - (\sigma_0/\sigma_e)^{2/3}] &= a_0/2t. \end{aligned}\right\} \tag{2.11}$$

The first equation corresponds to the case of one-sided yield, and the second to the case of two-sided yield. If an arbitrary value l_* is used instead of the value of l indicated above, the magnitude σ_e in (2.11) is to be multiplied by $0.5[(l/l_*)^2 + (l_*/l)^2]$ and the magnitude a_0 by $2[1 + (l_*/l)^4]^{-1}$.

We shall now establish the criterion of elastic or inelastic buckling. The following value of the shell slenderness is adopted as criterion:

$$(R/t)^* = 0.18\, E/\sigma_T\,. \tag{2.12}$$

If the initial curvature satisfies the conditions $a_0/t < 1$, $l_*/l \geqslant \sqrt{2}$, it can be shown that at $R/t > 2\,(R/t)^*$ the buckling will be elastic. Such shells will be called "very slender." On the other hand, at $R/t < (R/t)^*$ plastic buckling will predominate. It is difficult to establish a clear-cut boundary between the two regions, since the effect of the transition beyond the elastic limit on the lower critical stress has not been studied so far. The value $(2/3)\,(R/t)^*$ can be adopted on the basis of experimental data as such boundary. In the range $(2/3)\,(R/t)^* < R/t < 2\,(R/t)^*$ (shells of medium slenderness) both buckling modes are possible, depending on the value of R/t and the character of the initial curvature.

It should be stressed that the boundary above which stability is lost by elastic buckling is a function of the value of a_0/t; thus, at $a_0/t = 2$ this boundary is $3(R/t)^*$.

It follows from the above that in practical calculations it is not sufficient to calculate the lower critical stress of an elastic shell. It is also necessary to find the critical stress σ_0 corresponding to the second mode of buckling of an elastic-plastic shell.

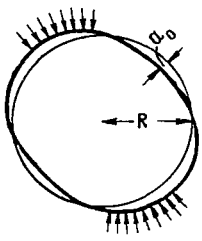

FIGURE 3.

The influence of the initial curvature on the lower critical stress of the shell was ignored in the determination of $(R/t)^*$. If this influence is taken into account and we write $(R/t)^* = \alpha\, E/\sigma_T$ $(\alpha < 0.18)$, the upper boundary of the zone of plastic buckling is shifted to the left.

All this is true for loads for which the upper and lower critical stresses of an ideal elastic shell differ substantially in magnitude. If the difference is small, it is sufficient to determine the upper critical stress, allowing for the initial curvature. Consider as an example a cylindrical shell subjected to an external uniform pressure normal to the middle surface (Figure 3). The results for an infinitely long shell having an initial curvature $w_0 = a_0 \sin 2\theta$ (θ is the central angle) similar to the mode of buckling of an ideal shell, are given below. The following alterations are introduced in equations (2.11): σ_0 is understood to be the critical circumferential stress; $a_0/2$ is replaced by a_0; $\sigma_e = Et^2/[4R^2(1-\nu^2)]$. The influence of the initial curvature decreases with the increase of R/t. Thus, at $R/t = 30$ and $a_0/t = 1$ the critical load is 60% lower than in the case of an ideal shell; at $R/t = 50$ and $a/t \leqslant 1$ the difference is of 40% only.

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BENDING OF SANDWICH PLATES ALLOWING FOR TEMPERATURE STRESSES

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(Novosibirsk)

The paper deals with the static derivation of the linear equations of longitudinal-transverse bending of nonsymmetrical sandwich plates with rigid filler allowing for a nonuniform heating $t = f(x, y, z)$.

The three layers of the plate are made of different isotropic materials. The moduli of elasticity of the faces E_1, E_2 are assumed to be constant. The moduli of elasticity of the filler E, G vary through the thickness.

It is assumed that the Poisson ratios are constant and equal to each other ($\mu_1 = \mu_2 = \mu$). The broken-line hypothesis is used for the distribution of stresses (u, v) through the plate thickness.

The reference surface was selected at a distance from the middle surface such that $B_k = 0$; the equations obtained as a result are simpler than those which follow from /1/, where the reference surface was selected equidistant from the faces.

The following relationships between the displacements and the stresses in the filler exist under the hypotheses made:

$$
\begin{gathered}
u = \left(\bar{u} + \frac{\delta}{2h}\frac{\partial w}{\partial x}\right) z + \left(\bar{\eta} + \frac{h_1\delta_2 - h_2\delta_1}{4h}\frac{\partial w}{\partial x}\right); \\
v = \left(\bar{v} + \frac{\delta}{2h}\frac{\partial w}{\partial y}\right) z + \left(\bar{\xi} + \frac{h_1\delta_2 - h_2\delta_1}{4h}\frac{\partial w}{\partial y}\right); \\
\sigma_x = E_\Gamma \left[\frac{\partial u}{\partial x} + \mu\frac{dv}{\partial y} - \alpha(1+\mu)t\right]; \\
\tau_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right); \qquad \tau_{xy} = \frac{1-\mu}{2}E_\Gamma\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{gathered}
\tag{1}
$$

etc.

For the faces

$$
\begin{gathered}
u_{1,2} = \bar{\eta} \mp \bar{u}h_{1,2} - (z \pm H_{1,2})\frac{\partial w}{\partial x}; \qquad v_{1,2} = \bar{\xi} \mp \bar{v}h_{1,2} - (z \pm H_{1,2})\frac{\partial w}{\partial y}; \\
\sigma_{x1,2} = E_{\Gamma 1,2}\left[\frac{\partial u_{1,2}}{\partial x} + \mu_{1,2}\frac{\partial v_{1,2}}{\partial y} - \alpha_{1,2}(1+\mu_{1,2})t\right]; \\
\tau_{xy1,2} = \frac{1-\mu_{1,2}}{2}E_{\Gamma 1,2}\left(\frac{\partial u_{1,2}}{\partial y} + \frac{\partial v_{1,2}}{\partial x}\right);
\end{gathered}
\tag{2}
$$

$$
M_{y1,2} = -D_{1,2}\left(\frac{\partial^2 w}{\partial x^2} + \mu_{1,2}\frac{\partial^2 w}{\partial y^2}\right) - (1+\mu_{1,2})\alpha_{1,2}(D_{t1,2} \pm H_{1,2}B_{t1,2}),
$$

etc.

The forces and moments in the plate are

$$N_{x\mathrm{pl}} = \left(B_k + \frac{\delta_1 B_1 - \delta_2 B_2}{2}\right)\left(\frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y}\right) + B_{\mathrm{pl}}\left(\frac{\partial \bar{\eta}}{\partial x} + \mu \frac{\partial \bar{\xi}}{\partial y}\right) +$$

$$+ \frac{1}{2h}\left(\delta B^{xx} + \frac{\delta_2 h_1 - \delta_1 h_2}{2} B^x\right)\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial w}{\partial y^2}\right) - B_t;$$

$$N_{xy\mathrm{pl}} = \frac{1-\mu}{2}\left[\left(B_k + \frac{\delta_1 B_1 - \delta_2 B_2}{2}\right)\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) + B_{\mathrm{pl}}\left(\frac{\partial \bar{\eta}}{\partial y} + \frac{\partial \bar{\xi}}{\partial x}\right) +\right.$$

$$\left. + \frac{1}{h}\left(\delta B^{xx} + \frac{\delta_2 h_1 - \delta_1 h_2}{2} B^x\right)\frac{\partial^2 w}{\partial x \partial y}\right];$$

$$Q_{x\mathrm{pl}} = 2hG^x\left(\bar{u} + \frac{H}{h}\right)\frac{\partial w}{\partial x} + \frac{\delta_1 h_1 B_1 + \delta_2 h_2 B_2}{2}\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 \bar{u}}{\partial y^2} +\right.$$

$$\left. + \frac{1+\mu}{2}\frac{\partial^2 \bar{v}}{\partial x \partial y}\right) - \frac{\delta_1 B_1 - \delta_2 B_2}{2}\left(\frac{\partial^2 \bar{\eta}}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 \bar{\eta}}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2 \bar{\xi}}{\partial x \partial y}\right) -$$

$$- 2\bar{D}\frac{\partial}{\partial x}\nabla^2 w - \frac{\partial}{\partial x} d_t; \qquad (3)$$

$$M_{y\,\mathrm{pl}} = D_{\mathrm{pl}}\left(\frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y}\right) + B_k\left(\frac{\partial \bar{\eta}}{\partial x} + \frac{\partial \bar{\xi}}{\partial y}\right) +$$

$$+ \left[\frac{1}{2h}\left(\delta D^x + \frac{\delta_2 h_1 - \delta_1 h_2}{2} B^{xx}\right) - 2\bar{D}\right]\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) - D_t;$$

$$H_{x\,pl} = H_{y\mathrm{pl}} = \frac{1-\mu}{2}\left\{D_{\mathrm{pl}}\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) + B_k\left(\frac{\partial \bar{\eta}}{\partial y} + \frac{\partial \bar{\xi}}{\partial x}\right) +\right.$$

$$\left. + 2\left[\frac{1}{2h}\left(\delta D^x + \frac{\delta_2 h_1 - \delta_1 h_2}{2} B^{xx}\right) - 2\bar{D}\right]\frac{\partial^2 w}{\partial x \partial y}\right\},$$

etc., where

$$\bar{u} = \frac{\bar{u}_2 - \bar{u}_1}{2h}; \qquad \bar{v} = \frac{\bar{v}_2 - \bar{v}_1}{2h}; \qquad \bar{\eta} = \frac{\bar{u}_2 h_1 + \bar{u}_1 h_2}{2h}; \qquad \bar{\xi} = \frac{\bar{v}_2 h_1 + \bar{v}_1 h_2}{2h};$$

$$B_{\mathrm{pl}} = B^x + B_1 + B_2; \qquad d_{\mathrm{pl}} = D^x + B_1 h_1^2 + B_2 h_2^2 - \frac{(\delta_1 B_1 - \delta_2 B_2)^2}{4B_{\mathrm{pl}}};$$

$$D_{\mathrm{pl}} = D^x + B_1 h_1 H_1 + B_2 h_2 H_2; \qquad \bar{D}_{\mathrm{pl}} = 2\bar{D} + B_1 H_1\left(h + \frac{\delta}{2}\right)\frac{h_1}{h} +$$

$$+ B_2 H_2\left(h + \frac{\delta}{2}\right)\frac{h_2}{h} + D^x - \frac{(h_1\delta_2 - h_2\delta_1)(B_1 h_1 - B_2 h_2)}{4h};$$

$$B_k = B^{xx} - B_1 H_1 + B_2 H_2; \qquad \delta = \frac{\delta_1 + \delta_2}{2}, \qquad \bar{D} = \frac{D_1 + D_2}{2}; \qquad (4)$$

$$B_{1,2} = E_{\mathrm{r}\,1,2}\,\delta_{1,2}; \qquad D_{1,2} = \frac{E_{\mathrm{r}\,1,2}\,\delta_{1,2}^3}{12}; \qquad B^x = \int_{h_1}^{+h_2} E_{\mathrm{r}}\,dz;$$

$$D^x = \int_{-h_1}^{+h_2} z^2 E_{\mathrm{r}}\,dz; \qquad G^x = \frac{1}{2h}\int_{-h_1}^{+h_2} G\,dz;$$

$$B^{xx} = \int_{-h_1}^{+h_2} z E_{\mathrm{r}}\,dz; \qquad E_{\mathrm{r}\,1,2} = \frac{E_{1,2}}{1-\mu_{1,2}^2}; \qquad E_{\mathrm{r}} = \frac{E}{1-\mu^2};$$

$$B_t = (1+\mu)(B_t^x + \alpha_1 B_{t1} + \alpha_2 B_{t2}); \qquad D_t = (1+\mu)(D_t^x + \alpha_1 D_{t1} + \alpha_2 D_{t2});$$

$$d_t = (1+\mu)\,[\alpha_1 (h_1 B_{t1} + D_{t1}) - \alpha_2 (h_2 B_{t2} - D_{t2})];$$

$$d_t' = (1+\mu)\,[\alpha_1 (H_1 B_{t1} + D_{t1}) - \alpha_2 (H_2 B_{t2} - D_{t2})];$$

$$b_t = (1+\mu)\,(D_t^x - \alpha_1 h_1 B_{t1} + \alpha_2 h_2 B_{t2}); \qquad B_{t1} = E_{T1} \int_{-(h_1+\delta_1)}^{-h_1} t\,dz;$$

$$B_{t2} = E_{T2} \int_{h_2}^{h_2+\delta_2} t\,dz; \qquad B_t^x = \int_{h_1}^{+h_2} \alpha E_T\, t\,dz; \qquad D_{t1} = E_{T1} \int_{-(h_1+\delta_1)}^{-h_1} z t\,dz; \tag{4 (cont'd)}$$

$$D_{t2} = E_{T2} \int_{h_2}^{h_2+\delta_2} z t\,dz; \qquad D_t^x = \int_{-h_1}^{+h_2} \alpha E_T\, t z\,dz,$$

Here $u_{1,2}$, $v_{1,2}$ are displacements of the middle surfaces of the faces; α_1, α_2, α are coefficients of linear temperature expansion; H_1, H_2 are the distances between the reference surface ($z = 0$) and the middle surface of the faces (figure).

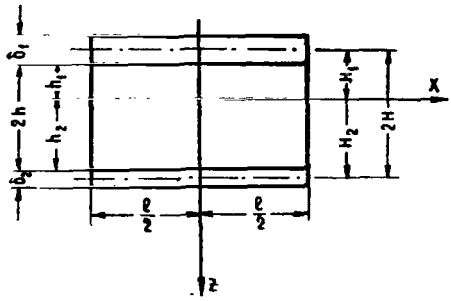

FIGURE.

The following five equations for $\bar{u}$, $\bar{v}$, $\bar{\eta}$, $\bar{\xi}$ ($\bar{u}_{1,2}$, $\bar{v}_{1,2}$) and w are obtained with the aid of the equations of equilibrium of a plate element and expressions (3):

$$\left(2\bar{D} - \frac{\delta D^x \psi}{2h}\right)\nabla^2\nabla^2 w - D_{pl}\left(\frac{\partial}{\partial x}\nabla^2\bar{u} + \frac{\partial}{\partial y}\nabla^2\bar{v}\right) -$$

$$- B_k\left(\frac{\partial}{\partial x}\nabla^2\bar{\eta} + \frac{\partial}{\partial y}\nabla^2\bar{\xi}\right) - 2N_{xy\,pl}\frac{\partial^2 w}{\partial x \partial y} -$$

$$- N_{x\,pl}\frac{\partial^2 w}{\partial x^2} - N_{y\,pl}\frac{\partial^2 w}{\partial y^2} = q - \nabla^2 D_t;$$

$$\frac{\delta D^x \theta}{2h}\frac{\partial}{\partial x}\nabla^2 w + d_{pl}'\left(\frac{\partial^2\bar{u}}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{u}}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2\bar{v}}{\partial x \partial y}\right) +$$

$$+ B_k\left(\frac{\partial^2\bar{\eta}}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{\eta}}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2\bar{\xi}}{\partial x \partial y}\right) - 2hG^x\left(\bar{u} + \frac{h+\frac{\delta}{2}}{h}\frac{\partial w}{\partial x}\right) = \tag{5}$$

$$= \frac{\partial}{\partial x}\left(b_t - \frac{B_t}{B_{pl}}\frac{\delta_1 B_1 - \delta_2 B_2}{2}\right);$$

$$\frac{\delta D^x \theta}{2h}\frac{\partial}{\partial y}\nabla^2 w + d_{pl}'\left(\frac{\partial^2\bar{v}}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{v}}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2\bar{u}}{\partial x \partial y}\right) +$$

$$+ B_k\left(\frac{\partial^2\bar{\xi}}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{\xi}}{\partial x^2}\ \frac{1+\mu}{2}\frac{\partial^2\bar{\xi}}{\partial x\partial y}\right) - 2hG^x\left(\bar{v} + \frac{h+\frac{\delta}{2}}{h}\frac{\partial w}{\partial y}\right) =$$

$$= \frac{\partial}{\partial y}\left(b_t - \frac{B_t}{B_{\text{pl}}}\frac{\delta_1 B_1 - \delta_2 B_2}{2}\right);$$

$$\left(B_k + \frac{\delta_1 B_1 - \delta_2 B_2}{2}\right)\left(\frac{\partial^2\bar{u}}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{u}}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2\bar{v}}{\partial x\partial y}\right) +$$

$$+ B_{\text{pl}}\left(\frac{\partial^2\bar{\eta}}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{\eta}}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2\bar{\xi}}{\partial x\partial y}\right) +$$

$$+ \frac{1}{2h}\left(\delta B^{xx} + \frac{\delta_2 h_1 - \delta_1 h_2}{2}B^x\right)\frac{\partial}{\partial x}\nabla^2 w = \frac{\partial}{\partial x}B_t; \qquad (5)\ \text{(cont'd)}$$

$$\left(B_k + \frac{\delta_1 B_1 - \delta_2 B_2}{2}\right)\left(\frac{\partial^2\bar{v}}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{v}}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2\bar{u}}{\partial x\partial y}\right) +$$

$$+ B_{\text{pl}}\left(\frac{\partial^2\bar{\xi}}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2\bar{\xi}}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2\bar{\eta}}{\partial x\partial y}\right) +$$

$$+ \frac{1}{2h}\left(\delta B^{xx} + \frac{\delta_2 h_1 - \delta_1 h_2}{2}B^x\right)\frac{\partial}{\partial y}\nabla^2 w = \frac{\partial}{\partial y}B_t,$$

where

$$\theta = \psi - \left(\delta B^{xx} + \frac{\delta_2 h_1 - \delta_1 h_2}{2}B^x\right)\frac{\delta_1 B_1 - \delta_2 B_2}{2B_{\text{pl}}D^x\delta};$$

$$\psi = 1 + \frac{\delta_2 h_1 - \delta_1 h_2}{2}\frac{B^{xx}}{\delta D^x}; \qquad d_{\text{pl}} = d_{\text{pl}} - \frac{\delta_1 B_1 - \delta_2 B_2}{2B_{\text{pl}}}B_k.$$

System (5) can be simplified if plane $z = 0$ is set at a distance from the faces such that $B_k = 0$ (cf. formulas (4)) instead of being set at the middle of the filler ($h_1 = h_2$).

If the temperature varies only along the plate thickness ($t = f(z)$), the temperature terms in equations (5) vanish. In this case (at $B_k = 0$) the first three equations coincide in form with the equations of bending of sandwich plates with faces of equal rigidity, so that the results given in /2/ can be used in many cases in their solution.

We shall give as an example the values of the deflections at $t = f(z)$ and the values of the critical forces at $t = 0$ for simply supported sandwich plates under conditions of cylindrical bending.

It should be noted that when the filler is assumed to be undeformable in the transverse direction the redistribution of stresses between the layers caused by the temperature gradient through the plate thickness does not have an influence on the critical loads for the plate. The deflections in the case of a uniformly distributed transverse load q are given by the formulas

$$[w]_{x=0} = \frac{5}{384}\frac{qe^4}{\bar{D}_{\text{pl}}}\left(1 + \frac{48K}{5\pi^2}C_1\right) + \frac{l^2}{8}\frac{D_t}{D_{\text{pl}}} - C_t;$$

$$T_{\text{cr}} = \frac{\pi^2}{l^2}\frac{\bar{D}_{\text{pl}} + C_2}{1+K}, \qquad (6)$$

where

$$C_1 = \left(1 - \frac{2hG^x}{k^2 d_{\text{pl}}}\right)\left[1 - \frac{8}{k^2 l^2}\left(1 - \operatorname{sech}\frac{kl}{2}\right)\right];$$

$$C_2 = \frac{\pi^2}{l^2 h G^x}\left(\bar{D}_\text{г}\, d_\text{pl} + \frac{\delta^2 D_\text{г}\, \bar{B}_\text{г}}{4}\right);$$

$$C_t = \frac{l^2 d_t}{16\bar{D} D_\text{pl}}\left(2\bar{D} + \frac{B_1 H_1 \delta_1 + B_2 H_2 \delta_2}{2}\right) + \frac{1 - \dfrac{2hG^x}{k^2 d_\text{pl}}}{1 + \dfrac{\bar{B}_\text{г}\, h\delta}{d_\text{pl}}}\left[\frac{D_t}{D_\text{pl}} - \right.$$

$$\left. - \frac{d_t'}{2\bar{D} D_\text{pl}}\left(2\bar{D} + \frac{B_1 H_1 \delta_1 + B_2 H_2 \delta_2}{2}\right) - \frac{d_t'}{2\bar{D}}\right]\left[\frac{1}{k^2}\left(1 - \operatorname{sech}\frac{kl}{2}\right) + \right.$$

$$\left. + \frac{\bar{B}_\text{г}\, h l^2}{4 d_\text{pl}}\, \frac{\dfrac{\delta}{2} + \dfrac{G^x}{k^2 \bar{B}_\text{г}}}{1 - \dfrac{2hG^x}{k^2 d_\text{pl}}}\right];$$

$$k = \sqrt{\frac{4\bar{D}_\text{pl} h G^x}{4\bar{D}_\text{г}\, d_\text{pl} + \delta^2 D_\text{г}\, \bar{B}_\text{г}}}; \qquad K = \frac{\pi^2 d_\text{pl}}{2hG^x l^2}; \qquad \varphi = \frac{(B_1\delta_1 - B_2\delta_2)^2}{B_\text{pl}};$$

$$\bar{D}_\text{г} = \bar{D} - \frac{\varphi\left(h + \dfrac{\delta}{2}\right)}{8h}; \qquad \bar{B}_\text{г} = \frac{1}{2h\delta}\left(B_1 h_1 \delta_1 + B_2 h_2 \delta_2 + \frac{1}{2}\varphi\right);$$

$$D_\text{г} = D^x - \frac{1}{2\delta}\left[\varphi\left(h + \frac{\delta}{2}\right) - (h_1\delta_2 - h_2\delta_1)(B_1 h_1 - B_2 h_2)\right].$$

These formulas are considerably simplified if the simplifications made in /3/ are used. In this case the values of the deflections and the critical loads coincide in form with expressions (6), with $C_1 = 1$; $C_2 = C_t = 0$.

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STABILITY OF STRUCTURALLY ORTHOTROPIC CYLINDRICAL PANELS UNDER THE ACTION OF SHEARING AND NORMAL FORCES AND INTERNAL PRESSURE

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(Saratov)

1. Consider a shallow structurally orthotropic cylindrical panel stretched along the generatrix, with a middle surface radius equal to R.

The problem of the bending of shallow cylindrical panels at large deflections reduces to the integration of the following two equations /2/:

$$Dn_2\Delta_1\Delta_2 w - \frac{\partial^2\varphi}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\right) - \frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - q = 0, \tag{1.1}$$

$$\Delta_3\Delta_4\varphi - Em_2\delta\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\right)\right] = 0. \tag{1.2}$$

Here

$$\Delta_k = \frac{\partial^2}{\partial x^2} + \alpha_k\frac{\partial^2}{\partial y^2}, \qquad (k=1,\ 2,\ 3,\ 4)$$

$$\alpha_1\alpha_2 = n^2, \quad \alpha_1 + \alpha_2 = \beta = 2\left[\mu + \frac{2G(n^2-\mu^2)}{En_2}\right], \tag{1.3}$$

$$\alpha_3\alpha_4 = m^2, \quad \alpha_3 + \alpha_4 = \alpha = \frac{Em_2}{G} - 2\nu,$$

w is the panel deflection, φ, the stress function, q, the external load, and δ, the shell thickness.

The elasticity relationships for a structurally orthotropic shell are

$$\begin{gathered} T_1 = Bm_2(\varepsilon_1 + \nu\varepsilon_2), \qquad T_2 = Bm_2(m^2\varepsilon_2 + \nu\varepsilon_1), \qquad S = G\delta\omega, \\ M_1 = Dn_2(x_1 + \mu x_2), \qquad M_2 = Dn_2(n^2x_2 + \mu x_1), \qquad H = -2D_k\tau, \\ B = \frac{E\delta}{m^2 - \nu^2}, \qquad D = \frac{E\delta^3}{12(n^2-\mu^2)}, \qquad D_k = \frac{G\delta^3}{12}. \end{gathered} \tag{1.4}$$

The coefficients $m^2 = m_1m_2$, $n^2 = n_1n_2$ are determined from the expressions

$$\begin{gathered} E_1 = \frac{E}{m_1}, \quad E_2 = m_2E, \quad \nu_1 = \frac{\nu}{m^2}, \quad \nu_2 = \nu, \\ E_1' = \frac{E}{n_1}, \quad E_2' = n_2E, \quad \nu_1' = \frac{\mu}{n^2}, \quad \nu_2' = \mu, \end{gathered} \tag{1.5}$$

where E_1, E_2, ν_1, ν_2 are Young's moduli and the Poisson ratios respectively of the shell at extension along the coordinate lines, and E_1', E_2', ν_1', ν_2' are

the same magnitudes at bending, E, ν and μ are the reduced Young's modulus and the Poisson ratios, ε_1, ε_2, ω, $\varkappa_1$, $\varkappa_2$, and τ are strains of the middle surface, connected with the displacements by the formulas

$$\varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varkappa_1 = \frac{\partial^2 w}{\partial x^2},$$

$$\varepsilon_2 = \frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad \varkappa_2 = \frac{\partial^2 w}{\partial y^2}, \tag{1.6}$$

$$\omega = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \qquad \tau = \frac{\partial^2 w}{\partial x \partial y}.$$

2. In order to find the critical combination of the acting loads, represent the deflection and the stress function in (1.1) and (1.2) in the form

$$w = w_1 + w_2, \qquad \varphi = \varphi_1 + \varphi_2, \tag{2.1}$$

where w_1 and φ_1 are the deflection and the stress function in the precritical state, and w_2 and φ_2 are the additional deflection and the function of the additional stresses appearing with oblique waves.

Only the moment of appearance of oblique waves in the panel will interest us in what follows; we shall therefore assume that w_2 is small compared with w_1, and shall discard in the stability equations all the terms containing squares and products of the derivatives of w_2. All magnitudes relative to the precritical state are assumed to be known.

By subtracting from equations (1.1) and (1.2), in which w has been replaced by w_1 and φ by φ_1, the same equations but with w and φ replaced according to (2.1), we obtain the following differential equations of stability of a cylindrical panel subjected to the action of the forces T_1, T_2, S, and q

$$Dn_2\Delta_1\Delta_2 w_2 - \frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2\varphi_1}{\partial y^2}\frac{\partial^2 w_2}{\partial x^2} + 2\left(\frac{\partial^2\varphi_1}{\partial x\partial y}\frac{\partial^2 w_2}{\partial x\partial y} + \right.$$

$$\left. + \frac{\partial^2\varphi_2}{\partial x\partial y}\frac{\partial^2 w_1}{\partial x\partial y}\right) - \left[\frac{\partial^2\varphi_1}{\partial x^2}\frac{\partial^2 w_2}{\partial y^2} + \frac{\partial^2\varphi_2}{\partial x^2}\left(\frac{\partial^2 w_1}{\partial y^2} - \frac{1}{R}\right)\right] = 0, \tag{2.2}$$

$$\Delta_3\Delta_4\varphi_2 + Em_2\delta\left[\frac{\partial^2 w_2}{\partial x^2}\left(\frac{\partial^2 w_2}{\partial y^2} - \frac{1}{R}\right) + \frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_2}{\partial y^2} - 2\frac{\partial^2 w_1}{\partial x\partial y}\frac{\partial^2 w_2}{\partial x\partial y}\right] = 0. \tag{2.3}$$

The system of equations (2.2) and (2.3) is linear and homogeneous relative to the unknowns w_2 and φ_2.

We shall determine the upper boundary of the critical loads, since w_2 is considered as small and is determined from equations (2.2) and (2.3) as a function of the stresses and strains of the precritical state, caused by the load q. In order to determine the lower boundary of the critical load it is necessary to consider w_2 as large, and this leads to nonlinear equations.

3. Consider a structurally orthotropic cylindrical panel stretched along the generatrix $(a \gg b)$ and loaded by the forces T_1, S, and q (Figure 1).

The panel can be considered as a strip, and its deflection in the precritical state is therefore $w_1 = w_1(y)$.

The problem of the bending of a structurally orthotropic panel reduces in this case to the integration of the following system:

$$Dn_2 n^2 \frac{d^4 w_1}{dy^4} - \frac{\partial^2\varphi_1}{\partial x^2}\left(\frac{d^2 w_1}{dy^2} - \frac{1}{R}\right) - q = 0, \tag{3.1}$$

$$\Delta_3\Delta_4\varphi = 0.$$

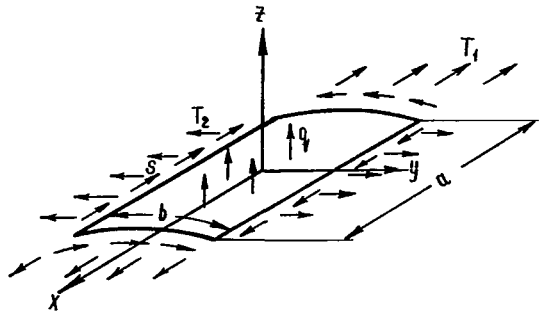

FIGURE 1.

Since $\frac{\partial T_1}{\partial x} = \frac{\partial T_2}{\partial x} = \frac{\partial S}{\partial x} = 0$, system (3.1), allowing for the equations of equilibrium, can be reduced to the one equation

$$Dn_2n^2 \frac{d^4w_1}{dy^4} - T_2\left(\frac{d^2w_1}{dy^2} - \frac{1}{R}\right) - q = 0 \tag{3.2}$$

and the conditions

$$\frac{\partial S}{\partial y} = \frac{\partial T_2}{\partial y} = \frac{\partial^2 T_1}{\partial y^2} = 0. \tag{3.3}$$

Conditions (3.3) mean that in the precritical state the stresses τ, σ_1, and σ_2 are constant, and that the last two depend only on the values of the forces at the edges, and are independent of the load q.

The equations of stability for a strip are simpler, and reduce to

$$\begin{gathered} Dn_2\Delta_1\Delta_2 w_2 - T_1 \frac{\partial^2 w_2}{\partial x^2} - T_2 \frac{\partial^2 w_2}{\partial y^2} - 2S \frac{\partial^2 w_2}{\partial x \partial y} - \frac{\partial^2 \varphi_2}{\partial x^2}\left(\frac{\partial^2 w_1}{\partial y^2} - \frac{1}{R}\right) = 0, \\ \Delta_3\Delta_4\varphi_2 + Em_2\delta \frac{\partial^2 w_2}{\partial x^2}\left(\frac{\partial^2 w_1}{\partial y^2} - \frac{1}{R}\right) = 0. \end{gathered} \tag{3.4}$$

We shall now solve equation (3.2) for the following conditions:

$$y = \pm \frac{b}{2}, \qquad \tau = \frac{S}{\delta} = \text{const}, \qquad w_1 = \frac{d^2 w_1}{dy^2} = 0, \qquad v_1 = 0. \tag{3.5}$$

All the conditions for w_1 will be satisfied by representing it in the form

$$w_1 = f_1 \cos \frac{\pi y}{b}. \tag{3.6}$$

The substitution of (3.6) in (3.2) and the integration of the expression obtained by the Galerkin method yield

$$Dn_2n^2 f_1 \left(\frac{\pi}{b}\right)^4 + T_2\left[f_1\left(\frac{\pi}{b}\right)^2 + \frac{4}{\pi R}\right] - \frac{4}{\pi} q = 0. \tag{3.7}$$

To determine T_2 we make use of the condition $v_1 = 0$ at $y = \pm \frac{b}{2}$

$$\int_{-b/2}^{b/2} \frac{dv_1}{dy} dy = \int_{-b/2}^{b/2} \left[\frac{1}{Em_2\delta} (T_2 - \nu T_1) - \frac{w_1}{R} - \frac{1}{2} \left(\frac{\partial w_1}{\partial y} \right)^2 \right] dy = 0,$$

or after integration

$$T_2 - \nu T_1 = Em_2\delta \left[\frac{2f_1}{\pi R} + \left(\frac{f_1 \pi}{2b} \right)^2 \right]. \tag{3.8}$$

Define the dimensionless magnitudes

$$\sigma_1^* = \frac{T_1 b^2}{\pi^2 D n_2 n^2}, \qquad \sigma_2^* = \frac{T_2 b^2}{\pi^2 D n_2 n^2}, \qquad f^* = \frac{f}{\delta} = f_1^* + f_0^*,$$
$$f_0^* = \frac{4}{\pi^3} \frac{b^2}{R\delta}, \qquad q^* = \frac{16q}{\pi^5 E} \left(\frac{b}{\delta} \right)^4. \tag{3.9}$$

The substitution of (3.9) in (3.7) and (3.8) yields

$$\sigma_2^* = \nu \sigma_1^* + \frac{3}{n_2 n^2} (n^2 - \mu^2)(f^{*2} - f_0^{*2}), \tag{3.10}$$

$$\left[f^{*3} - f^* f_0^{*2} + \frac{n_2 n^2 (f^* - f_0^*)}{3(n^2 - \mu^2)} \right] + \frac{n_2 n^2 \nu \sigma_1^*}{3(n^2 - \mu^2)} f^* = q^*. \tag{3.11}$$

The first term in (3.11) is a function of f^* and f_0^* only, and can be represented in q^* and f^* coordinates by a family of curves with parameter f_0^* = const. The second term gives the correction due to σ_1^* and is likewise a function of f^* and q^* only; it can likewise be represented by a family of curves with parameter σ_1^* = const.

Consider as an example a plywood panel for which

$$E = 0.6 \cdot 10^5 \text{ kg/cm}^2 \quad G = 0.07 \cdot 10^5 \text{ kg/cm}^2 \quad \nu = \mu = 0.036,$$
$$m_1 = n_1 = 0.5, \quad m_2 = n_2 = 1, \quad m^2 = n^2 = 0.5.$$

Equation (3.11) is represented graphically in Figure 2.

4. Proceed to the problem of the stability of a cylindrical strip stretched along the generatrix. Equations (3.4) are to be integrated for the following boundary conditions:

$$\text{at } y = \pm \frac{b}{2}, \qquad w_2 = \frac{\partial^2 w_2}{\partial y^2} = 0. \tag{4.1}$$

It will be assumed in addition that the longitudinal beams are absolutely rigid at bending, i.e., that they do not bend with the appearance of oblique waves.

This means that $v_2 = 0$ at $y = \pm \frac{b}{2}$, i.e.,

$$\frac{\partial^3 \varphi_2}{\partial y^3} + \frac{1}{m^2} \left(\frac{Em_2}{G} - \nu \right) \frac{\partial^3 \varphi_2}{\partial x^2 \partial y} = 0. \tag{4.2}$$

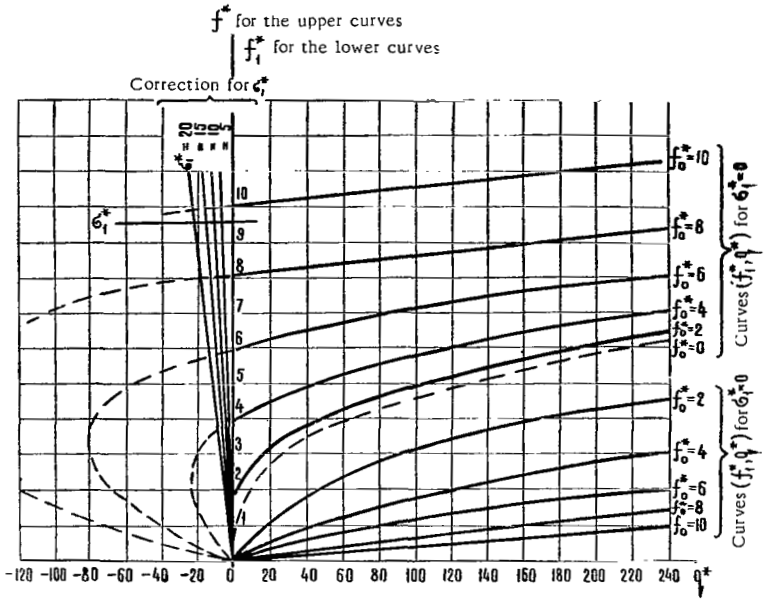

FIGURE 2.

The second additional displacement u_2 at the edges must likewise be equal to zero. These conditions are difficult to satisfy, and we shall accordingly restrict ourselves to satisfying condition $u_2 = 0$ on the average over the wave length λ along the generatrix /3/

$$\int_x^{x+2\lambda} \frac{\partial u_2}{\partial x}\, dx = 0. \tag{4.3}$$

All boundary conditions will be satisfied by taking w_2 and φ_2 in the form

$$\begin{aligned} w_2 &= f_2\left[\sin\frac{\pi x}{\lambda}\sum_{i=2,4\ldots}^{\infty} A_i \sin\frac{i\pi y}{b} + \cos\frac{\pi x}{\lambda}\sum_{j=1,3\ldots}^{\infty} B_j \cos\frac{j\pi y}{b}\right], \\ \varphi_2 &= \frac{Em_2 f_2 \lambda^2}{\pi^2}\left[\sin\frac{\pi x}{\lambda}\sum_{k=1,3\ldots}^{\infty} C_k \sin\frac{k\pi y}{b} + \cos\frac{\pi x}{\lambda}\sum_{k=0,2\ldots}^{\infty} D_k \cos\frac{k\pi y}{b}\right]. \end{aligned} \tag{4.4}$$

Substitute (4.4) in (3.4) and integrate by the Galerkin method from $-\lambda$ to λ and from $-\frac{b}{2}$ to $\frac{b}{2}$. After the calculation of all the integrals we obtain

$$\begin{aligned} &A_i\left[\left(\frac{\pi}{\lambda}\right)^4 (1+\beta\gamma_i^2+n^2\gamma_i^4) Dn_2 + T_1\left(\frac{\pi}{\lambda}\right)^2 + T_2\left(\frac{i\pi}{b}\right)^2\right] - \\ &8S\frac{\pi}{\lambda b}\sum_{j=1,3,5\ldots}^{\infty} B_j(-1)^{\frac{-i-j-1}{2}} \frac{ij}{i^2-j^2} - \frac{Em_2}{R}\frac{4}{\pi}\sum_{k=1,3\ldots}^{\infty} C_k(-1)^{\frac{i-k-1}{2}} \times \\ &\times \frac{i}{i^2-k^2} - Em_2 f_1\left(\frac{\pi}{b}\right)^2 \frac{1}{2}(C_{i-1}+C_{i+1}) = 0, \quad (i=2,4,\ldots), \end{aligned} \tag{4.5}$$

$$B_j\left[Dn_2(1+\beta\gamma_j^2+n^2\gamma_j^4)\left(\frac{\pi}{\lambda}\right)^4+T_1\left(\frac{\pi}{\lambda}\right)^2+T_2\left(\frac{j\pi}{b}\right)^2\right]-$$

$$-8S\frac{\pi}{\lambda b}\sum_{i=2,4\ldots}^{\infty}A_i(-1)^{\frac{j-i-1}{2}}\frac{ij}{j^2-i^2}-\frac{Em_2}{R}\frac{4}{\pi}\sum_{k=0,2\ldots}^{\infty}D_k(-1)^{\frac{j-k-1}{2}}\times \tag{4.5 (cont'd)}$$

$$\times\frac{j}{j^2-k^2}-\left(\frac{\pi}{b}\right)^2Em_2f_1\frac{1}{2}(D_{j-1}+D_{j+1})=0,\qquad (j=3,5,\ldots),$$

$$\underset{k=1,3\ldots}{C_k}=-\frac{\delta}{1+\alpha\gamma_k^2+m^2\gamma_k^4}\left[\frac{4}{\pi R}\sum_{i=2,4\ldots}^{\infty}A_i(-1)^{\frac{k-i-1}{2}}\frac{i}{k^2-i^2}+\right.$$

$$\left.+\frac{1}{2}f_1\left(\frac{\pi}{b}\right)^2(A_{k-1}+A_{k+1})\right],$$

$$\underset{k=2,4\ldots}{D_k}=-\frac{\delta}{1+\alpha\gamma_k^2+m^2\gamma_k^4}\left[\frac{4}{\pi R}\sum_{j=1,3\ldots}^{\infty}B_j(-1)^{\frac{k-j-1}{2}}\frac{j}{k^2-j^2}+\right. \tag{4.6}$$

$$\left.+f_1\left(\frac{\pi}{b}\right)^2\frac{1}{2}(B_{k-1}+B_{k+1})\right].$$

Here

$$\gamma=\frac{\lambda}{b},\qquad \gamma_k=\frac{\lambda k}{b}. \tag{4.7}$$

By restricting ourselves in what follows in a first approximation to two terms in the series for w_2 $(i=2,\ j=1)$ and four terms in the series for φ_2 with coefficients $(D_0,\ D_2,\ C_1,\ C_3)$ we reduce equations (4.5) required for a determination of coefficients A_2 and B_1 to

$$L_1B_1-\frac{16n^2}{3\pi}\tau^*A_2=0,\quad -\frac{16n^2}{3\pi}\tau^*B_1+L_2A_2=0, \tag{4.8}$$

where

$$L_1=\frac{1}{\gamma^3}(1+\beta\gamma^2+n^2\gamma^4)+n^2\left(\frac{1}{\gamma}\sigma_1^*+\gamma\sigma_2^*\right)+$$

$$+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma\left[(1+\varkappa)^2+\frac{\left(1-\frac{\varkappa}{3}\right)^2}{1+4\alpha\gamma^2+16m^2\gamma^4}\right]f^{*2},$$

$$L_2=\frac{1}{\gamma^3}(1+4\beta\gamma^2+16n^2\gamma^4)+n^2\left(\frac{1}{\gamma}\sigma_1^*+4\gamma\sigma_2^*\right)+ \tag{4.9}$$

$$+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma\left[\frac{\left(1+\frac{\varkappa}{3}\right)^2}{1+\alpha\gamma^2+m^2\gamma^4}+\frac{\left(1-\frac{\varkappa}{5}\right)^2}{1+9\alpha\gamma^2+81m^2\gamma^4}\right]f^{*2},$$

τ^* is the relative shearing stress and $\frac{1}{\varkappa}=\frac{f^*}{f_0^*}$ is a parameter characterizing the ratio of the total deflection f^* to the rise of the panel $0\leqslant\varkappa\leqslant 1$.

System (4.8) has a nontrivial solution only if its determinant is equal to zero:

$$\left(\frac{16n^2}{3\pi}\tau^*\right)^2=\left\{\frac{1}{\gamma^3}(1+\beta\gamma^2+n^2\gamma^4)+n^2\left(\frac{1}{\gamma}\sigma_1^*+\gamma\sigma_2^*\right)+\right. \tag{4.10}$$

$$+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma f^{*2}\left[(1+\varkappa)^2+\frac{\left(1-\frac{\varkappa}{3}\right)^2}{1+4\alpha\gamma^2+16m^2\gamma^4}\right]\Bigg\}\times$$

$$\times\left\{\frac{1}{\gamma^3}(1+4\beta\gamma^2+16n^2\gamma^4)+n^2\left(\frac{1}{\gamma}\sigma_1^*+4\gamma\sigma_2^*\right)+\right. \qquad (4.10)\ \text{(cont'd)}$$

$$\left.+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma f^{*2}\left[\frac{\left(1+\frac{\varkappa}{3}\right)^2}{1+\alpha\gamma^2+m^2\gamma^4}+\frac{\left(1-\frac{\varkappa}{5}\right)^2}{1+9\alpha\gamma^2+81m^2\gamma^4}\right]\right\}.$$

Equation (4.10) determines the critical value of the force $S(\tau^*)$ at variable parameters σ_1^*, σ_2^*, f^*, and $\varkappa$.

The relative wavelength γ is selected so as to make τ^* *minimum* at constant values of all other parameters, i.e., we must add to (4.10) the equation

$$\frac{d\tau^*}{d\gamma}=\left\{-3-\beta\gamma^2+n^2\gamma^4-n^2\gamma^2(\sigma_1^*+\gamma^2\sigma_2^*)+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma^4 f^{*2}\left[(1+\varkappa)^2+\right.\right.$$

$$\left.\left.+\frac{\left(1-\frac{\varkappa}{3}\right)^2}{(1+4\alpha\gamma^2+16m^2\gamma^4)^2}(1-4\alpha\gamma^2-48m^2\gamma^4)\right]\right\}\left\{\frac{1}{\gamma^3}(1+4\beta\gamma^2+16n^2\gamma^4)+\right.$$

$$\left.+n^2\left(\frac{1}{\gamma}\sigma_1^*+4\gamma\sigma_2^*\right)+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma f^{*2}\left[\frac{\left(1+\frac{\varkappa}{3}\right)^2}{1+\alpha\gamma^2+m^2\gamma^4}+\right.\right.$$

$$\left.\left.+\frac{\left(1-\frac{\varkappa}{5}\right)^2}{1+9\alpha\gamma^2+81m^2\gamma^4}\right]\right\}+\left\{3-4\beta\gamma^2+16n^2\gamma^4-n^2\gamma^2(\sigma_1^*+4\gamma^2\sigma_2^*)+\right.$$

$$\left.+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma^4 f^{*2}\left[\frac{\left(1+\frac{\varkappa}{3}\right)^2}{(1+\alpha\gamma^2+m^2\gamma^4)^2}(1-\alpha\gamma^2-3m^2\gamma^4)+\right.\right.$$

$$\left.\left.+\frac{\left(1-\frac{\varkappa}{5}\right)^2}{(1+9\alpha\gamma^2+82m^2\gamma^4)^2}(1-9\alpha\gamma^2-243m^2\gamma^4)\right]\right\}\left\{\frac{1}{\gamma^3}(1+\beta\gamma^2+n^2\gamma^4)+\right.$$

$$\left.+n^2\left(\frac{1}{\gamma}\sigma_1^*+\gamma\sigma_2^*\right)+3(n^2-\mu^2)\frac{m_2}{n_2}\gamma f^{*2}\left[(1+\varkappa)^2+\frac{\left(1-\frac{\varkappa}{3}\right)^2}{1+4\alpha\gamma^2+km^2\gamma^4}\right]\right\}. \qquad (4.11)$$

The number of independent variables in equations (4.10) and (4.11) can be reduced with the aid of equality (3.10).

Equations (4.10) and (4.11) thus give a relationship between the critical values of the loads σ_1^*, σ_2^*, τ^* and the geometrical characteristics of the panel f^* and $\varkappa$ in a parametric form.

The dependence of the critical shearing stress τ^* on the total deflection of the shell f^* for a plywood panel is represented in Figure 3.

The following conclusions can be drawn from the preceding treatment:

1. The results of the first approximation turn out to be overestimated, just as is the case of an isotropic panel /1/. This follows from the

formulas of the second approximation (three terms were taken for w_2, and five terms for φ_2), which are not given here because they are very cumbersome.

2. The numerical values of τ^* for a plywood panel turned out to be somewhat lower than for an isotropic panel /1/.

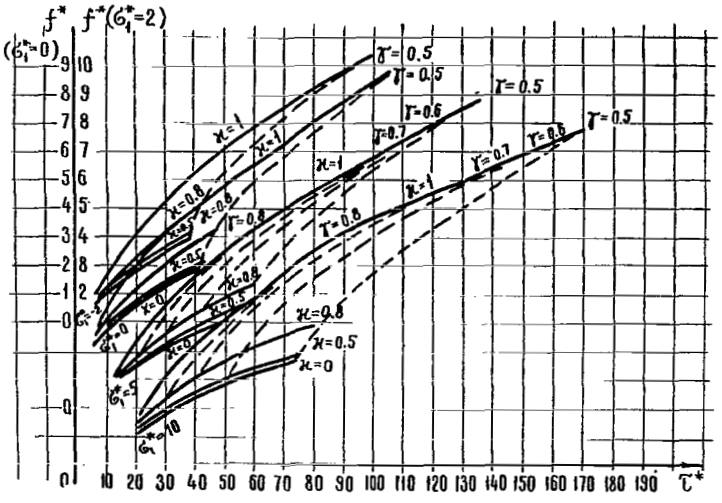

FIGURE 3.

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APPLICATION OF THE METHOD OF LAGRANGE MULTIPLIERS IN THE STUDY OF THE STABILITY OF PLATES AND SHELLS

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When the Ritz method is used for solving stability problems of plates and shells, the displacements of the middle surface points are usually represented in the form

$$u = \sum_{n=1}^{n} a_n u_n; \qquad v = \sum_{n=1}^{n} b_n v_n; \qquad w = \sum_{n=1}^{n} c_n w_n; \tag{1}$$

where u_n, v_n, w_n are some functions of the coordinates which satisfy the boundary conditions. No mathematical restrictions are imposed in the general case on these functions. The solution is, however, considerably simplified and leads to satisfactory results when the coordinate functions u_n, v_n, and w_n are orthogonal.

In many cases it is impossible to form a system of orthogonal functions which satisfy all the boundary conditions of the problem. It can be required then that expressions (1) themselves satisfy the conditions at the edges. By substituting (1) in the relationships representing the given boundary conditions we obtain for the parameters a_n, b_n, and c_n some k equations of the form

$$f_r(a_n, b_n, c_n) = 0 \qquad (r = 1, 2, \cdots k) \tag{2}$$

which must be satisfied.

The substitution of (1) in the expression for the total energy yields, after the necessary integration, an expression for the energy U as a quadratic function of the parameters a_n, b_n, and c_n:

$$U = \Phi(a_n, b_n, c_n). \tag{3}$$

If (1) is considered as a solution of an extremal problem, the parameters a_n, b_n, and c_n must be selected so that expression (3) will have a stationary value and conditions (2) will be satisfied. The problem reduces thus to finding the relative extremum of the function (3) whose arguments are the parameters a_n, b_n, c_n, linked by the additional conditions (2). Mathematically this is realized by means of the method of undeterminate Lagrange multipliers. Form the function

$$\overline{U} = \Phi(a_n, b_n, c_n) - \sum_{r=1}^{k} \lambda_r f_r(a_n, b_n, c_n), \tag{4}$$

where λ_r are undeterminate multipliers, which must be selected so that the values found for the parameters a_n, b_n, and c_n satisfy equations (2). Expression (4) must in addition have a stationary value, ie.,

$$\frac{\partial \bar{U}}{\partial a_n} = 0; \quad \frac{\partial \bar{U}}{\partial b_n} = 0; \quad \frac{\partial \bar{U}}{\partial c_n} = 0. \tag{5}$$

Equations (5) determine, together with conditions (2), the values of the parameters a_n, b_n, c_n, and the multipliers λ_r.

The method exposed here has not been used so far in Soviet literature. A very brief description of the method is found in the treatise /1/, together with references to hardly accessible sources.

The solution of several problems of elastic stability by means of the method of Lagrange multipliers is given below.

1. STABILITY OF A BAR WITH ONE END BUILT-IN AND THE OTHER HINGED

Consider a slightly bent bar loaded by an axial compressive force P (Figure 1). The total energy of the bar will be

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{P}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx. \tag{6}$$

Let its elastic curve in the slightly bent state be

$$y = \sum_{m=1}^{m} a_m \sin\frac{m\pi x}{l}, \qquad (m > 2). \tag{7}$$

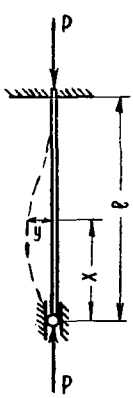

FIGURE 1.

It is easily seen that every coordinate function $y_m = \sin\frac{m\pi x}{l}$ satisfies the boundary conditions of the hinged bar $(y = 0, y'' = 0)$, but does not satisfy

the boundary condition $y' = 0$ at $x = l$. In order that this boundary condition be satisfied it is necessary to have

$$\sum_{m=1}^{m} (-1)^m m a_m = 0. \tag{8}$$

This is the additional condition imposed on the parameters a_m.

Form the function $\overline{U}$, making use of (4):

$$\overline{U} = \frac{EI}{4l^3} \sum_{m=1}^{m} m^4 a_m^2 - \frac{P\pi^2}{4l} \sum_{m=1}^{m} m^2 a_m - \lambda \sum_{m=1}^{m} (-1)^m m a_m. \tag{9}$$

The condition $\frac{\partial \overline{U}}{\partial a_m} = 0$ leads to the following expression for the parameter a_m:

$$a_m = \frac{(-1)^m 2l}{\pi^2 m} \cdot \frac{\lambda}{m^2 P_E - P_{cr}}, \tag{10}$$

where $P_E = \frac{\pi^2 EI}{l^2}$ is the Euler critical force for a hinged bar.

By substituting (10) in (8) we find that (8) has a nontrivial solution λ only if

$$\sum_{m=1}^{m} \frac{1}{m^2 P_E - P_{cr}} = 0. \tag{11}$$

Condition (11) determines the value of the critical load. By substituting successively $m = 2, 3$, etc. we obtain respectively $P_{cr} = 2.5 P_E$, $2.33 P_E$, etc. The solution converges to the exact value $P_{cr} = 2 P_E$.

A solution to the problem of the stability of a bar built-in at both ends is given in /1/. The stability equation obtained is

$$\sum_{m=1}^{m} \frac{1}{(2m-1)^2 P_E - P_{cr}} = 0. \tag{12}$$

The solution converges to the exact value $P_{cr} = 4P_E$.

2. STABILITY OF A RECTANGULAR PLATE

Consider a plate built-in along two opposite edges and simply supported along the two other, compressed in the direction of the supported edges by the uniformly distributed forces N_x (Figure 2). The deflection function will be represented in the form

$$w = \cos \frac{(2n-1)\pi y}{b} \sum_{m=1}^{m} a_m \cos \frac{(2m-1)\pi x}{a}. \tag{13}$$

The clamping condition $\frac{\partial w}{\partial x} = 0$ is not satisfied by this function on the edges $x = \pm \frac{a}{2}$. The edge conditions will be satisfied by imposing on the

parameters a_m the additional condition

$$\sum_{m=1}^{m} a_m (2m-1)(-1)^m = 0. \tag{14}$$

By solving the problem along the same lines as above we obtain the following stability condition:

$$\sum_{m=1}^{m} \frac{1}{k_m \frac{D\pi^2}{b^2} - N_x^{cr}} = 0, \tag{15}$$

where

$$k_m = \frac{\delta^2}{(2m-1)^2}\left[\frac{(2m-1)^2}{\delta^2} + (2n-1)^2\right]^2;$$

$$\delta = \frac{a}{b}; \qquad D = \frac{Eh^3}{12(1-\nu^2)};$$

h is the plate thickness; ν is the Poisson ratio.

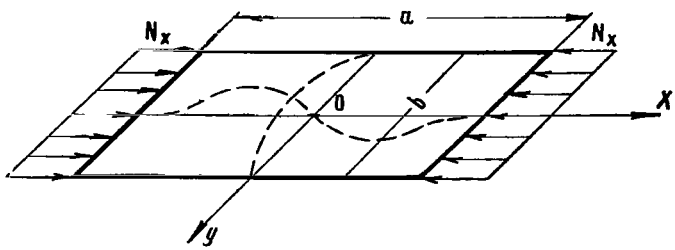

FIGURE 2.

The calculations conducted for several values of δ and b have shown that the solution reduces to the values of the critical load given in /2/.

3. STABILITY OF AN ORTHOTROPIC CYLINDRICAL SHELL WITH BUILT-IN EDGES UNDER THE ACTION OF AN EXTERNAL PRESSURE

Introduce the following designations: R, L, h are radius, length, and thickness, respectively, of the shell; ε_{11}, ε_{22}, ε_{12}, $\varkappa_{11}$, $\varkappa_{22}$, $\varkappa_{12}$ are strains of the middle surface; φ, x, z are coordinates of points at the middle surface; E_1, E_2, G, ν_1, ν_2 are elastic constants along the principal elasticity axes, which are assumed to coincide with the lines of principal curvatures; $B_1 = \frac{E_1 h}{1-\nu_1\nu_2}$ $B_2 = \frac{E_2 h}{1-\nu_1\nu_2}$, $B_{12} = Gh$, $D_1 = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}$, $D_2 = \frac{E_2 h^3}{12(1-\nu_1\nu_2)}$, $D_{12} = \frac{Gh}{12}$ are rigidities of the shell; n is the number of waves formed in the peripheral direction at buckling; p is the external pressure.

The displacement functions will be represented in the form:

$$u = \sin n\varphi \sum_{m=1}^{m} a_m \sin\frac{(2m-1)\pi x}{L}; \tag{16}$$

$$v = \cos n\varphi \sum_{m=1}^{m} b_m \cos \frac{(2m-1)\pi x}{L};$$
$$w = \sin n\varphi \sum_{m=1}^{m} c_m \cos \frac{(2m-1)\pi x}{L}. \qquad (16)\ (\text{cont'd})$$

To satisfy the fundamental clamping condition $\frac{\partial w}{\partial x} = 0$ at the shell edges $\left(x = \pm \frac{L}{2}\right)$, the parameters c_m must be linked by the following additional condition:

$$\sum_{m=1}^{m} (-1)^m c_m (2m-1) = 0. \qquad (17)$$

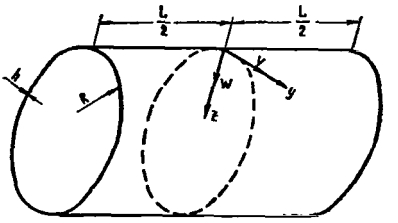

FIGURE 3.

The shell energy is equal to

$$U = U_1 + U_2 - A, \qquad (18)$$

where

$$U_1 = \frac{1}{2} \int_0^{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} (B_1\varepsilon_{11}^2 + B_2\varepsilon_{22}^2 + 2B_1\nu_2\varepsilon_{11}\varepsilon_{12} + B_{12}\varepsilon_{12}^2)\, R d\varphi dx; \qquad (19)$$

$$U_2 = \frac{1}{2} \int_0^{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} (D_1\varkappa_{11}^2 + D_2\varkappa_{22}^2 + 2D_1\nu_2\varkappa_{11}\varkappa_{22} + 4D_{12}\varkappa_{12}^2)\, R d\varphi dx: \qquad (20)$$

$$A = -p \frac{R}{2} \int_0^{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \varkappa_{22} w R d\varphi dx. \qquad (21)$$

The strain components will be written in the form

$$\varepsilon_{11} = \frac{\partial u}{\partial x}; \quad \varepsilon_{12} = \frac{\partial v}{R\partial\varphi} - \frac{w}{R}; \quad \varepsilon_{12} = \frac{\partial u}{R\partial\varphi} + \frac{\partial v}{\partial x};$$
$$\varkappa_{11} = \frac{\partial^2 w}{\partial x^2}; \quad \varkappa_{22} = \frac{1}{R^2}\left(w + \frac{\partial^2 w}{\partial\varphi^2}\right); \quad \varkappa_{12} = \frac{1}{R}\left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial\varphi}\right). \qquad (22)$$

Using these relationships, we can represent the function $\overline{U}$ in the form

$$\overline{U} = U - \lambda \sum_{m=1}^{m} (-1)^m c_m (2m-1), \tag{23}$$

where

$$U = \frac{\pi^2}{4}\Big[B_1\delta \sum_{m=1}^{m} a_m^2 (2m-1)^2 + B_2 \frac{n^2}{\delta} \sum_{m=1}^{m} b_m^2 + B_2 \frac{2n}{\delta} \sum_{m=1}^{m} b_m c_m +$$
$$+ B_2 \frac{1}{\delta} \sum_{m=1}^{m} c_m^2 - 2(B_1\nu_2 + B_{12})\, n \sum_{m=1}^{m} a_m b_m (2m-1) + B_{12} \frac{n^2}{\delta} \sum_{m=1}^{m} a_m^2 -$$
$$- 2B_1\nu_2 \sum_{m=1}^{m} a_m c_m (2m-1) + B_{12}\delta \sum_{m=1}^{m} b_m^2 (2m-1)^2\Big] +$$
$$+ \frac{\pi^2}{4R^2}\Big\{D_1\delta^3 \sum_{m=1}^{m} c_m^2 (2m-1)^4 + D_2 \frac{(n^2-1)^2}{\delta} \sum_{m=1}^{m} c_m^2 +$$
$$+ 2\,[D_1\nu_2 (n^2-1) + 2D_{12}n^2]\,\delta \sum_{m=1}^{m} c_m^2 (2m-1)^2 + 4D_{12}\delta \sum_{m=1}^{m} b_m^2 (2m-1)^2\Big\} -$$
$$- p\frac{\pi L}{4}(n^2-1) \sum_{m=1}^{m} c_m^2 \tag{24}$$

and

$$\delta = \frac{\pi R}{L}.$$

The stationarity condition (5) yields the following system of equations:

$$\begin{aligned} a_{11}a_m + a_{12}b_m + a_{13}c_m &= 0; \\ a_{21}a_m + a_{22}b_m + a_{23}c_m &= 0; \\ a_{31}a_m + a_{32}b_m + a_{33}c_m &= d; \end{aligned} \tag{25}$$

where

$$\begin{aligned} a_{11} &= B_1\delta (2m-1)^2 + B_{12}\frac{n^2}{\delta}; \\ a_{12} = a_{21} &= -(B_1\nu_2 + B_{12})\, n (2m-1) \\ a_{13} = a_{31} &= -B_1\nu_2 (2m-1); \\ a_{22} &= B_2\frac{n^2}{\delta} + B_{12}\delta (2m-1)^2; \\ a_{23} = a_{32} &= B_2\frac{n}{\delta}; \\ a_{33} &= B_2\frac{1}{\delta} + \frac{1}{\delta R^2}\{D_2(n^2-1)^2 + 2\,[D_1\nu_2(n^2-1) + 2D_{12}n^2]\,\delta^2 (2m-1)^2 + \\ &\quad + D_1\delta^4 (2m-1)^4\} - p\frac{L}{\pi}(n^2-1); \\ d &= \frac{4\lambda}{\pi^2}(-1)^m (2m-1). \end{aligned} \tag{26}$$

By solving system (25) we find c_m

$$c_m = \frac{a_{11}a_{22} - a_{12}^2}{\Delta}, \tag{27}$$

where Δ is the determinant of system (25).

By substituting (27) in (17) and allowing for (26) we find the following expressions for the determination of the critical pressure:

$$\sum_{m=1}^{m} \frac{(2m-1)^2}{\dfrac{\Theta_1}{R(n^2-1)\Theta_2} + \dfrac{\Theta_3}{R^3(n^2-1)} - p_{cr}} = 0, \tag{28}$$

where

$$\Theta_1 = B_1 B_2 B_{12} (1 - \nu_1\nu_2)\,\delta^4 (2m-1)^4;$$

$$\Theta_2 = B_2 B_{12} n^4 + B_1 B_2 (1 - \nu_1\nu_2)\, n^2\delta^2 (2m-1)^2 - $$
$$- 2B_1 B_{12}\nu_2\delta^2 n^2 (2m-1)^2 + B_1 B_{12}\delta^4 (2m-1)^4;$$

$$\Theta_3 = D_2 (n^2-1)^2 + 2\,[D_1\nu_2 (n^2-1) + 2D_{12} n^2]\,\delta (2m-1)^2 + D_1\delta^4 (2m-1)^4.$$

In the case of an isotropic shell (28) simplifies to

$$\sum_{m=1}^{m} \frac{(2m-1)^2}{p^* - p_{cr}} = 0, \tag{29}$$

where

$$p^* = \frac{Eh}{R(1-\nu^2)(n^2-1)} \left\{ \frac{(1-\nu^2)\delta^4 (2m-1)^4}{[n^2 + \delta^2 (2m-1)^2]^2} + \frac{h^2}{12R^2}\,[n^2 - 1 + \delta^2 (2m-1)^2]^2 \right\}.$$

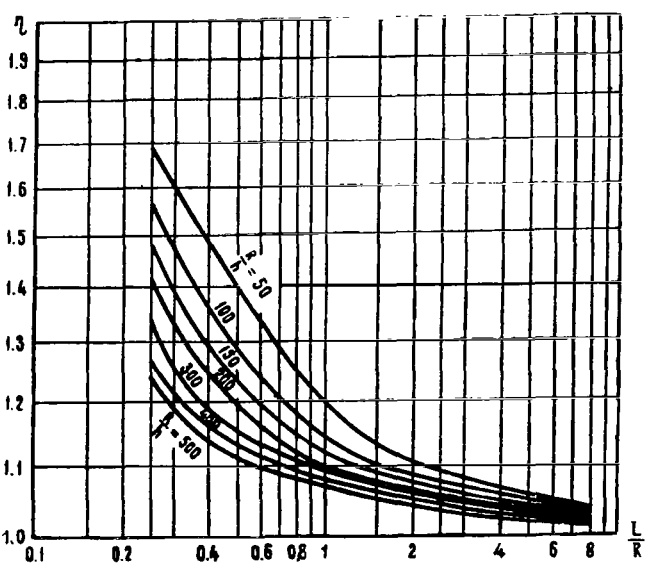

FIGURE 4.

Relationship (29) was programmed on the "Ural-2" electronic computer. The main results obtained are represented graphically in Figure 4, which represents in a double logarithmic scale the relationship $\eta = f\left(\frac{L}{R}\right)$ at different $\frac{R}{h}$. The parameter η represents the ratio of the critical pressure

for a built-in shell to the critical pressure of a hinged shell. The following relationship, similar to Mises' formula, was used in the latter case:

$$p_{cr} = \frac{Eh}{(1-\nu^2)R}\left[\frac{\delta^4(1-\nu^2)}{(n^2+\delta^2)^2(n^2-1)} + \frac{h^2}{12R^2}\frac{(\delta^2+n^2-1)^2}{(n^2-1)}\right]. \qquad (30)$$

It is seen from the data given that the influence of the clamping is small for shells of length $L = (2 \text{ to } 3)R$ and represents about 5 to 10 %, and that at $\frac{R}{h} > 200$ and $\frac{L}{R} > 4$ it is smaller than 5 % and can be neglected. In the case of relatively shorter shells, on the other hand, $\left(\frac{L}{R} < 1\right)$, the influence of the clamping is substantial; thus, at $\frac{R}{h} = 100$ and $\frac{L}{R} = 0.25$ it is more than 50 %.

Another interesting feature is the difference between the patterns of wave formation, which is illustrated by the values given in the following table.

TABLE

$\frac{L}{R}$	Number of waves in the peripheral direction			
	Hinged edges		Built-in edges	
	$R/h = 100$	$R/h = 500$	$R/h = 100$	$R/h = 500$
0.25	16	24	20	26
0.50	11	18	12	18
1.0	8	13	9	13
1.5	7	10	7	10
2.0	6	9	6	9
3.0	5	7	5	7
4.0	4	7	4	7
5.0	4	6	4	6
6.0	4	5	4	5
8.0	3	5	3	5

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STUDY OF MEDIUM AND LARGE DEFLECTIONS OF SHALLOW SPHERICAL SEGMENTS UNDER ARBITRARY AXISYMMETRICAL LOADS

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1. The problem of the axisymmetric elastic deformation of a shallow spherical segment can be reduced to the solution of the following system of equilibrium equations /1/:

$$\frac{d}{d\alpha_1}(\alpha_1 T_1) - T_2 = 0,$$
$$\frac{2Eh^3}{3(1-\nu^2)}\Delta\Delta w - \frac{1}{R}(T_1 + T_2) - (T_1\varkappa_1 + T_2\varkappa_2) - p(\alpha) = 0. \tag{1}$$

Here T_1 and T_2 are meridional and circumferential forces per unit length; $\varkappa_1$ and $\varkappa_2$ are variations of the curvature; $p(\alpha)$ is an arbitrary normal load; R is the radius, $2h$ is the shell thickness; $\Delta = \frac{1}{\alpha_1}\frac{d}{d\alpha_1}\left[\alpha_1 \frac{d(\,)}{d\alpha_1}\right]$; w is the normal deflection; E is the modulus of elasticity; ν is the Poisson ratio.

By writing $\alpha_1 = r_0\alpha$, where $r_0^4 = \frac{R^2h^2}{3(1-\nu^2)}$, and expressing the magnitudes T_1, T_2, $\varkappa_1$, $\varkappa_2$ through w and the meridional displacement u with the aid of the relationships

$$T_1 = \frac{2Eh}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2), \qquad T_2 = \frac{2Eh}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1),$$
$$\varepsilon_1 = \frac{du}{d\alpha_1} - \frac{w}{R} + \frac{1}{2}\left(\frac{dw}{d\alpha_1}\right)^2, \qquad \varkappa_1 = \frac{d^2w}{d\alpha_1^2}, \tag{2}$$
$$\varepsilon_2 = \frac{u}{\alpha_1} - \frac{w}{R}, \qquad \varkappa_2 = \frac{1}{\alpha_1}\frac{dw}{d\alpha_1},$$

we obtain a system of two nonlinear equations relative to the displacements u and w. The dimensionless coordinate α varies from zero at the top of the segment to α_0 at its edge.

Consider a particular case of a rigidly fixed edge, and assume that the segment does not have a cut in the pole. We have then the following boundary conditions:

$$u = 0, \quad \frac{dw}{d\alpha} = 0, \quad \frac{d}{d\alpha}\left(\frac{d^2w}{d\alpha^2} + \frac{1}{\alpha}\frac{dw}{d\alpha}\right) = 0 \quad \text{for} \quad \alpha = 0; \tag{3}$$

$$u = 0, \quad w = 0, \quad \frac{dw}{d\alpha} = 0 \qquad \text{for} \quad \alpha = \alpha_0. \tag{4}$$

The third condition of (3) means that there is no concentrated force at the top of the dome; were such a force present it should have been replaced by the condition of statics /2/.

System (1) being written in displacements, we can study the problem of small, medium, and large deflections of a shallow spherical segment. We shall adhere to the classification proposed in /1/: $\omega \ll 1$ at small deflections, $\omega^2 \ll 1$ at medium deflections, and $\omega \sim 1$ at large deflections, where ω is the rotation of a linear element as a result of the bending.

2. By twice integrating the first of the equilibrium equations (1) after substituting in it the displacements one obtains u as a function of w. The two integration constants are determined from the conditions: $u = 0$ at $\alpha = 0, \alpha_0$. Substituting the value found for u in the second equilibrium equation leads to the following nonlinear integral-differential equation relative to the normal deflection w:

$$L(w) = \frac{p(\alpha) R^2}{2Eh} + \frac{1}{h} N_1(w) + \frac{1}{h^2} N_2(w). \tag{5}$$

The operators figuring in this formula have the following values:

$$L(\cdots) = (\nabla^2\nabla^2 + 1)(\cdots) + \frac{2(1+\nu)}{\alpha_0^2(1-\nu)} \int_0^{\alpha_0} \alpha(\cdots)\, d\alpha;$$

$$\nabla^2(\cdots) = (\cdots)'' + \frac{1}{\alpha}(\cdots)', \quad (\cdots)' = \frac{d(\cdots)}{d\alpha}, \quad (\cdots)'' = \frac{d^2(\cdots)}{d\alpha^2};$$

$$N_1(w) = -\sqrt{3(1-\nu^2)} \left\{ -\frac{1}{1-\nu}\chi_3^0(w) + \frac{w \cdot w'}{\alpha} + \frac{1+\nu}{1-\nu}\chi_1^0(w) \cdot \nabla^2 w + \right.$$

$$\left. + \chi_1(w)\left(w'' - \frac{1}{\alpha}w'\right) + \frac{1}{2}\chi_2(w) \right\};$$

$$N_2(w) = \frac{3}{2}\left\{ (1+\nu)\chi_3^0(w) \cdot \nabla^2 w + \right.$$

$$\left. + (1-\nu)\left[\chi_3(w)\left(w'' - \frac{1}{\alpha}w'\right) - \chi_2(w)\left(w'' + \frac{\nu}{\alpha}w'\right)\right]\right\};$$

$$\chi_1(w) = \frac{1}{\alpha^2}\int_0^{\alpha} \alpha w\, d\alpha; \qquad \chi_2(w) = \int_0^{\alpha} \frac{(w')^2}{\alpha}\, d\alpha;$$

$$\chi_3(w) = \frac{1+\nu}{2\alpha^2}\int_0^{\alpha} \alpha (w')^2\, d\alpha + \frac{1-\nu}{2}\chi_2(w);$$

$$\chi_i^0(w) = \chi_i(w(\alpha_0)), \quad (i = 1, 2, 3).$$

At small deflections equation (5) reduces to the following linear equation:

$$L(w) = \frac{p(\alpha) R^2}{2Eh}. \tag{6}$$

Its solution, which satisfies boundary conditions (3) and (4), is

$$w = A_1 J_1(\alpha) + A_2 J_2(\alpha) + C_1 + \sum_{i=1}^{4} G_i(\alpha) J_i(\alpha), \tag{7}$$

where

$$J_1(\alpha) = \operatorname{Re} J_0(\alpha\sqrt{i}),\ J_2(\alpha) = -\operatorname{Im} J_0(\alpha\sqrt{i}),$$
$$J_3(\alpha) = \operatorname{Re} H_0^{(1)}(\alpha\sqrt{i}),\ J_4(\alpha) = \operatorname{Im} H_0^{(1)}(\alpha\sqrt{i}).$$

Here $J_0(\alpha\sqrt{i})$ $H_0^{(1)}(\sqrt{i})$ are zero-order Bessel and Hankel functions, respectively, of the first kind,

$$G_i(\alpha) = \frac{\pi R^2}{4Eh}\int_0^{\alpha} \alpha p(\alpha) g_i(\alpha)\, d\alpha, \quad (i=1,\ 2,\ 3,\ 4)$$

$$g_1(\alpha) = -J_3(\alpha),\quad g_2(\alpha) = -J_4(\alpha),\quad g_3(\alpha) = J_1(\alpha),\quad g_4(\alpha) = J_2(\alpha).$$

The coefficients A_1, A_2, C_1 are expressed by the formulas

$$A_1 = -G_1(\alpha_0) + k(\alpha_0)(-\xi J_2' + G_3\eta_1 + G_4\eta_2)\big|_{\alpha=\alpha_0};$$

$$A_2 = -G_2(\alpha_0) + k(\alpha_0)\left[\xi J_1' + G_3\eta_2 + G_4\left(\frac{2}{\pi\alpha_0} - \eta_1\right)\right]\Bigg]_{\alpha=\alpha_0};$$

$$C_1 = -\left(A_1 J_1(\alpha_0) + A_2 J_2(\alpha_0) + \sum_{i=1}^{4} G_i(\alpha_0) J_i(\alpha_0)\right);$$

where

$$\frac{1}{k} = J_2 J_1' - J_1 J_2' + \frac{1+\nu}{\alpha}(J_1^2 + J_2'^2), \qquad \xi = \frac{1+\nu}{2Eh}\cdot\frac{R^2}{\alpha_0^2}\int_0^{\alpha_0} \alpha p(\alpha)\, d\alpha;$$

$$\eta_1 = J_3 J_2' - J_2 J_3' + \frac{1+\nu}{\alpha}(J_2' J_4' - J_1' J_3');$$

$$\eta_2 = J_4 J_2' - J_2 J_4 - \frac{1+\nu}{\alpha}(J_1' J_4' + J_2' J_3').$$

3. Equation (5) can be solved for medium deflections by making use of the fact that the external load $p(\alpha)$ is bounded and by applying the method of successive approximations. It will be assumed that $w = \lim_{n\to\infty} w_n$, where w_n satisfies conditions (3) and (4) and the recurrent equation by means of which every successive approximation can be determined from the preceding one:

$$L(w_n) = \frac{p(\alpha) R^2}{2Eh} + \frac{1}{h} N_1(w_{n-1}) + \frac{1}{h^2} N_2(w_{n-1}). \tag{8}$$

The function w_n is determined by formula (7), with the actual load $p(\alpha)$ replaced by the fictitious load

$$p^*(\alpha) = p(\alpha) + \frac{2E}{R^2}\left(N_1(w_{n-1}) + \frac{1}{h} N_2(w_{n-1})\right).$$

We shall represent $p(\alpha)$ in the form $p(\alpha) = p_0 \cdot \tilde{p}(\alpha)$, where p_0 has the dimension of stress. The process of successive approximations converges if p_0 does not exceed some definite value, which depends on the geometrical

dimensions of the shell and the law of variation of the load $\bar{p}(\alpha)$. In the general case the limiting value of p_0 for which the process of successive approximations is still convergent can be found by direct calculations.

We have solved as an example the problem of a load varying by the law $p(\alpha) = p_0 \cos \frac{\pi\alpha}{2\alpha_0}$. It was found that for $\alpha_0 = 6$ the process of successive approximations converges for $t \leqslant 2$ and diverges for $t \geqslant 3$, where $t = \frac{p_0}{E}\left(\frac{R}{2h}\right)^2 \sqrt{3(1-\nu^2)}$. The maximum deflections for the converging process attained the order of two shell thicknesses.

4. In order to study large deflections it is necessary to apply to equation (5) a method of solution which does not make an important use of the fact that the load $p(\alpha)$ is bounded. It is possible in particular to use the Bubnov-Galerkin method and to look for a solution of (5) in the form

$$w = h(Aw_1 + Bw_2), \tag{9}$$

where w_1 and w_2 are known functions satisfying conditions (3) and (4), and A and B are constants depending on the amplitude of the load p_0 which remain to be determined. In order to obtain the best approximation to the solution of the initial problem we selected w_1 and w_2 in a special way.

We select as w_1 a function satisfying (up to a constant factor) the linearized equation of equilibrium of the shell considered at given loading law and given boundary conditions

$$L(w_1) = \bar{p}(\alpha), \qquad \begin{array}{lll} w_1' = 0, & \frac{d}{d\alpha}\nabla^2 w_1 = 0 & \text{for } \alpha = 0, \\ w_1 = 0, & w_1' = 0 & \text{for } \alpha = \alpha_0. \end{array} \tag{10}$$

Such a selection of the function w_1 not only satisfies all the boundary conditions, but also makes it possible to allow for the specific law of distribution of the external load.

As w_2 we select a function proportional to the deflection of the segment considered under the action of a circumferential concentrated load applied along a circle of radius $r_0\alpha_1$. The parameter w_2 represents an indeterminate constant, which can be varied in order to obtain the minimum value of the critical load. The boundary conditions which the function w_2 satisfies are the same as for the function w, i.e., (3) and (4). The range of variation of the parameter α_1 is $0 < \alpha_1 < \alpha_0$.

The function w_1 can be expressed explicitly with the aid of formula (7), with $p(\alpha)$ replaced by $\frac{2Eh}{R^2}\bar{p}(\alpha)$.

The function w_2 can be represented in the form

$$w_2 = C_1 + A_1 J_1 + A_2 J_2 - \left\{ \begin{array}{l} 0, \ 0 \leqslant \alpha \leqslant \alpha_1 \\ \sum_{i=1}^{4} D_i J_i, \ \alpha_1 \leqslant \alpha \leqslant \alpha_0 \end{array} \right\}, \tag{11}$$

where

$$D_1 = J_3(\alpha_1), \quad D_2 = J_4(\alpha_1), \quad D_3 = -J_1(\alpha_1), \quad D_4 = -J_2(\alpha_1),$$

$$A_1 = J_3(\alpha_1) + k(\alpha_0)\left[-\frac{2(1+\nu)}{\pi\alpha_0^2} J_2'(\alpha_0) + J_1(\alpha_1)\eta_1(\alpha_0) + J_2(\alpha_1)\eta_2(\alpha_0)\right],$$

$$A_2 = J_4(\alpha_1) + k(\alpha_0)\left[\frac{2(1+\nu)}{\pi\alpha_0^2} J_1'(\alpha_0) + \right.$$

$$\left. + J_1(\alpha_1)\eta_2(\alpha_0) + J_2(\alpha_1)\left(\frac{2}{\pi\alpha_0} - \eta_1(\alpha_0)\right)\right],$$

$$C_1 = \frac{2(1+\nu)}{\pi\alpha_0^2} k(\alpha_0)\{J_1'(\alpha_0)[J_2(\alpha_1) - J_2(\alpha_0)] - J_2'(\alpha_0)[J_1(\alpha_1) - J_1(\alpha_0)]\}.$$

The function w_2 corresponds to the influence function for the linearized problem of bending of the shallow segment considered.

By using the Bubnov-Galerkin method we obtain a system of two equations for determining the constants A and B:

$$\int_0^{\alpha_0}\left[AL(w_1) + BL(w_2) - \frac{p(\alpha)R^2}{2Eh^2} - N_1(Aw_1 + Bw_2) - \right.$$

$$\left. - N_2(Aw_1 + Bw_2)\right] w_i \alpha d\alpha = 0, \quad (i = 1, 2). \tag{12}$$

By using the properties of the functions w_1 and w_2 we can write the following equalities:

$$\int_0^{\alpha_0} L(w_1) w_i \alpha d\alpha = \int_0^{\alpha_0} p(\alpha) w_i \alpha d\alpha; \quad \int_0^{\alpha_0} L(w_2) w_i \alpha d\alpha = \frac{2}{\pi} w_i(\alpha_1), \ (i=1, 2). \tag{13}$$

The explicit form of expressions $N_1(Aw_1 + Bw_2)$ and $N_2(Aw_1 + Bw_2)$ is:

$$\begin{gathered} N_1(Aw_1 + Bw_2) = A^2 N_1(w_1) + B^2 N_1(w_2) + ABM(w_1, w_2), \\ N_2(Aw_1 + Bw_2) = A^3 N_2(w_1) + B^3 N_2(w_2) + \\ + \frac{3(1-\nu^2)}{4} AB[A\Phi(w_1; w_2) + B\Phi(w_2; w_1)], \end{gathered} \tag{14}$$

where

$$M(w_1, w_2) = -\sqrt{3(1-\nu^2)}\left\{\frac{1}{\alpha}(w_1 w_2' + w_2 w_1') + \chi_1(w_1)\left(w_2'' - \frac{1}{\alpha} w_2'\right) + \right.$$

$$+ \chi_1(w_2)\left(w_1'' - \frac{1}{\alpha} w_1'\right) + \int_{\alpha_0}^{\alpha} \frac{w_1' w_2'}{\alpha} d\alpha + \frac{1+\nu}{1-\nu}\left[\chi_1(w_1(\alpha_0)) \cdot \nabla^2 w_2 + \right.$$

$$\left.\left. + \chi_1(w_2(\alpha_0)) \cdot \nabla^2 w_1 - \frac{1}{\alpha_0^2}\int_0^{\alpha_0} \alpha w_1' w_2' d\alpha\right]\right\};$$

$$\Phi(w_1; w_2) = 2\nabla^2 w_1 \cdot \left[\frac{1+\nu}{(1-\nu)\alpha_0^2}\int_0^{\alpha_0} \alpha w_1' w_2' d\alpha + \int_{\alpha}^{\alpha_0} \frac{w_1' w_2'}{\alpha} d\alpha\right] +$$

$$+ 2\left(w_1'' - \frac{1}{\alpha} w_1'\right)\frac{1}{\alpha^2}\int_0^{\alpha} \alpha w_1' w_2' d\alpha + \left(w_2'' - \frac{1}{\alpha} w_2'\right)\frac{1}{\alpha^2}\int_0^{\alpha} \alpha (w_1')^2 d\alpha +$$

$$+ \nabla^2 w_2 \cdot \left[\chi_2(w_1(\alpha_0)) - \chi_2(w_1) + \frac{1+\nu}{(1-\nu)\alpha_0^2}\int_0^{\alpha} \alpha (w_1')^2 d\alpha\right].$$

The function $\Phi(w_2; w_1)$ is obtained from $\Phi(w_1; w_2)$ by interverting w_1 and w_2.

The substitution of (13) and (14) in (12) yields

$$\left(A-\frac{p_0R^2}{2Eh^2}\right)\int_0^{\alpha_0}\tilde{p}(\alpha)\,w_i\alpha d\alpha+B\frac{2}{\pi}w_i(\alpha_1)=A^2\int_0^{\alpha_0}N_1(w_1)\,w_i\alpha d\alpha+$$

$$+B^2\int_0^{\alpha_0}N_1(w_2)\,w_i\alpha d\alpha+AB\int_0^{\alpha_0}M(w_1,w_2)\,w_i\alpha d\alpha+A^3\int_0^{\alpha_0}N_2(w_1)\,w_i\alpha d\alpha+$$

$$+B^3\int_0^{\alpha_0}N_2(w_2)\,w_i\alpha d\alpha+A^2B\frac{3(1-\nu^2)}{4}\int_0^{\alpha_0}\Phi(w_1;w_2)\,w_i\alpha d\alpha+$$

$$+AB^2\frac{3(1-\nu^2)}{4}\int_0^{\alpha_0}\Phi(w_2;w_1)\,w_i\alpha d\alpha,\quad (i=1,2). \tag{15}$$

By calculating the coefficients of the different powers of A and B we obtain a system of two equations from which we obtain expressions for these constants as functions of the load parameter p_0.

The results of the solution of system (15) can be represented in the form of the curve $p_0=p_0[w(\alpha_x)]$, where α_x designates the coordinate of some characteristic point, whose displacements characterize the local stability of the shell. As such point can be taken the dome top or the point where the load maximum is attained.

5. System (15) can be split in two equations in a first approximation. By writing $A\neq 0$, $B=0$, we obtain an equation for determining A:

$$\left(A-\frac{p_0R^2}{2Eh^2}\right)\int_0^{\alpha_0}\tilde{p}(\alpha)\,w_1\alpha d\alpha=A^2\int_0^{\alpha_0}N_1(w_1)\,w_1\alpha d\alpha+A^3\int_0^{\alpha_0}N_2(w_1)\,w_1\alpha d\alpha; \tag{16}$$

by writing $A=0$, $B\neq 0$, we obtain an equation for determining B:

$$-\frac{p_0R^2}{2Eh^2}\int_0^{\alpha_0}\tilde{p}(\alpha)\,w_2\alpha d\alpha+B\frac{2}{\pi}w_2(\alpha_1)=B^2\int_0^{\alpha_0}N_1(w_2)\,w_2\alpha d\alpha+B^3\int_0^{\alpha_0}N_2(w_2)\,w_2\alpha d\alpha. \tag{17}$$

In this case, when finding the relationship between p_0 and $w(\alpha_x)$ from (16) one should take $w(\alpha_x)=Ahw_1(\alpha_x)$ and from (17): $w(\alpha_x)=Bhw_2(\alpha_x)$.

A second-approximation solution can also be found for other forms of the functions w_1 and w_2. Only the left-hand parts of equations (15) are altered as a result. If in particular the function w_2 is left unchanged and the function w_1 is replaced by this same function w_2 but with a different value of the parameter α_1, for instance α_1^*, the left-hand part of (15) becomes

$$A\frac{2}{\pi}w_i(\alpha_1^*)+B\frac{2}{\pi}w_i(\alpha_1)-\frac{p_0R^2}{2Eh^2}\int_0^{\alpha_0}\tilde{p}(\alpha)\,w_i\alpha d\alpha,\quad (i=1,2), \tag{18}$$

where $w_2(\alpha)$ is determined as before by formulas (11), and $w_1(\alpha)$ differs from $w_2(\alpha)$ only in that α_1 is replaced by α_1^*.

6. We have examined as an application of the Bubnov-Galerkin method the case of loading of the segment by a circumferential distributed pressure.

If we designate by F the total force acting on the shell, the function $p(\alpha)$ becomes of the form

$$p(\alpha)=\frac{F\delta(\alpha)}{4\pi r_0^2\alpha_s\varepsilon}, \text{ where } \delta(\alpha)=\begin{cases}0, & 0\leqslant\alpha\leqslant\alpha_s-\varepsilon\\ 1, & \alpha_s-\varepsilon\leqslant\alpha\leqslant\alpha_s+\varepsilon\\ 0, & \alpha_s+\varepsilon\leqslant\alpha\leqslant\alpha_0\end{cases}. \tag{19}$$

It follows from (19) that the load acts along a ring of outer radius $r_0(\alpha_s+\varepsilon)$ and inner radius $r_0(\alpha_s-\varepsilon)$. As p_0 we took the value $p_0=\frac{F}{4\pi r_0^2}$ and as $\tilde{p}(\alpha)$, $\tilde{p}(\alpha)=\frac{\delta(\alpha)}{\alpha_s\varepsilon}$. A number of calculations were conducted for the following numerical values: $\alpha_0=10$; $\alpha_s=2.2$; $\varepsilon=0.6$. The parameter α_1 was varied, and was taken in turns as equal to $\alpha_1=0.2$; 0.6; 1.7; 1.8; 2.1; 2.4; 3. The solutions were obtained in a first and second approximation.

In addition, the problem of the action of a concentrated circumferential load was also considered. It was assumed in this case that its diameter is equal to $2r_0\alpha_s$. The solution was found from the equations for the circumferential distributed load by passing to the limit for $\varepsilon\to 0$.

7. Note in conclusion that although only the case of a normal load was considered here, the same reasoning applies to the case of the combined action of a normal pressure and meridional distributed forces.

In this case the first equation of (1) will contain a component of the shearing load, which will influence in what follows only the expression of $p(\alpha)$. Since, however, no restrictions were imposed on the form of the function $p(\alpha)$, all methods of solution remain unchanged.

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ITERATIVE ALGORITHMS AND NUMERICAL PROBLEMS OF THE THEORY OF PLATES AND SHELLS

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The development of high-speed computers enabled new approaches to the problems of the theory of elasticity. The traditional methods of calculating structures do not take into account the possibilities of modern computers. There appear many new problems in regions of study which were considered completed not long ago.

Of course, in order to be able to solve a problem on a computer it is not sufficient to know a method of solution realizable by man. Additional logical formulations are necessary in order to formalize all stages of the problem solution.

The calculation of structures on digital computers will become in the future an independent branch of the engineering science. Today the literature on the subject is scarce, and the necessity of exchanging information on the subject is obvious.

We shall consider here the problem of determining the algorithm of the numerical solution of a wide class of systems of discrete equations of the theory of elasticity in such a way as to use the resources of modern digital computers efficiently. The method proposed takes into account the limited volume of the computer memory and makes maximum use of the possibilities offered by the high speed of the calculations.

Methods of solution in which the equations are not transformed in the course of the calculations turned out to be best suited for problem solving by means of computers. In this case the separate equations are not stored in the computer memory, but each of them is formed automatically at the moment when it has to be used. Iterative methods must be considered as suitable from this point of view. The iteration (continued repetitions of similar calculations) is a most natural method of work for digital computers.

A program based on the iterative method was worked out, suitable for solving a wide circle of problems, differing by the type of equations, the configuration of the region, the type of boundary conditions, and other initial parameters. The program makes an efficient use of the computer memory. The input and output data are represented in a compact form. The program has been split into blocks on the basis of a detailed study of the algorithm structure, each block fulfilling a specific function. The set of such standard blocks enables the solution for a wide circle of similar problems to be programmed easily.

The program described below is based on the class of iterative algorithms called descent methods. A brief geometrical exposition of the method used follows.

Consider a positively determined form $\Phi(\vec{\xi})$, where $\vec{\xi}$ is a vector in the n-dimensional Euclidean space. The equation

$$\Phi(\vec{\xi}) = C \tag{1}$$

defines a family of hypersurfaces. Let it be required to determine the "center" of the system of hypersurfaces (1), i.e., to find a vector $\vec{\xi}$ for which $\Phi(\vec{\xi})$ is equal to zero.* Take an arbitrary vecotr $\vec{\xi}_1$ and calculate $\Phi(\vec{\xi}_1) = C_1$. The iterative methods which consist in passing at each step from the vector $\vec{\xi}_1$ to a vector $\vec{\xi}_2$ for which $\Phi(\vec{\xi}_2) = C_2 < C_1$ are called descent methods. The transition from $\vec{\xi}_1$ to $\vec{\xi}_2$ will be represented as motion from the straight line given by the parametric equation

$$\vec{\xi} = \vec{\xi}_1 + \tau\vec{u}, \tag{2}$$

where $\vec{u}$ is a vector characterizing the direction of descent, and τ is a parameter. The multidimensional problem of the minimization of $\Phi(\vec{\xi})$ has been reduced thus to a series of one-dimensional problems of determining the minimum of $\varphi(\tau) = \Phi(\vec{\xi}_1 + \tau\vec{u})$.

The vector $\vec{\xi}_2$ is determined by assuming a certain value of the parameter $\tau = t$:

$$\vec{\xi}_2 = \vec{\xi}_1 + t\,\vec{u}. \tag{3}$$

A set of different descent methods is obtained by assuming different conditions for determining the vector $\vec{u}$ and the number t. The step t can be determined in particular from the condition that the minimum of $\Phi(\vec{\xi})$ is attained on the straight line (2). This minimum is attained at the point where the hyperline (2) is tangential to one of the hypersurfaces of family (1). The condition of tangency is

$$\vec{u}\,\mathrm{grad}\,\Phi(\vec{\xi}_1 + t\,\vec{u}) = 0. \tag{4}$$

The problem of solving the functional equation

$$A\,\vec{\xi} = \vec{b}, \tag{5}$$

where A is an operator in the n-dimensional Euclidean space, $\vec{\xi}$ is the unknown vector, and $\vec{b}$ is a known vector, is equivalent to the minimization of the form

$$\Phi(\vec{\xi}) = (A\,\vec{\xi} - \vec{b})(A\,\vec{\xi} - \vec{b}). \tag{6}$$

Relationship (4) then becomes of the form

$$\vec{u}\,\mathrm{grad}\,(A\,\vec{\xi}_2 - \vec{b})\,(A\,\vec{\xi}_2 - \vec{b}) = 0. \tag{7}$$

If the vector $\vec{u}$ is selected at each step parallel to one of the coordinate axes, we obtain the relaxation method.

* It is assumed that the problem has a unique solution.

If we select $\vec{u} = \vec{r}_1$, where $\vec{r}_1 = A\vec{\xi}_1 - \vec{b}$ is the so-called "discrepancy" vector, we obtain the method of descent by discrepancy.

If we select $\vec{u} = -\operatorname{grad}(A\vec{\xi}_1 - \vec{b})(A\vec{\xi}_1 - \vec{b})$ we obtain the method of fastest descent.

The results of the preceding calculations can be allowed for in the selection of the vector $\vec{u}$. The allowance for the preceding step imposes one constraint on $\vec{u}$, the allowance for the preceding two steps imposes two constraints, etc. The method of imposing the constraints can be the following. Let the value $\xi = \vec{\xi}_2$ have been attained during the descent along the vector $\vec{u}_1$. Equation (4) shows that the vector grad $\Phi(\vec{\xi}_2)$ is orthogonal to $\vec{u}_1$. We shall look for $\min \Phi(\xi)$ among the vectors of the type $\alpha \vec{u}_1 + \beta \operatorname{grad} \Phi(\vec{\xi}_2)$ which leads to the following conditions for calculating α and β:

$$\begin{gathered} \vec{u}_1 \operatorname{grad} \Phi(\vec{\xi}_3) = 0, \\ \operatorname{grad} \Phi(\vec{\xi}_2) \operatorname{grad} \Phi(\vec{\xi}_3) = 0. \end{gathered} \tag{8}$$

Equation (4) is nonlinear in the case of nonlinear problems. The transition from the problem of the minimization of $\varphi(r)$ to equation (4) can turn out to be only a reformulation of the problem, so that it is best to find directly by a numerical method the minimum value of the concave function $\varphi(r)$ without calculating its derivative. The essence of such methods is the introduction in the number of basic mathematical operations of the process of finding the minimum number from a finite table.

The methods described above are based on reaching the minimum of $\Phi(\vec{\xi})$ on the line (2) at each step.

We shall give now an algorithm which makes it possible to have the the final aim, the vector $\vec{\xi}_0$, in view at each step. Introduce the designation $E_k^{(i)}$ for a k-dimensional space. Our object will be to determine $E_0^{(0)}$, the end of vector $\vec{\xi}_0$, as the intersection of n mutually orthogonal subspaces $E_{n-1}^{(1)}$, $E_{n-1}^{(2)}$ $\cdots$, $E_{n-1}^{(n)}$. The system of subspaces is built in the following way. We select a certain direction of descent $\vec{u}_1$:

$$\vec{\xi}_2 = \vec{\xi}_1 + t_1 \vec{u}_1. \tag{9}$$

The step t_1 is selected from the condition

$$(\vec{\xi}_2 - \vec{\xi}_1)(\vec{\xi}_2 - \vec{\xi}_0) = 0. \tag{10}$$

The space $E_{n-1}^{(1)}$ is selected so that the vector $(\vec{\xi}_2 - \vec{\xi}_0)$ lies in it and $E_{n-1}^{(1)}$ is orthogonal to $\vec{u}_1$. Obviously $E_0^{(0)} \subset E_{n-1}^{(1)}$.

The vector $\vec{u}_2$ is selected from among the vectors $\in E_{n-1}^{(1)}$

$$\vec{\xi}_3 = \vec{\xi}_2 + t_2 \vec{u}_2. \tag{11}$$

The value of t_2 is determined from the condition

$$(\vec{\xi}_3 - \vec{\xi}_2)(\vec{\xi}_3 - \vec{\xi}_0) = 0. \tag{12}$$

The subspace $E_{n-1}^{(2)}$ is selected in such a way that the vector $(\vec{\xi}_3 - \vec{\xi}_0)$ lies in it and $E_{n-1}^{(2)}$ is orthogonal to $\vec{u}_2$, and therefore also $E_{n-1}^{(2)} \cdot E_0^{(0)} \subset E_{n-1}^{(2)}$, whence $E_0^{(0)} \subset E_{n-1}^{(1)} \cap E_{n-1}^{(2)}$.

This process is continued until we obtain at the n-th step

$$\vec{\xi}_n = \vec{\xi}_{n-1} + t_{n-1}\vec{u}_{n-1} = \vec{\xi}_0 , \tag{13}$$

with

$$\vec{u}_{n-1} \subset E_{n-1}^{(1)} \cap E_{n-1}^{(2)} \cap \cdots \cap E_{n-1}^{(n-1)} ,$$

$$E_0^{(0)} = E_{n-1}^{(1)} \cap E_{n-1}^{(2)} \cap \cdots \cap E_{n-1}^{(n-1)} \cap E_{n-1}^{(n)} .$$

This method is applicable in those cases when it is possible to satisfy equations (10), (12), by eliminating from them $\vec{\xi}_0$. In the case when $\Phi(\vec{\xi}) = (A\vec{\xi} - \vec{b})(A\vec{\xi} - \vec{b})$, where A is a linear operator, the iterative formula can be represented in the following form:

$$\vec{\xi}_{l+1} = \vec{\xi}_l + t_l A'\vec{v}_l , \tag{14}$$

where

$$t = \frac{\vec{r}_l \cdot \vec{v}}{A'\vec{v}_l \cdot A\vec{v}} ,$$

$$\vec{v} = \sum_{j=1} a_{lj}\vec{v}_j + a_{ll}\vec{r}_l ,$$

$$\vec{r}_l = A\vec{\xi} - \vec{b} ,$$

$$a_{ll} = \frac{\vec{r}_l \cdot AA'\vec{r}_l}{AA'\vec{r}_l \cdot AA'\vec{r}} , \qquad a_{lj} = -a_{ll}\frac{AA'\vec{r}_l\vec{v}_j}{A'\vec{v}_j \cdot A'\vec{v}_j} .$$

We obtain the method of conjugate gradients, which our studies have shown to be very efficient in solving a number of problems.

We shall now describe the program. Its distinctive feature is that the structure of the equations, the form of the grid region and the type of the boundary conditions do not figure in the program itself, but are introduced with the input data.

A set of similar blocks d_0, d_1, d_2, d_3...* are set apart in the computer memory for storing the input data and the results of the calculations. Block d_1 serves for storing the approximate values of the unknowns, d_2 for storing the "discrepancies", d_3 for storing the loads, etc. The number of blocks set apart for intermediate calculations is determined by the variant of the descent method used.

The cells in all the blocks are numbered in the same way. The address of a given cell is obtained by adding to its number a the constant d_j which defines the position of the block in the memory. These blocks are associated with the concept of a rectangular grid. A grid region of arbitrary shape considered in the problem is inserted in these blocks, and forms a part of them.

The method used for distributing memory space, based on the iterative method, results in a considerable saving in memory space as compared with methods requiring the filling of matrices. Thus, $4n$ cells are needed

* Block d_j starts from cell d_{j+1}.

in a problem with n variables as against n^2 cells necessary for storing a matrix of order n.

Information on the form of the region and on the types of boundary conditions is fed to the computer in the following way. A special copy d_0 of the grid has been provided, on which are marked the codes describing the type of the given point and corresponding to inner points of the region, to points which do not belong to the given region, to points lying on the contour with an arbitrary support, to "forecontour" points, etc. To each code, i.e., to each type of point, there corresponds a specific structure of the equations.

The algorithms proposed have been written in the address-programming language /1/.

We shall use the following symbols: $'i$ is the iteration number, * $'i^0$ is the total number of iterations, $'a$ is the number of the grid node, $'a_0$ is the total number of nodes.

We start with the following commands:

$$1:1 => i,$$
$$2:1 => a.$$

The calculation will then proceed along different paths depending on the coded position of the point a relative to the boundaries of the elastic region. This is realized by means of the multivalued predicative formula

$$3:\ Q['(d_0 + 'a) = 1,2,\cdots,\ k,\ 0;\ 4,5,\cdots,\ k+3,\ k+4],$$

which means that at $'(d_0 + 'a) = 1,2,\cdots,\ k,\ 0$ control is transferred respectively to the commands $4,5,\ \cdots k+3,\ k+4$.

The commands

$$4:\ f_4[\ \ ] => d_2 + 'a,\quad k+4;$$
$$5:\ f_2[\ \ ] => d_2 + 'a\quad k+4;$$
$$\cdots\cdots\cdots\cdots$$
$$k+3:\ f_k[\ \ ] => d_2 + 'a,\ k+4$$

calculate the discrepancy in node a of the grid, with subsequent jump to command $k+4$.

The functions $f_{(d_1+a)}$ [ ] depend on several magnitudes of the type $'(d_1 + 'a + j)$.

As a result of this arrangement of the program the equation for each node of the grid is not stored in the memory, but is formed automatically every time when needed. Note that commands 3 to $k+3$ form a separate block and are fed to the computer with the input data, and not with the program.

$$k+4:\ '(d_2 + 'a) + '(d_3 + 'a) => d_2 + 'a.$$

The following two commands serve for by-passing the grid

$$k+5:\ \ 'a + 1 => a,$$
$$k+6:\ \ p\ ('a < 'a_0)\,3.$$

Command $k+6$ checks the condition $'a < 'a_0$. If it is fulfilled, control is transferred to command 3, and in the contrary case to the next command.

* The symbol "$'i$" means "the number contained in cell i".

$k+7$: calculation of the descent step,

$k+10$: calculation of new approximate values of the unknowns.

The form of the program blocks $k+7$, $k+10$ depends on the iterative method used. The by-passing of the grid and the sequence of operations inside operators $k+7$, $k+10$ as a function of $'(d_0+'a)$ are the same as illustrated above in the calculation of the discrepancies.

The following commands ensure the fulfillment of $'i_0$ iterations and the printing of the results.

$k+11$: $'i+1 => i$,

$k+12$: $p('i < 'i_0)$,

$k+13$: printing of the results. The computer stops.

This program can be repeated several times in succession. By transfering control to command 1 at the end of the program, the computer will perform $'i_0$ more iterations, using the results obtained before as initial approximation, and then stop. We shall give as an illustration the form of commands 3, ..., for the case of bending of a rectangular plate:

$$'(d_1+'a)-'(d_1+2+'a) => d_1+'a;\quad k_1+4,$$
$$'(d_1+'a)-'(d_1-2+'a) => d_1+'a;\quad k_1+4,$$
$$'(d_1+'a)-'(d_1+2N+'a) => d_1+'a;\quad k_1+4,$$
$$'(d_1+'a)-'(d_1-2N-'a) => d_1+'a;\quad k_1+4,$$
$$20\,'(d_1+'a)-8\,[\,'(d_1-1+'a)+'(d_1+1+'a)+'(d_1+N+'a)+$$
$$+\,'(d_1-N+'a)]+2\,[\,'(d_1-N-1+'a)+'(d_1-N+1+'a)+$$
$$+\,'(d_1+N-1+'a)+'(d_1+N+1+'a)]+'(d_1-2N+'a)+'(d_1-$$
$$-2+'a)+'(d_1+2N+'a)+'(d_1+2+'a) => d_1+'a;\quad k_1+4.$$

It is important to note that only the contents of memory block d_0 vary with the passage from a problem relative to a region of given shape to a problem relative to a different shape or to a problem with different boundary conditions. If one passes to problems containing different difference equations, it is necessary to replace only the program block consisting of commands 3 to $k+3$. With the transition to a different iterative method the only program blocks to change are $k+7$ and $k+16$.

If the difference equations contain variable parameters, such as a variable grid step, a variable rigidity, the values of these magnitudes are stored in special memory blocks, similar to those indicated above.

A group of profiles were calculated for the case of torsion, and the Dirichlet problem was solved for a set of 625 nodes. A plate loaded by a concentrated transverse force was calculated for a grid of 100 nodes. A cylindrical panel was also calculated. The two-dimensional problem for a double-connected region was solved by iterations. The problem of calculating warped plate systems was programmed. The algorithms and programs considered remain unchanged in the case of physically discrete systems, such as statically undeterminate frames or guy nets. The same methods can also be applied to three-dimensional problems of the theory of elasticity and to nonlinear problems, where they are particularly efficient.

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VIBRATIONS OF SHALLOW SPHERICAL SHELLS

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1. The system of equations of transverse vibrations of a shallow spherical shell can be written in the form

$$\nabla^2\nabla^2\varphi + \frac{Eh}{R}\nabla^2 w = 0;$$

$$\nabla^2\nabla^2 w - \frac{1}{RD}\nabla^2\varphi + \frac{\gamma h}{D}\frac{\partial^2 w}{\partial t^2} = \frac{P(t)\,\delta(\bar{r}-\bar{r}_0)}{D}; \qquad (1)$$

where φ is a stress function; w is the deflection; R and h are the radius and thickness of the shell; γ and E are the specific gravity and modulus of elasticity of the shell material; $D = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity; $P(t)\,\delta(\bar{r}-\bar{r}_0)$ is the disturbing force normal to the surface; ∇^2 is the Laplacian in polar coordinates ρ, θ; t is time.

We shall consider harmonic vibrations. Then, writing $P(t) = P_0 e^{i\omega_0 t}$ and therefore $\varphi = \bar{\varphi} e^{i\omega_0 t}$, $w = \bar{w} e^{i\omega_0 t}$ we reduce system (1) to

$$\nabla^2\nabla^2\bar{\varphi} + \frac{Eh}{R}\nabla^2\bar{w} = 0;$$

$$\nabla^2\nabla^2\bar{w} - \frac{1}{RD}\nabla^2\varphi - \frac{\gamma h}{D}\omega_0^2\bar{w} = \frac{P_0\,\delta(\bar{r}-\bar{r}_0)}{D}. \qquad (2)$$

In order to obtain separate equations for φ and w introduce the auxiliary functions F and ψ /1/

$$\bar{\varphi} = F - \frac{Eh}{R}\psi, \qquad \bar{w} = \nabla^2(\psi - \lambda F). \qquad (3)$$

If we write $\lambda = (\gamma h \omega_0^2 R)^{-1}$, we obtain from (2) the following two independent equations for determining these auxiliary functions:

$$\nabla^2\nabla^2 F = 0;$$

$$\nabla^2(\nabla^2\nabla^2 - \varkappa^4)\,\psi = \frac{P_0\,\delta(\bar{r}-\bar{r}_0)}{D}; \qquad (4)$$

where

$$\varkappa^4 = \frac{h(R^2\omega_0^2\gamma - E)}{DR^2}.$$

2. The solution of the first equation of (4) can be considered as known; as to the second equation, its solution can be represented in the form

$$\psi = \psi_p + \psi_0 + \psi_1 + \psi_2, \tag{5}$$

where ψ_p is a particular solution of the nonhomogeneous equation; the other three functions are the general solutions of the following equations respectively:

$$\begin{aligned} \nabla^2\psi_0 &= 0; \\ \nabla^2\psi_1 + \varkappa^2\psi_1 &= 0; \\ \nabla^2\psi_2 - \varkappa^2\psi_2 &= 0. \end{aligned} \tag{6}$$

There is no need to dwell on the solution of the Laplace equation. The general solution of the second equation is

$$\psi_1 = [C_{1n}J_n(\varkappa\rho) + D_{1n}H_n(\varkappa\rho)]\sin n\theta + [\tilde{C}_{1n}J_n(\varkappa\rho) + \tilde{D}_{1n}H_n(\varkappa\rho)]\cos n\theta, \tag{7}$$

where C and D are integration constants and J_n and H_n are n-th order Bessel and Hankel functions of the first kind. The solution of the third equation is of the same form, the only difference being that the factor $i = \sqrt{-1}$ appears in the arguments of the Bessel and Hankel functions.

The next step consists in finding a particular solution ψ_p. Using the representation of the δ-function by a Fourier integral

$$\delta(\bar{r} - \bar{r}_0) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{i\bar{k}(\bar{r}-\bar{r}_0)}(d\bar{k}),$$

where $(d\bar{k}) = dndm$, $\bar{k}(\bar{r} - \bar{r}_0) = n(x - x_0) + m(y - y_0)$, we can also represent ψ_p as such an integral:

$$\psi_p = -\frac{P_0}{4\pi^2 D}\iint_{-\infty}^{\infty}\frac{e^{i\bar{k}(\bar{r}-\bar{r}_0)}(d\bar{k})}{(n^2+m^2)[(n^2+m^2)^2 - \varkappa^4]}. \tag{8}$$

Integral (8) can be calculated by the methods of the theory of functions. The integration contour is to be selected as for a source. Following /2/, we obtain then

$$\psi_p = -\frac{P_0}{4\pi D\varkappa^4}\left\{\ln|\bar{r} - \bar{r}_0|^2 + \frac{\pi i}{2}[H_0(i\varkappa|\bar{r} - \bar{r}_0|) + H_0(\varkappa|\bar{r} - \bar{r}_0|)]\right\}.$$

In polar coordinates ρ, θ we can, by using the series expansions for the logarithm and the cylindrical functions /3/, represent ψ_p in the following form:

$$\psi_p = -\frac{P_0}{2\pi D\varkappa^4}\left[\ln\rho_0 - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{\rho_0}\right)^n\cos n(\theta - \theta_0)\right] -$$

$$-\frac{iP_0}{8D\varkappa^4}\sum_{n=0}^{\infty}{}'[J_n(i\varkappa\rho)H_n(i\varkappa\rho_0) + J_n(\varkappa\rho)H_n(\varkappa\rho_0)]\cos n(\theta - \theta_0); \qquad (\rho < \rho_0)$$

$$\psi_p = -\frac{P_0}{2\pi D\varkappa^4}\left[\ln\rho - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho_0}{\rho}\right)^n\cos n(\theta - \theta_0)\right] -$$

$$-\frac{iP_0}{8D\varkappa^4}\sum_{n=0}^{\infty}{}'[J_n(i\varkappa\rho_0)H_n(i\varkappa\rho) + J_n(\varkappa\rho_0)H_n(\varkappa\rho)]\cos n(\theta - \theta_0).$$

The apostrophe on the summation sign means that the first term of the sum is to be multiplied by 2.

3. In the particular case of axial symmetry, when the disturbing force is applied at the top of the shell, we have $\rho_0 = 0$, and also $D_{1n} = \widetilde{D}_{1n} = D_{2n} = \widetilde{D}_{2n} = 0$. Based on the relationship between the functions φ and w and the functions F and ψ we can write

$$\varphi e^{-i\omega_0 t} = a\rho^2 + \frac{EhP_0}{2\pi D\varkappa^4 R}\ln\rho - \frac{Eh}{R}[CJ_0(\varkappa\rho) + dJ_0(i\varkappa\rho)] + \\ + \frac{iEhP_0}{4D\varkappa^4 R}[H_0(\varkappa\rho) + H_0(i\varkappa\rho)], \tag{9}$$

$$we^{-i\omega_0 t} = -4\lambda a - \varkappa^2 CJ_0(\varkappa\rho) + \varkappa^2 dJ_0(i\varkappa\rho) + \frac{iP_0}{4D\varkappa^2}[H_0(\varkappa\rho) - H_0(i\varkappa\rho)].$$

Note that the solution obtained is arbitrary, both in this particular case and in the general case, and that it is possible to satisfy any boundary conditions (on two contours in the general case). It is seen that in the particular case of axial symmetry the solution is obtained in a closed form.

The arbitrary constants a, c, d in the expression for φ and w are determined from the boundary conditions. Thus, in the case of rigid fixation of the shell edge, the boundary conditions

$$u = w = \frac{dw}{d\rho} = 0 \quad \text{for} \quad \rho = b$$

yield a system of three linear algebraic equations for determining these constants:

$$CJ_0'(\varkappa b) + dI_0'(\varkappa b) + \frac{2Rb}{(1+\nu)\varkappa}\left(\frac{1-\nu}{Eh} - \frac{2\lambda}{R}\right)a = \\ = \frac{P_0 R}{2\pi D\varkappa^5 b} + \frac{iP_0}{4D\varkappa^4}\left[H_0'(\varkappa b) - \frac{2i}{\pi}K_0'(\varkappa b)\right]; \\ CJ_0(\varkappa b) - dI_0(\varkappa b) + \frac{4\lambda}{\varkappa^2}a = \frac{iP_0}{4D\varkappa^4}\left[H_0(\varkappa b) + \frac{2i}{\pi}K_0(\varkappa b)\right]; \\ CJ_0'(\varkappa b) - dI_0'(\varkappa b) = \frac{iP_0}{4D\varkappa^4}\left[H_0'(\varkappa b) + \frac{2i}{\pi}K_0'(\varkappa b)\right]. \tag{10}$$

Here I_0 and K_0 are modified Bessel functions of the first and second kind. The solution of system (10) yields the natural frequencies of the shell for the boundary conditions considered. By equating the determinant of the system to zero we obtain the frequency equation

$$J_0(\varkappa b)I_1(\varkappa b) + J_1(\varkappa b)I_0(\varkappa b) + \frac{4(1+\nu)\lambda J_1(\varkappa b)I_1(\varkappa b)}{R\varkappa b\left(\frac{1-\nu}{Eh} - \frac{2\lambda}{R}\right)} = 0. \tag{11}$$

From the value of $\varkappa b$ we determine the fundamental frequency by means of the formula

$$\omega^2 = \frac{E}{\gamma}\left[\frac{\varkappa^4 h}{12(1-\nu^2)} + \frac{1}{R^2}\right]. \tag{12}$$

In the case of boundary conditions

$$T_\rho = w = \frac{dw}{d\rho} = 0 \quad \text{for} \quad \rho = b$$

the frequency equation will be:

$$J_0(\varkappa b) I_1(\varkappa b) + J_1(\varkappa b) I_0(\varkappa b) - \frac{4\lambda Eh}{R\varkappa b} J_1(\varkappa b) I_1(\varkappa b) = 0.$$

Consider a shell with $R = 2$ m, $h = 0.2$ cm, $E = 7.2 \cdot 10^5$ kg/cm^2, $\nu = 0.3$, $b = 40$ cm, $\gamma g = 2200$ kg/m^3. We obtain for it $\varkappa b = 7.62$, $\omega^2 = 9573$, or $\omega_1 = 15.6$ cycles/sec. For a plate with the same characteristics $\omega_1 = 14.3$. It is seen therefore that the curvature does not have a considerable effect on the natural frequencies.

It follows from the formulas obtained that at $\omega \to \omega_0$, the amplitudes of vibrations increase indefinitely, and therefore the analysis of the shell vibrations near $\omega = \omega_0$ is realized allowing for the dissipation of energy inside the material. In particular, a satisfactory approximation of the process for certain plastics at room temperature is obtained by assuming a force of internal friction proportional to the rate of transverse deflection./4/.

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ON THE THEORY OF CALCULATING PLATES ON AN ELASTIC FOUNDATION

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The theory of calculating plates on an elastic foundation with two elastic characteristics is being widely used lately. This theory is free of many of the shortcomings of the Winkler theory; it is also much simpler mathematically than the methods of solution based on the representation of the elastic foundation as a homogeneous isotropic elastic half-space. The theory has been considerably developed from the point of view of practical applications /2/; the question of the determination of the values of the two foundation moduli k and t entering in the expression for the elastic reaction of the foundation $q = kw - 2t\nabla^2 w$ has not been sufficiently elucidated. It was proposed in /1/ to determine these coefficients experimentally, by applying an eccentric force on a rigid punch placed on the elastic foundation. In /2/ the elastic foundation was replaced by an elastic layer of thickness H and the foundation moduli were determined by semi-empirical formulas containing, in addition to Young's modulus E_0 and the Poisson ratio μ_0, the layer thickness H and a parameter γ, called the coefficient of damping of the depression with depth. No sound rule was given, however, for the selection of H and γ in different specific cases.

In the present paper the mathematical frame of the theory with two foundation moduli is preserved, but a different method is advanced for determining their values. It is assumed that the foundation is an elastic half-space, and the parameters k and t are determined from the condition that the results of the calculation diverge as little as possible from the results given by the theory based on the elastic half-space hypothesis. This idea is not new, as applied to a Winkler foundation, but has not been recognized in the literature only because with the one Winkler foundation modulus k_0 it is impossible to ensure a close coincidence of the results relative to both the deflections and the bending moments. The introduction of two foundation moduli widens the possibilities of using this idea, as will be shown below on the example of a circular plate of radius a lying on an elastic foundation and loaded by an arbitrary axisymmetrical load.

Consider a cylindrical circular punch of the same radius a with a spherical base of radius R. When such a punch is pressed in the elastic half-space by a force P, the magnitude of the latter can be selected in such a way that elastic reactions of infinite magnitude appear on the punch edges (Figure 1). The elastic reaction q and the deflection w at any

arbitrary point under the punch can then be determined by the formulas

$$q(\rho) = p_0 \left(\frac{\frac{1}{2} - \frac{2}{3}\delta}{\sqrt{1-\rho^2}} + 2\delta\sqrt{1-\rho^2} \right), \tag{1}$$

$$w(\rho) = \frac{p_0}{k_0}\left(1 + \frac{2}{3}\delta - \delta\rho^2\right), \tag{2}$$

where p_0 is the mean value of the pressure: $p_0 = P : \pi a^2$; ρ is a relative coordinate: $\rho = r : a$;

$$k_0 = \frac{E_0}{1-\mu_0^2} \cdot \frac{2}{\pi a}, \quad \delta = \frac{E_0}{1-\mu_0^2} \cdot \frac{a^3}{P \cdot R}. \tag{3}$$

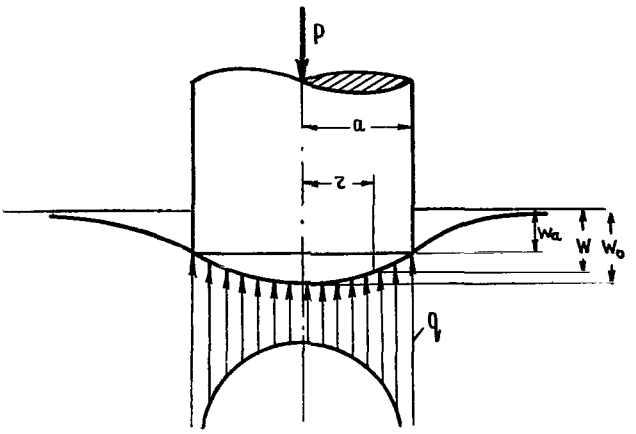

FIGURE 1.

If the flexible circular plate lying on the elastic half-space is loaded from above by a transverse load $p(\rho)$ distributed by law (1), and along the contour by uniformly distributed bending moments M_0 given by the formula

$$M_0 = \frac{D(1+\mu)}{R} = \frac{2\delta}{a^2} \cdot \frac{p_0}{k_0} D(1+\mu) \tag{4}$$

(D is plate rigidity and μ is the Poisson ratio), then such a plate will be in a state of pure bending and its deflection will be determined by (2).

Assume now that the plate lies not on an elastic half-space, but on an elastic foundation characterized by two parameters k and t (Figure 2). The soil reaction and the contour fictitious forces will then be determined by the formulas /2/:

$$q = kw - 2t\nabla w$$

$$Q^{\Phi} = 2t\left[\sqrt{\frac{k}{2t}} \times \frac{K_1\left(\sqrt{\frac{k}{2t}}\,a\right)}{K_0\left(\sqrt{\frac{k}{2t}}\,a\right)} w_a - \frac{a}{R}\right],$$

where K_1 ( ) and K_0 ( ) are modified first-order and zero-order Bessel functions of the second kind; w_α is the deflection on the plate contour.

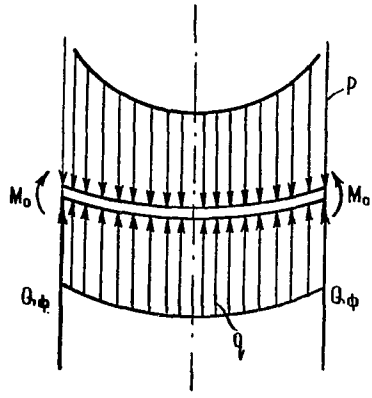

FIGURE 2.

Introducing the designations

$$\varkappa = \frac{k}{k_0}; \quad \tau = \frac{t}{k_0 a^2}; \quad \alpha = \frac{K_1\left(\sqrt{\frac{\varkappa}{2\tau}}\right)}{K_0\left(\sqrt{\frac{\varkappa}{2\tau}}\right)}, \tag{5}$$

we can rewrite the expressions for q and Q_ϕ in the following form:

$$q = p_0\left[\varkappa\left(1+\frac{2}{3}\delta\right)+8\tau\delta-\varkappa\delta\rho^2\right], \tag{6}$$

$$Q_\phi = p_0 \cdot a\left[\alpha\left(1-\frac{1}{3}\delta\right)\sqrt{2\varkappa\tau}-4\tau\delta\right]. \tag{7}$$

Obviously, in the case of an elastic half-space, when the reaction q has the value (1), we obtain $p-q=0$. If q has the value (6), then $p-q\neq 0$ and therefore the plate will bend under the action of a transverse load $p \quad q$. The problem now is to determine the values of the coefficients $\varkappa$ and τ for which the potential energy of bending of the plate under the load $p-q$ will be minimum. It can be asserted that the values of $\varkappa$ and τ determined in this way solve the problem of determining the two foundation moduli in such a way that the solution obtained by the model of an elastic foundation with two moduli differs little from the solution by the elastic half-space model.

We now have to solve the equation of bending of a plate

$$D\nabla^4 w = p - q = p_0\left[\frac{\frac{1}{2}-\frac{2}{3}\delta}{\sqrt{1-\rho^2}}+2\delta\sqrt{1-\rho^2}+\varkappa\delta\rho^2-\right.$$

$$\left.-\varkappa\left(1+\frac{2}{3}\delta\right)-8r\delta\right] \tag{8}$$

with conditions at the edges and the center of the plate

$$w\Big|_{\rho=1}=0 \qquad \frac{dw}{d\rho}\Big|_{\rho=1}=0 \qquad \frac{dw}{d\rho}\Big|_{\rho=0}=0 \qquad Q_\rho\Big|_{\rho=0}=0\ . \tag{9}$$

The second condition of (9) means that the plate is built-in along the contour. This condition must be adopted for two reasons: the variation of the slope on the contour will vary the fictitious loads Q_ϕ, and in addition the contour moments M_0 will do an additional work, which will alter the values of the potential energy.

To determine the potential energy

$$U=\pi a^2 D\int_0^1 (\nabla^2 w)^2\,\rho\, d\rho \tag{10}$$

it is sufficient to determine $\nabla^2 w$ from (8):

$$\nabla^2 w=\frac{p_0 a^4}{D}\left[-\frac{1}{2}\sqrt{1-\rho^2}+\frac{1}{2}\ln\left(1+\sqrt{1-\rho^2}\right)-\frac{2\delta}{9}\sqrt{(1-\rho^2)^3}+\right.$$

$$\left.+\frac{\varkappa\delta}{16}\rho^4-\frac{\varkappa+\frac{2}{3}\varkappa\delta+8\tau\delta}{4}\rho^2+\left(\frac{\varkappa}{8}+\frac{\varkappa\delta}{16}+\tau\delta+\frac{1}{12}+\frac{4}{45}\delta\right)\right]. \tag{11}$$

The substitution of (11) in (10) gives for the potential energy a relationship of the type $U=U(\varkappa,\tau)$, where $\varkappa$ and τ are not independent, but are related by the equilibrium condition $\sum Z=0$:

$$\int_0^1 (p-q)\,2\pi\,(a\rho)\,d\,(a\rho)-2\pi a\cdot Q_\phi=0,$$

or, in expanded form

$$F(\varkappa,\tau)=\varkappa\left(1+\frac{1}{6}\delta\right)+2\sqrt{2\varkappa\tau}\left(1-\frac{1}{3}\delta\right)\alpha-1=0. \tag{12}$$

The problem is thus reduced to finding the minimum of the function $U=U(\varkappa,\tau)$ under condition (12). It follows that the values of $\varkappa$ and τ corresponding to the extremum of the function U are determined from the system

$$\frac{\partial U}{\partial\varkappa}\cdot\frac{\partial F}{\partial\tau}-\frac{\partial U}{\partial\tau}\cdot\frac{\partial F}{\partial\varkappa}=0, \qquad F(\varkappa,\tau)=0. \tag{13}$$

Equations (13) are reduced after simplification to the form

$$2\sqrt{\frac{\varkappa}{2\tau}}\cdot N\left[2\alpha-(\alpha^2-1)\sqrt{\frac{\varkappa}{2\tau}}\right](a\varkappa+b\tau-c)-$$

$$-[M+N(\alpha^2-1)](d\varkappa+e\tau-f)=0, \qquad M\varkappa+2N\alpha\sqrt{2\varkappa\tau}-1=0, \tag{14}$$

where

$$N=1-\frac{1}{3}\delta;\quad M=1+\frac{1}{6}\delta;\quad a=\frac{1}{384}\left(1+\frac{5}{6}\delta+\frac{8}{45}\delta^2\right);$$

$$b=\frac{\delta}{48}\left(1+\frac{5}{12}\delta\right);\quad c=\frac{1}{640}\left(1+\frac{122}{63}\delta+\frac{368}{567}\delta^2\right); \tag{15}$$

$$d=b;\quad e=\frac{\delta^2}{6};\quad f=\frac{\delta}{80}\left(1+\frac{32}{21}\delta\right). \tag{15}$$
(cont'd)

It is seen from (5) and (15) that all the coefficients in (14) depend only on the one dimensionless magnitude δ, and therefore $\varkappa=\varkappa(\delta)$; $\tau=\tau(\delta)$.

By analyzing (14) it is seen that the coefficient $\varkappa$ increases with the increase of δ, and that τ decreases. Taking a lower limit $\tau = 0$ for τ, i.e., passing to the case of a Winkler foundation, we obtain an upper limit for δ equal to 0.582; for this value $\alpha = 1$, $\varkappa = 0.91$. The lower limit of δ is determined from the condition of the soil reaction at the plate center $q(0)=0$. We obtain from (1) for $\rho = 0$ that $\delta_{min}=-0.375$, for which $\alpha = 1.20$, $= 0.48$, $\tau = 0.047$. We thus note that for the problem considered the parameter δ can vary within the limits $0.582 \geqslant \delta \geqslant -0.375$. The corresponding ranges of variation of $\varkappa$, τ, α will be

$$0.91 \geqslant \varkappa \geqslant 0.48;\quad 0 \leqslant \tau \leqslant 0.047;\quad 1 \leqslant \alpha \leqslant 1.2.$$

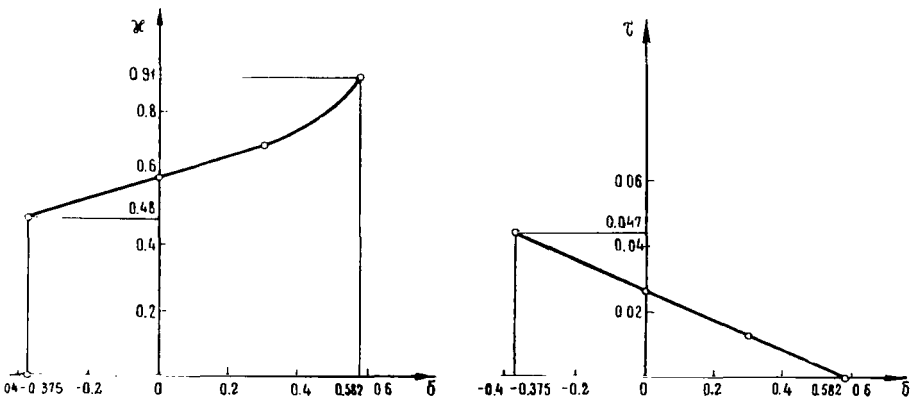

FIGURE 3.

The curves given in Figure 3 represent the values of the coefficients $\varkappa$ and τ as a function of δ which ensure the minimum of the potential energy of plate bending (10) under the action of the load $p-q$ represented in the right-hand part of (8). The results given above correspond to the particular case of pure bending. If the plate is subjected to an arbitrary axisymmetrical transverse load p, the values of the parameters k and t are determined as follows. By writing $p=p_0 f(\rho)$, where p_0 is the mean value of the load, and $f(\rho)$ is a dimensionless function, we approximate the plate deflection under a load p by the deflection at pure bending. This means that the elastic reaction q and the deflection w are taken by formulas (1) and (2), but that the parameter δ, related to the radius R by formula (3), is selected in a suitable way depending on the type of load p.

If δ acquires an increment $d\delta$, then according to the principle of virtual displacements

$$dA_p + dA_q - dU = 0, \tag{16}$$

where dA_p is the elementary work of the external load p, dA_q is the elementary work of the elastic reaction of the foundation, $-dU$ is the elementary work of the internal forces of the plate, equal to the increment of the potential energy dU with opposite sign. The elementary works dA_p and dA_q are calculated by the formulas

$$dA_p = \int_0^0 p \cdot dw \cdot 2\pi\ (a\rho)\ d\ (a\rho); \quad dA_q = -\int_0^1 q \cdot dw \cdot 2\pi\ (a\rho)\ d\ (a\rho).$$

where

$$dw = \frac{p_0}{k_0}\left(\frac{2}{3} - \rho^2\right) d\delta.$$

The potential energy of pure bending can be presented in the form

$$U = \frac{1}{2}\ M_0 \cdot \varphi_0 \cdot 2\pi a,$$

where

$$M_0 = \frac{D\ (1+\mu)}{R}; \quad \varphi_0 = \frac{a}{R}\ .$$

Using relationship (3) between R and δ, we obtain from (16) the following expression for δ:

$$\delta = \frac{12}{\frac{32}{15} + \frac{1}{\sigma}} \int_0^1 f\ (\rho) \left(\frac{2}{3} - \rho^2\right) \rho d\rho, \tag{17}$$

where the coefficient σ, called the relative flexibility of the plate, is determined by the formula

$$\sigma = \frac{k_0\ a^4}{48D(1+\mu)} = \frac{E_0}{E} \cdot \frac{1-\mu}{1-\mu_0^2} \cdot \frac{a^3}{h^3} \cdot \frac{2}{2\pi}\ . \tag{18}$$

Note that σ differs from the similar magnitude S used in /3/ by the factor $6\pi(1+\mu)$.

In the case of concentrated forces the integral in the right-hand part of (17) is to be understood in the Stieltjes sense. Thus, for a uniformly distributed load, when $f(\rho) = 1$, we obtain $\delta = 1 : \left(\frac{32}{15} + \frac{1}{\sigma}\right)$. For a force concentrated at the center we have $\delta = 4 : \left(\frac{32}{15} + \frac{1}{\sigma}\right)$ and for a concentrated load distributed along the contour $\delta = -2 : \left(\frac{32}{15} + \frac{1}{\sigma}\right)$.

Thus, in order to determine the parameters k and t in the case of an axisymmetrically loaded circular plate, we find σ and k_0 by (18) and (3) from the initial data E_0, μ (for the elastic foundation) and E, μ, a, h (for the plate); we then determine δ by (17) from the given load; finally, we determine from the curves of Figure 3 the dimensionless foundation parameters $\varkappa$ and τ, and therefore $k = k_0 \cdot \varkappa$ and $t = k_0 \cdot a^2 \cdot \tau$.

It follows that the parameters k and t should not be considered as constant for the same foundation, since they depend not only on the foundation characteristics E_0 and μ_0, but also on the plate outline, its relative flexibility σ, and the type of the applied load. Several calculations were conducted with the values of k and t found by this method, and their results were compared with the data given in /3/, pp. 260-270. The diagrams of the bending moments and the deflections were very similar to the diagrams obtained from the tabulated data in /3/. The discrepancies were largest at the center of the plate; the value of these discrepancies was lower than 3% for the bending moments, and lower than 8% for the deflections.

Note in conclusion that by using the method described for the determination of k and t it is possible to refine the computing model of the foundation as well. It is thus possible to represent the foundation reaction in the form $q = kw - 2t\nabla^2 w + n \cdot \nabla^4 w$. The equation of plate bending will then be written in the form

$$(D + n)\ \nabla^4 w - 2t\nabla^2 w + kw = p. \tag{19}$$

The parameters k, t, and n can be determined from the minimum condition of the potential energy, which now will depend on k, t, n: $U = U\ (k,\ t,\ n)$, while from the condition of equilibrium we obtain the relationship $F\ (k,\ t,\ n) = 0$. We obtain as a result instead of (14) three equations with three unknowns.

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ON THE SOLUTION OF THE PROBLEM OF STRESS CONCENTRATION AROUND A CIRCULAR HOLE IN A SPHERICAL SHELL IN THE ELASTIC-PLASTIC DOMAIN

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The numerical solution proposed for the problem of the axisymmetrical elastic-plastic stressed and strained state around a reinforced circular hole in a thin spherical shell is based on the theory of small elastic-plastic deformations and the method of elastic solutions /1/. The differential equations are written in displacements and are integrated by the method of finite differences. It is assumed that the shell material is work-hardening to an arbitrary extent.

§ 1.

Without dwelling on the static and geometrical equations of the problem /2/, we only remind one that the tangential displacements are neglected in the formulas for the curvature variation, and therefore the shearing force is neglected in the equilibrium conditions in the direction of the tangent to the meridian /3/.

The physical equations of the problem are obtained by starting from the laws of the theory of small elastic-plastic deformations. Using the relationship between the intensity of the stresses and the strains, written with the aid of Il'yushin's function ω_i /1/, one obtains the following expressions for the meridional and circumferential components of the stresses at an arbitrary point through the shell thickness:

$$\sigma_1 = \frac{E}{1-\nu^2}(e_1 + \nu e_2) - \frac{E\omega_i}{3(1-\nu^2)}[(2-\nu)\, e_1 - (1-2\nu)e_2 - (1+\nu)\, e_3],$$

$$\sigma_2 = \frac{E}{1-\nu^2}(e_2 + \nu e_1) - \frac{E\omega_i}{3(1-\nu^2)}[(2-\nu)\, e_2 - (1-2\nu)\, e_1 - (1+\nu)\, e_3],$$

where

$$e_3 = -\mu_i(e_1 + e_2), \qquad \mu_i = \frac{3\nu + (1-2\nu)\,\omega_i}{3(1-\nu) - 2(1-2\nu)\,\omega_i},$$

e_1, e_2, e_3 are components of the tensile strains in the directions of the tangents to the meridian and parallel, and of the normal to the middle surface, E is the modulus of elasticity, and ν is the coefficient of transverse deformation.

In the range of small elastic-plastic deformations, ν increases from a value equal to the Poisson ratio to a value near 0.5 /5/. It is however very difficult to allow for the variability of ν. At negligible plastic deformations it is therefore simpler to assume that ν is equal to the Poisson ratio, and at developed plastic deformations to the condition of incompressibility of the material. In the latter case the physical relationships are considerably simplified.

The meridional and circumferential normal forces T_1, T_2, the bending moments G_1, G_2, and the meridional shearing forces Q_1, are determined by the formulas

$$T_1 = \frac{Eh}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) + \Delta T_1,$$

$$T_2 = \frac{Eh}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) + \Delta T_2,$$

$$G_1 = -D(\varkappa_1 + \nu\varkappa_2) + \Delta G_1,$$

$$G_2 = -D(\varkappa_2 + \nu\varkappa_1) + \Delta G_2,$$

$$Q_1 = \frac{1}{R}[\operatorname{ctg}\varphi\,(G_2 - G_1) - G_1'],$$

where

$$\Delta T_1 = -\frac{4D}{R^2}[\eta^2(M_1\varepsilon_1 + L_1\varepsilon_2) + R\eta(M_2\varkappa_1 + L_2\varkappa_2)],$$

$$\Delta T_2 = -\frac{4D}{R^2}[\eta^2(L_1\varepsilon_1 + M_1\varepsilon_2) + R\eta(L_2\varkappa_1 + M_2\varkappa_2)],$$

$$\Delta G_1 = \frac{4D}{R}[\eta(M_2\varepsilon_1 + L_2\varepsilon_2) + R(M_3\varkappa_1 + L_3\varkappa_2)],$$

$$\Delta G_2 = \frac{4D}{R}[\eta(L_2\varepsilon_1 + M_2\varepsilon_2) + R(L_3\varkappa_1 + M_3\varkappa_2)];$$

$$M_n = (2-\nu)I_n + (1+\nu)F_n,\quad L_n = -(1-2\nu)I_n + (1+\nu)F_n,\quad n = 1, 2, 3;$$

$$I_1 = \int_{-0.5}^{0.5}\omega_l\,d\xi,\quad I_2 = \int_{-0.5}^{0.5}\omega_l\,\xi\,d\xi,\quad I_3 = \int_{-0.5}^{0.5}\omega_l\,\xi^2\,d\xi;$$

$$F_1 = \int_{-0.5}^{0.5}\mu_l\,\omega_l\,d\xi,\quad F_2 = \int_{-0.5}^{0.5}\mu_l\,\omega_l\,\xi\,d\xi,\quad F_3 = \int_{-0.5}^{0.5}\mu_l\,\omega_l\,\xi^2\,d\xi.$$

In these formulas ε_1, ε_2, $\varkappa_1$, $\varkappa_2$ are the tensile strains and curvature-variation of the middle surface; h is the shell thickness; D is the flexural rigidity; φ is the meridional angle (see figure); η is the ratio of the shell radius R to the thickness; ξ is the relative coordinate along the shell thickness.

When the problem is formulated in displacements, the following system of two ordinary differential equations is obtained:

$$\begin{cases} m_{10}u + m_{11}u^{\mathrm{I}} + m_{12}u^{\mathrm{II}} + n_{11}w^{\mathrm{I}} + R^4D^{-1}X + \Omega_u = 0, \\ m_{20}u + m_{21}u^{\mathrm{I}} + n_{20}w + n_{21}w^{\mathrm{I}} + n_{22}w^{\mathrm{II}} + n_{23}w^{\mathrm{III}} - w^{\mathrm{IV}} + R^4D^{-1}Z + \\ \qquad + \Omega_w = 0, \end{cases}$$

$$m_{10} = -12\eta^2(\operatorname{ctg}^2\varphi + \nu),\quad m_{11} = 12\eta^2\operatorname{ctg}\varphi,\quad m_{12} = 12\eta^2,$$

$$n_{11} = 12(1+\nu)\eta^2,\quad m_{20} = -n_{11}\operatorname{ctg}\varphi,\quad m_{21} = -n_{11},$$

$$n_{20} = -2\, n_{11},\ n_{21} = -(2-\nu)\operatorname{ctg}\varphi - \operatorname{ctg}^3\varphi,\ n_{22} = 1+\nu+\operatorname{ctg}^2\varphi,$$
$$n_{23} = -2\operatorname{ctg}\varphi,$$
$$\Omega_u = R^3 D^{-1}\left[\operatorname{ctg}\varphi\ (\Delta T_1 - \Delta T_2)) + \Delta T_1'\right],$$
$$\Omega_w = R^3 D^{-1}\ (\Delta Q_1' + \operatorname{ctg}\varphi\,\Delta Q_1 - \Delta T_1 - \Delta T_2),$$

where u, w are components of the displacements of the middle surface in the direction of the x, z axes of the mobile xz coordinate system (figure); X, Z are components of the surface load intensity.

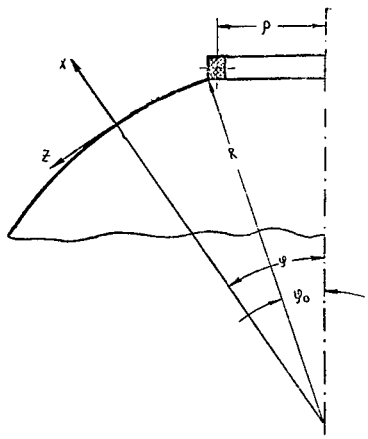

FIGURE.

The expressions Ω_u, Ω_w contain the nonlinear terms of the equations and are used in solving the problem by the method of elastic solutions /1/ as components of the fictitious load.

The solution of the differential equations of the boundary-value problem in an elastic formulation by the method of finite differences reduces to solving a system of linear algebraic equations. When the differential equations are approximated by equations in finite differences, a uniform step of the independent variable φ is used, and the approximate expressions for the derivatives by φ are written in central differences with an error of the order of the step square.

§ 2.

We shall now consider the elastic-plastic state in a shell weakened by a large circular hole, and loaded by a uniform pressure of intensity p. The hole is reinforced by a thin elastic ring of rectangular cross section and of small dimensions compared with the hole radius. The effect of the

internal pressure on the lid is equivalent to a vertical load P_0, transmitted to the ring and distributed uniformly over a circle passing through the centroid of the ring section. The boundary conditions for the shell edge considered at $\varphi = \varphi_0$ are /4/

$$\frac{w^1}{R} = \frac{\rho^2}{E_k I_k} (G_1 + Q_1 r_1 + T_1 r_2),$$

$$w \sin \varphi_0 + u \cos \varphi_0 = \frac{\rho^2}{E_k F_k} (Q_1 \sin \varphi_0 + T_1 \cos \varphi_0),$$

$$Q_1 \cos \varphi_0 - T_1 \sin \varphi_0 + P_0 = 0,$$

where G_1, T_1, Q_1 are the bending moment and forces in the section of connection of the shell and ring; r_1, r_2 are the arms of the forces T_1, Q_1 relative to the center of gravity of the ring cross section; I_k is the principal moment of inertia of the ring cross section relative to the horizontal axis; F_k is the ring section area; E_k is the modulus of elasticity of the ring.

A membrane state of strain is assumed to exist at a sufficient distance from the hole.

The calculation was conducted for a shell with $R = 170$ cm, $h = 0.3$ cm, $\varphi_0 = 0.1380$ rad. The shell was made of $AMr6$ alloy /6/, and the ring of duralumin.

The elastic-plastic state about the hole was investigated at an internal pressure $p = 4.25$ kg/cm^2, which produced stresses near to the limit of proportionality ($\sigma_n = 1300$ kg/cm^2) in the shell far from the hole membrane. In order to elucidate the influence of the supporting-ring rigidity on the concentration of stresses about the hole, the shell was calculated for several values of the ring rigidity, with a considerable variation of the flexural rigidity. A reinforcing ring of infinite flexural and tensile rigidities was also investigated.

We shall give here the values of the stresses for the elastic-plastic state of the shell in the connection section with the ring for the two extreme cases only: the case of a shell reinforced by the least rigid ring ($I_k = 0.539 \cdot 10^{-8} R^4$, $F_k = 2.08 \cdot 10^{-4} R^2$, $\rho = 0.132 R$, $r_1 = 0.704 \cdot 10^{-2} R$, $r_2 = 0.793 \cdot 10^{-2} R$) and the case of a shell reinforced by an infinitely rigid ring ($I_k = F_k = \infty$, $\rho = 0.138 R$, $r_1 = r_2 = 0$).

The values of the stresses in kg/cm^2 on the outer, middle, and inner surfaces of the shell are given in the following table.

TABLE

ξ	Case 1		Case 2	
	σ_1	σ_2	σ_1	σ_2
0.5	2020	1230	1030	950
0	1330	1520	1230	1010
−0.5	−900	860	1430	1070

The results of the calculations have shown that the increase in the ring rigidity is accompanied by a decrease in the local disturbance of the stressed and strained state, caused by the presence of the hole. Thus, with the increase in the flexural rigidity of the ring the plastic region at the place of connection of the shell with the ring becomes elastic-plastic, until it finally disappears at an infinitely rigid ring.

The coefficient of stress concentration, i.e., the ratio between the maximum stress intensity and the membrane stress intensity, is equal to 1.5 in the elastic-plastic state considered; when the problem is solved in an elastic formulation, this ratio turns out to be equal to 2.5. With the increase in the ring rigidity the coefficient of concentration drops, becoming equal to 1.1 for a ring of infinite flexural and tensile rigidities. It follows that the concentration of stresses can be considerably reduced by reinforcing the hole with a ring of large rigidity.

The study showed that the propagation of the zone of disturbance of the stressed and strained state is restricted and bears a local character in both the elastic and the elastic-plastic states of the structure. A considerable deviation from the fundamental membrane state of the shell was observed in the plastic and the elastic-plastic regions. These regions propagated in the case of the ring of lowest rigidity (from among those examined) in the meridional direction along the shell at a distance from the section of connection with the ring corresponding to a meridional angle of about 5°. The stressed and strained state in the elastic region of the shell is near to a membrane state.

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SOME PROBLEMS OF CALCULATING LARGE-PANEL EDIFICES AND EDIFICES MADE OF VOLUME ELEMENTS AS THREE-DIMENSIONAL SYSTEMS

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The increasing emphasis on multistory buildings made from large panels and volume elements has made it necessary to develop methods for their calculation as three-dimensional systems. Standard brick buildings, and large-panel buildings in particular (and their volume elements), undergo considerable strains and stresses under the action of temperature variations, seismic effects, wind pressure, and also nonuniform settling (when the foundations are laid on yielding soil).

Methods of calculation of buildings as three-dimensional systems can be developed on the basis of the following three trends of the theory of structures and the theory of elasticity:

a) on the basis of the theory of calculation of thin-walled three-dimensional systems /1, 2/;

b) on the basis of the theory of compound bars;

c) on the basis of the theory of elasticity of an anisotropic body and anisotropic plates.

We consider in this paper the calculation of buildings by the methods of the theory of calculation of thin-walled three-dimensional systems. These methods are based on the theory of shells and the variational method, created and developed by Prof. Vlasov, Corresponding Member of the AN SSSR.

The general method of calculation proposed here takes into account the strains of the transverse compression of the longitudinal and lateral walls /2/, in addition to the shearing strains and the longitudinal elongations taken into account in the general variational method of Vlasov.

1. Method of calculation. The building is considered (Figure 1) as a prismatic body of multiconnected section, built in the general case on a yielding foundation. Its vertical and horizontal sides represent continuous plates of reduced thickness δ_{1r} at longitudinal extension-compression, δ_{2r} at shear in their own plane, and δ_{3r} at transverse compression.

At the intersection between the different faces of the building there can be longitudinal elements, or stringers, which work only in the longitudinal direction at extension-compression. These elements can represent special reinforced-concrete belts or beams of prestressed framework, stiffening the building in the longitudinal direction. In the absence of special reinforced-concrete belts or frameworks in the longitudinal direction, the panels of the walls and ceilings in buildings made from large panels are considered as longitudinal stringers of reduced area ΔF_r,

calculated allowing for their actual rigidity and for the joints between them when subjected to longitudinal strain.

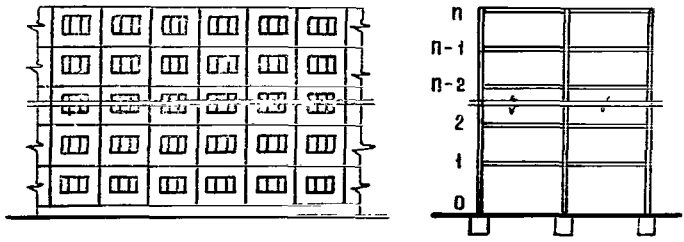

FIGURE 1.

The reduced thicknesses δ_{1r}, δ_{2r}, δ_{3r} of the faces of the building body are likewise determined allowing for the actual behavior of the walls and ceilings in the elementary strain states of tension-compression in the longitudinal and lateral directions and shear. In the case of large-panel buildings and buildings made from volume elements these thicknesses must be determined allowing for the yielding of the joints which connect the panels bearing the longitudinal walls and the panels of the ceilings between stories and the transverse walls.

The computing scheme is presented in Figure 2.

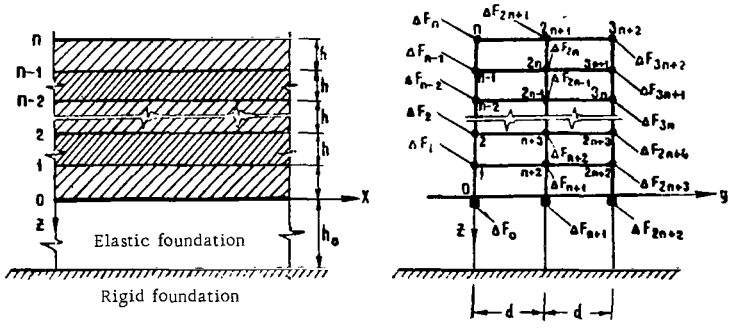

FIGURE 2.

The system of differential equations for calculating the building box as a three-dimensional system is established allowing for the joint response under strain of the system building + elastic foundation, with the elastic foundation considered as an elastic layer /1,3/ of finite height h_0.

The underlying layer is assumed to be absolutely rigid. Instead of Winkler's hypothesis we use a more accurate geometrical hypothesis /1, 3/, which allows for the foundation rigidity not only in compression, but in shear as well; the tensile strains along the height of the elastic layer h_0 are assumed to be constant, and the longitudinal displacements of the elastic foundation are everywhere taken as equal to zero.

The scheme of an n-story building with three longitudinal bearing walls is given in Figure 1, the computing scheme of an n-story building with three longitudinal bearing walls and an elastic-foundation layer of height h_c, in Figure 2.

The system of differential equations which describe the stressed state of the n-story building considered, allowing for the strains of wall pressing, the shears, and the longitudinal elongations, is given in the table. The system of differential equations was obtained on the basis of Vlasov's general variational method /1/, allowing in addition for the transverse strains /2/. There are in this system of ordinary differential equations three groups of fundamental unknown functions, which depend on the one variable x only. To the first group of unknowns belong the longitudinal displacements of the ribs at the intersection of the walls with the ceilings (along the lines of intersection of the vertical plates with the horizontal ones)

$$U_0(x), \cdots, U_n(x), U_{n+1}(x), \cdots, U_{2n+1}(x), U_{2n+2}(x), \cdots, U_{3n+2}(x).$$

TABLE

	U_0	...	U_{3n+2}	V_0	...	V_n	W_0	...	W_{3n+2}	Load terms
0	$BU_0''a_{00} - CU_0 a_{00}'$		$BU_{3n+2}''a_{0\,3n+2} - CU_{3n+2}a_{0\,3n+2}'$	$-CV_0' b_{00}$		$-CV_n' b_{0n}$	$(\nu BC_{00} - Ce_{00})W_0'$		$(\nu BC_{0;3n+2} - Ce_{0;3n+2})W_{3n+2}'$	q_{0x}
...	...	...	...	...	...	...	...	...	...	...
$3n+2$	$BU_0'' a_{3n+2;0} - CU_0 a_{3n+2;0}$		$BU_{3n+2}'' a_{3n+2;3n+2} - CU_{3n+2} a_{3n+2;3n+2}$	$-CV_0' b_{3n+2,0}$		$-CV_n' b_{3n+2;n}$	$(\nu BC_{3n+2;0} - Ce_{3n+2;0})W_0'$		$(\nu BC_{3n+2;3n+2} - Ce_{3n+2;3n+2})W_{3n+2}'$	$q_{3n+2;x}$
0	$CU_0' b_{00}$		$CU_{3n+2}' b_{0;3n+2}$	$CV_0'' f_{00} - KV_0 g_{00}$		$CV_n'' f_{0n} - KV_n g_{0n}$	$-KW_0 m_0$		$-KW_{3n+2} m_{0;3n+2}$	q_{0y}
...	...	...	...	...	...	...	...	...	...	...
	$CU_0' b_{n0}$		$CU_{3n+2}' b_{n;3n+2}$	$CV_0'' f_{n0} - KV_0 g_{n0}$		$CV_n'' f_{nn} - KV_n g_{nn}$	$-KW_0 m_{n0}$		$-KW_{3n+2} m_{n;3n+2}$	q_{ny}
0	$-(\nu BC_{00} - Ce_{00})U_0'$		$-(\nu BC_{0;3n+2} - Ce_{0;3n+2})U_{3n+2}'$	$-KV_0 m_{00}$		$-KV_n m_{0n}$	$-BW_0 p_{00}' + CW_0 p_{00} - (K+L)W_0 \tau_{00}' + E_0(\delta_{00}'' W_0'' - \tau_{00}^{\circ} W_0)$		$-BW_{3n+2} p_{0;3n+2}' + CW_{3n+2}'' p_{0;3n+2} - (K+L)W_{3n+2} \tau_{0;3n+2}'$	q_{0z}
...	...	...	...	...		...	...		...	...
$3n+2$	$-(\nu BC_{3n+2;0} - Ce_{3n+2;0})U_0'$		$-(\nu BC_{3n+2;3n+2} - Ce_{3n+2;3n+2})U_{3n+2}'$	$-KV_0 m_{3n+2;0}$		$-KV_n m_{3n+2;n}$	$-BW_0 p_{3n+2;0} + CW_0'' p_{3n+2;0} - (K+L)W_0 \tau_{3n+2;0}'$		$-BW_{3n+2} p_{3n+2;3n+2}' + CW_{3n+2}' p_{3n+2;3n+2} - (K+L)W_{3n+2} \tau_{3n+2;3n+2}'$	$q_{3n+2;z}$

To the second group of unknowns belong the horizontal displacements of the building in the planes of the floors (in the planes of the horizontal plates)

$$V_0(x), \cdots, V_n(x).$$

To the third group of unknowns belong the vertical displacements of the longitudinal ribs of the building

$$W_0(x), \cdots, W_n(x), W_{n+1}(x), \cdots, W_{2n+1}(x), W_{2n+2}(x), \cdots, W_{3n+2}(x).$$

The longitudinal displacement $u(x, s)$ at an arbitrary point (x, s) of the

cross section of the building considered is represented in the general case by the expansion

$$u(x, s) = \sum_{0}^{3n+2} U_i(x)\,\zeta_i(s), \qquad (i=0,1,\cdots,3n+2), \tag{1}$$

where $U_i(x)$ are the unknown longitudinal displacements of the building body ribs; $\zeta_i(s)$ are the functions selected for describing the longitudinal displacements between the ribs of the building body in both the horizontal and the vertical directions; s represents the contour coordinate of the building cross section, coinciding with z for the vertical elements of the building (the walls), and with y for the horizontal plates (ceilings).

With this selection of the unknown longitudinal displacements $U_i(x)$ the functions $\zeta_i(s)$ have a very simple form: each of the functions $\zeta_i(s)$ differs from zero only along the straight portions of the contour of the building cross section connected by joint i. Within the limits of each of these portions the function $\zeta_i(s)$ varies linearly, becoming equal to unity at the i-th joint of the building cross section. In all the other joints and portions of the cross-sectional contour the function $\zeta_i(s)$ is identically zero.

The diagrams of the functions $\zeta_i(s)$ for the building considered are given in Figure 3. The method selected for forming the functions $\zeta_i(s)$ is not unique and depends on the selection of the unknown functions $U_i(x)$. If any $3n+2$ independent magnitudes are taken as unknown functions $U_i(x)$, to each such magnitude will correspond a definite linear combination of independent functions $\zeta_i(s)$. The functions $\zeta_i(s)$ thus obtained will likewise be continuous on the entire multiconnected contour, and will be expressed along separate parts of the cross-sectional contour of the building body by a linear diagram.

We can select as functions $\zeta_i(s)$, corresponding to the first state, functions which satisfy the law of plane sections. The other $3n-1$ terms of expression (1) will then determine the deviation of the longitudinal displacements from the law of plane sections (deplanation of the building cross section).

The horizontal displacements $v(x, y)$ at an arbitrary point (x,y) of the cross section of the building considered are represented in the general case by the expansion

$$v(x, y) = \sum_{0}^{n} V_i(x)\,\eta_i(y), \quad (i=0,1,\cdots,n), \tag{2}$$

where $V_i(x)$ are the unknown horizontal displacements of the building ceilings (the horizontal plates of the building box); $\eta_i(y)$ are previously selected functions of distribution of the horizontal displacements in the plane of the building cross section.

The system of functions $\eta_i(y)$ $(i = 0, 1, 2, \cdots, n)$ defines the strained state of the building in the horizontal direction in the cross-sectional plane. If the interstory ceilings are inextensible in the horizontal direction (in direction $0y$), the deformation of the cross-sectional contour caused by the horizontal displacements is determined by the set of functions $V_i(x)$, whose total number is equal to the number n of ceilings in the building. The functions $\eta_i(y)$ are determined from a consideration of the elementary horizontal displacements of the building cross section in the cross-sectional plane at $V_i = 1$ and $V_k = 0$ $(k \neq i)$.

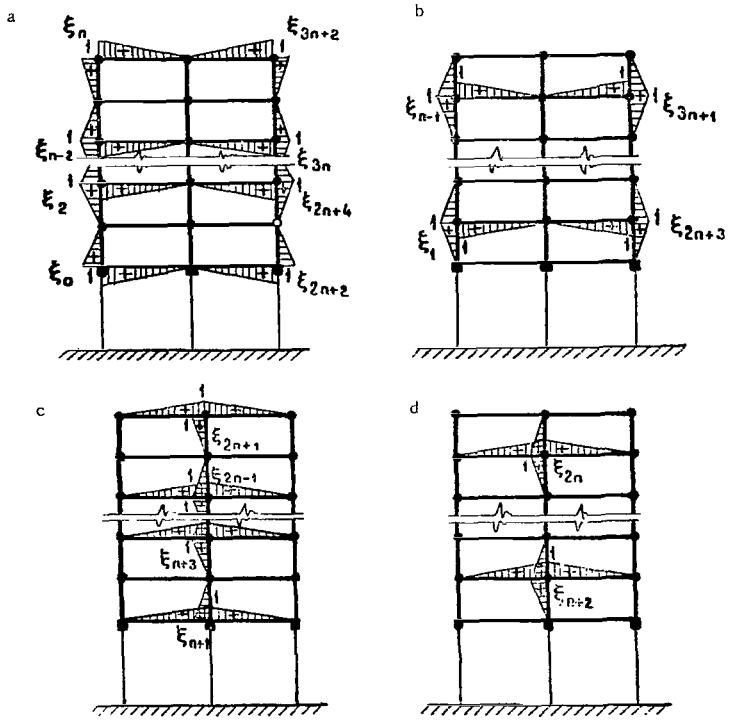

FIGURE 3.

The function $\eta_i(y)$ is constant within the limits of each ceiling (is independent of y) and corresponds to the horizontal displacement of the ceiling in the cross-sectional plane. The diagrams of the functions $\eta_i(y)$ for the building considered are given in Figure 4. The method considered for forming the functions $\eta_i(y)$ is not unique, but depends on the selection of the unknown functions $V_i(x)$.

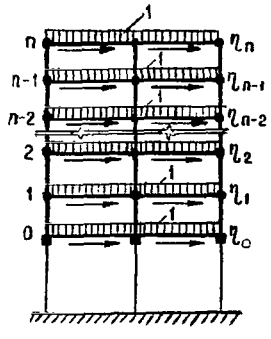

FIGURE 4.

If we select as unknown functions $V_i(x)$ any n mutually independent magnitudes, to each such magnitude will correspond a definite linear combination of independent functions $\eta_i(y)$. The functions $\eta_i(y)$ obtained will likewise be continuous and constant over each ceiling (horizontal plate) of the building.

The vertical displacements $w(x,z)$ at an arbitrary point (x, z) of the cross section of the building considered are represented as expansions

$$w(x, z) = \sum_{0}^{3n+2} W_i(x)\ \gamma_i(z), \quad (i=0,1,\cdots,3n+2), \tag{3}$$

where $W_i(x)$ are the unknown vertical displacements of the ribs of the building box, $\gamma_i(z)$ are previously selected functions of distribution of the vertical displacements between the ribs of the building box in the vertical direction (along the z axis).

When the compressibility of the walls in the vertical direction is allowed for, and when the above-defined functions $W_i(x)$ are used, each of the functions $\gamma_i(s)$ differs from zero only on the straight portions of the cross-sectional contour connected by the i-th joint. Within the limits of each such portion the function $\gamma_i(z)$ varies linearly with z, attaining the value unity at the joint.

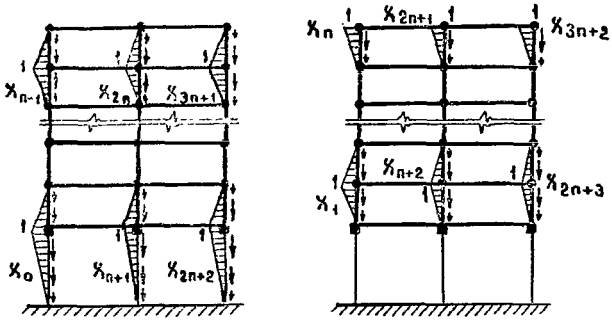

FIGURE 5.

The function $\gamma_i(z)$ is identically zero at all other joints and portions of the cross-sectional contour of the building. The diagram of the functions $\gamma_i(z)$ for the building considered is given in Figure 5.

The method selected for determining the functions $\gamma_i(z)$ is not unique and depends on the selection of the unknown functions $W_i(x)$.

If we adopt as unknown functions $W_i(x)$ any $3n+2$ mutually independent magnitudes, to each such magnitude there will correspond a definite linear combination of independent functions $\gamma_i(z)$. The functions $\gamma_i(z)$ obtained will likewise be continuous on the entire multiconnected contour, and on separate parts of the cross-sectional contour will be expressed by a linear diagram.

The constants B, C, K, L in the table have the following values:

$$B = \frac{E\delta}{1-\nu^2}, \quad C = \frac{E\delta}{2(1+\nu)}, \quad K = \frac{2Gd}{l_n}, \quad L = \frac{Eh}{l_n}, \tag{4}$$

where E is the modulus of elasticity at extension-compression of the material of the walls and ceilings; G is the shear modulus; δ is the thickness of the walls and ceilings at extension-compression; ν is the Poisson ratio; l_n is the distance between the transverse walls of the building.

The coefficients of the first group of equations of the table are calculated by the formulas

$$a_{ik}=\int \zeta_i\,\zeta_k\,ds;\quad a_{ik}=\int \zeta_i'\,\zeta_k'\,ds;\quad b_{ik}=\int \zeta_i'\eta_k\,dy;$$
$$C_{ik}=\int \zeta_i\,\gamma_k\,dz;\quad e_{ik}=\int \zeta_i'\,\gamma_k\,dz. \tag{5}$$

The formulas for calculating the coefficients of the second group of equations of the table are

$$f_{ik}=\int \eta_i\,\eta_k\,dy;\quad g_{ik}=\int \eta_i'\,\eta_k'\,\delta_n\,dz;\quad m_{ik}=\int \gamma_i'\,\eta_k'\,\delta_n\,dz. \tag{6}$$

The coefficients of the third group of equations of the table are calculated by the formulas

$$p_{ik}=\int \gamma_i\,\gamma_k\,dz;\quad p_{ik}'=\int \gamma_i'\,\gamma_k'\,dz;\quad r_{ik}'=\int \gamma_i'\,\gamma_k'\,\delta_n\,dz. \tag{7}$$

The coefficients s_{00}^0 and t_{00}^0 allow for the strain of the elastic layer of the foundation and are calculated by the formulas /1/

$$s_{00}^0=\frac{E_0\,h_0}{12(1+\nu_0)}b_b;\quad t_{00}^0=\frac{E_0\,b_\Phi}{h_0\,(1-\nu_0^2)}\left[1+\frac{h_0}{b_b}\cdot\frac{\sqrt{6(1-\nu_0)}}{3}\right]. \tag{8}$$

Here E_0 and ν_0 are elastic constants of the foundation material and b_b is the width of the building base.

2. Example of calculation. As an example of the method of calculation we give below the calculation of a building at bending by the beam scheme with a design span l = 27.3 m. This example is also useful in estimating the influence of the pressing of the longitudinal walls along the building height on the stressed and strained state of the building in the longitudinal direction.

We analyze the stressed state of this building as a function of the variation of the reduced thickness at shear and at pressing of the building wall δ and of the influence of the load distribution along the height. It was assumed that the longitudinal forces are taken only by the floors, considered as stringers along the building height.

The panels of the longitudinal walls being connected with each other and the ceilings at the corners only, they were assumed to respond only to shearing strains, with a reduced thickness δ_2, and to tensile-compressive strains in the vertical plane, with a reduced thickness δ_3.

In the example considered the reduced thickness of the panels at shear and the reduced thickness at extension-compression along the building height were taken as equal to each other and as equal to 17.3 cm in the first variant of the calculation, and to 1 cm in the second variant.

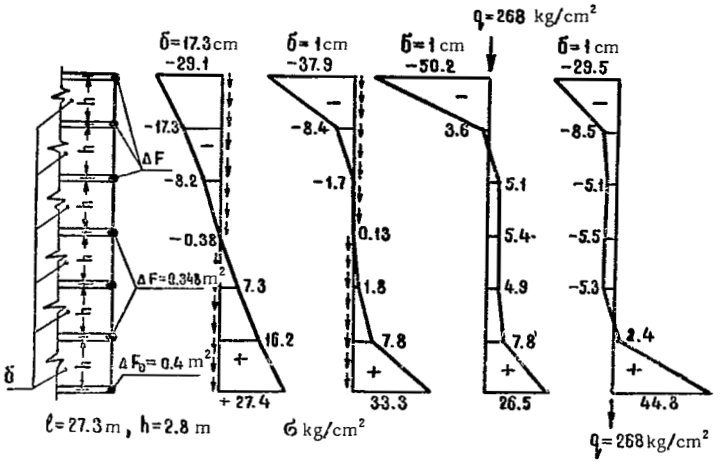

FIGURE 6.

The strain of the transverse walls was not allowed for in the given case, since the calculation was conducted in the longitudinal direction under the assumption of equal deflection of all the longitudinal walls of the building. Figure 6 presents diagrams of longitudinal normal stresses σ (kg/cm^2) for the middle cross section of the building:

at a reduced thickness δ = 17.3 cm and a uniformly distributed load over the building height;

at a thickness δ = 1.0 cm and a uniformly distributed load over the height;

at a thickness δ = 1.0 cm and a load applied along the upper stringer of the building only; and, finally,

at a thickness δ = 1.0 cm and a load applied along the lower stringer of the building only.

The following conclusions can be drawn from the results:

1) the longitudinal normal stresses at a reduced thickness of the longitudinal walls δ = 17.3 cm and a uniformly distributed load along the building height deviate very little from the law of plane sections for the entire building;

2) at a thickness of the longitudinal walls δ = 1.0 cm and a load uniformly distributed along the building height the longitudinal normal stresses deviate considerably from the law of plane sections, but the deviations display an antisymmetrical pattern;

3) when the load is applied along the upper or the lower stringer of the building only, a considerable deviation of the longitudinal normal stresses from the law of plane sections is observed, and the diagram of longitudinal normal stresses has a strongly displayed nonsymmetrical pattern; accordingly, the pressing of the building walls along its height is to be taken into account when calculating buildings in the case of nonuniform settling of the foundation.

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ON ONE SOLUTION OF THE THEORY OF BENDING

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The general relationships for the displacements and the stresses determined by the method of initial functions /1, 2, 3/ are

$$\begin{aligned} U &= L_{UU}U_0 + L_{UV}V_0 + \cdots + L_{UX}X_0, \\ V &= L_{VU}U_0 + L_{VV}V_0 + \cdots + L_{VX}X_0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_{xy} &= C_U U_0 + C_V V_0 + \cdots + C_X X_0. \end{aligned} \tag{1}$$

Here, following /1/, the letters X, Y, Z designate the components of the stress vector τ_{xz}, τ_{yz}, σ_z; the letters U, V, W are the magnitudes proportional to the actual displacements u, v, w: $U = Gu$, $V = Gv$, $W = Gw$, where G is the shear modulus. The values of these six functions at $z = 0$, i.e., the initial functions, are designated by the subscript zero.

The formulas for the operators $L_{UU}, \cdots, C_X$ are given in /1/. Thus, the operators L_{UU} and L_{UV} are written in symbolical transcendental notation as follows:

$$L_{UU} = \cos\gamma z - \frac{1}{2(1-\nu)}\frac{\alpha^2 z}{\gamma}\sin\gamma z, \qquad L_{UV} = -\frac{1}{2(1-\nu)}\frac{\alpha\beta z}{\gamma}\sin\gamma z, \tag{2}$$

where ν is the Poisson ratio. In order to reach the representation of (1) in the form of series it is necessary to expand the trigonometric functions in (2) in formal power series, to replace α, β and γ^2 by $\partial/\partial x$, $\partial/\partial y$, $\partial^2/\partial x^2 + \partial^2/\partial y^2$, respectively, and to differentiate the initial functions in accordance with (1).

It was shown in /1/ that the problem of the equilibrium of a two-dimensional plate subjected to arbitrary surface normal loads $Z^*(x, y)$ applied along the boundary planes of the plate $z = \pm h/2$ and antisymmetrical relative to the middle surface of the plate $z = 0$ (Figure 1) reduces to determining the resolving function $F(x, y)$ from the following nonhomogeneous differential equation in partial derivatives of an infinitely high order:

$$\frac{1}{2(1-\nu)}\gamma\cos\frac{h}{2}\gamma(h\gamma - \sin h\gamma)F = Z^*. \tag{3}$$

The initial functions W_0, X_0, and Y_0 are determined by the following formulas through the function $F(x, y)$:

$$W_0 = \frac{1}{2}\left[1 + \cos h\gamma - \frac{h\gamma}{4(1-\nu)}\sin h\gamma\right]F, \tag{4}$$

$$X_0 = -\frac{h}{4(1-\nu)}\alpha\gamma \sin h\gamma F, \qquad Y_0 = -\frac{h}{4(1-\nu)}\beta\gamma \sin h\gamma F. \qquad (4)$$
(cont'd)

The remaining initial functions U_0, V_0, and Z_0 are identically zero, since the state of equilibrium of the plate is considered as completely antisymmetrical relative to the middle surface $z = 0$.

If the plate is subjected to the action of one shearing load $X^*(x, y)$ or $Y^*(x, y)$, the equation for the second resolving function $F(x, y)$ can be obtained in the same way. This equation will be identical with (3), except for the right-hand part, in which Z^* will be replaced by X^* or Y^*.

Methods for obtaining particular solutions of (3) have been described in /3/. We can for instance represent the load $Z^*(x, y)$ in the form of a polynomial series in x, y with definite coefficients, and the function $F(x, y)$ as a similar series with unknown coefficients. We next substitute both series in (3) and equate the coefficients before the similar terms on both sides of this identity. This gives us all the unknown coefficients, and as a result the function $F(x, y)$ is completely determined.

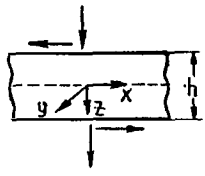


FIGURE 1.

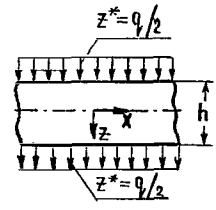


FIGURE 2.

We shall now find a particular solution for a uniformly distributed normal load q = const, i.e., $Z^* = q/2$ (Figure 2). The plate will be assumed to be in a state of plane strain in plane xz. To find this particular solution of (3), where in the given case $\alpha = \gamma = \partial/\partial x$, it is sufficient to find a particular solution of the equation

$$h^3\alpha^4 F = 6(1-\nu)q.$$

Such a solution is

$$F = \frac{6(1-\nu)q}{h^3}\frac{x^4}{4!}.$$

By formulas (4) and (1) we then find the displacements and stresses corresponding to this solution

$$Gu = \frac{1}{4}\frac{q}{h}xz\left[-4(1-\nu)\frac{x^2}{h^2} + 4(2-\nu)\frac{z^2}{h^2} - 3\nu\right],$$

$$Gw = \frac{3}{8}\frac{q}{h}\left[\frac{2}{3}(1-\nu)\frac{x^4}{h^2} - (2-\nu)x^2 + 4\nu\frac{x^2z^2}{h^2} + \right.$$

$$\left. + (1-\nu)z^2 - \frac{2(1+\nu)}{3}\frac{z^4}{h^2}\right], \qquad (5)$$

$$\sigma_z = \frac{qz}{2h}\left(3 - 4\frac{z^2}{h^2}\right), \qquad \sigma_x = \frac{4qz}{h^3}\left(z^2 - \frac{3}{2}x^2\right),$$

$$\sigma_y = \frac{\nu q z}{2h}\left[3 + \frac{4(z^2 - 3x^2)}{h^2}\right], \quad \tau_{xz} = \frac{3qx}{2h}\left(\frac{4z^2}{h^2} - 1\right). \qquad (5) \text{ (cont'd)}$$

A particular solution can also be obtained in the form of trigonometric series. By representing the given load $Z^*(x, y)$ in the form of a double Fourier series, and the function $F(x, y)$ in the form of a similar series with unknown coefficients, and by substituting both series in (3), we again determine the coefficients of $F(x, y)$. The particular solution in double trigonometric series has its independent importance, since it is the general solution of the theory of elasticity in double trigonometric series /4/. In the case of plane strain we obtain a solution in simple trigonometric series, belonging to Feylon and Rivière and corresponding to the antisymmetrical problem considered.

The particular solutions do not satisfy the boundary conditions at the lateral surface of the plate. In order to satisfy these conditions it is necessary to obtain the general solution of the homogeneous equation corresponding to (3), and corresponding to homogeneous static conditions at $z = \pm h/2$.

It is easily seen in particular that the arbitrary biharmonic function $F(x, y)$ is a solution of the homogeneous differential equation corresponding to (3). We shall now find the expressions for the initial functions determined by this function. By representing the differential operators in (4) as series we obtain

$$W_0 = \left[1 - \frac{3 - 2\nu}{8(1 - \nu)} h \gamma^2\right] F,$$
$$X_0 = -\frac{h^2}{4(1 - \nu)} \alpha\gamma^2 F, \quad Y_0 = -\frac{h^2}{4(1 - \nu)} \beta\gamma^2 F, \qquad (6)$$

from which we find

$$F = W_0 + \frac{3 - 2\nu}{8(1 - \nu)} h^2 \gamma^2 F. \qquad (7)$$

The substitution of (7) in (6) yields

$$X_0 = -\frac{h^2}{4(1 - \nu)} \alpha\gamma^2 W_0, \quad Y_0 = -\frac{h^2}{4(1 - \nu)} \beta\gamma^2 W_0, \qquad (8)$$

where, in accordance with (6), $W(x, y)$ is an arbitrary biharmonic function. The substitution of relationships (8) in the general relationships of the method of initial functions yields finally

$$u = -z\left\{1 + \frac{1}{4(1 - \nu)} h^2 \gamma^2 \left[1 - \frac{2(2 - \nu)}{3}\left(\frac{z}{h}\right)^2\right]\right\} \alpha w_0,$$
$$v = -z\left\{1 + \frac{1}{4(1 - \nu)} h^2 \gamma^2 \left[1 - \frac{2(2 - \nu)}{3}\left(\frac{z}{h}\right)^2\right]\right\} \beta w_0,$$
$$w = \left[1 + \frac{\nu}{2(1 - \nu)} z^2 \gamma^2\right] w_0, \quad \sigma_z = 0, \qquad (9)$$
$$\tau_{xz} = -\frac{6D}{h^3}\left(\frac{h^2}{4} - z^2\right) \alpha\gamma^2 w_0, \quad \tau_{yz} = -\frac{6D}{h^3}\left(\frac{h^2}{4} - z^2\right) \beta\gamma^2 w_0,$$
$$\sigma_x = -\frac{12D}{h^3} z\left\{\alpha^2 + \nu\beta^2 + \frac{h^2}{4}\left[1 - \frac{2(2 - \nu)}{3}\left(\frac{z}{h}\right)^2\right] \alpha^2 \gamma^2\right\} w_0,$$

$$\sigma_y = -\frac{12D}{h^3} z \left\{ \beta^2 + \nu\alpha^2 + \frac{h^2}{4} \left[1 - \frac{2(2-\nu)}{3} \left(\frac{z}{h}\right)^2 \right] \beta^2 \gamma^2 \right\} w_0, \qquad (9) \text{ (cont'd)}$$

$$\tau_{xy} = -\frac{12D(1-\nu)}{h^3} z \left\{ 1 + \frac{1}{4(1-\nu)} h^2 \gamma^2 \left[1 - \frac{2(2-\nu)}{3} \left(\frac{z}{h}\right)^2 \right] \right\} w_0.$$

We have now reached the actual displacements u, v, w and have introduced the flexural rigidity D.

Relationships (9) represent the well-known solution of the theory of elasticity /3, 5/ determined by a biharmonic function and corresponding to the problem of equilibrium of a plane plate with homogeneous static conditions on planes $z = \pm h/2$, when the plate is loaded on the lateral surface by antisymmetrical loads relative to the middle surface of the plate. All the formulas of (9) consist of a fundamental and an additional group of terms, with the exception of the expressions for the shearing stresses τ_{xz} and τ_{yz}, which contain terms of the fundamental group only. In the case of thin plates the second group of terms can be neglected compared with the first group, and we obtain the Kirchhoff theory for the case of the homogeneous problem. At the same time, the two groups of terms taken together give a rigorous solution for the displacements and stresses, which satisfies all the relationships of the three-dimensional problem of the theory of elasticity.

The biharmonic function w_0 has remained so far arbitrary. This arbitrariness must be removed by means of the boundary conditions on the lateral surface of the plate. As in the theory of bending of plates, two independent boundary conditions must obviously be formulated on each edge of the lateral surface, to be represented as known functions of the coordinates x, y of the lateral surface of the plate.

In the case of plane strain of the plate in plane xz the general solution for the biharmonic function w_0 is

$$w_0 = C_1 + C_2 x + C_3 x^2 + C_4 x^3,$$

where C_i are arbitrary constants.

Substituting the expression for w_0 in (9), we obtain

$$u = -z(C_2 + 2C_3 x + 3C_4 x^2) - \frac{3}{2(1-\nu)} C_4 h^2 z \left[1 - \frac{2(2-\nu)}{3} \left(\frac{z}{h}\right)^2 \right],$$

$$w = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + \frac{\nu}{1-\nu} z^2 (C_3 + 3C_4 x), \quad \sigma_z = 0, \qquad (10)$$

$$\tau_{xz} = -\frac{36D}{h^3} C_4 \left(\frac{h^2}{4} - z^2\right), \quad \sigma_x = -\frac{24D}{h^3} z(C_3 + 3C_4 x), \quad \sigma_y = \nu\sigma_x.$$

By adding solution (10) to the corresponding particular solution which satisfies the given static boundary conditions on the plate planes $z = \pm h/2$ we can, remaining in the class of solution (10), determine the stressed and strained state of the plate in the case of plane strain at any nonhomogeneous static conditions at the plate planes $z = \pm h/2$. Consider thus the problem of the equilibrium of a plate supported at the edges $x = 0$ and $x = a$ and acted upon by a uniformly distributed surface normal load $Z^* = q/2$. Take the sum of solutions (5) and (7) (general solution of the nonhomogeneous problem). The arbitrary constants are determined from the

conditions of plate support. By introducing the values of the constants in the general solution of the nonhomogeneous problem we obtain the required formulas for the displacements and stresses

$$w = \frac{q}{24D}\left\{x(a^3 - 2ax^2 + x^3) + \frac{3}{10}\frac{8-3\nu}{1-\nu}h^2x(a-x) + \right.$$

$$\left. + z^2\left[\frac{3(5-3\nu)}{10(1-\nu)}h^2 + \frac{6\nu}{1-\nu}x(x-a) - \frac{1+\nu}{1-\nu}z^2\right]\right\},$$

$$u = \frac{q}{24D}z\left\{6ax - 4x^3 - a^3 + (a-2x)\left[\frac{3(2+3\nu)}{10(1-\nu)}h^2 - \frac{2(2-\nu)}{1-\nu}z^2\right]\right\}, \tag{11}$$

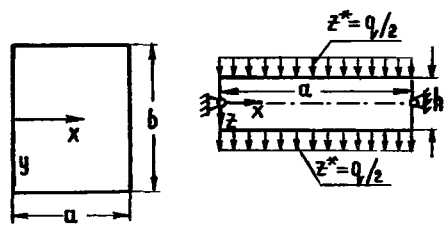
$$\sigma_z = \frac{qz}{2h}\left(3 - 4\frac{z^2}{h^2}\right), \quad \sigma_y = \frac{\nu qz}{h}\left[\frac{6}{h^2}x(a-x) + \frac{9}{10} + 2\frac{z^2}{h^2}\right],$$

$$\sigma_x = \frac{qz}{h}\left[6\frac{x}{h^2}(a-x) - \frac{3}{5} + 4\frac{z^2}{h^2}\right], \quad \tau_{xz} = \frac{3q}{4h}(a-2x)\left[1 - 4\left(\frac{z}{h}\right)^2\right].$$

We have described the method of obtaining solutions for the two-dimensional problem of the theory of elasticity corresponding to problems of equilibrium of flat plates with arbitrary static conditions on planes $z = \pm h/2$, with the boundary conditions on the lateral surface of the plate satisfied in the Saint-Venant sense. It should be noted that we have thus obtained all the solutions of the theory of elasticity in polynomials, corresponding to the antisymmetrical state of equilibrium of the plate relative to plane $z = 0$. We use here the direct method of obtaining these solutions. A similar class of solutions can likewise be obtained in the case of the axisymmetrical problem of bending of a plane circular plate /3, 6/.

The solutions of the one-dimensional problems of bending of plates are obtained from the exact solutions by neglecting the secondary terms, whose order $(h/a)^2$ is small compared with the order of the fundamental terms, taken as unity. The latter follows in particular also from the solution (11). Starting however from (9) it is possible to find a solution of the boundary-value problem in the Saint-Venant sense in the three-dimensional case as well. For thin plates these solutions correspond to the two-dimensional problem of bending of plates.

Consider as an example a rectangular plate supported along the contour and subjected to the action of a normal uniformly distributed load $Z^* = q/2$ (Figure 3).



a b

FIGURE 3.

$$
\begin{aligned}
&z = \pm h/2 \quad \sigma_z = q/2; \quad \tau_{xz} = \tau_{yz} = 0, \\
&x = 0 \text{ and } x = a \qquad w_0 = M_x = 0, \\
&y = \pm b/2 \qquad\qquad w_0 = M_y = 0.
\end{aligned} \tag{12}
$$

The solution is obtained in this case by supporting two solutions: solution "I", determined by formulas (11), and solution "II", determined by relationships (9), where the biharmonic function w_0^{II} is represented by the series

$$
w_0^{II} = \sum_{m=1}^{\infty} (A_m \operatorname{ch} \alpha_m y + B_m y \operatorname{sh} \alpha_m y) \sin \alpha_m x, \qquad \alpha_m = \frac{m\pi}{a}. \tag{13}
$$

A_m and B_m here are arbitrary constants as yet.

By adding the two solutions we obtain expressions for the displacements and stresses in the plate which satisfy the boundary conditions (12) at $z = \pm h/2$, $x = 0$ and $x = a$. The constants A_m and B_m must be determined so as to satisfy the boundary conditions along the edges $y = \pm b/2$ as well. We thus represent the expressions for the deflection w_0^I and the bending moment M_y^I by the following trigonometric series:

$$
w_0^I = \sum_{m=1,3,5,\ldots}^{\infty} a_m \sin \alpha_m x, \qquad M_y^I = \sum_{m=1,3,5,\ldots}^{\infty} b_m \sin \alpha_m x,
$$

$$
a_m = \frac{2}{a} \int_0^a w_0^I \sin \alpha_m x dx = \frac{4qa^4}{(m\pi)^5 D} \left[1 + \frac{8 - 3\nu}{40(1-\nu)} (m\pi)^2 \left(\frac{h}{a}\right)^2 \right],
$$

$$
b_m = \frac{2}{a} \int_0^a M_y^I \sin \alpha_m x dx = \frac{4\nu}{(m\pi)^2} qa^2 \left[1 + 0.1 (m\pi)^2 \left(\frac{h}{a}\right)^2 \right].
$$

By substituting $y = b/2$ in the formulas for w_0^{II} and M_y^{II} and equating the sums $w_0^I + w_0^{II}$ and $M_y^I + M_y^{II}$ to zero we obtain for each value of m a system of two algebraic equations relative to A_m and B_m.

Having determined from here the constants A_m and B_m we find the required state of equilibrium of the plate. The following expression is obtained in particular for the deflection of the middle surface of the plate w_0:

$$
\begin{aligned}
w_0 = {} & \frac{q}{24D} \left[x(x^3 - 2ax^2 + a^3) + \frac{3}{10} \frac{8 - 3\nu}{1 - \nu} x(a - x) h^2 - \right. \\
& - \frac{4qa^4}{D\pi^5} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^5 \operatorname{ch} \frac{m\pi b}{2a}} \left\{ \left[1 + \frac{8 - 3\nu}{40(1-\nu)} (m\pi)^2 \left(\frac{h}{a}\right)^2 + \right. \right. \\
& \left. \left. + \frac{m\pi b}{4a} \operatorname{th} \frac{m\pi b}{2a} \right] \operatorname{ch} \frac{m\pi y}{a} - \frac{m\pi y}{2a} \operatorname{sh} \frac{m\pi y}{a} \right\} \sin \frac{m\pi x}{a}.
\end{aligned} \tag{14}
$$

It is easily seen that relationship (14) differs from the similar relationship for the deflection determined by the Kirchhoff theory /7/ by the presence of additional terms of order $(h/a)^2$ compared with the fundamental terms, whose order is taken as unity (a and b are magnitudes of the same order). By the method exposed it is obviously possible to obtain in the class of solutions (9) rigorous solutions of the problem of bending in trigonometric series for any nonhomogeneous static conditions on the plate planes $z = \pm h/2$, if the given normal or shearing load can be represented in the form of a polynomial or trigonometric series. By neglecting in

these series the terms of order $(h/a)^2$ we obtain in the case of a normal load the corresponding solution of the Kirchhoff theory.

It is also possible to formulate the Kirchhoff theory as a whole on the basis of the general relationships of the method of initial functions. One should then neglect in differential equation (3) and in relationships (4), (1), (9) all the terms of order $(h/a)^2$ and higher compared with the fundamental terms, whose order is equal to unity; in other words, one should preserve in the nonhomogeneous equation (3) only the biharmonic operator, and in relationships (9), where the initial function w_r satisfies a nonhomogeneous biharmonic equation, only the first fundamental group of terms. In fact, all the other neglected terms contain as a factor h^{2n}, where n is an integer equal to or larger than unity, and the order of the derivatives in these terms is likewise equal to $2n$. When solving in series this leads as a result of the differentiation to the factors $(h/a)^{2n}$ $(n = 1, 2, 3, \cdots)$.

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STRESSED-STRAINED STATE OF A TOROIDAL SHELL AT BENDING, APPLICABLE TO CALCULATING BELLOWS

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This paper is devoted to the study of several particular cases for which it is possible to develop a simple algorithm of the calculation of a closed toroidal shell with two edges, one of which contains a singular line, subjected to a "wind-type" load. The development of a method for calculating toroidal shells is of interest both for its own sake and for calculating bellows-type shells.

Many papers have been devoted to the solution of this problem /1, 2, 3/. However, it seems advisable to us to amplify the existing studies in several points.

I. Statement of the problem and derivation of the resolving equation

Consider the torus quadrant represented in Figure 1; the two torus edges correspond to $\theta = 0$ and $\theta = \frac{\pi}{2}$.

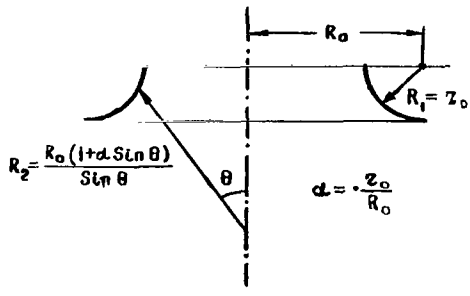
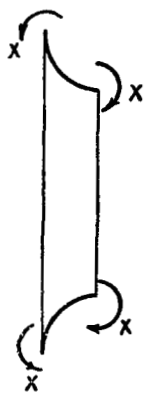


FIGURE 1.

As known, the radius of curvature $R_2 \to \infty$ on the line $\theta = 0$. This is what causes the peculiar difficulties in calculating a torus, both in the axisymmetrical problem and in the problem considered here.

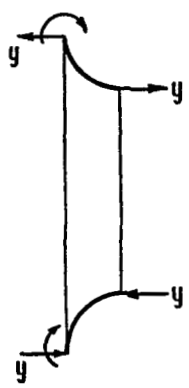
The problem formulated will be solved only for the types of end loads represented in Figures 2, 3, and 4.



$C_1 \neq 0$

$C_2 = C_3 = C_4 = 0$

FIGURE 2.



$C \neq 0$

$C_1 = C_3 = C_4 = 0$

FIGURE 3.

$C_3 \neq 0, \; C_4 \neq 0$

$C_1 = C_2 = 0$

FIGURE 4.

We shall use a method similar to the one used in /4/ for the axisymmetrical problem, i.e., the method of the so-called standard solutions, consisting in reducing the system of equations to one equation of the same order and the same singularities. Many papers have been devoted to this subject /5, 6/. While this method has many advantgages, at the same time it imposes definite restrictions on the problem stated.

The present paper is based on the equations given in Gol'denveizer's treatise /7/. All the symbols, and the sign rule for the forces, moments, and displacements are taken over from this treatise. For brevity's sake we shall not give the fundamental equations and relationships here, but shall only indicate their number in /7/.

The system of seventeen initial independent equations can be represented in the form (12.6), (20.11), (23.3), (24.1) of part 1 of /7/. For loads of the so-called "wind" type we can represent all forces, moments, slopes, strains, and stress functions in the form $T_1 = t_1 \cos\beta, G_1 = g_1 \cos\beta$, etc.

The coefficients depending on the coordinate θ only will be designated by $t_1, t_2, n_1, n_2, s_1, s_2$ for the forces, and by g_1, g_2, h_1, h_2 for the moments. These coefficients for the displacements, slopes, and stress functions will be designated by the same letter, but with a tilde above it, such as

$$\gamma_1 = \tilde{\gamma}_1 \cos\theta, \qquad \varepsilon_1 = \tilde{\varepsilon}_1 \cos\theta, \qquad v = \tilde{v} \sin\theta.$$

The substitution of this expansion, corresponding to the "wind-type" load, in the initial system of equations yields a system of equations in ordinary derivatives for the homogeneous problem, namely:

equilibrium equations

$$\frac{d}{d\theta}(R_2 \sin\theta t_1) - R_1 s_1 - R_1 \cos\theta t_2 - R_2 \sin\theta n_1 = 0;$$

$$\frac{d}{d\theta}(R_2 \sin\theta s_1) - R_1 t_2 - R_1 \cos\theta s_2 - R_1 \sin\theta n_2 = 0;$$

$$R_2 \sin\theta t_1 + R_1 \sin\theta t_2 + \frac{d}{d\theta}(R_2 \sin\theta n_1) + R_1 n_2 = 0; \qquad (1)$$

$$\frac{d}{d\theta}(R_2 \sin\theta h_1) + R_1 g_2 - R_1 \cos\theta h_2 + R_1 R_2 \sin\theta n_2 = 0;$$

$$\frac{d}{d\theta}(R_2 \sin\theta g_1) + R_1 h_2 - R_1 \cos\theta g_2 - R_1 R_2 \sin\theta n_1 = 0,$$

equations of compatibility

$$\frac{d}{d\theta}(R_2 \sin\theta \tilde{\varkappa}_2) - R_1 \tilde{\tau}^{(2)} - R_1 \cos\theta \tilde{\varkappa}_1 + R_2 \sin\theta \tilde{\zeta}_2 = 0;$$

$$\frac{d}{d\theta}(R_2 \sin\theta \tilde{\tau}^{(2)}) - R_1 \tilde{\varkappa}_1 - R_1 \cos\theta \tilde{\tau}^{(1)} - R_1 \sin\theta \tilde{\zeta}_1 = 0;$$

$$R_2 \sin\theta \tilde{\varkappa}_2 + R_1 \sin\theta \tilde{\varkappa}_1 - \frac{d}{d\theta}(R_2 \sin\theta \tilde{\zeta}_2) + R_1 \tilde{\zeta}_1 = 0;$$

$$\frac{d}{d\theta}\left(R_2 \sin\theta \frac{\tilde{\omega}}{2}\right) + R_1 \tilde{\varepsilon}_1 + R_1 \cos\theta \frac{\tilde{\omega}}{2} + R_1 R_2 \sin\theta \tilde{\zeta}_1 = 0; \qquad (2)$$

$$\frac{d}{d\theta}(R_2 \sin\theta \tilde{\varepsilon}_2) - R_1 \frac{\tilde{\omega}}{2} - R_1 \cos\theta \tilde{\varepsilon}_1 + R_1 R_2 \sin\theta \tilde{\zeta}_2 = 0;$$

$$\tilde{\tau}^{(2)} + \tilde{\tau}^{(1)} - \frac{\tilde{\omega}}{2R_1} + \frac{\tilde{\omega}}{2R_2} = 0,$$

elasticity relationships

$$t_1 = \frac{2Eh}{1-\sigma^2}(\tilde{\varepsilon}_1 + \sigma\tilde{\varepsilon}_2);$$

$$t_2 = \frac{2Eh}{1-\sigma^2}(\tilde{\varepsilon}_2 + \sigma\tilde{\varepsilon}_1);$$

$$g_1 = -\frac{2Eh^3}{3(1-\sigma^2)}(\tilde{\varkappa}_1 + \sigma\tilde{\varkappa}_2); \qquad (3)$$

$$g_2 = -\frac{2Eh^3}{3(1-\sigma^2)}(\tilde{\varkappa}_2 + \sigma\tilde{\varkappa}_1);$$

$$s_1 + s_2 = 0;$$

$$h_1 + h_2 = 0.$$

Introduce the stress functions $\tilde{\varphi}$, $\tilde{\psi}$, $\tilde{\nu}$. The relationship between the stress functions and the forces is given in (13.7) of part 1 of /7/; these relationships transform the first three equations of (1) into identities.

We shall express the strains $\tilde{\varkappa}_1$, $\tilde{\varkappa}_2$, $\tilde{\tau}^{(1)}$, $\tilde{\tau}^{(2)}$, $\tilde{\zeta}_1$, $\tilde{\zeta}_2$ through the slopes $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\delta}$ ((19.4) of part 1 of /7/); the first three continuity equations of (2)

are then transformed into identities. Instead of the fourth equilibrium equation we write an integrable combination, representing the condition of equilibrium of a finite portion of the shell, limited by the two lines $\theta_1 = \text{const}$; $\theta_2 = \text{const}$. which in a finite form is written as follows:

$$\begin{gathered}-h_1\cos\theta - g_1 + [\sin\theta(R_0 + r_0\sin\theta) - r_0\cos\theta(1-\cos\theta)]\,t_1 + \\ + [\cos\theta(R_0 + r_0\sin\theta) + r_0\sin\theta(1-\cos\theta)]\,n_1 + \\ + s_1 r_0(1-\cos\theta) + C_1 = 0. \end{gathered} \tag{4}$$

A similar relationship can be written instead of the fourth equation of continuity on the strength of the static-geometric analogy ((30.3) of part 1 of /7/).

We thus obtain a system of six equations with six unknowns, four of which are found algebraically, and the other two from two differential equations, whose structure is symmetrical according to the statical-geometrical analogy /7/ and which are easily reduced to one complex resolving equation (8). The unknown functions $\tilde{\varphi}$, $\tilde{\nu}$, $\tilde{\gamma}_1$, $\tilde{\delta}$ are finally expressed in the following way through the two fundamental functions $\tilde{\psi}$, $\tilde{\gamma}_2$:

$$\begin{gathered}\tilde{\nu} = \frac{R_2\sin\theta}{(R_2+R_1)\sin\theta}\frac{\partial\tilde{\psi}}{\partial\theta} - \\ - \frac{2EhR_1^2R_2\sin\theta}{(R_2+R_1)^2\sin^2\theta}\cos\theta\tilde{\gamma}_2 - \chi_{1\nu}C_1 + \chi_{2\nu}C_2 - \chi_{3\nu}C_3 - \chi_{4\nu}C_4; \\ \tilde{\varphi} = \frac{2EhR_1R_2\sin\theta}{(R_2+R_1)\sin\theta}\cos\theta\tilde{\gamma}_2 - \cos\theta\tilde{\psi} - \chi_{1\varphi}C_1 + \chi_{2\varphi}C_2 - \chi_{3\varphi}C_3 - \chi_{4\varphi}C_4; \\ \tilde{\gamma}_1 = -\frac{3(1-\sigma^2)R_1R_2\sin\theta}{2Eh^3(R_2+R_1)\sin\theta}\cos\theta\tilde{\psi} - \cos\theta\tilde{\gamma}_2 + \chi_{2\gamma_1}C_1 - \chi_{2\gamma_1}C_2 + \\ + \chi_{3\gamma_1}C_3 + \chi_{4\gamma_1}C_4; \\ \tilde{\delta} = \frac{R_2\sin\theta}{(R_2+R_1)\sin\theta}\frac{\partial\tilde{\gamma}_2}{\partial\theta} - \\ - \frac{3(1-\sigma^2)R_1^2R_2\sin\theta}{2Eh^3(R_2+R_1)^2\sin^2\theta}\cos\theta\tilde{\psi} + \chi_{1\delta}C_1 - \chi_{2\delta}C_2 + \chi_{3\delta}C_3 + \chi_{4\delta}C_4. \end{gathered} \tag{5}$$

In relationships (5) $C_1 \cdots C_4$ are arbitrary constants, which are determined from the conditions of equilibrium of the torus parts represented in Figures 2, 3, and 4. The functions by which the arbitrary constants are multiplied are found by the formulas

$$\begin{gathered}\chi_{1\nu} = \frac{3(1-\sigma^2)R_1^4}{h^2(R_2+R_1)\sin\theta}\cos\theta\int_0^\theta\int_0^\theta\frac{1}{(R_2+R_1)\sin\theta}\,d\theta d\theta; \\ \chi_{2\nu} = \frac{3(1-\sigma^2)ER_1^6}{h(R_2+R_1)\sin\theta}\cos\theta\int_0^\theta\int_0^\theta\frac{\cos\theta}{(R_2+R_1)\sin\theta}\int\frac{1}{(R_2+R_1)\sin\theta}\,d\theta d\theta d\theta; \\ \chi_{3\nu} = \frac{3(1-\sigma^2)R_1^4}{h^2(R_2+R_1)\sin\theta}\cos\theta\int_0^\theta\int_0^\theta\frac{\cos\theta}{(R_2+R_1)\sin\theta}\,d\theta d\theta; \\ \chi_{4\nu} = -\frac{4EhR_1^2}{(R_2+R_1)\sin\theta}\theta\cos\theta. \end{gathered} \tag{6}$$

The coefficients with subscripts φ have the same structure as the coefficients with subscripts ν:

$$\chi_{15} = \frac{3(1-\sigma^2)R_1^2}{2Eh^3}\int_0^\theta \frac{1}{(R_2+R_1)\sin\theta}\,d\theta;$$

$$\chi_{25} = -\frac{3(1-\sigma^2)R_1^4}{h^2}\int_0^\theta \frac{\cos\theta}{(R_2+R_1)\sin\theta}\int_0^\theta \frac{1}{(R_2+R_1)\sin\theta}\,d\theta d\theta; \qquad (7)$$

$$\chi_{35} = -\frac{3(1-\sigma^2)R_1^2}{2Eh^3}\int_0^\theta \frac{\cos\theta}{(R_2+R_1)\sin\theta}\,d\theta, \qquad \chi_{45} = 1.$$

The coefficients with subscripts γ_1 have the same structure as the coefficients with subscripts δ.

The main unknowns $\tilde{\Psi}$ and $\tilde{\gamma}_2$ are found from the resolving equation, which coincides in structure with the resolving equation for the axisymmetrical problem /4/.

$$\varepsilon\frac{d}{d\theta}\left[p\frac{d}{d\theta}V\right] + \frac{R_1}{(R_2+R_1)\sin\theta}V - \sin\theta V = i[f_2C_1 + f_2C_2 + f_3C_3 + f_4C_4] - [\chi_1C_1 + \chi_2C_2 + \chi_3C_3 + \chi_4C_4]. \qquad (8)$$

Here

$$\varepsilon = i\frac{R_0}{r_0^2}\sqrt{\frac{h^2}{3(1-\sigma^2)}}; \qquad (9)$$

$$p = (1 + \alpha\sin\theta);$$

$$V = \left(\sqrt{\frac{h^2}{3(1-\sigma^2)}}\,2Eh\tilde{\gamma}_2 + i\tilde{\psi}\right);$$

$$f_1 = -1 + \frac{\pi}{4}\,\frac{(R_2+R_1)^2\sin^2\theta + 4R_1^2}{R_2\sin\theta}\int_0^\theta\int_0^\theta \frac{1}{(R_2+R_1)\sin\theta}\,d\theta d\theta - \frac{R_1(R_2+R_1)}{R_2}\cos\theta\left(\sigma + \frac{\pi}{8}\right)\int_0^\theta \frac{1}{(R_2+R_1)\sin\theta}\,d\theta;$$

$$f_2 = -\frac{4EhR_1^2}{\pi}\cos\theta\int_0^\theta \frac{1}{(R_2+R_1)\sin\theta}\,d\theta + \frac{EhR_1^2}{R_2\sin\theta}\left[(R_2+\sigma R_1)(R_2+R_1)\sin^2\theta + \frac{4}{\pi}R_1^2\right]\int_0^\theta\int_0^\theta \frac{\cos\theta}{(R_2+R_1)\sin\theta}\int_0^\theta \frac{1}{(R_2+R_1)\sin\theta}\,d\theta d\theta d\theta + \frac{EhR_1^3(R_2+R_1)}{R_2}\cos\theta\left(1 - \frac{4}{\pi}\sigma\right)\times \int_0^\theta \frac{\cos\theta}{(R_2+R_1)\sin\theta}\int_0^\theta \frac{1}{(R_2+R_1)\sin\theta}\,d\theta d\theta; \qquad (10)$$

$$f_3 = \frac{4}{\pi}\cos\theta -$$

$$- \frac{\pi(R_2 + \sigma R_1)(R_2 + R_1)\sin^2\theta + 4R_1^2}{4R_2\sin\theta} \int_0^\theta\int_0^\theta \frac{\cos\theta}{(R_2 + R_1)\sin\theta}\, d\theta d\theta -$$

$$- \frac{R_1(R_2 + R_1)}{R_2}\,\frac{8\sigma - \pi}{2\pi}\cos\theta \int_0^\theta \frac{\cos\theta}{(R_2 + R_1)\sin\theta}\, d\theta;$$

$$f_4 = \frac{Eh^3}{12(1-\sigma^2)R_1^2R_2\sin\theta}\{[2\pi R_2(R_2 + R_1)\sin^2\theta + 2\pi\sigma R_1(R_2 + R_1)\sin^2\theta +$$

$$+ 8R_1^2]\theta + (\pi - 8\sigma)R_1(R_2 + R_1)\sin\theta\cos\theta\};$$

$$\chi_1 = \frac{\sqrt{3(1-\sigma^2)}}{h}\,\frac{R_1^2(R_1\cos^2\theta - R_2\sin^2\theta)}{R_2\sin\theta}\int_0^\theta\int_0^\theta \frac{1}{(R_2 + R_1)\sin\theta}\, d\theta d\theta; \qquad (10)$$

(cont'd)

$$\chi_2 = \frac{4\sqrt{3(1-\sigma^2)}}{\pi}\,\frac{ER_1^4(R_1\cos^2\theta - R_2\sin^2\theta)}{R_2\sin\theta} \times$$

$$\times \int_0^\theta\int_0^\theta \frac{\cos\theta}{(R_2 + R_1)\sin\theta}\int_0^\theta \frac{1}{(R_2 + R_1)\sin\theta}\, d\theta d\theta d\theta;$$

$$\chi_3 = \frac{4\sqrt{3(1-\sigma^2)}}{\pi h}\,\frac{R_1^2(R_1\cos^2\theta - R_2\sin^2\theta)}{R_2\sin\theta}\int_0^\theta\int_0^\theta \frac{\cos\theta}{(R_2 + R_1)\sin\theta}\, d\theta d\theta;$$

$$\chi_4 = \frac{2Eh^2}{\sqrt{3(1-\sigma^2)}}\,\frac{(R_1\cos^2\theta - R_2\sin^2\theta)}{R_2\sin\theta}\,\theta.$$

II. Study of the particular cases represented in Figures 2, 3, and 4

In the case $C \neq 0$, $C_2 = C_3 = C_4 = 0$ (Figure 2) we have

$$t_1 = \frac{4}{\pi}\,\frac{3(1-\sigma^2)R_1^2}{h^2(R_2 + R_1)}\,|\rho|\lambda(\theta)\cos^2\theta(\chi_1C_1 + C_1 + X)\operatorname{Im} l_0(\rho u_0);$$

$$t_2 = \frac{12}{\pi h^2}\,\frac{R_1R_2}{(R_2 + R_1)}\,|\rho|^2 u_0'\lambda(\theta)\sin\theta(\chi_1C_1 + C_1 + X)\operatorname{Im} l_0'(\rho u_0);$$

$$n_1 = -\frac{4}{\pi}\,\frac{3(1-\sigma^2)R_1^2}{(R_2 + R_1)h^2}\,|\rho|\lambda(\theta)\cos\theta\Big[\sin\theta -$$

$$- \frac{R_1}{(R_2 + R_1)\sin\theta}\Big](\chi_1C_1 + C_1 + X)\operatorname{Im} l_0(\rho u_0); \qquad (11)$$

$$g_1 = -\frac{h\cos\theta}{R_1\sqrt{3(1-\sigma^2)}}\,|\rho|^2 u_0'\lambda(\theta)\,[f_1C_1\operatorname{Re} l_0'(\rho u_0) -$$

$$-(\chi_1C_1 + \sin\theta C_1 + \sin\theta X)\operatorname{Im} l_0'(\rho u_0)] - \sin\theta X - \sin\theta C_1 + C_1.$$

The arbitrary constant C_1 is determined from the equilibrium conditions of the shell represented in Figure 2.

In the case $C_2 \neq 0$, $C_1 = C_3 = C_4 = 0$ (Figure 3) we have

$$t_1 = -\frac{\sin\theta}{R_2}|\rho|\lambda(\theta)\,[(f_2C_2 + 2Ehr_0\sin^2\theta Y)\,\mathrm{Im}\,l_0(\rho u_0) +$$

$$+\chi_2C_2\,\mathrm{Re}\,l_0(\rho u_0)] - \frac{\sin^2\theta}{R_2}2Ehr_0Y - \frac{\mathrm{ctg}\,\theta}{R_2}\chi_{2\varphi}C_2;$$

$$t_2 = \frac{1}{R_1}\cos\theta\,|\rho|^2u_0'\lambda(\theta)\,[(f_2C_2 + 2Ehr_0\sin^2\theta Y)\,\mathrm{Im}\,l_0'(\rho u_0) +$$

$$+\chi_2C_2\,\mathrm{Re}\,l_0'(\rho u_0)] + \frac{1}{R_1}\cos^2\theta\,2Ehr_0Y - \frac{1}{R_1}\frac{d}{d\theta}\chi_{2\varphi}C_2; \qquad (12)$$

$$n_1 = -\frac{1}{(R_2+R_1)\sin\theta}|\rho|^2u_0'\lambda(\theta)\,[(f_2C_2 + 2Ehr_0\sin^2\theta Y)\,\mathrm{Im}\,l_0(\rho u_0) +$$

$$+\chi_2C_2\,\mathrm{Re}\,l_0'(\rho u_0) - \frac{2Ehr_0\cos\theta}{(R_2+R_1)\sin\theta}Y - \frac{2Ehr_0\sin\theta\cos\theta}{R_2}Y + \frac{1}{R_2}\chi_{2\varphi}C_2;$$

$$g_1 = -\frac{4R_2\cos\theta}{\pi(R_2+R_1)}\{|\rho|^2u_0'\lambda(\theta)\,[(f_2C_2 + 2Ehr_0\sin^2\theta Y)\,\mathrm{Im}\,l_0'(\rho u_0) +$$

$$+\chi_2C_2\mathrm{Re}\,l_0'(\rho u_0')] + 2\cos\theta\,Ehr_0Y\} + \frac{8}{\pi}Eh\alpha r_0\theta\cos\theta C_2.$$

The arbitrary constant C_2 is determined from the condition of equality of t_1 at $\theta = \pi/2$ to the given value X.

In the case $C_3 \neq 0$; $C_4 \neq 0$; $C_1 = C_2 = 0$ we have

$$t_1 = \frac{\sin\theta}{R_2}|\rho|\lambda(\theta)\,[(f_3C_3 + f_4C_4)\,\mathrm{Im}\,l_0(\rho u_0) + (\chi_3C_3 + \chi_4C_4)\,\mathrm{Re}\,l_0(\rho u_0) +$$

$$+2ZR_2\sin^2\theta\cos\theta\,\mathrm{Im}\,l_0(\rho u_0)] - 2Z\sin^2\theta\cos\theta + \frac{\mathrm{ctg}\,\theta}{R_2}(\chi_{3\varphi}C_3 + \chi_{4\varphi}C_4);$$

$$t_2 = \frac{1}{R_1}\cos\theta\,|\rho|^2u_0'\lambda(\theta)\,[(f_3C_3 + f_4C_4)\,\mathrm{Im}\,l_0'(\rho u_0) + (\chi_3C_3 + \chi_4C_4)\,\mathrm{Re}\,l_0'(\rho u_0)] -$$

$$-2\frac{R_2\sin^2\theta}{R_1}\cos^2\theta\,|\rho|\lambda(\theta)\,Z\,\mathrm{Im}\,l_0'(\rho u_0) + 2Z\cos\theta\,(R_1\cos^2\theta - R_2\sin^2\theta) -$$

$$-\frac{d}{d\theta}\chi_{3\varphi}C_3 - \frac{d}{d\theta}\chi_{4\varphi}C_4;$$

$$n_1 = -\frac{1}{R_2\sin\theta}|\rho|^2\lambda(\theta)\,[(f_3C_3 + f_4C_4)\,\mathrm{Im}\,l_0'(\rho u_0) + \qquad (13)$$

$$+(\chi_3C_3 + \chi_4C_4)\,\mathrm{Re}\,l_0'(\rho u_0) + 2R_2\sin^2\theta\cos\theta Z\,\mathrm{Im}\,l_0'(\rho u_0)]\,\frac{R_2}{(R_2+R_1)} -$$

$$-2\frac{1}{(R_2+R_1)\sin\theta}Z(R_1\cos^2\theta - R_2\sin^2\theta) +$$

$$+\frac{1}{R_2\sin\theta}(\chi_{3\nu}C_3 + \chi_{4\nu}C_4) - 2Z\cos^2\theta - \frac{1}{R_2}(\chi_{3\varphi}C_3 + \chi_{4\varphi}C_4);$$

$$g_1 = -\frac{R_2}{(R_2+R_1)}|\rho|^2\lambda(\theta)\cos\theta\,[(f_3C_3 + f_4C_4)\,\mathrm{Im}\,l_0' +$$

$$+(\chi_3C_3 + \chi_4C_4)\,\mathrm{Re}\,l_0'(\rho u_0) - 2ZR_2\sin^2\theta\cos\theta\,\mathrm{Im}\,l_0'(\rho u_0)] -$$

$$-\frac{2R_2}{(R_2-R_1)}\cos\theta\,Z(R_1\cos^2\theta - R_2\sin^2\theta) + \chi_{3\tau_1}C_3 + \chi_{4\tau_1}C_4.$$

The arbitrary conditions are determined: C_3 from the condition of equality of the shearing force at $\theta = \frac{\pi}{2}$ to the given value Z, and C_4 from the condition of equality to zero of the bending moment at the edge $\theta = \frac{\pi}{2}$.

The values of $|\rho|$ (meaning the real part of ρ) appearing in formulas (11)-(13), and the values of u_0, $\operatorname{Im} l_0(\rho u_0)$, $\operatorname{Re} l_0(\rho u_0)$, their derivatives, and the magnitudes $\lambda(0) = \frac{1}{u_0' \sqrt{p u_0'}}$ correspond to the values obtained in /6/. We have spoken only of the forces and moments for brevity, although it is easy to obtain formulas for the displacements and strains also.

Brief conclusions

The method proposed for calculating a toroidal shell subjected to a "wind-type" load in the case of an edge coinciding with a singular line is correct for $\varepsilon \ll 1$. This case is of practical interest in the calculation of bellows. It should be emphasized that the presence of an edge coinciding with a singular line introduces a considerable distortion in the stressed-strained state of the shell in the region near to this edge. In addition, the stressed states near this edge become mixed, and their separation is no longer possible. Thus, the state of pure bending stress (Figure 2) near $\theta = 0$ includes a membrane stress state which is dampened rapidly; on the other hand, the state of membrane stress (Figures 3, 4) at the edge $\theta = 0$ becomes a moment state, which likewise is dampened inside the region. These properties are specific for the given problem.

Having obtained the solutions for the problems represented schematically in Figures 2, 3, and 4, it becomes possible to give a solution to the more complex problem of the connection of two toroidal shells with an annular plate or cone, i.e., to the problem of the stressed-strained state of bellows-type shells.

With this static problem solved, it is likewise possible to examine the problem of the stability of bellows according to Euler. Additional studies to that end are not necessary. These problems are however of independent interest, and are not given here in view of the brevity of the exposition.

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CONTACT PROBLEM FOR FOUNDATIONS SUBJECTED TO BENDING STRAINS

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The paper deals with the action of an elastic axisymmetrical punch on a layer of finite thickness h supported along a ring. The friction forces between the punch and the layer are neglected. An approximate solution is obtained, correct for h not smaller than the diameter $2a$ of the contact zone.

1. STATEMENT OF THE PROBLEM AND DERIVATION OF THE FUNDAMENTAL INTEGRAL EQUATION

Let an elastic punch having a base of shape $z = \xi(\rho)$ penetrate into an elastic layer supported along a ring of finite thickness ε. The punch is loaded by a force P acting along the axis of symmetry. We first assume that the pressure $p_1(\rho)$ between the punch and the layer is known, and that the pressure distribution $p_2(\rho)$ under the supporting ring is likewise known. Consider the solution of the problem of equilibrium of the layer under the action of the forces given on the boundary planes. The corresponding relationships are given in /1/. The following formulas can be obtained on the basis of these relationships:

$$W_1 = \frac{K_1}{a}\int_0^\infty q_1 J_0(t\eta)\,dt, \tag{1.1}$$

$$W_{2s} = -\frac{K_2}{a}\int_0^\infty \frac{q_1+q_2}{2}\,\frac{\operatorname{ch} t\dfrac{h}{a} - 1}{\operatorname{sh} t\dfrac{h}{a} + t\dfrac{h}{a}}\,J_0(t\eta)\,dt, \tag{1.2}$$

$$W_{3s} = \frac{K_2}{a}\int_0^\infty \frac{q_1+q_2}{2}\,\frac{\operatorname{ch} t\dfrac{h}{a} - 1}{\operatorname{sh} t\dfrac{h}{a} + t\dfrac{h}{a}}\,J_0(t\eta_0)\,dt, \tag{1.3}$$

$$W_{2t} = \frac{K_2}{a}\int_0^\infty \frac{q_1-q_2}{2}\,\frac{\operatorname{ch} t\dfrac{h}{a} + 1}{\operatorname{sh} t\dfrac{h}{a} - t\dfrac{h}{a}}\,[J_0(t\eta) - J_0(t\eta_0)]\,dt. \tag{1.4}$$

In these formulas $W_1(\eta)$ is the displacement of the punch points under the action of the pressure $p_1(\rho)$. In the calculation of $W_1(\eta)$ it was assumed that the region of contact is small, so that the punch can be considered as a semispace; W_{2s} and W_{2l} are the compression and bending displacements respectively of the layer points under the punch at $z = \frac{h}{2}$; W_{3s} is the displacement of the layer points under the support at $z = -\frac{h}{2}$; q_1 and q_2 are the Fourier-Hankel transforms of the pressures $p_1(\rho)$ and $p_2(\rho)$

$$\eta = \frac{\rho}{a}, \qquad \eta_1 = \frac{\rho_1}{a}, \qquad \eta_2 = \frac{\rho_2}{a}, \qquad \rho_2 = \rho_1 + \varepsilon;$$

ρ_2 and ρ_1 are the outer and inner radii of the supporting ring; $K_i = \frac{m_i - 1}{m_i G_i}$, where m_1, G_1 and m_2, G_2 are the elastic characteristics of the punch and layer, respectively.

The following relationships are easily obtained for q_1 and q_2:

$$q_1(t) = a^2 \int_0^1 p_1(\eta a)\, \eta J_0(t\eta)\, d\eta; \tag{1.5}$$

$$q_2(t) = \frac{P}{2\pi} \frac{J_1(t\eta_2)\,\eta_2 - J_1(t\eta_1)\,\eta_1}{t\,\frac{\eta_2^2 - \eta_1^2}{2}} = \frac{P}{2\pi}\varphi(t). \tag{1.6}$$

Note further that if ε is considerably smaller than ρ_1, the form of $p_2(\rho)$ cannot sensibly influence the problem solution. It is therefore possible to take $p_2(\rho)$ in (1.6) as constant and equal to $\frac{P}{\pi(\rho_2^2 - \rho_1^2)}$.

The condition of contact between the punch and the layer dictates the following relationships:

$$-W_{2s} + W_{3s} + W_{2l} + W_l = \delta - \xi(\eta a), \qquad \eta \leqslant 1, \tag{1.7}$$

$$\sigma_z = 0, \qquad \eta > 1, \tag{1.8}$$

where δ is the total displacement of the punch under the action of the force P. Since

$$\sigma_z = -\frac{1}{a^2}\int_0^\infty q_1(t)\, tJ_0(t\eta)\, dt, \tag{1.9}$$

and using (1.1) - (1.6), formulas (1.7) and (1.8) can be represented as follows:

$$\int_0^\infty q_1(t)\, J_0(t\eta)\, dt = g(\eta), \qquad \eta \leqslant 1; \tag{1.10}$$

$$\int_0^\infty q_1(t)\, tJ_0(t\eta)\, dt = 0. \qquad \eta > 1. \tag{1.11}$$

Here

$$g(\eta) = \frac{a(\delta - \xi)}{K_1 + K_2} + \frac{K_3}{2}\int_0^\infty (q_1 + q_2) f\left(t\frac{h}{a}\right)[J_0(t\eta) + J_0(t\eta_0)]\,dt -$$

$$-K_3\int_0^\infty q_2 J_0(t\eta_0)\,dt - \frac{K_3}{2}\int_0^\infty (q_1 - q_2)\psi\left(t\frac{h}{a}\right)[J_0(t\eta) - J_0(t\eta_0)]\,dt; \qquad (1.12)$$

$$K_3 = \frac{K_2}{K_1 + K_2}, \quad f(t) = \frac{1 + t - e^{-t}}{\operatorname{sh} t + t}, \quad \psi(t) = \frac{1 + t + e^{-t}}{\operatorname{sh} t - t}. \qquad (1.13)$$

The problem formulated has thus been reduced to solving the dual integral equations (1.10) and (1.11). It is further known /2/ that equations (1.10) and (1.11) are equivalent to the relationship

$$q_1(p) = \frac{2}{\pi}\cos p\int_0^1 \frac{g(y)\,dy}{\sqrt{1 - y^2}} + \frac{2}{\pi}\int_0^1\int_0^1 \frac{yg(uy)\,pu\sin pu}{\sqrt{1 - y^2}}\,du\,dy. \qquad (1.14)$$

This relationship can be considered as an integral relative to $q_1(p)$. It will be assumed below that

$$\xi(\rho) = A_n\rho^n.$$

Since in addition

$$\int_0^\infty q_1 J_1(t)\,dt = \frac{P}{2\pi}, \qquad \int_0^1 \frac{y^{n+1}\,dy}{\sqrt{1 - y^2}} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)},$$

$$\int_0^1 \frac{yJ_0(uy)\,dy}{\sqrt{1 - y^2}} = \frac{\sin u}{u},$$

formula (1.14) can be represented in the form:

$$q_1(p) = \frac{P}{2\pi}\,\frac{\sin p}{p} - T\frac{n}{n+1}\,\frac{\sin p}{p} + T\frac{n}{p}\int_0^1 u^{n-1}\sin pu\,du -$$

$$-\frac{K_3}{\pi}\int_0^\infty (q_1 + q_2) f\left(t\frac{h}{a}\right) S(p,t)\,dt + \frac{K_3}{\pi}\int_0^\infty (q_1 - q_2)\psi\left(t\frac{h}{a}\right) S(p,t)\,dt, \qquad (1.15)$$

where

$$T = A_n a^{n+1}\frac{n+1}{\sqrt{\pi}}\,\frac{1}{K_1 + K_2}\,\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)},$$

$$S(p, t) = \frac{\sin t}{t}\,\frac{\sin p}{p} - \int_0^1 \cos pv\cos vt\,dt. \qquad (1.16)$$

Since the right-hand part of (1.15) contains terms of the type $\frac{\sin p}{p}$, the pressure $p_1(\rho)$ will be nonregular in the general case. Accordingly, two cases are considered below.

I. The punch shape is such that the stresses under it are not bounded; in this case the problem is to be solved starting directly from (1.15).

II. The stresses in the region of contact are bounded; in this case the radius of the zone of contact is unknown, and has to be determined from the relationship

$$\frac{P}{2\pi} = T\frac{n}{n+1} - \frac{K_3}{\pi}\int_0^\infty (q_1+q_2) f\left(t\frac{h}{a}\right)\left(\cos t - \frac{\sin t}{t}\right) dt +$$

$$+ \frac{K_3}{\pi}\int_0^\infty (q_1 - q_2)\psi\left(t\frac{h}{a}\right)\left(\cos t - \frac{\sin t}{t}\right) dt, \qquad (1.17)$$

which ensures the regularity of the pressure.

Allowing for (1.17) the equation for $q_1(p)$ becomes

$$q_1(p) = T\frac{n}{p}\int_0^1 u^{n-1}\sin pu\,du - \frac{K_3}{\pi}\int_0^\infty (q_1+q_2) f\left(t\frac{h}{a}\right) R(p,t)\,dt +$$

$$+ \frac{K_3}{\pi}\int_0^\infty (q_1 - q_2)\psi\left(t\frac{h}{a}\right) R(p,t)\,dt. \qquad (1.18)$$

where

$$R(p,t) = \int_0^1 \cos pv\,(\cos t - \cos vt)\,dv.$$

2. APPROXIMATE SOLUTION OF EQUATION (1.15) AND STUDY OF THE PROBLEM

Consider first case (I). Equation (1.15) can be represented in the following operatorial form:

$$q_1(p) = q_{10}(p) + A(q_1), \qquad (2.1)$$

where

$$q_{10}(p) = \frac{P}{2\pi}\frac{\sin p}{p} - T\frac{n}{n+1}\frac{\sin p}{p} + T\frac{n}{p}\int_0^1 u^{n-1}\sin pu\,du -$$

$$- \frac{K_3}{\pi}\int_0^\infty \left(q_2 + \frac{P}{2\pi}\right) f\left(t\frac{h}{a}\right) S(p,t)\,dt + \frac{K_3}{\pi}\int_0^\infty \left(\frac{P}{2\pi} - q_2\right)\psi\left(t\frac{h}{a}\right) S(p,t)\,dt,$$

$$A(q_1) = \frac{K_3}{\pi}\int_0^\infty \left(q_1 - \frac{P}{2\pi}\right)\left[\psi\left(t\frac{h}{a}\right) - f\left(t\frac{h}{a}\right)\right] S(p,t)\,dt.$$

We shall solve (2.1) by successive approximations by the scheme

$$q_1^{(n)} = q_{10}(p) + A[q_1^{(n-1)}]. \qquad (2.2)$$

If we consider the space of the functions $C[0, \infty)$ (continuous and bounded on the semiaxis), then the following inequality is satisfied in this space:

$$\|A(q_1)\| \leqslant \|q_1\| \frac{2K_3}{\pi^3}\left(\frac{a}{h}\right)^3\left[26 + 9.3\left(\frac{a}{h}\right)\right]. \qquad (2.3)$$

In the derivation of this inequality it was taken into account that P is expressed in terms of q_1 on the strength of the equilibrium condition of the punch, and that ψ, f, and $S(p,t)$ are given by formulas (1.13) and (1.16).

It follows from (2.3) that if $K_3 = 1$, then $A(q_1)$ will be the operator of approach at $\frac{a}{h} \leqslant 0.77$. We can proceed in the same way in case (II).

Here equation (1.15) can be written in the following form:

$$q_1(p) = q_0(p) + B(q_1), \tag{2.4}$$

where

$$q_0(p) = T\frac{n}{p}\int_0^1 u^{n-1}\sin pu\,du - \frac{K_3}{\pi}\int_0^\infty \left(q_2 + \frac{P}{2\pi}\right) f\left(t\frac{h}{a}\right) R(p,t)\,dt +$$

$$+ \frac{K_3}{\pi}\int_0^\infty \left(\frac{P}{2\pi} - q_2\right)\psi\left(t\frac{h}{a}\right) R(p,t)\,dt, \tag{2.5}$$

$$B(q_1) = \frac{K_3}{\pi}\int_0^\infty \left(q_1 - \frac{P}{2\pi}\right)\left[\psi\left(t\frac{h}{a}\right) - f\left(t\frac{h}{a}\right)\right] R(p,\ t)\,dt. \tag{2.6}$$

Operator $B(q_1)$ turns out to be the operator of approach when the following inequality is fulfilled:

$$\frac{2K_3}{3\pi}\left[26 + 9.3\left(\frac{a}{h}\right)\right]\left(\frac{a}{h}\right)^3 < 1. \tag{2.7}$$

In the case $K_3 = 1$ operator $B(q_1)$ will be the operator of approach of

$$\frac{a}{h} \leqslant 0.53.$$

The integral equation (1.15) can thus be solved by successive approximations when the layer thickness is not very small.

3. STUDY OF A PUNCH WITH FLAT BASE

We shall calculate the successive approximations for σ_z under the assumption $\varepsilon = 0$. We have

$$\sigma_z^{(0)}(\eta) = -\frac{P}{2\pi a^2}\left\{B_0 - \frac{K_3}{\pi}\int_0^\infty [1 + J_0(t)] f\left(t\frac{h}{a}\right) S_1(\eta,t)\,dt +\right.$$

$$\left. + \frac{K_3}{\pi}\int_0^\infty [1 - J_0(t)]\psi\left(t\frac{h}{a}\right) S_1(\eta,\ t)\,dt\right\}, \tag{3.1}$$

$$\sigma_z^{(1)}(\eta) = \sigma_z^{(0)}(\eta) + \frac{K_3}{\pi}\int_0^\infty \left[q_1^{(0)} - \frac{P}{2\pi}\right]\left[\psi\left(t\frac{h}{a}\right) - f\left(t\frac{h}{a}\right)\right] S_1(\eta,\ t)\,dt. \tag{3.2}$$

where

$$B = (1-\eta^2)^{-1/2},$$

$$S_1(\eta, t) = (1-\eta^2)^{-1/2} t^2 \left[\left(\eta^2 - \frac{2}{3}\right) - \frac{t^2}{9}\left(\eta^4 - \frac{1}{2}\eta^2 - \frac{1}{5}\right) + \right.$$

$$\left. + \frac{t^4}{225}\left(\eta^6 - \frac{\eta^4}{2} - \frac{\eta^2}{8} - \frac{3}{28}\right)\right] \text{ etc.}$$

In order to simplify the use of relationships (3.1), (3.2) it is possible to use asymptotic representations of the right-hand parts of (3.1), (3.2) at large $\left(\frac{h}{a}\right)$. The following approximate formula is obtained:

$$\sigma_z^{(1)} = -\frac{P}{2\pi a^2 \sqrt{1-\eta^2}} \left\{1 - \frac{K_3}{\pi}\lambda_2 0.667\left(\frac{a}{h}\right)^3 + \right.$$

$$+ \frac{K_3}{\pi}(0.0222\lambda_4 + 9.78)\left(\frac{a}{h}\right)^5 - \frac{K_3}{\pi}(0.000476\lambda_6 + 10.2)\left(\frac{a}{h}\right)^7 +$$

$$+ \frac{K_3}{\pi}\eta^2\left[\lambda_2 + (0.0555\lambda_4 - 14.67)\left(\frac{a}{h}\right)^2 - (0.000555\lambda_6 + 1.05)\left(\frac{a}{h}\right)^4\right] \times$$

$$\times \left(\frac{a}{h}\right)^3 - \eta^4 \frac{K_3}{\pi}\left[0.111\lambda_4 + (0.00222\lambda_6 - 20.4)\left(\frac{a}{h}\right)^2\right]\left(\frac{a}{h}\right)^5 +$$

$$\left. + \eta^6 \frac{K_3}{\pi}\lambda_6 0.00444\left(\frac{a}{h}\right)^7\right\}, \tag{3.3}$$

where

$$\lambda_2 = T_2 - P_2 - 12.7, \quad \lambda_4 = T_4 - P_4 - 258, \quad \lambda_6 = T_6 - P_6 - 10900,$$

and

$$T_n = \int_0^\infty [1 - J_0(tl_0)]\,\psi(t)\,t^n dt, \quad P_n = \int_0^\infty J_0(tl_0) f(t)\,t^n dt, \quad l_0 = \frac{\rho_0}{h}. \tag{3.4}$$

Formula (3.3) gives an exact representation of the first seven terms of the expansion of σ_z in a power series in $\left(\frac{a}{h}\right)$. The subsequent approximations $\sigma_z^{(2)}$, $\sigma_z^{(3)}$, etc., only refine the terms in $\left(\frac{a}{h}\right)$ and above. The following approximate relationships for λ_k are obtained for $l_0 \geqslant 5$:

$$\lambda_2 = 23.6 + 12\ln\frac{1 + \sqrt{1 + \left(\frac{l_0}{1.348}\right)^2}}{2} - \frac{8.5}{\sqrt{1.82 + l_0^2}} -$$

$$- \frac{33.2}{\sqrt{(1.82 + l_0^2)^3}}, \qquad \lambda_4 = 90, \ \lambda_6 = 119. \tag{3.5}$$

Formula (3.3) gives the stresses with a maximum error which do not exceed 3% when $\left(\frac{a}{h}\right) \leqslant 0.4$ and $l_0 \leqslant 15$. Curves of σ_z as a function of η have been plotted in Figure 2 for different values of l_0. As seen from these curves, a redistribution of the contact pressures over the region of contact is obtained at bending: the contact pressures drop sharply at the center of the region of contact, and increase in the boundary ring,

which is easily explained by the influence of the bending. At sufficiently large values of l_0 negative pressures can appear in the central part of the region of contact, and a separation of the punch can take place. Thus, when $\left(\frac{a}{h}\right) = 0.47$ the pressure in the zone of contact becomes equal to zero when $l_0 = 15$. For large l_0 the zone of contact will represent a ring, whose inner radius depends on l_0 and $\left(\frac{a}{h}\right)$. This case was not treated by us.

Figure 3 represents curves of the contact pressure at different h. Figure 4 represents the pressure at the middle of the zone of contact as a function of the supporting ring radius for different h.

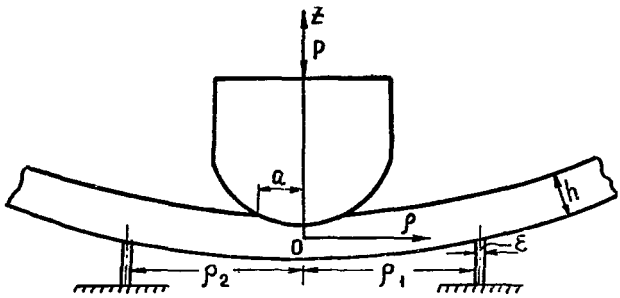

FIGURE 1.

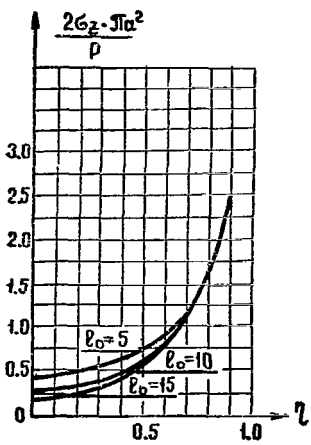

FIGURE 2.

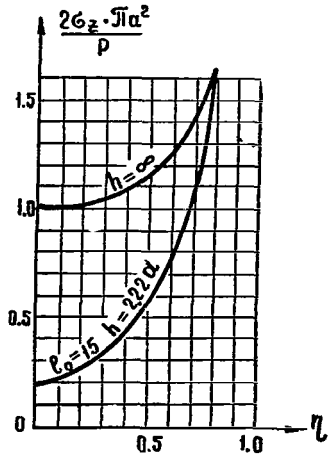

FIGURE 3.

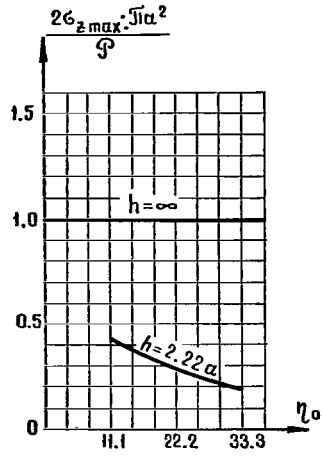

FIGURE 4.

4. STUDY OF A PUNCH OF SHAPE $\xi(\rho) = A_n\rho^n$

The following series expansions by the powers of $\left(\frac{a}{h}\right)$ are obtained in this case for the contact pressure:

$$\sigma_z = -\frac{2\sqrt{1-\eta^2}}{a^2}\left\{T\frac{n}{2}\int_0^1 \frac{u^{n-1}du}{\sqrt{u^2-\eta^2}}(1-\eta^2)^{-1/2} - \right.$$

$$-\frac{P}{2\pi}\frac{K_3}{\pi}\left[0.5\lambda_2 - 0.0278\lambda_4\left(\frac{a}{h}\right)^2 + 0.000835\lambda_6\left(\frac{a}{h}\right)^4 - \right.$$

$$\left. - \eta^2 0.0555\lambda_4\left(\frac{a}{h}\right)^2 + \eta^2 0.00111\lambda_6\left(\frac{a}{h}\right)^4 + \eta^4 0.000222\lambda_6\left(\frac{a}{h}\right)^4\right]\left(\frac{a}{h}\right)^3 +$$

$$\left. + T\frac{K_3}{\pi}\left(\frac{a}{h}\right)^5\left[\frac{7.33n}{n+3} - \left(\frac{a}{h}\right)^2\left(\frac{4.57n}{n+5} + \frac{5.09n}{n+3} + \eta^2\frac{10.2n}{n+3}\right)\right]\right\}, \qquad (4.1)$$

where λ_2, λ_4, λ_6 are given by relationships (3.5), and

$$\int_0^1 \frac{nu^{n-1}du}{\sqrt{u^2-\eta^2}}(1-\eta^2)^{-1/2} = 2\sum_{s=0}^{s=\frac{n}{2}-1} \frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}-s\right)\Gamma\left(\frac{3}{2}+s\right)} \times$$

$$\times(-1)^s(1-\eta^2)^s.$$

The connection between the force and the radius of the contact zone is given by the relationship

$$\frac{P}{2\pi} = T\frac{n}{n+1}\frac{1+\frac{n+1}{n+3}4.9\frac{K_3}{\pi}\left(\frac{a}{h}\right)^5 - \left(\frac{a}{h}\right)^7\frac{K_3}{\pi}\left[3.06\frac{n+1}{n+5} + 6.1\frac{n+1}{n+3}\right]}{1+0.333\lambda_2\frac{K_3}{\pi}\left(\frac{a}{h}\right)^3 - 0.0333\lambda_4\frac{K_3}{\pi}\left(\frac{a}{h}\right)^5 + 0.00119\frac{K_3}{\pi}\lambda_6\left(\frac{a}{h}\right)}, \qquad (4.2)$$

where T is defined by (1.16).

Formulas (4.1), (4.2) give the expansion with an accuracy up to $\left(\frac{a}{h}\right)^8$. They are best used in the following sequence: we start from a given value a of the zone of contact, and then determine T from (1.16), P from (4.2), and the contact pressure from (4.1). The case $n = 2$ was studied by the relationships given above. The contact pressures were determined in this case with an accuracy of 3%, for $l_0 \leqslant 15$ and $\left(\frac{a}{h}\right) \leqslant 0.4$. Curves of the contact pressure at different values of l_0 are given in Figure 5. The stress drop at the center of the contact zone is observed here too. We did not succeed in establishing whether a drop of the contact pressure to zero at $\eta = 0$ and a separation of the punch are possible for the given value of n, since the formulas derived are insufficient, and asymptotic expansions with higher-order approximations are necessary.

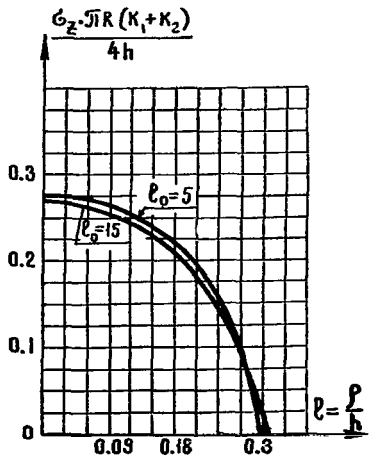

FIGURE 5.

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CALCULATION OF A CYLINDRICAL SHELL REINFORCED BY ELASTIC STIFFENERS UNDER THE ACTION OF A CONCENTRATED FORCE

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1. The problem of the stressed-strained state of a circular cylindrical shell reinforced by stiffeners, with the end stiffener more rigid than the other ones, was treated in /1/. When the shell edge is loaded by an arbitrary system of shearing and axial forces, the solution can be represented according to /1/ in the form (Figure 1)

$$
\left.
\begin{aligned}
U_k = {} & \frac{l^2}{2ER\delta}\left[2(m^2-k^2) - \sum_{i=k+1}^{m}(2i-1)\right]\frac{P_z\cos\varphi + P_y\sin\varphi}{\pi R} - \\
& - \frac{(m-k)\,l}{ER\delta}\left(\frac{M_y\cos\varphi + M_z\sin\varphi}{\pi R} + \frac{P_x}{2\pi}\right) + \\
& + \frac{l^2}{2ER\delta}\sum_{n=2}^{\infty}\frac{\beta_{1n}\beta_{3n}\left(\frac{\beta_{1n}+1}{\beta_{1n}-1}\beta_{1n}^{-k} - \frac{\beta_{3n}+1}{\beta_{3n}-1}\beta_{3n}^{-k}\right)}{n(\beta_{1n}-\beta_{3n})\frac{\bar{E}_0 I_0}{\bar{E}I}\left(\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1\right)}(C_n\sin n\varphi - F_n\cos n\varphi) - \\
& - \frac{l}{2E\delta}\sum_{n=2}^{\infty}\frac{(\beta_{1n}-1)(\beta_{3n}-1)}{\beta_{1n}-\beta_{3n}}\left\{\frac{\left(1-\frac{\bar{E}I}{\bar{E}_0 I_0}\right)\left(\frac{\beta_{1n}+1}{\beta_{1n}-1}\beta_{1n}^{-k} - \frac{\beta_{3n}+1}{\beta_{3n}-1}\beta_{3n}^{-k}\right)}{\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1} - \right. \\
& \left. - \left[\frac{\beta_{1n}+1}{(\beta_{1n}-1)^2}\beta_{1n}^{-k} - \frac{\beta_{3n}+1}{(\beta_{3n}-1)^2}\beta_{3n}^{-k}\right]\right\}(B_n\sin n\varphi - D_n\cos n\varphi); \\
\Phi_k = {} & \frac{M_{kp}}{2\pi R^2} + \frac{P_z\sin\varphi - P_y\cos\varphi}{\pi R} + \sum_{n=2}^{\infty}\frac{\beta_{1n}\beta_{3n}[(\beta_{1n}-1)\beta_{1n}^{-k} - (\beta_{3n}-1)\beta_{3n}^{-k}]}{(\beta_{1n}-\beta_{3n})\frac{\bar{E}_0 I_0}{\bar{E}I}\left(\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1\right)}\times \\
& \times(C_n\cos n\varphi + F_n\sin n\varphi) - \frac{R}{l}\sum_{n=2}^{\infty}\frac{(\beta_{1n}-1)(\beta_{3n}-1)}{n(\beta_{1n}-\beta_{3n})}\times \\
& \times\left\{\frac{\left(1-\frac{\bar{E}I}{\bar{E}_0 I_0}\right)[(\beta_{1n}-1)\beta_{1n}^{-k} - (\beta_{3n}-1)\beta_{3n}^{-k}]}{\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1} - (\beta_{1n}^{-k} - \beta_{3n}^{-k})\right\}\times \\
& \times(B_n\cos n\varphi + D_n\sin n\varphi);
\end{aligned}
\right\}\quad(1)
$$

$$
\left.
\begin{aligned}
V_k = &- \sum_{i=k+1}^{m} (a_i \cos\varphi + c_i \sin\varphi + b_i) - \\
&- \frac{l}{E\delta} \sum_{n=2}^{\infty} \frac{\beta_{1n}\beta_{3n}\left[\frac{(\beta_{1n}-1)^2}{\beta_{1n}}\beta_{1n}^{-k} - \frac{(\beta_{3n}-1)^2}{\beta_{3n}}\beta_{3n}^{-k}\right]}{\lambda_{1n}(\beta_{1n}-\beta_{3n})\frac{\bar{E}_0 I_0}{\bar{E}I}\left(\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1\right)} (C_n \cos n\varphi + F_n \sin n\varphi) + \\
&+ \frac{l}{E\delta} \sum_{n=2}^{\infty} \frac{(\beta_{1n}-1)(\beta_{3n}-1)}{\lambda_{1n}\frac{ln}{R}(\beta_{1n}-\beta_{3n})} \left\{ \frac{\left(1 - \frac{\bar{E}I}{\bar{E}_0 I_0}\right)\left[\frac{(\beta_{1n}-1)^2}{\beta_{1n}}\beta_{1n}^{-k} - \frac{(\beta_{3n}-1)^2}{\beta_{3n}}\beta_{3n}^{-k}\right]}{\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1} - \right. \\
&\left. - \left(\frac{\beta_{1n}-1}{\beta_{1n}}\beta_{1n}^{-k} - \frac{\beta_{3n}-1}{\beta_{3n}}\beta_{3n}^{-k}\right)\right\} (B_n \cos n\varphi + D_n \sin n\varphi).
\end{aligned}
\right\} \quad \begin{matrix}(1)\\(\text{cont'd})\end{matrix}
$$

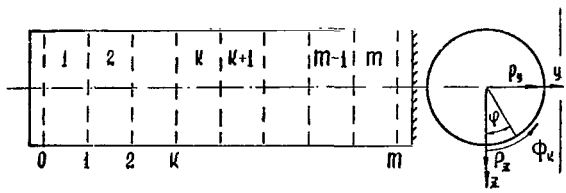

FIGURE 1.

The second expression of (1) is correct at all $k \geqslant 1$. At $k = 0$ it is to be replaced by

$$
\Phi_0 = \frac{M_{kp}}{2\pi R^2} + \frac{P_z \sin\varphi - P_y \cos\varphi}{\pi R} + \sum_{n=2}^{\infty} (C_n \cos n\varphi + F_n \sin n\varphi). \qquad (2)
$$

Here: δ, R are the thickness and radius of the shell; I is the moment of inertia of the stiffeners; I_0 is the moment of inertia of the end stiffener; E, $\bar{E}$ are moduli of normal elasticity of the sheathing and the stiffeners; $\bar{E}_0$ is the modulus of normal elasticity of the end stiffener; G is the modulus of shear of the sheathing; U_k, V_k are longitudinal and tangential displacements of the sheathing in the shell section by the k-th stiffener; Φ_k are fluxes of shearing forces in the k-th portion of the shell, located between frames $k-1$, k. The remaining symbols are understood from Figure 1.

$$
\begin{aligned}
a_k = &- \frac{l^2}{2ER\delta}\left\{\frac{l}{2R}\left[4(m^2-k^2) + 2k - 1 - 2\sum_{i=k+1}^{m}(2i-1)\right] + \right. \\
&\left. + 2\frac{E}{G}\frac{R}{l}\left(1 + \frac{Gl^2}{12ER^2}\right)\right\}\frac{P_y}{\pi R} + \frac{l^2}{2ER\delta}[2(m-k)+1]\frac{M_z}{\pi R^2}, \qquad (3)
\end{aligned}
$$

$$
b_k = \frac{l}{R}\frac{M_{kp}}{2\pi G R\delta}, \qquad (4)
$$

$$c_k = \frac{l^2}{2ER\delta}\left\{\frac{l}{2R}\left[4(m^2-k^2)+2k-1-2\sum_{i=k+1}^{m}(2i-1)\right]+\right.$$

$$\left.+\frac{2E}{G}\frac{R}{l}\left(1+\frac{Gl^2}{12ER^2}\right)\right\}\frac{P_z}{\pi R}-\frac{l^2}{2ER\delta}[2(m-k)+1]\frac{M_y}{\pi R^2}. \qquad (5)$$

C_n, F_n, D_n, B_n are coefficients of the Fourier series, with C_n, F_n determined from the series expansions of the flux of shearing forces at the shell edge $k = 0$, and B_n, D_n from the series expansions of the axial forces per unit length at the shell edge.

The symbols β_{1n}, β_{3n} in expressions (1) designate the roots of the algebraic equation

$$\left(\beta_n+\frac{1}{\beta_n}\right)^2-\left(\beta_n+\frac{1}{\beta_n}\right)[\lambda_{1n}(\lambda_2-\lambda_{3n})+4]+2[\lambda_{1n}(\lambda_2+2\lambda_{3n})+2]=0, \qquad (6)$$

which satisfy the conditions $|\beta_{1n}|>1$, $|\beta_{3n}|>1$.

The following symbols have been used here:

$$\lambda_{1n}=cn^2(n^2-1)^2, \qquad \lambda_2=\frac{E}{G}, \qquad \lambda_{3n}=\frac{d^2n^2}{6}, \qquad c=\frac{\bar{E}Il}{ER^4\delta}, \quad d=\frac{l}{R}. \qquad (7)$$

The solution given was obtained under the following assumptions. It is assumed that the sheathing behaves as a thin shell acted upon by membrane stresses subjected to axial $T_k(\varphi)$ and shearing $\Phi_k(\varphi)$ forces; the Poisson ratio of the sheathing is taken as equal to zero; the stiffeners have a finite flexural rigidity in their plane and a zero flexural rigidity out of their plane. The stiffeners are assumed to be continuously connected along the contour with the sheathing; the radius of the neutral axis of the stiffener is equal to the radius of the middle surface of the sheathing.

The functions U_k, V_k, Φ_k define completely the stressed and strained state of a circular cylindrical shell reinforced through equal intervals by elastic stiffeners. Thus, the axial forces per unit length appearing in the sheathing over the section of location of the k-th stiffener, and the bending moments in the section of the k-th stiffener, are determined by the formulas

$$T_k=\frac{E\delta}{l}(U_{k+1}-U_k)+\frac{l}{2R}\frac{d\Phi_{k+1}}{d\varphi},$$
$$M_k=-\frac{\bar{E}I}{R^2}\left(\frac{d^3V_k}{d\varphi^3}+\frac{dV_k}{d\varphi}\right). \qquad (8)$$

Solution (1) makes it possible to study the stressed and strained state of a circular reinforced cylindrical shell for different cases of loading of the shell edge, including the action of concentrated forces. It is sufficient to find the expansion coefficients C_n, F_n, B_n, D_n of the given functions in Fourier series.

2. Consider thus a shell whose edge $k = 0$ is loaded only by a concentrated axial force P at point $\varphi=\varphi_0$. The representation of a concentrated force in a Fourier series is, as known

$$T_0=\frac{P}{\pi R}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\varphi-\varphi_0)\right]. \qquad (9)$$

Series (9) converges at all points except point $\varphi = \varphi_0$.

The fluxes of axial forces acting on the shell edge can be obtained from expressions (8) and (1) by substituting $k = 0$. After the necessary transformations the expression for the axial forces per unit length acting on the shell edge becomes

$$T_0 = \frac{P}{2\pi R} + \frac{M_z \cos\varphi + M_y \sin\varphi}{\pi R^2} + \sum_{n=2}^{\infty} (B_n \sin n\varphi - D_n \cos n\varphi). \qquad (10)$$

The comparison of the coefficients in series (9) and (10) yields

$$M_z = PR\cos\varphi_0; \quad M_y = PR\sin\varphi_0; \quad B_n = \frac{P}{\pi R}\sin n\varphi_0; \quad D_n = -\frac{P}{\pi R}\cos n\varphi_0. \qquad (11)$$

The following expressions for the fluxes of axial and shearing forces are obtained on the strength of (11):

$$T_k = \frac{P}{\pi R}\left\{\frac{1}{2} + \cos(\varphi - \varphi_0) - \sum_{n=2}^{\infty} \frac{(\beta_{1n} - 1)(\beta_{3n} - 1)}{\beta_{1n} - \beta_{3n}} \left[\frac{\left(1 - \frac{\bar{E}I}{\bar{E}_0 I_0}\right)(\beta_{1n}^{-k} - \beta_{3n}^{-k})}{\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1} - \right.\right.$$

$$\left.\left. - \frac{\beta_{1n}^{-k}}{\beta_{1n} - 1} + \frac{\beta_{3n}^{-k}}{\beta_{3n} - 1}\right] \cos n(\varphi - \varphi_0)\right\}; \qquad (12)$$

$$\Phi_k = \frac{P}{\pi R}\frac{R}{l}\sum_{n=2}^{\infty} \frac{(\beta_{1n} - 1)(\beta_{3n} - 1)}{n(\beta_{1n} - \beta_{3n})} \times$$

$$\times \left\{\frac{\left(1 - \frac{\bar{E}I}{\bar{E}_0 I_0}\right)[(\beta_{1n} - 1)\beta_{1n}^{-k} - (\beta_{3n} - 1)\beta_{3n}^{-k}]}{\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1} - (\beta_{1n}^{-k} - \beta_{3n}^{-k})\right\} \sin n(\varphi - \varphi_0). \qquad (13)$$

The bending moments in the stiffeners will then be equal to

$$M_k = -\frac{P}{\pi}\frac{R^2}{l}\sum_{n=2}^{\infty} \frac{(\beta_{1n} - 1)(\beta_{3n} - 1)}{n^2(n^2 - 1)(\beta_{1n} - \beta_{3n})} \times$$

$$\times \left\{\frac{\left(1 - \frac{\bar{E}I}{\bar{E}_0 I_0}\right)\left[\frac{(\beta_{1n} - 1)^2}{\beta_{1n}}\beta_{1n}^{-k} - \frac{(\beta_{3n} - 1)^2}{\beta_{3n}}\beta_{3n}^{-k}\right]}{\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1} - \right.$$

$$\left. - \left(\frac{\beta_{1n} - 1}{\beta_{1n}}\beta_{1n}^{-k} - \frac{\beta_{3n} - 1}{\beta_{3n}}\beta_{3n}^{-k}\right)\right\} \cos n(\varphi - \varphi_0). \qquad (14)$$

Series (12) converges very slowly. Its convergence can be improved by noting that at

$$n \to \infty \quad \beta_{1n} \to -(2 + \sqrt{3}), \quad \beta_{3n} \to -\infty.$$

Therefore we add to and subtract from (12) the term

$$-(2 + \sqrt{3})\sum_{n=1}^{\infty} \cos n(\varphi - \varphi_0),$$

taking into account that $\sum_{n=1}^{\infty} \cos n(\varphi - \varphi_0) = -\frac{1}{2}$ for all $\varphi \neq \varphi_0$.

The expression for T_k becomes, after certain transformations,

$$T_k = \frac{P}{\pi R}\left\{[1-(-1)^{-k}(2+\sqrt{3})^{-k}]\left[\frac{1}{2}+\cos(\varphi-\varphi_0)\right]\right\} +$$

$$+\sum_{n=2}^{\infty}\left\{\frac{(\beta_{1n}-1)(\beta_{3n}-1)}{\beta_{1n}-\beta_{3n}}\times\right.$$

$$\times\left[\frac{\left(1-\frac{\bar{E}I}{\bar{E}_0 I_0}\right)(\beta_{1n}^{-k}-\beta_{3n}^{-k})}{\frac{\bar{E}I}{\bar{E}_0 I_0}+\beta_{1n}\beta_{3n}-1}-\frac{\beta_{1n}^{-k}}{\beta_{1n}-1}+\frac{\beta_{3n}^{-k}}{\beta_{3n}-1}\right]\left.-(-1)^{-k}(2+\sqrt{3})^{-k}\right\}\times$$

$$\times\cos n(\varphi-\varphi_0). \qquad (15)$$

The stressed state of a reinforced shell loaded at the edge by a concentrated axial force is described therefore by formulas (13), (14), and (15). Formulas for the shell displacements U_k, V_k can be similarly obtained.

If we write in expressions (13), (14), and (15) $\frac{\bar{E}I}{\bar{E}_0 I_0}=1$, we obtain the corresponding formulas for a shell reinforced all over by stiffeners of identical rigidity.

If we substitute $c \to \infty$ in (5), and $\beta_{3n} \to \infty$, $\frac{\bar{E}I}{\bar{E}_0 I_0} = 1$ in (13), (14), and (15), we obtain formulas for a shell reinforced all over by stiffeners absolutely rigid in their plane. These formulas are:

$$\left.\begin{aligned}
T_k &= \frac{P}{\pi R}\left\{[1-(-1)^{-k}(2+\sqrt{3})^{-k}]\left[\frac{1}{2}+\cos(\varphi-\varphi_0)\right]+\right.\\
&\left.+\sum_{n=2}^{\infty}[\beta_{1n}^{-k}-(-1)^{-k}(2+\sqrt{3})^{-k}]\cos n(\varphi-\varphi_0)\right\};\\
\Phi_k &= \frac{P}{\pi R}\frac{R}{l}\sum_{n=2}^{\infty}\frac{(\beta_{1n}-1)\beta_{1n}^{-k}}{n}\sin n(\varphi-\varphi_0);\\
M_k &= -\frac{P}{\pi}\frac{R^2}{l}\sum_{n=2}^{\infty}\frac{(\beta_{1n}-1)^2\beta_{1n}^{-k}}{n^2(n^2-1)\beta_{1n}}\cos n(\varphi-\varphi_0);
\end{aligned}\right\} \qquad (16)$$

where β_{1n} designate the roots of the algebraic equation

$$\beta_n^2 - \frac{2(\lambda_2+2\lambda_{3n})}{\lambda_2-\lambda_{3n}}\beta_n + 1 = 0, \qquad (17)$$

satisfying the condition $|\beta_{1n}| > 1$.

3. We can similarly obtain a solution for a shell loaded at the edge by a concentrated force Q tangential to the contour.

The representation of this concentrated force in a Fourier series is*

$$\Phi_0 = \frac{Q}{\pi R}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\varphi-\varphi_0)\right]. \qquad (18)$$

* The necessary calculations for this case of shell loading were conducted by the engineer Lyashenko under the author's guidance.

The comparison of the coefficients of series (18) and (2) yields

$$M_{kp} = QR; \quad P_y = -Q\cos\varphi_0; \quad P_z = Q\sin\varphi_0;$$
$$C_n = \frac{Q}{\pi R}\cos n\varphi_0; \qquad F_n = \frac{Q}{\pi R}\sin n\varphi_0. \tag{19}$$

The following expressions for the fluxes of shearing and axial forces are obtained with the aid of (19):

$$\left.\begin{aligned}
\Phi_k &= \frac{Q}{\pi R}\left\{\frac{1}{2} + \cos(\varphi-\varphi_0) + \right.\\
&\left. + \sum_{n=2}^{\infty} \frac{\beta_{1n}\beta_{3n}\left[(\beta_{1n}-1)\beta_{1n}^{-k} - (\beta_{3n}-1)\beta_{3n}^{-k}\right]}{\frac{\bar{E}_0 I_0}{\bar{E}I}\left(\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1\right)(\beta_{1n}-\beta_{3n})}\cos n(\varphi-\varphi_c)\right\};\\
T_k &= \frac{Q}{\pi R}\frac{l}{R}\left\{k\sin(\varphi-\varphi_0) - \sum_{n=2}^{\infty}\frac{\beta_{1n}\beta_{3n}(\beta_{1n}^{-k}-\beta_{3n}^{-k})\, n\sin n(\varphi-\varphi_0)}{\frac{\bar{E}_0 I_0}{\bar{E}I}\left(\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1\right)(\beta_{1n}-\beta_{3n})}\right\}.
\end{aligned}\right\} \tag{20}$$

The bending moments will be

$$M_k = \frac{Q}{\pi R}\sum_{n=2}^{\infty}\frac{\beta_{1n}\beta_{3n}\left[(\beta_{1n}-1)^2\beta_{1n}^{-(k+1)} - (\beta_{3n}-1)^2\beta_{3n}^{-(k+1)}\right]\sin n(\varphi-\varphi_0)}{\frac{\bar{E}_0 I_0}{\bar{E}I}\left(\frac{\bar{E}I}{\bar{E}_0 I_0} + \beta_{1n}\beta_{3n} - 1\right)(\beta_{1n}-\beta_{3n})\, n(n^2-1)}. \tag{21}$$

If we write $\frac{\bar{E}I}{\bar{E}_0 I_0} = 1$ in expressions (20) and (21), we obtain, after suitable transformations, the corresponding expressions for a shell reinforced by stiffeners of equal rigidity.

In the case of stiffeners absolutely rigid in their plane, expressions (20) and (21) reduce to

$$\Phi_k = \frac{Q}{\pi R}\left[\frac{1}{2} + \cos(\varphi-\varphi_0)\right], \quad T_k = \frac{Q}{\pi R}\frac{l}{R}k\sin(\varphi-\varphi_0), \quad M_k = 0. \tag{22}$$

As easily seen, these expressions correspond to the elementary solution for a cantilever shell-beam subjected to bending and torsion.

4. Curves of the distribution of the axial forces per unit length over the sections in the case of loading of the shell edge by a longitudinal concentrated force applied at point $\varphi_0 = 0$ have been drawn in Figures 2, 3, and 4.

The calculation was conducted for a weakly reinforced shell of parameters: $c = 0.1\times10^{-7}$, $d = 0.064$, for three different values of the rigidity $\bar{E}_0 I_0$ of the end stiffener. The continuous lines represent the curves of distribution of the axial forces per unit length at the different sections of the shell; the dotted line represents the curve of distribution of the axial forces plotted on the basis of the calculation of the shell as a cantilever beam in a state of bending strain.

It is seen from these curves that:

a) in the first portions of the shell ($k = 1, 2$) a high concentration of stresses is observed near the point of application of the concentrated force $\varphi = 0$. The concentration decreases slowly with the increase in the

distance from the edge; thus, at a distance from the edge equal to the shell diameter (corresponding to section $k = 30$) the maximum stresses exceed by a factor of 2 to 2.8 (depending on the rigidity of the end stiffener) the values of the stresses calculated by the elementary formulas of strength of materials;

b) the part of the shell contour which is in a state of strain increases with the increase of the distance from the loaded edge. Thus, at a distance from the edge equal to the shell diameter, the part of the contour which is in a state of strain has a central angle of 90 to 120° (depending on the rigidity of the end stiffener);

c) when the shell edge is loaded by a longitudinal concentrated force, the increase in the rigidity of the end stiffener does not considerably reduce the maximum normal stresses in the corresponding sections of the shell.

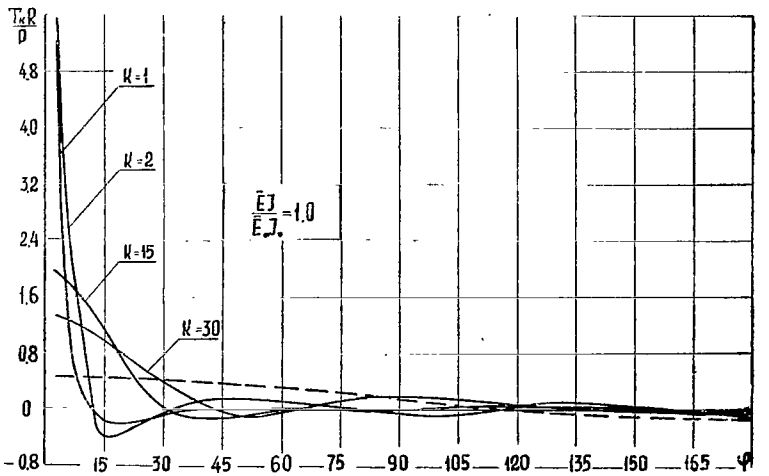

FIGURE 2.

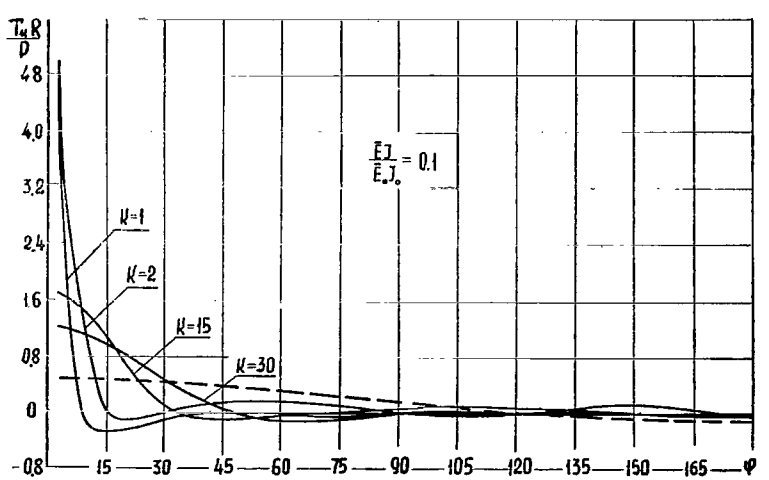

FIGURE 3.

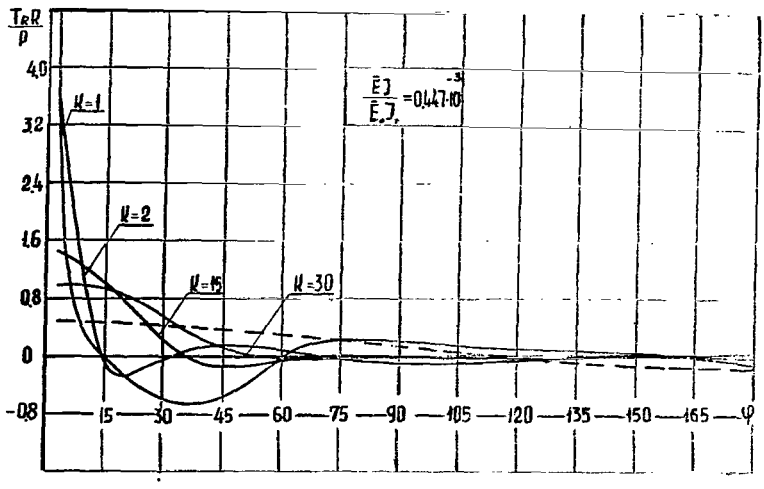

FIGURE 4.

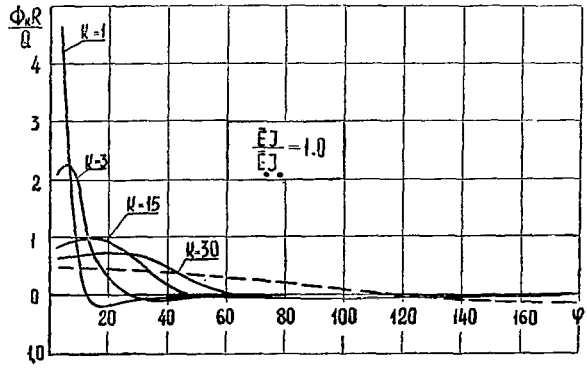

FIGURE 5.

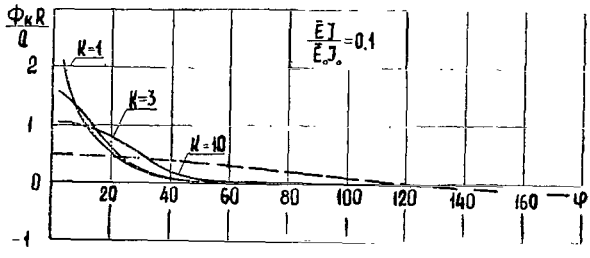

FIGURE 6.

Curves of the distribution of the fluxes of shearing forces over the shell section for the case of edge loading by a concentrated force tangential to the contour have been plotted in Figures 5, 6, and 7. The calculation was conducted for a weakly reinforced shell having parameters $c = 0.1\times10^{-7}$, $a = 0.064$. The dotted line represents as before the curve of distribution

of the fluxes of shearing forces, plotted by calculating the shell as a cantilever beam subjected to bending and torsion.

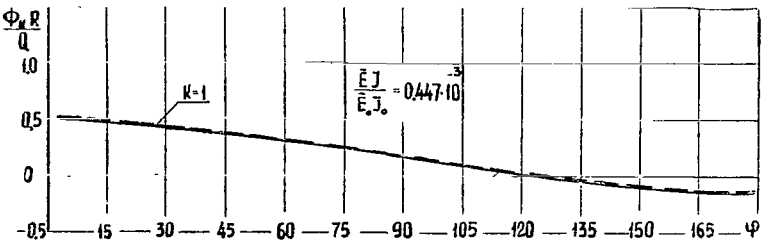

FIGURE 7.

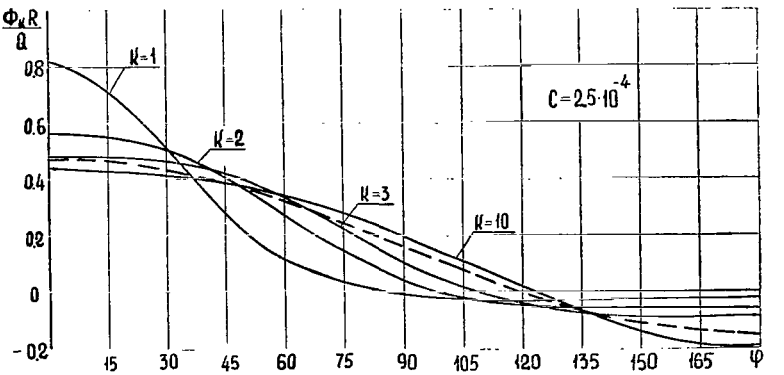

FIGURE 8.

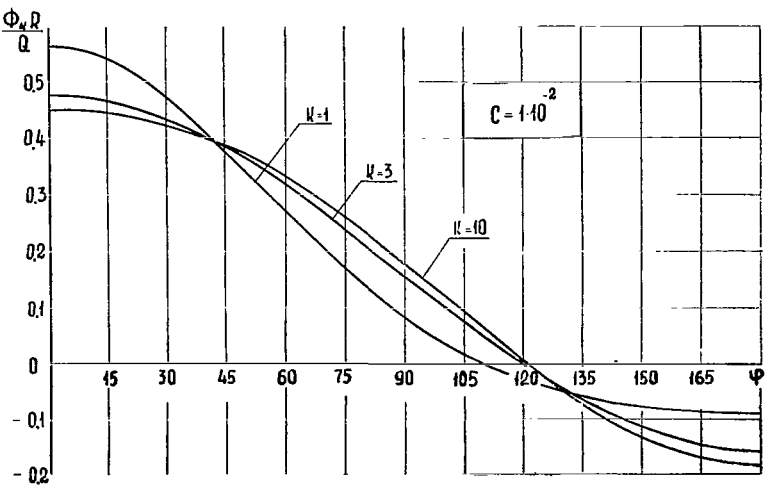

FIGURE 9.

It is seen from the curves given that the increase in the rigidity of the end stiffener has a considerable influence on both the pattern of distribution of the fluxes of shearing stresses over the shell sections and the magnitude of the maximum stresses. Thus, at $\frac{\bar{E}I}{\bar{E}_0 I_0} = 0.447 \times 10^{-3}$, no noticeable concentration of the stresses is observed in the extreme portion $k = 1$, as can be seen from Figure 9.

Figures 8 and 9 show similar curves, plotted for shells with more rigid stiffeners ($C = 2.5 \times 10^{-4}$, $c = 10^{-2}$, $d = 0.5$).

It is seen from these curves that, with the increase in the rigidity of the stiffener, the state of strain of a shell loaded at the edge by a concentrated force tangential to the contour propagates more uniformly over the shell.

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PROBLEMS OF THE THEORY OF PNEUMATIC SHELLS

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Closed pneumatic shells are being used extensively lately as bearing elements of building structures and special-purpose structures. When such shells are designed from very extensible materials, one of the main problems of their calculation is the determination of the deformations and displacements of the shell surface. Two different problems are usually considered:

1. The determination of the geometrical shape which the shell surface acquires when inflated (by increasing the internal pressure up to the rated value).

2. The determination of the deformations and displacements of the shell surface under an external load.

This report deals with the first of these two problems.

We shall consider the problem of determining the geometrical shape of a closed singly connected pneumatic shell of revolution made of a strongly extensible ideally elastic material under a statically increasing internal pressure. The axis of revolution is the coordinate axis z.

The initial shape of the shell surface (corresponding to zero excess pressure) is considered as given. We shall consider it to be such that only tensile stresses appear in it under the action of the internal pressure; this condition is imposed on the strength of the fact that pneumatic shells made from soft material cannot take compressive stresses, which cause local buckling. We consider a momentless state of stress of the shells — the values of the tensile stresses are constant through the thickness.

The differential equations of state of shells of revolution made from an ideally elastic material are

$$0.5\psi\gamma^{-1}\rho(1+\dot{\rho}^2\dot{\zeta}^{-2})^{0.5}[1-2\nu+\nu\rho(1+\dot{\rho}^2\dot{\zeta}^{-2})^{0.5}(\ddot{\rho}\dot{\zeta}-\dot{\rho}\ddot{\zeta})(\dot{\rho}^2+\dot{\zeta}^2)^{-1.5}]$$
$$-(\ddot{\rho}_0\dot{\zeta}_0-\dot{\rho}_0\ddot{\zeta}_0)(\dot{\rho}_0^2+\dot{\zeta}_0^2)^{-1.5}(\dot{\rho}^2+\dot{\zeta}^2)^{0.5}+1=0; \qquad (1)$$

$$0.5\psi\gamma^{-1}\rho(1+\dot{\rho}^2\dot{\zeta}^{-2})^{0.5}[2-\nu-\rho(1+\dot{\rho}^2\dot{\zeta}^{-2})^{0.5}(\ddot{\rho}\dot{\zeta}-\dot{\rho}\ddot{\zeta})\times$$
$$\times(\dot{\rho}^2+\dot{\zeta}^2)^{-1.5}]-\rho_0^{-1}\rho+1=0; \qquad (2)$$

$$0.5\psi\gamma^{-1}\rho(1+\dot{\rho}^2\dot{\zeta}^{-2})^{0.5}\nu[3-\rho(1+\dot{\rho}^2\dot{\zeta}^{-2})^{0.5}(\ddot{\rho}\dot{\zeta}-\dot{\rho}\ddot{\zeta})\times$$
$$\times(\dot{\rho}^2+\dot{\zeta}^2)^{-1.5}]+\gamma-1=0. \qquad (3)$$

Equations (1) and (2) relate the geometrical expressions of the meridional and circumferential strains through the displacements with their physical

expressions through the stresses and forces, found from the equilibrium conditions for the deformed shell surface.

Equation (3) determines the variation of the shell thickness caused by deformation of its surface.

In equations (1), (2), and (3):

$\psi = \frac{q}{E}$ is the ratio of the excess internal pressure to the modulus of elasticity of the shell material; ν is the Poisson ratio; $\gamma = \gamma(\varphi_0) = \frac{\delta(\varphi_0)}{\delta_0}$ is the ratio of the thickness of the deformed shell at an arbitrary point of its surface to the initial thickness (at $\psi = 0$);

$$\rho_0 = \rho_0(\varphi_0) = \frac{r_0(\varphi_0)}{\delta_0}; \quad \zeta_0 = \zeta_0(\varphi_0) = \frac{z_0(\varphi_0)}{\delta_0}$$

are the dimensionless rectangular coordinates of the meridional section of the initial shell surface;

$$\rho = \rho(\varphi_0) = \frac{r(\varphi_0)}{\delta_0}; \quad \zeta = \zeta(\varphi_0) = \frac{z(\varphi_0)}{\delta_0}$$

are dimensionless rectangular coordinates of the meridional section of the deformed shell surface; φ_0 is the angle between the normal to the meridional section of the shell surface at an arbitrary physically fixed point and the z axis.

The dots in (1), (2), and (3) designate differentiation by φ_0.

The functions $\rho_0(\varphi_0)$, $\zeta_0(\varphi_0)$ are known.

The functions $\rho(\varphi_0)$, $\zeta(\varphi_0)$, $\gamma(\varphi_0)$ have to be determined. We shall assume that

$$\dot{\zeta}_0(0) = \dot{\zeta}_0(\pi) = 0. \tag{4}$$

It follows from the condition of symmetry relative to the z axis that

$$\left.\begin{aligned} \rho(-\varphi_0) &= -\rho(\varphi_0) \\ \zeta(-\varphi_0) &= \zeta(\varphi_0) \end{aligned}\right\}. \tag{5}$$

The functions $\rho(\varphi_0)$ and $\zeta(\varphi_0)$ must satisfy the following conditions as well:

$$\left.\begin{aligned} &\rho(0) = \rho(\pi) = 0 \\ &\zeta(\tilde{\varphi}_0) = \zeta_0(\tilde{\varphi}_0) = 0 \\ &\dot{\zeta}(0) = \dot{\zeta}(\pi) = 0 \end{aligned}\right\}, \tag{6}$$

where $\tilde{\varphi}_0$ determines, on the meridional section of the shell, a point whose displacement in the direction of the z axis is equal to zero. The displacements of the points of the meridional section of the shell in the direction of the r and z axes are respectively equal to

$$\left.\begin{aligned} v &= v(\varphi_0) = [\rho(\varphi_0) - \rho_0(\varphi_0)]\,\delta_0 \\ u &= u(\varphi_0) = [\zeta(\varphi_0) - \zeta_0(\varphi_0)]\,\delta_0 \end{aligned}\right\}. \tag{7}$$

We shall consider the problem of the deformation of a shell whose initial surface is formed by the rotation of an ellipse relative to one of its

major axes, i.e., represents an ellipsoid of revolution. We have

$$\left.\begin{aligned}\rho_0 &= (\alpha_0^2\cos^2\varphi_0 + \beta_0^2\sin^2\varphi_0)^{-0.5}\beta_0^2\sin\varphi_0\\ \zeta_0 &= (\alpha_0^2\cos^2\varphi_0 + \beta_0^2\sin^2\varphi_0)^{-0.5}\alpha_0^2\cos\varphi_0\end{aligned}\right\}, \tag{8}$$

where $\alpha_0 = \frac{a_0}{\delta_0}$ and $\beta_0 = \frac{b_0}{\delta_0}$ are dimensionless semiaxes of the ellipsoid. The z axis coincides with the direction of the semiaxis a_0, and the r axis with the direction of the semiaxis b_0. The condition of absence of compressive stresses in the shell is

$$\beta_0 \leqslant 2^{0.5}\alpha_0. \tag{9}$$

We shall look for a solution of system (1), (2) in the form

$$\left.\begin{aligned}\rho &= (\alpha^2\cos^2\varphi + \beta^2\sin^2\varphi)^{-0.5}\beta^2\sin\varphi\\ \zeta &= (\alpha^2\cos^2\varphi + \beta^2\sin^2\varphi)^{-0.5}\alpha^2\cos\varphi\end{aligned}\right\}. \tag{10}$$

Solution (10) starts from the assumption that the shell surface remains an ellipsoid of revolution in the process of deformation, and that only the values of the semiaxes and their ratio change with the increase of the internal pressure.

In (10) $\alpha = \frac{a}{\delta_0}$ and $\beta = \frac{b}{\delta_0}$ are dimensionless semiaxes of the deformed ellipsoid; φ is the angle between the normal to the meridional section of the deformed shell surface at an arbitrary physically fixed point and the z axis; $\varphi_0 = \frac{\pi}{2}$. Obviously

$$\varphi = \varphi(\varphi_0). \tag{11}$$

Relationship (11) is unknown and must be determined.

Introduce the following designations:

$$\left.\begin{aligned}\tau &= \tau(\varphi_0) = \sin\varphi\cdot\sin^{-1}\varphi_0\\ K &= K(\varphi) = (\alpha^2\cos^2\varphi + \beta^2\sin^2\varphi)^{0.5}\\ K_0 &= K_0(\varphi_0) = (\alpha_0^2\cos^2\varphi_0 + \beta_0^2\sin^2\varphi_0)^{0.5}\end{aligned}\right\}. \tag{12}$$

Using (8), (10), and (12), system (1), (2) can be written in the form

$$\left.\begin{aligned}&0.5\psi\gamma^{-1}\beta^2K^{-1}(1 - 2\nu + \nu\alpha^{-2}K^2) - \alpha_0^{-2}\beta_0^{-2}K_0^3\alpha^2\beta^2K^{-3}\varphi + 1 = 0\\ &0.5\psi\gamma^{-1}\beta^2K^{-1}(2 - \nu - \alpha^{-2}K^2) - \beta_0^{-2}K_0\beta^2K^{-1}\tau + 1 = 0\end{aligned}\right\}, \tag{13}$$

where the function $\gamma = \gamma(\varphi_0)$ is determined from (3) and (10) by the expression

$$\gamma = 0.5 + [0.5^2 - 0.5\psi\nu\beta^2K^{-1}(3 - \alpha^{-2}K^2)]^{0.5}. \tag{14}$$

Obviously, for points $\varphi_0 = 0$ and $\varphi_0 = \frac{\pi}{2}$, always $\varphi = \varphi_0 = 0$ and $\varphi = \varphi_0 = \frac{\pi}{2}$.

By separating the variables in the first (differential) equation of system (13) and integrating it, and by writing the second (algebraic) equation of system (13) for the characteristic point $\varphi = \varphi_0 = \frac{\pi}{2}$, we obtain a system of two equations for determining the unknowns α and β:

$$\left.\begin{aligned} \alpha_0^2\beta_0^2\int\limits_{\varphi_0=0}^{\varphi_0=\frac{\pi}{2}}\frac{d\varphi_0}{K_0^3} &= \alpha^2\beta^2\int\limits_{\varphi=0}^{\varphi=\frac{\pi}{2}}\frac{d\varphi}{K^3[1+0.5\psi\gamma^{-1}\beta^2K^{-1}(1-2\nu+\nu\alpha^{-2}K^2)]} \\ 0.5\psi\gamma^{-1}\left(\frac{\pi}{2}\right)\beta(2-\nu-\alpha^{-2}\beta^2)-\beta_0^{-1}\beta+1&=0 \end{aligned}\right\}, \tag{15}$$

where in the second equation

$$\gamma\left(\frac{\pi}{2}\right)=0.5+[0.5^2-0.5\psi\nu\beta(3-\alpha^{-2}\beta^2)]^{0.5}. \tag{16}$$

Relationship (11), which determines the displacements of the physically fixed points of the shell surface, can be found from the first equation of system (13):

$$\alpha_0^2\beta_0^2\int\limits_0^{\varphi_0}\frac{d\varphi_0}{K_0^3}=\alpha^2\beta^2\int\limits_0^{\varphi}\frac{d\varphi}{K^3[1+0.5\psi\gamma^{-1}\beta^2K^{-1}(1-2\nu+\nu\alpha^{-2}K^2)]}. \tag{17}$$

The ratio $\chi=\frac{\beta}{\alpha}$ approaches unity with the increase of ψ, i.e., the surface of the ellipsoid becomes more and more spherical. It is easily proved that the function

$$\psi=\Psi(\lambda)=\Psi\left(\frac{\beta}{\beta_0}\right) \tag{18}$$

has a maximum, which determines the limiting internal pressure. It follows from (15) and (16) that the limiting values of the parameters ψ and λ (ψ_m and λ_m) are respectively equal to

$$\left.\begin{aligned} 0.5\psi_m\beta_0 &= \frac{\lambda_m-1}{\lambda_m}\cdot\frac{(2-\chi_m^2-\nu)-(3-\chi_m^2)\nu(\lambda_m-1)}{(2-\chi_m^2-\nu)^2} \\ \lambda_m &= \left(\frac{2-\chi_m^2}{3-\chi_m^2}\cdot\frac{1+\nu}{\nu}\right)^{0.5} \end{aligned}\right\}, \tag{19}$$

where $\chi_m=\frac{\beta_m}{\alpha_m}$ is the ratio between the semiaxes at the moment when the limiting pressure is reached.

The values of χ_m for ellipsoid surfaces at

$$0.5<\chi_0=\frac{\beta_0}{\alpha_0}<2^{0.5} \tag{20}$$

are usually very near unity, and relationships (19) can be written in the form

$$\left.\begin{aligned} 0.5\psi_m\beta_0 &= \frac{\lambda_m-1}{\lambda_m}\cdot\frac{(1-\nu)-2\nu(\lambda_m-1)}{(1-\nu)^2} \\ \lambda_m &= \left(\frac{1+\nu}{2\nu}\right)^{0.5} \end{aligned}\right\}. \tag{21}$$

At

$$\nu\to 0:\ \lambda_m\to\infty,\quad 0.5\psi_m\beta_0\to 1.$$

In the general case the value of χ_m is determined with the aid of the first relationship of system (15).

FREE VIBRATIONS OF CIRCULAR CONIC SHELLS

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It was established by the experimental and theoretical studies of numerous authors /1, 2, 3/ that to free nonsymmetrical vibrations of shells of zero curvature there generally corresponds a multiwave pattern of deformation. Based on this, it is possible to use the equations of shells with large index of variability in both principal directions as initial equations of the problem of free vibrations of a shell of zero curvature /4/:

$$\frac{1}{Eh}\,\Delta\Delta\varphi - \Delta_k w = 0,$$
$$\Delta_k \varphi + D\Delta\Delta w - \frac{\gamma h \omega^2}{g}\, w = 0, \tag{1}$$

where

$$\Delta = \frac{1}{B}\left[\frac{\partial}{\partial x}\left(B\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial}{\partial \beta}\right)\right],$$
$$\Delta_k = \frac{1}{B}\left[\frac{\partial}{\partial x}\left(\frac{B}{R_2}\frac{\partial}{\partial x}\right)\right], \tag{2}$$

where φ is a stress function; w is a function of the normal displacement; x and β are curvilinear coordinates of the middle surface; B is the Lamé coefficient; ω is the frequency of the natural vibrations; R_2 is the principal radius of curvature; $D = \dfrac{Eh^3}{12(1-\nu^2)}$.

In the case of a circular conic shell (figure)

$$B = x\cdot\sin\varkappa_0, \quad R_2 = x\,\mathrm{tg}\,\varkappa_0. \tag{3}$$

In the case of a closed circular conic shell we shall look for a solution to (1) in the form

$$w_{(x,\beta)} = W_{(x)}\cos m\beta, \quad \varphi_{(x,\beta)} = \Phi_{(x)}\cos m\beta. \tag{4}$$

Equations (1) can then be written in the form

$$\frac{1}{Eh}\left[\frac{d^4\Phi}{d\alpha^4} + \frac{2}{\alpha}\frac{d^3\Phi}{d\alpha^3} - \frac{1}{\alpha^2}\left(1+\frac{2m^2}{\sin^2\varkappa_0}\right)\frac{d^2\Phi}{d\alpha^2} + \frac{1}{\alpha^3}\left(1+\frac{2m^2}{\sin^2\varkappa_0}\right)\frac{d\Phi}{d\alpha} + \right.$$
$$\left. + \left(\frac{m^4}{\sin^4\varkappa_0} - \frac{4m^2}{\sin^2\varkappa_0}\right)\frac{\Phi}{\alpha^4}\right] - \frac{l}{\alpha\,\mathrm{tg}\,\varkappa_0}\frac{d^2W}{d\alpha^2} = 0, \tag{5}$$

$$\frac{l}{\alpha \operatorname{tg} x_0}\frac{d^2\Phi}{d\alpha^2}+D\left[\frac{d^4W}{d\alpha^4}+\frac{2}{\alpha}\frac{d^3W}{d\alpha^3}-\frac{1}{\alpha^2}\left(1+\frac{2m^2}{\sin^2 x_0}\right)\frac{d^2W}{d\alpha^2}+\right. \tag{5 (cont'd)}$$

$$\left.+\frac{1}{\alpha^3}\left(1+\frac{2m^2}{\sin x_0}\right)\frac{dW}{d\alpha}+\left(\frac{m^4}{\sin^4 x_0}-\frac{4m^2}{\sin^2 x_0}\right)\frac{W}{\alpha^4}\right]-\frac{\gamma h l^4\omega^2}{g}W=0,$$

$$\left(\alpha=\frac{x}{l}\right).$$

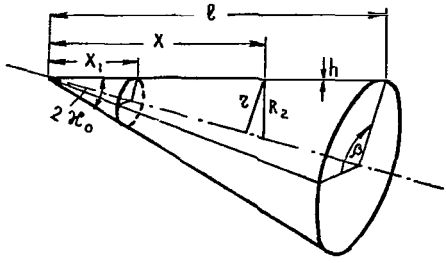

FIGURE.

Equations (5) are solved by the Galerkin method, by approximating the unknown functions $\Phi_{(\alpha)}$ and $W_{(\alpha)}$ by the series

$$\Phi_{(\alpha)}=\sum_{n=1}^{\infty} A_n \sin a_n(\alpha-\alpha_1), \qquad a_n=\frac{n\pi}{\alpha_2-\alpha_1},$$
$$W_{(\alpha)}=\sum_{n=1}^{\infty} C_n \sin a_n(\alpha-\alpha_1). \tag{6}$$

Only the kinematic boundary conditions are satisfied by these representations /5/. The smaller the angle x_0, the more accurately the static conditions will be satisfied. Functions (6) satisfy the static boundary conditions only approximately, under the assumption that the Poisson ratio is equal to zero on the curved contour /3/.

Multiplying equations (5) by α^4, we solve the system obtained by the Galerkin method.

The following equation for the frequencies is obtained then in a first approximation:

$$\omega_{mn}^2=\frac{gE}{\gamma l^2}\frac{h^{*2}\left\{a_n^4\frac{1-\alpha_1^5}{10}+a_n\left(1+\frac{2m^2}{\sin^2 x_0}\right)\left(a_n\frac{1-\alpha_1^3}{6}-\frac{1-\alpha_1}{2a_n}\right)+\right.}{\left\{a_n^4\frac{1-\alpha_1^5}{10}+a_n\left(1+\frac{2m^2}{\sin^2 x_0}\right)\left(a_n\frac{1-\alpha_1^3}{6}-\frac{1-\alpha_1}{2a_n}\right)+\right.}$$

$$\frac{\left.+\left(\frac{m^4}{\sin^4 x_0}-\frac{4m^2}{\sin^2 x_0}\right)\frac{1-\alpha_1}{2}\right\}^2+\frac{a_n^4}{\operatorname{tg}^2 x_0}\left[\frac{1-\alpha_1^4}{8}-\frac{3(1-\alpha_1^2)}{8a_n^2}\right]^2}{\left.+\left(\frac{m^4}{\sin^4 x_0}-\frac{4m^2}{\sin^2 x_0}\right)\frac{1-\alpha_1}{2}\right\}\left[\frac{1-\alpha_1^5}{10}-\frac{1-\alpha_1^3}{2a_n^2}+\frac{3(1-\alpha_1)}{8a_n^4}\right]} \tag{7}$$

$$\left(h^{*2}=\frac{h^2}{12l^2(1-\nu^2)}\right).$$

Using equation (7) it is possible to find the frequencies of natural vibrations of conic shells at any number of half-waves in the longitudinal

direction ($n=1, 2, \cdots$) and a sufficiently large number of half-waves in the transverse direction ($2m=4, 6, 8, \cdots$).

We shall now find the spectrum of frequencies of natural vibrations to which corresponds a multiwave pattern of deformations in the transverse direction, with only one half-wave in the longitudinal direction. These frequencies can be found directly from (7), by taking $n=1$. A different approach to the solution of this problem will however be of interest.

Since our aim is to form the frequency equations at different indices of the function variability in the longitudinal and transverse directions, the first term in differential operator Δ (2) can be neglected. Equations (1) reduce then to

$$\begin{gathered}\frac{1}{Eh}\frac{m^4}{\sin^4\varkappa_0}\frac{\Phi}{\alpha^4}-\frac{l}{\alpha\operatorname{tg}\varkappa_0}\frac{d^2W}{d\alpha^2}=0,\\ \frac{l}{\alpha\operatorname{tg}\varkappa_0}\frac{d^2\Phi}{d\alpha^2}+D\frac{m^4}{\sin^4\varkappa_0}\frac{W}{\alpha^4}-\gamma\frac{hl^4\omega^2}{g}W=0.\end{gathered}\tag{8}$$

Such equations, in the absence of inertial forces, were used by Gol'denveizer /4/ as equations of the nondegenerate edge effect.

By solving equations (8) by the Galerkin method, we obtain the following frequency equation in a first approximation:

$$\omega_{mn}^2=\frac{gE}{\gamma l^2}\frac{h^{*2}\left(\frac{m^4}{\sin^4\varkappa_0}\frac{1-\alpha_1}{2}\right)^2+\frac{a_n^4}{\operatorname{tg}^2\varkappa_0}\left[\frac{1-\alpha_1^4}{8}-\frac{3(1-\alpha_1^2)}{8a_n^2}\right]^2}{\left(\frac{m^4}{\sin^4\varkappa_0}\frac{1-\alpha_1}{2}\right)\left[\frac{1-\alpha_1^5}{10}-\frac{1-\alpha_1^3}{2a_n^2}+\frac{3(1-\alpha_1)}{4a_n^4}\right]}.\tag{9}$$

The lowest frequency is obtained for $a_n=a_1$ and m determined from the condition

$$m_{\min}^2=a_1\sin^2\varkappa_0\sqrt[4]{\frac{\left[\frac{1-\alpha_1^4}{8}-\frac{3(1-\alpha_1^2)}{8a_1^2}\right]^2}{h^{*2}\operatorname{tg}^2\varkappa_0\left(\frac{1-\alpha_1}{2}\right)^2}}.\tag{10}$$

In the second approximation we start from the expressions

$$\begin{gathered}\Phi_{(\alpha)}=A_1\sin a_1(\alpha-\alpha_1)+A_2\sin 2a_1(\alpha-\alpha_1),\\ W_{(\alpha)}=C_1\sin a_1(\alpha-\alpha_1)+C_2\sin 2a_1(\alpha-\alpha_1).\end{gathered}\tag{11}$$

We then obtain the second frequency equation:

$$\begin{gathered}z_1^2(z_5z_6-z_7^2)k^2+z_1[(z_5+z_6)(z_1z_8+4z_4^2)-5z_4z_7(z_2+z_3)+\\ +z_3^2z_5+z_2^2z_6]k+[z_1z_8(z_1z_8+z_2^2+z_3^2+8z_4^2)+z_2z_3(z_2z_3-8z_4^2)+\\ +16z_4^4]=0,\end{gathered}\tag{12}$$

where

$$\begin{gathered}z_1=\frac{m^4}{Eh\sin^4\varkappa_0}\frac{1-\alpha_1}{2},\\ z_2=\frac{la_1^2}{\operatorname{tg}\varkappa_0}\left[\frac{1-\alpha_1^4}{8}-\frac{3(1-\alpha_1^2)}{8a_1^2}\right],\\ z_3=\frac{4la_1^2}{\operatorname{tg}\varkappa_0}\left[\frac{1-\alpha_1^4}{8}-\frac{3(1-\alpha_1^2)}{32a_1^2}\right],\end{gathered}\tag{13}$$

$$z_4 = \frac{l}{\operatorname{tg}\varkappa_0}\left[\frac{160}{27a_1^2} - \frac{4(1+\alpha_1^2)}{3}\right],$$

$$z_5 = -\left[\frac{1-\alpha_1^5}{10} - \frac{1-\alpha_1^3}{2a_1^2} + \frac{3(1-\alpha_1)}{4a_1^4}\right],$$

$$z_6 = -\left[\frac{1-\alpha_1^5}{10} - \frac{1-\alpha_1^3}{8a_1^2} + \frac{3(1-\alpha_1)}{64\,a_1^4}\right], \qquad (13)\ \text{(cont'd)}$$

$$z_7 = \frac{16(1+\alpha_1^3)}{9a_1^2} + \frac{320\,(1+\alpha_1)}{27\,a_1^4},$$

$$z_8 = \frac{Dm^4}{\sin^4\varkappa_0}\,\frac{1-\alpha_1}{2}, \quad k = \frac{\gamma h l^4 \omega^2}{g}.$$

The results obtained were checked experimentally by subjecting a number of conical shells to special tests on an electromagnetic vibrostand. A comparison between the experimentally determined frequencies and the values obtained by formulas (7), (9), and (12) for the same shells is given in the table.

TABLE

m	Experimental results ω^* cycles/sec	By formula (7) ω^* cycles/sec	By formulas (9) and (12) ω^* cycles/sec
Shell No. 1	$\varkappa_0 = 16°$,	$l = 92.5$ cm,	$\alpha_1 = 0.22$, $h = 0.25$ cm
3	372	321	323 (+ 40)
4	450	395	390 (+ 73)
5	567	581	575 (— 56)
6	700	826	819 (—197)
Shell No. 2	$\varkappa_0 = 6°$,	$l = 236$ cm,	$\alpha_1 = 0.62$, $h = 0.25$ cm
3	245	250.5	256
4	276	266.4	262
5	377.5	374	368
6	510	529	521
Shell No. 3	$\varkappa_0 = 15°\ 40$	$l = 92.5$ cm,	$\alpha_1 = 0$, $h = 0.25$ cm
2	315	306	308 (+7)
3	357,5	289	288 (+145)
4	440	466	465 (—61)
5	555	719	720 (—216)
6	700	1034	1034 (—221)
Shell No. 4	$\varkappa_0 = 70°\ 30'$,	$l = 52.5$ cm,	$\alpha_1 = 0$, $h = 0.25$ cm
3	247	251	294
4	203	189	192
5	286	210	207
6	342	275	272

Note: The correction introduced by the second approximation (12) is indicated between parentheses with the corresponding sign.

It is seen from the table that formulas (7) and (9) give almost identical results. This ought to be expected, since the calculations were conducted for $n = 1$. The second approximation (12) gives a correction of the order of 10% to the first one (9). As to the experimental results, the divergence between them and the calculated values lies in most cases between 2 and 16%.

The Galerkin method for determining the free vibrations of a conic shell has been used by other authors as well /5, 6/. They, however, used

different equations as initial equations. Thus, paper /6/ is based on the equations of the semimembrane theory of shells. The papers mentioned contain no information on experimental studies.

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CLINGING OF PANELS UNDER THE ACTION OF RANDOM FORCES

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This paper deals with the vibratory motion of flexible shells under the action of random forces. Problems of this type are essential first of all for the theory of shell stability. It is known that in many cases the shell has several stable equilibrium conditions in the load range between the upper and lower critical values. The transition from one such state to another one is possible under the action of any kind of perturbing factors; this is a case of buckling which cannot be expected on the basis of the classical theory. Hence it is necessary to introduce considerable corrections into the ordinary methods of stability calculations, and it can be of interest in this respect to know the probability characteristics of such a buckling, obtained on the basis of the statistical data on normal perturbations. No less interesting are the cases when the deflection is restricted by suitable devices, and the buckling is not destructive, but the frequency transitions to a perturbed state under the action of a time-dependent load can turn out to be dangerous from the point of view of accumulation of fatigue damages. Such a load frequently displays a purely random pattern (such is the case with acoustic and aerodynamic fatigue). It is necessary in this case to determine the statistical characteristics of multiple buckling (such as the average number of bucklings during a fixed time interval). Such phenomena are usually considered in a linear formulation; the case of small deflections is however far from being the only possible situation.

The theory of Markow processes is used in solving the problem. This approach to the statistical theory of stability of shells was developed in the papers of Vorovich /1/ and the author /2-4/. It is however not the only possible one. Some other possibilities were considered in /5/.

1. Consider a lengthened shallow cylindrical panel of radius R, width b, thickness h, elastic constants E and ν, with edges $\beta = 0$, b (the coordinate β is measured along the cross section arc) hinged to fixed ribs. The panel is subjected to a uniformly distributed lateral pressure $q(t)$.

Restricting ourselves to the case of panels with small curvature parameter

$$k = \frac{b^2}{Rh}, \tag{1.1}$$

we assume /6/ that the deflection is approximated sufficiently accurately by the expression

$$w = h\zeta \sin\frac{\pi\beta}{b}. \tag{1.2}$$

The stochastic equation of motion is obtained by the ordinary methods in the form

$$\ddot{\zeta} = -\beta\rho^{-1}\dot{\zeta} - \frac{2Eh^3}{b^4(1-\nu^2)\rho}\frac{d\Theta^*}{d\zeta}. \tag{1.3}$$

Here ρ is mass per unit area of the middle surface, Θ^* is a dimensionless parameter of the total potential energy

$$\Theta^* = \frac{\pi^4}{32}\zeta^4 - \frac{\pi}{2}k\zeta^3 + \left(\frac{\pi^4}{48} + \frac{2k^2}{\pi^2}\right)\zeta^2 - \frac{2}{\pi}\lambda\zeta,$$
$$\lambda = \frac{q}{E}\left(\frac{b}{h}\right)^4(1-\nu^2). \tag{1.4}$$

The coefficient $\beta\rho^{-1}$ can be expressed through the decrement Δ and the frequency ω of natural linear vibrations about the undeformed state

$$\beta\rho^{-1} = \frac{\omega\Delta}{\pi}. \tag{1.5}$$

It is easily seen that

$$\omega^2 = \frac{Eh^3}{b^4(1-\nu^2)\rho}\left(\frac{\pi^4}{12} + \frac{8k^2}{\pi^2}\right). \tag{1.6}$$

We shall represent the load in the following form:

$$q = Mq + \xi(t), \quad Mq = \text{const}, \tag{1.7}$$

where Mq is the constant part of the load, and ξ its disturbances, representing a centered stationary random process of known spectral density $f(\omega)$. Formula (1.4) can now be written in the form

$$\ddot{\zeta} = -\beta\rho^{-1}\dot{\zeta} - \frac{dV}{d\zeta} + \frac{4}{\pi h\rho}\xi(t),$$
$$V = \frac{2Eh^3}{b^4(1-\nu^2)\rho}\Theta^*(M\lambda). \tag{1.8}$$

Our object is the study of the random process $\zeta(t)$, which can be considered as caused by the passage of process $\xi(t)$ through the inertial nonlinear system, described by equations (1.8). In the general case such a problem is exceedingly complex and has an obvious solution only if certain assumptions are made relative to $\xi(t)$.

Introduce into consideration the correlation time /7/ of the disturbances $\xi(t)$:

$$\tau_k = \frac{\int_0^\infty d\tau \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\omega)\, d\omega}{2\int_0^\infty f(\omega)\, d\omega}. \tag{1.9}$$

If condition $\tau_k \ll T$ is satisfied, where T is the time constant characterizing the period of variation of ζ (for instance the period of natural vibrations $T = 2\pi\omega^{-1}$), then ζ and $\dot{\zeta}$ form a two-dimensional Markow process. Numerous real random interactions have a small correlation time: the acoustic pressure of the jet-engine jet, the pressure of the turbulent atmosphere, etc. This case is of considerable importance, and we shall restrict ourselves to it.

2. At $\lambda_- < M\lambda < \lambda_+$ where λ_- and λ_+ are the lower and upper critical values respectively, the panel has three equilibrium states $\zeta_1, \zeta_2, \zeta_3$ ($\zeta_1 < \zeta_2 < \zeta_3$). States ζ_1 and ζ_3 are stable, and state ζ_2 unstable. Assume that at the initial moment $t = 0$ the panel was in the undisturbed state ζ_1, and that it then started to perform random vibrations under the action of disturbances $\xi(t)$, attaining at some random moment t_* the state ζ_2 and passing into the vicinity of state ζ_3 by overcoming the potential barrier $H = V(\zeta_2) - V(\zeta_1)$. Our problem will be the study of the characteristic of the random magnitude t_*.

This problem is solved by Kolmogorov's equations, satisfied by the probability $p(t, x, x, y, \dot{y})$ of the transition during time t from the state $\zeta = x$, $\dot{\zeta} = \dot{x}$ to the state $\zeta = y$, $\dot{\zeta} = \dot{y}$. The first application of these equations to a problem similar to our own was made in papers /2, 3/; their form in the case considered is

$$\frac{\partial p}{\partial t} = -\dot{y}\frac{\partial p}{\partial y} - \frac{\partial}{\partial \dot{y}}\left[\left(-\beta\rho^{-1}\dot{y} - \frac{dV}{dy}\right)p\right] + a\frac{\partial^2 p}{\partial \dot{y}^2}, \tag{2.1}$$

$$\frac{\partial p}{\partial t} = \dot{x}\frac{\partial p}{\partial x} - \left(\beta\rho^{-1}\dot{x} + \frac{dV}{dx}\right)\frac{\partial p}{\partial \dot{x}} + a\frac{\partial^2 p}{\partial \dot{x}^2}, \tag{2.2}$$

$$a = \frac{16f}{\pi\rho^2 h^2}.$$

Here f is the value of the spectral density of the disturbances $\xi(t)$, corresponding to the frequency ω.

We shall first determine the mean value of $Mt_* = M$. It is a function of the initial values $\zeta = x$, $\dot{\zeta} = \dot{x}$

$$M = \int_0^\infty t\frac{\partial p(x, \dot{x})}{\partial t}\,dt. \tag{2.3}$$

Differentiating (2.2) by time, multiplying by t, and integrating, we easily obtain the equation which $M(x_1 \dot{x})$ satisfies (cf. /2,3, 8/):

$$\dot{x}\frac{\partial M}{\partial x} - \left(\beta\rho^{-1}\dot{x} + \frac{dV}{dx}\right)\frac{\partial M}{\partial \dot{x}} + a\frac{\partial^2 M}{\partial \dot{x}^2} = -1. \tag{2.4}$$

The solution of equation (2.4) represents a very difficult problem; we shall accordingly solve it at this stage under some simplifying assumptions.

It is known /1, 2/ that when we are interested in large-scale information on the variations of ζ, and transient effects of the order of $\tau_н = \rho\beta^{-1}$ can be neglected, $\zeta(t)$ can be approximately treated as a Markow process. The transitional probability $p(t, x, y)$ satisfies in this case the simplified equations

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial y}\left(\frac{dV}{dy}p\right)\tau_н + a\tau_н^2\frac{\partial^2 p}{\partial y^2}, \tag{2.5}$$

$$\frac{\partial p}{\partial t} = -\tau_н \frac{dV}{dx}\frac{\partial p}{\partial x} + a\tau_н^2 \frac{\partial^2 p}{\partial x^2}. \tag{2.6}$$

The equation for $M(x)$ is likewise simplified to

$$-\tau_н \frac{dV}{dx}\frac{\partial M}{dx} + a\tau_н^2 \frac{\partial^2 M}{dx^2} = -1. \tag{2.7}$$

Equation (2.7) diverges substantially from (2.4) only in the region $M \lesssim \tau_н$. The fact that for real Δ $\tau_н$ represents no more than several tens of periods of natural vibrations makes it possible to assert that equation (2.7) is sufficiently accurate for studying the stability of shells.

In addition to (2.7), $M(x)$ must also satisfy the boundary conditions

$$M(-\infty) = M(\zeta_3) = 0. \tag{2.8}$$

The ordinary differential equation (2.7) is easily integrated. The following expression is obtained after the determination of the integration constants from conditions (2.8):

$$M(x) = (a\tau_н^2)^{-1} \int_x^{\zeta_3} d\beta \int_{-\infty}^{\beta} \exp\{(a\tau_н)^{-1}[V(\beta) - V(\alpha)]\, d\alpha\}. \tag{2.9}$$

The integral in (2.9) is calculated in the general case by numerical methods. The situation is more convenient when the disturbances $\xi(t)$ can be considered as weak, i.e., when the condition $a\tau_н \ll H$ is fulfilled. It is possible then to calculate the integral by Laplace's method /9/: the integration interval is contracted up to the vicinity of the extreme points ζ_1, ζ_2, where the function V is represented by three terms of a Taylor series. Simple calculations lead to the result

$$M = 2\pi\tau_н^{-1}\left| -\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_1}\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}\right|^{-\frac{1}{2}} \exp[(a\tau_н)^{-1} H]. \tag{2.10}$$

It was assumed here that $\zeta = \zeta_1$ at $t = 0$.

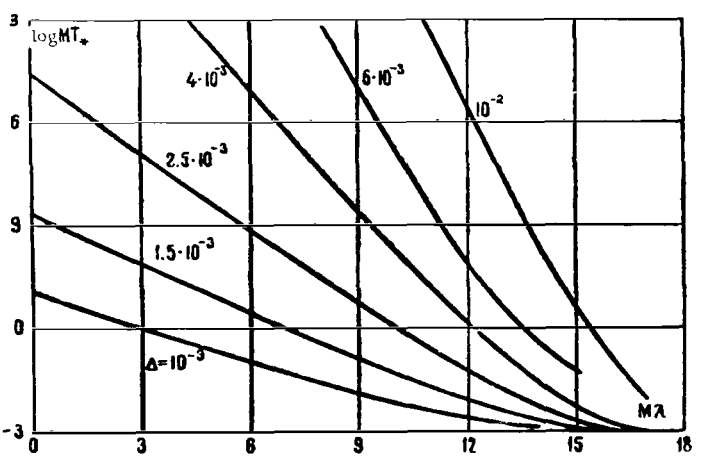

FIGURE 1.

Figure 1 gives the results of the calculations conducted by (2.10) for an aluminum panel at $k = 10$; $hb^{-1} = 0.5\times10^{-2}$; $h = 1.6$ mm, $\lambda_+ = 18, \lambda_- < 0$, situated in the field of acoustic radiation /10/. The assessment of the correlation time of this radiation showed that it does not exceed 6.3×10^{-4} sec, which is considerably less than the period of natural vibrations (7.8×10^{-3} sec). The condition $a\tau_н \ll H$ is likewise fulfilled.

It is seen from Figure 1 that a sharp drop of the clinging time, down to fractions of a second, occurs at a certain value $M\lambda = \lambda_*$. At $M\lambda > \lambda_*$ the panel motion displays the pattern of vibrations about three equilibrium states $\zeta_1, \zeta_2, \zeta_3$. As already noted, the consideration of such cases is of interest not from the point of view of the theory of stability of shells, but for studying the accumulation of fatigue damages. Formulas (2.9) and (2.10) are of interest in this respect too. It is possible to determine from them the number N of bucklings per unit time: $N = M^{-1}$. We stress once more that these formulas are correct only for $N \ll \tau_н^{-1}$; at $N \geqslant \tau_н^{-1}$ they give an overestimated result /4/.

The variation of N as a function of $M\lambda$ is shown in Figure 2. With the increase of $M\lambda$ above the lower critical value the number of bucklings increases, reaching a maximum at a value of $M\lambda$ near the critical load of "equal energies" /6/. The panel then starts to "linger" in the vicinity of the disturbed equilibrium state ζ_3, which leads to the decrease of N. The calculations for this portion of the curve are likewise conducted by (2.10), with the following alterations: instead of H use $H_1 = V(\zeta_2) - V(\zeta_3)$ and instead of $\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_1}$ use $\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_3}$.

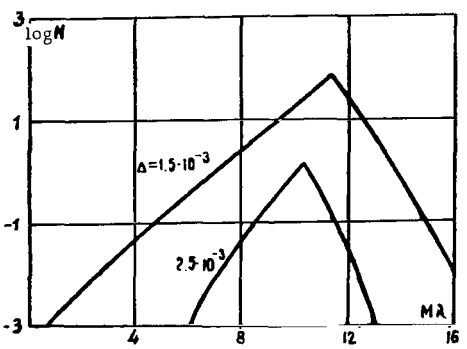

FIGURE 2.

3. The random magnitude t_* is completely described by its distribution function $p(\Delta t)$, which represents the probability that at least one buckling will take place during the time interval Δt. For weak disturbances (at $a\tau_н \ll H$) and small values of Δt the function $P(\Delta t)$ was determined in /3/:

$$P(\Delta t) = \Delta t\, \tau_н\, (2\pi)^{-1} \sqrt{-\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_1}\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}}\, \exp\left[-(a\tau_н)^{-1}H\right], \qquad (3.1)$$

$$P(\Delta t)=\frac{\Delta t}{2\pi}\sqrt{-\frac{\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_1}}{\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}}}\left[\sqrt{(2\tau_H)^{-2}-\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}}-(2\tau_H)^{-1}\right]\times$$

$$\times\exp\left[-(a\tau_H)^{-1}H\right]. \qquad (3.2)$$

Formula (3.1) corresponds to approximation (2.1), (2.2), and formula (3.2) to approximation (2.5), (2.6). We shall give here additional considerations relative to their limits of applicability.

The distribution densities $p(t,\zeta)$ and $p(t,\zeta,\dot\zeta)$ satisfy equations (2.5) and (2.1). According to these equations the initial distributions $p(0,\zeta)=\delta(\zeta-\zeta_1)$ and $p(0,\zeta,\dot\zeta)=\delta(\zeta-\zeta_1)\delta(\dot\zeta)$ pass over into the stationary distributions

$$p_0=C\exp\left[-(a\tau_H)^{-1}V\right],\quad p_0=C\exp\left[-(a\tau_H)^{-1}\left(\frac{\dot\zeta^2}{2}+V\right)\right]. \qquad (3.3)$$

In the process of this transition a part of the systems of the ensemble given by the distribution density pass from the vicinity of ζ_1 to be vicinity of ζ_3. The function $P(\Delta t)$ was determined as follows:

$$P(\Delta t)=\frac{j\Delta t}{n}. \qquad (3.4)$$

Here j is the stream of this diffusion per unit time and n is the number of systems of the ensemble in the vicinity of ζ_1.

Two important assumptions were made in the calculation of j and n: the ensemble density in ζ_3 was assumed to be small compared with the density in ζ_1, and n was calculated by the equilibrium distribution (3.3).

We shall now examine the correctness of these assumptions.

We note first /11/ that at small disturbances the stationary ensemble (3.3) is near microcanonical: at the end of the relaxation time τ all the systems of the ensemble assemble near the absolute minimum $V_{(\zeta)}$. At $M\lambda<\lambda_0$ (λ_0 is the critical load of "equal energies") such a point is ζ_1; at $M\lambda>\lambda_0$, it is ζ_3. The following pattern of the establishment of the equilibrium distributions (3.3) follows from here. At $M\lambda<\lambda_0$ an equilibrium distribution is established in the vicnity of ζ_1 in the course of a very short time $\tau_1\ll Mt_*$. The ensemble density $b\zeta_3$ remains negligibly small, at least for $\Delta t\lesssim Mt_*$. At $M\lambda>\lambda_0$ an equilibrium distribution is established in the vicinity of ζ_1 in the course of time τ_1, after which the ensemble systems jump over the potential barrier during time τ_2. The ensemble density in ζ_3 is negligibly small only for $\Delta t\ll Mt_*$.

The value of τ_2 is found from the obvious relationship

$$Mt_*=\int_0^{\tau_2}tj(t)\,dt\cdot\left\{\int_0^{\tau_2}j(t)\,dt\right\}^{-1}. \qquad (3.5)$$

The stream $j(t)$ decreases with time from the maximum value j_0 found in /2, 4/ to nearly zero. Since the equilibrium state in the vicinity of ζ_1 is established almost instantaneously, the form of the distribution does not vary subsequently, and it is easily shown (cf. /7/, pp. 479-490) that the

stream through ζ_2 decreases by the exponential law

$$j(t) = j_0 e^{-\alpha t}. \tag{3.6}$$

By definition

$$\varepsilon j_0 = j_0 e^{-\alpha \tau}, \quad j(t) = j_0 e^{\ln \varepsilon \frac{t}{\tau}}, \tag{3.7}$$

where ε is the a priori error. The substitution of (3.7) in (3.5) yields, after the corresponding calculations:

$$\tau_2 = M\,|\ln \varepsilon|. \tag{3.8}$$

In conclusion, in the case $M\lambda < \lambda_0$ the expression $j_0 \Delta t\, n^{-1}$ represents the probability of finding the panel at moment Δt in the vicinity of point ζ_3, if $\Delta t \lesssim Mt_*$. If $M\lambda > \lambda_0$, this takes place at $\Delta t \ll Mt_*\,|\ln \varepsilon|$.

The probability indicated coincides with $P(\Delta t)$ (the probability of at least one buckling) if the probability of reverse bucklings can be neglected. In the case $M\lambda > \lambda_0$ this is possible for $\Delta t \ll Mt_*\,|\ln \varepsilon|$, when the ensemble density in ζ_3 is small.

At $M\lambda < \lambda_0$ this fact is already unsufficient, since reverse bucklings are realized much more easily. In this case $P(\Delta t) = j_0 \Delta t\, n^{-1}$, if $\Delta t \ll Mt_*$. These inequalities fix the boundaries of the region of applicability of (3.2).

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ON THE BEHAVIOR OF FABRIC SHELLS UNDER LOAD

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This report deals with some pecularities of the behavior of fabric shells — systems made from two families of intersecting flexible threads. Such a system can be realized structurally from two families of ropes, two families of bars or wire bundles from fabric, and so on.

Since in real structures the flexible threads form as a rule a dense net, the shell material will be considered as compact and resisting only to forces acting along the threads.

The shell will be examined under the following simplifying assumptions:

1. The relative elongations of the material are negligibly small compared with unity.
2. The horizontal displacements of the shell points are so small that they can be neglected when studying the equilibrium of a shell element.
3. The rotations of the elements of the shell during its deformation are so small that the squares of the angles of rotation can be neglected when studying the equilibrium of the elements, and the higher powers of these angles when studying the geometrical relationships.

While the first of these assumptions is fairly general, the two others considerably restrict the class of problems considered. For example, assumption 2 is realized in the case of shells shallow relative to the horizontal plane, if each thread has at least one point fixed in the longitudinal direction, and assumption 3 requires a sufficient rigidity. Similar requirements are made relative to building structures, in which the rigidity is of considerable importance for the normal functioning of the roofing.

The assumptions made are thus applicable to the study of shallow fabric shells, such as are used in building.

A system of differential equations describing the behavior of a tissue shell under a static load was obtained on the strength of these assumptions.

A. GEOMETRICAL EQUATIONS

$$\frac{\varepsilon_1 A_1}{C_1^2} = \frac{\partial v_1}{\partial u_1} + P_1 v_2 - P_1 v_1 \cos\chi + \frac{1}{A_1}\left[\frac{\partial z}{\partial u_1}\cdot\frac{\partial w}{\partial u_1} + \frac{1}{2}\left(\frac{\partial w}{\partial u_1}\right)^2\right]; \qquad (1)$$

$$\frac{\varepsilon_2 A_2}{C_2^2} = \frac{\partial v_2}{\partial u_2} + P_2 v_1 - P_2 v_2 \cos\chi + \frac{1}{A_2}\left[\frac{\partial z}{\partial u_2}\cdot\frac{\partial w}{\partial u_2} + \frac{1}{2}\left(\frac{\partial w}{\partial u_2}\right)^2\right]. \qquad (2)$$

B. STATIC EQUATIONS

$$\frac{\partial H_1}{\partial u_1} + P_1 H_1 \cos\chi - P_2 H_2 + A_1 A_2 q_1 \sin\chi = 0; \tag{3}$$

$$\frac{\partial H_2}{\partial u_2} + P_2 H_2 \cos\chi - P_1 H_1 + A_2 A_1 q_2 \sin\chi = 0; \tag{4}$$

$$\frac{\partial}{\partial u_1}\left[\frac{H_1}{A_1} \cdot \frac{\partial (z+w)}{\partial u_1}\right] + \frac{\partial}{\partial u_2}\left[\frac{H_2}{A_2} \cdot \frac{\partial (z+w)}{\partial u_2}\right] + A_1 A_2 q \sin\chi = 0 \tag{5}$$

C. PHYSICAL EQUATIONS

$$\varepsilon_1 = \frac{H_1 - H_1^0}{C_1 E_1 F_1}; \tag{6}$$

$$\varepsilon_2 = \frac{H_2 - H_2^0}{C_2 E_2 F_2}. \tag{7}$$

The system is written in u_1, u_2 coordinates. The coordinate lines of this system are obtained as projections of the threads of families 1 and 2 on the horizontal plane. A curvilinear oblique-angled system of coordinates is thus obtained in the horizontal plane. The Lamé parameters of this system are A_1 and A_2, and the angle between the positive directions of the coordinate lines is χ.

The shell surface is determined by giving the vertical coordinate z as a functions of u_1 and u_2. C_1 and C_2 are the cosines of the angles made by the threads of families 1 and 2 with the corresponding coordinate lines.

The system of equations contains seven unknown magnitudes:

H_1, H_2 are projections on the plan plane of the forces in the threads of families 1 and 2 respectively per unit variation of the coordinate. Such a system of stresses is convenient since an equal number of threads is situated between two coordinate lines, independent of the value of the second coordinate.

ε_1, ε_2 are relative elongations of the threads of families 1 and 2 respectively.

v_1, v_2 are projections of the displacements of the shell points on the coordinate axes.

w is the vertical component of the displacement.

The loads per unit area of the shell plan are represented by their components q_1, q_2, and q in directions u_1, u_2, and z.

E_1, E_2 are moduli of elasticity of the thread material for families 1 and 2 respectively.

F_1, F_2 are the areas of the threads per unit variation of the coordinates for families 1 and 2 respectively.

H_1^0, H_2^0 are prestresses of the threads for families 1 and 2 respectively in their projection on the plan plane.

The following symbols are also used in these equations:

$$P_1 = \frac{\dfrac{\partial A_1}{\partial u_2} - \dfrac{\partial}{\partial u_1}(A_2 \cos\chi)}{A_2 \sin^2\chi}; \quad P_2 = \frac{\dfrac{\partial A_2}{\partial u_1} - \dfrac{\partial}{\partial u_2}(A_1 \cos\chi)}{A_1 \sin^2\chi}.$$

The solution of the nonlinear system of differential equations (1) to (7) is very difficult, all the more so since the linear system obtained by neglecting the nonlinear terms has no solution at all in certain cases, and in other cases is satisfied at different values of the unknown functions. Some results can however be obtained by considering the linear system.

We shall consider several classes of surfaces.

Let the fabric shell have the form of a transfer surface, defined by the equation

$$z = z_1(u_1) + z_2(u_2); \tag{8}$$

and let the threads be projected on the horizontal plane in straight lines so that A_1 = const, A_2 = const, χ = const. We also assume that $C_1 \approx C_2 \approx 1$, and we restrict ourselves to considering a vertical load. System (1) to (7) with neglected nonlinear terms becomes then of the form

$$\frac{H_1 - H_1^0}{G_1} = \frac{\partial v_1}{\partial u_1} + \frac{z_1'(u_1)}{A_1} \cdot \frac{\partial w}{\partial u_1}; \tag{9}$$

$$\frac{H_2 - H_2^0}{G_2} = \frac{\partial v_2}{\partial u_2} + \frac{z_2'(u_2)}{A_2} \cdot \frac{\partial w}{\partial u_2}; \tag{10}$$

$$\frac{\partial H_1}{\partial u_1} = 0; \quad \frac{\partial H_2}{\partial u_2} = 0; \tag{11}$$

$$\frac{\partial}{\partial u_1}\left[\frac{H_1}{A_1} z_1'(u_1)\right] + \frac{\partial}{\partial u_2}\left[\frac{H_2}{A_2} z_2'(u_2)\right] + A_1 A_2 q \sin\chi = 0. \tag{12}$$

Here ε_1 and ε_2 are eliminated, $G_1 = \frac{E_1 F_1}{A_1}$; $G_2 = \frac{E_2 F_2}{A_2}$.

We shall first examine which is the prestress which the shell shape allows. It is thus necessary to solve equations (11) and (12) for $q = 0$, $H_1 = H_1^0$, $H_2 = H_2^0$.

The solution of the system of equilibrium equations contains the arbitrary constant K^0

$$H_1^0 = K^0 A_1 z_2''(u_2); \quad H_2^0 = -K^0 A_2 z_1''(u_1). \tag{13}$$

It should be noted that not for all shell shapes can a solution of the equilibrium equations containing an arbitrary constant be found. The absence of such a solution means that no stressed state different from zero can exist in the unloaded shell.

Assume now that $q \neq 0$. By substituting in (12) the solution of equations (11)

$$H_1 = F_2(u_2); \quad H_2 = F_1(u_1), \tag{14}$$

we conclude that it cannot be satisfied under an arbitrary load. It is necessary that the load satisfy the condition

$$q = \varphi_1(u_1) z_2''(u_2) + \varphi_2(u_2) z_1'(u_1), \tag{15}$$

where $F_1(u_1)$, $F_2(u_2)$, $\varphi_1(u_1)$, $\varphi_2(u_2)$ are arbitrary functions.

Naturally, in order that (15) be satisfied it is necessary that the edge restraints of the shell allow the possibility of existence of arbitrary strains (14) on the contour. Thus, in the case of a complete restraint of the shell on a rigid contour

$$\begin{aligned} u_1 &= 0; \quad v_1 = 0; \quad w = 0; \\ u_1 &= a_1; \quad v_1 = 0; \quad w = 0; \end{aligned} \tag{16}$$

$$u_2=0;\quad v_2=0;\ w=0; \tag{16}$$
$$u_2=a_2;\quad v_2=0;\ w=0 \qquad \text{(cont'd)}$$

these conditions are fulfilled. It is assumed here and below that the contour represents a quadrangle formed by the coordinate lines.

Consider a different case of edge conditions:

$$\begin{aligned} &u_1=0;\ v_1=0;\quad w=0;\\ &u_1=a_1;\ v_1=0;\quad w=0;\\ &u_2=0;\ v_2=0;\quad w=0;\\ &u_2=a_2;\ H_2=H_2\ (u_1);\ w=0. \end{aligned} \tag{17}$$

In this case H_2 cannot be expressed by an arbitrary function of u_1, but will be equal to $H_2\ (u_1)$ in accordance with the edge conditions. The class of loads taken by the shell without considerable deformations will then be

$$q=\varphi_2\ (u_2)\ z_1'\ (u_1). \tag{18}$$

It was naturally taken into account here that the edge condition $H_2=H_2\,(u_1)$ satisfies (13), since otherwise the shape of the unloaded shell would differ from (8).

If

$$\begin{aligned} &u_1=0;\ v_1=0; &&w=0;\\ &u_1=a_1;\ H_1=H_1\ (u_2) &&w=0;\\ &u_2=0;\ v_2=0; &&w=0;\\ &u_2=a_2;\ H_2=H_2(u_1); &&w=0, \end{aligned} \tag{19}$$

then the last arbitrariness drops out of the load expression, from which it follows that in this case any load will cause considerable displacements and cannot be considered without allowing for the nonlinear terms of the equations.

Consider once more the edge conditions (16), which are not contradicted by (15). When the shell is loaded in accordance with (15), the general solution of equations (11) and (12) contains one arbitrary constant K:

$$H_1=A_1\ [K z_2'\ (u_2)-\varphi_2\ (u_2)\ A_1A_2\sin\chi]; \tag{20}$$

$$H_2=-A_2\ [K z_1'\ (u_1)+\varphi_1\ (u_1)\ A_1A_2\sin\chi], \tag{21}$$

for the determination of which we use equations (9) and (10). By substituting in them (13), (20), and (21) and integrating twice allowing for (16), we obtain two algebraic equations with two unknown constants, one of which is K. By solving this system we obtain

$$K=K^0+\frac{[A_1^2a_1J_2-A_2^2a_2J_1]\ A_1A_2\sin\chi}{A_1^2\,a_1\displaystyle\int_0^{a_2}\frac{z_2''^2\ (u_2)}{G_1}\,du_2+A_2^2\,a_2\int_0^{a_1}\frac{z_1''^2\ (u_1)}{G_2}\,du_1}, \tag{22}$$

where

$$J_2=\int_0^{a_2}\frac{z_2'\,(u_2)\,\varphi_2\,(u_2)}{G_1}\,du_2,\qquad J_1=\int_0^{a_1}\frac{z_1'\ (u_1)\,\varphi_1(u_1)}{G_2}\,du_1.$$

The same result could have been obtained by a more general and clearer method as well. The fact that the forces determined from the equilibrium

equations contain one indeterminate constant indicates that the system considered is statically indeterminate. We shall calculate it by the method of forces. We select as redundant unknown the indeterminate constant K, for which we write the equation

$$\delta_{11} K + \delta_{(1p)} = 0. \tag{23}$$

The coefficients of this equation are easily determined by Mohr's formula, which is in the given case

$$\delta_{(ik)} = \iint \frac{H_{1(i)} H_{1(k)}}{C_1^3 E_1 F_1} A_1 du_1 du_2 + \iint \frac{H_{2(i)} H_{2(k)}}{C_2^3 E_2 F_2} A_2 du_1 du_2 . \tag{24}$$

$H_{1(i)}$; $H_{2(i)}$ are stresses in state (i); $H_{1(k)}$; $H_{2(k)}$ are stresses in state (k). The integration is extended over the entire shell surface.

A great advantage of this method is that it makes it possible to assess the influence of the supporting structures on the shell behavior. It is necessary, therefore, when determining the displacements by Mohr's formula, to extend the integration over the supporting structures. It is thus possible to allow for the boundary conditions, which frequently are difficult even to formulate.

Consider now fabric shells having the shape of a surface of revolution:

$$z = z\ (u_1). \tag{25}$$

The threads of the first family form the meridians, and those of the second family the parallels (Figure 1). Write $A_1 = \text{const}$; $A_2 = bu_1$; $\cos\chi = 0$; $C_1 \approx C_2 \approx 1$. The linear part of system (1)-(7) reduces then to

$$\frac{H_1 - H_1^0}{G_1} = \frac{\partial v_1}{\partial u_1} + \frac{1}{A_1}\left[z'(u_1)\frac{\partial w}{\partial u_1}\right]; \tag{26}$$

$$\frac{H_2 - H_2^0}{G_2} = \frac{\partial v_2}{\partial u_2} + \frac{b v_1}{A_1}; \tag{27}$$

$$\frac{\partial H_1}{\partial u_1} - \frac{b H_2}{A_1} = 0; \quad \frac{\partial H_2}{\partial u_2} = 0; \tag{28}$$

$$\frac{\partial}{\partial u_1}\left[\frac{H_1}{A_1} z'(u_1)\right] + A_1 b u_1 q = 0. \tag{29}$$

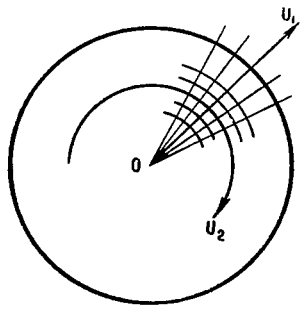

FIGURE 1.

It is assumed that the shell is loaded by a vertical load only; ε_1 and ε_2 are eliminated just as in equations (9)-(12);

$$G_1 = \frac{E_1 F_1}{A_1}; \qquad G_2 = \frac{E_2 F_2}{b u_1}.$$

The solution of the homogeneous equations (28) and (29) contains one arbitrary constant just as in the preceding case:

$$H_1 = K e^{-\int_0^{u_1} \frac{z^{\cdot}(u_1)}{z'(u_1)} du_1}; \tag{30}$$

$$H_2 = -K \frac{A_1}{b} \cdot \frac{z''(u_1)}{z'(u_1)} \cdot e^{-\int_0^{u_1} \frac{z''(u_1) du_1}{z'(u_1)}}. \tag{31}$$

This means that the shell of revolution allows prestress in the form (30) and (31). The solution of (28) being

$$H_1 = F_1(u_1) + F_2(u_2); \quad H_2 = \frac{A_1}{b} F_1'(u_1), \tag{32}$$

where $F_1(u_1)$ and $F_2(u_2)$ are arbitrary functions, we conclude that at boundary conditions allowing (32) the shell can take, without considerable deformations, a load of the form

$$q = \varphi_1(u_1) + \varphi_2(u_2) \frac{z^{\cdot}(u_1)}{u_1}, \tag{33}$$

where $\varphi_1(u_1)$ and $\varphi_2(u_2)$ are likewise arbitrary functions.

Let us calculate a shell fastened on a rigid contour under a load of the form of (33). The problem is statically indeterminate. with one redundant unknown. We select as redundant unknown the indeterminate parameter K. The stresses in the shell under a unit value of the parameter will then be represented by formulas (30) and (31). The stresses in some "primary" system under a load of the form of (33) will be equal to the particular solution of the system of equations (28) and (29):

$$H_1 = -A_1^2 b \left[\frac{\int_0^{u_1} \varphi_1(u_1) u_1 du_1}{z'(u_1)} + \varphi_2(u_2) \right]; \tag{34}$$

$$H_2 = \frac{A_1^3}{z'(u_1)} \left[\frac{z^{\cdot}(u_1) \int_0^{u_1} \varphi_1(u_1) u_1 du_1}{z'(u_1)} - \varphi_1(u_1) u_1 \right]. \tag{35}$$

The value of the parameter K is determined from equation (23), whose coefficients can be calculated by (24). There is no need to determine it in a general form, since the expression obtained is very bulky. It is much simpler to form equation (23) in each specific case.

Consider as an example a surface of revolution with a logarithmic curve as generatrix (Figure 2):

$$z = \frac{h}{\ln \frac{R}{r}} \cdot \ln u. \tag{36}$$

Take: $A_1 = r$; $b = r$. The values of the coordinates will vary then within the

following limits:

$$1 \leqslant u_1 \leqslant \frac{R}{r}\ ; \quad 0 \leqslant u_2 \leqslant 2\pi.$$

Let the shell be rigidly fastened on the outer and inner contours. It is required to determine the stresses appearing under a uniformly distributed load of intensity q (in addition to the initial stress).

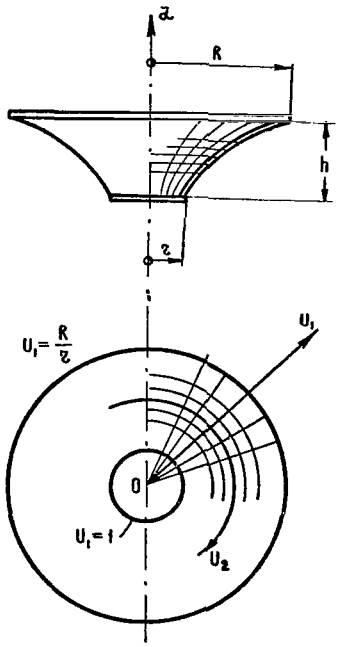

FIGURE 2.

As already noted, this problem is statically indeterminate, and has one redundant unknown. The stresses in the "primary" system under a unit value of the redundant unknown are

$$\bar{H}_1 = u_1; \quad \bar{H}_2 = 1. \tag{37}$$

The stresses caused by the load are

$$H_1 = Bqu_1^3;\ H_2 = 3Bq\,u_1^2\ , \tag{38}$$

where

$$B = \frac{r^3}{2h} \cdot \ln\ \frac{R}{r}\ .$$

Using formula (24), we determine the coefficients of equation (23):

$$\delta_{11} = 2\,\pi r\left[\frac{1}{3G_1}\left(\frac{R^3}{r^3} - 1\right) + \frac{1}{2G_2}\left(\frac{R^2}{r^2} - 1\right)\right]; \tag{39}$$

$$\delta_{(1p)}=2\pi r Bq\left[\frac{1}{5G_1}\left(\frac{R^5}{r^5}-1\right)+\frac{3}{4G_2}\left(\frac{R^4}{r^4}-1\right)\right], \tag{40}$$

after which we obtain

$$k=-\frac{\delta_{(1p)}}{\delta_{(11)}}. \tag{41}$$

The unknown forces are obtained by the formula:

$$H_1^* = Bqu_1^3+ku_1; \tag{42}$$

$$H_2^* = 3Bqu_1^2+k. \tag{43}$$

It is seen from the examples given that the linear formulation makes it possible to determine the stressed state of a shell subjected to the action of special types of load.

When considering an arbitrary load it is necessary to use nonlinear equations, but the methods considered here do not lose their interest. In fact, it is possible to remove from the arbitrary load the part which allows a linear solution of the problem. The corresponding solution represents as a rule an essential part of the solution of the nonlinear equations. It is therefore expedient to determine the linear solution, and then the correction to it; the approximate methods used are much more efficient in determining this correction than in determining the complete solution.

APPROXIMATE CALCULATION OF LINKED THICK CYLINDRICAL SHELLS AND PLATES

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Thick cylindrical shells linked to circular plates have recently been applied to the production of presses and other equipment. An example of such a structure is a hydraulic cylinder with bottom made of prestressed reinforced concrete. It is convenient to divide the approximate calculation of compound structures of this type in two stages: first the thick cylindrical shell and the circular plate are considered separately, and then the stresses and displacements in the two elements when linked are equated.

1. CALCULATION OF THICK CYLINDRICAL SHELLS

In the calculation of the thick cylindrical shell we shall make use of the solution of the axisymmetrical problem of the theory of elasticity for a shallow cylinder. We introduce the dimensionless coordinates $\rho = \frac{r}{r^0}$, $\zeta = \frac{z}{r^0}$, and the functions proportional to the displacements and stresses $u_1 = \frac{E}{(1+\nu)} u_r$, $u_2 = \frac{E}{(1+\nu)} u_z$, $u_3 = \sigma_r$, $u_4 = \tau_{rz}$, $u_5 = \sigma_\varphi$, $u_6 = \sigma_z$. We define also the following symbols: $\beta^n = \frac{\partial^n}{\partial \zeta^n}$, $k = \frac{r_1}{r^0}$, $\gamma = \rho - 1$, $\lambda = k - 1$.

The method of initial functions will be used in the derivation of approximate computing formulas /1, 2/. We take as initial surface the external lateral surface of the shell; all the magnitudes corresponding to this surface will be called initial magnitudes, and will be designated by the superscript ^{0}. The unknown displacements and stresses in the shell will be expressed in this case through four initial functions, i.e., two functions of the displacements and two functions of the stresses on the external surface of the shell. The transition from the initial functions to the unknown ones is accomplished by means of linear differential operators:

$$u_n = \sum_{m=1}^{4} L_{nm} u_m^0 , \qquad (1)$$

where L_{nm} are the differential operators given in Table 1. The operators are expressed through the functions φ_i.

TABLE 1

$2n$-multiple operators ($n = 1 - \nu$)

m / n	L_{nm}	1	2	3	4
1	$\beta\rho\varphi_1$	$-\beta$	0	0	1
	$\beta\rho\varphi_2$	1	β	1	0
	$\beta\varphi_3$	$3-2\nu$	β	1	0
	φ_4	β^2-4n	$-2n\beta$	$-4n$	$-\beta$
2	φ_1	$(1-2\nu)\beta$	$-\beta^2$	$-\beta$	$-4n$
	$\beta\varphi_2$	$-\beta$	$-2n$	0	1
	$\beta\rho\varphi_3$	$-\beta$	0	0	1
	$\beta\rho\varphi_4$	1	β	1	0
3	$\beta\varphi_1$	2β	β^2	β	$1-2\nu$
	$\beta\varphi_2$	$\beta^2-(3-2\nu)$	$-\beta$	$-(3-2\gamma)$	$-\beta$
	$\frac{\beta}{\rho}\varphi_3$	$\rho^2\beta^2-(3-2\nu)$	$-\beta$	-1	$-\rho^2\beta$
	$\frac{1}{\rho}\varphi_4$	$4n-(1+\rho^2)\beta^2$	$\beta(2n-\rho^2\beta^2)$	$4n-\rho^2\beta^2$	β
4	$\beta^2\rho\varphi_1$	$-\beta$	0	0	1
	$\beta\rho\varphi_2$	β	β^2	β	0
	$\beta\varphi_3$	β	β^2	β	$2n$
	$\beta\varphi_4$	$-2n+\beta^2$	0	$-2n$	$-\beta$
5	$\beta\varphi_1$	$-(1-2\nu)\beta$	0	0	$1-2\nu$
	$\beta\varphi_2$	$1-2\nu$	$(1-2\nu)\beta$	$1-2\nu$	0
	$\frac{\beta}{\rho}\varphi_3$	$3-2\nu$	β	1	0
	$\frac{1}{\rho}\varphi_4$	$-4n+\beta^2$	$-2n\beta$	$-4n$	$-\beta$
6	$\beta\varphi_1$	β	$-\beta^2$	$-\beta$	$-2(2-\nu)$
	$\beta\varphi_2$	$-2\nu-\beta^2$	-2β	-2ν	β
	$\beta^2\rho\varphi_3$	$-\beta$	0	0	1
	$\beta^2\rho\varphi_4$	1	β	1	0

$$\varphi_1 = J_0 N_0^1 - N_0 J_0^1; \quad \varphi_3 = J_1 N_0^1 - N_1 J_0^1;$$
$$\varphi_2 = J_0 N_1^1 - N_0 J_1^1; \quad \varphi_4 = J_1 N_1^1 - N_1 J_1^1; \tag{2}$$

where $J_0 = J_0(\rho\beta)$, $J_0^1 = J_0(\beta)$, $J_1 = J_1(\rho\beta)$, $J_1^1 = J_1(\beta)$ are Bessel functions of the zero and first order, $N_0 = N_0(\rho\beta)$, $N_0^1 = N_0(\beta)$, $N_1 = N_1(\rho\beta)$ $N_1^1 = N_1(\beta)$ are Neumann functions of the zero and first order. It should be taken into account that in the given case these functions have a meaning as operational series only and represent a symbolic notation of differential operators of infinitely high order.

Since we are considering a hydraulic cylinder, the calculation of this structure for the case of internal pressure is of greatest interest. The external initial surface is not loaded in this case, and therefore two initial functions out of the four are identically equal to zero:

$$u_3^0 = u_4^0 \equiv 0. \tag{3}$$

Instead of the other two initial functions it is possible to introduce one resolving function and to express through it the unknown displacements and stresses:

$$u_n = (L_{n1}\ L_{42}^k - L_{n2}\ L_{41}^k)\ F. \tag{4}$$

The superscripts k mean that these operators must be taken at $\rho = k$. The resolving function F is determined from the condition that the stress u_3 at the shell surface $\rho = k$ is equal to the internal pressure $q(\zeta)$.

$$(L_{31}^k\ L_{42}^k - L_{32}^k\ L_{41}^k)\ F = -q(\zeta). \tag{5}$$

The shearing stress u_4 is identically zero on both the internal and the external surfaces of the shell.

In order to obtain approximate computing formulas it is necessary to expand the cylindrical functions in the operators into power series, and then to expand the expressions obtained in Taylor series by the powers of ρ. By neglecting then the terms of order γ and λ compared with unity, we obtain the following formulas for the first approximation:

$$\begin{gathered}
u_1 = -\frac{2\lambda\beta^2}{1-\nu}\ F; \quad u_2 = \frac{\lambda\beta}{1-\nu}\ [2\nu + (2\gamma - \lambda)\beta^2]\ F; \\
u_3 = \frac{\gamma\lambda\beta^2}{(1-\nu)^2}[-4\ (1-\nu^2) + 2\nu\ (\gamma - \lambda)\beta^2 + \left(\frac{2}{3}\gamma - \lambda\right)\gamma\beta^4]\ F; \\
u_4 = \frac{2\gamma\lambda}{(1-\nu)^2}\ (\lambda - \gamma)\beta^5 F; \qquad u_6 = \frac{2\lambda\beta^4}{(1-\nu)^2}\ (2\gamma - \lambda) F; \\
u_5 = \frac{2\lambda\beta^2}{(1-\nu)^2}\ [-2\ (1-\nu^2) + (2\gamma - \nu\lambda)\ \beta^2]\ F.
\end{gathered} \tag{6}$$

The function F is determined from the ordinary fourth-order differential equation

$$[4\ (1-\nu^2) + \frac{1}{3}\ \lambda^2\beta^4]\ \lambda^2\beta^2\ F = (1-\nu)^2\ q(\zeta). \tag{7}$$

The substitution of $\beta^2 F$ from (6) into (7) reduces the latter to the differential equation of the moment theory of cylindrical shells.

In a second approximation we neglect the terms of order λ^2 compared with unity, and we obtain the more accurate theory of cylindrical shells. The resolving equation in the second approximation will be of order eight, and the error will be of the order of the square of the relative thickness of the shell compared with unity. In a third approximation the equation will be of order twelve, and the error will be of order λ^3 compared with unity.

The solution of equation (7) contains four arbitrary constants, which make it possible to satisfy two boundary conditions on each of the shell ends. When a thin cylindrical shell is linked to a plate the boundary conditions for the stresses are satisfied with an accuracy of up to the stresses created by the bending moment and the shearing force. Using the formulas of the second and third approximations it is possible to satisfy, with an accuracy sufficient for practical needs, the boundary conditions at the ends of a thick cylindrical shell.

2. APPROXIMATE CALCULATION OF A THICK CIRCULAR PLATE

In the calculation of a thick circular plate we shall make use of the set of homogeneous solutions for a continuous cylinder with an unloaded lateral surface and axisymmetrically loaded ends. The method for obtaining such solutions has been exposed in /2/.

The homogeneous solutions can be obtained in the following form:

$$u_r = \frac{(1+\nu)r^0}{E\beta^2}\,[-2\,(1-\nu)\,J_1 J_1^1 - \psi]\,\Phi; \quad u_z = \frac{(1+\nu)\,r^0}{E\beta^2}\,[-2(1-\nu)J_0 J_1^1 + \varphi]\Phi,$$

$$\sigma_\varphi = \frac{1}{\rho\beta^2}\,[-2\,(1-\nu)\,J_1 J_1^1 - \psi - 2\,\nu\rho\beta J_0 J^1]\,\Phi; \qquad \tau_{rz} = -\frac{1}{\rho}\,\psi\,\Phi; \tag{8}$$

$$\sigma_r = \frac{1}{\rho\beta^2}\,[2\,(1-\nu)\,J_1 J_1^1 + \psi - \rho\beta\,\varphi]\,\Phi; \qquad \sigma_z = \frac{1}{\beta}\,[\varphi - 2\,J_0\,J_1^1]\,\Phi,$$

where

$$\varphi = \beta[J_0 J_0^1 + \rho\,J_1\,J_1^1]; \quad \psi = \beta\,[J_1 J_0^1 - \rho\,J_0\,J_1^1]. \tag{9}$$

Here, as in the preceding case, the Bessel functions represent a symbolic notation of the differential operators.

The function Φ must be determined from the ordinary differential equation of infinitely high order

$$\left[\frac{2\,(1-\nu) - \beta^2}{\beta^2}\,(J_1^1)^2 - (J_0^1)\right]\,\Phi = 0, \tag{10}$$

whose general solution can be written in the form

$$\Phi = \sum_1^\infty C_n\,e^{t_n\zeta}\,. \tag{11}$$

When (11) is substituted in the left-hand part of (10), the symbolic Bessel functions entering in the equation are reduced to ordinary Bessel functions of the argument , determined from the equation

$$t_n^2\,(1+\delta^2) = 2\,(1-\nu); \quad \delta = \frac{J_0(t_n)}{J_1(t_n)}\,. \tag{12}$$

The transitional equation obtained for determining t_n has an infinite number of complex conjugate roots. The first three roots of the equation, at $\nu = 0.15$ for a reinforced concrete plate and at $\nu = 0.30$ for a steel plate, are given in Table 2.

TABLE 2

Roots $p_A\,e^{i\varphi}$ of the homogeneous solutions

n	$\nu=0.15$		$\nu=0.30$	
	p_1	φ	p_1	φ
1	2.982	27.50°	3.044	26.58°
2	6.252	15.19°	6.277	15.12°
3	9.429	11.18°	9.445	11.16°

The arbitrary constants in the solution are determined from the boundary conditions at the plate edges. By using the first group of four roots it is possible to satisfy two boundary conditions on each edge; the second and third groups of four roots make it possible to satisfy the boundary conditions at the edges with an accuracy sufficient for practical purposes.

3. EQUATING THE DISPLACEMENTS AND STRESSES IN THE CASE OF A LINK

The solution for a thick cylindrical shell and a plate used in the calculation contain arbitrary constants, which have to be determined from the conditions of linking of the elements by the ends. In this connection any method leading to a system of algebraic equations for determining the unknown arbitrary constants can be used for calculating the link.

The method of orthogonalization satisfies this requirement. We shall treat the orthogonalization of the displacements and stresses to a system of functions ρ^n. In this case to any function, such as the displacement or the stress, there corresponds a numerical sequence, whose terms represent definite integrals of the form

$$\int_0^1 f(\rho)\rho d\rho, \quad \int_0^1 f(\rho)\,\rho^2 d\rho, \cdots \int_0^1 f(\rho)\,\rho^n\, d\rho. \tag{13}$$

Such a sequence will be called in what follows the sequence of the moments of the function $f(\rho)$ and will be designated by $[f(\rho)]$.

It is easily seen that if all the moments of any two continuous functions in the interval $0 \leqslant \rho \leqslant 1$ are equal, then these functions are identically equal. It can be shown also that the moment of the continuous function $f(\rho)$ decreases with the increase of its order.

Thus, instead of equating the displacements and stresses themselves in the linked elements, it is sufficient to require the term-by-term equality of the moments of these functions. In an approximate calculation one equates not all the moments, but only some of them and in this case the assessment of the calculation error is of great importance. We shall use for that assessment polynomials with unit moments of the form

$$P_k^n = \frac{(n+k+1)!}{(n-k)!\,(k!)^2} \sum_{m=0}^{n} \frac{(m+n+1)!(k+m)!\; x^m\,(-1)^{m+k}}{(m+k+1)!\;(n-m)!\;(m!)^2}. \tag{14}$$

The superscript n represents the degree of the polynomial, and the subscript k its order. The main property of polynomials with unit moments is that the moment of order k of the polynomial P_k^n is equal to unity, and the other moments of this polynomial up to the moment of order n are equal to zero.

Any function F continuous or piecewise continuous and bounded in the interval $0 \leqslant x \leqslant 1$ can be approximately represented as a finite sum

$$F \approx \sum_0^n M_k P_k^n. \tag{15}$$

where

$$M_k = \int_0^1 F x^n \, dx; \qquad k = 0, 1, 2, \cdots n. \tag{16}$$

The approximation can be realized with any required accuracy, the error of the approximation being determined by the number of moments used.

Consider as an example the approximation by polynomials of the function

$$y(x) = \begin{cases} 1 \text{ for } 0 \leqslant x \leqslant \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \leqslant 1. \end{cases} \tag{17}$$

The approximation will be realized with an accuracy of up to four moments

$$y(x) \approx \frac{1}{16} (13 + 60x - 210x^2 + 140x^3). \tag{18}$$

With an accuracy of up to six moments we shall have

$$y(x) \approx \frac{1}{32} (37 - 210x + 1890x^2 - 5880x^3 + 6930x^4 - 2772x^5). \tag{19}$$

Curves of the function $y(x)$ and of its approximation by polynomials are represented in the figure. The number of points in which the approximation coincides with the exact value is equal to the number of moments with which the approximation was conducted. The difference between the function $y(x)$ and its approximations has the shape of a curve intersecting the axis at several points. In the linked elements, if we equate the stresses, the error of the solution will decrease rapidly with the distance from the line of connection, it being known that stresses whose diagrams are bounded by curves intersecting the axis at many points possess a high degree of self-equilibration.

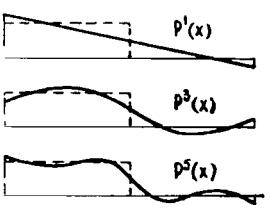

FIGURE. Approximation of the function $y(x)$ by polynomials with unit moments.

Let the calculation of the connection between the thick cylindrical shell and the plate be conducted with an accuracy of up to three moments; in other words, instead of requiring the exact equality of the displacements and stresses at the link, we equate only three moments of each function. Designate the magnitudes corresponding to the cylindrical shell by the superscript c, and those corresponding to the plate by the superscript n. The conditions of monolithic link will then be

$$\begin{gathered} [\sigma_z^c]_3 = [\sigma_z^n]_3; \quad [\tau_{rz}^c]_3 = [\tau_{rz}^n]_3; \\ [u_r^c]_3 = [u_r^n]_3; \quad [(u_z^c)']_3 = [(u_z^n)']_3. \end{gathered} \tag{20}$$

The subscript 3 means that the sequences of moments contain three terms each. The stroke designates derivatives of the vertical displacements by the radius. Equating the moments of the derivatives of the vertical displacements simplifies the calculation, since in this case the shell and plate can have different coordinate axes. In this case only the shapes of the surfaces of the shell and plate will coincide in the link, and not the vertical displacements; this has however no effect on the distribution of the displacements and stresses.

When calculating with an accuracy of up to three moments, third-approximation formulas are formed for the shell, and three homogeneous solutions are used for the plate. The moments of the displacements and stresses will contain unknown arbitrary constants, which enter in the functions Φ and F. By equating term by term the moments in the link, we obtain a system of twelve linear algebraic equations with twenty-four unknown arbitrary constants. The remaining twelve equations can be obtained by equating the moments of the stresses at the ends of the shell and plate to the moments of the external load. Having determined the arbitrary constants, it is possible to calculate the displacements and stresses at an arbitrary point of the shell and plate.

The order of the system of algebraic equations can be lowered four times by using the method of forces. In this case the orthogonalization to a system of functions ρ^n is wholly justified, since the first terms of the sequences of moments of stresses and displacements acquire the clear meaning of the usual concepts of the method of forces: bending moment, shearing force, radial displacement, and angle of rotation.

When calculating a link by the method of forces, each of the elements must be calculated first for the unit moments of the stresses in the link and for the external load. The sequence of moments of the stresses in the link between the shell and plate will be written in the form

$$[\sigma_z] = M_1,\ M_2,\ M_3, \cdots \qquad (21)$$
$$[\tau_{rz}] = T_1,\ T_2,\ T_3, \cdots$$

It is easily seen that the first terms of the sequences are approximately equal to the bending moment and the shearing force.

$$M_1 = (r^0)^3 \int_0^1 \sigma_z \rho^2 d\rho; \quad T_1 = (r^0)^2 \int_0^1 \tau_{rz} \rho \, d\rho. \qquad (22)$$

The resultant of the normal stresses σ_z is identically zero, since the calculation of the link is realized only on a load self-equilibrated in the link surface.

By equating the sequence of moments of the radial displacements and the derivatives of the vertical displacements we obtain a system of algebraic equations relative to the unknown moments of the stresses in the link. In the right-hand parts of the equations the moments of the displacements due to the external load will appear. If the problem is solved in a first approximation, with an accuracy of up to the bending moment and the shearing force, we obtain two equations with two unknowns. These will be the ordinary canonical equations of the method of forces.

Having determined the moments of the stresses in the link, it is easy to calculate the stresses and displacements at any point of the shell and plate, since each of these elements was calculated beforehand for the unit moments of the stresses.

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SOME THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF SELF-OSCILLATIONS OF CURVED PANELS IN A GAS STREAM

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Many papers have been written on the subject of the self-oscillations of cylindrical shells and panels in a supersonic gas stream, both by Soviet (Bolotin, Movchan, Shveiko, etc.) and by non-Soviet (Miles, Hedgepeth, Voss, etc.) authors. In particular, the flutter of a supported shell was studied in detail by Shveiko /1/, who studied the case of relatively long shells (with ratio of length to radius larger than two).

The present paper is devoted to the peculiarities of the calculation of short shells and shells reinforced by elastic stiffeners, and to the first experimental results of the study of self-oscillations of cylindrical panels in a supersonic pipe obtained by Shandarov.

It is difficult to obtain an exact solution of the equations for a reinforced shell. We have accordingly used variational methods for solving this problem. The potential and kinetic energies of the shell and stiffeners can be written in the following form as a function of the displacements of the middle surface of the shell:

$$\Pi = \Pi\ (u,\ v,\ w),$$

$$T = T\ (\dot{w}).$$

The shell is considered as shallow, and the inertia in the direction lying in the shell surface is neglected. The stiffeners represent frames possessing tensile rigidity, torsional rigidity, and flexural rigidity in their plane. The displacements of the middle surface will be developed in series whose terms satisfy the support conditions

$$u = \sum_{j=1}^{N} U_j(t) \cos\frac{j\pi x}{l} \cos\frac{ky}{R}\ ;\quad v = \sum_{j=1}^{N} V_j(t) \sin\frac{j\pi x}{l} \sin\frac{ky}{R}\ ;$$

$$w = \sum_{j=1}^{N} W_j(t) \sin\frac{j\pi x}{l} \cos\frac{ky}{R}$$ (l, R are the length and radius of the shell).

We impose the restriction $k \geqslant 2$ on the number k of waves in the circumferential direction, i.e., we do not consider motions such as general bending of the shell ($k=1$) and axisymmetrical motions ($k=0$) connected with displacements of the ends as a whole. The coefficients of the series can be considered as generalized coordinates, and we can form the

equations of motion — Lagrange equations of the second kind:

$$\frac{\partial \Pi}{\partial U_j} = 0; \quad \frac{\partial \Pi}{\partial V_j} = 0; \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{W}_j} + \frac{\partial \Pi}{\partial W_j} = Q_j.$$

The generalized forces have the excess aerodynamic pressure

$$\Delta P = A_1 \frac{\partial w}{\partial x} + A_2 \frac{\partial w}{\partial t};$$

$$Q_j = -\iint_s \Delta p \sin\frac{j\pi x}{l} \cos\frac{ky}{R} dx\, dy.$$

Depending on the form of the coefficients, this expression corresponds to the two-dimensional stationary theory or to the linear variant of the piston theory. The solution of the equations of motion will be looked for in the form of harmonic oscillations, assuming the frequency of oscillations to be complex.

By introducing dimensionless parameters and performing several calculations we obtain the following algebraic system of homogeneous linear equations:

$$L_j\ (V_j,\ W_n,\ \alpha,\ \delta,\ \Omega,\ EI^*, \cdots) = 0; \qquad (j=1,\ 2 \cdots N)$$
$$M_j\ (V_n,\ W_n \cdots) = 0.$$

In this system the parameter α characterizes the pressure and velocity of the supersonic stream, δ is the damping, Ω is the frequency of oscillations, EI^* is the rigidity of the stiffener at bending in its plane. Note that the system of equations has exactly the same form for any number of stiffeners (equal and equidistant from each other and from the ends): the only difference will be in the numerical values of the coefficients.

The next step is to determine the critical parameter α above which exist values of Ω corresponding to unstable motion ($\mathrm{Im}\Omega < 0$). We have examined both a nonreinforced shell and a shell reinforced by one frame.

In the case of an unstiffened shell a coarse solution can be obtained by representing the displacements in the form of two terms (the number of waves in the circumferential direction and of half-waves in the longitudinal direction is arbitrary in each term).

The condition for small influence of the damping δ is, for steel shells:

$$\Omega_j^2 - \Omega_{j-1}^2 > 0.73 \cdot 10^{-5} \frac{R}{h}.$$

Therefore, the damping need be taken into account only in the case of very thin shells subjected to longitudinal compression. Without allowing for the damping it is possible to find the values of the minima α_{kp} by sorting out the possible values of the number of waves in the circumferential direction and the number of half-waves in the longitudinal direction. There are two such minima, the first corresponding to short waves in the circumferential direction and long waves in the longitudinal direction, and the second to long waves in the circumferential direction and short waves in the longitudinal direction. The first is smaller for long shells, and the second for short shells.

It was established by Shveiko that in the case of long shells two terms are sufficient for an approximate solution, and four give a practically exact result. In the case of short shells, however, in which two forms with frequencies lying in the region of very dense frequency spectrum are allowed for in the two-term approximation, the influence of the neglected forms can be very considerable. The basic characteristic of the calculation of short shells is thus the necessity to take into account a larger number of terms, which leads to a higher-order determinant:

$$|a_{jn}| = 0$$

$$a_{jn} = \begin{cases} \Omega_j^2 - \Omega^2 & \text{for } n = j, \\ [1 - (-1)^{j+n}] \dfrac{jn}{j^2 - n^2} \alpha & \text{for } n \neq j. \end{cases}$$

The calculation of such a determinant was programmed and carried out on the digital computer of the Computer Center of the Siberian Division of AN SSSR. The calculation was conducted by two different methods. The first consisted in the direct determination of the frequencies Ω at given α. By sorting out the α it is possible to obtain an idea on the variation of the frequencies in the stream and to find α_{kp}. In view of the length of the calculations involved, only a few examples were calculated by this method.

The second method consists in determining α_{kp} and the corresponding value Ω_{kp} by successive approximations. We substitute in the determinant for all the terms except those corresponding to the two-term approximation, certain $\alpha^{(0)}$ and $\Omega^{(0)}$, and we find $\alpha^{(1)}$ and $\Omega^{(1)}$. The process is repeated until the assumed and the obtained values coincide with the accuracy required. No rigorous proof of the convergence of the method exists. Out of several hundreds of cases calculated on the computer, there were some ten cases in which the process did not converge. The method efficiency obviously depends on the selection of $\alpha^{(0)}$ and $\Omega^{(0)}$. By combining both methods it is possible to solve the probem in full.

The results obtained by the two methods coincide satisfactorily.

The results of the calculations, represented in the form of curves and surfaces passing through the calculated α_{kp}, corroborate that the second minimum of α_{kp} will be the smaller for a short shell. In the case of a shell with radius/thickness ratio 500, the first minimum becomes smaller for lengths larger than 1.3 radii.

In order to describe satisfactorily the form of oscillations of a shell stiffened by a frame it is absolutely necessary to take a large number of terms in the series expansions of the displacements. It is unfortunately difficult to use here an efficient method of successive approximations, since it is not clear which forms and frequencies are to be considered as fundamental.

The calculations by the direct method of the frequencies in the stream of a shell with unponderable frame having a large tensile rigidity and zero torsional rigidity show the following.

1. Certain forms of instability have upper and lower values of α_{kp}; the instability is replaced again by stability.

2. The increase in the flexural rigidity of the frame is not always accompanied by an increase in the critical speed.

3. It is necessary to conduct the calculation in a wide range of k, since it is very difficult to find the values of k characteristic for the minimum a_{kv} at arbitrary frame rigidites.

4. The damping must be taken into account at frame rigidities corresponding to close frequencies.

The experiments were conducted with a model representing a cylinder quadrant with sharpened inside edges and milled cavity. Above the cavity was fastened a panel with radius/thickness ratio of 2250. The back edge was connected to a pusher and an air cylinder by means of which extension and compression of the panel were realized. The extension and compression were controlled from the control board. The model was calibrated on a special bench under vibration conditions close to the vibrations of a pipe. The calculations were subsequently converted to pipe conditions.

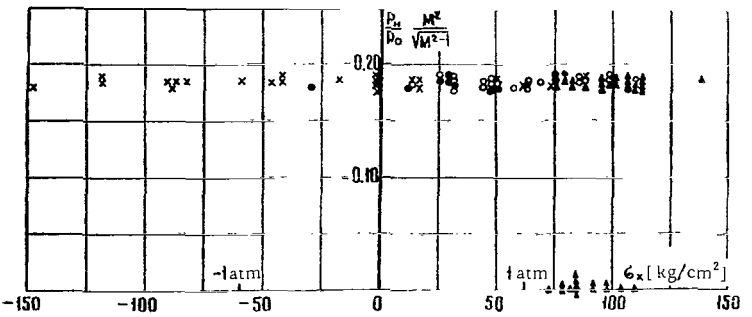

FIGURE 1.

After the model is installed in the pipe, the panel is extended by stresses ≈ 300 kg/cm^2 for the passage of the starting jump. The panel suffers intense vibrations until supersonic conditions are reached; after the passage of the starting jump the vibrations are interrupted. The following characteristic stages are observed with the variation of the longitudinal forces in the panel: 1) negligible local vibrations; 2) vibrations of the type of running waves, embracing the larger part of the panel. The appearance of these vibrations is taken as beginning of the self-oscillations; 3) intense oscillations of the entire panel, accompanied by deep nonlinear deformations; 4) static buckling.

The test results for batches of 23 panels each show that the beginning of each of these stages has a probabilistic character, apparently linked with the quality of the specimens. The comparison of the data of the static buckling with the results of the tests of 15 panels for static buckling on a vibrostand corroborates the conclusions of Bolotin and Shveiko /2/ on the stabilizing influence of the supersonic stream.

When comparing the results of the experiments and the calculations conducted with 16 terms of the series, one should specify that the conditions of support of the calculations do not correspond to the experimental conditions of support (building in along all edges).

A film was shot of the experiments; it contains sequences for several panels, and the mean values of the load are given in the captions.

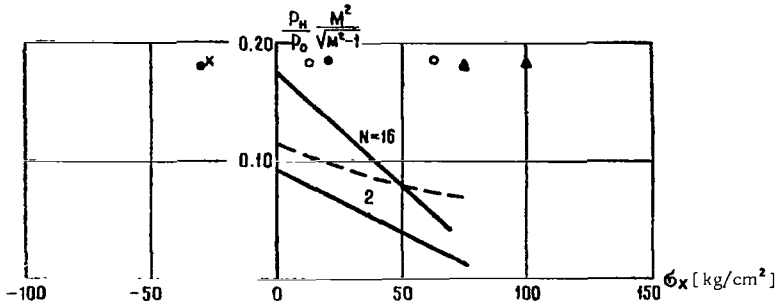

FIGURE 2.

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ON THE LARGE DEFLECTIONS AND STABILITY OF REINFORCED CONCRETE SHALLOW FLEXIBLE SHELLS ALLOWING FOR THE CREEP OF CONCRETE

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(Yerevan)

§ 1.

We shall consider a body made of reinforced concrete bars, plates, and shells as a sandwich (figure), made of an isotropic concrete layer and two isotropic symmetrical reinforcing faces fastened reliably to each other (without possibility of sliding along the surfaces of contact).

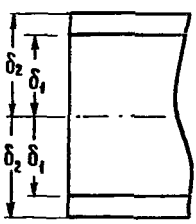

FIGURE.

Let: a) the faces behave elastically, and the concrete layer have a linear creep, with equal Poisson ratios of the elastic and inelastic parts of the deformation and equal modulus of elasticity and measure of creep of the concrete at extension and compression;

b) the generally used assumptions of the theory of calculation of flexible plates and shells /1/, based on the commensurability of their deflections with the thickness, be correct.

The connection between the stress $\sigma(t)$ and the unit elongation $\varepsilon(t)$ at axial extension-compression of the concrete, allowing for the creep, is:

$$\sigma(t) = \bar{E}\varepsilon(t) = E_b\left[\varepsilon(t) - \lambda\int_{\tau_1}^{t} K(t,\tau)\varepsilon(\tau)\,d\tau\right] = E_b(1-\lambda K^*)\varepsilon. \qquad (1.1)$$

The resolving equations of the corresponding elastic shells preserve their form when the creep of concrete is allowed for; it is only necessary to replace in them the modulus of elasticity of the concrete E by the elastic operator $\bar{E}$ /2, 3, 4/.

§ 2.

The fundamental equations of shallow flexible reinforced concrete shells allowing for the creep of concrete, on the strength of § 1 and for*

$$\mu_a = \mu_b = \mu \tag{2.1}$$

become

$$\begin{gathered}\bar{D}\nabla^2\nabla^2 w = L(w, \Phi) + k_x \frac{\partial^2\Phi}{\partial y^2} + k_y \frac{\partial^2\Phi}{\partial x^2} + q, \\ \bar{C}^{-1}(1-\mu^2)^{-1}\nabla^2\nabla^2\Phi = -\frac{1}{2}L(w, w) - k_x\frac{\partial^2 w}{\partial y^2} - k_y\frac{\partial^2 w}{\partial x^2},\end{gathered} \tag{2.2}$$

where

$$\bar{C} = \frac{2}{1-\mu^2}[\delta_1\bar{E} + (\delta_2 - \delta_1)E_a]; \bar{D} = \frac{2}{3(1-\mu^2)}[\delta_1^3\bar{E} + (\delta_2^3 - \delta_1^3)E_a] \qquad (\bigstar)$$

E_a, μ_a are the modulus of elasticity and the Poisson ratio of the faces; μ_b is the Poisson ratio of the concrete layer. The remaining symbols are explained in the figure and in /1/. Equations (2) correspond to equations (5.27) and (5.28) of /1/.

The bending and twisting moments and shearing forces are expressed through w by relationships obtained from the well-known formulas of the theory of elastic shells (cf. for instance /1/, p. 234) with D replaced by $\bar{D}$ in accordance with ($\bigstar$).

We similarly obtain for reinforced-concrete shallow flexible axisymmetrical shells

$$\begin{gathered}\rho\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial\psi}{\partial\rho} - \frac{\psi}{\rho} = \vartheta\left(\theta + \frac{\vartheta}{2}\right) + \frac{\bar{C}^{-1}}{1-\mu^2}\left[\frac{d}{dr}(r^2 q_t) + \mu r q_t\right]; \\ \rho\frac{\partial^2\vartheta}{\partial\rho^2} + \frac{\partial\theta}{\partial\rho} - \frac{\vartheta}{\rho} = -r_1^2(1-\mu^2)\bar{D}^{-1}[(\theta+\vartheta)\bar{C}\psi] - r_1\bar{D}^{-1}F,\end{gathered} \tag{2.3}$$

where

$$\psi = -\rho(1-\mu^2)^{-1}\bar{C}^{-1}T_1; \quad F = r[T_1(\theta+\vartheta) + Q];$$

C, D are operators ($\bigstar$). The remaining designations are the same as in /5/. Equations (3) correspond to equations (19) on p. 263 of /5/. In the case of a shallow spherical shell subjected to a load normal to the middle surface, equations (3) reduce to

$$\begin{gathered}\rho\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial\psi}{\partial\rho} - \frac{\psi}{\rho} = \vartheta\left(\theta_1\rho + \frac{\vartheta}{2}\right), \\ \bar{D}\left(\rho\frac{\partial^2\vartheta}{\partial\rho^2} + \frac{\partial\vartheta}{\partial\rho} - \frac{\vartheta}{\rho}\right) = -r_1^2(1-\mu^2)(\theta_1\rho + \vartheta)\bar{C}\psi + \frac{q r_1 r^2}{2},\end{gathered} \tag{2.4}$$

corresponding to equations (59) on p. 282 of /5/.

* At $\mu_a \neq \mu_b$ all the fundamental equations for reinforced concrete shells given below become somewhat different; no basic difficulties appear, however.

§ 3.

Following /5/, we assume*

$$\vartheta = (\rho^3 - \rho)\, y(t). \tag{3.1}$$

By performing calculations similar to those performed in /5/ we reduce system (2.4), in the case of a shell simply supported along the contour, to the following operatorial relationship:

$$(a_1 + a_2 y_1)(1 - K^*)\, y_1 + (a_4 + a_3 y_1)(1 - K^*)\, y_1^2 + \\ + a_5 y_1 + a_6 y_1^2 + a_7 y_1^3 + q_0 = 0, \tag{3.2}$$

where q_0 is a load parameter and $a_s\,(s = 1, \cdots, 7)$ are constants. In the case corresponding to a one-layer (homogeneous) concrete shell we shall have instead of (2):

$$(a_1 + a_2 y_1)(1 - K^*) y_1 + (a_4 + a_3 y_1)(1 - K^*) y_1^2 + q_0 = 0, \tag{3.3}$$

where $a_s\,(s = 1, \cdots, 4)$ are constants (somewhat different from those in (2)).

Relationship (3) corresponds to the cubic equation (63) on p. 284 of /5/.

The problems of determining the deflections of double-curvature panels rectangular in the plan and of lengthened cylindrical panels at uniformly distributed lateral load are likewise reduced to operatorial relationships, differing from (2) and (3) by the values of the constants only, on the basis of the first approximations of the solutions of the corresponding elastic problems (treated in /1/, chapter VI). When using the second approximations the problems reduce to systems of two operatorial relationships, similar to (2) and (3) and containing each two unknown functions of time, etc.

§ 4.

Following Maslov-Arutyunyan's linear theory of creep of concrete /7, 8/, we represent the kernel of integral equation (1.1) in the form**

$$K(t,\tau) = \gamma\varphi(\tau) E - D(\tau)\, e^{\eta(\tau)} \int_\tau^t e^{-\eta(z)}\, dz, \tag{4.1}$$

where

$$\eta(\tau) = \gamma \int_{\tau_1}^{\tau} [1 + \varphi(z) E]\, dz, \tag{4.2}$$

$$D(\tau) = \eta''(\tau) + [\eta'(\tau)]^2 - \gamma\eta'(\tau), \tag{4.3}$$

$$\varphi(\tau) = C_0 + \frac{A_1}{\tau}, \tag{4.4}$$

where γ, C_0, A_1 are empirically determined parameters.

Taking (1) through (4) into account, (3.3)† can be reduced to the

* Paper /5/ dealt with the elastic problem; here, in view of the presence of creep, the unknown constant C has been replaced (following /6/) by an unknown function of time $y_1(t)$.

** We have assumed for simplicity's sake only that E_b = const. The same method of derivation can be used in the case $E_b = E_b(t)$ as well.

† If we start from (3.2), the formulas obtained will be more cumbersome. No other complications will appear, however.

following system of differential equations:

$$\frac{dy_1}{dt} = y_2; \qquad \frac{dy_2}{dt} = Y(t, y_1, y_2). \tag{4.5}$$

Let

$$(1 - K^*)y_1^2 = y_3. \tag{4.6}$$

By assuming for t fixed values which are successively increasing and close to each other $\tau_1 < t_1 < t_2 < \cdots < t_n < \cdots$, we obtain from (3.3) and (4.6), taking (4.1) through (4.4) into account, the following approximate formulas (on the basis of Bogolyubov-Krylov's method /9/):

at $t = \tau_1$

$$y_{30} = y_{10}^2.$$

at $t = t_1$

$$y_3(t_1) = y_1^2(t_1) - C(t_1\tau_1)\, y_{30} - [y_3(t_1) - y_{30}]\, C\left(t_1, \frac{\tau_1 + t_1}{2}\right);$$

at $t = t_2$

$$y_3(t_2) = y_1^2(t_2) - y_{30}C(t_2,\tau_1) - [y_3(t_1) - y_{30}]\, C\left(t_2, \frac{\tau_1 + t_1}{2}\right) -$$

$$- [y_3(t_2) - y_3 t_1]\, C\left(t_2, \frac{\tau_1 + t_2}{2}\right)$$

. . . . . .

and the following algebraic equations:

at $t = \tau_1$

$$q_0 + a_1 y_{10} + (a_2 + a_4) y_{10}^2 + a_3 y_{10}^3 = 0; \tag{4.7}$$

at $t = t_1$

$$q_0 + [a_1 + a_2 y_1(t_1)]\left[1 + C\left(t_1, \frac{\tau_1 + t_1}{2}\right)\right]^{-1}\left\{\left[C\left(t_1, \frac{\tau_1 + t_1}{2}\right) - C(t_1,\tau_1)\right] y_{10} + y_1(t_1)\right\} + [a_4 + a_3 y_1(t_1)]\left[1 + C\left(t_1, \frac{\tau_1 + t_1}{2}\right)\right]^{-1}\left\{y_1^2(t_1) + \left[C\left(t_1, \frac{\tau_1 + t_1}{2}\right) - C(t_1,\tau_1)\right] y_{10}^2\right\} = 0; \tag{4.8}$$

at $t = t_2$

$$q_0 + [a_1 + a_2 y_1(t_2)]\left[1 + C\left(t_2, \frac{t_1 + t_2}{2}\right)\right]^{-1}\left\{\left(C\left(t_2, \frac{\tau_1 + t_1}{2}\right) - C(t_2,\tau_1)\right] y_{10} + \left[C\left(t_2, \frac{t_1 + t_2}{2}\right) - C\left(t_2, \frac{\tau_1 + t_1}{2}\right)\right] \times \left[1 + C\left(t_1, \frac{\tau_1 + t_1}{2}\right)\right]^{-1}\left\{y_1(t_1) + \left[C\left(t_1, \frac{\tau_1 + t_1}{2}\right) - C(t_1,\tau_1)\right] y_{10}\right\} + y_1(t_2)\right\} + [a_4 + a_3 y_1(t_2)]\left[1 + C\left(t_2, \frac{t_1 + t_2}{2}\right)\right]^{-1} \times \left\{\left[C\left(t_2, \frac{\tau_1 + t_1}{2}\right) - C(t_2,\tau_1)\right] y_{10}^2 + \left[C\left(t_2, \frac{t_1 + t_2}{2}\right) - C\left(t_2, \frac{\tau_1 + t_1}{2}\right)\right]\left[1 + C\left(t_1, \frac{\tau_1 + t_1}{2}\right)\right]^{-1}\left\{y_1^2(t_1) + \left[C\left(t_1, \frac{\tau_1 + t_1}{2}\right) - C(t_1, \tau_1)\right] y_{10}^2\right\} + y_1^2(t_2)\right\} = 0 \tag{4.9}$$

. . . . . .

where

$$C(t,\tau)=\left(C_0+\frac{A_1}{\tau}\right)[1-e^{-\gamma(t-\tau)}]E.$$

Once system (5) or (7) through (9) is solved, the determination of the design magnitudes of the shell presents no difficulties.

§ 5.

The creep of concrete leads to the increase of the displacements of reinforced-concrete structures with time under a constant load. Depending on the value of the load, this process can, in thin-walled structures after a certain time, lead to the loss of stability (buckling) of the original equilibrium form.

We shall show that, on the basis of Lyapunov's theory of stability of motion /10/, it is possible without solving system (4.5) to determine the maximum of all values $q_0=q_{01}$, for which the corresponding particular solution of system (4.5) will be stable at any $t\geqslant\tau_1$, i.e., to determine the critical value of the load parameter under conditions of creep. By taking in the final results values of the parameters caused by creep of the material equal to zero, we can obtain the critical value $q_0=q_{0*}$ of the load parameter in the case of elastic behavior of the material.

Let

$$(1-K^*)\,y_1^2=y_3, \tag{5.1}$$

then (3.3) reduces to

$$(1-K^*)\,y_1=F, \tag{5.2}$$

where

$$F=-\frac{y_3(a_4+a_2y_1)+q_0}{a_1+a_2y_1}. \tag{5.3}$$

On the strength of (4.1) through (4.4), (2) can be rewritten in the form

$$y_2+\gamma y_2=F''+\eta'F', \tag{5.4}$$

where

$$y_2=y_1, \tag{5.5}$$

$$\eta'=\gamma(1+\varphi E), \tag{5.6}$$

$$F'=\frac{a_2y_2}{(a_1+a_2y_1)^2}[(a_4+a_3y_1)\,y_3+q_0]-\frac{1}{a_1+a_2y_1}[a_3y_2y_3+(a_4+a_3y_1)\,y_3'], \tag{5.7}$$

$$\begin{aligned}F''=&\frac{a_2y_2'(a_1+a_2y_1)-2a_2^2y_2^2}{(a_1+a_2y_1)^3}[(a_4+a_2y_1)\,y_3+q_0]+\\&+\frac{2a_2y_2}{(a_1+a_2y_1)^2}[a_3y_2y_3+(a_4+a_3y_1)\,y_3']-\\&-\frac{1}{a_1+a_2y_1}[a_3y_2'y_3+2a_3y_2y_3'+(a_4+a_3y_1)\,y_3'']\end{aligned} \tag{5.8}$$

We obtain from (1)

$$y_3' = 2y_1y_2 - K(t,\ t)\, y_1^2(t) - \int_{\tau_1}^{t} \frac{\partial}{\partial t} K(t,\ \tau)\, y_1^2(\tau)\, d\tau; \tag{5.9}$$

$$y_3'' = 2(y_2^2 + y_1y_2') - \frac{dK(t,\ t)}{dt}\, y_1^2 - 2K(t,\ t)\, y_1y_2 -$$

$$- y_1^2 \frac{\partial K(t,\tau)}{\partial t}\Big|_{\tau=t} - \int_{\tau_1}^{t} \frac{\partial^2}{\partial t^2} K(t,\ \tau)\, y_1^2(\tau)\, d\tau, \tag{5.10}$$

where

$$K(t,\ t) = \gamma\left(C_0 + \frac{A_1}{t}\right) E, \tag{5.11}$$

$$\frac{dK(t,\ t)}{dt} = -\frac{\gamma A_1 E}{t^2}, \tag{5.12}$$

$$\frac{\partial K(t,\ \tau)}{\partial t}\Big|_{\tau=t} = -D(\tau)\, e^{\eta(\tau)-\eta(t)}\Big|_{\tau=t} = -D(t), \tag{5.13}$$

$$\frac{\partial^2 K(t,\ \tau)}{\partial t^2} = D(\tau)\, e^{\eta(\tau)-\eta(t)}\, \eta'(t). \tag{5.14}$$

Rewrite (8) and (10) in the form

$$F'' = B(t, y_1)\, y_2' + C(t,\ y_1,\ y_2). \tag{5.15}$$

$$y_3'' = 2y_1y_2' + A(t,\ y_1\ y_2). \tag{5.16}$$

where

$$A(t,\ y_1,\ y_2) = 2[y_2^2 - K(t,\ t)\, y_1y_2] + \left[\frac{\gamma A_1 E}{t^2} + D(t)\right] y_1^2 -$$

$$- \int_{\tau_1}^{t} \frac{\partial^2}{\partial t^2} K(t,\ \tau)\, y_1^2(\tau)\, d\tau. \tag{5.17}$$

$$B(t,\ y_1) = \frac{a_1}{(a_1 + a_2y_1)^2}[(a_4 + a_3y_1)\, y_3 + q_0] - \frac{1}{a_1 + a_2y_1}[a_3y_3 + 2a_1(a_4 + a_3y_1)]. \tag{5.18}$$

$$C(t,\ y_1,\ y_2) = -\frac{2a_2^2 y_2^2}{(a_1 + a_2y_1)^3}[(a_4 + a_3y_1)\, y_2 + q_0] +$$

$$+ \frac{2a_2y_2}{(a_1 + a_2y_1)^2}[a_3y_2y_3 + (a_4 + a_3y_1)\, y_3'] -$$

$$- \frac{1}{a_1 + a_2y_1}[2a_3y_2y_3' + (a_4 + a_3y_1)\, A(t,\ y_1,\ y_2)]. \tag{5.19}$$

Rewrite (4) in the form

$$\frac{dy_2}{dt} = \frac{M_1(t,\ y_1,\ y_2)}{M(t,\ y_1)}, \tag{5.20}$$

where

$$M = 1 - B(t,\ y_1), \tag{5.21}$$

$$M_1 = \eta' F' + C(t,\ y_1,\ y_2) - \gamma y_2. \tag{5.22}$$

To the given fixed value of the load parameter

$$q_0 \leqslant q_{01} \tag{5.23}$$

let there correspond the particular solution of system (5) and (20)

$$y_1(t) = f_1(t); \qquad y_2(t) = f_2(t). \tag{5.24}$$

Assuming that solution (24) is stable according to Lyapunov, we determine q_{01}.

Following Lyapunov and transforming the system of equations (5) and (20) to new variables

$$x_s(t) = y_s(t) - f_s(t), \qquad (s = 1, 2),$$

where $x_s(t)$ are the disturbances, we write

$$\frac{dx_1}{dt} = x_2, \tag{5.25}$$

$$\frac{dx_2}{dt} = \frac{M_1(t, f_1 + x_1, f_2 + x_2)}{M(t, f_1 + x_1)} - \frac{M_1(t, f_1, f_2)}{M(t, f_1)}. \tag{5.26}$$

The problem of the stability of the particular solution (24) of the system of equations (5) and (20) is equivalent to the stability problem of the trivial solution $x_1 = x_2 = 0$ of system (25), (26). By singling out the linear terms, we rewrite (26) in the form

$$\frac{dx_2}{dt} = p_1(t)\, x_1 + p_2(t)\, x_2 + X, \tag{5.27}$$

where $p_1(t)$ and $p_2(t)$ are determined by the well-known formulas

$$p_1(t) = \left[\frac{\partial}{\partial x_1} \frac{M_1(t, f_1 + x_1, f_2 + x_2)}{M(t, f_1 + x_1)}\right]_0; \tag{5.28}$$

$$p_2(t) = \left[\frac{\partial}{\partial x_2} \frac{M_1(t, f_1 + x_1, f_2 + x_2)}{M(t, f_1 + x_1)}\right]_0. \tag{5.29}$$

The subscript 0 here designates that after the differentiation one is to write $x_1 = x_2 = 0$.

As was shown by Lyapunov (/10/, p.56), "when solving stability problems it is sufficient to consider values of t larger than any arbitrarily large T and to replace the examination of the initial values of the functions x_s by the examination of their values corresponding to $t = T$".

In accordance with this, we shall consider in all subsequent calculations in this section t to be larger than an arbitrarily large T.

As known, the growth of the displacements in conditions of creep is dampened at $t \to \infty$ in stable reinforced-concrete structures /11/. Therefore, taking (24) into account, we can write

$$f_1(t) = f_1 = \text{const}, \tag{5.30}$$

$$f_2(t) = 0. \tag{5.31}$$

We have in this case

$$M(t, f_1 + x_1) = 1 - \frac{a_2}{[a_1 + a_2(f_1 + x_1)]^2} \{[a_4 + a_3(f_1 + x_1)](1 - K^*)(f_1 +$$

$$+ x_1)^2 + q_0\} + \frac{1}{a_1 + a_2(f_1 + x_1)} \{a_3(1 - K^*)(f_1 + x_1)^2 +$$

$$+ 2(f_1 + x_1)[a_4 + a_3(f_1 + x_1)]\}. \tag{5.32}$$

$$M_1(t, f_1 + x_1, f_2 + x_2) = 0. \tag{5.33}$$

$$\left[\frac{\partial}{\partial x_1} M_1(t, f_1 + x_1, f_2 + x_2)\right]_0 = 0, \tag{5.34}$$

$$\left[\frac{\partial}{\partial x_2} M_1(t, f_1 + x_1, f_2 + x_2)\right]_0 = 0. \tag{5.35}$$

The substitution of (32) through (35) in (28) and (29) yields

$$p_1(t) = 0, \tag{5.36}$$

$$p_2(t) = p = \gamma\{(1 + EC_0)[a_2 q_0 + (a_2 a_4 - a_1 a_3)(1 - K^*) f_1^2] -$$

$$- (a_1 + a_2 f_1)[a_1 + (a_2 + 2a_4) f_1 + 2a_3 f_1^2]\}\{[(a_1 a_3 - a_2 a_4)\times$$

$$(1 - K^*) f_1^2 - a_2 q_0] + (a_1 + a_2 f_1)[a_1 + (a_2 + 2a_4) f_1 + 2a_3 f_1^2]\}^{-1} \tag{5.37}$$

It can be shown that at $t > T$ the nonlinear addition X in (27) likewise does not depend explicitly on t. The problem is thus reduced to studying the trivial solution $x_1 = x_2 = 0$ of the system of differential equations

$$\frac{dx_1}{dt} = x_2; \quad \frac{dx_2}{dt} = px_2 + X, \tag{5.38}$$

where the right-hand parts do not contain t explicitly; in other words, the problem is reduced to the problem of the stability of steady motion.

In accordance with Lyapunov's well-known theorem, at

$$p > 0, \tag{5.39}$$

the trivial solution of system (38) is unstable, independent of the linear addition X. Note that all the terms in the expansion of X contain x_2, since the terms which do not contain x_2 are equal to zero. Following Lyapunov (cf. /10/, pp. 100 to 118), and taking

$$z_1 = x_1; \quad z = -px_1 + x_2, \tag{5.40}$$

we reduce system (38) to the form

$$\frac{dz}{dt} = X(z_1, z); \quad \frac{dz_1}{dt} = pz_1 + z. \tag{5.41}$$

Taking

$$pz_1 + z = 0,$$

we obtain

$$z_1 = -\frac{z}{p}. \tag{5.42}$$

Substituting (42) in the right-hand part of the first equation of system (41), we note that it becomes identically zero, whence, in accordance with

Lyapunov's theorem (cf. /10/, p.118), we conclude that for

$$p<0 \tag{5.43}$$

the trivial solution of system (38) is stable. Since we have instability for (39) and stability for (43), q_{01} must obviously be determined from the condition

$$p=0. \tag{5.44}$$

From here, allowing for (37), we obtain

$$q_{01}=\frac{1}{(1+EC_0)a_2}\{(a_1+a_2f_{kp})[a_1+(a_2+2a_4)f_{kp}+2a_3f_{kp}^2]+$$
$$+(1+EC_0)(a_1a_3-a_2a_4)(1-K^*)f_{kp}^2\}. \tag{5.45}$$

Here f_{kp} designates the value of f_1 at $q_0=q_{01}$.

Let

$$(1-K^*)f_1^2=f_3. \tag{5.46}$$

Since we are considering the case $t>T$, we can write approximately

$$\varphi(\tau)=C_0. \tag{5.47}$$

Substituting (47) in (46), differentiating by t, and taking into account that

$$f_1'=f_3'=0, \tag{5.48}$$

we obtain

$$f_3=\frac{f_1^2}{1+EC_0}. \tag{5.49}$$

We obtain in the same way

$$(1-K^*)f_1=\frac{f_1}{1+EC_0}. \tag{5.50}$$

On the strength of (24), (49), and (50), the initial equation (3.3) can be rewritten in the form

$$q_0=-\frac{1}{1+EC_0}[a_3f_1^3+(a_2+a_4)f_1^2+a_1f_1]. \tag{5.51}$$

On the strength of (46) and (49), (45) can be rewritten in the form

$$q_{01}=\frac{1}{(1+EC_0)a_2}\{(a_1+a_2f_{kp})[a_1+(a_2+2a_4)f_{kp}+2a_3f_{kp}^2]+$$
$$+(a_1a_3-a_2a_4)f_{kp}^2\}. \tag{5.52}$$

We have thus obtained two cubic equations (51) and (52), the first of which was obtained directly from (3.3), and the second corresponding to the boundary of the stability region.

By equating the left-hand and right-hand parts of (51) and (52) we obtain

$$f_{kp}^3+\left[\frac{2}{3a_3}(a_2+a_4)+\frac{a_1}{a_2}\right]f_{kp}^2+\left(\frac{a_1}{a_3}+\frac{2a_1a_4}{3a_2a_3}\right)f_{kp}+\frac{a_1^2}{3a_2a_3}=0,$$

whence, taking into account that

$$f_{kp} + \frac{a_1}{a_2} \neq 0,$$

we obtain finally

$$f_{kp}^2 + \frac{2}{3a_3}(a_2 + a_4) f_{kp} + \frac{a_1}{3a_3} = 0. \qquad (5.53)$$

If the roots of equation (53) are real, the system will buckle at a sufficiently large value of q_0. By substituting the (real) roots of (53) in (51) or (52) we can determine the upper and lower values of q_{01}.

The problem of determining the critical load q_{01} under conditions of creep has thus been solved. It is very important, however, that we can also determine the critical load q_{0*}, by taking $C_0 = 0$ in (51), which corresponds to the case of absence of creep in the material. This equation will be designated by (51'). The modern procedure for determining q_{0*} (in a first approximation by the Bubnov-Galerkin method) is to obtain the cubic equation (51'), to differentiate it by f_1, and to equate the expression obtained to zero; this yields a quadratic equation coinciding with (53). The roots of this equation are substituted in (51') (provided they are real), and q_{0*} is obtained as a result /1, 5/.

Such a method for obtaining equation (53) is not based directly on the theory of stability. In the above calculations equation (53) was obtained directly on the basis of Lyapunov's theory of the stability of motion; this merits attention in the author's opinion.

The method exposed for determining q_{01} and q_{0*} can obviously also be applied in calculations using the second, third, etc., approximations in the Bubnov-Galerkin method.

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STRESSED STATE OF A CIRCULAR THICK DISK IN A FIELD OF CENTRIFUGAL FORCES

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§1. SOLUTION OF THE PROBLEM

The rigorous solution of the problem of the stressed state of a circular thick disk in a field of centrifugal forces necessitates finding a solution of Lamé's equations

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + M\frac{\partial \theta}{\partial r} + \rho\frac{\omega^2 r_0^2}{G} r = 0,$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial w}{\partial z^2} + M\frac{\partial \theta}{\partial z} = 0, \qquad (1.1)$$

which satisfies the boundary conditions:

$$\text{at } z = h \quad \sigma_z = \tau_{rz} = 0; \quad \text{at } r = 1 \quad \sigma_r = \tau_{rz} = 0. \qquad (1.2)$$

We use a cylindrical system of coordinates (r, z), with the plane $z = 0$ coinciding with the middle plane of the disk. The following sy ˆbols are used in the formulas: $M = \frac{m}{m-2}$, where m is the Poisson ratio; ρ is the density of the disk material; ω is angular velocity; θ is volumetric expansion, related to the radial u and the axial w components of the displacement vector by the relationship

$$\theta = \frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{\partial w}{\partial z}. \qquad (1.3)$$

To obtain a solution of (1.1) with a degree of arbitrariness sufficient for satisfying the boundary conditions (1.2), we use a method inspired from Lamé's paper on the parallelepiped. Such an approach to the solution of tridimensional problems of the theory of elasticity for regions bounded by perpendicular surfaces has lately been developed in /2-6/.

The specific problem considered here can, in our opinion, show the elegance of this approach not only from working out the solution, but also from obtaining numerical results.

On the strength of the symmetry of the displacement field relative to plane $z = 0$, we represent the problem solution in the following form:

$$u = Ar^3 + Cr + \left[\frac{D_0}{2} - \frac{m-2}{m} B_0\right] r + \sum_{n=1}^{\infty} \left\{\frac{D_n}{k_n} I_1(k_n r) + \right.$$

$$\left. + B_n \left[r I_0(k_n r) - \frac{4}{k_n} \frac{m-1}{m} I_1(k_n r) \right] \right\} \cos k_n z -$$

$$- \sum_{j=1}^{\infty} \left[A_j \lambda_j \operatorname{ch} \lambda_j z + C_j \frac{3m-4}{m} \operatorname{ch} \lambda_j z + C_j \lambda_j z \operatorname{sh} \lambda_j z \right] \frac{J_1(\lambda_j r)}{\lambda_j}, \qquad (1.4)$$

$$w = -D_0 z - \sum_{n=1}^{\infty} \left\{ \frac{D_n}{k_n} I_0(k_n r) + B_n r I_1(k_n r) \right\} \sin k_n z +$$

$$+ \sum_{j=1}^{\infty} \left[A_j \operatorname{sh} \lambda_j z + C_j z \operatorname{ch} \lambda_j z \right] J_0(\lambda_j r),$$

where

$$A = -\frac{\rho \omega^2 r_0^2}{8(1+M)}, \qquad C = \frac{(3m-2)\rho\omega^2 r_0^2}{8m(1+M)};$$

D_0, B_0, D_n, B_n, A_j, C_j are arbitrary constants, to be determined so as to satisfy the boundary conditions (1.2); $k_n = n\frac{\pi}{h}$ $(n=1, 2\cdots)$, λ_j are roots of the equation $J_1(\lambda)=0$.

The following expressions for the stresses are obtained from Hooke's law:

$$\frac{1}{2G}\sigma_z = -D_0 - \frac{2}{m} B_0 + \frac{4Ar^2}{m-2} + \frac{2C}{m-2} + \sum_{j=1}^{\infty} \left[A_j \lambda_j \operatorname{ch} \lambda_j z + \right.$$

$$\left. + C_j \frac{m-2}{m} \operatorname{ch} \lambda_j z + C_j \lambda_j z \operatorname{sh} \lambda_j z \right] J_0(\lambda_j r) - \sum_{n=1}^{\infty} \left\{ D_n I_0(k_n r) + \right.$$

$$\left. + B_n \left[\frac{2}{m} I_0(k_n r) + k_n r I_1(k_n r) \right] \right\} \cos k_n z,$$

$$\frac{1}{2G}\sigma_r = \frac{D_0}{2} - B_0 + \frac{3m-2}{m-2} A r^2 + \frac{m}{m-2} C + \sum_{j=1}^{\infty} \left[A_j \lambda_j \operatorname{ch} \lambda_j z + \right.$$

$$\left. + C_j \frac{3m-4}{m} \operatorname{ch} \lambda_j z + C_j \lambda_j z \operatorname{sh} \lambda_j z \right] \frac{J_1(\lambda_j r)}{\lambda_j r} - \sum_{j=1}^{\infty} \left[A_j \lambda_j \operatorname{ch} \lambda_j z + \right. \qquad (1.5)$$

$$\left. + C_j \frac{3m-2}{m} \operatorname{ch} \lambda_j z + C_j \lambda_j z \operatorname{sh} \lambda_j z \right] J_0(\lambda_j r) +$$

$$+ \sum_{n=1}^{\infty} \left\{ D_n \left[-\frac{1}{k_n r} I_1(k_n r) + I_0(k_n r) \right] + B_n \left[k_n r I_1(k_n r) - \right. \right.$$

$$\left. \left. - \frac{3m-2}{m} I_0(k_n r) + \frac{4}{k_n r} \frac{m-1}{m} I_1(k_n r) \right] \right\} \cos k_n z,$$

$$\frac{1}{2G}\tau_{rz} = -\sum_{j=1}^{\infty} \left[A_j \lambda_j \operatorname{sh} \lambda_j z + C_j 2 \frac{m-1}{m} \operatorname{sh} \lambda_j z + C_j \lambda_j z \operatorname{ch} \lambda_j z \right] J_1(\lambda_j r) -$$

$$- \sum_{n=1}^{\infty} \left\{ D_n I_1(k_n r) + B_n \left[k_n r I_0(k_n r) - 2 \frac{m-1}{m} I_1(k_n r) \right] \right\} \sin k_n z.$$

The boundary conditions for the normal stresses yield the following relationships between the integration constants:

$$
\begin{aligned}
&-D_0-\frac{2}{m}B_0+\frac{4Ar^2}{m-2}+\frac{2C}{m-2}+\sum_{j=1}^{\infty}\Big[A_j\lambda_j\operatorname{ch}\lambda_j h+\frac{m-2}{m}C_j\operatorname{ch}\lambda_j h+\\
&+C_j\lambda_j h\operatorname{sh}\lambda_j h\Big]I_0(\lambda_j r)-\sum_{n=1}^{\infty}(-1)^n\Big\{D_n I_0(k_n r)+\\
&+B_n\Big[\frac{2}{m}I_0(k_n r)+k_n r I_1(k_n r)\Big]\Big\}=0,\\
&\frac{D_0}{2}-B_0-\sum_{j=1}^{\infty}\Big[A_j\lambda_j\operatorname{ch}\lambda_j z+C_j\frac{3m-2}{m}\operatorname{ch}\lambda_j z+\\
&+C_j\lambda_j z\operatorname{sh}\lambda_j z\Big]I_0(\lambda_j)+\sum_{n=1}^{\infty}\Big\{D_n\Big[-\frac{1}{k_n}I_1(k_n)+I_0(k_n)\Big]+\\
&+B_n\Big[k_n I_1(k_n)-\frac{3m-2}{m}I_0(k_n)+\frac{4}{k_n}\frac{m-1}{m}I_1(k_n)\Big]\Big\}\cos k_n z=0.
\end{aligned}
\tag{1.6}
$$

The boundary conditions for the shearing stresses yield in turn the following relationships:

$$
\begin{aligned}
A_j\lambda_j&=-C_j\Big[2\frac{m-1}{m}+\lambda_j h\operatorname{ch}\lambda_j h\Big],\\
D_n&=-B_n\Big[k_n\frac{I_0(k_n)}{I_1(k_n)}-2\frac{m-1}{m}\Big].
\end{aligned}
\tag{1.7}
$$

Using the expansions

$$
\begin{aligned}
r^2&=\frac{1}{2}+\sum_{j=1}^{\infty}\frac{4J_0(\lambda_j r)}{J_0(\lambda_j)\lambda_j^2},\\
I_0(k_n r)&=2\frac{I_1(k_n)}{k_n}+2k_n I_1(k_n)\sum_{j=1}^{\infty}\frac{J_0(\lambda_j r)}{(\lambda_j^2+k_n^2)J_0(\lambda_j)},
\end{aligned}
\tag{1.8}
$$

$$
k_n r I_1(k_n r)=2\frac{I_1(k_n)}{k_n}+2k_n I_1(k_n)\sum_{j=1}^{\infty}\frac{J_0(\lambda_j r)}{(\lambda_j^2+k_n^2)J_0(\lambda_j)}-4k_n^3 I_1(k_n)\sum_{j=1}^{\infty}\frac{J_0(\lambda_j r)}{(\lambda_j^2+k_n^2)^2J_0(\lambda_j)}
$$

and relationships (1.7), we obtain from the first equality of (1.6)

$$
\begin{aligned}
&-D_0-\frac{2}{m}B_0+\frac{2(A+C)}{m-2}=0,\\
&-C_j\Big[\operatorname{ch}\lambda_j h+\frac{\lambda_j h}{\operatorname{sh}\lambda_j h}\Big]-\sum_{n=1}^{\infty}(-1)^n B_n\frac{4k_n\lambda_j^2 I_1(k_n)}{(k_n^2+\lambda_j^2)^2J_0(\lambda_j)}+\frac{16A}{(m-2)\lambda_j^2J_0(\lambda_j)}=0.
\end{aligned}
\tag{1.9}
$$

Similarly, using the expansions

$$
\begin{aligned}
\operatorname{ch}\lambda_j z&=\frac{\operatorname{sh}\lambda_j h}{\lambda_j h}+\frac{2\lambda_j\operatorname{sh}\lambda_j h}{h}\sum_{n=1}^{\infty}\frac{(-1)^n\cos k_n z}{(k_n^2+\lambda_j^2)},\\
\lambda_j z\operatorname{sh}\lambda_j z&=\operatorname{ch}\lambda_j h-\frac{\operatorname{sh}\lambda_j h}{\lambda_j h}+\Big(\frac{2\lambda_j\operatorname{sh}\lambda_j h}{h}+2\lambda_j^2\operatorname{ch}\lambda_j h\Big)\times\\
&\times\sum_{n=1}^{\infty}\frac{(-1)^n\cos k_n z}{k_n^2+\lambda_j^2}-\frac{4\lambda_j^3\operatorname{sh}\lambda_j h}{h}\sum_{n=1}^{\infty}\frac{(-1)^n\cos k_n z}{(k_n^2+\lambda_j^2)^2},
\end{aligned}
\tag{1.10}
$$

we obtain from the second equality of (1.6), allowing for (1.7):

$$\frac{D_0}{2} - B_0 = 0,$$

$$-B_n\left[k_n \frac{I_0^2(k_n)}{I_1(k_n)} - k_n I_1(k_n) - \frac{2}{k_n}\cdot\frac{m-1}{m} I_1(k_n)\right] -$$

$$-\sum_{j=1}^{\infty} C_j J_0(\lambda_j) \frac{(-1)^n 4\lambda_j k_n^2 \operatorname{sh}\lambda_j h}{h(k_n^2+\lambda_j^2)^2} = 0. \tag{1.11}$$

The simultaneous solution of the first equations of (1.9) and (1.11) yields

$$D_0 = \frac{2m(A+C)}{(m+1)(m-2)}, \quad B_0 = \frac{m(A+C)}{(m+1)(m-2)}.$$

Introducing new variables, defined by the relationships

$$\begin{aligned} (-1)^n B_n 4 k_n I_1(k_n) &= -x_n \frac{16A}{m-2}, \\ C_j J_0(\lambda_j) \frac{4\lambda_j \operatorname{sh}\lambda_j h}{h} &= y_j \frac{16A}{m-2}, \end{aligned} \tag{1.12}$$

we transform the system of equations (1.9) and (1.11) to the following form:

$$\begin{aligned} S_j y_j &= \sum_{n=1}^{\infty} x_n \frac{\lambda_j^2}{(k_n^2+\lambda_j^2)^2} + \frac{1}{\lambda_j^2}, \\ \Delta_n x_n &= \sum_{j=1}^{\infty} y_j \frac{k_n^2}{(k_n^2+\lambda_j^2)^2}, \end{aligned} \tag{1.13}$$

where

$$\Delta_n = \frac{I_0^2(k_n) - I_1^2(k_n) - \frac{2}{k_n^2}\frac{m-1}{m} I_1^2(k_n)}{4 I_1^2(k_n)},$$

$$S_j = \frac{h\left[\operatorname{ch}\lambda_j h + \frac{\lambda_j h}{\operatorname{sh}\lambda_j h}\right]}{4\lambda_j \operatorname{sh}\lambda_j h}.$$

§2. STUDY OF THE INFINITE SYSTEM (1.13)

System (1.13) will be studied by Koyalovich's method /8/.

1. On the basis of the theorems of Hobson and Dirichlet relative to the convergence of Bessel-function series and trigonometric series, we obtain from expansions (1.8) and (1.10) the following equalities:

$$\sum_{n=1}^{\infty} \frac{\lambda_j^2}{S_j(k_n^2+\lambda_j^2)^2} = 1 - \frac{1}{2\lambda_j^2 S_j}, \quad \sum_{j=1}^{\infty} \frac{k_n^2}{\Delta_n(k_n^2+\lambda_j^2)^2} = 1 - \frac{m+1}{2m k_n^2 \Delta_n},$$

where the functions $\varphi(j) = \frac{1}{2\lambda_j^2 S_j}$ and $\psi(n) = \frac{m+1}{2m k_n^2 \Delta_n}$ are obviously positive definite and do not exceed unity.

2. After the function $\varphi(j)$ has been found it becomes obvious that the value $K = 2$ can be used in the inequality

$$\frac{1}{S_j \lambda_j^2 \varphi(j)} \leqslant K, \quad (j = 1, 2, \cdots)$$

3. It is likewise easy to show that the inequalities

$$l < \frac{k_n^2}{\Delta_n \psi(n) (k_n^2 + \lambda_j^2)^2} < L \quad \text{for } j < n,$$

$$l < \frac{\lambda_j^2}{S_j \varphi(j) (k_n^2 + \lambda_j^2)^2} < L \quad \text{for } n < j$$

will be fulfilled if we write $l = \frac{2m}{(m+1)\,25}$, $L = 2$.

4. In order to fulfill the inequalities

$$\frac{1}{2\lambda_j^2 S_j} + \sum_{n=1}^{p} \frac{\lambda_j^2}{S_j (k_n^2 + \lambda_j^2)^2} > \gamma \quad \text{for } j = p+1,\ p+2, \cdots,\ 2p,$$

$$\frac{m+1}{2m\, k_n^2 \Delta_n} + \sum_{j=1}^{p} \frac{k_n^2}{\Delta_n (k_n^2 + \lambda^2)^2} > \gamma \quad \text{for } n = p+1,\ p+2, \cdots,\ 2p$$

for any p, we must take as γ the lesser of the two numbers $\frac{1}{32 \lambda_1 S_1 \pi}$ and $\frac{1}{50 k_1 \Delta_1 \pi}$.

The fulfillment of the additional inequalities 3 and 4 indicates that the law of asymptotic expressions applies to system (1.13): the unknowns x_n and y_j tend to the same magnitude q with the increase of the index. Koyalovich's theory of infinite systems provides no means for determining the numerical value of this limit; nevertheless, its existence itself is of considerable importance in the numerical solution of systems of the form (1.13). It indicates in fact that Koyalovich's limitants, which bound the solution from above and below, will become arbitrarily close with the increase of the number of equations "cut" from system (1.13); it follows that the value of all the unknowns in system (1.13) can be determined with any desired accuracy by solving a finite system of algebraic equations. This is true of the general case of infinite systems obtained when considering the first and second boundary-value problems for a circular cylinder of finite length.

An additional conclusion relative to the possibilities of the approach described follows from the law just proved about the asymptotic expressions for the infinite system (1.13). An examination of expressions (1.5) for the stresses shows that the series in them diverge on the angular circumference of the cylinder.

Consider as an example the stressed state of a "cubic" cylinder, i.e., a cylinder whose thickness is equal to the diameter. The values of the unknowns of (1.13) with deficit $\tilde{x}_n$, $\tilde{y}_j$ and of the unknowns with surplus X_n, Y_j [i.e., the values of the approximations from below and above] are given in the table, together with the value of the error in % allowed when the middle value between the two is used in the calculations. The unknowns were calculated using the first nine equations in each infinite system in (1.13), which gave an accuracy of 2.5% in the determination of all unknowns. For lack of space we do not give here the bulky but actually elementary

calculations involved in the assessment of the limitant boundaries in successive approximations.

TABLE

No.	$\bar{x}_n$	X_n	δ_x %	$\bar{y}_j$	Y_j	δ_y %
1	0.26407	0.26461	0 10	1.20495	1.20589	0.04
2	0.60268	0.60458	0.16	0.88477	0.88707	0.13
3	0.70770	0.71103	0.22	0.80087	0.80463	0.23
4	0.74337	0.74814	0.32	0.77112	0.77635	0.34
5	0.75696	0.76320	0.41	0.75866	0.76535	0.46
6	0.76226	0.76996	0.50	0.75284	0.76078	0.52
7	0.76409	0.77319	0.59	0.74995	0.75950	0.64
8	0.76437	0.77482	0.68	0.74843	0.75931	0.73
9	0.76392	0.77566	0.76	0.74758	0.75974	0.81
>9	0.74389	0.77908	2.45	0.74389	0.77908	2.45

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APPROXIMATE SOLUTIONS OF SEVERAL PROBLEMS OF STRESS CONCENTRATION ABOUT HOLES IN ISOTROPIC AND ORTHOTROPIC SHELLS

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(Kiev)

One of the possible approaches to the study of stress concentration about holes was proposed by Savin /4/. The stressed state of the shell is represented as the sum of the stressed state of a shell without hole and the disturbances caused by the presence of a hole. It has been experimentally established that the disturbances are purely local. It is therefore assumed in /4/ that these disturbances can be described by the equations of the theory of shallow shells. This approach will be used here in studying the stress concentration about holes in shells.

§1. APPROXIMATE METHOD FOR STUDYING THE STRESS CONCENTRATION IN ISOTROPIC SHELLS ABOUT QUASI-CIRCULAR HOLES

Consider an arbitrary isotropic shell weakened by a hole. We shall represent the middle surface in the polar system of coordinates (r, θ), with the pole coinciding with the hole center. Following /4/, consider the plane of variation of the variables r and θ (plane z). Γ is the hole contour, ρ is the radius of the circular hole which is most similar to the hole considered, α and β are curvature lines, x and y are axes of the cartesian system of coordinates, coinciding with the axes of symmetry of the hole, ψ is the angle between the radial direction and the direction of the normal to Γ, $\vec{n}$ is the unit vector of the normal to Γ (Figure 1). We assume that the function $z = \omega(\zeta)$ $(z = re^{i\theta}; \zeta = r_1 e^{i\gamma})$ conformally maps an infinite plane with circular hole of unit radius into an infinite plane with hole limited by the contour Γ. We shall treat holes for which

$$\omega(\zeta) = \rho\left(\zeta + \frac{\varepsilon}{\zeta^N}\right), \quad (\varepsilon < 1). \tag{1}$$

The problem reduces /4/ to solving the equation

$$\nabla^2\nabla^2 \Phi - i\varkappa^2 \nabla_k^2 \Phi = 0 \tag{2}$$

for specified boundary conditions. The symbols used in (2) have the

following meaning:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{dr} + \frac{1}{r^2}\frac{\partial^2}{\partial y^2}, \qquad \nabla_k^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k^2}{r^2}.$$

We shall consider the boundary conditions in forces only

$$T_n|_\Gamma = F_1(\gamma, \varepsilon); \quad S_{ns}|_\Gamma = F_2(\gamma, \varepsilon); \quad G_n|_\Gamma = F_3(\gamma, \varepsilon); \quad \bar{Q}_n|_\Gamma = F_4(\gamma, \varepsilon), \qquad (3)$$

where $\Phi = w + in\varphi$, w is the deflection, φ is the stress function, and n and $\varkappa$ are constants.

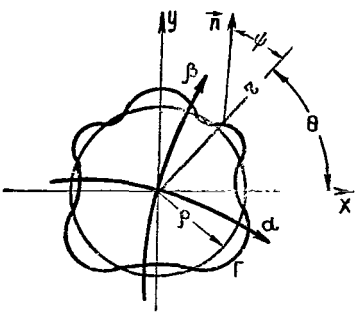

FIGURE 1.

The problem will be solved approximately, using the method of "boundary perturbation" /3/. Such a method was used by Lekhnitskii /2/ in solving the problem of anisotropic plates; after the solution had been represented through analytical functions of a complex variable. We shall proceed here as follows. Let the solution of equation (2) in the polar system of coordinates be represented by the Fourier series:

$$\Phi(r, \theta) = \sum_{k=0}^{\infty} f_{1k}(r)\cos k\theta + f_{2k}(r)\sin k\theta. \qquad (4)$$

The possibility of such a representation is suggested by the fact that a solution of the form of (4) was obtained for spherical and conical shells. Knowing $\Phi(r, \theta)$, we can determine $T_r, \cdots, \bar{Q}_r$; $T_n, \cdots, \bar{Q}_n$ on the contour Γ are then determined by the transformation formulas corresponding to rotation of the coordinate system through the angle ψ. Using (1), we can determine $r = r(\gamma, \varepsilon)$; $\theta = \theta(\gamma, \varepsilon)$. $\psi = \psi(\gamma, \varepsilon)$ on the contour Γ. Represent the components of the stressed state of the shell on contour Γ and $\Phi(r, \theta)$ in the entire region as series in powers of ε:

$$T_n|_\Gamma = \sum_{j=0}^{\infty} \varepsilon^j T_n^{(j)}; \cdots; \bar{Q}_n|_\Gamma = \sum_{j=0}^{\infty} \varepsilon^j \bar{Q}_n^{(j)}; \quad \Phi(r, \theta) = \sum_{j=0}^{\infty} \varepsilon^j \Phi_j(r, \theta). \qquad (5)$$

The boundary conditions (3) will be expanded likewise in series by the powers of ε. In order to determine, for instance, $T_{n\,\Gamma}$, we must determine

$T_r, \cdots, S_{r\theta}$, through $\Phi(r, \theta)$, and then use the formulas of transformation of the forces corresponding to rotation of the system of coordinates; we finally obtain

$$T_{n\tau} = f\,[\Phi(r, \theta), \psi]|_{r = r(\gamma, \varepsilon);\ \theta = \theta(\gamma, \varepsilon);\ \psi = \psi(\gamma, \varepsilon)}. \tag{6}$$

By substituting (5) in (2) and (4) and equating the coefficients of equal powers of ε we obtain

$$\Phi_j(r, \theta) = \sum_{k=0}^{\infty} f_{1k}^j(r) \cos k\theta + f_{2k}^j(r) \sin k\theta,$$

i.e., we can write directly the solution for the j-th approximation. In order to determine $T_n^{(j)}, \cdots, \widetilde{Q}_n^{(j)}$ we develop in series by ε expressions of the type of (6):

$$T_n^{(j)} = \frac{1}{n}\left(\frac{1}{\rho^2}\frac{\partial^2}{\partial\gamma^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\operatorname{Im}\Phi_j(\rho, \gamma) + \frac{1}{n}\sum_{m=0}^{j-1} L_1^{(j-m)} \operatorname{Im}\Phi_m(\rho, \gamma),$$

$$\widetilde{Q}_n^{(j)} = -D\left[\frac{\partial}{\partial\rho}\nabla^2 + (1-\nu)\frac{1}{\rho}\frac{\partial}{\partial\gamma}\frac{1}{\rho}\left(\frac{\partial^2}{\partial\rho\partial\gamma} - \frac{1}{\rho}\frac{\partial}{\partial\gamma}\right)\right]\operatorname{Re}\Phi_j(\rho, \gamma) -$$

$$-D\sum_{m=0}^{j-1} L_6^{(j-m)} \operatorname{Re}\Phi_m(\rho, \gamma), \tag{7}$$

where $L_i^{(j-m)}$ are differential operators, with $L_i^{(0)} \equiv 0$. These operators were obtained for the zero, first, and second approximations; we do not present them here however for lack of space. The following expression is obtained for $T_s|_\Gamma$:

$$T_s|_\Gamma = \frac{1}{n}\sum_{j=0}^{\infty} \varepsilon^j \frac{\partial^2}{\partial\rho^2}\operatorname{Im}\Phi_j(\rho, \gamma) + \frac{1}{n}\sum_{j=0}^{\infty}\sum_{m=0}^{j=1} \varepsilon^j L_2^{(j-m)} \operatorname{Im}\Phi_m(\rho, \gamma). \tag{8}$$

The only unknown in (7) is $\Phi_j(\rho, \gamma)$, while $\Phi_j(r, \theta)$ is already known from the preceding approximations. $\Phi_j(\rho, \gamma)$ is to be understood as $\Phi_j(r, \theta)$, in which r is replaced by ρ, and θ by γ. Comparing (7) with the known formulas for determining the stressed state components in the polar system of coordinates, we note that $T_n^{(j)}$ is expressed through $\Phi_j(\rho, \gamma)$ in the same way as T_r is expressed through $\Phi(r, \theta)$; as to the function $\Phi_j(\rho, \gamma)$ itself (or more accurately $\Phi_j(r, \theta)$), it is a solution of (2) in the polar system of coordinates. Formally, therefore, the problem of the successive approximations has been reduced to the problem for a circular hole of radius ρ in the ζ plane.

The convergence of the method was tested on problems involving stress concentration about elliptic, triangular, and square holes in plates. It turned out that the expansion of the concentration factor obtained by the rigorous solution coincides with an accuracy up to ε^2 (the problems were solved in the zero, first, and second approximations) with the value of the factor obtained by the approximation method. Thus, for an elliptic hole we obtain even for $\frac{a}{b} = 1.5$ an error of only 6% in the first approximation, and of 2% in the second approximation.

§2. STRESS CONCENTRATION ABOUT A CURVILINEAR HOLE IN A SPHERICAL SHELL

Consider a spherical shell loaded by a uniform pressure and weakened by a hole of the above-considered shape, covered by a lid which transmits only the shearing force /4/. We shall assume boundary conditions which are a generalization of the conditions adopted in /5/ for an elliptic hole. The functions giving the solution in the zero, first, and second approximations are respectively:

$$\Phi_0(r) = (c_2^{0,0} + id_2^{0,0}) H_0^{(1)}(r\varkappa\sqrt{-i}) . \tag{9}$$

For an elliptic hole: $\rho = \dfrac{a+b}{2}$; $\varepsilon = \dfrac{a-b}{a+b}$; $N=1$;

$$\Phi_1(r,\theta) = \{(c_1^{1,2} + id_1^{1,2}) r^{-2} + (c_2^{1,2} + id_2^{1,2}) H_2^{(1)}(r\varkappa\sqrt{-i})\}\cos 2\theta;$$

$$\Phi_2(r,\theta) = id_1^{2,0}\ln r + (c_2^{2,0} + id_2^{2,0}) H_0^{(1)}(r\varkappa\sqrt{-i}) + \{(c_1^{2,2} + id_1^{2,2}) r^{-2} + (c_2^{2,2} + i\,d_2^{2,2}) H_2^{1}(r\varkappa\sqrt{-i})\}\cos 2\theta + \{(c_1^{2,4} + i\,d_1^{2,4}) r^{-4} + (c_2^{2,4} + id_2^{2,4}) H_4^{(1)}(r\varkappa\sqrt{-i})\}\cos 4\theta. \tag{10}$$

For an equilateral triangular hole: $\varepsilon = 0.25$; $N=2$;

$$\Phi_1(r,\theta) = \{(c_1^{1,3} + id_1^{1,3}) r^{-3} + (c_2^{1,3} + i\,d_2^{1,3}) H_3^{(1)}(r\varkappa\sqrt{-i})\}\cos 3\theta;$$

$$\Phi_2(r,\theta) = id_1^{2,0}\ln r + (c_2^{2,0} + id_2^{2,0}) H_0^{(1)}(r\varkappa\sqrt{-i}) + \{(c_1^{2,3} + id_1^{2,3}) r^{-3} + (c_2^{2,3} + id_2^{2,3}) H_3^{(1)}(r\varkappa\sqrt{-i})\}\cos 3\theta + \{(c_1^{2,6} + id_1^{2,6}) r^{-6} + (c_2^{2,6} + id_2^{2,6}) H_6^{(1)}(r\varkappa\sqrt{-i})\}\cos 6\theta. \tag{11}$$

For a square hole: $\varepsilon = \pm\dfrac{1}{9}$ or $\pm\dfrac{1}{6}$, $N=3$,

$$\Phi_1(r,\theta) = \{(c_1^{1,4} + id_1^{1,4}) r^{-4} + (c_2^{1,4} + id_2^{1,4}) H_4^{(1)}(r\varkappa\sqrt{-i})\}\cos 4\theta;\ \Phi_2(r,\theta) = id_1^{2,0}\ln r + (c_2^{2,0} + id_2^{2,0}) H_0^{(1)}(r\varkappa\sqrt{-i}) + \{(c_1^{2,4} + id_1^{2,4}) r^{-4} + (c_2^{2,4} + i\,d_2^{2,4}) H_4^{(1)}(r\varkappa\sqrt{-i})\}\cos 4\theta + \{(c_1^{2,8} + id_1^{2,8}) r^{-8} + (c_2^{2,8} + id_2^{2,8}) H_8^{(1)}(r\varkappa\sqrt{-i})\}\cos 8\theta. \tag{12}$$

Here $H_k^{(1)}(r\varkappa\sqrt{-i})$ is a Hankel function of the first kind of order k /1/; $c_i^{k,l}$, $d_i^{k,l}$ are constants. Consider as an example the stress concentration about an elliptical hole:

$$R = 250\text{cm};\quad h = 0.3\text{cm};\quad \frac{a+b}{2} = 10.5\text{cm};\quad \nu = 0.3.$$

Using (8), (9), and (10), we obtain in a first approximation

$$k = 4.295 + 12.768\,\varepsilon\cos 2\gamma. \tag{13}$$

The advantage of the solution given here for an elliptic hole over that proposed in /5/ consists in the fact that in the final formula (13) there appears the parameter $\varepsilon = \dfrac{a-b}{a+b}$ (only $\rho = \dfrac{a+b}{2}$ is fixed). This fact makes it possible to calculate easily the concentration factor for an entire family of elliptic holes. The results of the calculations are given in Table 1.

TABLE 1

a, cm	b, cm	$\frac{a}{b}$	k	$k\|_{\gamma=0}$	$k\|_{\gamma=\frac{\pi}{2}}$
10.5	10.5	1	4.295	4.295	4.295
11.0	10.0	1.1	$4.295+0.613\cos 2\gamma$	4 908	3.628
11.4	9.6	1.2	$4.295+1.162\cos 2\gamma$	5.457	3.133
11.7	9.3	1.3	$4.295+1.660\cos 2\gamma$	5 955	2.635
12.3	8.7	1.4	$4.295+2.170\cos 2\gamma$	6.465	2.125
12.9	8.1	1.5	$4.295+2.554\cos 2\gamma$	6.849	1.741

§ 3. STRESS CONCENTRATION ABOUT A CIRCULAR HOLE REINFORCED BY AN ABSOLUTELY RIGID PIPE SOCKET IN A CYLINDRICAL ORTHOTROPIC SHELL

Consider a cylindrical orthotropic shell loaded by a uniform internal pressure p and weakened by a circular hole of radius r_0 reinforced by an absolutely rigid pipe socket. Let the basic stressed state be momentless, and let the elastically equivalent directions coincide with the directions of the generator and the directrix. The boundary conditions are written as follows:

$$(u+u^0)\Big|_{r=r_0}=0;\quad (v+v^0)\Big|_{r=r_0}=0;\quad \frac{\partial}{\partial r}(w+w^0)\Big|_{r=r_0}=0;$$
$$r_0\int_0^{2\pi}\left(Q_r+\frac{1}{r}\frac{\partial}{\partial\theta}H_{r\theta}\right)d\theta\Big|_{r=r_0}=-\pi r_0^2 p; \tag{14}$$

u^0, v^0, w^0 are displacements from the basic stressed state. The problem was solved by the Ritz method, using the relationships of the theory of shallow shells. The displacements, satisfying the conditions "at infinity" /4/, were selected in the following form:

$$u=r_0\left[u_{01}\left(\frac{r_0}{r}\right)+u_{03}\left(\frac{r_0}{r}\right)^3\right]+\left[u_{11}\left(\frac{r_0}{r}\right)+u_{13}\left(\frac{r_0}{r}\right)^3\right]r_0\cos 2\theta;$$
$$v=r_{01}\left[v_{11}\left(\frac{r_0}{r}\right)+v_{13}\left(\frac{r_0}{r}\right)^3\right]\sin 2\theta; \tag{15}$$
$$w=r_0\left[w_{03}\left(\frac{r_0}{r}\right)^3+w_{05}\left(\frac{r_0}{r}\right)^5\right]+r_0\left[w_{13}\left(\frac{r_0}{r}\right)^3+w_{15}\left(\frac{r_0}{r}\right)^5\right]\cos 2\theta.$$

Six constants were determined from the boundary conditions, and the remaining four from the condition of minimum of the total potential energy. A system of four algebraic equations was obtained for determining these constants; it is not given here for lack of space. Consider as an example the shell

$$E;\quad \nu=0.3;\quad R=100\text{cm};\quad h=0.5\text{ cm};\quad r_0=5\text{cm}.$$

The shell is stiffened by ribs in such a way that $E_x=2E$ and $E_s=E$, and is calculated as an orthotropic shell. Table 2 gives the maximum values of $k=\frac{\sigma_r}{p_0}\Big|_{r=r_0}$ (where $p_0=Rp/2h$), allowing for the membrane stresses and the bending stresses. Figure 2 represents graphically the distribution

of k along the hole contour as a function of the angle θ. It is seen from the table that the influence of the orthotropy on k is considerable at $\theta = 0°$ (it increases by 33% in an orthotropic shell), but that the maximum value of k increases by 5% in all.

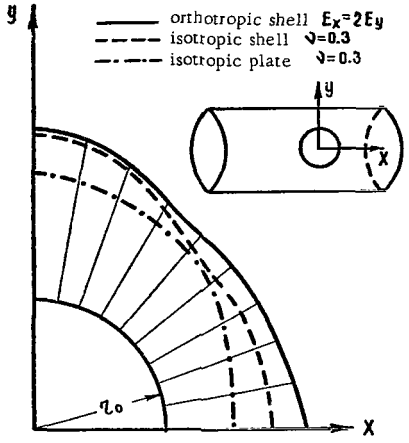

FIGURE 2.

TABLE 2

$\theta°$ deg	Isotropic plate	Isotropic shell	Orthotropic plate	Orthotropic shell
0	1.60	2.50	1.82	3.32
10	1.65	2.50	1.87	3.15
20	1.77	2.37	1.99	2.98
30	1.96	2.46	2.20	2.83
40	2.30	2.41	2.45	2.99
50	2.44	2.58	2.71	2.92
60	2.67	2.98	2.96	3.06
70	2.83	3.45	3.16	3.50
80	2.98	3.73	3.29	3.88
90	3.02	3.82	3.34	4.00

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ON THE NONLINEAR VIBRATIONS OF ELASTIC SHELLS UNDER RANDOM LOADS

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The problem of the random vibrations of plates and shallow shells is linked with the problem of estimating the fatigue life of many thin-walled structures loaded by random forces /1/. The question of the probability density of the extreme values of the deflections and stresses in the case of nonlinear vibrations was treated in /2/ for the case of a disturbing process of finite energy. Some of the results of /2/ are developed below, and the average number of "snap-throughs" of the shell under the action of random disturbances is determined.

1. The motion of an elastic shallow shell rectangular in the plan is described by a system of two equations in partial derivatives; the tangential inertia forces are neglected. The load $q\ (x,\ y,\ t)$ is considered as a normally distributed random function of all its variables, stationary in time, with a zero mean value at each point. Using Galerkin's method, the problem is reduced to a system of n ordinary second-order differential equations relative to the generalized coordinates of the normal deflection. The right-hand parts of these equations contain the normal generalized forces $Q_j(t)$, whose elements of the matrix of spectral densities are determined by the relationships

$$\Phi_{Q_jQ_k}(\omega)=$$

$$=\frac{1}{(\rho h)^2\nu_j\nu_k}\int_0^a\int_0^a\int_0^b\int_0^b\Phi_{qq}\,(\omega,\xi_1,\xi_2,\eta_1,\eta_2)\,\varphi_j(\xi_1,\eta_1)\varphi_k(\xi_2,\eta_2)\,d\xi_1d\xi_2d\eta_1d\eta_2. \qquad (1.1)$$

$$(j,\ k=1,\ \cdots,\ n).$$

Here Φ_{qq} is the spectral density of the load, φ_j are coordinate functions of the normal deflection (e.g., the modes of small natural oscillations of frequencies ω_j and norms ν_j); $\rho,\ h,\ a,\ b$ are respectively the material density, the thickness, and the length of the sides of the shell.

The shell motion is described in /2/ by the theory of multidimensional Markow processes and Fokker-Planck-Kolmogorov equations. It is assumed that the generalized forces $Q_j\ (t)$ are connected with the white noises by a system of several linear differential relationships. We shall examine this assumption more closely. A necessary condition for its fulfillment is the possibility of representing the spectral densities $\Phi_{Q_jQ_k}\ (\omega)$, determined by (1.1), in the form

$$\Phi_{Q_jQ_k}\ (\omega)=\frac{c_{jk}}{2\pi}\,Y_j\,(i\omega)Y_k^*\,(i\omega). \qquad (1.2)$$

Here c_{jk} are arbitrary mutual intensities of the white noises; the asterisk designates a complex conjugate function. We can write $c_{jj} = 1$, and then $0 \leqslant c_{jk} \leqslant 1$. The functions $Y_j(i\omega)$ are of the form

$$Y_j(i\omega) = \frac{a_{j0} + a_{j1}(i\omega) + \cdots + a_{js_j}(i\omega)^{s_j}}{b_{j0} + b_{j1}(i\omega) + \cdots + b_{jr_j}(i\omega)^{r_j}}, \tag{1.3}$$

where a, b are constants.

The question of representing the principal elements of the matrix $\Phi_{Q_j Q_k}(\omega)$ in such a form poses no basic difficulties. Any real spectrum which does not contain discrete components can be approximated with a sufficient accuracy (exceeding the measurement accuracy) by a fractional rational function. As to the secondary elements, they do not satisfy in the general case relationships (1.2). In many important particular cases these relationships are nevertheless fulfilled. We shall give several examples of this.

$$\text{a) } \Phi_{qq}(\omega, \xi_1, \xi_2, \eta_1, \eta_2) = \Psi(\omega) F(\xi_1, \xi_2, \eta_1, \eta_2). \tag{1.4a}$$

To this type belong the cases of a load determined and delta-correlated in space.

$$\text{b) } \Phi_{qq}(\omega, \xi_1, \xi_2, \eta_1, \eta_2) = \Psi(\omega) F(\xi_1, \eta_1) F^*(\xi_2, \eta_2). \tag{1.4b}$$

To this type belongs the important case of a "wave-type" load, corresponding to acoustic pressure in a remote field, to passage through a non-homogeneous medium, etc.

Generally speaking, there can exist types of load for which conditions (1.2) are not fulfilled for the secondary elements of the matrix. Equation (2.3) of /2/ remains however approximately correct for this case as well, since, as can be seen from the solution of the linear problem /1/, the part played by the secondary elements of the spectral density matrix of the generalized forces is small compared with the part played by the principal elements.

2. Let the normal deflection be approximated by one term of the series. Designate by ξ the ratio of the generalized coordinate f to the thickness. We have then in the general case the ordinary differential equation

$$\frac{d^2\xi}{dt^2} + 2\varepsilon\omega_0 \frac{d\xi}{dt} + \omega_0^2 [\xi + g(\xi)] = Q(t). \tag{2.1}$$

Here $g(\xi)$ is some nonlinear continuously differentiable function. For an elastic curved panel, taking Karman's equations as initial equations:

$$g(\xi) = \alpha\xi^3 + \beta\xi^2, \tag{2.2}$$

where α, β are constants. At very large deflections it is necessary to refine the Karman equations, and then the function $g(\xi)$ becomes more complex.

Let $Q(t)$ be an exponentially correlated random process of spectral density

$$\Phi_{QQ}(\omega) = \frac{D_0}{2\pi} \frac{1}{1 + \omega^2 \tau_0^2}, \tag{2.3}$$

where D_0, τ_0 are constants. We obtain from (2.1) and (2.3)

$$\tau_0\frac{d^3\xi}{dt^3}+(1+2\varepsilon\omega_0\tau_0)\frac{d^2\xi}{dt^2}+\omega_0^2\tau_0\left[1+\frac{2\varepsilon}{\omega_0\tau_0}+\frac{dg(\xi)}{d\xi}\right]+\omega_0^2\ [\xi+g(\xi)]=\psi(t), \qquad (2.4)$$

where $\psi(t)$ is white noise of spectral density $D_0/2\pi$. Let the excitation spectrum be wide-banded ($\omega_0\tau_0\ll 1$), and the damping small ($\varepsilon\ll 1$); the order of magnitude of the parameter $2\varepsilon/\omega_0\tau_0$ can be arbitrary. The second term in the parentheses before $d^2\xi/dt^2$ can then be neglected compared with unity, and (2.4) reduces to

$$\tau_0\frac{d^3\xi}{dt^3}+\frac{d^2\xi}{dt^2}+\omega_0^2\tau_0\left[1+\frac{2\varepsilon}{\omega_0\tau_0}+\frac{dg(\xi)}{d\xi}\right]+\omega_0^2\ [\xi+g(\xi)]=\psi(t). \qquad (2.5)$$

Introduce the designations $d\xi/dt=\eta$, $d\eta/dt=\zeta$ and form the Fokker-Planck-Kolmogorov stationary equation for the common probability density of the coordinates, velocities, and accelerations $p(\xi,\eta,\zeta)$:

$$\eta\frac{\partial p}{\partial\xi}+\zeta\frac{\partial p}{\partial\eta}=\frac{1}{\tau_0}\frac{\partial}{\partial\zeta}(p\zeta)+\omega_0^2\left\{\eta\left[1+\frac{2\varepsilon}{\omega_0\tau_0}+\frac{dg(\xi)}{d\xi}\right]+\right.$$
$$\left.+\frac{1}{\tau_0}\ [\xi+g(\xi)]\right\}\frac{\partial p}{\partial\zeta}+\frac{D_0}{2\tau_0^2}\frac{\partial^2 p}{\partial\zeta^2}. \qquad (2.6)$$

The solution of this equation is of the form

$$p(\xi,\eta,\zeta)=p_0(\xi,\zeta)\,p_1(\eta), \qquad (2.7)$$

$$p_0(\xi,\zeta)=\frac{1}{2\pi J\xi_0\zeta_0}\exp\left\{-\frac{1}{2\zeta_0^2}(\omega_0^2[\xi+g(\xi)]+\zeta)^2-\right.$$
$$\left.-\frac{1}{\xi_0^2}\left[\frac{\xi^2}{2}+G(\xi)\right]\right\},\quad G(\xi)=\int_0^\xi g(\chi)\,d\chi, \qquad (2.8)$$

$$p_1(\eta)=\frac{1}{\eta_0\sqrt{2\pi}}\exp\left(-\frac{\eta^2}{2\eta_0^2}\right). \qquad (2.9)$$

Here

$$\xi_0^2=\frac{D_0}{4\varepsilon\omega_0^3},\quad \eta_0^2=\frac{D_0}{4\varepsilon\omega_0},\quad \zeta_0^2=\frac{D_0}{2\tau_0}. \qquad (2.10)$$

A particular case of solution (2.7) through (2.9) for the function (2.2) was treated in /1/.

The probability density of the coordinate $p(\xi)$ is

$$p(\xi)=\int_{-\infty}^{\infty}p_0(\xi,\zeta)\,d\zeta=\frac{1}{J\xi_0\sqrt{2\pi}}\exp\left\{-\frac{1}{\xi_0^2}\left[\frac{\xi^2}{2}+G(\xi)\right]\right\}. \qquad (2.11)$$

The constant J is determined from the conditions of normalization of (2.11). Note that the product $p(\xi)\,p_1(\eta)$ coincides with the exact solution of the Fokker-Planck-Kolmogorov equation corresponding to excitation by white noise.

3. We shall now determine the density of the extrema $n(\xi_{max})$, $n(\xi_{min})$, i.e., the time averages of the number of extrema lying within the limits from ξ to $\xi+d\xi$. By substituting (2.7), (2.8), and (2.9) in Reiss'

formula /3/

$$n\,(\xi_{max}) = \frac{\omega_0}{4\pi J\xi_0}\left\{\frac{1}{\xi_0}[\xi + g\,(\xi)]\,(1 + \operatorname{erf}\frac{m}{\xi_0}\,[\xi + g(\xi)]) + \right.$$

$$\left. + \frac{1}{m\sqrt{\pi}}\exp\left(-\frac{m^2}{\xi_0^2}[\xi + g(\xi)]^2\right)\right\}\exp\left\{-\frac{1}{\xi_0^2}\left[\frac{\xi^2}{2} + G\,(\xi)\right]\right\}, \qquad (3.1)$$

$$n\,(\xi_{min}) = \frac{\omega_0}{4\pi J\xi_0}\left\{-\frac{1}{\xi_0}[\xi + g\,(\xi)]\,(1 - \operatorname{erf}\frac{m}{\xi_0}\,[\xi + g(\xi)]) + \right.$$

$$\left. + \frac{1}{m\sqrt{\pi}}\exp\left(-\frac{m^2}{\xi_0^2}[\xi + g(\xi)]^2\right)\right\}\exp\left\{-\frac{1}{\xi_0^2}\left[\frac{\xi^2}{2} + G\,(\xi)\right]\right\}, \qquad (3.2)$$

where $m = (\omega_0\tau_0/4\varepsilon)^{1/2}$. The parameter m characterizes the "width" of the spectrum of the function $\xi\,(t)$. The correlation time for the noise of aircraft engines is of the order of 10^{-4} sec /6/. The corresponding values of m are usually small. Thus, at $\omega_0 = 100\ \text{sec}^{-1}$, $\varepsilon = 0.025$, $m = 0.316$. At $m \gg 1$ (narrow-banded process) we can obtain from (3.1) and (3.2) the asymptotic relationships

$$n\,(\xi_{max}) \approx \frac{\omega}{2\pi J\xi_0}\,[\xi + g\,(\xi)]\,\exp\left\{-\frac{1}{\xi_0^2}\left[\frac{\xi^2}{2} + G(\xi)\right]\right\} \text{for } \xi + g(\xi) > 0, \qquad (3.3)$$

$$n\,(\xi_{max}) \approx 0 \qquad \text{for } \xi + g(\xi) < 0,$$

$$n(\xi_{min}) \approx -\frac{\omega_0}{2\pi J\xi_0}\,[\xi + g\,(\xi)]\,\exp\left\{-\frac{1}{\xi_0^2}\left[\frac{\zeta^2}{2} + G(\xi)\right]\right\} \text{for } \xi + g(\xi) < 0, \qquad (3.4)$$

$$n\,(\xi_{min}) \approx 0 \qquad \text{for } \xi + g(\xi) > 0.$$

The following results given below correspond to the case when $g(\xi)$ is determined by (2.2). An important part is played by the parameter of asymmetry of the nonlinearity $\gamma = \beta/\sqrt{\alpha}$. At $\gamma > 2$ the equation $dp(\xi)/d\xi = 0$ has three real roots. This means that the shell has two stable $(0, \xi_2)$ and one unstable (ξ_1) equilibrium positions

$$\zeta_{1,2} = -\frac{1}{2\sqrt{\alpha}}\left(\gamma \mp \sqrt{\gamma^2 - 4}\right). \qquad (3.5)$$

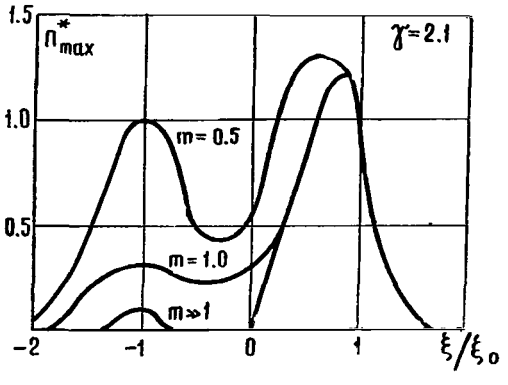

FIGURE 1.

Figures 1 and 2 represent curves of the relationships (3.1) and (3.2) at $\gamma = 2.1$, $\alpha\xi_0^2 = 1$ $(n^* = n \cdot 2\pi J\,\xi_0/\omega_0)$. It is easily seen that the shape of the

curves depends strongly on the parameter m, although the curves become similar at large ξ/ξ_0. The asymptotic relationship (3.3) was obtained in /4/, for the case $g(\xi)=\alpha\xi^3$, on the basis of different considerations, based on the assumption that the process $\xi(t)$ is narrow-banded, when $\varepsilon \ll 1$. Actually, as seen from the analysis, in order that the process be narrow-banded it is necessary that condition $m \gg 1$ be satisfied in addition to condition $\varepsilon \ll 1$. This is the reason for the difference between the curves represented in Figures 1 and 2.

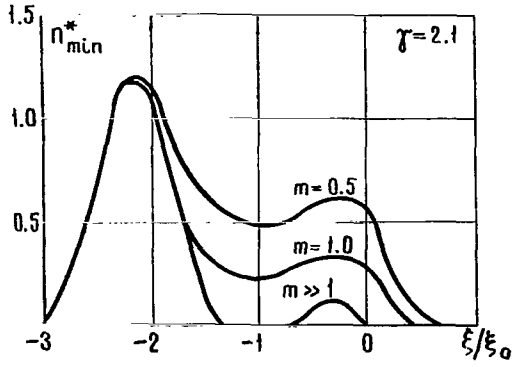

FIGURE 2.

4. In determining the average number of extrema of the stress S one uses the rules of calculation of the distribution of a function of a random magnitude /1/. The expression for $S(\xi)$ is of the form

$$S=\frac{S_0}{\xi_0}(\xi+r\xi^2). \tag{4.1}$$

This relationship is represented in Figure 3. Here S_0 is the mean-square stress of the linear problem, and r is the coefficient of membrane stresses.

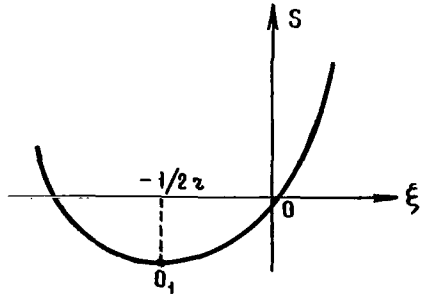

FIGURE 3.

The densities of the stress extrema were determined in /2/ for the case $\xi_0 r \ll 1/2$, i.e., when relationship $\xi(S)$ is single-valued for all cycles (except the cycles of very low probability). We shall renounce this assumption, but shall consider that $\gamma<2$ and $m \gg 1$. We shall make use of

relationships (3.3) and (3.4). Note that to the minima of S correspond the minima of ξ lying in segment $0_1 0$ of Figure 3, i.e., $-1/2r \leqslant \xi_{\min} \leqslant 0$, and that to the cycle ξ, for which $\zeta_{\min} < -1/2r$, correspond two minima of S of magnitude $-S_0/4r\xi_0$. Taking these remarks into account, we shall apply the formula for the distribution of a function of a random magnitude. Using (2.2), (3.3), (3.4), and (4.1), we obtain the following average number of minima smaller than S per unit time:

$$N(S_{\min} < S) = \frac{\omega_0}{2\pi J} \{\exp[-F_1(S)] + \exp(-F_0)\} \qquad (4.2)$$
$$(-S_0/4r\xi_0 \leqslant S_{\min} \leqslant 0)$$

and the following average number of maxima larger than S per unit time:

$$N(S_{\max} > S) = \frac{\omega_0}{2\pi J}\{\exp[-F_1(S)] + \exp[-F_2(S)]\} \quad (S>0),$$
$$N(S_{\max} > S) = \frac{\omega_0}{2\pi J}\{1 + \exp[-F_2(S)]\} \quad (-S_0/4r\xi_0 < S < 0). \qquad (4.3)$$

Here

$$F_{1,2}(S) = \frac{1 + 2r\xi_0 S/S_0 \mp \sqrt{1 + 4r\xi_0 S/S_0}}{4r^2\xi_0^2}\left[1 \pm \frac{\beta}{3r}\left(\sqrt{1 + 4r\xi_0 S/S_0} \mp 1\right) + \right.$$
$$\left. + \frac{\alpha}{4r^2}\left(1 + 2r\xi_0 S/S_0 \mp \sqrt{1 + 4r\xi_0 S/S_0}\right)\right],$$
$$F_0 = F_2(-S_0/4r\xi_0) = \frac{1}{8r^2\xi_0^2}\left(1 - \frac{\beta}{3r} + \frac{\alpha}{8r^2}\right). \qquad (4.4)$$

It is easily shown that at $\gamma < 2$ the function $F_2(S)$ is everywhere positive. Therefore at $\xi_0 r \ll 1/2$ the second terms in (4.2) and (4.3) are negligibly small. Differentiating these expressions, we obtain then the relationships for the densities of the stress extrema obtained earlier /2/.

5. The main part of fatigue damage appears frequently during the jump of the shell under the action of random disturbances from one stable position of equilibrium to another. A tentative estimate of the life in this case can be obtained by calculating the average number of such "snap-throughs" per unit time. In the case considered of a shallow shell at $\gamma > 2$ we assume that the number of "snap-throughs" is equal to the average number $n(\xi_1)$ of passages across the level ξ_1 corresponding to the unstable position of equilibrium. As seen from Figures 1 and 2, the representativeness of this characteristic of the life increases with the parameter m. In the limiting case of a narrow-banded process ($m \gg 1$) the function $n(\xi_1)$ coincides with the number of vibration cycles embracing both positions of stable equilibrium.

To determine $n(\xi_1)$ we shall make use of the well-known relationships of /1/, into which we substitute (2.8), (2.9), (2.2), and (3.5).

$$n(\xi_1) = 2p(\xi_1)\int_0^\infty \eta p_1(\eta)\, d\eta = \frac{\omega_0}{\pi J} e^{-u_1}, \qquad (5.1)$$

where

$$u_1 = \frac{1}{\xi_0^2}\left(\frac{\xi_1^2}{2} + \frac{\beta\xi_1^3}{3} + \frac{\alpha\xi_1^4}{4}\right) = \frac{1}{4\alpha\xi_0^2}\left\{\gamma^2 - 1 - \frac{\gamma}{6}\left[\gamma^3 - (\gamma^2 - 4)^{3/2}\right]\right\}. \qquad (5.2)$$

For the convenience of the calculation of u_1 we expand $(\gamma^2-4)^{3/2}$ in a

series in powers of $4/\gamma^2$. We obtain as a result

$$u_1 = \frac{1}{6\beta^2\xi_0^2}\left(1 + \frac{3}{2\gamma^2} + \frac{3}{\gamma^4} + \cdots\right). \tag{5.3}$$

Consider a cylindrical panel compressed by an axial load p. Using the results of /5/, it can be shown that the dependence of the panel parameters on the axial load is of the form

$$\beta = \beta_0\ (1 - p/p_*)^{-1}, \quad \omega_0 = \omega_{00}\ (1 - p/p_*)^{1/2},$$

$$\xi_0 = \xi_{00}\ (1 - p/p_*)^{-3/2}, \quad \gamma = \gamma_0\ (1 - p/p_*)^{-1/2}. \tag{5.4}$$

Here p_* is the upper critical load; the second subscript "0" corresponds to the absence of axial load. It can be seen from (5.1) and (5.3) that $n(\xi_1)$ increases with the axial load, passes through a maximum for a certain p_m, and then drops to zero as p approaches p_*. If we use in (5.3) one term of the expansion only, we obtain, by substituting (5.4) in (5.3) and (5.1) and using the condition $\partial n(\xi_1)/\partial p = 0$ the following value for p_m:

$$\frac{p_m}{p_*} = 1 - \left(\frac{6}{5}\beta_0^2\xi_{00}^2\right)^{\frac{2}{5}}. \tag{5.5}$$

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USE OF DIGITAL COMPUTERS IN THE CALCULATION OF SMOOTH AND RIBBED PLATES AND SHELLS, EITHER NOT STRESSED OR STRESSED BY HOLES

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(Kiev)

1. It was mentioned in /1/ and /2/ that the systems of equations obtained by the method of grids for calculating plates and shells are symmetrical and can be reduced to matrices of a three-cell or five-cell structure. In the case of problems involving plane stress of multiconnected regions these systems will be further bordered with additional columns and lines. The method of group elimination of the unknowns was used in the solution, by means of a digital computer, of systems of high-order difference equations bordered by one column and row.

This calculating method, which allows for the peculiar structure of the system of equations (the presence of a large number of cell zeroes and the matrix symmetry) turned out to be very efficient. Many simplifications were achieved by eliminating the groups of unknowns not through a preliminary calculation of the inverse matrices, but by a direct solution of the corresponding systems of equations with several columns of free terms. The cases treated were those of a rectangular region with a central rectangular hole and of a cylindrical shell weakened by concentric rectangular holes.

Proceeding further along this line, we have now programmed the method of group elimination of the unknowns, and have worked out two standard programs for the "BESM" computer, suitable for all types of problems relative to ribbed plates and shells. One of the programs corresponds to the case of symmetrical systems of equations of three-cell structure, either bordered by some additional columns and rows or unbordered; the second corresponds to systems of equations of five-cell structure. The systems of equations can be made not only of a definite number of cells of the same type, as in the problems treated earlier, but also of cells belonging to different types. Such types of systems of equations are the most general ones in the calculation of plates and shells.

The programs contain a special subprogram for the introduction of the input data —separate cells which are subsequently stored in a certain order in the computer memory. Only matrix elements different from zero are stored, while the zero elements, of which there is usually a large number, are dropped in order to reduce the bulk of the input data. Each cell must be accompanied by a scale (repeating cells — by the same scale) by means of which the law of alternation of nonzero and zero elements is coded. Such scales are formed very easily. After the different cells of the

system of difference equations have been formed, the solution of various problems by means of the programs necessitates no preparatory work. The following parameters are necessary for the calculation:

N is the order of the largest cell;

l_1 is the number of bordering columns;

l_2 is the number of columns of free terms;

s is the cell order of the matrix.

The following condition must be satisfied:

$$2N^2 + N(l_1 + l_2) \leqslant 1630$$

(if, for instance, $l_1 = 1$, $l_2 = 2$, then $N = 27$).

The computer program uses an external memory, such as a magnetic drum or a magnetic tape. In the first case the maximum order n of the system of equations which can be solved is given by the condition

$$n = \sum_{i=1}^{s} N_i,$$

$$S \leqslant \left[\frac{4710}{N^2 + N(l_1 + l_2) + 2} \right],$$

where N_i is the order of the i-th cell of the matrix. Thus, if $N_i = N = 20 (i = 1, 2, \cdots, s)$, $l_1 = 0$, $l_2 = 1$, then $s \leqslant 11$, whence $n \leqslant 220$.

In the second case the order is practically unlimited.

2. The following problems have been solved on the BESM computer by means of the standard programs:

a) a rectangular plate having a 3:2 ratio of sides with free short sides, one long side hinged and the other built-in (order of the system of equations $n = 110$);

b) a rectangular plate bordered by ribs — a panel of assembled reinforced concrete overlap with sides ratio 5:3, supported by the short sides ($n = 176$);

c) a smooth cylindrical shell of $l = 1.875\pi R$ with concentric rectangular holes subjected to longitudinal stresses distributed uniformly over the end section ($n = 139$);

d) the same shell under the action of longitudinal stresses, but with the condition that the end sections of the shell remain plane ($n = 143$);

e) a ribbed cylindrical shell subjected to internal pressure ($n = 168$).

3. The scope of this paper prevents us from examining all these problems. We shall accordingly consider the case of a rectangular plate bordered by ribs-beams only (Figure 1).

The boundary conditions for the rectangular end built-in in the bordering beam are

$$B \frac{\partial^4 w}{\partial s^4} = p_b^{(0)} + D \left[\frac{\partial^3 w}{\partial n^3} + (2 - \nu) \frac{\partial^3 w}{\partial n \partial s^2} \right], \tag{1}$$

$$C \frac{\partial^2 w}{\partial n \partial s^2} = D \left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial s^2} \right), \tag{2}$$

where B and C are the flexural and torsional rigidity of the beam, respectively, $p_b^{(0)}$ is the intensity of the external load per unit length applied to the beam.

In the x, y coordinate system expressions (1) and (2) have opposite signs for the opposite ends of the plate.

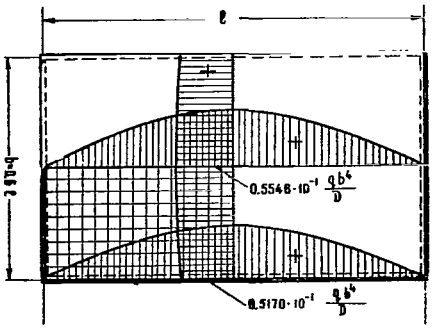

FIGURE 1.

The biharmonic equation in finite differences for the contour point ik (Figure 2) is, on the strength of condition (1):

$$(10+6\alpha)\ w_{ik} - 4\ (1+\alpha)\ (w_{i,\,k-1} + w_{i,\,k+1}) + (0.5+\alpha)\ (w_{i,\,k-2} + w_{i,\,k+2}) - 8\ w_{i-1,\,k} + 2\ (w_{i-1,\,k-1} + w_{i-1,\,k+1}) + w_{i-2,\,k} - K_{ik} + 0.5\ \nu\ (K_{i,\,k+1} - 2K_{ik} + K_{i,\,k-1}) - \frac{p_{b\,ik}^{(0)}\,h^3}{D} - \frac{P_{ik}\,h^2}{D} = 0, \tag{3}$$

where

$$\alpha = \frac{B}{Dh}, \tag{4}$$

$$K = 2h\,\frac{\partial w}{\partial n}, \tag{5}$$

P_{ik} is the load on the plate at node ik.

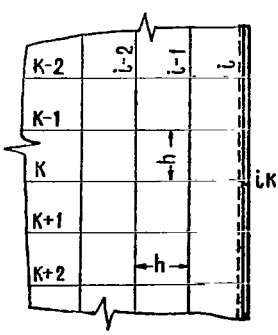

FIGURE 2.

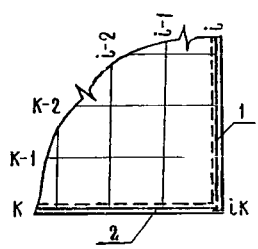

FIGURE 3.

The values of w for the first row of beyond-contour points are expressed by the extrapolation formula

$$w_{\text{beyond cont.}} = w_{\text{precont.}} + K_{\text{cont.}}, \tag{6}$$

so that the magnitudes K enter in the precontour equations as well. The

following equation is obtained for K_{ik} from condition (2):

$$w_{i-1,k} - (1+\nu)\, w_{ik} + 0.5\,\nu\,(w_{i,k+1} + w_{i,k-1}) + (0.5+\beta)\, K_{ik} - \\ -0.5\,\beta\,(K_{i,k+1} + K_{i,k-1}) = 0, \qquad (7)$$

where

$$\beta = \frac{C}{2Dh} \qquad (8)$$

Equation (7) reduces for a corner point where two bordering beams meet at a right angle (Figure 3) to

$$(0.5+\alpha_2)\, w_{i-1,k} - (0.5+0.5\,\nu+\alpha_2)\, w_{ik} + 0.5\nu\, w_{i,k-1} + \\ +(0.25+0.5\beta_1+0.5\,\alpha_2)\, K_{ik}^{(x)} - 0.5\beta_1\, K_{i,k-1} + 0.25\nu\, K_{ik}^{(y)} = 0. \qquad (9)$$

In this case the magnitude $K_{i,k+1}$, being beyond-contour, has been eliminated from the equations, and it has been taken into account that the torsional moment for beam 1 is equal to the bending moment for beam 2. It has also been taken into account that $w_{i,k+1} = w_{i,k-1} + K_{ik}^{(y)}$. The equation for $K_{ik}^{(y)}$ at a corner point is written by analogy with (9).

The plate represented in Figure 1 (bordered by beams) is supported on its short sides. The values of w at the internal nodes of the grid and at the points of the longitudinal edges are unknown, and so are the values of K at all contour points (two values of K correspond to each corner point). The total number of unknowns for a plate quadrant in the case of a load symmetrical relative to the plate axes is equal to 176 for the grid used (under the assumption of uniformly distributed load). The equations for w at the internal nodes are written in the usual way, and the equations for the contour unknowns by formulas (3), (7), and (9). The system of equations is represented in Table 1 in the form of a cellular matrix, the equations and unknowns being grouped in groups according to two horizontal rows of nodes. The matrices of the separate cells, with the exception of matrix Π_1 (the left half of which coincides with the left half of the matrix Π, while the right half contains only zeroes) and the matrices-columns of free terms (the free terms of the equations for w are equal to $P_{ik}\, h^2/D$), are given in Tables 2 to 5. The cells of type D of the main diagonal are represented by their symmetrical part only, as required by the program.

TABLE 1

m	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Free terms
1	D_0	Π							Δ_0
2	Π'	D	Π						Δ
3		Π	D	Π					Δ
4			Π'	D	Π				Δ
5				Π'	D	Π			Δ
6					Π'	D	Π		Δ
7						Π'	D	Π_1	Δ
8							Π_1'	D	Δ_1

TABLE 2

D_0

	$w_{0,0}$	$w_{1,0}$	$w_{2,0}$	.	.	$w_{8,0}$	$w_{9,0}$	$K_{9,0}$	$w_{0,1}$	$w_{1,1}$	$w_{2,1}$	.	.	$w_{8,1}$	$w_{9,1}$	$K_{9,1}$
1	5	−1	0.5						−4	2						
2		10.5	−4	0.5					2	−8	2					
3			10	−4	0.5					2	−8	2				
.				.	.	.					.	.	.			
8					10	−4	0.5					2	−8	2		
9						10.5	−4	0.5					2	−8	2	
10							$0.5c_1$	$0.5c_5$						2	c_2	c_6
11								$0.5c_7$							c_4	c_8
12									10.5	−8	1					
13										22	−8	1				
14											21	−8	1			
.												.	.	.		
19													21	−8	1	
20														22	−8	1
21															c_4	c_5
22																c_7

TABLE 3

D

	$w_{0,2(m-1)}$	$w_{1,2(m-1)}$	$w_{2,2(m-1)}$	.	.	$w_{8,2(m-1)}$	$w_{9,2(m-1)}$	$K_{9,2(m-1)}$	$w_{0,2m-1}$	$w_{1,2m-1}$	$w_{2,2m-1}$	.	.	$w_{8,2m-1}$	$w_{9,2m-1}$	$K_{9,2m-1}$
1	10	−8	1						−4	2						
2		21	−8	1					2	−8	2					
3			20	−8	1					2	−8	2				
.				.	.	.					.	.	.			
8					20	−8	1					2	−8	2		
9						21	−8	1					2	−8	2	
10							c_1	c_5						2	c_2	c_6
11								c_7							c_6	c_8
12									10	−8	1					
13										21	−8	1				
14											20	−8	1			
.												.	.	.		
19													20	−8	1	
20														21	−8	
21															c_1	c_5
22																c_7

TABLE 4

D_4

	$w_{0,14}$	$w_{1,14}$	$w_{2,14}$	.	.	$w_{8,14}$	$w_{9,14}$	$K_{9,14}$	$K_{0,15}$	$K_{1,15}$	$K_{2,15}$	.	.	$K_{8,15}$	$K_{9,15}^{(x)}$	$K_{9,15}^{(y)}$
1	10.5	−8	1						0.5							
2		22	−8	1						1						
3			21	−8	1						1					
.				.	.	.						.				
8					21	−8	1						1			
9						22	−8	1						1		
10							c_4	c_5							c_3	c_6
11								c_7								c_8
12									$0.5c_7$	c_8						
13										c_7	c_8					
14											c_7	c_8				
.												.	.	.		
19													c_7	c_8		
20														c_7	c_9	
21															c^6	c_{10}
22																c_9

TABLE 5

Π

	$w_{0,2m}$	$w_{1,2m}$	$w_{2,2m}$	.	.	$w_{8,2m}$	$w_{9,2m}$	$K_{9,2m}$	$w_{0,2m+1}$	$w_{1,2m+1}$	$w_{2,2m+1}$	.	.	$w_{8,2m+1}$	$w_{9,2m+1}$	$K_{9,2m+1}$
1	0.5															
2		1														
3			1													
.				.												
8					1											
9						1										
10							c_3									
11								—								
12	−4	2							0.5							
13	2	−8	2							1						
14		2	−8	2							1					
.				.	.							.				
19				2	−8	2							1			
20					2	−8	2							1		
21						2	c_2	c_6							c_3	
22							c_6	c_8								—

The symbols used in these tables have the following meaning:

$$c_1 = 10 + 6\alpha, \quad c_2 = -4 - 4\alpha, \quad c_3 = 0.5 + \alpha,$$
$$c_4 = c_1 + c_3 = 10.5 + 7\alpha, \quad c_5 = -(1 + \nu), \quad c_6 = 0.5\nu, \tag{10}$$
$$c_7 = 0.5 + \beta, \quad c_8 = -0.5\beta, \quad c_9 = 0.25 + 0.5\beta + 0.5\alpha,$$
$$c_{10} = 0.25\nu.$$

The calculation was conducted for $\alpha = 13.5$; $\beta = 2.9$; $\nu = 1/6$. Some deflection diagrams are given in Figure 1.

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A NONSTATIONARY TEMPERATURE PROBLEM RELATIVE TO A CYLINDRICAL SHELL STIFFENED BY A RING

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In the case of considerable temperature drops it frequently becomes necessary to allow for the variation of the physicomechanical properties of the material of structures and machine parts as a function of the heating temperature.

Structures such as thin-walled stiffened cylindrical shells working in a temperature field varying monotonically along the cylinder generator are finding an increasing application in engineering practice.

The present paper is devoted to the study of the state of stress and strain of structures of this type.

The problem is solved in a linear formulation in the frame of the generalized Hooke law, without allowing for the creep of the material.

§1. STATEMENT OF THE PROBLEM AND INITIAL ASSUMPTIONS

Consider, in a cylindrical system of coordinates x, θ, z (Figure 1), a thin orthotropic shell of revolution with uniform thickness h, whose material possesses cylindrical anisotropy of axis coinciding with the shell axis ox.

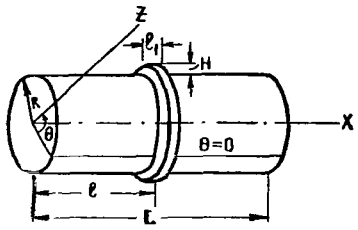

FIGURE 1.

The shell is assumed to be reinforced by a stiffener of constant cross section, and of physicomechanical properties different from those of the shell.

The law of variation of the temperature $T = T(x, t)$ along the cylinder generator and in time, and the dependence of the elastic constants of the shell material on the temperature are considered as known and given by monotonically varying functions in the strict sense. The temperature is assumed to be constant through the shell thickness.

It is assumed that the disturbances caused by the heat capacity of the annular stiffener do not have a considerable influence on the temperature distribution in the shell itself.

The following assumptions are used in the solution of the problem considered (/1/, /2/):

$$e_{xz} = 0, \quad e_z = \beta_z T, \quad \sigma_z = 0, \tag{1.1}$$

$$E_x = E_x(T), \quad E_\theta = E_\theta(T), \quad \mu_{\nu_j} = \mu_{\nu_j}(T) \tag{1.2}$$

$$\beta_i = a_i + b_i T \quad (i = x, \theta, z), \tag{1.3}$$

$$1 + \frac{h}{R} \approx 1, \tag{1.4}$$

where β_i, E_i are, respectively, the coefficients of temperature expansion and the moduli of elasticity along the coordinate lines i; $\mu_{\nu j}$ are the Poisson ratios.

§ 2. DERIVATION OF THE FUNDAMENTAL EQUATIONS

Using the generalized law of Hooke for the temperature problem /3/ and the well-known relationships for the relative strains (cf. /4/, p. 31), we obtain, allowing for (1.1), the following formulas for the stresses σ_x, σ_θ and the strains e_x, e_θ:

$$\sigma_x = B_{11}[e_x + \mu_{\theta x} e_\theta - (\beta_x + \mu_{\theta x}\beta_\theta) T],$$

$$\sigma_\theta = B_{11}\left[\mu_{\theta x} e_x + \frac{1}{k} e_\theta - \left(\mu_{\theta x}\beta_x + \frac{1}{k}\beta_\theta\right) T\right], \tag{2.1}$$

$$e_x = \varepsilon_1 + z\varkappa_1, \quad e_\theta = \varepsilon_2 + (a_z + b_z T)\frac{z}{R} T, \tag{2.2}$$

where /3/, /4/

$$B_{11} = \frac{E_x}{1 - k\mu_{\theta x}^2}, \quad k = \frac{\mu_{x\theta}}{\mu_{\theta x}},$$

$$\varepsilon_1 = \frac{du}{dx}, \quad \varepsilon_2 = \frac{w}{R}, \quad \varkappa_1 = -\frac{d\vartheta}{dx}, \tag{2.3}$$

u, w are displacements of the points of the middle surface along the generator and in the radial direction respectively, ϑ is one of Meissner's variables, defined in the case considered by the relationship

$$\vartheta = \frac{dw}{dx}$$

The substitution of (2.2) in (2.1) yields

$$\left.\begin{aligned} \sigma_x = B_{11}\Big[& \varepsilon_1 + z\varkappa_1 + \mu_{\theta x}\varepsilon_2 + \mu_{\theta x}(a_z + b_z T)\frac{z}{R} T - \\ & -(a_x + \mu_{\theta x} a_\theta) T - (b_x + \mu_{\theta x} b_\theta) T^2 \Big], \end{aligned}\right\} \tag{2.4}$$

$$\left.\begin{aligned}\sigma_\theta = B_{11}\Big[& \mu_{\theta x}\,\varepsilon_1 + \mu_{\theta x}\, z\varkappa_1 + \frac{1}{k}\,\varepsilon_2 + (a_z + b_z\,\mathrm{T})\frac{z}{kR}\,\mathrm{T} - \\ & -\left(\mu_{\theta x}\, a_x + \frac{a_\theta}{k}\right)\mathrm{T} - \left(\mu_{\theta x}\, b_x + \frac{b_\theta}{k}\right)\mathrm{T}^2\Big],\end{aligned}\right\} \quad \begin{matrix}(2.4)\\(\text{cont'd})\end{matrix}$$

$$\left.\begin{aligned}T_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\, dz = B_{11}\, h\, [\varepsilon_1 + \mu_{\theta x}\,\varepsilon_2 - (a_x + \mu_{\theta x}\, a_\theta)\,\mathrm{T} - (b_x + \mu_{\theta x} b_\theta)\,\mathrm{T}^2], \\ T_y &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta\, dz = B_{11}\, h\left[\mu_{\theta x}\,\varepsilon_1 + \frac{1}{k}\,\varepsilon_2 - \left(\mu_{\theta x} a_x + \frac{a_\theta}{k}\right)\mathrm{T} - \left(\mu_{\theta x} b_x + \frac{b_\theta}{k}\right)\mathrm{T}^2\right],\end{aligned}\right\} \quad (2.5)$$

$$\left.\begin{aligned}M_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz = \frac{B_{11}\, h^3}{12}\left[\varkappa_1 + \mu_{\theta x}\,(a_z + b_z\,\mathrm{T})\,\frac{\mathrm{T}}{R}\right], \\ M_\theta &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta z dz = \frac{B_{11} h^3}{12}\left[\mu_{\theta x}\,\varkappa_1 + (a_z + b_z\,\mathrm{T})\,\frac{\mathrm{T}}{kR}\right].\end{aligned}\right\} \quad (2.6)$$

Designating by X and Z the projections of the external load per unit area of the shell surface in the directions of the tangent to the meridian and the normal to the middle surface, we can write the equations of equilibrium of a shell element for the case considered in the form

$$R\frac{dN}{dx} - T_\theta = -RZ, \qquad \frac{dT_x}{dx} = -X, \qquad \frac{dM_x}{dx} - N = 0, \tag{2.7}$$

where N is the shearing force per unit circumferential length.

Introducing the second Meissner variable V, and writing /3/, /4/

$$T_\theta = \frac{dV}{dx}, \qquad RN = V - R\int_{x_0}^{x} Z dx, \qquad T_x = T_x^0 - \int_{x_0}^{x} X dx \tag{2.8}$$

(here $T_x^0 = C_0 = \text{const}$, cf. /1/, footnote on p. 129), we transform the first two equilibrium equations of (2.7) into identities.

Substituting (2.8) in (2.5) and solving the system of algebraic equations obtained relative to ε_1 and ε_2 yield

$$\begin{aligned}\varepsilon_1 &= -\frac{k\mu_{\theta x}}{E_x\, h}\,\frac{dV}{dx} - \frac{1}{E_x\, h}\int_{x_0}^{x} X\, dx + \frac{T_x^0}{E_x h} + a_x\,\mathrm{T} + b_x\,\mathrm{T}^2, \\ \varepsilon_2 &= \frac{k}{E_x\, h}\cdot\frac{dV}{dx} - \frac{k\mu_{\theta x}\,\mathrm{T}_x^0}{E_x\, h} + \frac{k\mu_{\theta x}}{E_x\, h}\int_{x_0}^{x} X\, dx + a_\theta\,\mathrm{T} + b_\theta\,\mathrm{T}^2.\end{aligned} \tag{2.9}$$

There remains to satisfy the third equilibrium equation (2.7), and the equation of strain compatibility /4/, /3/, which is represented in this case in the form

$$\frac{d\varepsilon_z}{dx} - \frac{\vartheta}{R} = 0. \tag{2.10}$$

Using (2.6) and (2.9), we obtain the following system of differential equations for the unknowns V and ϑ:

$$\begin{gathered}\frac{d}{dx}\left(B_{11}\frac{d\vartheta}{dx}\right) + \frac{12}{Rh^3}V = \frac{12}{h^3}\int_{x_0}^{x} Z dx + \frac{1}{R}\frac{d}{dx}\left[B_{11}\mu_{\theta x}(a_z + b_z \mathrm{T})\mathrm{T}\right],\\ \frac{d}{dx}\left(\frac{1}{E_x}\frac{dV}{dx}\right) - \frac{h}{kR}\vartheta = \left(T_x^0 - \int_{x_0}^{x} X dx\right)\frac{d}{dx}\left(\frac{\mu_{\theta x}}{E_x}\right) - \frac{\mu_{\theta x}}{E_x}X - \\ - \frac{h}{k}(a_\theta + 2b_\theta \mathrm{T})\frac{d\mathrm{T}}{dx}.\end{gathered} \tag{2.11}$$

This system is easily reduced to one fourth-order differential equation relative to V (or ϑ):

$$\frac{d}{dx}\left[B_{11}\frac{d^2}{dx^2}\left(\frac{1}{E_x}\frac{dV}{dx}\right)\right] + \frac{12}{kh^2R^2}V = \Phi\ (X,\ Z,\ \mathrm{T}). \tag{2.12}$$

The second unknown function ϑ (or V) is found from the second (or first) equation of system (2.11).

Equation (2.12) cannot be integrated in the general case. In particular cases (when the laws of variation of the physicomechanical properties of the material with the temperature, and of the temperature itself along the cylinder generator, are specified) it is possible to obtain approximate solutions of (2.12), and therefore of the problem of the stressed and strained state of a heated nonhomogeneous cylinder.

The study of (2.12) shows that the problem can be solved in a closed form by imposing some restrictions on the law of temperature distribution along the generator of the cylindrical shell and on the dependence of the physicomechanical properties of the shell material on the heating temperature.

We assume that the Poisson ratio is temperature-independent, and that the law of variation (along the cylinder generator) of Young's modulus can be approximated by the exponential function

$$E_x = Ee^{\alpha x}. \tag{2.13}$$

The following point should be noted relative to relationship (2.13). An analytical relationship between Young's modulus and the heating temperature can be obtained only by approximating experimental curves. Relationship (2.13) can be satisfied by suitably selecting the form of the approximating function at different laws of temperature distribution along the generator.

§ 3. SOLUTION OF THE PROBLEM OF AN UNSTIFFENED CYLINDRICAL SHELL

Assuming $\mu_{\theta x} = \mu = \text{const}$ and $E_x = Ee^{\alpha x}$ we can transform system (2.11) to the form

$$\frac{d^2\vartheta}{dx^2} + \alpha\frac{d\vartheta}{dx} + \frac{12(1-k\mu^2)}{Eh^3R}e^{-\alpha x}V =$$

$$= \frac{12(1-k\mu^2)}{Eh^3}e^{-\alpha x}\int_{x_0}^{x} Z\,dx + \frac{\alpha\mu}{R}(a_z + b_z\mathrm{T})\mathrm{T} + \frac{\mu}{R}(a_z + 2b_z\mathrm{T})\frac{d\mathrm{T}}{dx},$$

$$\frac{d^2V}{dx^2} - \alpha\frac{dV}{dx} - \frac{Eh}{kR}e^{\alpha x}\vartheta = -\mu\alpha\left(T_x^0 - \int_{x_0}^{x} X\,dx\right) - \mu X - \tag{3.1}$$

$$-\frac{Eh}{k}e^{\alpha x}(a_\theta + 2b_\theta\mathrm{T})\frac{d\mathrm{T}}{dx},$$

from which we obtain the following equations for determining the functions V and ϑ:

$$\frac{d^4V}{dx^4} - 2\alpha\frac{d^3V}{dx^3} + \alpha^2\frac{d^2V}{dx^2} + \frac{12(1-k\mu^2)}{kh^2R^2}V = \Phi_1(X, Z) + \Phi_2(\mathrm{T}), \tag{3.2}$$

$$\vartheta = \frac{kR}{Eh}e^{-\alpha x}\left[\frac{d^2V}{dx^2} - \alpha\frac{dV}{dx} + \mu\alpha\left(T_x^0 - \int_{x_0}^{x} X\,dx\right) + \mu X\right] +$$

$$+ R(a_\theta + 2b_\theta\mathrm{T})\frac{d\mathrm{T}}{dx}, \tag{3.3}$$

where

$$\Phi_1(X,Z) = \frac{12(1-k\mu^2)}{kh^2R}\int_{x_0}^{x} Z\,dx - \mu\alpha^2 X + 2\mu\alpha\frac{dX}{dx} - \mu\frac{d^2X}{dx^2},$$

$$\Phi_2(\mathrm{T}) = \frac{Eh}{kR^2}e^{\alpha x}\left[\alpha\mu(a_z + b_z\mathrm{T})\mathrm{T} + \mu(a_z + 2b_z\mathrm{T})\frac{d\mathrm{T}}{dx} -\right.$$

$$\left. -6R^2b_\theta\frac{d\mathrm{T}}{dx}\frac{d^2\mathrm{T}}{dx^2} - R^2(a_\theta + 2b_\theta\mathrm{T})\left(\alpha\frac{d^2\mathrm{T}}{dx^2} + \frac{d^3\mathrm{T}}{dx^3}\right) - 2R^2b_\theta\alpha\left(\frac{d\mathrm{T}}{dx}\right)^2\right].$$

The solution of the linear differential equation (3.2) can be represented in the form

$$V = V_0 + V_1 + V_2,$$

where V_0 is the general solution of the homogeneous equation

$$\frac{d^4V}{dx^4} - 2\alpha\frac{d^3V}{dx^3} + \alpha^2\frac{d^2V}{dx^2} + 12\frac{1-k\mu^2}{kh^2R^2}V = 0. \tag{3.4}$$

V_1 is a particular solution of the nonhomogeneous equation (3.2), corresponding to the case when its right-hand part consists of the load term $\Phi_1(X, Z)$ only, and V_2 is a particular solution of this same equation corresponding to the case when its right-hand part consists of the temperature term $\Phi_2(\mathrm{T})$ only.

The general solution V_0 of the homogeneous equation (3.4) is

$$V_0 = \sum_{j=1}^{4} C_j' e^{\rho_j x},$$

where C_j' are constants which remain to be determined, and ρ_j are roots of the characteristic equation

$$\rho^4 - 2\alpha\rho^3 + \alpha^2\rho^2 + \left(\frac{\alpha m}{2}\right)^4 = 0, \quad m^4 = \frac{192(1-k\mu^2)}{kh^2R^2\alpha^4}. \tag{3.5}$$

By solving (3.5) we obtain

$$\rho_{1,2,3,4} = \frac{\alpha}{2}\left[1 \pm (\delta_1 \pm i\delta_2)\right],$$

where

$$\delta_1 = \sqrt{\frac{\sqrt{1+m^4}+1}{2}}, \quad \delta_2 = \sqrt{\frac{\sqrt{1+m^4}-1}{2}}.$$

The general solution of (3.2) can thus be represented in the form

$$V = e^{\frac{\alpha x}{2}}\left[\left(C_1 \operatorname{sh}\frac{\alpha\delta_1 x}{2} + C_2 \operatorname{ch}\frac{\alpha\delta_1 x}{2}\right)\sin\frac{\alpha\delta_2 x}{2} + \left(C_3 \operatorname{sh}\frac{\alpha\delta_1 x}{2} + \right.\right.$$
$$\left.\left. + C_4 \operatorname{ch}\frac{\alpha\delta_1 x}{2}\right)\cos\frac{\alpha\delta_2 x}{2}\right] + V_1 + V_2. \tag{3.6}$$

The particular solutions V_1 and V_2 can be found for specific external loads and temperature functions.

Having determined V, we find ϑ from (3.3), and then from (2.9), (2.3), (2.4), (2.6), and (2.8) we obtain the expressions for the displacements u, w, the stresses σ_x, σ_θ, the forces per unit length and the moments T_x, T_θ, N, M_x, M_θ.

Note that, in addition to the four constants C_i $(i = 1, 2, 3, 4)$ entering in (3.6), there enters in the last expression of (2.8) a constant $T_x^0 = C_0$, and an additional constant C_5 appears in the determination of u from the first relationship of (2.3). The values of these constants are determined by the usual procedure as a function of the conditions at the cylinder ends.

§ 4. THE NONSTATIONARY TEMPERATURE PROBLEM FOR A STIFFENED CYLINDRICAL SHELL

The displacements, forces, moments, and stresses in the unstiffened shell will be considered in what follows as known, and will be designated by the superscript zero (i.e., w^0, ϑ^0, etc.).

The displacements, moments, and forces corresponding to the ring (stiffener) will be designated by the superscript "к" ($w^к$, $\vartheta^к$ etc.), and those corresponding to the shell stiffened by the ring by the superscript п ($w^п$, $\vartheta^п$, etc.).

By neglecting the effect of heat radiation by the stiffener /5/ in the formation of the equations of heat balance of the shell and stiffener, we obtain (cf. Figure 1)

$$\rho^{\kappa} c_p^{\kappa} l_1 H \frac{dT^{\kappa}}{dt} = \frac{\lambda^0 h}{\pi R} (T^0 - T^{\kappa}), \tag{4.1}$$

where ρ^{κ}, c_p^{κ}, T^{κ} are, respectively, the density, specific heat, and temperature of the ring (stiffener); λ^0 is the heat conductivity of the shell; T^0 is the shell temperature at section $x = l$.

We obtain from (4.1) the following equation for determining $T^{\kappa} = T^{\kappa}(t)$:

$$\int_{T_0^{\kappa}}^{T^{\kappa}} \frac{dT^{\kappa}}{T^0 - T^{\kappa}} = \omega t, \quad \omega = \frac{\lambda^0 h}{\pi R \rho^{\kappa} c_p^{\kappa} l_1 H}, \tag{4.2}$$

where T_0^{κ} is the temperature of the ring (stiffener) at the initial moment ($t = 0$). This equation can be integrated only after a specific relationship has been given for the dependence of the temperature on time.

In the particular case B = const and b = const we obtain from (4.2):

$$T^{\kappa} = T^0 - (T^0 - T_0^{\kappa})\, e^{-\omega t}. \tag{4.3}$$

We shall consider the interaction between the shell and ring.

Assume that the ribs are sufficiently thin and narrow. This gives us reason to consider that the contact forces and moments are uniformly distributed along a circumference lying in the middle plane of the rib, and also to use the approximation $R + \frac{h}{2} + \frac{H}{2} \approx R$.

The variation of the radius of curvature and slope of the rib cross sections relative to its axis under the action of uniformly distributed forces per unit length and moments P and M /6/, /7/, and of the heating $T^{\kappa} = T^{\kappa}(t)$, are then determined by the following formulas (cf. Figure 2):

$$w^{\kappa} = \frac{PR^2}{E\, l_1 H} + R \beta^{\kappa} T^{\kappa}, \qquad \vartheta^{\kappa} = \frac{MR^2}{E^{\kappa} J_z^{\kappa}}, \tag{4.4}$$

where β^{κ}, E^{κ} are, respectively, the coefficient of temperature expansion and the modulus of elasticity of the ring material; J_z^{κ} is moment of inertia of the rib cross section relative to the radial axis z'.

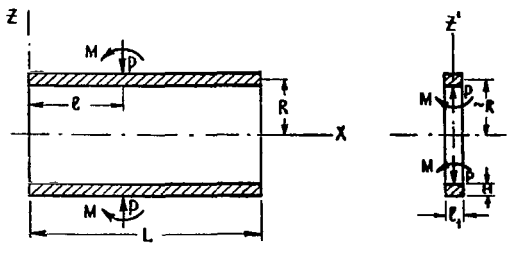

FIGURE 2.

The state of stress and strain of a shell nonhomogeneous along its generator and subjected to the action of forces and moments distributed uniformly along the circumference of the cross section (Figure 2) can be obtained by superposing the solutions of the two homogeneous problems* represented schematically on Figures 3 and 4, while fulfilling the conditions:

a) the conditions at the ends of the shell represented in Figure 2 are fulfilled at the ends $x = 0$ (Figure 3) and $x = L$ (Figure 4);

b) $N_1 - N_2 = -P, \quad M_1 - M_2 = -M;$

c)** $w_r^0 = w_l^0, \quad \vartheta_r^0 = \vartheta_l^0$

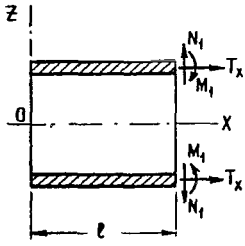

FIGURE 3

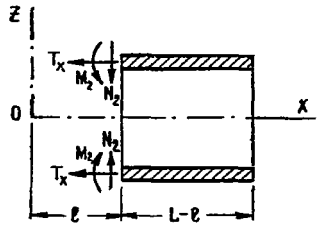

FIGURE 4.

All the magnitudes $(\bar{\sigma}_x, \bar{\sigma}_\theta, \bar{T}_x, \bar{T}_\theta, \bar{N}, \bar{M}_x, \bar{M}_\theta, \bar{u}, \bar{w})$† of the shell represented in Figure 2 are thus found as a function of x, P, M.

We obtain in particular

$$\bar{w} = \bar{w}\ (x,\ P,\ M), \qquad \bar{\vartheta} = \bar{\vartheta}\ (x,\ P,\ M). \tag{4.5}$$

By substituting $x = l$ in (4.5), we obtain from (4.4) and the conditions of compatibility of the shell and stiffener the following system of two equations:

$$\frac{PR^2}{E^\kappa l_1 H} + R\beta^\kappa T^\kappa = \bar{w}\ (P_1\ M),$$

$$\frac{MR^2}{E^\kappa J_z^\kappa} = \bar{\vartheta}\ (P_1\ M).$$

By solving it we find the unknowns P and M, which ends the solution of the problem of the state of stress and strain of a nonhomogeneous shell subjected to the load represented in Figure 2.

Turning back to the problem of a heated stiffened cylindrical shell, we obtain finally

$$w^n = w^0 + \bar{w}, \quad \vartheta^0 = \vartheta^0 + \bar{\vartheta}$$

etc., where, as mentioned above, the superscript zero designates the results obtained in § 3.

Note that even in the case B = const, b = const the stresses, strains, internal forces, and moments will be functions of time, due to the disturbances introduced by the heated ring (stiffener).

* The solution of the homogeneous problem is easily obtained from the results of § 3, by writing in the final formulas $V_1 = V_2 = 0$.

** The subscript "r" corresponds to the right end of the shell, represented in Figure 3, and the subscript "l" to the left end, represented in Figure 4.

† The magnitudes corresponding to the shell of Figure 2 are represented by a bar above the corresponding letter.

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THE TWO-DIMENSIONAL PROBLEM FOR A HELICOIDAL SHELL

V.K. Zalesskii

(Khar'kov)

1. Consider a shell of uniform thickness and helicoidal middle surface. The helicoid equations in dimensionless coordinates are

$$\left.\begin{aligned} x &= \operatorname{sh}(\alpha+\beta)\cos(\alpha-\beta), \\ y &= \operatorname{sh}(\alpha+\beta)\sin(\alpha-\beta), \\ z &= \alpha-\beta, \end{aligned}\right\} \tag{1}$$

where α and β are curvilinear coordinates coinciding with the lines of curvature of the middle surface of the shell.

We shall write the fundamental equations of the theory of thin shells /1/, /2/, /3/, substituting in them the geometrical parameters of the helicoid (1). Transferring to the right the terms of the equations of equilibrium containing the shearing forces and the load components, we obtain

$$\frac{\partial T_1}{\partial\alpha} - \frac{\partial S_2}{\partial\beta} + \operatorname{th}(\alpha+\beta)(T_1-T_2) + \operatorname{th}(\alpha+\beta)(S_1-S_2) = \frac{N_1}{\sqrt{2}\operatorname{ch}(\alpha+\beta)} - X\sqrt{2}\operatorname{ch}(\alpha+\beta), \tag{2}$$

$$\frac{\partial S_1}{\partial\alpha} + \frac{\partial T_2}{\partial\beta} + \operatorname{th}(\alpha+\beta)(T_2-T_1) + \operatorname{th}(\alpha+\beta)(S_1-S_2) = \frac{N_2}{\sqrt{2}\operatorname{ch}(\alpha+\beta)} - Y\sqrt{2}\operatorname{ch}(\alpha+\beta), \tag{3}$$

$$T_2 - T_1 = \frac{1}{\sqrt{2}}\left\{\frac{\partial}{\partial\alpha}[N_1\operatorname{ch}(\alpha+\beta)] + \frac{\partial}{\partial\beta}[N_2\operatorname{ch}(\alpha+\beta)]\right\} + Z\operatorname{ch}^2(\alpha+\beta), \tag{4}$$

$$\frac{\partial H_1}{\partial\alpha} - \frac{\partial G_2}{\partial\beta} + \operatorname{th}(\alpha+\beta)(G_1-G_2) + \operatorname{th}(\alpha+\beta)(H_1-H_2) = -N_2\sqrt{2}\operatorname{ch}(\alpha+\beta), \tag{5}$$

$$\frac{\partial G_1}{\partial\alpha} + \frac{\partial H_2}{\partial\beta} + \operatorname{th}(\alpha+\beta)(G_1-G_2) + \operatorname{th}(\alpha+\beta)(H_2-H_1) = N_1\sqrt{2}\operatorname{ch}(\alpha+\beta), \tag{6}$$

where the components X, Y, and Z of the distributed load are given functions of the variables α and β.

We assume that the load is sufficiently smooth and that there are no distributed moments or concentrated interactions.

The sixth equilibrium equation is an identity /1/, /3/, and is discarded as usual.

We write the equations of compatibility of strains in forces per unit length and moments in Novozhilov's form /3/:

$$\frac{\partial}{\partial\alpha}(G_1+G_2)-\sqrt{2}(1+\sigma)\,\mathrm{ch}\,(\alpha+\beta)\,N_1=0, \tag{7}$$

$$\frac{\partial}{\partial\beta}(G_1+G_2)-\sqrt{2}(1+\sigma)\,\mathrm{ch}\,(\alpha+\beta)\,N_2=0, \tag{8}$$

$$G_1-G_2=\frac{h^2}{24\,(1+\sigma)}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)(T_1+T_2)+$$
$$+\frac{h^2}{12\sqrt{2}}\left\{\frac{\partial}{\partial\alpha}[X\mathrm{ch}\,(\alpha+\beta)]+\frac{\partial}{\partial\beta}[Y\,\mathrm{ch}\,(\alpha+\beta)]\right\}. \tag{9}$$

The equations of elasticity will be written in the simplest form /2/, /3/:

$$T_1=\frac{2Eh}{1-\sigma^2}(\varepsilon_1+\sigma\varepsilon_2),\quad T_2=\frac{2Eh}{1-\sigma^2}(\varepsilon_2+\sigma\,\varepsilon_1), \tag{10}$$

$$S_1=-S_2=S=\frac{Eh}{1+\sigma}\,\omega, \tag{11}$$

$$G_1=-\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_1+\sigma\,\varkappa_2),\quad G_2=-\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_2+\sigma\varkappa_1), \tag{12}$$

$$H_1=-H_2=H=\frac{2Eh^3}{3\,(1+\sigma)}\,\tau. \tag{13}$$

We shall also need the following formulas connecting the displacements with the strains:

$$\varepsilon_1=\frac{1}{\sqrt{2}\,\mathrm{ch}(\alpha+\beta)}\left[\frac{\partial u}{\partial\alpha}+\mathrm{th}\,(\alpha+\beta)\,v+\frac{\sqrt{2}}{\mathrm{ch}\,(\alpha+\beta)}\,w\right], \tag{14}$$

$$\varepsilon_2=\frac{1}{\sqrt{2}\,\mathrm{ch}(\alpha+\beta)}\left[\frac{\partial v}{\partial\beta}+\mathrm{th}\,(\alpha+\beta)\,u-\frac{\sqrt{2}}{\mathrm{ch}\,(\alpha+\beta)}\,w\right], \tag{15}$$

$$\omega=\frac{1}{\sqrt{2}\,\mathrm{ch}(\alpha+\beta)}\left[\frac{\partial v}{\partial\alpha}+\frac{\partial u}{\partial\beta}-\mathrm{th}\,(\alpha+\beta)\,(u+v)\right], \tag{16}$$

$$\varkappa_1=L_1,\quad \varkappa_2=L_2,\quad \tau=L_3, \tag{17}$$

where L_1, L_2, L_3 represent the linear differential operators of the components u, v, w of the displacement vector.

2. Consider the bending of a helicoidal shell. On the strength of the analysis of the equations of compatibility given in /4/ and /5/, it is possible in several cases of considerable practical importance (when the shearing components of the load and the forces per unit length on the contour bounding the part of the shell considered are small) to neglect the right-hand side of equation (9). We obtain:

$$G_1=G_2=G. \tag{18}$$

The first and second compatibility equations (7) and (8) are then written as follows:

$$N_1=\frac{\sqrt{2}}{(1+\sigma)\,\mathrm{ch}\,(\alpha+\beta)}\frac{\partial G}{\partial\alpha},\quad N_2=\frac{\sqrt{2}}{(1+\sigma)\,\mathrm{ch}\,(\alpha+\beta)}\frac{\partial G}{\partial\beta}. \tag{19}$$

We shall determine the moments, forces per unit length, and displacements characterizing the state of stress and strain of the helicoidal shell considered under these assumptions.

3. Determine the moments G, H and the shearing forces N_1, N_2. By simplifying the equilibrium equations (5) and (6) with the aid of the compatibility equations (18) and (19), and the elasticity equation (13), we obtain:

$$\lambda \frac{\partial G}{\partial \alpha} + \frac{\partial H}{\partial \beta} + 2\mathrm{th}(\alpha + \beta)\, H = 0, \tag{20}$$

$$\lambda \frac{\partial G}{\partial \beta} + \frac{\partial H}{\partial \alpha} + 2\mathrm{th}(\alpha + \beta)\, H = 0, \tag{21}$$

where

$$\lambda = \frac{1 - \sigma}{1 + \sigma}. \tag{22}$$

Subtracting (20) from (21) and passing to new variables ξ and η according to the formulas

$$\xi = \alpha + \beta, \quad \eta = \alpha - \beta, \tag{23}$$

we obtain after integration by η

$$H - \lambda G = f_1(\xi), \tag{24}$$

where $f_1(\xi)$ is an arbitrary integration function.

Using (24), we can write (21) in the form

$$\lambda \frac{\partial G}{\partial \beta} + \frac{\partial H}{\partial \alpha} + 2\,\mathrm{th}(\alpha + \beta)\, \lambda G = -2\mathrm{th}(\alpha + \beta)\, f_1(\xi). \tag{25}$$

By adding (20) and (25) and passing to the variables ξ and η according to (23) we obtain after integration by ξ

$$H + \lambda G = \frac{1}{\mathrm{ch}\xi}\left[f_2(\eta) - \int \mathrm{sh}\xi\, f_1(\xi)\, d\xi\right], \tag{26}$$

where $f_2(\eta)$ is an arbitrary integration function.

By solving the system of finite linear differential equations (24) and (26) simultaneously we obtain the unknown moments G and H. By substituting the bending moment G in (19) we obtain the shearing forces N_1 and N_2.

4. We shall now determine the forces per unit length T_1, T_2, and S. By substituting in the equilibrium equations (2), (3), and (4) the known values of the shearing forces N_1, N_2 by (19), eliminating T_2 from (2) and (3) with the aid of (4), and using the elasticity equation (11), we obtain the system of equations

$$\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} + 2\mathrm{th}(\alpha + \beta)\, S = \varphi(\alpha, \beta), \tag{27}$$

$$\frac{\partial S}{\partial \alpha} + \frac{\partial T_1}{\partial \beta} + 2\mathrm{th}(\alpha + \beta)\, S = \psi(\alpha, \beta), \tag{28}$$

where $\varphi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ are known functions of the variables α and β:

$$\varphi(\alpha, \beta) = \operatorname{ch}(\alpha+\beta)\,[\operatorname{sh}(\alpha+\beta)\,Z - \sqrt{2}\,X] + \frac{N_1}{\sqrt{2}\operatorname{ch}(\alpha+\beta)} + \frac{\operatorname{th}(\alpha+\beta)}{\sqrt{2}}\left\{\frac{\partial}{\partial\alpha}[N_1 \operatorname{ch}(\alpha+\beta)] + \frac{\partial}{\partial\beta}[N_2 \operatorname{ch}(\alpha+\beta)]\right\}, \tag{29}$$

$$\psi(\alpha, \beta) = -\operatorname{ch}(\alpha+\beta)\,[\operatorname{sh}(\alpha+\beta)\,Z + \sqrt{2}\,Y] + \frac{N_2}{\sqrt{2}\operatorname{ch}(\alpha+\beta)} - \frac{\operatorname{th}(\alpha+\beta)}{\sqrt{2}}\left\{\frac{\partial}{\partial\alpha}[N_1 \operatorname{ch}(\alpha+\beta)] + \frac{\partial}{\partial\beta}[N_2 \operatorname{ch}(\alpha+\beta)]\right\}. \tag{30}$$

By applying to the system of equations (27) and (28) the scheme of integration of the system of equations (20) and (21) and passing to the variables (23) we obtain

$$T_1 - S = \int F_1(\xi, \eta)\, d\eta + f_3(\xi), \tag{31}$$

where

$$F_1(\xi, \eta) = \frac{1}{2}\left[\varphi\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) - \psi\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)\right], \tag{32}$$

and $f_3(\xi)$ is an arbitrary integration function;

$$T_1 + S = \frac{1}{\operatorname{ch}\xi}\left[\int F_2(\xi, \eta) \operatorname{ch}\xi\, d\xi + f_4(\eta)\right], \tag{33}$$

where

$$F_2(\xi, \eta) = \frac{1}{2}\left[\varphi\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) + \psi\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)\right] + \operatorname{th}\xi\left[\int F_1(\xi, \eta)\, d\eta + f_3(\xi)\right], \tag{34}$$

and $f_4(\eta)$ is an arbitrary integration function.

By solving the system of equations (4), (31), and (33) simultaneously we determine the unknown functions T_1, T_2, S.

5. We proceed to the determination of the displacement vector components u, (2). Having the forces per unit length T_1, T_2, S, we calculate from the elasticity equations (10) and (11) the values of ε_1, ε_2, ω, and substitute them in equations (14), (15), and (16). We obtain the system

$$\frac{\partial u}{\partial\alpha} + \operatorname{th}(\alpha+\beta)\, v + \frac{\sqrt{2}}{\operatorname{ch}(\alpha+\beta)}\, w = \Phi_1(\alpha, \beta), \tag{35}$$

$$\frac{\partial v}{\partial\beta} + \operatorname{th}(\alpha+\beta)\, u - \frac{\sqrt{2}}{\operatorname{ch}(\alpha+\beta)}\, w = \Phi_2(\alpha, \beta), \tag{36}$$

$$\frac{\partial v}{\partial\alpha} + \frac{\partial u}{\partial\beta} - \operatorname{th}(\alpha+\beta)(u+v) = \Phi_3(\alpha, \beta), \tag{37}$$

where

$$\Phi_1(\alpha, \beta) = \frac{\sqrt{2}}{Eh}(T_1 - \sigma T_2)\operatorname{ch}(\alpha+\beta), \tag{38}$$

$$\Phi_2(\alpha, \beta) = \frac{\sqrt{2}}{Eh}(T_2 - \sigma T_1)\,\mathrm{ch}(\alpha+\beta), \tag{39}$$

$$\Phi_3(\alpha, \beta) = \frac{2\sqrt{2}}{Eh}(1+\sigma)\,S\,\mathrm{ch}(\alpha+\beta), \tag{40}$$

are known functions of the variables α and β.

By adding equations (35), (36), and (37) and passing to the new variables ξ and η defined by (23) we obtain after integration by ξ:

$$u+v = \int F_3(\xi, \eta)\,d\xi + f_5(\eta), \tag{41}$$

where

$$F_3(\xi, \eta) = \frac{1}{2}\left[\Phi_1\left(\frac{\xi+\eta}{2}, \frac{\zeta-\eta}{2}\right) + \right.$$
$$\left. + \Phi_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) + \Phi_3\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)\right], \tag{42}$$

and $f_5(\eta)$ is an arbitrary integration function.

By adding equations (35) and (36) and subtracting from their sum equation (37), passing to new variables according to formulas (23) and (41), and integrating the result of all these operations by η, we obtain

$$u - v = \int F_4(\xi, \eta)\,d\eta + f_6(\xi), \tag{43}$$

where

$$F_4(\xi, \eta) = -\,\mathrm{th}\,\xi\left[\int F_3(\xi, \eta)\,d\xi + f_5(\eta)\right] +$$
$$+ \frac{1}{2}\left[\Phi_1\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) + \Phi_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) - \Phi_3\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)\right], \tag{44}$$

and $f_6(\xi)$ is an arbitrary integration function.

The displacement vector components u, v, w are determined from the system of equations (35), (41), and (43).

A general solution to the problem formulated has thus been obtained. It contains six arbitrary integration functions, which have to be determined from the boundary conditions.

6. If we assume that the right-hand parts of (35), (36), and (37) are equal to zero, i.e.,

$$\Phi_1(\alpha, \beta) = \Phi_2(\alpha, \beta) = \Phi_3(\alpha, \beta) = 0, \tag{45}$$

from which it follows that

$$\varepsilon_1 = \varepsilon_2 = \omega = 0, \tag{46}$$

and

$$T_1 = T_2 = S = 0, \tag{47}$$

this will correspond to a deformation of the middle surface of the shell without elongations or shears, i.e., to a pure moment state of stress; the components of the displacement vector for this case will be, according to (35), (41), and (43),

$$u=\frac{1}{2}\left[A(\eta)-\operatorname{th}\xi\int A(\eta)\,d\eta+B(\xi)\right], \tag{48}$$

$$v=\frac{1}{2}\left[A(\eta)+\operatorname{th}\xi\int A(\eta)\,d\eta-B(\xi)\right], \tag{49}$$

$$w=\frac{1}{2}\left\{\left[A'(\eta)+B'(\xi)\right]\operatorname{ch}\xi+\frac{\operatorname{sh}^2\xi+1}{\operatorname{ch}\xi}\int A(\eta)\,d\eta-B(\xi)\operatorname{sh}\xi\right\}, \tag{50}$$

where $A(\eta)$ and $B(\xi)$ are arbitrary integration functions.

With u, v, w known, the relative variations of the curvature and torsion $\varkappa_1, \varkappa_2, \tau$ are determined by (17), and the bending and torsional moments G_1, G_2, H_1, H_2 are calculated by (12) and (13).

7. If we write in the equilibrium equations (2), (3), (4), (5), and (6)

$$H_1=H_2=G_1=G_2=N_1=N_2=0 \tag{51}$$

and make use of the elasticity relationships (11), we obtain a system of equations of equilibrium of the momentless theory of shells, which coincides, after the elimination of T_2, with the system of equations (27) and (28), the difference being that the functions φ and ψ will contain only the terms underlined in the right-hand parts of (29) and (30).

The forces T_1, T_2, S per unit length can be found from the system of equations (4), (31), and (33), in which the conditions (51) are allowed for. The displacement vector components u, v, w are determined from the system of equations (35), (41), and (43), with the values of T_1, T_2, S obtained from the membrane problem considered substituted in the functions $\Phi_1(\alpha, \beta)$, $\Phi_2(\alpha, \beta)$, $\Phi_3(\alpha, \beta)$.

The displacements u, v, w calculated thus will contain as additional components the displacements of the pure moment state of stress (48), (49), (50), which must be discarded in order to obtain the pure membrane state of strain.

In this case the equilibrium equations (5) and (6) and the equations of strain compatibility (7), (8), and (9) drop out, while the load components X, Y, Z and the forces per unit length T_1, T_2, S will be subjected to the general conditions of existence of the membrane state of stress /1/, /3/. The assumption that X, Y and T_1, T_2, S are small drops out.

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DETERMINING GREEN'S FUNCTION FOR A PLATE OF VARIABLE THICKNESS, RECTANGULAR IN THE PLAN

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Plates of variable thickness are widely used in engineering structures of the most diverse types, such as bunkers, reinforced-concrete breast walls, rectangular reservoirs, buttress dams, etc. In ship design, the ship bottom is frequently considered as a complex plate of variable thickness. The skin of airplanes is likewise considered as a plate of variable thickness.

One of the basic methods now being used in structural mechanics for calculating rectangular plates of variable thickness is the method of grids, developed in the works of the Soviet scientists Dinnik, Panov, Varvak, Mikeladze, etc. The difference method requires protracted calculations, but is characterized by its remarkable simplicity.

The shortcomings of the grid method, as of all numerical methods in general, are that in many cases it is impossible to obtain the expression of the unknown function in an analytical form, and that the calculations are difficult (this is being gradually eliminated with the introduction of high-speed computers). The method of grids necessitates considering points outside the contour, which are treated very negligently. In many cases the increase in the number of equations does not lead to an increase in accuracy, due to the accumulation of errors.

We present here a method for determining Green's function for a plate of variable thickness rectangular in the plan, based on the application of the reciprocal theorem.

Using the reciprocal theorem it is possible to determine Green's function with a singularity enclosed in a function known beforehand. This function is the deflection of a rectangular plate of uniform thickness.

Consider the problem of the equilibrium of a rectangular plate of variable thickness (cf. the figure). Let a unit concentrated force be applied at some point $N(x', y')$ of the plate.

In other words, the load density $q(x, y)$ is equal to zero at all points $M(x, y)$ different from $N(x', y')$. The following condition must be satisfied in the vicinity of point N itself:

$$\lim_{\omega \to 0} \iint_{(\omega)} q(\rho, \varphi)\, \rho d\rho d\varphi = 1. \qquad (1)$$

Select two systems of loads and displacements. Let the fundamental system of loads correspond to the problem of the equilibrium of a plate

of variable thickness rectangular in the plan. Designate by $w(x, y, x', y')$ the fundamental displacement $M(x, y)$ of a plate of variable thickness under the action of a unit concentrated force applied at point $N(x', y')$.

As auxiliary displacement we select the static deflection $v(x, y, x_1, y_1)$ of a rectangular plate of uniform thickness under the action of a unit concentrated force applied at point $L(x_1, y_1)$.

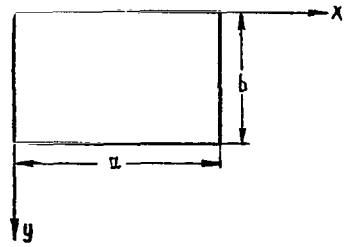

FIGURE.

The fundamental displacement corresponds to load (1). We shall now determine the auxiliary system of loads. To that end, the displacement $v(x, y, x_1, y_1)$, which satisfies the differential equation of static bending of a plate of uniform thickness

$$D_1 \Delta\Delta v = q_1, \tag{2}$$

is substituted in the differential equation of static bending of a rectangular plate of variable thickness

$$D\Delta\Delta w + 2\frac{\partial D}{\partial x}\cdot\frac{\partial}{\partial x}\Delta w + 2\frac{\partial D}{\partial y}\cdot\frac{\partial}{\partial y}\Delta w + \Delta D\Delta w -$$

$$-(1-\mu)\left(\frac{\partial^2 D}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2}\right) = q, \tag{3}$$

where w is the deflection of the middle plane of the rectangular plate of variable thickness. We obtain

$$D\Delta\Delta v + 2\frac{\partial D}{\partial x}\cdot\frac{\partial}{\partial x}\Delta v + 2\frac{\partial D}{\partial y}\cdot\frac{\partial}{\partial y}\Delta v + \Delta D\Delta v -$$

$$-(1-\mu)\left(\frac{\partial^2 D}{\partial x^2}\cdot\frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\cdot\frac{\partial^2 v}{\partial x^2}\right) = q. \tag{4}$$

The following condition is satisfied in the vicinity of the point $L(x_1, y_1)$ of application of the unit concentrated force on a plate of uniform thickness:

$$\lim_{\omega_1\to 0}\iint\limits_{(\omega_1)} q_1(\rho, \varphi)\,\rho d\rho d\varphi = 1. \tag{5}$$

We have from (2) $\Delta\Delta v = \frac{q_1}{D_1}$.

From (2) and (5) we obtain

$$\lim_{\omega_1\to 0}\iint\limits_{(\omega_1)} \Delta\Delta v\rho d\rho d\varphi = \frac{1}{D_1}, \tag{6}$$

whence

$$\lim_{\omega_1 \to 0} \iint_{(\omega_1)} D\Delta\Delta v \rho d\rho d\varphi = \frac{D_L(x_1, y_1)}{D_1}, \tag{7}$$

where $D_L(x_1, y_1)$ is the value of the flexural rigidity of the plate of variable thickness at point $L(x_1, y_1)$.

The auxiliary system of loads consists thus of the concentrated force

$$P = \frac{D_L(x_1, y_1)}{D_1}, \tag{8}$$

applied at point $L(x_1, y_1)$ and of a distributed load of intensity

$$2\frac{\partial D}{\partial x} \cdot \frac{\partial}{\partial x} \Delta v + 2\frac{\partial D}{\partial y} \cdot \frac{\partial}{\partial y} \Delta v + \Delta D \Delta v -$$
$$-(1-\mu)\left(\frac{\partial^2 D}{\partial x^2} \cdot \frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y} \cdot \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2} \cdot \frac{\partial^2 v}{\partial x^2}\right). \tag{9}$$

The application of the reciprocal theorem to the fundamental and auxiliary systems of loads and displacements yields

$$w(x_1, y_1, x', y') \frac{D_L(x_1, y_1)}{D_1} = v(x', y', x_1, y_1) -$$
$$- \iint_{(s)} w(x, y, x', y') \left\{ 2\frac{\partial D}{\partial x} \cdot \frac{\partial}{\partial x} \Delta v + 2\frac{\partial D}{\partial y} \cdot \frac{\partial}{\partial y} \Delta v + \right.$$
$$+ \Delta D \Delta v - (1-\mu)\left(\frac{\partial^2 D}{\partial x^2} \cdot \frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y} \cdot \frac{\partial^2 v}{\partial x \partial y} + \right.$$
$$\left. \left. + \frac{\partial^2 D}{\partial y^2} \cdot \frac{\partial^2 v}{\partial x^2} \right) \right\} dx dy - A, \tag{10}$$

where A is the work of the additional forces and moments applied to the contour of the middle surface of the plate (which depend on the auxiliary displacements) and performed during the actual displacements of the contour points of the middle surface of the plate of variable thickness. The work done in the auxiliary displacements by the forces and moments appearing under the action of the fundamental system of loads is usually equal to zero. In fact, it is always possible to select identical boundary conditions for the plates of uniform and variable thickness. The form of the work A depends on the specific boundary conditions of the problem.

Consider the case of a simply supported rectangular plate of variable thickness. We select as auxiliary state a simply supported rectangular plate of uniform thickness. We then obtain an integral equation for determining the deflection of a rectangular plate of variable thickness.

Let the plate thickness vary in the direction of the coordinate y. The equation for the deflection of a rectangular plate of variable thickness will then be

$$w(x_1, y_1, x', y') \frac{D_L(y_1)}{D_1} = v(x', y', x_1, y_1) -$$
$$- \iint_{(s)} w(x, y, x', y') \left\{ 2\frac{\partial D}{\partial y} \cdot \frac{\partial}{\partial y} \Delta v(x, y, x_1, y_1) + \right.$$
$$\left. + \frac{\partial^2 D}{\partial y^2} \Delta v(x, y, x_1, y_1) - (1-\mu) \right) \frac{\partial^2 D}{\partial y^2} \cdot \frac{\partial^2 v(x, y, x_1, y_1)}{\partial x^2} \Bigg\} dx dy. \tag{11}$$

Consider the kernel of this integral equation:

$$K(x, y, x_1, y_1) = 2\frac{\partial D}{\partial y} \cdot \frac{\partial}{\partial y} \Delta v(x, y, x_1, y_1) +$$

$$+ \frac{\partial^2 D}{\partial y^2} \Delta v(x, y, x_1, y_1) - (1-\mu) \frac{\partial^2 D}{\partial y^2} \cdot \frac{\partial^2 v(x, y, x_1, y_1)}{\partial x^2}. \qquad (12)$$

Using Dzhanelidze's papers /4/ and /5/, where Navier's solution is obtained by means of theta-functions, we obtain the following expression for the kernel:

$$k(x, y, x_1, y_1) = \mathrm{Re}\left\{ f_1(y) \ln \frac{\vartheta_1\left[\frac{i}{2b}(\bar z_1 + \bar z)\right] \cdot \vartheta_1\left[\frac{i}{2b}(\bar z_1 - \bar z)\right]}{\vartheta_1\left[\frac{i}{2b}(\bar z_1 + z)\right] \cdot \vartheta_1\left[\frac{i}{2b}(\bar z_1 - z)\right]} + \right.$$

$$+ f_2(x, y) \frac{\vartheta_1'}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 + \bar z)\right] - f_3(x, y) \frac{\vartheta_1'}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 - \bar z)\right] +$$

$$+ f_4(x, y) \frac{\vartheta_1'}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 + z)\right] - f_5(x, y) \frac{\vartheta_1'}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 - z)\right] -$$

$$- f_6(y)\left(\frac{\vartheta_1''}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 + \bar z)\right] + \frac{\vartheta_1''}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 - \bar z)\right] - \right.$$

$$\left.\left. - \frac{\vartheta_1''}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 + z)\right] - \frac{\vartheta_1''}{\vartheta_1}\left[\frac{i}{2b}(\bar z_1 - z)\right]\right)\right\}, \qquad (13)$$

where

$$z = x + iy; \qquad z_1 = x_1 + iy_1;$$

$$f_1(y) = \frac{\mu - 2}{2\pi D_1} \cdot \frac{\partial^2 D}{\partial y^2}; \quad f_2(x, y) = \alpha(y) + i\beta(y)(x + x_1);$$

$$f_3(x, y) = \alpha(y) - i\beta(y)(x - x_1); \quad f_4(x, y) = \alpha(y) - i\beta(y)(x + x_1); \qquad (14)$$

$$f_5(x, y) = \alpha(y) + i\beta(y)(x - x_1); \quad f_6(y) = \frac{a(1-\mu)}{16\pi^2 b D} \cdot \frac{\partial^2 D}{\partial y^2};$$

$$\alpha(y) = \frac{1}{2\pi b D_1} \frac{\partial D}{\partial y^2}; \quad \beta(y) = \frac{1-\mu}{8\pi b D_1} \cdot \frac{\partial^2 D}{\partial y^2}.$$

The representation of the kernel through theta-functions, which are expressed by rapidly converging series, makes it possible to avoid the use of the diverging series obtained in Navier's solution.

We shall transform kernel (12), using the expansion of the log and the logarithmic derivative functions of ϑ_1 in series /6/, and restricting ourselves to two terms. After the separation of the real part and some elementary transformations we then obtain the following approximate expression for the kernel:

$$k(x, y, x_1, y_1) = \frac{\mu - 2}{2\pi D_1} \cdot \frac{\partial^2 D}{\partial y^2} \times$$

$$\times \left[\frac{1}{2} \ln \frac{\left(\sin^2\pi \frac{y_1 + y}{2b} + \mathrm{sh}^2\pi \frac{x_1 + x}{2b}\right)\left(\sin^2\pi \frac{y_1 - y}{2b} + \mathrm{sh}^2\pi \frac{x_1 - x}{2b}\right)}{\left(\sin^2\pi \frac{y_1 - y}{2b} + \mathrm{sh}^2\pi \frac{x_1 + x}{2b}\right)\left(\sin^2\pi \frac{y_1 + y}{2b} + \mathrm{sh}^2\pi \frac{x_1 - x}{2b}\right)} \right.$$

$$\left. - \frac{2e^{-\pi\frac{a}{b}}}{\mathrm{sh}\pi\frac{a}{b}} \cdot \cos\pi\frac{y_1 + y}{b} \cdot \mathrm{sh}\,\pi\frac{x_1}{b} \cdot \mathrm{sh}\,\pi\frac{x}{b} \right] +$$

$$+\frac{4}{bD_1}\frac{\partial D}{\partial y}\cdot\frac{e^{-\pi\frac{a}{b}}}{\operatorname{sh}\pi\frac{a}{b}}\cdot\sin\pi\frac{y_1}{b}\cdot\operatorname{sh}\pi\frac{x_1}{b}\cdot\cos\pi\frac{y}{b}\cdot\operatorname{sh}\pi\frac{x}{b}+$$

$$+\frac{1-\mu}{8bD_1}\cdot\frac{\partial^2 D}{\partial y^2}\left[(x+x_1)\cdot\operatorname{cth}\pi\frac{x+x_1}{2b}+(x-x_1)\operatorname{cth}\pi\frac{x-x_1}{2b}+\right.$$

$$\left.+\frac{8e^{-\pi\frac{a}{b}}}{\operatorname{sh}\pi\frac{a}{b}}\cdot\sin\pi\frac{y_1}{b}\cdot\sin\pi\frac{y}{b}\left(x\operatorname{sh}\pi\frac{x_1}{b}\cdot\operatorname{ch}\pi\frac{x}{b}+x_1\operatorname{ch}\pi\frac{x_1}{b}\cdot\operatorname{sh}\pi\frac{x}{b}\right)\right]-$$

$$-\frac{a(1-\mu)}{bD_1}\cdot\frac{\partial^2 D}{\partial y^2}\cdot\frac{e^{-\pi\frac{a}{b}}}{\operatorname{sh}\pi\frac{a}{b}}\cdot\sin\pi\frac{y_1}{b}\cdot\sin\pi\frac{y}{b}\cdot\operatorname{ch}\pi\frac{x+x_1}{b}. \qquad (15)$$

The determination of Green's function for a rectangular plate of variable thickness reduces to solving the integral equation

$$w(x_1, y_1, x', y')\frac{D_L(y_1)}{D_1}=v(x_1, y_1, x', y')-$$

$$-\iint\limits_{(s)} w(x, y, x', y')\cdot k(x, y, x_1, y_1)\,dx\,dy \qquad (16)$$

with kernel (15).

The integral in the right-hand part of (16) contains no singularities, and equation (16) can accordingly be solved by approximate methods.

The advantage of the method of integral equations for determining Green's function consists in the fact that the integral equations represent a means for determining the interpolation formulas through which the unknown function can be approximately represented. We shall look for a solution of (16) of the form*:

$$w=\sum_{i=1}^{n}F_i(x', y')\,\varphi_i(x_1, y_1), \qquad (17)$$

where $F_i(x', y')$ are some unknown functions of the point of application of the unit concentrated force; $\varphi_i(x_1, y_1)$ are arbitrary functions, preferably satisfying the boundary conditions of the problem. We shall require that function (17) satisfy equation (16) at a discrete set of points lying inside and on the contour of the rectangular plan of the plate. We obtain then a system of algebraic equations for determining the unknown functions $F_i(x', y')$. The larger the number of terms taken in (17) (and therefore the larger the number of points), the higher the accuracy of the solution. Consider as an example a rectangular plate of rigidity varying according to the linear law

$$D=D_0+cy. \qquad (18)$$

We shall look for a solution of (17) as the sum of two terms. In the case of a simply supported plate the functions $\varphi_1(x_1, y_1)$ and $\varphi_2(x_1, y_1)$ can

* The method was proposed by N. A. Kil'chevskii.

be selected in the following form:

$$\varphi_1(x_1, y_1) = \sin\pi\frac{x_1}{a}\sin\pi\frac{y_1}{b}, \quad \varphi_2(x_1, y_1) = \sin 2\pi\frac{x_1}{a}\sin 2\pi\frac{y_1}{b}. \tag{19}$$

Substitute (17) in (16) and require that it be satisfied at two points: at the plate center and at the point of application of the unit concentrated force. We obtain then the following system of two algebraic equations with two unknowns:

$$\begin{gathered} F_1(x', y')\varphi_1\left(\frac{a}{2}, \frac{b}{2}\right) + F_2(x', y')\cdot\varphi_2\left(\frac{a}{2}, \frac{b}{2}\right) = v\left(\frac{a}{2}, \frac{b}{2}, x', y'\right) - \\ - \int_0^a\int_0^b k\left(\frac{a}{2}, \frac{b}{2}, x, y\right)[F_1(x', y')\varphi_1(x, y) + F_2(x', y')\varphi_2(x, y)]\,dxdy; \\ F_1(x', y')\cdot\varphi_1(x', y') + F_2(x', y')\varphi_2(x', y') = v(x', y') - \\ - \int_0^a\int_0^b k(x', y', x, y)[F_1(x', y')\varphi_1(x, y) + F_2(x', y')\varphi_2(x, y)]\,dxdy. \end{gathered} \tag{20}$$

Having determined the functions $F_1(x', y')$ and $F_2(x', y')$ and substituted them in (17), we obtain the approximate value of Green's function in a closed form. This value is then substituted in the integrand of (16) and, using the fact that the integral is little sensitive to errors, we find a more accurate value of Green's function. In the case of variable rigidity (18) the approximate expression of Green's function is of the form

$$\begin{gathered} w(x, y, x', y') = \frac{D_1}{D_L(y_1)}\left[v(x_1, y_1, x', y') - \right. \\ \left. - \frac{16b^2c}{3\pi^3 D_1} e^{-\kappa\frac{c}{b}}\cdot\sin\pi\frac{y_1}{b}\cdot\operatorname{sh}\pi\frac{x_1}{b}\cdot F_2(x', y')\right]. \end{gathered} \tag{21}$$

This generalized interpolation-iterative process makes it possible to determine Green's function for a plate of variable thickness rectangular in the plan in a closed form.

In order to obtain a more accurate solution it is necessary to take more terms in the sum (17) and to increase the number of iterative steps.

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STRESS CONCENTRATION IN THIN ELASTIC SPHERICAL SHELLS

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This paper deals with the case of stress concentration about an arbitrarily located nonreinforced circular hole in a thin elastic spherical shell with a built-in edge passing along a parallel of the geographical system of coordinates.

1. The Ukrainian school of mechanics considers (cf. Savin's survey /1/) that the stress concentration decreases rapidly with the increase of the distance from the hole, so that it is possible to use the equations of a shallow shell.

2. This same problem is solved below by the method of resolving the stressed state. It is assumed that the stressed state in the shell can be resolved into a membrane stressed state, a pure moment stressed state, and a simple edge effect. It follows from the results of /2/, /3/ that in order to be able to calculate the shell by the membrane theory it is necessary that the load applied vary not too rapidly, and that the shell itself be sufficiently rigid. It follows from the theory of surfaces that a doubly connected shell of positive curvature will be formally rigid if it is fastened sufficiently rigidly along an arbitrarily small portion of its boundary. In order that the membrane state of stress be the predominant one, however, this portion should not be too small /4/. Fastenings of this type alone will be considered in what follows. The shell will then be in a membrane state of stress at a sufficient distance from the built-in edge, since the stresses due to the edge effect by the hole will be small compared with the stresses of the membrane stressed state. Furthermore, a specially conducted study showed that for a considerably large class of loads and a wide range of hole dimensions the stresses of the pure moment stressed state are likewise small compared with the stresses of the membrane stressed state, which will interest us below and for the determination of which we shall make use of analytical functions /3/.

3. Consider thus a domelike spherical shell subjected to an arbitrary load. Let (P_1) be the stressed state of a shell without hole subjected to the same load. This problem will be solved by the procedure used in the theory of stress concentration. We superpose on the stressed state (P_1) the stressed state (P_2) corresponding to zero surface load, which would have eliminated the discordance in the boundary conditions at the hole. In accordance with the preceding reasoning, this stressed state (P_2) must be membrane near the hole. The mathematical problem corresponding to the determination of the stressed state (P_2) reduces to determining a

function analytical in an annular region, if its real and imaginary parts are given on one of the contours.

This problem cannot be solved in the general case. This stressed state (P_2) can however be determined with an accuracy of up to a rapidly varying stressed state /5/: $(P_2) = (P_2') + (P_2'')$. (P_2') is the membrane stressed state which eliminates the discordance in the boundary conditions on the hole with an accuracy of up to the rapidly varying stressed state (P_2''), which is the integral of a homogeneous elliptic equation with strongly fluctuating boundary conditions. As follows from the paper of Vishik and Lyusternik/6/, this stressed state (P_2'') is usually damped rapidly with the increase of the distance from the hole. Furthermore, if the applied load does not vary too rapidly, (P_2'') will have an intensity considerably lower than (P_2'). Since very general and simple methods have been evolved for determining this stressed state /3, 6, 7/, we shall not deal here with this determination, considering instead the more difficult and practically more important problem of determining the stressed state (P_2').

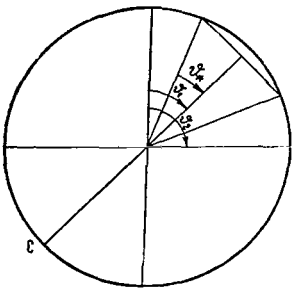

FIGURE.

4. The determination of the stressed state (P_2') is illustrated in the example of a dome-shaped shell with a hole (see figure) subjected to its dead weight. We take as stressed state (P_1) the stressed state of a full spherical shell subjected to the force of its dead weight, parallel to the polar axis, and fastened at the south pole of this system of coordinates. In the given case this stressed state differs in nothing, at a sufficient distance from the built-in end, from the earlier considered stressed state of the shell (P_1). If we pass to a new system of coordinates, in which the point corresponding to the hole center will be the origin of coordinates, and the point corresponding to point C will be at infinity, the solution can be written in this system in the form

$$T_{1,2} = \mp \operatorname{Re}\,[\psi_1(z_1\bar{z})]\left(\frac{1+z\bar{z}}{2\sqrt{z\bar{z}}}\right)^2 + \frac{R\gamma}{2}\,\tau(z,\bar{z}),$$

$$S_1 = -S_2 = \operatorname{Im}[\psi_1(z_1\bar{z})]\left(\frac{1+z\bar{z}}{2\sqrt{z\bar{z}}}\right)^2,$$

where

$$\psi_1(z,\bar{z}) = \frac{R\gamma}{2}\,[2+3\tau(z,\bar{z})-\tau^3(z,\bar{z})]\,\frac{(1+\rho_1^2)^2 z^2}{(1-\rho_1 z)^2(\rho_1+z)^2},$$

$$\tau(z_1\bar{z}) = \frac{(1-\rho_1^2)(z\bar{z}-1)+2\rho_1(z+\bar{z})}{(1+\rho_1^2)(1+z\bar{z})},$$

$$z = e^{\alpha_1+i\beta_1}, \qquad \rho_1 = \operatorname{tg}\frac{\vartheta_1}{2},$$

α_1, β_1 are isothermal coordinates on the inverted sphere, in which the hole contour coincides with the parallel of the geographical system of coordinates. It is easy to show that the function which removes the discordance in the boundary conditions at the hole is of the form

$$\psi(z) = -\psi_1\left(z, \frac{\rho_*^2}{z}\right) + \frac{R\gamma}{2}\,\tau\left(z, \frac{\rho_*^2}{z}\right)\left(\frac{2\rho_*}{1+\rho_*^2}\right)^2.$$

This function is not analytical in the entire region of the complex variable, since at points $z = 0$ and $z = \infty$ it has a pole of the first order, and at points $z = -\rho_1$ and $z = 1/\rho_1$ a pole of the second order; it is true that the points $z = 0$ and $z = 1/\rho_1$ do not belong to the region corresponding to the middle surface of the shell. If the remaining two points likewise do not belong to this region, the function $\psi(z)$ will be analytical, and in it will be realized the stressed state (P_2) considered in the preceding section. If the point $z = -\rho_1$, which corresponds to the north pole of the initial geographical system of coordinates, belongs to the shell, $\psi(z)$ will no longer be analytical in the entire considered region, but will be analytical in the vicinity of the hole contour. It follows that $\psi(z)$ can be expanded in a MacLaurin series in a certain annular region adjoining the hole contour. This is not necessary in practice, it being sufficient to develop the factor $1/(z+\rho_1)^2$ in a Taylor series in the vicinity of point $z = 0$. Thus, the stressed state (P_2') will be determined by the analytical function of the forces

$$\psi_*(z) = -\frac{R\gamma}{2}\left[2+3\tau\left(z, \frac{\rho_*^2}{z}\right) - \tau^3\left(z, \frac{\rho_*^2}{z}\right)\right] \times$$

$$\times \frac{(1+\rho_1^2)^2 z^2}{(1-\rho_1 z)^2} \sum_{n=0}^{N} (-1)^n (n+1)\left(\frac{z}{\rho_1}\right)^n + \frac{R\gamma}{2}\tau\left(z, \frac{\rho_*^2}{z}\right)\left(\frac{2\rho_*}{1+\rho_*^2}\right)^2,$$

where N is a sufficiently large number, not exceeding, however $\sqrt{\frac{R}{h}}$.

In view of the convergence of the Taylor series, the boundary conditions remaining unsatisfied will have a large variability index and a low intensity, and therefore the stressed state (P_2') determined by them need not be taken into account. The case when $z = \infty$ belongs to the shell is reduced to the preceding one by means of the transformation described in /8/.

5. We determined as an example the concentration of the circumferential force per unit length T_2 of a spherical shell characterized by the following values of the parameters $\vartheta_1 = 15°$, $\vartheta_2 = 90°$, $\vartheta_* = 20°$ for the forces of the dead weight and of an uniform internal pressure. The results obtained for the concentration factor k $(\varphi = 0)$ are represented in the table.

Load \ ϑ	20°	25°	30°	35°	40°
Dead weight	2	1.68	1.49	1.40	1.34
Internal pressure	2	1.65	1.46	1.35	1.28

Here ϑ is the geographical coordinate on the inverted sphere.

CONCLUSIONS

The problem of stress concentration near a circular hole of a spherical shell built-in at a sufficient distance from it is solved by finding the membrane stressed state (P_2). (In the case of the dead weight forces this state is determined by the analytical function of the forces ψ_*.) The

following factors are neglected in view of the low intensity of the forces produced by them:

1) the purely moment stressed state (a special study was conducted to that end);

2) the edge effect by the hole, and

3) the rapidly varying stressed state. The latter can be neglected when the load does not vary too rapidly. This approach is equivalent from the physical point of view to resolving the stress concentration into two parts: a relatively slowly decreasing part (P_2) and a rapidly decreasing part. The specially conducted analysis has established that in most cases of practical importance the first part (determined by the membrane stressed state) plays the main role.

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ON THE CALCULATION OF NONLINEAR VIBRATIONS OF FLEXIBLE PLATES AND SHALLOW SHELLS BY THE SMALL PARAMETER METHOD

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1. Mathematical difficulties which have not as yet been surmounted make it necessary to look for approximate solutions of the nonlinear problems of the theory of plates and shells, based on the first terms of the series expansions of the unknown magnitudes by the systems of some selected coordinate functions. As a result, the distributed mechanical system (plate, shell) with infinite number of degrees of freedom is replaced by a discrete system (which frequently has one degree of freedom only, in view of achieving a maximum simplification of the problem). The problem solution itself reduces to solving nonlinear equations, algebraic in statics and ordinary differential in dynamics.

The selection of the coordinate functions predetermines the distribution of stresses in the shell (plate). Such approximate solutions are accordingly usually suitable for calculating the deflections only, and not the stresses. This is true in particular of the dynamic problems considered below, when the form of the deformation depends on the frequency of the vibrations and on the particular mode of nonlinear vibrations realized from among the possible ones.

It should also be borne in mind that in the case of nonlinear boundary conditions it is frequently very difficult to find coordinate functions which satisfy them (the usually used modes of natural vibrations of the corresponding linear problem are not suitable in this case).

In order to calculate the stresses more accurately (which is of particular importance when estimating the fatigue strength) it is necessary to have solutions in which the form of the deformation is not selected, but is found from the equations allowing for all the linear and nonlinear boundary conditions. Such solutions of greater accuracy can be obtained by the method of the small parameter. A scheme for calculating the periodic vibrations of flexible shallow shells and plates, similar to that used earlier for calculating the nonlinear vibrations of bars /1, 2/, is exposed below.

2. The dimensionless nonlinear equations of the vibrations of shallow shells (the subscripts ξ, η, and s stand for differentiation by the dimensionless arguments, Δ represents the Laplacian):

$$\left.\begin{aligned} \varepsilon^2\Delta\Delta\bar{w} + \bar{w}_{ss} &= \bar{w}_{\xi\xi}\bar{\varphi}_{\eta\eta} + \bar{w}_{\eta\eta}\bar{\varphi}_{\xi\xi} - 2\bar{w}_{\xi\eta}\bar{\varphi}_{\xi\eta} + \bar{k}_1\bar{\varphi}_{\eta\eta} + \bar{k}_2\bar{\varphi}_{\xi\xi} + \bar{p} \\ \Delta\Delta\bar{\varphi} &= \bar{w}_{\xi\eta}^2 - \bar{w}_{\xi\xi}\bar{w}_{\eta\eta} - \bar{k}_1\bar{w}_{\eta\eta} - \bar{k}_2\bar{w}_{\xi\xi} \end{aligned}\right\} \qquad (2.1)$$

were obtained from the dimensional equations by means of the following transition formulas:

$$\left.\begin{aligned} &w=\frac{l}{c}\bar{w}, \quad \varphi=\frac{El^2}{c^2}\bar{\varphi}, \quad k_{1,2}=\frac{\pi^2}{cl}\bar{k}_{1,2}, \quad x=\frac{l}{\pi}\xi, \quad y=\frac{l}{\pi}\eta, \\ &t=\frac{cl}{\pi^2}\sqrt{\frac{\rho}{E}}\,s, \quad p=\frac{\pi^4 Eh}{c^3 l}\bar{p}, \quad c=\sqrt{12(1-\nu^2)}, \quad \varepsilon=\frac{1}{j}=\frac{h}{l} \end{aligned}\right\}. \tag{2.2}$$

The linear magnitude l characteristic of the shell is either one of the radii of curvature or the length of one of the edges. In the case of a plane plate $k_1=k_2=0$.

Various magnitudes can be used as small parameter μ. Independent of its nature it is expedient to write

$$\bar{w}=\sqrt{\mu}\,\bar{w}, \quad \bar{\varphi}=\mu\bar{\varphi}, \quad \bar{k}_{1,2}=\sqrt{\mu}\,\bar{k}_{1,2}. \tag{2.3}$$

This formal transition to new magnitudes, while not absolutely necessary, considerably simplifies the solution.

Equations (2.1) become then

$$\begin{gathered} \varepsilon^2\Delta\Delta\bar{w}+\bar{w}_{ss}=\mu(\bar{w}_{\xi\xi}\bar{\varphi}_{\eta\eta}+\bar{w}_{\eta\eta}\bar{\varphi}_{\xi\xi}-2\bar{w}_{\xi\eta}\bar{\varphi}_{\xi\eta}+\bar{k}_1\bar{\varphi}_{\eta\eta}+\bar{k}_2\bar{\varphi}_{\xi\xi})+\frac{\bar{p}}{\sqrt{\mu}} \\ \Delta\Delta\bar{\varphi}=\bar{w}_{\xi\eta}^2-\bar{w}_{\xi\xi}\bar{w}_{\eta\eta}-\bar{k}_1\bar{w}_{\eta\eta}-\bar{k}_2\bar{w}_{\xi\xi}. \end{gathered} \tag{2.4}$$

We shall look for a solution of system (2.4) in the form of the series expansions

$$\bar{w}=\sum_{i=0}^{\infty}\mu^i w_i, \quad \bar{\varphi}=\sum_{i=0}^{\infty}\mu^i\varphi_i. \tag{2.5}$$

The boundary conditions for w_i and φ_i are obtained from the given conditions by expanding the latter in series by the powers of μ with the aid of (2.5) and of expressions of the type

$$\left.\begin{aligned} &\bar{M}_1=\frac{c^3 l\mu^{-\frac{1}{2}}}{\pi^2 Eh^3}M_1=-\sum_{i=0}^{\infty}\mu^i(w_{i\xi\xi}+\nu w_{i\eta\eta}); \\ &\bar{N}_1=\frac{c^3 l^2\mu^{-\frac{1}{2}}}{\pi^3 Eh^3}N_1=-\Delta\sum_{i=0}^{\infty}\mu^i w_{i\xi}; \\ &\bar{r}_1=\frac{c^3 l^2\mu^{-\frac{1}{2}}}{\pi^3 Eh^3}r_1=-\sum_{i=0}^{\infty}\mu^i[w_{i\xi\xi\xi}+(2-\nu)w_{i\xi\eta\eta}]; \\ &\bar{T}_1=\bar{\sigma}_x=\frac{c^2\mu^{-1}}{\pi^2 Eh}T_1=\bar{\varphi}_{\eta\eta}=\sum_{i=0}^{\infty}\mu^i\varphi_{i\eta\eta}; \\ &\bar{S}=\bar{\sigma}_{xy}=\frac{c^2\mu^{-1}}{\pi^2 Eh}S=-\bar{\varphi}_{\xi\eta}=-\sum_{i=0}^{\infty}\mu^i\varphi_{i\xi\eta}; \\ &\bar{u}_\xi=\sum_{i=0}^{\infty}\mu_i(\varphi_{i\eta\eta}-\nu\varphi_{i\xi\xi}+\bar{k}_1 w_i-\frac{1}{2}\sum_{r+q=i}w_r w_{q\xi}). \end{aligned}\right\} \tag{2.6}$$

Equations (2.4) describe at $\mu=\bar{p}=0$ free vibrations of a shell of simpler properties than the considered one. This system can be called a generating

system, just as in Poincaré's method of a small parameter /3/, since its vibrations (as will be seen below) generate in fact in a certain sense the vibrations of the shell considered.

If, without using the formal representations (2.3), we try to find $\bar{w}$ and $\bar{\varphi}$ directly as series expansions by μ, the generating system will be an ordinary linear shallow shell with a resolving equation of order eight. The deflections of this shell will be independent of the shearing forces per unit length, while these forces themselves will depend on the deflections. To this shell correspond simpler differential equations of order four, whose solutions can be found in successive steps. The influence of the shearing forces on the deflections of the shell considered are allowed for in w_1, w_2, etc.

When selecting a generating system one should keep in mind that the nearer this system to the considered one, the smaller the number of terms which it will be necessary to keep in series (2.5) in order to achieve the required accuracy. On the other hand, it is usually much more difficult to integrate the equations of such a closer generating system.

3. Let, for simplicity, $p(x, y, t) = p_*(x, y)\sin\omega t$ (if p is an arbitrary periodic function of t, it must be expanded in a Fourier series in an interval equal to the period). When calculating the forced steady vibrations of the shell one should distinguish between the resonance and the nonresonance cases, corresponding to $\omega \approx \lambda^0_{mn}$ and $\omega \neq \lambda^0_{mn}$, respectively, where λ^0_{mn} (m, $n = 0, 1, 2, \cdots$) are frequencies of the free vibrations of the generating system.

In the nonresonance case, introducing the designations

$$\bar{\omega} = \frac{c}{\pi^2} l\omega \sqrt{\frac{\rho}{E}}, \quad \|\bar{p}_*\| = \frac{1}{\Sigma_p}\sqrt{\iint\limits_{\Sigma_p} \bar{p}_*^2\, dxdy}, \quad \bar{p}_* = \frac{\bar{p}_*}{\|\bar{p}_*\|}, \tag{3.1}$$

(where Σ_p is the area of the part of the shell surface loaded by the pressure p), writing $\mu = \|\bar{p}_*\|^2$ (usually $\|\bar{p}_*\| \ll 1$), and substituting (2.5) in (2.4), we obtain, by equating the coefficients of equal powers to zero, the following infinite sequence of systems of differential equations:

$$\left.\begin{aligned} &\varepsilon^2\Delta\Delta w_0 + w_{0ss} = \bar{p}_* \sin\bar{\omega}s, \\ &\Delta\Delta\varphi_0 = -\bar{k}_1 w_{0\eta\eta} - \bar{k}_2 w_{0\xi\xi} + w_{0\xi\eta}^2 - w_{0\xi\xi}w_{0\eta\eta}, \end{aligned}\right\} \tag{3.2$_0$}$$

$$\left.\begin{aligned} &\varepsilon^2\Delta\Delta w_1 + w_{1ss} = \bar{k}_1 \varphi_{0\eta\eta} + \bar{k}_2 \varphi_{0\xi\xi} + w_{0\xi\xi}\varphi_{0\eta\eta} + w_{0\eta\eta}\varphi_{0\xi\xi} - 2w_{0\xi\eta}\varphi_{0\xi\eta}, \\ &\Delta\Delta\varphi_1 = -\bar{k}_1 w_{1\eta\eta} - \bar{k}_2 w_{1\xi\xi} + 2w_{0\xi\eta}w_{1\xi\eta} - w_{0\xi\xi}w_{1\eta\eta} - w_{0\eta\eta}w_{1\xi\xi}, \end{aligned}\right\} \tag{3.2$_1$}$$

$$\left.\begin{aligned} &\varepsilon^2\Delta\Delta w_2 + w_{2ss} = \bar{k}_1\varphi_{1\eta\eta} + \bar{k}_2\varphi_{1\xi\xi} + w_{0\xi\xi}\varphi_{1\eta\eta} + w_{1\xi\xi}\varphi_{0\eta\eta} + \\ &\qquad + w_{0\eta\eta}\varphi_{1\xi\xi} + w_{1\eta\eta}\varphi_{0\xi\xi} - 2(w_{0\xi\eta}\varphi_{1\xi\eta} + w_{1\xi\eta}\varphi_{0\xi\eta}), \\ &\Delta\Delta\varphi_2 = -\bar{k}_1 w_{2\eta\eta} - \bar{k}_2 w_{2\xi\xi} + w_{1\xi\eta}^2 + 2w_{0\xi\eta}w_{2\xi\eta} - w_{0\xi\xi}w_{2\eta\eta} - \\ &\qquad - w_{1\xi\xi}w_{1\eta\eta} - w_{0\eta\eta}w_{2\xi\xi}, \end{aligned}\right\} \tag{3.2$_2$}$$

$$\vdots \quad \vdots$$

In the resonance case we assume $\mu = \|p_*\|^{2/3}$ since the small parameter must be selected so as to make the harmonic load appear in the equations of the first approximation (and not of the zero approximation). Otherwise, the solution of the generating system, containing secular terms, would not have been periodic. In addition, in order that the vibrations might be

calculated near a strict resonance as well, it is necessary to introduce the frequency discrepancy /3/ $\delta\bar{\omega} = \bar{\omega} - \bar{\lambda}^0_{mn}$. The frequency $\bar{\lambda}^0_{mn}$ does not appear, however, explicitly in equations (2.4). The frequency discrepancy is accordingly introduced obliquely, through the variation of the coefficient ε^2 corresponding to it:

$$\varepsilon^2 = \varepsilon_*^2 - \mu a, \tag{3.3}$$

where ε_* is the value of ε for which the mn-th frequency of free vibrations of the generating system is exactly equal to the given frequency ω. The coefficient a is calculated from (3.3) for the given μ and ε, and for ε_* found from the condition $\bar{\lambda}^0_{mn} = \bar{\omega}$.

Due to these two peculiarities, the first equations of system (3.2) have the following form in the resonance case:

$$\varepsilon^2 \Delta\Delta w_0 + w_{0ss} = 0; \tag{3.4$_0$}$$

$$\varepsilon^2 \Delta\Delta w_1 + w_{1ss} = \bar{p}_* \sin\bar{\omega} s + a\Delta\Delta w_0 + \bar{k}_1 \varphi_{0\eta\eta} + \bar{k}_2 \varphi_{0\xi\xi} + w_{0\xi\xi}\varphi_{0\eta\eta} + w_{0\eta\eta}\varphi_{0\xi\xi} - 2 w_{0\xi\eta}\varphi_{0\xi\eta}, \tag{3.4$_1$}$$

$$\varepsilon^2 \Delta\Delta w_2 + w_{2ss} = a\Delta\Delta w_1 + \bar{k}_1 \varphi_{1\eta\eta} + \bar{k}_2 \varphi_{1\xi\xi} + w_{0\xi\xi}\varphi_{1\eta\eta} + w_{1\xi\xi}\varphi_{0\eta\eta} + w_{0\eta\eta}\varphi_{1\xi\xi} + w_{1\eta\eta}\varphi_{0\xi\xi} - 2(w_{0\xi\eta}\varphi_{1\xi\eta} + w_{1\xi\eta}\varphi_{0\xi\eta}). \tag{3.4}$$

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The second equations of system (3.2) remain unaltered.

Let the shell edges be the coordinate lines $x=0$, $x=l$, $y=0$, $y=b$ or, if the shell is closed along coordinate y, only $x=0$, $x=l$. It is assumed in the first case that

$$w = \frac{\partial^2 w}{\partial y^2} = u = \frac{\partial^2 \varphi}{\partial x^2} = 0 \quad \text{at} \quad y = 0,\ b. \tag{3.5}$$

The calculation of the nonresonance vibrations of a nonclosed shell is conducted according to the following scheme.

The pressure $\bar{p}_*$ (ξ, η) is expanded in a Fourier series:

$$\bar{p}_* = \sum_{n=1}^{\infty} p_n(\xi) \sin \alpha_n \eta, \qquad \alpha_n = n\, l/b, \tag{3.6}$$

after which the first equation of $(3.2)_0$ becomes of the form

$$\varepsilon^2 \Delta\Delta w_0 + w_{0ss} = \sin\bar{\omega} s \sum_{n=1}^{\infty} p_n \sin \alpha_n \eta. \tag{3.7}$$

The solution of (3.7) satisfying conditions (3.5) is of the form

$$w_0 = \sin\bar{\omega} s \sum_{n=1}^{\infty} X_{0n}(\xi) \sin \alpha_n \eta, \tag{3.8}$$

where

$$X_{0n} = X_0^{(p)} + \sum_{r=1}^{4} C_r X_{0nr} \tag{3.9}$$

is the solution of the equation

$$X_{0n}^{\mathrm{IV}} - 2\alpha_n^2 X_{0n}'' + (j^2 - \bar{\omega}^2) X_{0n} = p_n, \qquad n = 1, 2, \cdots, \tag{3.10}$$

where $X_0^{(p)}$ is a particular solution corresponding to p_n, and X_{0nr} are linearly independent particular solutions of the corresponding homogeneous equation. The integration constants C_r are determined from the non-homogeneous system of four linear equations obtained from the boundary conditions for w_0 at the edges $x = 0$, $x = l$. The determinant of this system of equations is the frequency equation of the generating system. Since all $\bar{\lambda}_{mn}^0 \neq \bar{\omega}$ by hypothesis, this determinant is different from zero and there exists a unique solution x_{0n}.

To the periodic solution w_0 found there corresponds the periodic right-hand part of the second equation of (3.2). Therefore, φ_0 will likewise be a periodic function having the same period $\frac{2\pi}{\omega}$. After its determination the right-hand part of the first equation of $(3.2)_1$ becomes known. After finding its solution similarly to the solution (3.8) we can calculate φ_1, etc. At each step of the calculation w_i and φ_i will be periodic functions of period $\frac{2\pi}{\omega}$, all of them being determined in a one-valued manner on the strength of the condition $\bar{\omega} \neq \bar{\lambda}_{mn}^0$.

In the case of boundary conditions at the edges $y = 0$, b, different from (3.5), the variables in the equations can no longer be separated. In this case the solution can be represented in the form of a series by the corresponding beam functions, obtaining equations for X_{in} of the form of (3.10) by orthogonalizing the equations for w_i relative to each of the beam functions.

If the shell is closed along the coordinate y, then w_i is to be represented as a cosine and sine Fourier series. In this case equations of the type of (3.10) will once more be strict.

The scheme of calculation of resonance vibrations is somewhat different.

To the solution of $(3.4)_0$ for a closed shell

$$w_0 = A_0 X_m^0(\xi) \sin \alpha_n \eta \sin \bar{\lambda}_{mn}^0 s, \tag{3.11}$$

where $X_m(\xi) \sin \alpha_n \eta$ is the mode of the free vibrations of the generating system of frequency $\bar{\lambda}_{mn}^0$, and A_0, an arbitrary amplitude at this stage, corresponds the solution of the second equation of $(3.2)_0$:

$$\varphi_0 = \varphi_{01}(\xi, \eta, A_0) \sin \bar{\lambda}_{mn}^0 s + \varphi_{02}(\xi, \eta, A_0) \sin^2 \bar{\lambda}_{mn}^0 s, \tag{3.12}$$

A_0 figures in φ_{02} at the power of two.

The substitution of w_0 and φ_0 in the right-hand part of $(3.4)_1$ yields

$$\varepsilon^2 \Delta\Delta w_1 + w_{1ss} = [p_n(\xi) + P_1(\xi, A_0)] \sin \alpha_n \eta \sin \bar{\lambda}_{mn}^0 s + Q_1(\xi, \eta, s, A_0). \tag{3.13}$$

The right-hand part is split in two components such that Q_1 does not contain terms proportional to $\sin \alpha_n \eta \sin \bar{\lambda}_{mn}^0 s$. The part of the solution of w_1 corresponding to Q_1 will accordingly be nonresonance and can be found by the nonresonance scheme exposed above for any value of A_0.

The amplitude A_0 is found from the condition of periodicity of w_{1n}, properly speaking of the resonance part w_1 corresponding to the first term of the right-hand part of (3.13). The function w_{1n} will be a periodic

function of the form

$$w_{1n} = X_{1n}(\xi) \sin \alpha_n \eta \sin \bar{\lambda}^0_{mn} s, \tag{3.14}$$

provided there exists a solution X_{1n} of the equation

$$X^{IV}_{1n} - 2\alpha_n^2 X''_{1n} + (j_*^2 - \bar{\lambda}^{02}_{mn}) X_{1n} = p_n + P_1, \tag{3.15}$$

which satisfies the boundary conditions following from the conditions for w_1 at the edges $x=0$ and $x=l$.
The solution

$$X_{1n} = X^{(p)}_{1n} + \sum_{r=1}^{4} C_r X_{1nr} \tag{3.16}$$

is similar to solution (3.9). In this case, however, the determinant of the system of algebraic equations relative to C_r is equal to zero, since λ^{02}_{mn} is the eigenvalue of the corresponding homogeneous boundary-value problem. This system will therefore have a nonlinear solution only for those values of A_0 for which the expansion of the matrix of the equations coefficients by the addition to it of a column formed of the right-hand parts does not alter its rank. This fact should be used for forming the equation which determines the values of A_0 to which there correspond periodic values w_1 (the equation is nonlinear relative to A_0). But, according to the theory of ordinary differential equations, the condition of equality of ranks of the fundamental and expanded matrices is identical to the condition of orthogonality on segment $[0, l]$ of the right-hand part of (3.15) and of the solutions of the conjugated homogeneous boundary-value problem. This latter condition is much more easily formed, in particular in the case when the boundary conditions for X_{1n} are self-conjugated (the differential operator of the left-hand part of (3.15) is self-conjugated). In this case the equation for the determination of A_0

$$\int_0^l (p_n + P_1) X_m \, d\xi = 0 \tag{3.17}$$

is the condition of orthogonality of the right-hand part of (3.15) and the eigenfunction X_m corresponding to the frequency $\bar{\lambda}^0_{mn}$.

The solution of w_1 remains indeterminate even after A_0 has been calculated, since when $p_n + P_1$ satisfies condition (3.17) or a condition equivalent to it the constants C_r are determined up to some constant factor A_1. The indeterminate term $A_1 X_m(\xi) \sin \alpha_n \eta \sin \bar{\lambda}^0_{mn} s$ will appear as a result in w_1. The magnitude A_1 is determined only in the next stage of calculations, from the condition of periodicity of w_2, just as A_0 was determined from the condition of periodicity of w_1. It is important to note that the algebraic conditions for determining $A_1, A_2, \cdots$ are linear. Therefore, when conducting the calculations for each of the roots A_{0k} of the nonlinear equation (3.17) we do not obtain new bifurcations of the solution.

4. The scheme of calculation of resonance vibrations with small variations can be used for elucidating the influence of the amplitude of the free shell vibrations on the frequency. To that end one passes from the time s to the natural time τ by the formula

$$\tau = \lambda_N s, \tag{4.1}$$

where $\bar{\lambda}_N$ is the frequency of the free nonlinear vibrations, expanded in a series (the subscripts mn of the frequencies have been omitted)

$$\lambda_N^2 = \bar{\lambda}^{02} (1 + \beta_1 \mu + \beta_2 \mu^2 + \cdots) \tag{4.2}$$

with unknown coefficients β_i. It is in addition convenient to write $\mu = A_0^2$, where A_0 is the amplitude of the initial displacement.

Then, if we single out from the stresses the initial statistical membrane stresses σ_x^0, σ_y^0, σ_{xy}^0 the first equation of (2.4) will be written in the form

$$\varepsilon^2 \Delta\Delta \bar{w} + \lambda_N^2 w_{\tau\tau} = \mu \, [\bar{w}_{\xi\xi} (\bar{\sigma}_x^0 + \bar{\varphi}_{\eta\eta}) + \bar{w}_{\eta\eta} (\bar{\sigma}_y^0 + \bar{\varphi}_{\xi\xi}) + 2\bar{w}_{\xi\eta} (\bar{\sigma}_{xy}^0 - \bar{\varphi}_{\xi\eta}) + \\ + \bar{k}_1 \bar{\varphi}_{\eta\eta} + \bar{k}_2 \bar{\varphi}_{\xi\xi}], \tag{4.3}$$

while the second one will remain unchanged. In this case, however, $\bar{\varphi}$ determines the dynamic stresses only.

The use of series (2.5) and (4.2) yields the equations

$$\varepsilon^2 \Delta\Delta w_0 + \bar{\lambda}^{02} w_{0\tau\tau} = 0; \tag{4.4$_0$}$$

$$\varepsilon^2 \Delta\Delta w_1 + \bar{\lambda}^{02} w_{1\tau\tau} = -\beta_1 \bar{\lambda}^{02} w_{0\tau\tau} + \bar{k}_1 \varphi_{0\eta\eta} + \bar{k}_2 \varphi_{0\xi\xi} + w_{0\xi\xi} (\bar{\sigma}_x^0 + \varphi_{0\eta\eta}) + \\ + w_{0\eta\eta} (\bar{\sigma}_y^0 + \varphi_{0\xi\xi}) + 2 w_{0\xi\eta} (\bar{\sigma}_{xy}^0 - \varphi_{0\xi\eta}); \tag{4.4$_1$}$$

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Having found the solution of the second equation of $(3.2)_0$, corresponding to the solution $w_0 = A_0 X_m(\xi) \sin \alpha_n \eta \cos \tau$ of equation $(4.4)_0$, and having substituted it in $(4.4)_1$, it remains, just as in the resonance case, to form the condition of periodicity of the solution w_1. With its aid β_1 is represented as function of the initial amplitude. The arbitrary number A_1 entering in w_1 is found from the condition that at $\tau = 0$ $w_1 = 0$ at the point where $w_0 = A_0$ at this moment. At the next stage we similarly find β_2 and A_2, etc.

All that was said in § 3 relative to the calculation of a closed shell and relative to the case when the boundary conditions at two opposite edges are different from (3.5) remains correct for the resonance case and for free vibrations.

5. We shall now give the results of the solution of the simplest of the above-considered problems.

In the case of free vibrations of a closed circular cylindrical shell with boundary conditions at the edges

$$w = \frac{\partial^2 w}{\partial x^2} = v = \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{for } x = 0, \ l, \tag{5.1}$$

loaded by a constant internal pressure p ($\sigma_x^0 = 1/2\, \sigma_y^0 = \sigma/2 = pR/2h$, $\sigma_{xy}^0 = 0$) the dimensionless frequency $\bar{\lambda}_N$ is connected in the following way with the dimensionless initial amplitude $\bar{A}_0 = cA_0/l$:

$$\bar{\lambda}_N^2 = \varepsilon^2 (m^2 + \alpha_n^2)^2 + \frac{m^4 \bar{k}_2^2}{(m^2 + \alpha_n^2)^2} + \bar{\sigma}\left(\frac{m^2}{2} + \alpha_n^2\right) + \\ + \frac{3}{64} \bar{A}_0^2 \left[3\left(\frac{m^4}{2} + \alpha_n^4\right) - \frac{m^4}{2}\left(\frac{3}{\alpha_n \pi} + \operatorname{th} \alpha_n \pi\right) \operatorname{th} \alpha_n \pi \right], \quad \alpha_n = \frac{nl}{\pi R}. \tag{5.2}$$

The first three terms correspond exactly to the linear theory of shallow shells. It follows that the use of a shell with simpler properties as generating system has not led to an error in the fundamental terms.

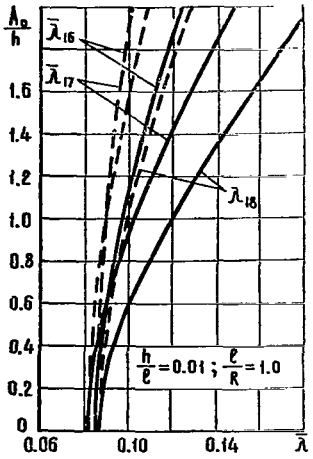

FIGURE 1.

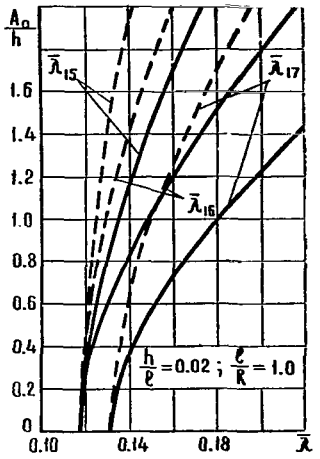

FIGURE 2.

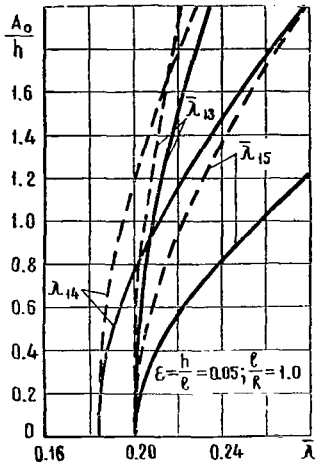

FIGURE 3.

Formula (5.2) was obtained by satisfying all the conditions (5.1) at each point of the edge. If the last two conditions are satisfied at the edges in an integral form (as in most solutions of nonlinear problems), the correction to the linear frequency is considerably modified.

Figures 1 to 3 represent the relationship between the lower three frequencies of the three shells and the ratio A_0/h for $p = 0$. The continuous curves were plotted by formula (5.2), while the dotted lines were obtained by satisfying the condition $v = \partial^2\varphi/\partial y^2 = 0$ on the average over the edges.

It is interesting that at a certain value of A_0/h the lowest frequency corresponds to a number n different from that predicted by the linear theory. This should be taken into account when calculating forced vibrations.

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LOAD-CARRYING CAPACITY OF CIRCULAR CYLINDRICAL SHELLS AT COMPRESSION

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(Khar'kov)

The loss of load-carrying capacity of a shell can be a consequence of buckling or of a strength disturbance (caused for instance by the appearance of stresses equal to the yield stress of the material). In spite of the importance of the study of the stability of thin circular cylindrical shells and of the large number of papers devoted to this question, there exists to date no generally agreed upon explanation of the mechanism of shell buckling. Different authors give various explanations of the considerable discrepancy between the experimental data and the results of theoretical studies based on the classical linear theory. Many scientists (Karman, Mushtari, Vol'mir, Donnell) are today of the opinion that the critical stresses of the shell are to be determined on the basis of the nonlinear theory, allowing for the considerable radial displacements of the system elements. Many papers begin from the existence of initial deflections of the shell surface. The complexity and inaccuracy of the solutions obtained on the basis of the nonlinear theory seem to us unjustified, not to mention the impossibility of explaining some of the experimental results.

The load-carrying capacity can be determined in a manner which is both physically more convincing and mathematically sufficiently accurate on the basis of the linear theory, allowing for small initial deflections of the surface, by reasoning in the following way.

The totality of initial imperfections of the middle surface of the system can be divided into two groups, corresponding to buckling in an axisymmetrical and a nonaxisymmetrical mode, respectively. We shall be interested in what follows only in the surface shapes for which the shell deformations can increase rapidly under the action of the compressive stresses. These shapes correspond to the minimum critical stress of the classical linear theory of buckling.

The longitudinal elements of the shell bend longitudinally-laterally under the action of the external compressive forces. Calculations show that, all other conditions being equal, the axisymmetrical mode of initial deflection is the more dangerous one. The load-carrying capacity of the shell can thus be determined from the condition of strength, starting from the external compressive forces under whose action the total internal stresses reach the yield point of the material. This procedure is, however, suitable only at relatively low values of the ratio of the shell radius to its thickness (R/δ) .* What happens at large R/δ?

* To this type of shells usually belong reinforced and corrugated structures.

The longitudinal bending moments of the separate strips at longitudinal-lateral bending can be considered as due not to the true initial axisymmetrical curvature of the surface, but to an equivalent fictitious axisymmetrical radial load under whose action the shell buckles in a nonaxisymmetrical mode. Since the flexural rigidity of the shell elements is proportional to the cube of its thickness, buckling will be very probable at relatively large R/δ, i.e., at small thicknesses, the compressive axial external load contributing actively to this phenomenon. It therefore becomes understandable why a surface deformation in the form of protuberances and depressions according to a nonaxisymmetrical mode is observed as a rule in thin shells at the last moment preceding collapse only. Buckling is usually accompanied in this case by snap-throughs.

We proceed to determine the formulas for calculating the load-carrying capacity of the shell on the basis of the phenomena described above.

STABILITY OF A SMOOTH SHELL

The initial deflection of the middle surface will be represented in the form

$$w_0 = f_0 \sin \frac{m_0 \pi x}{L}, \tag{1}$$

where f_0 is the amplitude of the initial deflection; L is the shell length; x is the coordinate along the system axis; m_0 is the number of half-waves into which an ideally smooth shell buckles in an axisymmetrical mode.

In the case of elastic radial displacements of the elements we have*

$$w = \frac{f_0 \bar{\sigma}_x}{1-\bar{\sigma}_x} \sin \frac{m_0 \pi x}{L}, \tag{2}$$

where $\left(\frac{m_0 \pi}{L}\right)^2 = \sqrt{\frac{E\delta}{DR^2}}$ is determined from the condition of minimum of the critical stresses σ_{cr} of an ideally smooth shell; $\bar{\sigma}_x = \frac{\sigma_x}{\sigma_{cr}}$ is the ratio of the collapse compressive load to the critical load

$$\left(\sigma_{cr} = \sqrt{4\frac{ED}{R^2\delta}} = 0.6\frac{E}{R/\delta}\right).$$

The longitudinal bending moments in the separate strips are

$$m_x = -D\frac{d^2w}{dx^2} = \frac{f_0 \delta \sigma_x}{2(1-\bar{\sigma}_x)} \sin \frac{m_0 \pi x}{L}. \tag{3}$$

* Cf. S. N. Kan. Prochnost', ustoichivost' i nesushchaya sposobnost' konstruktivno ortotropnykh obolochek (Strength, Stability, and Load-Carrying Capacity of Structurally Orthotropic Shells).—In: Sbornik "Raschet prostranstvennykh konstruktsii", No. 8, Gosstroizdat. 1962.

To these bending moments there corresponds a radial load of maximum intensity

$$p = -\frac{d^2 m_x}{dx^2}\bigg|_{\max} + \frac{\sigma_\varphi \delta}{R}\bigg|_{\max} = 2\frac{f_0 \delta \bar{\sigma}_x E}{(1-\bar{\sigma}_x)R^2}, \tag{4}$$

where $\sigma_\varphi = \frac{w}{R} E$ is the circumferential normal stresses in the skin.

In order to simplify the solution we shall assume in what follows that the intensity p is constant along the generator and directed toward the shell center, since such a direction of a constant load can lead to buckling. The influence of the variability of p on the critical stresses is allowed for by artificially lowering the initial deflection amplitude f_0 in (4). While this simplification is very convenient, it is not a necessary condition for solving the problem.

The collapse axial stress corresponding to the nonaxisymmetrical buckling mode will be, allowing for the influence of the constant radial load p:

$$\sigma_x = \sigma_{cr}\sqrt{1 - \frac{pR^3}{D(n^2-1)}}, \tag{5}$$

where n is the whole number of waves in the circumferential direction into which the shell buckles under the combined action of the axial forces and the radial pressure. The necessary minimum value of n must correspond to the whole number of half-waves along the generator, which is expressed by the following relationship:

$$\left(\frac{m\pi R}{nL}\right)^4 = \frac{D(n^2-1)^2}{R^2 E\delta}\left[1 - \frac{pR^3}{D(n^2-1)}\right]. \tag{6}$$

On the strength of (5) and (4) we obtain

$$\bar{\sigma}_x = \sqrt{1 - \frac{ER}{D(n^2-1)}\frac{2f_0 \delta \sigma_x}{(1-\sigma_x)}} \tag{7}$$

or

$$\frac{(1-\bar{\sigma}_x^2)(1-\bar{\sigma}_x)}{\bar{\sigma}_x} = \frac{24(1-\mu^2)}{(n^2-1)}\frac{R}{\delta}\bar{f}_0, \tag{8}$$

where

$$\bar{f}_0 = \frac{f_0}{\delta}$$

and μ is the Poisson ratio. Relationship (8) represents the equation for calculating $\bar{\sigma}_x$.

The analysis of (8) shows first of all that the relative magnitude of the load-carrying capacity $\bar{\sigma}_x$ is a function of the initial imperfections $\bar{f}_0$. In the case of high-grade surface finish ($\bar{f}_0$ tends to zero), σ_x approaches σ_{cr}.

It is necessary further on to note the substantial influence of R/δ on the value of $\bar{\sigma}_x$. A drop of $\bar{\sigma}_x$ takes place with the increase of R/δ, which agrees with the experimental data.

On the strength of (8) and (6) it is possible to explain why the material characteristics influence the value of $\bar{\sigma}_x$, a fact found so strange by Donnell. In fact, all other conditions being equal, the increase of the modulus of elasticity of the material E causes, according to (6), a reduction in the number of waves n, which in turn causes, in accordance with (8), a certain drop in $\bar{\sigma}_x$.

Relationship (6) finally elucidates why the values of L/m obtained in tests with real shells (allowing for p) are higher than the values calculated for ideal shells ($p = 0$).

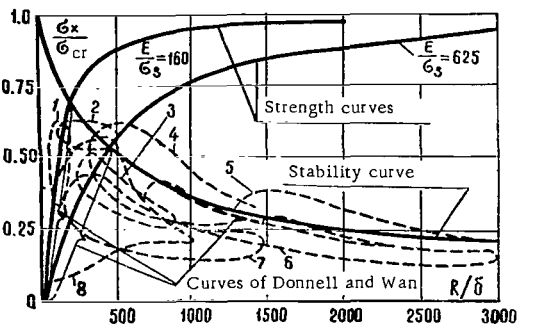
Practically, the utilization of (8) for structures of a certain grade of finish is envisaged in the following way. The value of $\bar{\sigma}_x$ is known from the data of tests with given value of R/δ. This makes it possible to calculate the constant parameter

$$A = \frac{24(1-\mu^2)}{(n^2-1)}, \quad f_0 = \frac{(1-\bar{\sigma}_x^2)(1-\bar{\sigma}_x)}{\bar{\sigma}_x R/\delta},$$

whose value is then substituted in (8):

$$\frac{(1-\bar{\sigma}_x^2)(1-\bar{\sigma}_x)}{\bar{\sigma}_x} = A\frac{R}{\delta}. \tag{9}$$

From (9) we determine $\bar{\sigma}_x$, and therefore σ_x, for any arbitrary value of R/δ. Such a sequence of using relationship (8) coincided with the results of the



Comparison of the results of calculation by (9) and (14) with the data of the experiments and of the theory of Donnell and Wan. The thin dotted lines correspond to the experimental data of various authors: 1 — celluloid (Flügge); 2 — steel (Robertson); 3 — steel, brass (Bridget); 4 — duralumin (Lundquist); 5 — steel (Boley); 6 — steel (GALCIT); 7 — steel, brass (Donnell); 8 — steel (Wilson, Newmark).

experiments and calculations given in the paper of Donnell and Wan* (see figure). When plotting the stability curve we took as initial point the point $\bar{\sigma}_x = 0.5$ lying on Donnell's curve.

CONDITIONS OF SHELL STABILITY

In the presence of an initial deflection (1), the compressive forces per unit length σ_x cause the appearance of bending moments (3), which lead to the appearance of longitudinal bending normal stresses

$$\sigma_{m_x} = \frac{m_x}{\delta^2/6}$$

or, in relative parameters,

$$\bar{\sigma}_{m_x} = \frac{\sigma_{m_x}}{\sigma_{cr}} = \frac{3\bar{f}_0\,\bar{\sigma}_x}{(1-\bar{\sigma}_x)} \sin\frac{m_0\pi x}{L}. \tag{10}$$

The elastic radial displacement (2) is accompanied by circumferential normal stresses

$$\sigma_\varphi = \frac{w}{R}E = \frac{E}{R}\frac{f_0\,\bar{\sigma}_x}{(1-\bar{\sigma}_x)} \sin\frac{m_0\pi x}{L}$$

or, in relative magnitudes,

$$\bar{\sigma}_\varphi = \frac{\sigma_\varphi}{\sigma_{cr}} = \frac{E}{\sigma_{cr}}\frac{\delta}{R}\bar{f}_0\frac{\bar{\sigma}_x}{(1-\bar{\sigma}_x)} \sin\frac{m_0\pi x}{L} = \frac{\bar{f}_0}{0.6}\frac{\bar{\sigma}_x}{(1-\bar{\sigma}_x)} \sin\frac{m_0\pi x}{L}. \tag{11}$$

Finally, the circumferential bending normal stresses at a Poisson ratio $\mu = 0.3$ are

$$\bar{\sigma}_{m_\varphi} = \mu\,\bar{\sigma}_{m_x} = 0.3\frac{3\bar{f}_0\,\bar{\sigma}_x}{(1-\bar{\sigma}_x)} \sin\frac{m_0\pi x}{L}. \tag{12}$$

We shall determine the value of σ_x which gives, in conjunction with σ_{m_x}, σ_φ and σ_{m_φ} by the third theory of strength, an equivalent stress equal to the yield stress of the material σ_s and leading to the collapse of the shell**

$$\sigma_x + (\sigma_{m_x})_{max} + (\sigma_\varphi)_{max} - (\sigma_{m_\varphi})_{max} = \sigma_s$$

or in relative magnitudes (expressing each term in σ_{cr} units):

$$\bar{\sigma}_x + (\bar{\sigma}_{m_x})_{max} + (\bar{\sigma}_\varphi)_{max} - (\bar{\sigma}_{m_\varphi})_{max} = \bar{\sigma}_s. \tag{13}$$

Substituting here $\bar{\sigma}_{m_x}$, $\bar{\sigma}_\varphi$, $\bar{\sigma}_{m_\varphi}$ and taking $\sin\frac{m_0\pi x}{L} = 1$ we obtain the following

* Donnell, L. and J. Wan. Vliyanie nepravil'nostei v forme na ustoichivost' sterzhnei i tonkostennykh tsilindrov pri osevom szhatii.—In: Sbornik "Mekhanika," No. 4. 1951. [Original title: Study of the Influence of Imperfections on the Process of Buckling.—J. Appl. Mech., Vol. 17, p. 73. 1950.]

** The condition of equality of the resultant stresses and the yield stress of the material at the extreme fibers does not exhaust the load-carrying capacity, and is arbitrary in this sense. The safety margin obtained as a result, connected with the variation of the stresses σ_{m_x} over the section, compensates for the initial deviations which are not allowed for in the calculation.

quadratic equation relative to the unknown magnitude $\bar{\sigma}_x$:

$$\bar{\sigma}_x + \frac{3.8\bar{f}_0\,\bar{\sigma}_x}{(1-\bar{\sigma}_x)} = \bar{\sigma}_s = \frac{1}{0.6}\,\frac{\sigma_s}{E}\,\frac{R}{\delta}\,. \qquad (14)$$

According to (14), the load-carrying capacity ($\bar{\sigma}_x$) is determined by the following parameters: the relative magnitude of the initial deflection $\bar{f}_0 = f_0/\delta$, the relative geometry of the shell section R/δ, and the material characteristic E/σ_s. The increase of $\bar{f}_0$ and E/σ_s leads to a reduction in $\bar{\sigma}_x$. The increase of R/δ leads to an increase in σ_x.

Strength curves $\bar{\sigma}_x = \varphi\,(R/\delta)$ have been plotted in the figure by (14) for materials with $E/\sigma_s = 160$ and = 625. The amplitude of the initial axisymmetrical deflection for these curves was taken as $f_0 = 0.14\,\delta$.

The analysis of the curves of the figure shows that at values of R/δ smaller than a certain magnitude it is necessary to use the strength curve for determining σ_x, while at higher values of R/δ the stability curve is to be used to that end.

CONCLUSION

The physical clarity of the proposed interpretation of the structure behavior, the simplicity of the derivation of the computing relationships and of the plotting of the curves of strength and stability of the shell, and the practically perfect coincidence with the results of the experiments and calculations by the relatively complex method of Donnell and Wan speak for themselves in favor of the solution proposed.

It should be noted also that, since the lateral load is equivalent, from the point of view of the stressed state, to some initial deflection, the method for calculating the load-carrying capacity proposed here applies directly to this type of load as well. The lateral load can sharply reduce the load-carrying capacity of the shell in those cases when the initial deflection equivalent to it approaches the "critical" harmonic. It is at the same time obvious that large local dents can influence only weakly the load-carrying capacity, if they have an irregular pattern; in fact, their Fourier series expansions have very small amplitudes for the harmonics near to the "critical" one.

FREE VIBRATIONS OF A CYLINDRICAL SHELL WITH STIFFENING RIBS

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(Kiev)

Numerous studies have been devoted to the question of the vibrations of reinforced shells. Despite the outstanding scientific and practical importance of these studies, no efficient method for solving this problem in the engineering practice has been evolved. The existing methods are either coarsely approximate or reduce the solution of the problem to solving an infinite system of algebraic equations. The frequency equation is obtained in the form of infinite determinants, which are of little use even for analyzing the spectrum of frequencies of a reinforced shell. As a result, almost no studies have as yet been conducted on the optimum parameters of the stiffeners and of their optimum disposition on the shell, and of the design of structures of minimum weight.

A method is proposed below for solving the problem of free vibrations of a shell reinforcement by stringers, allowing for the discrete arrangement of the latter on the shell. The frequency equation is obtained in the form of a transcendental equation which can be easily analyzed.

STATEMENT OF THE PROBLEM AND METHOD FOR SOLUTION

Consider a circular cylindrical shell reinforced by stringers of identical geometrical and elastic characteristics, arranged on the shell at equal angular distances $\alpha = \frac{2\pi}{k}$, where k is the number of stringers. Let the shell rest by its ends on diaphragms absolutely rigid in their plane and easily yielding to lateral deformations, and let it be loaded by an axial static force P distributed uniformly over its ends. The vibrations are small and the deformations lie within the limits of proportionality.

The problem of the free vibrations of a reinforced shell is treated as a contact problem of the theory of vibrations of shells and bars. It is assumed that the contact between the shell and the i-th stringer is such that it does not prevent the free sliding of the stringer along the shell and its free twisting relative to the line of contact, but ensures equal displacements of the points in contact along the normal to the middle surface of the shell.

Assume that the components of the displacements of the middle surface points have been specified in some way, defining thus the deformed state

of the shell. The shell and the stringer being in contact, this defines the deformed state of the stringer as well, Using the equations of vibrations of the stringer, we determine the load on the stringer which would give it this deformation. According to the scheme of contact adopted, the latter must be directed along the normal to the middle surface of the shell. We mentally remove the stringer, and replace its action on the shell by this load with reversed sign.

As follows from this statement of the problem, we start in the solution of the problem of the free vibrations of a reinforced shell from the equations of vibrations of a smooth shell /1/ and the equations of vibrations of a bar in a local system of coordinates (/2/, p. 302). The latter are of the form

$$\left\{\frac{1}{r^2}\frac{\partial^2}{\partial\xi^2}-\frac{\gamma}{gE_c}\frac{\partial^2}{\partial t^2}\right\}\left\{Fu-\frac{S_y}{r}\frac{\partial}{\partial\xi}v-\frac{S_z}{r}\frac{\partial}{\partial\xi}w-\frac{S_\Omega}{r}\frac{\partial}{\partial\xi}\theta\right\}=0;$$

$$\left\{\frac{1}{r^3}\frac{\partial^3}{\partial\xi^3}-\frac{\gamma}{gE_c r}\frac{\partial^3}{\partial\xi\partial t^2}\right\}\left\{S_y u-\frac{J_{yy}}{r}\frac{\partial v}{\partial\xi}-\frac{J_{zy}}{r}\frac{\partial w}{\partial\xi}-\frac{J_{\Omega y}}{r}\frac{\partial\theta}{\partial\xi}\right\}-$$

$$-\left\{\frac{\gamma}{gE_c}\frac{\partial^2}{\partial t^2}+\frac{N}{E_c r^2}\frac{\partial^2}{\partial\xi^2}\right\}\{Fv-S_z\theta\}=0;$$

$$\left\{\frac{\partial^3}{\partial\xi^3}\frac{1}{r^3}-\frac{\gamma}{gE_c r}\frac{\partial^3}{\partial\xi\partial t^2}\right\}\left\{S_z u-\frac{J_{zy}}{r}\frac{\partial v}{\partial\xi}-\frac{J_{zz}}{r}\frac{\partial w}{\partial\xi}-\frac{J_{z\Omega}}{r}\frac{\partial\theta}{\partial\xi}\right\}-$$

$$-\left\{\frac{\gamma}{gE_c}\frac{\partial^2}{\partial t^2}+\frac{N}{E_c r^2}\frac{\partial^2}{\partial\xi^2}\right\}\{Fw+S_y\theta\}=-\frac{g_z}{E_c};$$

$$\left\{\frac{1}{r^3}\frac{\partial^3}{\partial\xi^3}-\frac{\gamma}{gE_c r}\frac{\partial^3}{\partial\xi\partial t^2}\right\}\left\{S_\Omega u-\frac{J_{\Omega y}}{r}\frac{\partial v}{\partial\xi}-\frac{J_{\Omega z}}{r}\frac{\partial w}{\partial\xi}-\frac{J_\Omega}{r}\frac{\partial\theta}{\partial\xi}\right\}-$$

$$-\left\{\frac{\gamma}{gE_c}\frac{\partial^2}{\partial t^2}+\frac{N}{E_c r^2}\frac{\partial^2}{\partial\xi^2}\right\}\{S_z v-S_y w-(J_{yy}+J_{zz})\theta\}+\frac{GJ_\alpha}{E_c r^2}\frac{\partial^2\theta}{\partial\xi^2}=0; \qquad (1)$$

$$N=\frac{P}{4\pi rh+kF}.$$

The physical meaning of the magnitudes entering in these equations is explained in /2/ (p. 302) and in /3/. We only changed the designations of the coordinate axes, according to the following rule: x to y, y to z, z to x, and introduced the relative coordinate $\xi=\frac{x}{r}$, where r is the radius of the middle surface of the shell.

The problem reduces to determining the nontrivial solutions of the initial equations in the following form:

$$u(\xi,\beta,t)=\sum_{m=0}^{\infty}(u_{mc}\cos m\beta+u_{ms}\sin m\beta)\cos\lambda_n\xi\sin(\omega t+\alpha);$$

$$v(\xi,\beta,t)=\sum_{m=0}^{\infty}(v_{mc}\cos m\beta+v_{ms}\sin m\beta)\sin\lambda_n\xi\sin(\omega t+\alpha); \qquad (2)$$

$$w(\xi,\beta,t)=\sum_{m=0}^{\infty}(w_{mc}\cos m\beta+w_{ms}\sin m\beta)\sin\lambda_n\xi\sin(\omega t+\alpha);$$

$$u_{cl}(\xi,t)=A_l\cos\lambda_n\xi\sin(\omega t+\alpha);$$

$$v_{cl}(\xi,t)=B_l\sin\lambda_n\xi\sin(\omega t+\alpha); \qquad (3)$$

$$w_{ci}(\xi, t) = \sum_{m=0}^{\infty} (w_{mc}\cos m\beta_i + w_{ms}\sin m\beta_i)\sin\lambda_n\xi\sin(\omega t + \alpha);$$

$$\theta(\xi, t) = D_i \sin\lambda_n\xi \sin(\omega t + \alpha); \qquad (3) \text{ (cont'd)}$$

where u, v, w and u_{ci}, v_{ci}, w_{ci} are the components of the displacements of the points of the middle surface of the shell and the points of the i-th stringer, lying on the line of contact; θ is the angle of twist of the stringer relative to the line of contact; $\lambda_n = \frac{n\pi r}{l}$; l is the shell length; ω and α are the unknown frequency and phase angle of vibrations. Substituting (2) and (3) in the initial equations, we obtain after elementary transformations a system of linear algebraic equations relative to w_{mc} and w_{ms} whose solution yields:

$$w_{mc} = r^2 \frac{1-\sigma^2}{2E_0 h} \frac{\Delta_{33}(\lambda, m)}{D(\lambda, m)} Z_{cm}; \qquad w_{ms} = r^2 \frac{1-\sigma^2}{2E_0 h} \frac{\Delta_{33}(\lambda, m)}{D(\lambda, m)} Z_{sm}; \qquad (4)$$

where

$$Z_{cm} = \frac{kD_c E_c}{2\pi r D_{44}} \left\{ \sum_{r=1}^{\infty} w_{c(kr-m)}\cos r\pi + \sum_{r=0}^{\infty} w_{c(kr+m)}\cos r\pi \right\};$$

$$Z_{sm} = \frac{kD_c E_c}{2\pi r D_{44}} \left\{ \sum_{r=1}^{\infty} w_{s(kr+m)}\cos r\pi - \sum_{r=1}^{\infty} w_{s(kr-m)}\cos r\pi \right\}. \qquad (5)$$

Here $D(\lambda, m)$ is a known third-order operator (the natural frequencies of the smooth shell are determined from the equation $D(\lambda, m) = 0$); Δ_{33} is its minor corresponding to the element a_{33}; D_c is a known fourth-order determinant (the natural frequencies of the bar are determined from the equations $D_c = 0$); D_{44} is its minor, corresponding to the element a_{44}; E_0, σ, and h are, respectively, the modulus of elasticity, the Poisson ratio, and the half-thickness of the shell; E_c is the modulus of elasticity of the bar. Formulas (4) and (5) yield an infinite system of algebraic equations for determining w_{mc} and w_{ms}, obtained by substituting in (4) the values of Z_{cm} and Z_{sm} from (5). This is the procedure usually adopted, and as a result a frequency equation in the form of an infinite determinant is obtained. Some of the possible modes of vibration are dropped out in the course of the derivation; these are the modes of vibration corresponding to some natural frequencies of the nonreinforced shell.

There is, however, another possibility of forming the frequency equation, also starting from (4) and (5). Having determined $w_{c(kr-m)}$, $w_{c(kr+m)}$ and $w_{s(kr-m)}$, $w_{s(kr+m)}$ from (4) and substituted the values obtained in (5), we conclude that nontrivial solutions of the initial equations in the form of (2) and (3) are possible if the following alternative conditions are satisfied:

$$Z_{cm} = 0; \quad Z_{sm} = 0; \qquad (6)$$

$$\frac{(1-\sigma^2) rkD_c E_c}{4\pi h D_{44} E_0} \left\{ \sum_{r=1}^{\infty} \frac{\Delta_{33}(\lambda, kr-m)}{D(\lambda, kr-m)} + \sum_{r=0}^{\infty} \frac{\Delta_{33}(\lambda, kr+m)}{D(\lambda, kr+m)} \right\} = 1, \qquad (7)$$

$$\left(m = 0, 1, 2, \cdots, r \leqslant \frac{k}{2}\right).$$

We make use here of equalities of the type

$$Z_{cm}\cos r\pi = Z_{c(kr-m)} = Z_{c(kr+m)};$$

$$Z_{sm}\cos r\pi = -Z_{s(kr-m)} = Z_{s(kr+m)}$$

$$(m=0,\ 1,\ 2,\cdots,\ k).$$

It can be established that when equalities (6) are fulfilled the reinforced shell vibrates with the frequency of the nonreinforced shell. The middle surface of the shell is then split in the lateral direction in a number of half-waves which is a multiple of the number of stringers. The stringers are located at the nodes of the natural modes of vibrations corresponding to these frequencies. If we exclude these modes of vibrations from consideration, there remains equation (7) as frequency equation. Equation (7) is transcendental, as can be easily shown by summing the series in it. It will then be written in the following form:

$$\frac{D_{44}}{D_c} = \frac{r(1-\sigma^2)E_c}{2E_0h}\sum_{j=1}^{4}\frac{D_{33}(\lambda, P_j)}{\overline{D}'(\lambda, P_j)}\,\frac{\operatorname{sh} p_j\frac{2\pi}{k}}{\cos m\frac{2\pi}{k} - \operatorname{ch} p_j\frac{2\pi}{k}}, \tag{8}$$

where $D(\lambda, p) = D(\lambda, ip)$, $D_{33}(\lambda, p) = \Delta_{33}(\lambda, ip)$, $i = \sqrt{-1}$, $\overline{D}'$ is the derivative of $D(\lambda, p)$ by p, p_j are the roots of the equation $D(\lambda, p) = 0$ with positive real part. Such a notation is convenient for numerical calculations.

In order to be able to study conveniently each of the equations of (7), we write them in the form

$$f(\omega_n) = \varphi(\omega_n), \tag{9}$$

where

$$f(\omega_n) = \frac{kE_c}{2\pi r\rho}\left\{\sum_{r=0}^{\infty}\frac{1}{C^2_{n(kr+m)} - \omega_n^2} + \sum_{r=1}^{\infty}\frac{1}{C^2_{n(kr-m)} - \omega_n^2}\right\},$$

$$\varphi(\omega_n) = \sum_{i=1}^{3}\frac{D_{44}(\omega_i^2)}{D_c'(\omega_i^2)}\,\frac{1}{\omega_n^2 - \omega_i^2}, \tag{10}$$

$\omega_i\,(i = 1, 2, 3)$ are the natural frequencies of the bar; C_{nm} are the natural frequencies of the nonreinforced shell, whose vibrations are accompanied by the division of the middle surface of the shell into n and m half-waves in the longitudinal and lateral directions respectively; ρ is the density of the shell material; D_c' is the derivative of D_c by ω_n^2.

It can be easily shown that in the intervals of continuity (for f and φ these intervals lie respectively between the natural frequencies of the bar and the natural frequencies of the nonreinforced shell) $f(\omega_n)$ and $\varphi(\omega_n)$ have derivatives of the same sign and that $f(0) > 0$ while $\varphi(0) < 0$. At the ends of the continuity intervals these functions suffer discontinuities of the type $+\infty$ and $-\infty$. Taking into account the above-mentioned properties of the functions $f(\omega_n)$ and $\varphi(\omega_n)$, and allowing for the fact that their continuity intervals overlap either completely or partially, we reach the following conclusion.

The roots of each of equations (7) lie within intervals bounded by the magnitudes $\omega_i\,(i = 1, 2, 3)$, $C_{n(kr+m)}$, $C_{n(kr-m)}$, arranged in an increasing order, each interval containing one root only. Thus, if $\omega_1 < C_{nm} < C_{n(k-m)} < \omega_2 < \cdots$,

the first root of the equation will be larger than or equal to ω_1, but smaller than or equal to C_{nm}, the second will be larger than or equal to C_{nm}, but smaller than or equal to $C_{n(k-m)}$, etc. The magnitudes ω_l, $C_{n(kr+m)}$, $C_{n(kr-m)}$ representing the natural frequencies of the bar and shell and the frequency spectra of the latter being discrete, the roots of each equation of (7) form a discrete spectrum of numbers. In other words, in every interval bounded by two finite numbers a and b there exists a finite number of simple roots of these equations. It follows in particular from here that the frequency spectrum of a reinforced shell for which the middle surface of the shell and the stringer axis are divided into n half-waves in the longitudinal direction is discrete, and that the minimum frequency will be the smallest of the magnitudes ω_l, $C_{n(kr+m)}$, $C_{n(kr-m)}$.

As known, the larger the number of half-waves formed along the shell (bar) length during the vibrations, the higher the natural frequency of these vibrations. It follows that, at any finite number $N>0$, only a finite number of frequencies of the reinforced shell is contained in the inerval $(0, N)$. In fact, those frequencies to which there correspond n half-waves along the reinforced shell, the smallest of the magnitudes ω_i, $C_{n(kr-m)}$, $C_{n(kr+m)}$ being larger than N turn out to lie outside the limits of this interval.

The following conclusions can be made from these studies.

1. The spectrum of natural frequencies of a shell reinforced by stringers is discrete.

2. If the fundamental frequency of the stringers reinforcing the shell is larger (smaller) than the fundamental frequency of the shell which is to be reinforced, the fundamental frequency of the resulting reinforced shell will likewise be larger (smaller) than the latter.

3. The larger the number of stringers, the nearer the fundamental frequency of the reinforced shell to the fundamental frequency of the reinforcing stringer.

4. The frequency spectrum of the reinforced shell always contains the natural frequency of the shell without the stringers, if

a) the reinforcing stringers are arranged at the nodes of the natural mode corresponding to this frequency;

b) the frequency spectra of the stringers contain this frequency.

5. The spectrum of natural frequencies of the shell is sufficiently dense, and it becomes even denser after its reinforcement by the stringers.

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POST-BUCKLING STATE AND STRESS CONCENTRATION IN ANNULAR PLATES

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Consider an annular plate of inner radius b and outer radius a, with inner contour free, and outer contour either hinged or rigidly clamped. Let a normal compressive force per unit length P be applied to the outer contour. We shall study the stress concentration on the free contour of the plate in the post-buckling elastic axisymmetrical state.

The problem reduces to integrating the equations /1/

$$D\nabla^2\nabla^2 w - \frac{1}{r}\frac{d}{dr}\left(rT_r^*\frac{dw}{dr}\right) = 0, \tag{1}$$

$$r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\left(r^2 T_r^*\right) + \frac{Eh}{2}\left(\frac{dw}{dr}\right)^2 = 0 \tag{2}$$

$$\left(\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)$$

with boundary conditions:

1) outer contour hinged:

$$r = a \quad T_r^* = -P, \quad w = 0, \quad \frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} = 0, \tag{3}$$

$$r = b \quad T_r^* = 0, \quad \frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} = 0, \quad \frac{d}{dr}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = 0; \tag{4}$$

2) outer contour rigidly clamped:

$$r = a \quad T_r^* = -P, \quad w = 0, \quad \frac{dw}{dr} = 0; \tag{5}$$

$$r = b \quad T_r^* = 0, \quad \frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} = 0, \quad \frac{d}{dr}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = 0. \tag{6}$$

In accordance with /2/, we represent P, T_r^* in the following form:

$$P = P^0 + \tilde{P}, \quad T_r^* = T_r^0 + \tilde{T}_r + T_r^k \tag{7}$$

and introduce the following dimensionless parameters and symbols:

$$t_\rho^* = \frac{T_r^*}{P^0}, \quad t_\rho^0 = \frac{T_r^0}{P^0}, \quad \tilde{t}_\rho = \frac{\tilde{T}_r}{\tilde{P}}, \quad t_\rho^k = \frac{T_r^k}{P^0},$$

$$\rho = \frac{r}{a}, \quad \bar{w} = \frac{w}{h}, \quad \varphi = \frac{d\bar{w}}{d\rho}, \quad \alpha = \frac{b}{a}, \quad \varepsilon = \frac{\tilde{P}}{P^0}. \tag{8}$$

$$\lambda^2 = \frac{a^2P^0}{D(1-\alpha^2)}, \quad \delta = \frac{6(1-\nu^2)}{\lambda^2},$$

where $P^0, \tilde{P}$ are, respectively, the critical and the additional contour load: $T_r^0, \tilde{T}_r$ are the internal forces per unit length corresponding to these loads; T_r^k is the corrective force per unit length.

Integrate equation (1) allowing for the conditions for the shearing force in (4) and (6) and introduce designations (7) and (8); equations (1) and (2) become then of the form:

$$L\varphi = (1-\alpha^2)\lambda^2 (\varepsilon \tilde{t}_\rho + t_\rho^k)\varphi, \tag{9}$$

$$Nt_\rho^k = -\delta\varphi^2, \tag{10}$$

where

$$L = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \left(\lambda^2 - \frac{q^2}{\rho^2}\right),$$

$$N = \rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho^2, \quad q^2 = 1 + \alpha^2\lambda^2.$$

The boundary conditions are:

$$1.\ \rho = 1, \quad t_\rho^k = 0, \quad \frac{d\varphi}{d\rho} + \frac{\nu}{\rho}\varphi = 0; \tag{11}$$

$$\rho = \alpha, \quad t_\rho^k = 0, \quad \frac{d\varphi}{d\rho} + \frac{\nu}{\rho}\varphi = 0; \tag{12}$$

$$2.\ \rho = 1, \quad t_\rho^k = 0, \quad \frac{d\varphi}{d\rho} = 0; \tag{13}$$

$$\rho = \alpha, \quad t_\rho^k = 0, \quad \frac{d\varphi}{d\rho} + \frac{\nu}{\rho}\varphi = 0. \tag{14}$$

The second formula of (7) becomes

$$t_\rho^* = t_\rho^0 + \varepsilon\tilde{t}_\rho + \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 t_{\rho,1}^k + \left(\frac{\varepsilon}{1+\varepsilon}\right)^4 t_{\rho,2}^k + \cdots \tag{15}$$

Equations (9) and (10) will be solved by the small-parameter method. In order to increase the region of applicability of this method we shall express the parameter ε through another parameter η by the formula

$$\varepsilon = \frac{\eta}{1-\eta} = \eta + \eta^2 + \eta^3 + \cdots$$

We then find the solution of equations (9) and (10) in the form /3/

$$\varphi = \eta\varphi^0 + \eta^3\varphi_1 + \cdots = \left(\frac{\varepsilon}{1+\varepsilon}\right)\varphi_0 + \left(\frac{\varepsilon}{1+\varepsilon}\right)^3\varphi_1 + \cdots, \tag{16}$$

$$t_\rho^k = \eta^2 t_{\rho,1}^k + \eta^4 t_{\rho,2}^k + \cdots = \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 t_{\rho,1}^k + \left(\frac{\varepsilon}{1+\varepsilon}\right)^4 t_{\rho,2}^k + \cdots \tag{17}$$

By substituting (16) and (17) in (9) and (10) and equating the terms on the left and right with the same power of η we obtain the following equations:

for η

$$L\varphi_0 = 0; \tag{18}$$

for η^2

$$Nt_{\rho,1}^k = 0; \tag{19}$$

for η^3

$$L\varphi_1 = (1-\alpha^2)\lambda^2(\tilde{t}_\rho + t_{\rho,1}^k)\varphi_0; \tag{20}$$

for η^4

$$Nt^k_{\rho,2} = -2\delta\varphi_0\,\varphi_1; \tag{21}$$

for η^5

$$L\varphi_3 = (1-\alpha^2)\,\lambda^2\,[(\bar{t}_\rho + t^k_{\rho,1})\,\varphi_1 + (\bar{t}_\rho + t^k_{\rho,2})\,\varphi_0], \tag{22}$$

etc.

The boundary conditions for these equations are (11) to (14) as before, with φ replaced by $\varphi_0, \varphi_1, \cdots$, and t^k_ρ by $t^k_{\rho,1}$, $t^k_{\rho,2}, \cdots$

The solution of (18) will be represented in the form

$$\varphi_0 = A\,\psi_0\,(\lambda\rho), \tag{23}$$

where A is a constant, $\psi_0\,(\lambda\rho)$ is an eigenfunction.

It then follows from (10) that the solution of (19) will be proportional to A^2. The value of A will then be determined from the condition of existence of a solution of equation (20)

$$\int_\alpha^1 \rho\psi_0^2\,(\bar{t}_\rho + t^k_{\rho,1})\,d\rho = 0. \tag{24}$$

With A known, the solution of (20) will be

$$\varphi_1 = B\,\psi_0 + \overset{\text{n}}{\varphi_1}, \tag{25}$$

where B is a constant and $\overset{\text{n}}{\varphi_1}$ is a particular integral of the nonhomogeneous equation.

Having found the solution of equation (21), we determine the constant B (allowing for (25)) from the condition of existence of a solution of (22):

$$\int_\alpha^1 \rho\,\psi_0[\,(\bar{t}_\rho + t^k_{\rho,1})\,\varphi_1 + (\bar{t}_\rho + t^k_{\rho,2})\,\varphi_0]\,d\rho = 0. \tag{26}$$

Condition (26) will be linear relative to B, while (25) will be nonlinear relative to A.

This process for obtaining solutions can be extended further.

The eigenfunction $\psi_0\,(\rho\lambda)$ in (23) for annular plates is of the form

$$\psi_0\,(\rho\lambda) = J_q(\rho\lambda) - \gamma\,N_q\,(\rho\lambda), \tag{27}$$

where J_q, N_q are Bessel functions and γ is a constant, depending on the boundary conditions.

At small α $(q \simeq 1)$ the functions J_q, N_q can be approximately represented thus:

$$\begin{gathered} J_q(z) = J_1\,(z) + \delta^* \lim_{q\to 1}\frac{\partial J_q}{\partial q} + \cdots \approx J_1\,(z)\left(1 + \delta^*\,ln\,\frac{z}{2}\right) + \delta^*\,i_1\,(z), \\ N_q\,(z) = N_1\,(z) + \delta^* \lim_{q\to 1}\frac{\partial N_q}{\partial q} + \cdots \approx N_1, \quad z = \rho\lambda, \end{gathered} \tag{28}$$

where

$$i_1\,(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{z}{2}\right)^{2k+1}\frac{d}{dt}\,\frac{1}{\Gamma(t)}\bigg|_{t=k+2}, \quad \delta^* = \frac{1}{2}\,\alpha^2\,\lambda^2.$$

Using the expression for N_1, formula (27) becomes, after some simplifications,

$$\psi_0(z) = J_1(z)\left(1 - \frac{2\gamma c}{\pi}\right) + \delta^* i_1(z) - \gamma n_1(z) \qquad (c = 0.57\cdots), \tag{29}$$

where $i_1(z)$, $n_1(z)$ can be approximately represented by the following formulas:

$$\begin{aligned} i_1(z) &\approx -0.43\left(\frac{z}{2}\right) + 0.46\left(\frac{z}{2}\right)^3 - 0.10\left(\frac{z}{2}\right)^5 + 0.01\left(\frac{z}{2}\right)^7, \\ n_1(z) &\approx -\frac{0.63}{z} - 0.31\left(\frac{z}{2}\right) + 0.39\left(\frac{z}{2}\right)^3 - 0.08\left(\frac{z}{2}\right)^5 + 0.009\left(\frac{z}{2}\right)^7. \end{aligned} \tag{30}$$

We shall consider an example.

1. The outer edge is hinged ($\alpha = 0.1$). The value of γ is then determined from the condition

$$\gamma = \left.\frac{\lambda J_{q-1}(z) + \frac{\nu - q}{\rho} J_q(z)}{\lambda N_{q-1}(z) + \frac{\nu - q}{\rho} N_q(z)}\right|_{\rho=\alpha}$$

for $\alpha = 0.1$, $q \approx 1$. Using the value $\lambda_1 = 1.97$ for λ_1, we obtain $\gamma = -0.06$, $\delta^* = 0.02$ and

$$\varphi_0 = A\left(\frac{0.09}{\rho} + 0.98\rho - 0.48\rho^3 + 0.07\rho^5\right), \quad (A = \pm 5.5), \tag{31}$$

$$\varphi_1 = B\psi_0 + A\lambda_1^2\left(-\frac{6\cdot 10^{-4}}{\rho} - 0.06\rho + 0.41\rho^3 - 0.26\rho^5 - 0.01\rho^7\right) \quad (B = \pm 2.75). \tag{32}$$

The following expressions are obtained for $t^*_{\rho,1}, t^*_{\rho,2}$:

$$t^*_{\rho,1} = 84\cdot 10^{-4}\frac{\ln\rho}{\rho^2} - \frac{2.94\cdot 10^{-2}}{\rho^2} - 0.84\ln\rho + 2.94 - 4.62\rho^2 + 1.68\rho^4, \tag{33}$$

$$t^*_{\rho,2} = 10^{-2}\frac{\ln\rho}{\rho^2} - \frac{10^{-2}}{\rho^2} - 0.68\ln\rho + 1.74 - 2.66\rho^2 - 1.22\rho^4 + 2.9\rho^6 - 0.64\rho^8. \tag{34}$$

Using the formula for the circumferential force per unit length $t^*_\varphi = \frac{d}{d\rho}(\rho t^*_\rho)$ we calculate, with the aid of (15), (33), and (34), the value of $t^*_\varphi|_{\rho=0.1}$ for different values of ε (see table). The stress concentration about the free hole (the outer edge is hinged) decreases with the increase of the post-buckling load.

TABLE

ε	$t^*_\varphi\|_{\rho=0.1}$	
	Outer edge hinged	Outer edge built-in
0.1	−2.13	−2.2
0.3	−2.12	−2.41
0.5	−1.92	−2.67
0.7	−1.69	−2.90

2. The outer edge is built-in (α = 0.12). Taking /4/ into account, and determining γ by the formula

$$\gamma = \left.\frac{J_q\,(\rho\lambda_1)}{N_q\,(\rho\lambda_1)}\right|_{\rho=1}$$

for λ_1 = 3.68, we obtain = 0.25 and

$$\varphi_0 = A\left(\frac{0.045}{\rho} + 1.78\rho - 3.28\rho^3 + 1.92\rho^5 - 0.56\rho^7 + 0.03\rho^9\right) \quad (A = \pm 4.5). \qquad (35)$$

The solution of (25) is of the form

$$\varphi_1 = B\psi_0\,(z) + A\lambda_1^2\left(\frac{3.5\cdot 10^{-3}}{\rho} + 0.18\rho - 0.01\rho^2 - 0.05\rho^3 - 0.22\rho^4 - \right.$$

$$\left. -0.30\rho^5 + 0.27\rho^6 + 0.45\rho^7 - 0.18\rho^8 - 0.44\rho^9 + 0.24\rho^{11} - 0.10\rho^{13}\right) \qquad (36)$$

$$B = -4.7 \quad (A > 0), \quad B = 4.7 \quad (A < 0).$$

We obtain for $t^k_{\rho,1}$, $t^k_{\rho,2}$

$$t^k_{\rho,1} = 7.2\cdot 10^{-3}\frac{\ln\rho}{\rho^2} - \frac{1.8\cdot 10^{-2}}{\rho^2} - 0.63\ln\rho + 1.08 - 3.24\rho^2 + 4.32\rho^4 -$$

$$-3.24\rho^6 + 1.62\rho^8 - 0.54\rho^{10},$$

$$t^k_{\rho,2} = -0.013\frac{\ln\rho}{\rho^2} + 0.013\frac{\ln\rho}{\rho} - \frac{0.035}{\rho^2} - 1.3\ln\rho - 2.56 +$$

$$+2.89\rho^2 + 2.20\rho^4 - 2.86\rho^5 - 6.77\rho^6 + 4.54\rho^7 + 8.12\rho^8 -$$

$$-3.59\rho^9 - 6.71\rho^{10} + 2.39\rho^{11} + 3.53\rho^{12} - 0.71\rho^{13} - 0.47\rho^{14}. \qquad (37)$$

The value of $t^*_\varphi|_{\rho=0.12}$ for different ε is calculated in the same way as in point 1. The data in the table indicate an increase in the stress concentration about the free hole (the outer contour is built-in) in the post-buckling stage.

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STUDY OF THE STRESSED STATE OF RIBBED SHELLS WITH ZERO GAUSSIAN CURVATURE

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Consider a shell of zero Gaussian curvature subjected to a symmetrical load. The shell is provided with ribs arranged along the generators at equal distance from each other along the directrix. The stressed state of this shell will be represented as a superposition of the symmetrical stressed state of a shell without ribs under the given external load and of the nonsymmetrical stressed state, i.e., of the disturbance introduced by the ribs.

We shall use the following symbols:

$$\begin{aligned} &\text{tensile forces per unit length } T_1 = T_{1c};\ T_2 = T_{2c} + T_{2p};\\ &\text{shearing force per unit length } N_2 = N_{2p};\\ &\text{tangential force per unit length } S_1 = S_2 = S = 0;\\ &\text{bending moment } G_2 = G_{2p}. \end{aligned} \tag{1}$$

We assume that the disturbance introduced by the ribs only slightly alters the symmetrical stressed state along the generators; it can therefore be considered that the forces and moments T_1, N_1, G_1 are determined by the symmetrical stressed state, and the influence of the ribs on their magnitude, can be neglected.

It remains to examine the influence of the ribs on the magnitude of the forces and moments T_2, N_2, G_2. Let us cut out one of the ribs and the two sections adjoining it (Figure 1).

In order to ensure the joint deformation of the rib and the shell along the entire rib, the following conditions of strain compatibility must be satisfied: the shell deflection must be equal to the deflection of the rib; the relative angle of rotation γ_2 and the relative displacement v of one section relative to the adjacent one must be equal to zero.

$$1.\ w_{sh} = w_r;\quad 2.\ \gamma_2 = 0;\quad 3.\ v = 0. \tag{2}$$

The shell deflection will be represented as the sum of the deflections of the symmetrical and nonsymmetrical stressed states.

$$w_{sh} = w_{sh,s} + w_{sh,ns}. \tag{3}$$

The deflection $w_{sh,ns}$, the angle of rotation γ_2, and the displacement will be represented by the influence coefficients in the following form:

$$
\begin{aligned}
w_{\text{sh,ns}} &= T_2 c_{11} + G_2 c_{12} + N_2 c_{13};\\
\gamma_2 &= T_2 c_{21} + G_2 c_{22} + N_2 c_{23};\\
v &= T_2 c_{31} + G_2 c_{32} + N_2 c_{33}.
\end{aligned}
\tag{4}
$$

We have taken here as directions 1, 2, and 3 the directions of the deflection w, the angle of rotation γ_2, and the displacement v, respectively.

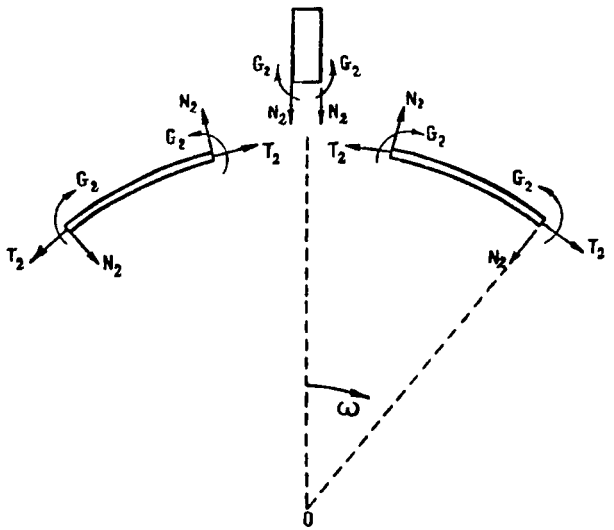

FIGURE 1.

The following three cases of loading are considered:

$$
\begin{aligned}
&1.\ N_2 = 1; \quad G_2 = T_2 = S = 0;\\
&2.\ G_2 = 1; \quad T_2 = N_2 = S = 0;\\
&3.\ T_2 = 1; \quad N_2 = G_2 = S = 0.
\end{aligned}
\tag{5}
$$

The deflection $w_{\text{sh,s}}$ is known for various cases of loading and edge conditions; it is proposed, e.g. /2/, to take its value as a function of the external load and the edge conditions of the ribbed shell.

We shall represent $w_{\text{sh,s}}$ as a series by the functions $\varphi_i(\xi)$, where $\varphi_i(\xi)$ are the eigenfunctions of the equation

$$
\frac{d^4\varphi_i}{d\xi^4} - \varkappa_i^4 \varphi_i = 0 \ \text{(for a cylinder)} \tag{6}
$$

for the same boundary conditions as at the edges of the ribbed shell.

$\varphi_i(\xi)$ are given in Gol'denveizer's paper /1/ for various edge conditions. Thus,

$$
w_{\text{sh,s}}(\xi) = \sum_{k_i = k_1}^{k_i = k_n} A_{k_i} \cdot \varphi_i(\xi), \tag{7}
$$

where

$$A_{k_i} = \frac{\int_0^l w_{\text{sh,s}}(\xi)\cdot\varphi_i(\xi)\,d\xi}{\int_0^l \varphi_i^2(\xi)\,d\xi}. \tag{8}$$

Here $\varkappa_i = k_i R$ are the eigenvalues of equation (7); $\xi = \frac{x}{R}$ is a dimensionless coordinate; R is the shell radius; l is the shell length.

$\int_0^l \varphi_i^2(\xi)\,d\xi$ was calculated by Rayleigh and given in /1/ for various conditions at the edges. $\int_0^l w_{\text{sh,s}}(\xi)\cdot\varphi_i(\xi)\,d\xi$ must be calculated for every type of external load and edge condition.

Expansion (7) will contain odd terms only, since $A_{k_i} = 0$ for $k_i = k_{2m}$.

Two terms of the expansion give a satisfactory approximation.

Having expanded the forces and moments in series by the eigenfunctions $\varphi_i(\xi)$, and using (3), (4), and (7), we write the shell deflection in the form

$$w_{\text{sh,s}} = \sum_{k_1}^{k_2} A_{k_i}\cdot\varphi_i(\xi) + \sum_{k_1}^{k_2} T_{2i}\cdot\varphi_i(\xi)\cdot C_{11} + \sum_{k_1}^{k_2} G_{2i}\cdot\varphi_i(\xi)\cdot C_{12} + \\ + \sum_{k_1}^{k_2} N_{2i}\cdot\varphi_i(\xi)\cdot C_{13}. \tag{9}$$

But, according to the first condition of strain compatibility (2),

$$w_{\text{sh}} = w_{\text{r}}.$$

The rib deflection is found as deflection of a beam:

$$\frac{1}{2} E_{\text{r}} J_{\text{r}}\, w_{\text{r}}^{\text{IV}} = N_{2\text{r}}. \tag{10}$$

$$\frac{1}{2} E_{\text{r}} J_{\text{r}} \left[\sum_{k_1}^{k_2} A_{k_i} \frac{d^4\varphi_i}{d\xi^4} + \sum_{k_1}^{k_2} T_{2i}\cdot C_{11} \frac{d^4\varphi_i}{d\xi^4} + \sum_{k_1}^{k_2} G_{2i}\cdot C_{12} \frac{d^4\varphi_i}{d\xi^4} + \right. \\ \left. + \sum_{k_1}^{k_2} N_{2i}\cdot C_{13} \frac{d^4\varphi_i}{d\xi^4} \right] = \sum_{k_1}^{k_2} N_{2i}\cdot\varphi_i. \tag{11}$$

Replacing $\frac{d^4\varphi_i}{d\xi^4}$ by $\varkappa_i^4\varphi_i$, and introducing the designation $D_i = \frac{2}{\varkappa_i^4 E_{\text{r}} J_{\text{r}}}$ we write equation (9) in the following form:

$$\sum_{k_1}^{k_2} T_{2i}\cdot C_{11}\cdot\varphi_i + \sum_{k_1}^{k_2} G_{2i}\cdot C_{12}\cdot\varphi_i + \sum_{k_1}^{k_2} N_{2i}\cdot C_{13}\cdot\varphi_i - \sum_{k_1}^{k_2} N_{2i}\cdot D_i\cdot\varphi_i = \\ = -\sum_{k_1}^{k_2} A_{k_i}\cdot\varphi_i.$$

Thus, all the conditions of strain compatibility (2) can be written in the form of two systems of equations

$$\begin{aligned} T_{2i}C_{11} + G_{2i}C_{12} + N_{2i}(C_{13} - D_i) &= -A_{k_i}; \\ T_{2i}C_{21} + G_{2i}C_{22} + N_{2i}C_{23} &= 0; \\ T_{2i}C_{31} + G_{2i}C_{32} + N_{2i}C_{33} &= 0. \end{aligned} \tag{12}$$

By solving system (12) we obtain the expansion coefficients T_{2i}, G_{2i}, N_{2i}.

$$N_{2i} = -\frac{A_{k_i}}{d_1 C_{11} - d_2 C_{12} + C_{13} + D_i}; \quad G_{2i} = -N_{2i} d_2; \quad T_{2i} = N_{2i} d_1; \tag{12a}$$

where

$$d_1 = \frac{C_{22}C_{33} - C_{23}^2}{C_{12}C_{23} - C_{13}C_{22}}; \quad d_2 = \frac{C_{12}C_{33} - C_{13}C_{23}}{C_{12}C_{23} - C_{13}C_{22}}.$$

The forces and moments are determined as follows:

$$\begin{aligned} T_2 &= T_{2c} + \sum_{k_1}^{k_2} T_{2i} \cdot \varphi_i; \\ G_2 &= \sum_{k_1}^{k_2} G_{2i} \varphi_i; \\ N_2 &= \sum_{k_1}^{k_2} N_{2i} \varphi_i. \end{aligned} \tag{13}$$

$\varphi_i(\xi)$ determines the stress variation along the shell generators.

To calculate the influence coefficients C_{ij}, consider the nonsymmetrical stressed state. Take out a shell section between any two adjacent ribs, loaded along the meridional edges by unit forces per unit length and moments $N_2 = 1$, $G_2 = 1$, $T_2 = 1$, $S = 0$ (Figure 2). On the other two edges are given the same boundary conditions as on the ribbed shell. Using the theory of the undegenerated edge effect exposed in /1/, we reduce the determination of the stressed state to the solution of the differential equation

$$\frac{d^8 p}{d\omega^8} + \lambda_i^8 p = 0, \tag{14}$$

where

$$\lambda_i = \sqrt{\varkappa_i} \cdot \sqrt[4]{\frac{R}{h}} \cdot \sqrt[8]{3(1-\nu^2)};$$

$2h$ is the shell thickness; R is the mean radius of curvature of the shell cross section; ν is the Poisson ratio; ω is the running angle; $\varkappa_i$ are eigenvalues; $p(\omega)$ determines the variation of the stresses and strains along the shell directrix.

The solution of (14) is

$$p = \sum_0^7 B_j Y_j, \tag{15}$$

where Y_j are fundamental functions of the solution; B_j are arbitrary integration constants.

We shall make use of the method of initial parameters. We therefore measure the angle ω for each section of the shell from the rib adjoining the section on the left, and require that the fundamental function Y_j satisfy

the following conditions:

$$Y_0(0)=1;\quad Y_1(0)=0;\quad Y_2(0)=0,\cdots Y_7(0)=0;$$
$$Y_0'(0)=0;\quad Y_1'(0)=1;\quad Y_2'(0)=0;\cdots Y_7'(0)=0;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Y_0^{\mathrm{VII}}(0)=0;\quad Y_1^{\mathrm{VII}}(0)=0;\quad Y_2^{\mathrm{VII}}(0)=0;\cdots Y_7^{\mathrm{VII}}(0)=1.$$

The derivatives of Y_j are related to Y_j in the following way:

$$Y_j'=Y_{j-1}-t_{8-j}\cdot Y_j;$$
$$Y_0'=-\lambda_i^8 Y_7;$$

where

$$t_0=1;\quad t_1=0;\quad t_2=0;\cdots t_6=0;\quad t_7=\lambda_i^8;$$
$$j=1;\ 2;\cdots 7.$$

The fundamental functions are of the form

$$Y_0=\frac{1}{2}(\operatorname{ch}a\omega\cdot\cos b\omega+\operatorname{ch}b\omega\cdot\cos a\omega);$$

$$Y_1=\frac{1}{2\lambda_i}\left[\cos\frac{\pi}{8}(\operatorname{sh}a\omega\cdot\cos b\omega+\operatorname{ch}b\omega\cdot\sin a\omega)+\right.$$
$$\left.+\sin\frac{\pi}{8}(\operatorname{ch}a\omega\cdot\sin b\omega+\operatorname{sh}b\omega\cdot\cos a\omega)\right];$$

$$Y_2=\frac{1}{2\sqrt{2}\,\lambda_i^2}(\operatorname{ch}a\omega\cdot\cos b\omega-\operatorname{ch}b\omega\cdot\cos a\omega+\operatorname{sh}a\omega\cdot\sin b\omega+\operatorname{sh}b\omega\cdot\sin a\omega);$$

$$Y_3=\frac{1}{2\lambda_i^3}\left[\sin\frac{\pi}{8}(\operatorname{sh}a\omega\cdot\cos b\omega-\operatorname{ch}b\omega\cdot\sin a\omega)+\right.$$
$$\left.+\cos\frac{\pi}{8}(\operatorname{ch}a\omega\cdot\sin b\omega-\operatorname{sh}b\omega\cdot\cos a\omega)\right];$$

$$Y_4=\frac{1}{2\lambda_i^4}(\operatorname{sh}a\omega\cdot\sin b\omega-\operatorname{sh}b\omega\cdot\sin a\omega);\tag{16}$$

$$Y_5=\frac{1}{2\lambda_i^5}\left[\cos\frac{\pi}{8}(\operatorname{ch}a\omega\cdot\sin b\omega+\operatorname{sh}b\omega\cdot\cos a\omega)-\right.$$
$$\left.-\sin\frac{\pi}{8}(\operatorname{sh}a\omega\cdot\cos b\omega+\operatorname{ch}b\omega\cdot\sin a\omega)\right];$$

$$Y_6=\frac{1}{2\sqrt{2}\,\lambda_i^6}(\operatorname{ch}b\omega\cdot\cos a\omega-\operatorname{ch}a\omega\cdot\cos b\omega+\operatorname{sh}a\omega\cdot\sin b\omega+\operatorname{sh}b\omega\cdot\sin a\omega);$$

$$Y_7=\frac{1}{2\lambda_i^7}\left[\sin\frac{\pi}{8}(\operatorname{ch}a\omega\cdot\sin b\omega-\operatorname{sh}b\omega\cdot\cos a\omega)-\right.$$
$$\left.-\cos\frac{\pi}{8}(\operatorname{sh}a\omega\cdot\cos b\omega-\operatorname{ch}b\omega\cdot\sin a\omega)\right];$$

where

$$a=\lambda_i\cos\frac{\pi}{8}$$

The forces and displacements are easily expressed through $p(\omega)$ and $\varphi_i(\xi)$:

$$T_2 = \frac{\varkappa_i^2}{R} \cdot p(\omega) \cdot \varphi_i(\xi); \quad S_2 = -\frac{1}{R\varkappa_i^2} \cdot \frac{dp}{d\omega} \cdot \frac{d^3\varphi_i}{d\xi^3};$$

$$G_2 = -\frac{h^2}{3(1-\nu^2)R^2} \cdot \frac{d^2q}{d\omega^2} \cdot \varphi_i; \quad N_2 = -\frac{h^2}{3(1-\nu^2)R^3} \cdot \frac{d^3q}{d\omega^3} \cdot \varphi_i; \tag{17}$$

$$2Eh\,v = -\frac{1}{\varkappa_i^2} \cdot \frac{d^2p}{d\omega^3} \cdot \varphi_i; \quad 2Eh\,w = q\varphi_i; \quad 2Eh\gamma_2 = -\frac{1}{R} \cdot \frac{dq}{d\omega} \cdot \varphi_i;$$

where

$$q = -\frac{1}{\varkappa_i^2} p^{\mathrm{IV}}.$$

We shall calculate in what follows not the forces and deformations themselves, but the coefficients of their expansions in series by the functions $\varphi_i(\xi)$; the arbitrary constants B_i are determined from the following boundary conditions respectively for the three types of loading

1. $N_2 = 1; \quad G_2 = T_2 = S_2 = 0;$

at the edge $\quad \omega = 0$ and $\omega = \theta_0 \quad p = p' = p^{\mathrm{VI}} = 0; \quad p^{\mathrm{VII}} = g.$

2. $G_2 = 1; \quad T_2 = S_2 = N_2 = 0;$

at the edge $\quad \omega = 0$ and $\omega = \theta_0 \quad p = p' = p^{\mathrm{VII}} = 0; \quad p^{\mathrm{VI}} = gR.$

3. $T_2 = 1; \quad S_2 = N_2 = G_2 = 0;$ (18)

at the edge $\quad \omega = 0$ and $\omega = \theta_0 \quad p' = p^{\mathrm{VI}} = p^{\mathrm{VII}} = 0; \quad p = f,$

where $\quad \theta_0 = \frac{2\pi}{z}$; z-number of ribs;

$$g = \frac{\varkappa_i^2 \cdot R \cdot 3(1-\nu^2)}{h}; \qquad f = \frac{R}{\varkappa_i^2}.$$

Using expressions (17) for w, γ_2, and v, we obtain the following expressions for the loading coefficients C_{ij} for each of the three types of loading:

$$C_{1j} = -\frac{1}{2Eh\varkappa_i^2} p^{\mathrm{IV}};$$

$$C_{2j} = \frac{1}{2Eh\varkappa_i^2} \cdot \frac{p^{\mathrm{V}}}{R}; \tag{19}$$

$$C_{3j} = -\frac{1}{2Eh\varkappa_i^2} p^{\mathrm{III}}.$$

Here $j = 1, 2, 3$.

The fulfillment of the reciprocal theorem

$$C_{ij} = C_{ji}.$$

serves as test of the correctness of the solution obtained.

The reasoning exposed above was based on the assumption that the middle surface of the shell is a cylinder; the same method is, however, applicable to the case of a conical shell as well.

The integration of (14) and the determination of the influence coefficients (19) is independent of the shape of the middle surface of the shell (whether it is a cylinder or a cone). The geometrical meaning of λ_i, ω, and R is, however, different for the cylinder and the cone.

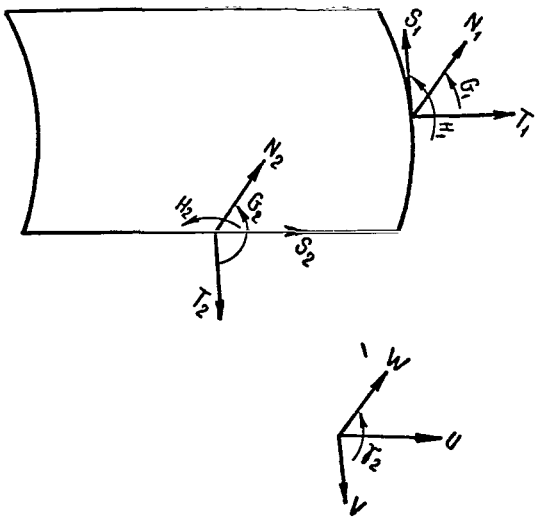

FIGURE 2.

The following rule is proposed for passing from the calculation of a cylindrical shell to the calculation of a conical one:

In the calculation of a cylindrical shell $\varkappa_i$ represents the eigenvalues and $\varphi_i(\xi)$ the eigenfunctions of the equation of vibrations of a beam of uniform cross section $\frac{d^4\varphi_i}{d\xi^4} - \varkappa_i^4\varphi_i = 0$ with boundary conditions corresponding to the conditions at the edges of the ribbed shell; in the calculation of a conical shell $\varkappa_i$ and $\varphi_i(\xi)$ are, respectively, the eigenvalues and eigenfunctions of the equation of vibrations of a beam of variable cross section $\xi^3\frac{d^2}{d\xi^2}\left(\xi^3\frac{d^2\varphi_i}{d\xi^2}\right) - \varkappa_i^4\varphi_i = 0$ with the same boundary conditions. The equation was thoroughly discussed in Mikhlin's paper /3/.

The expressions for the forces and displacements (17) for a conical shell are

$$
\begin{aligned}
T_2 &= \frac{\varkappa_i^2}{R}\cdot p(\omega)\cdot\frac{\varphi_i(\xi)}{\xi^3};\\
S_2 &= -\frac{1}{R\varkappa_i^2}\cdot\frac{dp}{d\omega}\cdot\frac{1}{\sin\chi_0}\cdot\frac{d}{d\xi}\left(\xi^2\frac{d^2\varphi_i}{d\xi^2}\right);\\
G_2 &= -\frac{h^2}{3(1-\nu^2)R^2}\cdot\frac{d^2q}{d\omega^2}\cdot\frac{1}{\sin\chi_0}\cdot\frac{\varphi_i}{\xi};\\
N_2 &= -\frac{h^2}{3(1-\nu^2)R^3}\cdot\frac{d^3q}{d\omega^3}\cdot\frac{1}{\sin^3\chi_0}\cdot\frac{\varphi_i}{\xi^3};\\
2Ehv &= -\frac{1}{\varkappa_i^2}\cdot\frac{d_i^2p}{d\omega^3}\cdot\frac{1}{\sin^3\chi_0}\cdot\varphi_i;\\
2Ehw &= q\varphi_i;\\
2Eh\gamma_{12} &= -\frac{1}{R}\cdot\frac{dq}{d\omega}\cdot\frac{1}{\sin\chi_0}\cdot\varphi_i,
\end{aligned}
\tag{20}
$$

where χ_0 is the cone angle.

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INTEGRAL-DIFFERENTIAL EQUATIONS OF SHELL EQUILIBRIUM WITH FOCUSING KERNELS

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The idea of forming integral-differential equations with focusing kernels arose in the course of the development of numerical methods for solving the boundary-value problems of the theory of shells. Various methods are possible for forming integral-differential equations with focusing kernels. One of these is exposed below* /1/.

In our development of Somilian's method we make use of the reciprocal theorem as proposed in /1-4/.**

The reciprocity theorem leads to integral-differential equations with focusing kernels if the auxiliary displacements /1,2/ are selected in the form of products of the displacements of a plate whose middle surface maps the middle surface of the shell, by a positive exponential factor which decreases rapidly with increasing distance r between the point of displacement and the point of application of the auxiliary concentrated force. It can be shown that the introduction of this factor does not alter the analytical singularity in the expressions for the auxiliary displacements at the point of application of a concentrated force.

The advantage of the system of equations with focusing kernels over those treated in /1-4/ lies in the fact that when this system is approximately replaced by a system of algebraic equations it becomes possible to pick out the dominant terms in these equations and to work out approximate methods of solution, similar to Gauss' method of solving systems of linear algebraic equations. When the focusing exponential factor decreases sufficiently rapidly, the matrix of the coefficients of the system of algebraic equations which replaces approximately the system of integral-differential equations will contain many terms practically equal to zero. This makes it possible to develop efficient methods for the numerical solution of boundary-value problems of the theory of shells.

§1.

Following the ideas of /1, 2/, we communicate to the shell

* Other methods for forming integral-differential equations with focusing kernels of the statics and dynamics of shells are indicated in Kil'chevskii's treatise "Fundamentals of the Analytical Mechanics of Shells" (in print).

** We shall make use in what follows of the terms and symbols used in these papers.

the displacement

$$\bar{v}_{(\alpha)\beta}(N;M)=e^{-kf[r(N;M)]}v_{(\alpha)\beta}(N;M) \qquad (1.1)$$
$$(\alpha,\beta=1,2,3),$$

where $v_{(\alpha)\beta}(N;M)$ is Green's tensor for the plate whose middle surface maps the shell considered /2/; $f(r(N;M)]$ is an arbitrary function of distance r, with $f(0)=0$; k is a positive arbitrarily large number.

The displacements (1.1) in the shell are caused by a concentrated force at point M, a load $\bar{K}_{(\alpha)\beta}(Q,M)$ distributed over the middle surface, and forces per unit length $\bar{S}_{(\alpha)\beta}(L;M)$ and moments $\bar{M}_{\alpha(\gamma)}(L;M)$ applied on the contour of the middle surface of the shell (Q is an arbitrary point of the middle surface of the shell, L is an arbitrary point of the contour).* The load $\bar{K}_{(\alpha)\beta}$ will be determined by the methods indicated in /1-4/, and $\bar{S}_{(\alpha)\beta}$ and $\bar{M}_{(\alpha)\gamma}$ by the well-known methods of the theory of shells.

We have:

$$\bar{K}_{(\alpha)\beta}=e^{-kf(r)}(K_{(\alpha)\beta}+kX_{(\alpha)\beta}),$$
$$\bar{S}_{(\alpha)\beta}=e^{-kf(r)}(S_{(\alpha)\beta}+kY_{(\alpha)\beta}), \qquad (1.2)$$
$$\bar{M}_{(\alpha)\gamma}=e^{-kf(r)}(M_{(\alpha)\gamma}+kZ_{(\alpha)\gamma}) \qquad (\alpha,\beta=1,2,3;\ \gamma=1,2).$$

Here $K_{(\alpha)\beta}$, $S_{(\alpha)\beta}$, $M_{(\alpha)\gamma}$ is the load distributed over the middle surface of the shell and its contour, corresponding to the displacement $v_{(\alpha)\beta}$; $X_{(\alpha)\beta}$, $Y_{(\alpha)\beta}$, $Z_{(\alpha)\gamma}$ is the additional load on the middle surface of the shell and its contour, due to the presence of the focusing factor $e^{-kf(r)}$. The character of this load depends on the properties of the function $f(r)$. Let the displacements (1.1) and the load in the shell (1.2) causing the displacements determine the auxiliary state of the shell. The fundamental (unknown) state of the shell will be determined by Green's tensor $u_{(\beta)\alpha}(M;N)$, the forces per unit length $T_{(\beta)\alpha}(L;N)$, and the moments $L_{(\beta)\gamma}(L;N)$ on the contour of the middle surface of the shell.

Consider a shell with Euclidean matrix of the middle surface. The reciprocal theorem, applied to the two states of the shell, yields**

$$\begin{aligned}u_{(\beta)\alpha}(M;N)&=e^{-kf[r(N;M)]}v_{(\alpha)\beta}(N;M)-\\&-\iint\limits_{(S)}[K_{(\alpha)i}(Q;M)+kX_{(\alpha)i}(Q;M)]\,e^{-kf[r(Q;M)]}u_{(\beta)i}(Q;N)\,dS_Q+\\&+\oint\limits_{(l)}\{T_{(\beta)i}(L;N)\,v_{(\alpha)i}(L;M)+L_{(\beta)\gamma}(L;N)\,\bar{\varphi}_{(\alpha)\gamma}(L;M)-\\&-[S_{(\alpha)i}(L;M)+kY_{(\alpha)i}(L;M)]\,u_{(\beta)i}(L;N)-[M_{(\alpha)\gamma}(L;M)+\\&+kZ_{(\alpha)\gamma}(L;M)]\,\omega_{(\beta)\gamma}(L;N)]\,e^{-kf[r(L;M)]}\,dl_L\\&(\alpha,\beta,i=1,2,3;\quad\gamma=1,2),\end{aligned} \qquad (1.3)$$

where S is the area of the middle surface of the shell; l is the length of the arc of its contour; $\omega_{(\beta)\gamma}$, $\bar{\varphi}_{(\alpha)\gamma}$ are the angles of rotation of the shell at point L in the fundamental and auxiliary states respectively. We have

$$\bar{\varphi}_{(\alpha)\gamma}=\varphi_{(\alpha)\gamma}+k\psi_{(\alpha)\gamma}, \qquad (1.4)$$

* It is assumed here and below that the torsional moments on the contour are replaced by a statically equivalent system of shearing stresses.

** Here and below we omit the sign of summation by a "mute" index. We give the name of "mute" indices to repeating subscripts, since the covariant and contravariant components of tensor magnitudes coincide in the metrics of the middle surface.

where $\varphi_{(\alpha)\gamma}$ is the angle of rotation corresponding to the displacement $v_{(\alpha)\beta}$; $\psi_{(\alpha)\gamma}$ is the additional angle due to the presence of the factor $e^{-kf(r)}$ in (1.1).

Equations (1.3) form a system of nine integral-differential equations with focusing kernels relative to the component of Green's tensor $u_{(\beta)\gamma}$. They describe the equilibrium of a shell of arbitrary contour subjected to a concentrated force of unit components under arbitrary boundary conditions, provided the metrics of the middle surface of the shell is Euclidean (shallow shells, shells of zero Gaussian curvature). If the metrics of the middle surface of the shell is not Euclidean, it must be taken into account that a unit concentrated force in a plate is transformed into a concentrated (nonunit) force in a shell /2, 4/. The structure of equations (1.3) is not disturbed substantially as a result; there appears to the left of $u_{(\beta)\alpha}$ a factor independent of $e^{-kf(r)}$.

Analyzing the structure of equations (1.3), we can conclude that the presence of an exponential focusing factor $e^{-kf(r)}$ makes it possible to simplify its numerical solution. In fact, the number k and the function $f(r)$ can be selected in such a way as to make the kernel of (1.3) arbitrarily small outside a certain sufficiently small region enclosing the point M. Using the formulas of the mechanical quadratures we obtain a system of algebraic equations which approximately replaces system (1.3), and in which the unknowns are the values of the unknown functions at the nodes of the interpolation grid. This system will have a matrix with elements small in absolute value, equal to the coefficients of the unknowns at the nodes of the grid outside the above-mentioned region enclosing point M.

§ 2.

Consider as an example the equilibrium of a circular cylindrical panel with contour elastically supported at four corner points under the action of a normal concentrated force.

The fundamental state satisfies the following boundary conditions:

$$T_1^1=0,\ T_2^1=0,\ T_3^1=0,\ L_1^1=1\ (x=0,\ 0<s<b;\ x=a,\ 0<s<b);$$
$$T_1^2=0,\ T_2^2=0,\ T_3^2=0,\ L_2^2=0\ (s=0,\ 0<x<a;\ s=b,\ 0<x<a). \tag{2.1}$$

In order to obtain the conditions at a corner point, we assume that this point moves along the axis of the spring which represents schematically an elastic support, and that the displacement vector lies in a plane normal to the panel generators. We assume further that the hinge at the spring end excludes the possibility of appearance of concentrated reactive moments. Then:

$$(u_1)_j=0,\ (R_2)_j=c_j(u_2)_j,\ (R_3)_j=c_j(u_3)_j,\ (M_{R1})_j=(M_{R2})_j=(M_{R3})_j=0. \tag{2.2}$$

Here j is the support number, c_j is the rigidity factor of the j-th support, $(R_i)_j$ $(M_{Ri})_j$ are the components of the reaction and the reactive moment. Naturally, depending upon the type of support, different formulations of the boundary conditions at the corner point and its vicinity are also possible.

In selecting the auxiliary state we shall use the fact that the work of the forces and moments of the fundamental state in the displacements of the auxiliary state must be different from zero.

Since the shell considered possesses a sufficient mobility at the ends, we select the auxiliary displacements in the form (1.1), where $v_{(\alpha)\beta}$ are the known solutions of the problem of the action of a concentrated force with unit components on an unlimitedly thin elastic shell. All the characteristics of the stressed and strained auxiliary state on the shell contour are in this case different from zero. The auxiliary load of the shell is determined by formulas (1.2), where $K_{(\alpha)\beta}$, $X_{(\alpha)\beta}$, $Y_{(\alpha)\beta}$, $Z_{(\alpha)\gamma}$ are given by:

$$K_{(\gamma)i} = 0, \quad K_{(\gamma)3} = \frac{Eh}{(1-\nu^2)R}\left[\nu\frac{\partial v_{(\gamma)1}}{\partial x} + \frac{\partial v_{(\gamma)2}}{\partial s}\right],$$

$$K_{(3)1} = -\frac{Eh}{1-\nu^2}\cdot\frac{\nu}{R}\frac{\partial v_{(3)3}}{\partial x}; \quad K_{(3)2} = -\frac{Eh}{1-\nu^2}\cdot\frac{1}{R}\frac{\partial v_{(3)3}}{\partial s}, \tag{2.3}$$

$$K_{(3)3} = \frac{Eh}{1-\nu^2}\frac{v_{(3)3}}{R^2}; \quad X_{(\alpha)\gamma} = -\frac{Eh}{1-\nu^2}\bar{X}_{(\alpha)\gamma}, \quad X_{(\alpha)3} = \frac{Eh}{1-\nu^2}\bar{X}_{(\alpha)3},$$

$$Y_{(\alpha)\beta} = -\frac{Eh}{1-\nu^2}\bar{Y}_{(\alpha)\beta}, \qquad Z_{(\alpha)\gamma} = \frac{Eh^3}{12(1-\nu^2)}\bar{Z}_{(\alpha)\gamma},$$

in which

$$X_{(\gamma)1} = v_{(\gamma)}\left[k\left(\frac{\partial f}{\partial x}\right)^2 - \frac{\partial^2 f}{\partial x^2}\right] - 2\frac{\partial v_{(\gamma)1}}{\partial x}\frac{\partial f}{\partial x} + \frac{1-\nu}{2}\left\{v_{(\gamma)1}\left[k\left(\frac{\partial f}{\partial s}\right)^2 - \right.\right.$$
$$\left. - \frac{\partial^2 f}{\partial s^2}\right] - 2\frac{\partial v_{(\gamma)1}}{\partial s}\frac{\partial f}{\partial s}\Big\} + \frac{1+\nu}{2}\left\{v_{(\gamma)2}\left[k\frac{\partial f}{\partial s}\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial x\partial s}\right] - \right.$$
$$\left. - \frac{\partial v_{(\gamma)2}}{\partial s}\frac{\partial f}{\partial s} - \frac{\partial v_{(\gamma)2}}{\partial x}\frac{\partial f}{\partial s}\right\}, \tag{2.4}$$

$$\bar{X}_{(3)1} = -\frac{1}{R}v_{(3)3}\frac{\partial f}{\partial s}, \tag{2.5}$$

$$\bar{Y}_{(\alpha)1} = v_{(\alpha)1}\frac{\partial f}{\partial x} + \nu v_{(\alpha)2}\frac{\partial f}{\partial s}, \tag{2.6}$$

$$\bar{Z}_{(\alpha)1} = v_{(\alpha)3}k\left(\frac{\partial f}{\partial x}\right)^2 - v_{(\alpha)3}\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial v_{(\alpha)3}}{\partial x}\frac{\partial f}{\partial x} +$$
$$+ \nu\left[v_{(\alpha)3}k\left(\frac{\partial f}{\partial s}\right)^2 - v_{(\alpha)3}\frac{\partial^2 f}{\partial s^2} - 2\frac{\partial v_{(\alpha)3}}{\partial s}\frac{\partial f}{\partial s}\right] \tag{2.7}$$

$$(\gamma, i = 1, 2; \quad \alpha = 1, 2, 3).$$

The analytical expressions of the remaining functions will not be given here.

Allowing for (2.1) and (2.2), system (1.3) becomes of the form

$$u_\gamma = -\iint\limits_{(s)}[kX_{(\gamma)\beta}u_\beta + (K_{(\gamma)3} + kX_{(\gamma)3})u_3]e^{kf(r)}ds -$$
$$-\oint\limits_{(l)}[S_{(\gamma)\beta} + kY_{(\gamma)\beta}]u_\beta e^{-kf(r)}dl + \sum_{j=1}^{4}e^{-kf(r_j)}(R_\beta)_j(v_{(\gamma)\beta})_j, \tag{2.8}$$

$$u_3 = v_{(3)3} - \iint\limits_{(s)}[K_{(3)i} + kX_{(3)i}]u_i e^{-kf(r)}dS - \oint\limits_{(l)}[(S_{(3)3} + kY_{(3)3})u_3 +$$
$$+ M_{(3)\gamma}\omega_\gamma]e^{-kf(r)}dl + \sum_{j=1}^{4}e^{-kf(r_j)}(R_3)_j(v_{(3)3})_j, \tag{2.9}$$

where r_j is the distance of the j-th support from the point $M(x, s)$,

$$(\beta, \gamma = 1, 2, \quad i = 1, 2, 3).$$

We shall indicate the scheme of the numerical solution of system (2.8), (2.9). Let us select the function $f(r)$ in the form

$$f(r) = r^2, \text{where } r = \sqrt{(x-x_Q)^2 + (s-s_Q)^2}. \tag{2.10}$$

When $f(r)$ is selected in this way the functions $X_{(\alpha)\beta}$, $Y_{(\alpha)\beta}$, $Z_{(\alpha)\gamma}$ are regular everywhere except at the point of application of the auxiliary force. It follows that the kernels of the system (2.8), (2.9) are sufficiently rapidly decreasing functions, due to the presence of the factor $e^{-k/(r)}$.

When system (2.8), (2.9) is replaced approximately by a system of algebraic equations, we take into account the values of the kernels at the nodes of the grid enclosing point M. At $a = b$ and a grid with step $\lambda = \frac{a}{4}$ the systems of algebraic equations for the inner points have matrices in which 9 elements differ from zero, and 16 elements are equal to zero. The system of algebraic equations, formed for the boundary points and for the points adjoining the boundary, contains a smaller number of zeroes.

The case when the region of kernel determination lies within the first ring of nodes surrounding the point M is of interest. The system of algebraic equations for the inner points of the grid then splits into quasi-independent systems, determining all three components of the displacement at the given point. As an example, the system of algebraic equations corresponding to the internal node of index i is

$$u_\gamma^i = -\frac{a^2}{16}\left[kX_{(\gamma)\beta}^i u_\beta^i + (K_{(\gamma)3}^i + kX_{(\gamma)3}^i)\, u_3^i\right],$$
$$u_3^i = v_{(3)3}^i - \frac{a^2}{16}(K_{(3)\alpha}^i + kX_{(3)\alpha}^i)\, u_\alpha^i \qquad (\alpha = 1.\ 2,\ 3;\ \ \beta,\ \gamma = 1,\ 2). \tag{2.11}$$

The solution obtained as a result must obviously be considered as an initial approximation, corresponding to Gauss' scheme, since the coefficients of the unknowns at the nodes of the interpolation grid lying outside the nearest vicinity of point M are only approximately equal to zero. Approximations of the form of (2.11) are independent of the boundary conditions, which is an additional fact pointing to their insufficiency. Subsequent approximations make it possible to introduce the boundary conditions in the solution. Due to lack of space, we shall not describe the method of forming equations with the boundary conditions elements.

In addition to the numerical methods, the so-called interpolation-iterative method can be used in solving system (2.8), (2.9).

The solution of this system will be represented in the form

$$u_j(M;\, N) = a_{jl}(N)\, \varphi_{(j)l}(M) \qquad (j = 1,\ 2,\ 3), \tag{2.12}$$

where $\varphi_{(j)l}(M)$ are arbitrary functions satisfying some boundary conditions; $a_{jl}(N)$ are functions determined from the system of algebraic equations derived from the conditions of fulfillment of (2.8), (2.9) at a discrete set of points of the middle surface of the shell, and also at several points of the contour. The number of components in (2.12) is a function of the number of selected points. In order to satisfy (2.8), (2.9) at the point of application of the fundamental concentrated force (N) and at four nodal points of the contour, it is necessary to substitute in (2.12) 11 unknown functions a_{jl}, whose determination necessitates the solution of a system of 11 algebraic equations. The presence of a focusing load in (2.8), (2.9) makes it possible to simplify the calculation of the coefficients of this system. It thus suffices to calculate the value of the integrand functions at the grid nodes adjoining point M. The first (interpolative) state of the calculation ends with the calculation of the functions a_{jl}.

The second (iterative) stage consists in substituting the functions (2.12) in (2.8), (2.9) and obtaining more refined values of the unknown functions u_i by integration. By increasing the number of points on the first stage and the number of iterations on the second it becomes possible to obtain more accurate solutions of the problem.

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DYNAMIC INSTABILITY OF CIRCULAR CYLINDRICAL SHELLS SUBJECTED TO LONGITUDINAL COMPRESSIVE STRESSES IN THE CASE OF THERMOPARAMETRIC RESONANCE

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The excitation of parametric vibrations is important, since it can lead either to a disturbance of the integrity of the structure or even (in the case of lengthy excitation) to its fatigue collapse.

The dynamic stability of elastic systems, and of shells in particular, was studied in detail in Bolotin's treatise /2/. Some problems involving parametric resonance of cylindrical shells were treated in /4/, /10/, and /11/; in particular, the case of thermoparametric resonance of the shell at equal frequencies of fluctuations of the pressure and the temperature field was treated in /4/.

An analysis is given below of the stability of cylindrical shells subjected to the simultaneous action of longitudinal compressive force and a temperature field, both periodic in time, in the case of temperature-dependent coefficient of linear thermal expansion and modulus of elasticity. The analysis is based on the following assumptions: 1) the reverse thermoelastic effect is neglected; 2) the modulus of elasticity E and the coefficient of linear thermal expansion α are assumed to be linear functions of the temperature

$$E = E_0 - E'T; \qquad \alpha = \alpha_0 + \alpha' T; \tag{1}$$

where T is the running coordinate of the temperature, α_0 and E_0 are isothermal values of the corresponding magnitudes; 3) the Poisson ratio ν is assumed to be temperature-independent; 4) it is assumed that the temperature varies linearly through the shell thickness and periodically with time; 5) the analysis is based on the differential equations of motion of flexible shells, using Kirchhoff-Love's hypothesis /6/; 6) the problem is considered in a linear formulation only.

If a circular-cylindrical shell of radius R and length l is compressed by longitudinal forces P varying periodically with time

$$P = P_0 + P_1(t), \tag{2}$$

then, on the basis of assumptions 1 to 6, the equations of the disturbed motion of the shell will reduce, according to /4/ and /10/, to the following differential equation*:

* The system of coordinates is selected as follows: the z axis is directed along the normal to the middle syrface of the shell, and the x and y axes along the lines of principal curvatures; t is the time coordinate.

$$\frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)}\nabla^2\nabla^2\nabla^2\nabla^2 w + \frac{E_1}{k^2}\frac{\partial^4 w}{\partial x^4} +$$

$$+ P\nabla^2\nabla^2\frac{\partial^2 w}{\partial x^2} + \rho h\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2 w = 0 \qquad (3)$$

where $w(x, y, t)$ is the deflection function, ρ is the density of the material, h is the shell thickness, E_1, E_2, E_3 are the moduli of elasticity of the first, second, and third order, respectively, determined by the following relationships:

$$\left.\begin{aligned} E_1 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} E dz; \\ E_2 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} E z dz; \\ E_3 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} E z^2 dz. \end{aligned}\right\} \qquad (4)$$

Using the Galerkin method, the solution of the problem is presented in the form of the series

$$w(x, y, t) = \sum_{m,n} f_{mn}(t)\, w_{mn}(x, y), \qquad (5)$$

where, in accordance with /8/, w_{mn} are given functions, satisfying homogeneous boundary conditions determined by the type of fastening of the shell, and f_{mn} are functions which remain to be determined. If the shell is hinged at the edges and free in the circumferential direction, the boundary conditions will be

$$\left.\begin{aligned} w\Big|_{\substack{x=0\\x=l}} &= 0; \\ M_x\Big|_{\substack{x=0\\x=l}} &= 0; \\ T_x\Big|_{\substack{x=0\\x=l}} = T'_{xy}\Big|_{\substack{x=0\\x=l}} &= 0, \end{aligned}\right\} \qquad (6)$$

where T_x is the normal force per unit length, T'_{xy} is the reduced shearing force per unit length, and M_x is the bending moment, determined from the relationship

$$M_x\Big|_{\substack{x=0\\x=l}} = \left[\frac{E_1E_3 - E_2^2}{(1-\nu^2)E_1}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + \right.$$

$$\left. + \frac{1}{(1-\nu)}\left(M_t - \frac{E_2}{E_1}T_t\right)\right]_{\substack{x=0\\x=l}} = 0. \qquad (7)$$

The nonhomogeneity of the boundary condition (7), caused by the temperature field, can be removed by means of the following substitution:

$$w_1(x, y, t) = w(x, y, t) - \frac{1}{2}x(x-l)v(t), \qquad (8)$$

where the function $v(t)$ is determined as follows:

$$v(t) = (1+\nu)\frac{E_2T_t - E_1M_t}{E_1E_3 - E_2^2}. \qquad (8')$$

When the boundary conditions are (6), (7), the following system of orthogonal functions $w_{mn}(x, y)$ can be used /3/ as expansion functions in (5):

$$w_{mn}(x, y) = \sin\frac{m\pi x}{l}\sin\frac{ny}{R} \qquad (m, n = 1, 2, 3\cdots). \qquad (9)$$

After the substitution of relationships (5), (8), (8'), and (9) in equation (4) and the fulfillment of the generalized condition of orthogonality /8/ the problem reduces to studying a second-order differential equation with variable (on the strength of relationships (1), (2)) coefficients

$$\frac{d^2 f_{mn}}{dt^2} + [A_0 + \Phi_1(t) + \Phi_2(t)]f_{mn} = 0 \qquad (m, n = 1, 2, 3), \qquad (10)$$

where $\Phi_1(t) = -\left(\frac{m\pi}{l}\right)^2\frac{P_1(t)}{\rho h}$ is due to the variation of the compressive forces with time;

$$\Phi_2(t) = -\frac{1}{(1-\nu^2)}\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2\left(E'\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 T dz + \frac{E_2^2}{E_1}\right)\frac{1}{\rho h} -$$

$$-\frac{1}{\rho h}E\int_{-\frac{h}{2}}^{\frac{h}{2}} T dz\left(\frac{m\pi}{l}\right)^4 : R^2\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2$$ is due to the variation

of the temperature field;

$$A_0 = \frac{E_0}{\rho}\left\{\frac{h^2}{12(1-\nu^2)}\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2 + \frac{1}{R^2}\frac{\left(\frac{m\pi}{l}\right)^4}{\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2}\right\} -$$

$$-\frac{P_0}{\rho h}\left(\frac{m\pi}{l}\right)^2. \qquad (11)$$

The next stage of the study of the problem depends on the interrelationship

between the frequencies (or the periods) of the periodic functions $\Phi_1(t)$ and $\Phi_2(t)$. In the case of incommensurable frequencies ω_1 and ω_2 the sum of the functions $\Phi_1(t)$ and $\Phi_2(t)$ is, in accordance with /9/, a quasiperiodic function; when these frequencies are commensurable it is a periodic function. We consider here the latter case, when the differential equation (10) belongs to the Mathieu-Hill class and is reduced to the best-known type of equations of this class*

$$\frac{d^2 f_{mn}}{dt^2} + \left[\mu_0' + \sum_{k=1}^{\infty} \left(\mu_k + \nu_{\frac{k}{\beta}} \right) \cos k\omega t \right] f_{mn} = 0, \tag{12}$$

where $\omega_2 = \omega$; $\frac{\omega_1}{\omega_2} = \beta$ is an integer; μ_k, $\nu_{\frac{k}{\beta}}$ are coefficients of the Fourier series of the functions $\Phi_1(t)$ and $\Phi_2(t)$ respectively; $\mu_0' = \mu_0 + \nu_0$; $\nu_{\frac{k}{\beta}} \neq 0$ with k a multiple of β. The study was conducted by the method developed in /2/, according to which the boundaries of the stability and instability regions of the solution of Mathieu-Hill's equations are periodic solutions, corresponding to two groups of multiple roots of the characteristic equation of period $t_0 = \frac{2\pi}{\omega}$;

$$f_1(t) = \sum_{k=1,3,5} \left(a_k \sin \frac{k\omega t}{2} + b_k \cos \frac{k\omega t}{2} \right) \tag{13}$$

and of period $2t_0 = \frac{4\pi}{\omega}$

$$f_2(t) = \sum_{k=2,4,6} \left(a_k \sin \frac{k\omega t}{2} + b_k \cos \frac{k\omega t}{2} \right) + b_0, \tag{14}$$

where a_k and b_k are coefficients to be determined from the condition that relationships (13) and (14) satisfy equation (12). The determinants of the systems of linear homogeneous algebraic equations obtained represent the equations of critical frequencies. To solution (13) correspond the equations of critical frequencies of the odd regions of instability (15), and to solution (14) correspond the equations of critical frequencies of the even regions of instability (16), (17).

The question of the convergence of infinite determinants of the type of (15), (16), and (17) has been examined in papers dealing with the analysis of the Mathieu-Hill equations (such as /5/).

It follows from the results obtained that a temperature field varying periodically with time has the same action on the shell as a parametric load: it can cause the appearance of regions of dynamic instability for specific relationships between the parameters of the temperature field and the shell when the latter is subjected initially to uniform stresses in the middle surface. In the case of action on the shell of a longitudinal compressive parametric load, the temperature field, which varies periodically with time, increases the region of dynamic instability.

* It is assumed that the functions $\Phi_1(t)$ and $\Phi_2(t)$ can be expanded in cosine Fourier series:

$$\Phi_1(t) = \sum_{k=0}^{\infty} \nu_k \cos k\beta\omega t; \quad \Phi_2(t) = \sum_{k=0}^{\infty} \mu_k \cos k\omega t.$$

$$\Delta_{1,2}=\begin{vmatrix} \mu_0'-\frac{\omega^2}{4}\mp\frac{1}{2}\left(\mu_1+\nu_{\frac{1}{\beta}}\right); & \frac{1}{2}\left[\mu_1+\nu_{\frac{1}{\beta}}\mp\left(\mu_2+\nu_{\frac{2}{\beta}}\right)\right]; & \cdots\frac{1}{2}\left[\mu_{\frac{k-1}{2}}+\nu_{\frac{k-1}{2\beta}}\mp\left(\mu_{\frac{k+1}{2}}+\nu_{\frac{k+1}{2\beta}}\right)\right];\cdots \\ \frac{1}{2}\left[\mu_1+\nu_{\frac{1}{\beta}}\mp\left(\mu_2+\nu_{\frac{2}{\beta}}\right)\right]; & \mu_0'-\frac{9\omega^2}{4}\mp\frac{1}{2}\left(\mu_3+\nu_{\frac{3}{\beta}}\right);\cdots & \cdots\frac{1}{2}\left[\mu_{\frac{k+3}{2}}+\nu_{\frac{k+3}{2}}\mp\left(\mu_{\frac{k+3}{2}}+\nu_{\frac{k+3}{2\beta}}\right)\right];\cdots \\ \cdots & \cdots & \cdots \\ \frac{1}{2}\left[\mu_{\frac{k-1}{2}}+\nu_{\frac{k-1}{2\beta}}\mp\left(\mu_{\frac{k+1}{2}}+\nu_{\frac{k+1}{2\beta}}\right)\right]; & \frac{1}{2}\left[\mu_{\frac{k-3}{2}}+\nu_{\frac{k-3}{2}}\mp\left(\mu_{\frac{k+3}{2}}+\nu_{\frac{k+\beta}{2\beta}}\right)\right];\cdots & \mu_0-\frac{k^2\omega^2}{4}\mp\frac{1}{2}\left(\mu_k+\nu_{\frac{k}{\beta}}\right);\cdots \\ \cdots & \cdots & (k=1,3,5\cdots). \end{vmatrix}=0;\quad(15)$$

$$\Delta_3=\begin{vmatrix} \mu_0-\omega^2-\frac{1}{2}\left(\mu_1+\nu_{\frac{2}{\beta}}\right); & \frac{1}{2}\left[\mu_1+\nu_{\frac{1}{\beta}}-\left(\mu_3+\nu_{\frac{3}{\beta}}\right)\right];\cdots & \frac{1}{2}\left[\mu_{\frac{k-2}{2}}+\nu_{\frac{k-2}{2\beta}}-\left(\mu_{\frac{k+2}{2}}+\nu_{\frac{k+2}{2\beta}}\right)\right];\cdots \\ \frac{1}{2}\left[\mu_1+\nu_{\frac{1}{\beta}}-\left(\mu_3+\nu_{\frac{3}{\beta}}\right)\right]; & \mu_0-4\omega^2-\frac{1}{2}\left(\mu_4+\nu_{\frac{4}{\beta}}\right);\cdots & \frac{1}{2}\left[\mu_{\frac{k-4}{2}}+\nu_{\frac{k-4}{2\beta}}-\left(\mu_{\frac{k+2}{2}}+\nu_{\frac{k+2}{2\beta}}\right)\right];\cdots \\ \cdots & \cdots & \cdots \\ \frac{1}{2}\left[\mu_{\frac{k-2}{1}}+\nu_{\frac{k-2}{2\beta}}-\left(\mu_{\frac{k+2}{2}}+\nu_{\frac{k+2}{2\beta}}\right)\right]; & \frac{1}{2}\left[\mu_{\frac{k-4}{2}}+\nu_{\frac{k-4}{2\beta}}-\left(\mu_{\frac{k+4}{2}}+\nu_{\frac{k+4}{2\beta}}\right)\right];\cdots & \mu_0'-\frac{k^2\omega^2}{4}+\frac{1}{2}\left(\mu_{\frac{k}{2}}+\nu_{\frac{k}{2\beta}}\right);\cdots \\ \cdots & \cdots & (k=2,4,6\cdots). \end{vmatrix}=0;\quad(16)$$

$$\Delta_3=\begin{vmatrix} b_0\mu_0'; & \frac{1}{2}\left(\mu_1+\nu_{\frac{1}{\beta}}\right);\cdots & \frac{1}{2}\left(\mu_{\frac{k}{2}}+\nu_{\frac{k}{2}}\right);\cdots \\ \frac{1}{2}\left(\mu_1+\nu_{\frac{1}{\beta}}\right); & \mu_0'-\omega^2+\frac{1}{2}\left(\mu_2+\nu_{\frac{2}{\beta}}\right);\cdots & \frac{1}{2}\left(\mu_{\frac{k+2}{2}}+\nu_{\frac{k+2}{2\beta}}+\mu_{\frac{k-2}{2}}+\nu_{\frac{k-2}{2\beta}}\right);\cdots \\ \cdots & \cdots & \cdots \\ \frac{1}{2}\left(\mu_{\frac{k}{2}}+\nu_{\frac{k}{2\beta}}\right); & \frac{1}{2}\left(\mu_{\frac{k+2}{2}}+\nu_{\frac{k+2}{2\beta}}+\mu_{\frac{k-2}{2}}+\nu_{\frac{k+2}{2\beta}}\right);\cdots & \mu_0'-\frac{k^2\omega^2}{4}+\frac{1}{2}\left(\mu_{\frac{k}{2}}+\nu_{\frac{k}{2\beta}}\right);\cdots \\ \cdots & \cdots & (k=2,4,6\cdots). \end{vmatrix}=0.\quad(17)$$

We shall consider several particular cases. Let the temperature vary according to the following law:

$$T(z, t) = T_1\left(\frac{1}{2} + \frac{z}{h}\right)\cos\omega t + T_2. \tag{18}$$

This corresponds approximately to a periodic variation of the temperature on the surface of a thin shell. The "thinness" of the shell (from the point of view of its thermal properties) depends on, in the case of periodic variation of the temperature on its surface, the heat capacity of the material /12/. The shell thickness represents a "layer of sharp fluctuations" of the temperature, i.e., relationship (18) is correct only when the following restriction is fulfilled:

$$h \ll \sqrt{\frac{\varkappa}{\omega}}, \tag{19}$$

where $\varkappa$ is the thermal diffusivity of the shell. The boundaries of the principal region of dynamic instability of a shell subjected to the action of a periodic temperature field only are determined from the following relationship*:

$$\frac{\omega^2}{4} = \Omega^2\left\{1 \mp \frac{E'T_1}{2(E_0 - E'T_2)}\left[1 - \frac{2(z_1 + z_2) + z_2(z_2^2 + 3)}{18(z_2 - z_1)\gamma + 4z_2}\right]\right\},$$

where

$$\gamma = (1 - \nu^2)\frac{l^4}{R^2h^2} : \left\{(m\pi)^4\left[1 + \left(\frac{nl}{m\pi R}\right)^2\right]^4\right\}; \tag{20}$$

and z_1 and z_2 (with $|z_1| > |z_2|$) are the roots of the algebraic equation

$$z^2 - \frac{4(E_0 - E'T_2)}{E'T_1}z + 1 = 0, \tag{21}$$

obtained by the theory of deductions /1/ when equation (10) is reduced to form (12), if the temperature field is given by relationship (18). $\Omega = \sqrt{\mu_0^2}$ corresponds to the natural frequency of the shell at constant temperature and is determined as follows:

$$\Omega^2 = \frac{(E_0 - E'T_2)}{\rho h}\left\{\frac{h}{R^2}\frac{\left(\frac{m\pi}{l}\right)^4}{\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2} + \right.$$

$$\left. + \frac{h^3}{9(1 - \nu^2)}\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2\left[1 + \frac{2(E_0 - E'T_2)}{E'T_1(z_2 - z_1)}\right]\right\}. \tag{22}$$

* Relationship (19) can also be determined from the condition that relationship (18) corresponds approximately to the solution of the problem of heat conduction for a thin wall in the case of periodic variation of the temperature on the surface /7/.

The conditions under which a shell buckles when subjected only to a periodic temperature field, cannot be determined from the quasistatical theory, since the latter neglects the inertial terms.

Consider now a second example: Let the shell considered above be compressed by constant longitudinal forces distributed uniformly over the edges. Let the parameters of the temperature field be given, and let it be required to determine the magnitude of the critical forces at thermo-parametric resonance. In this case the boundaries of the minimum critical forces of the principal region of dynamic instability are approximately determined by the following relationships:

$$P_{cr}^{(d)} = 2\sqrt{AB} - \frac{\rho h \omega^2}{4\left(\frac{m\pi}{l}\right)^2}, \tag{23}$$

where

$$A = \frac{h}{R^2}\left(E_0 - E'T_2 \pm \frac{1}{4}E'T_1\right),$$

$$B = \frac{h^3}{9(1-\nu^2)}\left\{E_0 - E'T_2 \pm \frac{1}{8}E'T_1 + \frac{(E_0 - E'T_2)^2}{E'T_1(z_2 - z_1)}\left[2 \mp \frac{z^2}{2}\right]\right\},$$

z_1 and z_2 are roots of equations (21).

In the case of stationary variation of the temperature, if we take in relationship (18) the maximum value of the function $\cos \omega t$,

$$T = T_1\left(\frac{1}{2} + \frac{z}{h}\right) + T_2, \tag{24}$$

the value of the statical critical force per unit length is determined, assuming 1, 2, 3, 5, and 6, by the following formula:

$$P_{cr}^{(st)} = \frac{2}{R}\sqrt{\frac{E_1E_3 - E_2^2}{1-\nu^2}}. \tag{25}$$

A numerical example was calculated for a shell made of steel-35:

$$E_0 = 2\cdot 10^6 \frac{\text{kg}}{\text{cm}^2}; \quad E' = 680 \frac{\text{kg}}{\text{cm}^2\cdot\text{g}}; \quad T_1 = 20°\text{C}; \quad T_2 = 500°\text{C};$$

$$\omega = 100 \text{ sec}^{-1}.$$

The calculation by the dynamic theory yields the following boundaries of the minimum critical forces per unit length of the principal region of dynamic instability

$$1.419\cdot 10^6\,\theta h \leqslant P_{cr}^{(d)} \leqslant 1.423\cdot 10^6\,\theta h, \tag{26}$$

where

$$\theta = h : R.$$

The calculation by the quasistatical theory gives a lower value of the critical force per unit length

$$P_{cr}^{(st)} = 1.00054\cdot 10^6\theta h. \tag{27}$$

It is seen from formulas (26) and (27) that the parametric influence of the temperature field can increase the value of the critical forces per unit length (at least in the particular cases considered). The results obtained are obviously in need of further substantiation, both theoretical and experimental.

The examples considered above point to the existence of qualitatively new dynamic processes, depending on the inertial terms in the equations of thermoelasticity of the shells and on the temperature variation of the modulus of elasticity and the coefficient of linear thermal expansion. Taking into account the dependence of the modulus of elasticity and the coefficient of linear thermal expansion on the temperature, we obtain the possibility not only of allowing for the parametric influence of the temperature field in the dynamic equations, but also of generalizing the results obtained to the case of high heating temperatures (lying within the limits of a certain interval in which, according to /13/, the linear differential equations describe with sufficient accuracy the processes taking place).

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CALCULATION OF RECTANGULAR ORTHOTROPIC PLATES LYING ON AN ELASTIC FOUNDATION WITH TWO FOUNDATION MODULI AND SUBJECTED TO STATIC AND PULSATING LOADS

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§1. INTRODUCTION

Rectangular orthotropic plates lying on an elastic foundation with two foundation moduli (represented by the models of Prof. Pasternak, Prof. Vlasov, and others) and subjected to static and pulsating loads are calculated by the method of initial parameters. It is assumed that two opposite edges of the plate are hinged (Figure 1) or have sliding fastening while the fastening of the other two edges is arbitrary.

§2. DIFFERENTIAL EQUATION OF BENDING OF A PLATE AND EXPRESSIONS OF THE INTERNAL FORCES

The differential equation of bending of an orthotropic shell subjected to tensile forces in its middle plane and lying on an elastic foundation is

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} - H_x \frac{\partial^2 w}{\partial x^2} - H_y \frac{\partial^2 w}{\partial y^2} + \\ + c_1 w - c_2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = -q(x, y), \qquad (1)$$

where $w(x, y)$ is the vertical displacement of the points of the middle plane of the plate,

$$D_1 = \frac{E_1 h^3}{12(1 - \mu_1 \mu_2)}; \qquad D_2 = \frac{E_2 h^3}{12(1 - \mu_1 \mu_2)};$$

$$D_k = \frac{Gh^3}{12}; \qquad D_3 = D_1 \mu_2 + 2D_k = D_2 \mu_1 + 2D_k;$$

E_1, E_2, μ_1 and μ_2 are, respectively, the moduli of elasticity and Poisson ratios for the principal directions, coinciding with the coordinate axes, c_1 and c_2 are the foundation moduli, $q(x, y)$ is the load on the plate.

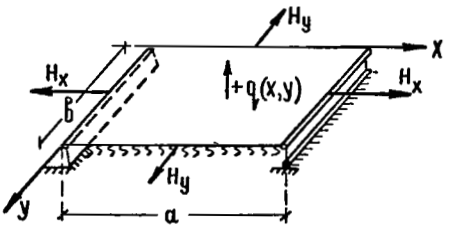

FIGURE 1.

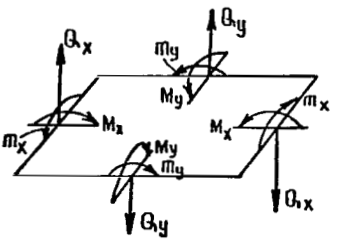

FIGURE 2.

The expressions for the internal forces (Figure 2) are

$$M_x = -D_1\left(\frac{\partial^2 w}{\partial x^2} + \mu_2 \frac{\partial^2 w}{\partial y^2}\right); \tag{2}$$

$$M_y = -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_1 \frac{\partial^2 w}{\partial x^2}\right); \tag{3}$$

$$m_x = -m_y = -2D_k \frac{\partial^2 w}{\partial x \partial y}; \tag{4}$$

$$Q_x = -D_1\left[\frac{\partial^3 w}{\partial x^3} + (D_3 : D_1)\frac{\partial^3 w}{\partial x \partial y^2}\right]; \tag{5}$$

$$Q_y = -D_2\left[\frac{\partial^3 w}{\partial y^3} + (D_3 : D_2)\frac{\partial^3 w}{\partial x^2 \partial y}\right]. \tag{6}$$

The reduced lateral forces (support reactions) are determined by the formulas

$$V_x = -D_1\left(\frac{\partial^3 w}{\partial x^3} + \bar{\varepsilon}\frac{\partial^3 w}{\partial x \partial y^2}\right); \tag{7}$$

$$V_y = -D_2\left(\frac{\partial^3 w}{\partial y^3} + \varepsilon\frac{\partial^3 w}{\partial x^2 \partial y}\right); \tag{8}$$

where

$$\varepsilon = \frac{D_1\mu_2 + 4D_k}{D_2}; \quad \bar{\varepsilon} = \frac{D_1\mu_2 + 4D_k}{D_1}.$$

In the case of isotropic plates we have

$$D_1 = D_2 = D_3 = D = \frac{Eh^3}{12(1-\mu^2)}; \quad \varepsilon = \bar{\varepsilon} = 2 - \mu; \quad 2D_k = D(1-\mu).$$

§ 3. SOLUTION OF THE DIFFERENTIAL EQUATION

We shall represent the solution of (1) for the case of hinged support of the edges parallel to the y-axis (Figure 1) in the form

$$w(x, y) = \sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a}. \tag{9}$$

The acting load $q(x, y)$ will be represented as follows:

$$q(x, y) = \sum_{m=1}^{\infty} q_m(y) \sin \frac{m\pi x}{a}, \tag{10*}$$

where

$$q_m(y) = \frac{2}{a} \int_0^a q(x, y) \sin \frac{m\pi x}{a} dx.$$

The substitution of (9) in (1) yields

$$Y_m^{\text{IV}} - 2\alpha_m^2 Y_m'' + \lambda_m^4 Y_m = -\frac{q_m(y)}{D_2}, \tag{11}$$

where

$$2\alpha_m^2 = \frac{2D_3\left(\frac{m\pi}{a}\right)^2 + H_y + c_2}{D_2},$$

$$\lambda_m^4 = \frac{D_1\left(\frac{m\pi}{a}\right)^4 + H_x\left(\frac{m\pi}{a}\right)^2 + c_1 + c_2\left(\frac{m\pi}{a}\right)^2}{D_2}. \tag{12}$$

We shall represent the solution of (11) in the form

$$Y_m = Y_m^0 + Y_m^*, \tag{13}$$

where $Y_m^0(y)$ is the general solution of the homogeneous equation corresponding to (11) and $Y_m^*(y)$ is a particular solution of (11).

The roots of the characteristic equation

$$r^4 - 2\alpha_m^2 r^2 + \lambda_m^4 = 0$$

are

$$r_1 = -r_2 = \sqrt{\alpha_m^2 + \sqrt{\alpha_m^4 - \lambda_m^4}}; \quad r_3 = -r_4 = \sqrt{\alpha_m^2 - \sqrt{\alpha_m^4 - \lambda_m^4}}. \tag{14}$$

First case: $\alpha_m^2 > 0$; $\lambda_m^4 > \alpha_m^4 > 0$.

In this case the roots will be complex conjugates

$$r_1 = -r_2 = \beta_m + \gamma_m i; \quad r_3 = -r_4 = \beta_m - \gamma_m i;$$

* In the case of sliding clamping $w(x, y) = \sum_{m=0}^{\infty} Y_m \cos \frac{m\pi x}{a}$;

$$q(x, y) = \sum_{m=0} q_m^*(y) \cos \frac{m\pi x}{a}; \quad q_m^*(y) = \frac{2}{a} \int_0^a q(x, y) \cos \frac{m\pi x}{a} dx;$$

$$q_0^*(y) = \frac{1}{a} \int_0^a q(x, y) dx.$$

where

$$\beta_m = \sqrt{\frac{\lambda_m^2 + \alpha_m^2}{2}}; \qquad \gamma_m = \sqrt{\frac{\lambda_m^2 - \alpha_m^2}{2}}.$$

The general integral of the homogeneous equation corresponding to (11) is

$$Y_m^0(y) = A \operatorname{ch} \beta_m y \cos \gamma_m y + B \operatorname{sh} \beta_m y \cos \gamma_m y + C \operatorname{ch} \beta_m y \sin \gamma_m y + \\ + D \operatorname{sh} \beta_m y \sin \gamma_m y.$$

We shall represent it by a combination of particular integrals in the form of functions satisfying the unit matrix

$$Y_m^0(y) = C_1 F_1(y) + C_2 F_2(y) + C_3 F_3(y) + C_4 F_4(y), \tag{15}$$

where

$$F_1(y) = \operatorname{ch} \beta_m y \cos \gamma_m y - \frac{\beta_m^2 - \gamma_m^2}{2\beta_m \gamma_m} \operatorname{sh} \beta_m y \sin \gamma_m y; \tag{16}$$

$$F_2(y) = \frac{3\beta_m^2 - \gamma_m^2}{2\beta_m(\beta_m^2 + \gamma_m^2)} \operatorname{sh} \beta_m y \cos \gamma_m y + \frac{3\gamma_m^2 - \beta_m^2}{2\gamma_m(\beta_m^2 + \gamma_m^2)} \cdot \operatorname{ch} \beta_m y \sin \gamma_m y; \tag{17}$$

$$F_3(y) = \frac{1}{2\beta_m \gamma_m} \operatorname{sh} \beta_m y \sin \gamma_m y; \tag{18}$$

$$F_4(y) = \frac{1}{2(\beta_m^2 + \gamma_m^2)} \left(\frac{\operatorname{ch} \beta_m y \sin \gamma_m y}{\gamma_m} - \frac{\operatorname{sh} \beta_m y \cos \gamma_m y}{\beta_m} \right). \tag{19}$$

The particular solution will be written by Cauchy's rule

$$Y_m^*(y) = -\int_0^y \frac{q_m(u)}{D_2} F_4(y-u)\, du. \tag{20}$$

We shall now apply the method of initial parameters. We therefore express the integration constants in (15) by the initial parameters at $y=0$; $Y_m(0)$; $Y'_m(0)$; $M_{ym}(0)$; $V_{ym}(0)$, two of which are either known (frequently equal to zero) or expressed through the two others. Since the particular integral (20) can be used in the case of a discontinuous load as well, the problem of the determination of Y_m will reduce to the determination of two and only two initial parameters at any load $q(x, y)$.

The expressions for $Y_m(y)$, $Y'_m(y)$, $M_{ym}(y)$, and $V_{ym}(y)$ on the n-th segment along the y-axis, for a discontinuous load distributed according to some arbitrary law and for concentrated-strip forces and moments are (Figures 3 and 4), omitting the derivation:

$$Y_{mn}(y) = Y_{m1}^0(y) + Y_{mn}^*(y), \tag{21}$$

where

$$Y_{m1}^0(y) = Y_m(0) \left[F_1(y) + \mu_1 \left(\frac{m\pi}{a} \right)^2 F_3(y) \right] + \\ + Y'_m(0) \left[F_2(y) + \varepsilon \left(\frac{m\pi}{a} \right)^2 F_4(y) \right] - \frac{M_{ym}(0)}{D_2} F_3(y) - \frac{V_{ym}(0)}{D_2} F_4(y);$$

$$Y^{*}_{mn}(y) = -\int_0^y \frac{q_{m1}(u)}{D_2} F_4(y-u)\,du -$$

$$-\sum_{i=1}^{n-1} \frac{\Delta M_{ym}(d_i)}{D_2} F_3(y-d_i) - \sum_{i=1}^{n-1} \frac{\Delta V_{ym}(d_i)}{D_2} F_4(y-d_i) -$$

$$-\sum_{i=1}^{n-1} \int_{d_i}^{y} \frac{\Delta q_{mi}(u)}{D_2} F_4(y-u)\,du,$$

for $d_i > 0$

$$\begin{aligned} &\Delta q_{mi}(u) = q_{m(i+1)}(u) - q_{mi}(u); \\ &\Delta V_{ym}(d_i) = \Delta Q_{ym}(d_i) \doteq P_{md_i}; \\ &\Delta M_{ym}(d_i) = \dot{M}_{ym}; \\ &Y'_{mn}(y) = Y^{0'}_{m1}(y) + Y^{*'}_{mn}(y). \end{aligned} \tag{22}$$

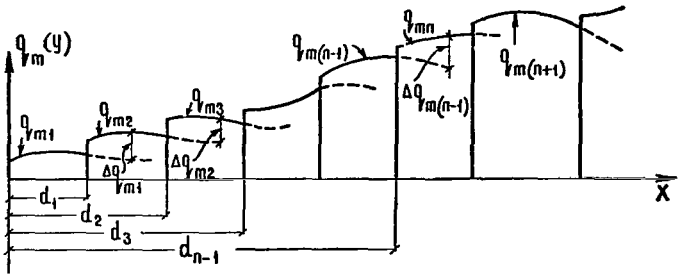

FIGURE 3.

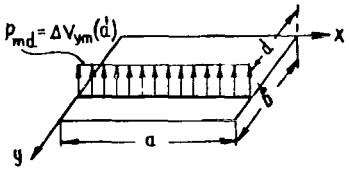

FIGURE 4.

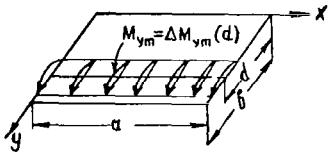

FIGURE 5.

Here

$$Y^{0'}_{m1}(y) = Y_m(0)\left[F'_1(y) + \mu_1\left(\frac{m\pi}{a}\right)^2 F'_3(y)\right] +$$

$$+ Y'_m(0)\left[F'_2(y) + \varepsilon\left(\frac{m\pi}{a}\right)^2 F'_4(y)\right] - \frac{M_{ym}(0)}{D_2} F'_3(y) - \frac{V_{ym}(0)}{D_2} F'_4(y);$$

$$Y^{*'}_{mn}(y) = -\int_0^y \frac{q_{m1}(u)}{D_2} F'_4(y-u)\,du -$$

$$-\sum_{i=1}^{n-1} \frac{\Delta M_{ym}(d_i)}{D_2} F'_3(y-d_i) - \sum_{i=1}^{n-1} \frac{\Delta V_{ym}(d_i)}{D_2} F'_4(y-d_i) - \sum_{i=1}^{n-1} \int_{d_i}^{y} \frac{\Delta q_{mi}(u)}{D_2} F'_4(y-u)\,du;$$

$$-\frac{M_{ymn}(y)}{D_2} = Y''_{mn}(y) - \mu_1\left(\frac{m\pi}{a}\right)^2 Y_{mn}(y) = -\frac{M^0_{ym1}(y)}{D_2} - \frac{M^*_{ymn}(y)}{D_2}, \qquad (23)$$

where

$$-M^0_{ym1}(y) = Y_m(0)\left\{\left[F_1^*(y) + \mu_1\left(\frac{m\pi}{a}\right)^2 F_3^*(y)\right] - \right.$$

$$\left. - \mu_1\left(\frac{m\pi}{a}\right)^2\left[F_1(y) + \mu_1\left(\frac{m\pi}{a}\right)^2 F_3(y)\right]\right\} +$$

$$+ Y'_m(0)\left\{\left[F_2^*(y) + \varepsilon\left(\frac{m\pi}{a}\right)^2 F_4(y)\right] - \right.$$

$$\left. - \mu_1\left(\frac{m\pi}{a}\right)^2\left[F_2(y) + \varepsilon\left(\frac{m\pi}{a}\right)^2 F_4(y)\right]\right\} -$$

$$-\frac{M_{ym}(0)}{D_2}\left[F_3^*(y) + \mu_1\left(\frac{m\pi}{a}\right)^2 F_3(y)\right] -$$

$$-\frac{V_{ym}(0)}{D_2}\left[F_4^*(y) - \mu_1\left(\frac{m\pi}{a}\right)^2 F_4(y)\right];$$

$$-\frac{M^*_{ymn}(y)}{D_2} = -\int_0^y \frac{q_{m1}(u)}{D_2}\left[F_4^*(y-u) - \mu_1\left(\frac{m\pi}{a}\right)^2 F_4(y-u)\right]du -$$

$$-\sum_{i=1}^{n-1}\frac{\Delta M_{ym}(d_i)}{D_2}\left[F_3^*(y-d_i) - \mu_1\left(\frac{m\pi}{a}\right)^2 F_3(y-d_i)\right] -$$

$$-\sum_{i=1}^{n-1}\frac{\Delta V_{ym}(d_i)}{D_2}\left[F_4^*(y-d_i) - \mu_1\left(\frac{m\pi}{a}\right)^2 F_4(y-d_i)\right] -$$

$$-\sum_{i=1}^{n-1}\int_{d_i}^{y}\frac{\Delta q_{mi}(u)}{D_2}\left[F_4^*(y-u) - \mu_1\left(\frac{m\pi}{a}\right)^2 F_4(y-u)\right]du;$$

$$-\frac{V_{ymn}(y)}{D_2} = V'''_{mn} - \varepsilon\left(\frac{m\pi}{a}\right)^2 Y'_{mn} = -\frac{V^0_{ym1}(y)}{D_2} - \frac{V^*_{ymn}(y)}{D_2}. \qquad (24)$$

Here

$$-\frac{V^0_{ym1}}{D_2} = Y_m(0)\left\{\left[F_1'''(y) + \mu_1\left(\frac{m\pi}{a}\right)^2 F_3'''(y)\right] - \right.$$

$$\left. - \varepsilon\left(\frac{m\pi}{a}\right)^2\left[F_1'(y) + \mu_1\left(\frac{m\pi}{a}\right)^2 F_3'(y)\right]\right\} +$$

$$+ Y'_m(0)\left\{\left[F_2'''(y) + \varepsilon\left(\frac{m\pi}{a}\right)^2 F_4'''(y)\right] - \right.$$

$$\left. - \varepsilon\left(\frac{m\pi}{a}\right)^2\left[F_2'(y) + \varepsilon\left(\frac{m\pi}{a}\right)^2 F_4'(y)\right]\right\} -$$

$$-\frac{M_{ym}(0)}{D_2}\left[F_3'''(y) - \varepsilon\left(\frac{m\pi}{a}\right)^2 F_3'(y)\right] -$$

$$-\frac{V_{ym}(0)}{D_2}\left]F_4'''(y) - \varepsilon\left(\frac{m\pi}{a}\right)^2 F_4'(y)\right];$$

$$-\frac{V^{*}_{ymn}(y)}{D_2}=-\int_0^y \frac{q_{m1}(u)}{D_2}\left[F_4'''(y-u)-\varepsilon\left(\frac{m\pi}{a}\right)^2 F_4'(y-u)\right]du-$$

$$-\sum_{i=1}^{n-1}\frac{\Delta M_{ym}(d_i)}{D_2}\left[F_3'''(y-d_i)-\varepsilon\left(\frac{m\pi}{a}\right)^2 F_3'(y-d_i)\right]-$$

$$-\sum_{i=1}^{n-1}\frac{\Delta V_{ym}(d_i)}{D_2}\left[F_4'''(y-d_i)-\varepsilon\left(\frac{m\pi}{a}\right)^2 F_4'(y-d_i)\right]-$$

$$-\sum_{i=1}^{n-1}\int_{d_i}^{y}\frac{\Delta q_{mi}(u)}{D_2}\left[F_4'''(y-u)-\varepsilon\left(\frac{m\pi}{a}\right)^2 F_4'(y-u)\right]du.$$

The integrals in (21) to (24) can be represented in an explicit form for linear loads of the type

$$q_{m1}(y)=q_{m1}(0)+q'_{m1}(0)\,y;$$

$$\Delta q_{mi}(y)=q_{m(i+1)}(y)-q_{mi}(y)=\Delta q_{mi}(d_i)+\Delta q'_{mi}(d_i)(y-d_i).$$

For the sake of brevity we shall give only those corresponding to (21):

$$-\int_0^y \frac{q_{m1}(u)}{D_2}F_4(y-u)\,du=-\frac{q_{m1}(0)}{D_2\lambda_m^4}[1-F_1(y)]-\frac{q'_{m1}(0)}{D_2\lambda_m^4}[y-F_2(y)] \quad (25)$$

$$-\int_{d_i}^{y}\frac{\Delta q_{mi}(u)}{D_2}F_4(y-u)\,du=-\frac{\Delta q_{mi}(d_i)}{D_2\lambda_m^4}[1-F_1(y-d_i)]-$$

$$-\frac{\Delta q'_{mi}(d_i)}{D_2\lambda_m^4}[(y=d_i)-F_2(y-d_i)]. \quad (26)$$

Second case: $\alpha_m^2>0$; $\alpha_m^4>\lambda_m^4>0$.

Here all the roots are real and different:

$$Y_m^0(y)=A\,\mathrm{ch}\,r_1y+B\,\mathrm{sh}\,r_1y+C\,\mathrm{ch}\,r_3y+D\,\mathrm{sh}\,r_3y.$$

The functions which satisfy the unit matrix will be

$$L_1(y)=-\frac{r_3^2}{2\sqrt{\alpha_m^4-\lambda_m^4}}\,\mathrm{ch}\,r_1y+\frac{r_1^2}{2\sqrt{\alpha_m^4-\lambda_m^4}}\,\mathrm{ch}\,r_3y; \quad (27)$$

$$L_2(y)=-\frac{r_3^2}{2r_1\sqrt{\alpha_m^4-\lambda_m^4}}\,\mathrm{sh}\,r_1y+\frac{r_1^2}{2r_3\sqrt{\alpha_m^4-\lambda_m^4}}\,\mathrm{sh}\,r_3y; \quad (28)$$

$$L_3(y)=\frac{\mathrm{ch}\,r_1y}{2\sqrt{\alpha_m^4-\lambda_m^4}}-\frac{\mathrm{ch}\,r_3y}{2\sqrt{\alpha_m^4-\lambda_m^4}}; \quad (29)$$

$$L_4(y)=\frac{\mathrm{sh}\,r_1y}{2r_1\sqrt{\alpha_m^4-\lambda_m^4}}-\frac{\mathrm{sh}\,r_3y}{2r_3\sqrt{\alpha_m^4-\lambda_m^4}}. \quad (30)$$

They replace the functions $F_1(y)$, $F_2(y)$, $F_3(y)$, and $F_4(y)$ in expressions (20) to (26), respectively.

Third case: $\alpha_m^2>0$; $\lambda_m^4=\alpha_m^4>0$.

The characteristic equation has multiple roots $r_1=-r_2=\alpha_m$; $r_3=-r_4=\alpha_m$,

and therefore

$$Y_m^0(y) = A \operatorname{ch} \alpha_m y + B \operatorname{sh} \alpha_m y + Cy \operatorname{ch} \alpha_m y + Dy \operatorname{sh} \alpha_m y.$$

The functions which satisfy the unit matrix and replace $F_1(y)$, $F_2(y)$, $F_3(y)$, and $F_4(y)$ in (20) to (26) will be

$$R_1(y) = \operatorname{ch} \alpha_m y - \frac{\alpha_m}{2} y \operatorname{sh} \alpha_m y; \tag{31}$$

$$R_2(y) = \frac{3}{2\alpha_m} \operatorname{sh} \alpha_m y - \frac{y}{2} \operatorname{ch} \alpha_m y; \tag{32}$$

$$R_3(y) = \frac{1}{2\alpha_m} y \operatorname{sh} \alpha_m y; \tag{33}$$

$$R_4(y) = -\frac{1}{2\alpha_m^3} \operatorname{sh} \alpha_m y + \frac{1}{2\alpha_m^2} y \operatorname{ch} \alpha_m y. \tag{34}$$

Fourth case: $\alpha_m^2 > 0$; $\lambda_m^4 < 0$.

The roots $r_1 = -r_2 = \sqrt{\alpha_m^2 + \sqrt{\alpha_m^4 - \bar{\lambda}_m^4}}$ are real, and the roots

$$r_3 = -r_4 = \sqrt{\alpha_m^2 - \sqrt{\alpha_m^4 - \bar{\lambda}_m^4}} = i\bar{r}_3$$

imaginary; here

$$\bar{\lambda}_m^4 = -\lambda_m^4 > 0, \qquad \bar{r}_3 = \sqrt{\sqrt{\alpha_m^4 + \bar{\lambda}_m^4} - \alpha_m^2} > 0;$$

and therefore

$$Y_m^0(y) = A \operatorname{ch} r_1 y + B \operatorname{sh} r_1 y + C \cos \bar{r}_3 y + D \sin \bar{r}_3 y.$$

The functions satisfying the unit matrix and replacing $F_1(y)$, $F_2(y)$, $F_3(y)$, and $F_4(y)$ in (20) to (26) will be

$$T_1(y) = \frac{\bar{r}_3^2 \operatorname{ch} r_1 y}{r_1^2 + \bar{r}_3^2} + \frac{r_1^2 \cos \bar{r}_3 y}{r_1^2 + \bar{r}_3^2}; \tag{35}$$

$$T_2(y) = \frac{\bar{r}_3^2 \operatorname{sh} r_1 y}{r_1 (r_1^2 + \bar{r}_3^2)} + \frac{r_1^2 \sin \bar{r}_3^2 y}{\bar{r}_3 (r_1^2 + \bar{r}_3^2)}; \tag{36}$$

$$T_3(y) = \frac{1}{r_1^2 + \bar{r}_3^2} (\operatorname{ch} r_1 y - \cos \bar{r}_3 y); \tag{37}$$

$$T_4(y) = \frac{\operatorname{sh} r_1 y}{r_1 (r_1^2 + \bar{r}_3^2)} - \frac{\sin \bar{r}_3 y}{\bar{r}_3 (r_1^2 + \bar{r}_3^2)}. \tag{38}$$

Fifth case: $\alpha_m^2 < 0$; $\lambda_m^4 < 0$.

The roots of the characteristic equations are:

$$r_1 = -r_2 = \sqrt{-\bar{\alpha}_m^2 + \sqrt{\bar{\alpha}_m^4 + \bar{\lambda}_m^4}} \tag{39}$$

(real roots) and

$$r_3 = -r_4 = \sqrt{-\bar{\alpha}_m^2 - \sqrt{\bar{\alpha}_m^4 + \bar{\lambda}_m^4}} = i\bar{r}_3 \tag{40}$$

(imaginary roots), where

$$\bar{r}_3 = \sqrt{\bar{\alpha}_m^2 + \sqrt{\bar{\alpha}_m^4 + \bar{\lambda}_m^4}}; \qquad \bar{\alpha}_m^2 = -\alpha_m^2 > 0; \ \bar{\lambda}_m^4 = -\lambda_m^4 > 0. \tag{41}$$

The fifth case reduces to the fourth, but with new values of the roots, given by (39) to (41).

Sixth case: $\alpha_m^2 < 0$; $\lambda_m^4 > 0$.

The roots are

$$r_1 = -r_2 = \sqrt{-\bar{\alpha}_m^2 + \sqrt{\bar{\alpha}_m^4 - \lambda_m^4}}\ ; \tag{42}$$

$$r_3 = -r_4 = \sqrt{-\alpha_m^2 + \sqrt{\bar{\alpha}_m^4 - \bar{\lambda}_m^4}}\ ; \tag{43}$$

where $$\bar{\alpha}_m^2 = -\alpha_m^2 > 0.$$

a) If $\bar{\alpha}_m^4 > \lambda_m^4$, then

$$r_1 = -r_2 = \sqrt{-\bar{\alpha}_m^2 + \sqrt{\bar{\alpha}_m^4 - \lambda_m^4}} = i\bar{r}_1 \tag{44}$$

are imaginary, and

$$r_3 = -r_4 = \sqrt{-\bar{\alpha}_m^2 - \sqrt{\bar{\alpha}_m^4 - \lambda_m^4}} = i\bar{r}_3 \tag{45}$$

are likewise imaginary; here

$$\bar{r}_1 = \sqrt{\bar{\alpha}_m^2 - \sqrt{\bar{\alpha}_m^4 - \lambda_m^4}} > 0;\quad \bar{r}_3 = \sqrt{\bar{\alpha}_m^2 + \sqrt{\bar{\alpha}_m^4 - \lambda_m^4}} > 0; \tag{46}$$

therefore:

$$Y_m^0(y) = A\cos\bar{r}_1 y + B\sin\bar{r}_1 y + C\cos\bar{r}_3 y + D\sin\bar{r}_3 y.$$

The functions which replace $F_1(y)$, $F_2(y)$, $F_3(y)$, and $F_4(y)$ in (20) to (26) will be:

$$U_1(y) = -\frac{\bar{r}_3^2}{\bar{r}_1^2 - \bar{r}_3^2}\cos\bar{r}_1 y + \frac{\bar{r}_1^2}{\bar{r}_1^2 - \bar{r}_3^2}\cos\bar{r}_3 y; \tag{47}$$

$$U_2(y) = -\frac{\bar{r}_3^2}{\bar{r}_1(\bar{r}_1^2 - \bar{r}_3^2)}\sin\bar{r}_1 y + \frac{\bar{r}_1^2}{\bar{r}_3(\bar{r}_1^2 - \bar{r}_3^2)}\sin\bar{r}_3 y; \tag{48}$$

$$U_3(y) = -\frac{1}{\bar{r}_1^2 - \bar{r}_3^2}(\cos\bar{r}_1 y - \cos\bar{r}_3 y); \tag{49}$$

$$U_4(y) = -\frac{\sin\bar{r}_1 y}{\bar{r}_1(\bar{r}_1^2 - \bar{r}_3^2)} + \frac{\sin\bar{r}_3 y}{\bar{r}_3(\bar{r}_1^2 - \bar{r}_3^2)}. \tag{50}$$

b) If $\lambda_m^4 > \bar{\alpha}_m^4 > 0$, then

$$r_1 = -r_2 = \sqrt{-\bar{\alpha}_m^2 + i\sqrt{\lambda_m^4 + \bar{\alpha}_m^4}} = \bar{\beta}_m + \bar{\gamma}_m i; \tag{51}$$

$$r_3 = -r_4 = \sqrt{-\bar{\alpha}_m^2 - i\sqrt{\lambda_m^4 - \bar{\alpha}_m^4}} = \bar{\beta}_m - \bar{\gamma}_m i; \tag{52}$$

where $$\bar{\beta}_m = \sqrt{(\lambda_m^2 - \bar{\alpha}_m^2)/2};\quad \bar{\gamma}_m = \sqrt{(\lambda_m^2 + \bar{\alpha}_m^2)/2}\ . \tag{53}$$

This case reduces to the first one, with new values of β_m and γ_m, given by (53).

§ 4. BOUNDARY CONDITIONS

The boundary conditions are written as usual, except for the conditions on the free edge, where the elastic foundation with two foundation moduli gives

concentration-strip reactions (Figure 6)

$$R_m(0) = c_2[\mathrm{tg}\varphi_m - Y'_m(0)]. \tag{54}$$

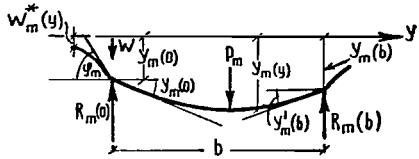

FIGURE 6.

The differential equation of displacements of the unloaded foundation surface is

$$\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} - \frac{c_1}{c_2} w^* = 0.$$

Writing $w^*(x, y) = \sum_{m=1}^{\infty} w^*_m(y) \sin \frac{m\pi x}{a}$ and using the conditions: 1) $y = -\infty$, $w^*_m = 0$; 2) $y = 0$, $w^*_n = Y_m(0)$, we obtain

$$w^*_m(y) = Y_m(0) \exp\left[\sqrt{\frac{c_1}{c_2} + \left(\frac{m\pi}{a}\right)^2} \cdot y\right].$$

We then find

$$R_m(0) = c_2\left[Y_m(0)\sqrt{\frac{c_1}{c_2} + \left(\frac{m\pi}{a}\right)^2} - Y'_m(0)\right], \tag{55}$$

and similarly

$$R_m(b) = c_2\left[Y_m(b)\sqrt{\frac{c_1}{c_2} + \left(\frac{m\pi}{a}\right)^2} + Y'_m(b)\right]. \tag{56}$$

§5. CALCULATION OF PLATES SUBJECTED TO A PULSATING LOAD

In the case of a pulsating load the differential equation (1) will contain an additional term representing the inertial forces:

$$D_1 \frac{\partial^4 \bar{w}}{\partial x^4} + 2D_3 \frac{\partial^4 \bar{w}}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 \bar{w}}{\partial y^4} - H_x \frac{\partial^2 w}{\partial x^2} - H_y \frac{\partial^2 w}{\partial y^2} +$$

$$+ c_1 \bar{w} - c_2\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + m^* \frac{\partial^2 w}{\partial t^2} = -q(x, y)(A \sin \theta t + B \cos \theta t), \tag{57}$$

where m^* is the mass of the plate and the attached foundation per unit area; $\bar{w}(x, y, t)$.

We shall consider steady vibration of the plate; in this case

$$\bar{w}(x, y, t) = w(x, y)(A \sin \theta t + B \cos \theta t). \tag{58}$$

The substitution of (58) in (57) yields the following differential equation:

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} - H_x \frac{\partial^2 w}{\partial x^2} -$$

$$- H_y \frac{\partial^2 w}{\partial y^2} + c_1^* w - c_2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = - q(x, y), \qquad (59)$$

where $$c_1^* = c_1 - m^* \theta^2.$$

This equation coincides in structure with the initial equation (1). All the results obtained earlier on the basis of the solution of (1) are suitable therefore in this case as well, it being only necessary to replace c_1 by c_1^*.

§ 6. FREE VIBRATIONS AND PLATE STABILITY

The question will be described in general lines only, for the sake of brevity. The frequencies of natural vibrations of the plates ω, and the critical loads H_x and H_y, can be found from the determinant of the equations, written for calculating the unknown initial parameters, in the absence of lateral load; to that end the determinant is equated to zero, and the frequency of the acting load θ or the loads H_x and H_y are considered unknown magnitudes.

In this case the factors before the arguments of the different functions (the roots of the characteristic equation) will be unknown magnitudes; herein lies the basic difficulty in solving problems of this type.

§ 7. SOME PARTICULAR CASES

Consider a trapezoidal load on a portion of the plate (Figure 7).*

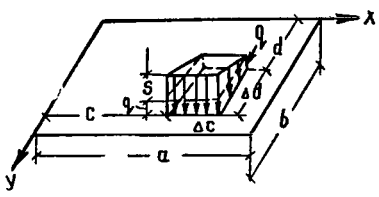

FIGURE 7.

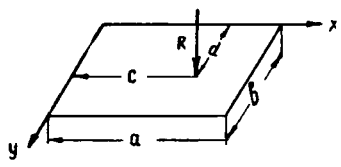

FIGURE 8.

First portion: $0 \leqslant y \leqslant d = d_1$, $q(x, y) = 0$, and therefore $q_{m1} = 0$.

According to (21) $$Y_{m1}(y) = Y_{m1}^0(y). \qquad (60)$$

Second portion: $d \leqslant y \leqslant d + \Delta d = d_2$.

Here, in the region $c \leqslant x \leqslant c + \Delta c$, $d \leqslant y \leqslant d + \Delta d$,

* The load can be arbitrary, either static or vibrating.

the load is determined by the equation $q(x, y) = -\left[q + \frac{s}{\Delta d}(y - d) \right]$.
According to (10)

$$q_{m2}(y) = -\frac{2}{a}\int_{c}^{c+\Delta c} \left[q + \frac{s}{\Delta d}(y-d) \right] \sin\frac{m\pi x}{a}\,dx =$$

$$= -\frac{2}{m\pi}\left[q + \frac{s}{\Delta d}(y-d) \right] A; \quad q'_{m2}(y) = -\frac{2}{m\pi}\frac{s}{\Delta d},$$

where
$$A = \cos\frac{m\pi c}{a} - \cos\frac{m\pi(c+\Delta c)}{a}.$$

The transition conditions from the first portion into the second one are

$$\Delta M_{ym1}(d_1) = 0; \quad \Delta V_{ym1}(d_1) = 0;$$

$$\Delta q_{m1}(d_1) = q_{m2}(d_1) - q_{m1}(d_1) = -\frac{2q}{m\pi}A;$$

$$\Delta q'_m(d_1) = q'_{m2}(d_1) - q'_{m1}(d_1) = -\frac{2s}{m\pi\Delta d}.$$

From (21) and (26) we obtain

$$Y_{m2}(y) = Y^0_{m1}(y) + Y^*_{m2}(y), \tag{61}$$

where

$$Y^*_{m2}(y) = \frac{2q}{m\pi}A\frac{1}{D_2\lambda^4_m}[1 - F_1(y-d)] +$$

$$+ \frac{2r}{m\pi\Delta d}\cdot A\frac{1}{D_2\lambda^4_m}[(y-d) - F_2(y-d)].$$

Third portion: $d_2 = d + \Delta d \leqslant y \leqslant b$; $q_3(x, y) = 0$, therefore $q_{m3}(y) = 0$.
The transition conditions from the second portion into the third one are

$$\Delta M_{ym2}(d_2) = 0; \quad \Delta V_{ym2}(d_2) = 0;$$

$$\Delta q_{m2}(d_2) = \frac{2}{m\pi}(q+s)A; \quad \Delta q'_{m2}(d_2) = \frac{2s}{m\pi\Delta d}A.$$

From (21) and (26) we obtain

$$Y_{m3}(y) = Y^0_{m1}(y) + Y^*_{m3}(y), \tag{62}$$

where

$$Y^*_{m3}(y) = \frac{2qA}{m\pi D_2\lambda^4_m}[1 - F_1(y-d)] + \frac{2sA}{m\pi\Delta d D_2\lambda^4_m}[(y-d) -$$

$$- F_2(y-d)] - \frac{2(q+s)A}{m\pi D_2\lambda^4_m}[1 - F_1(y-d-\Delta d)] -$$

$$- \frac{2sA}{m\pi\Delta d D_2\lambda^4_m}[(y-d-\Delta d) - F_2(y-d-\Delta d)].$$

It is possible to obtain from this case expressions for numerous other cases, corresponding to different combinations of the magnitudes c, d, Δc, Δd, $c = 0$, $d = 0$, $\Delta c = 0$, $\Delta d = 0$, $q = 0$, and $s = 0$. We also obtain the case of action of a concentrated force P (Figure 8). In this case we shall have two portions only (the second portion disappears, and the third one becomes second).

First portion: $0 \leqslant y \leqslant d$; $Y_{m1}(y) = Y_{m1}^{0-}(y)$.

Second portion: $d \leqslant y \leqslant b$; $Y_{m2}(y) = Y_{m1}^{0}(y) + Y_{m2}^{*}(y)$.
According to (62), for $s = 0$,

$$Y_{m2}^{*}(y) = \frac{2qA}{m\pi D_2 \lambda_m^4} \cdot [F_1(y - d - \Delta d) - F_1(y - d)].$$

Write

$$q = \frac{R}{\Delta c \cdot \Delta d},$$

$$Y_{m2}^{*}(y) = \frac{2R}{\Delta c \cdot \Delta d \cdot m\pi D_2 \lambda_m^4} \left[\cos\frac{m\pi c}{a} - \cos\frac{m\pi(c + \Delta c)}{a}\right] \times$$
$$\times [F_1(y - d - \Delta d) - F_1(y - d)].$$

At the limit $\Delta c \to 0$ and $\Delta d \to 0$,

$$Y_{m2}^{*}(y) = \frac{2R}{aD_2} \sin\frac{m\pi c}{a} F_4(y - d). \tag{63}$$

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ON THE STATE OF MEMBRANE STRESS OF SHELLS OF REVOLUTION WITH POSITIVE CURVATURE

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The state of membrane stress of a shell of revolution formed by a second-order surface is described by an analytical stress function. The more general case of a shell formed by an arbitrary surface of revolution of positive curvature is treated below. The corresponding homogeneous static equations of equilibrium in the geodesic system of coordinates ($r = r(z)$ is the equation of the generator; z = const is the equation of the parallel; β = const is the equation of the meridian) /1/ are reduced by means of the transformation of variables

$$\xi = \int \sqrt{-\frac{r''(z)}{r(z)}}\,dz, \quad \eta = \beta$$

to a system which defines a p-analytical function of a complex variable

$$\frac{\partial \Phi}{\partial \xi} = \frac{1}{p(\xi)} \frac{\partial \Psi}{\partial \eta}, \qquad \frac{\partial \Phi}{\partial \eta} = -\frac{1}{p(\xi)} \frac{\partial \Psi}{\partial \xi}, \tag{1}$$

where $p(z) = \sqrt{-\frac{1}{r''(z)\,r(z)}}$ is the nonnegative characteristic of the p-analytical function, defined by the shell geometry /*/.

The stresses appearing in a shell of revolution free of surface load are thus expressed through the real and imaginary parts of a stress function which is a p-analytical function of a complex variable /4/.

In the case of a shell of revolution whose characterstic p is either an exponential function or such that $\sqrt{p}$ is a harmonic function, the boundary-value problem arising for the p-analytical function can be reduced to the corresponding boundary-value problem for an analytical function, by using the known relationship between analytical and p-analytical functions /3/. In the case that $\sqrt{p}$ is a harmonic function, this connection is expressed by the following Theorem 1.

Let $\bar{f}(z) = \bar{u}(x, y) + i\bar{v}(x, y)$ be an analytical function, and let $\Delta\sqrt{p} = 0$

* [The reference is not given in the Russian text.]

Then

$$f(z) = u(x, y) + iv(x, y) = \frac{\tilde{u}(x, y)}{\sqrt{p}} + i(v(x, y)\sqrt{p} + R)$$

will be a p-analytical function, and $R(x, y)$ will satisfy the conditions

$$\frac{\partial R}{\partial x} = \tilde{u}(x, y)\frac{\partial\sqrt{p}}{\partial y} - \tilde{v}(x, y)\frac{\partial\sqrt{p}}{\partial x};$$

$$\frac{\partial R}{\partial y} = -\left[\tilde{u}(x, y)\frac{\partial\sqrt{p}}{\partial x} + \tilde{v}(x, y)\frac{\partial\sqrt{p}}{\partial y}\right].$$

This is the method used to solve in a closed form the problem of the state of membrane stress of a glass-shaped shell of revolution of generator

$$r(z) = \sqrt{4cz^2 + 2z}\ln\sqrt{\frac{2cz+1}{2cz}}, \tag{2}$$

where $c > 0$ is a numerical parameter /4/.

We consider next the case of a membrane shell of revolution subjected to a concentrated force. Integral equations of equilibrium were obtained for a finite part of the shell, similar to the conditions of equilibrium for a second-order shell of revolution. Using the apparatus of p-analytical functions developed in Polozhii's papers /2, 3/, these conditions are written as integrals by the conjugate variable of the p-analytical function of a complex variable $\Phi(\xi, \eta) + i\Psi(\xi, \eta)$:

$$P_z - iM_z = \int_\Gamma \Phi d\tilde{Z}_1 + i\Psi dZ_1, \quad Z_1 = -\eta + i\int_\Gamma \frac{d\xi}{p},$$

$$\tilde{Z}_1 = -\eta + i\int p d\xi,$$

$$P_x + iP_y = \int_\Gamma \Phi d\tilde{Z}_2 + i\Psi dZ_2, \quad Z_2 = -\frac{i}{r}l^{i\eta}, \quad \tilde{Z}_2 = ir'l^{i\eta} \tag{3}$$

$$M_y - iM_x = \int_\Gamma \Phi d\tilde{Z}_3 + i\Psi dZ_3, \quad Z_3 = -\frac{i}{r}zl^{i\eta},$$

$$\tilde{Z}_3 = i(r - zr')l^{i\eta},$$

where P is the resultant of the forces applied to the shell, M_x, M_y, M_z are its moments relative to the coordinate axes, Z_l and $\tilde{Z}_l$ $(l = 1, 2, 3)$ are conjugate variables corresponding to the given p-characteristic.

In the case of a glass-shaped shell (2) we obtain $P = \xi^3$, and the conjugate variables are

$$Z_1 = -\eta - \frac{i}{\xi};\ \tilde{Z}_1 = -\eta - i\frac{\xi^3}{3},$$

$$Z_2 = -\frac{i}{\xi}(l^{\xi+i\eta} - l^{-\xi+i\eta}),\ \tilde{Z}_2 = -i[(\xi+1)l^{-\xi+i\eta} + (\xi-1)l^{-\xi+i\eta}],$$

$$Z_3 = \frac{i}{\xi}l^{\xi+i\eta},\ \tilde{Z}_3 = i(\xi-1)l^{\xi+i\eta}. \tag{4}$$

The ξ^3-analytical function $\Phi(\xi, \eta) + i\Psi(\xi, \eta)$ is then defined in the region G

(the plane $\zeta = \xi + i\eta$) which has the shape of a left-hand unbounded horizontal semistrip of width 2π, and possesses at the point of application of the concentrated force a singularity in the form of a pole.

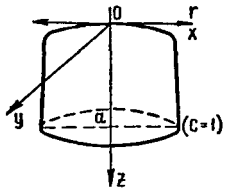

FIGURE 1.

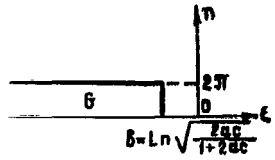

FIGURE 2.

We shall consider two cases of loading of the glass-shaped shell by a concentrated force.

1. Let the concentrated force be applied at the shell top. In accordance with Theorem 1, we form the ξ^2-analytical function of stresses, starting from an analytical function having a singularity at the point corresponding to the shell top. We obtain

$$f(\xi) = \Phi(\xi, \eta) + i\,\Psi(\xi, \eta) = l^{-\xi}\left[\frac{1}{\xi}(a_{-1}\cos\eta + b_{-1}\sin\eta) - \right.$$

$$\left. - i(\xi+1)(a_{-1}\sin\eta - b_{-1}\cos\eta)\right] + \left[b_0 + a_0\left(-i\eta + \frac{1}{\xi}\right)\right] +$$

$$+ l^{\xi}\left[\frac{1}{\xi}(a_1\cos\eta + b_1\sin\eta) + i(\xi-1)(a_1\sin\eta - b_1\cos\eta)\right] +$$

$$+ \sum_{m=2}^{\infty} l^{m\xi}\left[\frac{1}{\xi}(a_m\cos m\eta + b_m\sin m\eta) + \frac{i(m\xi-1)}{m}(a_m\sin m\eta - b_m\cos m\eta)\right]. \quad (5)$$

The coefficients of the regular part of the ξ^2-analytical function a_m, b_m ($m=2, 3, \cdots$) are determined from the boundary conditions, and the coefficients a_{-1}, b_{-1}, a_0, b_0, a_1, b_1 from the integral conditions of equilibrium (3), (4). By substituting (5) and (4) in (3) and taking the integrals by the conjugate variables we obtain

$$a_0 = -\frac{M_z}{2\pi^2}, \quad b_0 = -\frac{P_z}{2\pi}, \quad a_{-1} = -\frac{\sqrt{c}}{\pi}M_y, \quad b_{-1} = \frac{\sqrt{e}}{\pi}M_x, \quad (6)$$

$$a_1 = \frac{P_x + 2cM_y}{2\pi\sqrt{c}}, \quad b_1 = \frac{P_y - 2cM_x}{2\pi\sqrt{c}}. \quad (7)$$

Consider as an example the problem of determining the forces in a single-shear glass-shaped dome loaded at the top by a concentrated force, in the case when the end $z = a = \text{const}$ is connected with a support which does not take reactions in the tangential direction. This problem leads to the determination in region G of a ξ^2-analytical function with a singularity in the form of a pole at the shell top, from the boundary conditions

$$\Phi(z, \beta)|_{z=a} = \Phi(\xi, \eta)|_{\xi=-b} = 0. \quad (8)$$

We shall represent the solution in the form of the function (5), determining

the coefficients a_m, b_m by (6) and (7). We obtain

$$f(\xi)=\Phi(\xi,\eta)+i\,\Psi(\xi,\eta)=\frac{l^{-\xi}}{2\pi\sqrt{c}\,(l^{2b}+1)}\left[\frac{1}{\xi}(P_x\cos\eta+P_y\sin\eta)-\right.$$

$$\left.-i(\xi+1)(P_x\sin\eta-P_y\cos\eta)\right]+\frac{l^{-\xi}}{2\pi\sqrt{c}\,(l^{2b}+1)}\left[\frac{1}{\xi}(P_x\cos\eta+P_y\sin\eta)+\right.$$

$$\left.+i(\xi-1)(P_x\sin\eta-P_y\cos\eta)\right].$$

The following relationships must be satisfied from boundary conditions (8):

$$M_z=P_z=0,\quad M_y=\frac{P_x}{2c\,(l^{2b}+1)},\quad M_x=\frac{P_y}{2c\,(l^{2b}+1)}.$$

It is seen from these relationships that the force acting on the shell top cannot be specified arbitrarily.

2. Let now the concentrated force be applied at an arbitrary point $\zeta_0=\xi_0+i\eta_0$ of the shell. We make the transformation of variables $\gamma=t+is=l^{\xi}=l^{\xi+i\eta}$. The region G is transformed as a result into a circle of radius $R=l^{-b}$, and the p-analytical function with characteristic $p=\xi^2$ is transformed into a p-analytical function with characteristic $p=\ln^2\sqrt{t^2+s^2}$. Since $\sqrt{p}=\ln\sqrt{t^2+s^2}$ is a harmonic function, we form a p-analytical function with third-order pole at point $\gamma_0=l^{\xi_0}=t_0+is_0$, starting, in accordance with Theorem 1, from an analytical function. We obtain

$$f(\gamma)=\frac{1}{\ln\sqrt{r^2+2r\,(t_0\cos\varphi+s_0\sin\varphi)+t_0^2+s_0^2}}\sum_{k=1}^{3}\frac{A_k\cos k\varphi+B_k\sin k\varphi}{r^k}+$$

$$+i[\ln\sqrt{r^2+2r\,(t_0\cos\varphi+s_0\sin\varphi)+t_0^2+s_0^2}\sum_{k=1}^{3}\frac{-A_k\sin k\varphi+B_k\cos k\varphi}{r^k}+$$

$$+\frac{B_3(t_0\cos2\varphi-s_0\sin2\varphi)-A_3(t_0\sin2\varphi+s_0\cos2\varphi)}{2\,(t_0^2+s_0^2)\,r^2}+$$

$$+\frac{B_2(t_0\cos\varphi-s_0\sin\varphi)-A_2(t_0\sin\varphi+s_0\cos\varphi)}{(t_0^2+s_0^2)\,r}+$$

$$+\frac{A_3(t_0^2-s_0^2)\sin\varphi+2A_3t_0^2s_0^2\cos\varphi+2B_3t_0^2s_0^2\sin\varphi-B_3(t_0^2-s_0^2)\cos\varphi}{(t_0^2+s_0^2)\,r}\Bigg]+\mu(\gamma),$$

where $r=|\gamma-\gamma_0|$, $\varphi=\arg(\gamma-\gamma_0)$, and $\mu(\gamma)$ is a function regular in the vicinity of γ_0.

The function $\mu(\gamma)$ is determined from the boundary conditions. The coefficients A_k, B_k $(k=1,2,3)$ are determined from the integral conditions of equilibrium (3), (4), which yield

$$\int_\Gamma\Phi d\bar{Z}_k+i\Psi dZ_k=2\pi i\ln\sqrt{t_0^2+s_0^2}\left[\left(A_1\frac{\partial Z_k}{\partial t}-iB_1\frac{\partial Z_k}{\partial is}\right)+\right.$$

$$+\frac{A_2}{2}\left(\frac{\partial^2Z_k}{\partial t^2}+\frac{\partial^2Z_k}{\partial^2 is}\right)-iB_2\frac{\partial^2Z_k}{\partial t\partial is}+\frac{A_3}{8}\left(\frac{\partial^3Z_k}{\partial t^3}+3\frac{\partial^2Z_k}{\partial t\partial is^2}\right)-$$

$$\left.-i\frac{B_3}{8}\left(\frac{\partial^3Z_k}{\partial is^3}+3\frac{\partial^3Z_k}{\partial is\partial t^2}\right)\right]_{\gamma=\gamma_0}+\frac{\pi i}{t_0^2+s_0^2}\left\{A_2\left(t_0\frac{\partial Z_k}{\partial t}-s_0\frac{\partial Z_k}{\partial s}\right)+\right.$$

$$+B_2\left(s_0\frac{\partial Z_k}{\partial t}+t_0\frac{\partial Z_k}{\partial s}\right)+A_3\left[2t_0^2\,s_0^2\frac{\partial Z_k}{\partial s}-(t_0^2+s_0^2)\frac{\partial Z_k}{\partial t}+\right.$$

$$+ t_0\left(\frac{\partial^2 Z_k}{\partial t^2} - \frac{\partial^2 Z_k}{\partial s^2}\right) - 2s_0 \frac{\partial^2 Z_k}{\partial t \partial s}\Bigg] - B_3\Bigg[2t_0^2\, s_0^2 \frac{\partial Z_k}{\partial t} +$$

$$+ s_0\left(\frac{\partial^2 Z_k}{\partial t^2} - \frac{\partial^2 Z_k}{\partial s^2}\right) + 2t_0 \frac{\partial^2 Z_k}{\partial t \partial s}\Bigg]\Bigg\}_{\tau=\tau_0} \quad (k = 1, 2, 3).$$

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SOLUTION OF THE PROBLEM OF THE CYCLICALLY SYMMETRICAL DEFORMATION OF A CONIC SHELL IN DEGENERATE HYPERGEOMETRICAL FUNCTIONS

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The paper deals with the cyclical deformation of a conic shell of uniform thickness under the action of a force load and nonuniform heating, corresponding to the distribution along the parallel of contour and surface forces and purely thermal strains according to the law $\cos k\theta$ or $\sin k\theta$ ($k=2, 3$), where θ is the angle between the planes of the running and the initial meridians.

In this case, unlike the cases of axisymmetrical ($k=0$) and antisymmetrical ($k=1$) distributions of forces and thermal strains, the equations of the theory of shells cannot be integrated, and the problem is described by a resolving fourth-order differential equation in a complex form /3/.

The solution of this equation for a conical shell of arbitrary cone angle is found here in degenerate hypergeometrical functions, the fundamental properties of which are given in /3/.

The solution for all forces, moments, strains, and displacements is given in the same functions.

The earlier obtained solution in degenerate hypergeometrical functions of the problem of the cyclical deformation of a shallow conic shell /2/ is a particular case of the problem treated here.

The solutions in degenerate hypergeometrical functions, defined by series summed by means of rather simple recurrent formulas, are easily tabulated by digital computers; the tables of the solution for the k-th harmonic are obtained in the case of a conic shell of arbitrary cone angle with two inlets, and in the case of a shallow conic shell with one inlet.

1. We shall use the fundamental equations of the theory of shells and symbols given in /3/.

As unknown magnitudes we shall consider in what follows the amplitude values of the stress resultants, moments, strains, and displacements.

The resolving equation of the problem considered is

$$\left\{z\frac{d}{dz}\left(z\frac{d}{dz}+1\right)\left[z\frac{d}{dz}\left(z\frac{d}{dz}+1\right)-z-\frac{2k^2}{\sin^2\alpha}\right]+\frac{k^2(k^2-1)}{\sin^4\alpha}\right\}\bar{N}+Az^{\lambda}=0, \quad (1.1)$$

where

$$z=ix=a_0 l, \quad a_0=\frac{ic_0\operatorname{ctg}\alpha}{h}, \quad c_0=\sqrt{12(1-\nu^2)};$$

$$\bar{N}=\bar{N}_l+\bar{N}_\theta=\langle N_l+N_\theta+k_0(\varkappa_l+\varkappa_\theta), \quad k_0=i\frac{Eh^2}{c_0};$$

Table of values of A and λ

Type of load or nonuniform heating	A	λ
Surface load		
$q_l = q_m l^m$ ($q_m = \text{const}$; $m = 0, 1, \cdots$)	$\dfrac{i h q_m}{C_0 \operatorname{ctg}\alpha \, a_0^m} (m+3)$	$m+2$
Surface load		
$q_z = q_n l^n$ ($q_n = \text{const}$; $n = 0, 1, \cdots$)	$-\dfrac{ihq_n}{C_0 \operatorname{ctg}^2 \alpha \, a_0^n}\left(n+3-\dfrac{k}{\sin\alpha}\right)\left(n+3+\dfrac{k}{\sin\alpha}\right)$	$n+2$
Nonuniform heating		
$\varepsilon_T = \varepsilon_j l^j$, $\varkappa_T = \varkappa_j l^j$ ($\varepsilon_j \varkappa_j = \text{const}$; $j = 0, 1, \cdots$)	$-\dfrac{ih}{a_0^j}\left[i\varepsilon_j + \dfrac{(1+\nu)h}{C_0}\varkappa_j\right]\left[j^4 + 2j^3 + \left(1-\dfrac{2k^2}{\sin^2\alpha}\right)j^2 - \dfrac{2k^2}{\sin^2\alpha}j + \dfrac{k^2(k^2-1)}{\sin^4\alpha}\right]$	j

α is the angle between the meridian and the axis of the middle surface; h is the shell thickness; N_l and N_θ are the normal stress resultants in sections l = const and θ = const; $\varkappa_l$ and $\varkappa_\theta$ are changes of curvature in the meriodional and the other principal sections. The constants A and λ have the values given in the table.

We shall represent the general solution of (1.1) in the form

$$\tilde{N} = \sum_{n=1}^{4} \tilde{C}_n \tilde{N}^{(n)} + \tilde{N}^{(s)} , \tag{1.2}$$

where $\tilde{N}^{(n)}$ $(n=1, 2, 3, 4)$ is a particular solution of the homogeneous equation corresponding to (1.1); $\tilde{C}_n$ $(n=1, 2, 3, 4)$ are complex integration constants; $\tilde{N}^{(s)}$ is a particular solution of the nonhomogeneous equation (1.1).

2. By means of the substitution

$$\tilde{N} = z^{\rho_1} W, \tag{2.1}$$

where

$$\rho_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4k(k+1)}{\sin^2\alpha}} ,$$

we transform the homogeneous equation corresponding to (1.1) to the canonical form of a degenerate hypergeometrical fourth-order equation

$$\left[z\frac{d}{dz}\left(z\frac{d}{dz} + \beta_1 - 1\right)\left(z\frac{d}{dz} + \beta_2 - 1\right)\left(z\frac{d}{dz} + \beta_3 - 1\right) - \right.$$
$$\left. - z\left(z\frac{d}{dz} + \alpha_1\right)\left(z\frac{d}{dz} + \alpha_2\right)\right] W = 0, \tag{2.2}$$

whose parameters are equal to

$$\left.\begin{aligned} &\alpha_1 = 1 + \rho_1, \quad \alpha_2 = 2 + \rho_1, \\ &\beta_1 = 1 + \rho_1 - \rho_2, \\ &\beta_2 = 1 + \rho_1 - \rho_3, \\ &\beta_3 = 1 + \rho_1 - \rho_4, \end{aligned}\right\} \tag{2.3}$$

where

$$\rho_1, \rho_2, \rho_3, \rho_4 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4k(k \pm 1)}{\sin^2\alpha}} . \tag{2.4}$$

On the strength of the properties of the hypergeometrical equations /3/ we obtain the following particular solution of the homogeneous equation corresponding to (1.1):

$$\tilde{N}^{(n)} = z^{\rho_n}\, {}_2F_3\,(1 + \rho_n, 2 + \rho_n;\ 1 + \rho_n - \rho_1,\ {}^*1 + \rho_n - \rho_2,$$
$$1 + \rho_n - \rho_3,\ 1 + \rho_n - \rho_4;\ z) \quad (n = 1, 2, 3, 4), \tag{2.5}$$

where the asterisk designates that the expression $1 + \rho_n - \rho_m$ is to be discarded at $n = m$.

3. The substitution of (2.1) into (1.1) yields

$$\left[z\frac{d}{dz}\left(z\frac{d}{dz}+\beta_1-1\right)\left(z\frac{d}{dz}+\beta_2-1\right)\left(z\frac{d}{dz}+\beta_3-1\right)-\right.$$
$$\left.-z\left(z\frac{d}{dz}+\alpha_1\right)\left(z\frac{d}{dz}+\alpha_2\right)\right]W+A\,z^{\lambda-\rho_1}=0. \qquad (3.1)$$

Using the particular solution of equations such as (3.1) through a hypergeometric function given in /3/, and allowing for the substitution of (2.1), we find the following particular solution of the nonhomogeneous equation (1.1):

$$\tilde{N}^{(s)}=-\frac{Az^{\lambda}}{(\lambda-\rho_1)(\lambda-\rho_2)(\lambda-\rho_3)(\lambda-\rho_4)}\times$$
$$\times{}_3F_4\,(\lambda+1,\ \lambda+2,\ 1;$$
$$\lambda+1-\rho_1,\ \lambda+1-\rho_2,\ \lambda+1-\rho_3,\ \lambda+1-\rho_4;\ z), \qquad (3.2)$$

where at $s=q_l,\ q_z,\ \varepsilon_T,\ \varkappa_T$ the magnitudes A and λ acquire the values given in the table.

4. Knowing the solution for the function $\tilde{N}$, we find the solutions for

$$\left.\begin{aligned}
\tilde{N}_l&=N_l+k_0\varkappa_\theta,\\
\tilde{N}_\theta&=N_\theta+k_0\varkappa_l,\\
\tilde{S}&=S+k_0\,(-\varkappa_{l\theta}),\\
\tilde{M}_l&=M_l+k_0\,(-\varepsilon_\theta),\\
\tilde{M}_\theta&=M_\theta+k_0\,(-\varepsilon_l),\\
\tilde{M}_{l\theta}&=M_{l\theta}+k_0\left(\frac{1}{2}\varepsilon_{l\theta}\right).
\end{aligned}\right\} \qquad (4.1)$$

where the same symbols as in /3/ are used for the stress resultants, moments, and strains.

The formulas for finding these solutions are

$$\left.\begin{aligned}
\tilde{N}_l&=\tilde{N}-\tilde{N}_\theta,\\
\tilde{N}_\theta&=\frac{1}{z}\left(z\frac{d}{dz}z\frac{d}{dz}-\frac{k^2}{\sin^2\alpha}\right)\tilde{N}-\\
&-\frac{1}{l\operatorname{ctg}\alpha}\left(l\frac{d}{dl}l\frac{d}{dl}-\frac{k^2}{\sin^2\alpha}\right)\Phi+\frac{q_z l}{\operatorname{ctg}\alpha},\\
\tilde{S}&=\frac{\sin\alpha}{k}\left(\tilde{N}_\theta-\frac{d\tilde{N}_l z}{dz}\right)-\frac{q_l\,l\sin\alpha}{k},\\
\tilde{M}_l&=-\frac{ih}{c_0}\,[\tilde{N}-(1-\nu)\tilde{N}_l-2\nu\operatorname{Re}\tilde{N}_l]-\Phi,\\
\tilde{M}_\theta&=-\frac{ih}{c_0}\,[\tilde{N}-(1-\nu)\tilde{N}_\theta-2\nu\operatorname{Re}\tilde{N}_\theta]-\Phi,\\
\tilde{M}_{l\theta}&=-\frac{ih}{c_0}\,[(1-\nu)\tilde{S}+2\nu\operatorname{Re}\tilde{S}],
\end{aligned}\right\} \qquad (4.2)$$

where

$$\Phi=\frac{Eh^2}{c_0}\left[i\,\varepsilon_T+\frac{(1+\nu)h}{c_0}\varkappa_T\right].$$

The following formula is used in differentiating the products of exponential and hypergeometric functions:

$$\frac{d}{dz} z^{\lambda}{}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z) =$$
$$= \lambda z^{\lambda-1}{}_{p+1}F_{q+1}(\alpha_1, \alpha_2 \cdots, \alpha_p, \lambda+1; \beta_1, \beta_2, \cdots, \beta_q, \lambda; z). \qquad (4.3)$$

Replacing the complex integration constants $\bar{C}_n$ entering in (4.2) by the expressions

$$\bar{C}_n = \bar{C}_n' + i\, C_n^{\bullet} \quad (n = 1,2,3,4),$$

where C_n' and $C_n^{\bullet}$ are real constants, and separating the real and imaginary parts of solutions (4.2), we find the solutions for the stress resultants N_l, N_θ, S, the moments M_l, M_θ, $M_{l\theta}$, and the strains ε_l, ε_θ, $\varepsilon_{l\theta}$, $\varkappa_l$, $\varkappa_\theta$, $\varkappa_{l\theta}$.

These solutions contain eight integration constants C_n' $(n=1, 2, 3, 4)$ and $C_n^{\bullet}$ $(n=1, 2, 3, 4)$, which are determined from the boundary conditions on the inner and outer contours of the shell.

5. To determine the displacements, we integrate the first, second, and sixth relationships between the strains and the displacements /3/.

We obtain

$$\left.\begin{aligned} u &= \int \varepsilon_l\, dl + u_0, \\ v &= \frac{\sin\alpha}{\cos^2\alpha - k^2}\left[k\left(\int \varepsilon_l\, dl - l\varepsilon_\theta\right) + \cos\alpha\; l \int \varkappa_{l\theta}\, dl\right] + v_0, \\ w &= -\frac{\sin\alpha}{\cos^2\alpha - k^2}\left[\cos\alpha\left(\int \varepsilon_l\, dl - l\,\varepsilon_\theta\right) + kl \int \varkappa_{l\theta}\, dl\right] + w_0, \end{aligned}\right\} \qquad (5.1)$$

where

$$\left.\begin{aligned} u_0 &= C_1, \\ v_0 &= \frac{\sin\alpha}{\cos^2\alpha - k^2}\left(k\, C_1 + \cos\alpha\; l\, C_2\right), \\ w_0 &= -\frac{\sin\alpha}{\cos^2\alpha - k^2}\,(\cos\alpha\; C_1 + k\, l\, C_2). \end{aligned}\right\} \qquad (5.2)$$

The terms (5.2) in expressions (5.1) obviously represent the displacements which nullify the strains ε_l, ε_θ, $\varkappa_{l\theta}$.

The substitution of the displacements (5.2) in the third, fourth, and fifth relationships between the strains and the displacements yields

$$\left.\begin{aligned} \varepsilon_{l\theta} &= -\frac{k\,(1-k^2)}{(\cos^2\alpha - k)\sin\alpha}\,\frac{1}{l}\,C_1, \\ \varkappa_l &= 0, \quad \varkappa_\theta = \frac{k^2\,(1-k^2)}{(\cos^2\alpha - k^2)\sin\alpha}\,\frac{1}{l}\,C_2. \end{aligned}\right\} \qquad (5.3)$$

Displacements (5.2) make thus all the strains equal to zero, and represent the displacements of the shell as a rigid body for the axisymmetrical $(k=0)$ and the antisymmetrical $(k=1)$ stressed states only.

At $k \geqslant 2$ an undeformed state of the shell can exist only for $C_1 = 0$, $C_2 = 0$, and there are no displacements of the shell as a rigid body.

In this case we calculate the displacements by (5.1), substituting in them

$$u_0 = v_0 = w_0 = 0.$$

Using the formula

$$\int z^{\lambda}\ {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z)\,dz =$$

$$= \frac{z^{\lambda+1}}{\lambda+1}\,{}_{p+1}F_{q+1}\ (\alpha_1, \alpha_2, \cdots, \alpha_p, \lambda+1; \beta_1, \beta_2, \cdots, \beta_q, \lambda+2; z) + C, \qquad (5.4)$$

which is true for λ different from a negative integer, the integrals $\int \varepsilon_l\, dl$ and $\int \varkappa_{l\theta}\, dl$ can likewise be found in degenerate hypergeometrical functions.

6. In the case of a shallow conic shell we can write

$$\cos\alpha \approx \varphi, \ \sin\alpha \approx 1, \qquad (6.1)$$

where φ is the angle between the normal to the middle surface and the shell axis.

The magnitudes (2.4) are then equal to

$$\rho_1 = k, \ \rho_2 = -k-1, \ \rho_3 = k-1, \ \rho_4 = -k. \qquad (6.2)$$

The substitution of (6.1) in solution (2.5) shows that the second ($n=2$) and third ($n=3$) particular solutions of the homogeneous equation become meaningless.

In these cases we form, by passing to the limit, the following logarithmic solutions:

$$\begin{aligned} \tilde{N}^{(2)} &= z^k\ {}_2\Phi_3\ (1+k,\ 1+2k;\ 2,\ 1+2k;\ 2+2k;\ z), \\ \tilde{N}^{(3)} &= z^{-k}\ {}_2\Phi_3\ (1-k,\ 2-k;\ 2,\ 1-2k,\ 2-2k;\ z), \end{aligned} \qquad (6.3)$$

where the degenerate hypergeometrical function of the second kind

$${}_p\Phi_q\ (\alpha_1, \cdots, \alpha_p;\ 1+m,\ \beta_2, \cdots, \beta_q;\ z);\ (m=0,1,\cdots)$$

is determined by the series given in /3/.

If, in accordance with the theory of shallow shells /1/, we neglect (in addition to assumptions (6.1)) the small term containing Q_θ in the second equation of equilibrium, the small term containing $M_{\theta l}$ in the expression for the stress resultant S, and the small terms containing v in the relationships between the bending strains and the displacements, we can select as resolving function the function $\tilde{F}$, which, in the case of the homogeneous problem (without allowing for the surface forces and the temperature-dependent terms), is associated with the functions $\tilde{N}_l, \tilde{N}_\theta, \tilde{N}_{l\theta}$ by the simpler formulas

$$\tilde{N}_l = \frac{1}{l}\frac{d\tilde{F}}{dl} - \frac{k^2}{l^2}\tilde{F}, \quad \tilde{N}_\theta = \frac{d^2\tilde{F}}{dl^2},$$

$$\tilde{N}_{l\theta} = \tilde{N}_{\theta l} = l\frac{d}{dl}\left(\frac{\tilde{F}}{l}\right). \qquad (6.4)$$

With these additional assumptions the homogeneous problem reduces to solving the equation

$$\nabla^2\nabla^2\tilde{F} - \frac{i\varphi c_0}{h}\,\frac{1}{l}\,\frac{d^2\tilde{F}}{dl^2} = 0, \qquad (6.5)$$

where

$$\nabla^2 = \frac{d^2}{dl^2} + \frac{1}{l}\frac{d}{dl} - \frac{k^2}{l^2}.$$

The solution of this equation was treated in /2/.

The same method is used (cf. section 3 of this paper) for obtaining particular solutions of the honhomogeneous equations for a shallow conic shell.

Having formed in some specified form linear combinations of these solutions with the corresponding particular solutions of the homogeneous equations, we can obtain particular solutions of the nonhomogeneous equations in the form of polynomials. At $\lambda - k = -m$ $(m=0, 1, \cdots)$ the non-homogeneous equations have logarithmic particular solutions.

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EXPERIMENTAL STUDY OF THIN-WALLED STRUCTURES IN LIMITING STATES

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(Khar'kov)

An experimental study was conducted in order to test the accuracy of the computing models and the theoretical calculation used, and mainly to determine the weak places in which collapse occurs and which are not susceptible to theoretical calculation by the computing model adopted. A large number of gages were used in the experiment to measure the elongations, displacements, and stresses. The data accumulated in the course of the tests was so extensive, that it was impossible to process all of it manually within a practical amount of time. As a result, a part of the material obtained has remained unused (unprocessed).

The introduction of automatic processing equipment and high-speed computers in the processing of the experimental data seems therefore desirable.

Automation cannot be based, however, on the methods used in manual processing which depend on the direct participation of a specialist who can creatively solve many problems due to his knowledge and experience. Automation necessitates, first of all, a clear formulation of the criteria and estimates characterizing the stressed-strained state, and a method ensuring the highest degree of objectivity and reliability of the processing. It is desirable to include in the processing program the experience which scientists gradually accumulate. On this basis they are capable of making intuitive forecasts deduced from a small amount of information.

The possibilities of the modern high-speed computers should be taken into account when designing the method of automated processing of the experimental data; in fact, their high speed makes it possible not only to reduce the processing time, but also to increase the accuracy of the processing.

1. LIMITING STATES

When using automatic installations to process the experimental data it is necessary to restrict the volume of output data, so as not to reduce the speed of calculations and not to bother the researcher with a great amount of secondary and irrelevant data. The volume of output data can be significantly reduced by using the concept of limiting state in the analysis of the behavior of systems.

By limiting state we mean the stressed-strained state immediately preceding a state of the system which is dangerous in some sense. Examples of limiting states are:

1. The operating limiting state, in which the yield point is reached in at least one element.
2. The life limiting state, in which the fatigue limit (corresponding to a certain specified number of cycles in the case of repeated cyclical loads) is reached in at least one element.
3. The first collapse limiting state, in which collapse starts in at least one element.
4. The limiting state corresponding to the maximum load which the system can withstand at a given pattern of distribution and application of loads.
5. The operating limiting states according to the system rigidity. These are states in which the displacements (deflections, angles of rotation, etc.) attain unallowable (from the point of view of normal operation) magnitudes.

The use of the concept of limiting state in this sense greatly reduces the volume of output data.

Knowing the limiting loads and the zones determining the onset of the limiting states, the design engineer can take measures to strengthen the structure whenever necessary. The same method can obviously be applied as well to the determination of weakly loaded zones (elements), which lead to an unwarranted increase of the system weight.

The limiting states can similarly be used in theoretical calculations of systems /3/.

Since the limiting states considered can also occur in the elastic-plastic stage and in the post-buckling zone, the connection between the parameter P, which determines the value of the load, and the magnitudes measured (the elongations ε, the displacements v) can be nonlinear. In this case the functions $\varepsilon - P$, $v - P$ will be nonlinear and their curves should be processed by means of nonlinear functions.

In the practice of static tests the nonlinear graphs are sometimes discarded, while in thin-walled structures the use of such graphs can be necessary /4/.

2. PROCESSING THE EXPERIMENTAL DATA

Only linear displacements can be measured directly in static tests of thin-walled structures. All the other characteristics of the stressed-strained state are either measured indirectly (the elongations by means of strain gages) or calculated theoretically (the stresses, the angular displacements, the shears, the moments of internal forces, etc.).

It follows that the processing of experimental data in strength calculations is impossible without a theory. In fact, even such a simple characteristic as the stress represents an abstract magnitude, which has a meaning in the computing model of a continuous medium only. The theory of strains (strain tensor) and the theory of stresses (stress tensor) correspond to this computing model. Since the stresses are not physical

magnitudes which can be measured directly, their determination necessitates using the formulas of the theories of elasticity, plasticity, and creep.

The physical laws defining the relationship between the components of the stress and strain tensors are determined experimentally and have a statistical character.

It is thus seen that the processing of the experimental data is unthinkable without a theory. At the same time, however, only the basic tenets of theory are necessary in this processing, since the experimental material obtained as a result is more reliable than that obtained in the theoretical calculation of complex systems (which necessitates the use of computing models containing additional assumptions and hypotheses).

In the processing of the experimental data relative to tests of shells and plates we use only assumptions such as the "hypothesis" of plane normals and the assumption that the plane stressed state is the dominant one. When complex systems are theoretically calculated, on the other hand, it is necessary to introduce additional hypotheses: the "hypothesis" of plane sections, the law of sectorial areas, the hypothesis of absence of shears, etc.

The processing of the experimental data can be split in two basic stages: primary and secondary processing.

The primary processing includes the processing of the measurement results up to the determination of the stresses and displacements. The limiting state and the danger zones are determined in the course of this processing.

The secondary processing is directed toward the solution of more complex problems: comparison of the theoretical calculations with the experimental results, check-over of the additional hypotheses and assumptions, and experimental reconstruction of the probable loads in several complex cases of loading of the system elements. The secondary processing is treated in /5/, and we shall not dwell on it here.

Statistical methods are being widely used lately in the solution of strength problems. A thorough survey of the theoretical studies in this region has been made in Bolotin's paper /2/. The interest in probability methods stems from the character of the strength studies. The methods of mathematical statistics applicable to the problems of static tests of thin-walled structures subjected to a nonrepeated loading are treated in /6/.

Using the method exposed in /6/ it is possible to find some generalized polynomial approximating the experimental relationships with a specified reliability and with known confidence interval. The use of the methods of statistical processing considerably increases the computing work involved in the processing and to a large extent necessitates the use of automatic devices.

This paper makes no claims to thoroughness. Its scope prevents a discussion of the specific programs and installations ensuring the automatic processing of the experiment at a high speed and with a known reliability.

To illustrate the necessity of such an approach, we give one specific example of the behavior of an element of the skin of a thin-walled system in the post-buckling stage.

3. EXAMPLE

Due to the influence of numerous factors (such as the eccentricity at the places of joining of the elements, the nonrectilinearity of the axes, etc.), the longitudinal elements of a thin-walled framed structure are subjected to a bending stressed state which, while not always dangerous, can greatly affect the pattern of the experimentally determined relationships between the load parameter and the elongations (stresses) at the places where gages are set /4/. The pattern of these relationships (linearity, nonlinearity, curvature) provides a means for estimating the correctness of the selection of the computing model, and for elucidating and predicting the character of the collapse of the system elements.

Even more complex relationships are observed for gages set on a thin skin. We give below the results of tests recorded for one arbitrarily selected "rosette" of strain gages set at the middle of the wall of a two-belt thin-walled beam loaded by a concentrated force applied in the plane of the undeformed wall.

The scheme of installation and disposition of the "rosette" gages is represented in Figure 1. The same numbering of the gages is used for the left and right surfaces of the wall, but the numbers of the left surface are accompanied by a stroke.

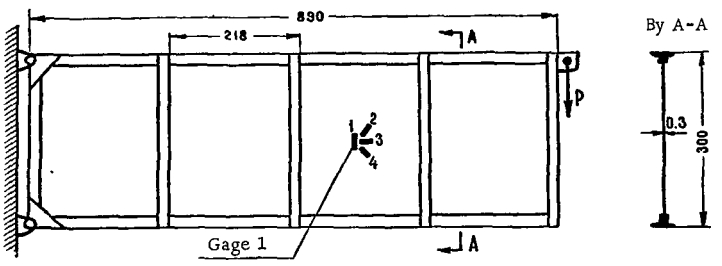

FIGURE 1.

In Figure 2 the experimental data (for gage No. 3) are represented by tiny circles. The relationship $\varepsilon - P$ is clearly nonlinear for the gage considered. The problem of the processing consists therefore in selecting the function best approximating the experimental relationship. The experimental data, however, contain some errors, which are random magnitudes in the statistical sense. There is therefore no need to look for a function passing exactly through all the points. The statistical approach will be more correct here; it consists in finding the most probable function, which best approximates the experimental relationship allowing for the statistical pattern of the experiment errors. If we first determine the dispersal of the measurements, then, by specifying some reliability, we can find even for a nonrepeated loading the confidence interval, and then plot by the method of least squares the relationship $\varepsilon = \varepsilon(P)$ which best approximates the experimentally determined function /6/. Such a curve has been drawn in Figure 2 (the unbroken line). The confidence interval for 95 % reliability is represented by the dotted line.

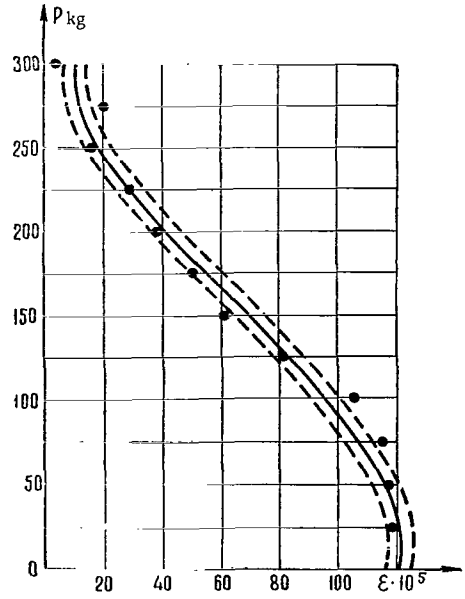

FIGURE 2.

This method was applied to all the gages of the "rosette." The approximating curves obtained for all the gages are represented in Figure 3 by unbroken lines. Eight values of the elongations do not define by themselves the stressed state at the place where the rosette is placed. In order to determine this state it is necessary to make use of the tenets of the theory of elasticity. Since the stress tensor is formed (just as the strain tensor) at a point of the body (in an infinitely small vicinity of the point), we must assume that the gages give values of the elongations differing only little from the values at the point of intersection of their axes. In other words, it is necessary to assume that the stress (strain) field is homogeneous or weakly varying, i.e., that the rosette dimensions are so small that in the region it occupies the increments of the elongations (stresses) are considerably smaller than the elongations themselves.

In this case the values obtained for the elongations can be considered as the elongations at the point of intersection of the axes of the gages. Since the gages are set on the plate surfaces only, the measured elongations can be used directly only under the assumption that the state of stress and strain is plane. In this case the sum of the elongations for any mutually perpendicular directions is constant /1/

$$\begin{aligned} \varepsilon_1 + \varepsilon_3 &= \varepsilon_2 + \varepsilon_4 = \text{const}, \\ \varepsilon_1' + \varepsilon_3' &= \varepsilon_2' + \varepsilon_4' = \text{const}. \end{aligned} \tag{1}$$

In the example considered it turned out, however, that

$$\begin{aligned} \Delta &= (\varepsilon_1 + \varepsilon_3) - (\varepsilon_2 + \varepsilon_4) \neq 0, \\ \Delta' &= (\varepsilon_1' + \varepsilon_3') - (\varepsilon_2' + \varepsilon_4') \neq 0, \end{aligned} \tag{2}$$

and, moreover, Δ and Δ' are of the same order as the sum of the positive (negative) magnitudes entering in the right-hand part of (2).

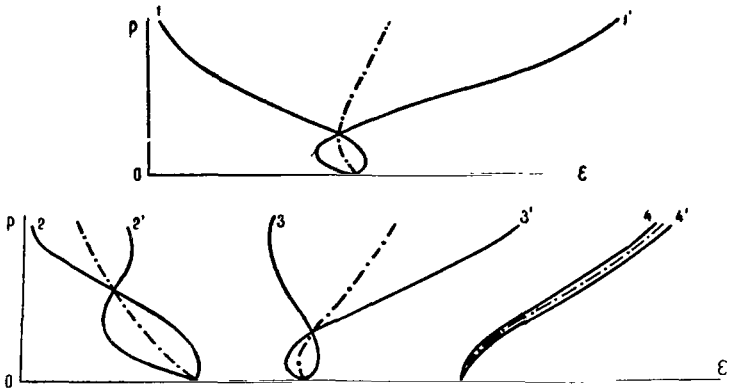

FIGURE 3.

At the same time, assuming the correctness of the hypothesis of plane normals and passing to the elongations in the middle surface (the dotted lines on Figure 3), we obtain

$$\Delta_0 = \left(\frac{\varepsilon_1' + \varepsilon_1}{2} + \frac{\varepsilon_3' + \varepsilon_3}{2}\right) - \left(\frac{\varepsilon_2' + \varepsilon_2}{2} + \frac{\varepsilon_4 + \varepsilon_4}{2}\right) \qquad (3)$$

In the example considered, the value of Δ_0 turned out to be, in the entire range of variation of the load parameter P, of the order of one-tenth the minimum value of the sum of the positive (negative) terms in (3). It follows that in the middle surface the stresses vary more smoothly than on the surfaces, and the readings of the gages can be used to reconstruct with a sufficient accuracy the stressed state corresponding to the membrane computing model. Since in thin-walled structures the membrane stressed state characterizes the basic components of the internal forces and moments equilibrating the external load, the experiment reveals with a sufficient accuracy these forces and moments.

Collapse starts when the local strength of the elements is exhausted. The setting of eight-gage rosettes does not ensure a reliable study of the bending stressed state. The magnitude and sign of the curvature of the function $\varepsilon = \varepsilon(P)$ can, however, characterize the onset of the bending stressed state; with the aid of some additional conditions they can also be used to control the criteria of onset of dangerous limiting states.

We note in conclusion that when the experimental data $\varepsilon - P$ were approximated by linear functions the value of Δ_0 in the example considered increased considerably, reaching the order of magnitude of the sum of terms of the same sign in (3).

CONCLUSION

1. The use of the methods of mathematical statistics increases the reliability of the processing of the experimental data. The resulting increase in computing work must be offset by the use of automatic high-speed processing installations.

2. The limiting states (§ 1) based on the criteria of dangerous stressed-strained states are formulated in a form suitable for experimental tests, and provide a means for establishing clear algorithms for the analysis of the experimental data. Their use contributes to the automation of the data processing, with its resulting speeding up of the calculations and increase of their accuracy and reliability.

3. The setting of eight-gage rosettes makes it possible to analyze successfully membrane computing models. The setting of gages on one surface only, or the use of rosettes with gages glued on less than three directions, is absolutely unacceptable in testing thin-walled structures, since the experimental results cannot be checked by means of such rosettes.

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STABILITY OF UNSTIFFENED AND STIFFENED LONGITUDINALLY CORRUGATED CYLINDRICAL SHELLS

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Corrugated elements are being used increasingly in instruments, ships, aircraft, etc. The calculating methods suitable to such elements have not been sufficiently developed so far, however, and only the particular case of a corrugated plate has been solved basically. The methods of calculation of corrugated plates so far published can be split in two groups: 1) calculation of corrugated plates as anisotropic plates. This method is used in instrument design in the calculation of corrugated membranes /1, 8, 10/; 2) calculation of corrugated plates as beams, by splitting the corrugated field in separate crimps and neglecting the interaction between them. This method is used in ship design in the calculation of corrugated bulkheads /5, 6, 9/.

The only works known to us on the subject of calculation of corrugated shells are Mushtari's treatise on the strength calculation of longitudinally corrugated cylindrical shells with sinusoidal corrugation subjected to axial compression /11/, and the paper /2/ on the determination of the stressed state of corrugated shells.

Several buckling modes can be observed in tests of longitudinally corrugated cylindrical shells at axial compression. The buckling mode is basically determined by the form and dimensions of the corrugation, i.e., by the increase of the flexural rigidity in the axial direction relative to the smooth shell. The ratio of the moments of inertia of the cross sections of the corrugated and smooth shells relative to the OX axis, $k = \frac{J_x^*}{y_x}$ (Figure 1) will be taken as a parameter characterizing the increase of the flexural rigidity of the corrugated shell in the axial direction. The OX axis passes through the center of gravity of the arc of the smooth-shell cross section having central angle $\frac{2\pi}{n}$, where n is the number of corrugation waves along the circumference of the cross section of the corrugated shell. The corrugated shells tested can be arbitrarily divided in three groups according to the value of the parameter k.

1. Corrugated shells of low flexural rigidity in the axial direction ($1 < k < 50$). These shells buckle like smooth shells, with the formation of a large number of half-waves along the shell length and in the plane of its cross section. The buckling is accompanied by snap-through.

2. Corrugated shells of medium flexural rigidity in the axial direction ($50 < k < 150$). When these shells buckle, two to three half-waves form along their length, and three to five waves in the plane of the cross section. The buckles are stretched along the generator. The buckling is accompanied by weakly expressed snap-through.

3. Corrugated shells of high flexural rigidity in the axial direction ($k > 150$). When these shells buckle, one half-wave forms along the length, and two to three half-waves in the cross-sectional plane. The buckling is not accompanied by snap-through.

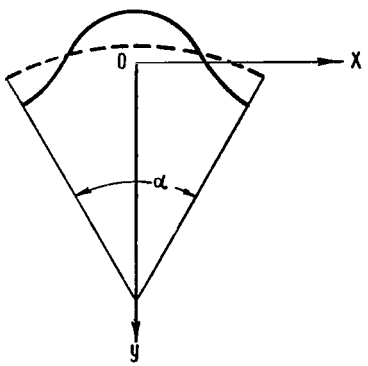

FIGURE 1.

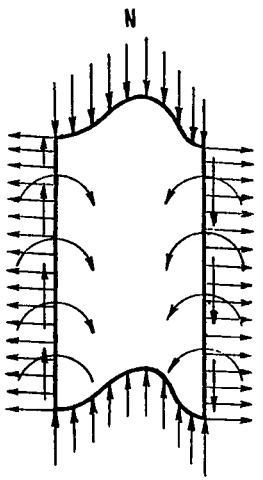

FIGURE 2.

Different theories must obviously be used to calculate the different buckling modes. Shells of the first group can be calculated by the theory of a structurally orthotropic shell. Shells of the third group can be calculated by Vlasov's theory of thin-walled bars /3/. Shells of the second group can be calculated by either one of these theories. It should, however, be remembered that the number of waves formed in the longitudinal and the circumferential directions during buckling is limited.

We shall consider the stability of longitudinally corrugated cylindrical shells of large and medium flexural rigidity in the axial direction, either stiffened or unstiffened.

Consider a longitudinally corrugated cylindrical shell subjected to the simultaneous action of uniformly distributed axial and lateral loads. Let n be the number of corrugation waves along the cross section circumference. Cut out a panel of central angle $\frac{2\pi}{n}$. The reaction of the discarded constraints will be replaced by a system of forces and moments which can be determined on the strength of Vlasov's theory of folded systems /4/.

We pass to the question of the stability of a thin-walled bar of arbitrary cross-sectional contour subjected to the action of uniformly distributed axial and lateral loads and a system of forces and moments applied at the

longitudinal ends of the panel (Figure 2). It shall be assumed that the bar is in a state of central lateral bending before buckling, i.e., that the bar cross section is not subjected to deplanation and suffers only translatory displacements in its plane. The general differential equations of stability for an arbitrary open profile subjected to the conditions of complex resistance in the case of central lateral bending accompanied by extension or compression are

$$EJ_y \xi^{IV} - [N(\xi' + a_y \theta')]' + (M_x \theta)'' = 0,$$

$$EJ_x \eta^{IV} - [N(\eta' - a_x \theta')]' + (M_y \theta)'' - (T_k \theta)' 2x_l = 0,$$

$$EY_\omega \theta^{IV} = [(r^2 N + 2\beta_y M_x)\theta']' + [q_x^0 (l_x - a_x) + q_y^0 (l_y - a_y)]\theta + m_k \theta -$$

$$- a_y (N\xi')' + a_x (N\eta')' + M_x \xi'' + M_y \eta'' - T_k \eta' 2x_l = 0. \qquad (1)$$

The unknown functions in (1) are the displacements $\xi = \xi(z)$, $\eta = \eta(z)$, and $\theta = \theta(z)$ appearing as a result of the change in the fundamental bending mode of equilibrium at buckling. The coefficients of differential equations (1) depend on the elastic constants E and G, the geometrical characteristics of the cross section J_x, J_y, J_ω, a_x, a_y, β_x, β_y, r, and the functions $N(z)$, $M_x(z)$, $M_y(z)$, $q_x^0(z)$, $q_y^0(z)$, which depend on the external (relative to the bar) load and on the conditions of support of the bar at the ends. Here $M_x(z)$ and $M_y(z)$ are the bending moments due to the external load, determined from the statical calculation allowing for the conditions of support.

It is usually assumed in stability calculations of elastic systems that the external load is given with an accuracy up to a general coefficient of proportionality characterizing the load intensity. The stability calculation reduces to finding the minimum critical value of the generalized load at given boundary conditions.

Let the longitudinally corrugated cylindrical shell be subjected to the action of a uniformly distributed axial compressive load only. The studies we have conducted have established that in this case the influence of the forces and moments applied at the longitudinal ends of the bar cut from the corrugated shell on the value of the critical load is negligible. The differential equations of stability of an open thin-walled bar of arbitrary profile with one axis of symmetry ($a_x = \beta_x = 0$) at central axial compression are

$$EJ_y \xi^{IV} + P\xi'' + a_y P\theta'' = 0, \quad EJ_x \eta^{IV} + P\eta'' = 0, \qquad (2)$$

$$EJ_\omega \theta^{IV} + r^2 P\theta'' + a_y P\xi'' = 0.$$

Equations (2) form a system of linear differential equations with constant coefficients. We shall consider the solution of the problem for the following end conditions of the bar:

$$\xi = \eta = \theta = 0, \ \xi'' = \eta'' = \theta' \quad (z = 0, \ z = l), \qquad (3)$$

where l is the bar length. Allowing for these end conditions, we select the displacements ξ, η, θ in the form

$$\xi = A \sin \frac{n_1 \pi z}{l}, \quad \eta = B \sin \frac{n_1 \pi z}{l}, \quad \theta = C\left(1 - \cos 2\frac{n_1 \pi z}{l}\right), \qquad (4)$$

where A, B, C are some constant coefficients, and n_1 is any positive integer ($n_1 = 1, 2, \cdots$).

System (2) will be solved by Bubnov-Galerkin's variational method. The following equations are obtained, after a series of transformations, for the determination of the critical value of the load:

$$EJ_x \lambda_{n_1}^2 - P = 0 \tag{5a}$$

$$\begin{vmatrix} EJ_y \lambda_{n_1}^2 - P & -\frac{16}{3\pi} a_y P \\ -\frac{16}{3\pi} a_y P & EJ_\omega (2\lambda_{n_1})^2 - r^2 P \end{vmatrix} = 0, \tag{5b}$$

where $\lambda_{n_1} = \frac{n_1 \pi}{l}$. The critical load will be equal to the smallest positive root of equations (5a) and (5b).

We proceed to the case of a stiffened corrugated shell. Let a longitudinally corrugated cylindrical shell with an arbitrary number m of stiffeners be subjected to the action of axial compressive forces. As before, we cut out a panel of central angle $\frac{2\pi}{n}$ from the shell, and consider the problem of the stability of a thin-walled bar stiffened by m stiffeners at axial compression. The stiffeners are bound to the bar by means of spot welding or riveting. We shall assume that the bar is elastically built-in at the points of its fastening to the stiffener. The elastic fastening of the bar prevents the variation of its state of strain to a certain extent, hence increases the value of the critical force. Let $k_{\xi i}^{(j)}$, $k_{\eta i}^{(j)}$, $k_{\theta i}^{(j)}$ be the coefficients of the elastic restraint of the bar, $h_{xi}^{(j)}$, $h_{yi}^{(j)}$ be the coordinates of the point of contact of the stiffener with the bar in the plane of the j-th stiffener ($j=1, 2\cdots, m$; $i=1,2\cdots, m_j$; m_j is the number of points of elastic restraint of the bar in the plane of the j-th stiffener). The coefficients $k_{\xi i}^{(j)}$, $k_{\eta i}^{(j)}$, $k_{\theta i}^{(j)}$ possess the following properties:

$$k_{\xi i}^{(j)}(z) = \begin{cases} 0 \text{ for } z \neq z_j \\ \infty \text{ for } z = z_j, \text{ but } k_{\xi i}^{(j)}(z)\, dz = k_{\xi i}^{(j)}, \\ \qquad\qquad dz \to 0 \end{cases}$$

$$k_{\eta i}^{(j)}(z) = \begin{cases} 0 \text{ for } z \neq z_j \\ \infty \text{ for } z = z_j, \text{ but } k_{\eta i}^{(j)}(z)\, dz = k_{\eta i}^{(j)}, \\ \qquad\qquad dz \to 0 \end{cases} \tag{6}$$

$$k_{\theta i}^{(j)}(z) = \begin{cases} 0 \text{ for } z \neq z_j \\ \infty \text{ for } z = z_j, \text{ but } k_{\theta i}^{(j)}(z)\, dz = k_{\eta i}^{(j)}. \\ \qquad\qquad dz \to 0 \end{cases}$$

The coefficients of elastic restraint are determined in each particular case by the methods of structural mechanics.

The differential equations of stability of such a bar at longitudinal compression are

$$EJ_y \xi^{IV} + P\xi'' + a_y P\theta'' + \sum_{j=1}^{m} \sum_{i=1}^{m_j} k_{\xi i}^{(j)} [\xi - (h_{yi}^{(j)} - a_y)\theta] = 0,$$

$$EJ_x \eta^{IV} + P\eta'' + \sum_{j=1}^{m} \sum_{i=1}^{m_j} k_{\eta i}^{(j)} [\eta + h_{xi}^{(j)}\theta] = 0. \tag{7}$$

$$a_y P\xi'' + EJ_\omega \theta^{IV} + r^2 P\theta'' - \sum_{j=1}^{m} \sum_{i=1}^{m_j} k_{\xi i}^{(j)} (h_{yi}^{(j)} - a_y)\xi + \sum_{j=1}^{m} \sum_{i=1}^{m_j} k_{\eta i}^{(j)} h_{xi}^{(j)} \eta +$$

$$+ \sum_{j=1}^{m} \sum_{i=1}^{m_j} [k_{\xi i}^{(j)} (h_{yi}^{(j)} - a_y)^2 + k_{\eta i}^{(j)} h_{xi}^{(j)2} + k_{\theta i}^{(j)}]\theta = 0.$$

The coefficients of equations (7) depend on the elastic constant E, the geometrical characteristics of the cross section J_x, J_y, J_ω, a_y, r, the elasticity coefficients of the stiffeners $k_{\xi i}^{(j)}$, $k_{\eta i}^{(j)}$, $k_{\theta i}^{(j)}$, the coordinates of the point of contact of the stiffener with the bar $h_{xi}^{(j)}$, $h_{yi}^{(j)}$, and the compressive force P.

We shall consider the case of boundary conditions (3). The expressions for the displacements ξ, η, θ, allowing for the boundary conditions (3) and the influence of the stiffeners on the mode of wave formation at buckling, will then be:

$$\xi = A_1 \sin\frac{\pi z}{l} + A_m \sin\frac{(m+1)\pi z}{l}, \quad \eta = B_1 \sin\frac{\pi z}{l} + B_m \sin\frac{(m+1)\pi z}{l},$$
$$\theta = C_1\left(1-\cos 2\frac{\pi z}{l}\right) + C_m\left(1-\cos 2\frac{(m+1)\pi z}{l}\right). \tag{8}$$

We shall again use Bubnov-Galerkin's variational method to solve the system of equations (7) and determine the critical value of the load. After a series of transformations and using the properties of the coefficients of elastic restraint of the bar (6) we obtain the following equations for finding the critical value of the load:

$$EJ_x(m+1)^2\lambda^2 - P = 0, \tag{9a}$$
$$\| a_{ik} \| = 0, \tag{9b}$$

where

$$a_{11} = EJ_y\lambda^4 - P\lambda^2 + \frac{2}{l}\sum_{j=1}^{m}\sum_{i=1}^{m_j} k_{\xi i}^{(j)} \sin^2\frac{\pi z_j}{l},$$

$$a_{14} = a_{41} = -\left[\frac{16}{3\pi} a_y \lambda^2 P + \frac{2}{l}\sum_{j=1}^{m}\sum_{i=1}^{m_j} k_{\xi i}^{(j)} (h_{yi}^{(j)} - a_y) \times \right.$$
$$\left. \times \left(1-\cos 2\frac{\pi z_j}{l}\right)\sin\frac{\pi z_j}{l}\right],$$

$$a_{15} = a_{51} = -\frac{16(m+1)^2}{[4(m+1)^2-1]\pi} a_y \lambda^2 P, \quad a_{22} = EJ_y(m+1)^4\lambda^4 - P(m+1)^2\lambda^2,$$

$$a_{24} = a_{42} = \frac{16(m+1)}{[(m+1)^2-4]\lambda} a_y \lambda^2 P, \quad a_{25} = a_{52} = -\frac{16(m+1)}{3\pi} a_y \lambda^2 P,$$
$$(m - \text{even}), \tag{10}$$
$$a_{24} = a_{42} = a_{25} = a_{52} = 0, \ (m - \text{odd})$$

$$a_{33} = EJ_x\lambda^4 - P\lambda^2 + \frac{2}{l}\sum_{j=1}^{m}\sum_{i=1}^{m_j} k_{\eta i}^{(j)} \sin^2\frac{\pi z_j}{l},$$

$$a_{34} = a_{43} = \frac{2}{l}\sum_{j=1}^{m}\sum_{i=1}^{m_j} k_{\eta i}^{(j)} h_{xi}^{(j)}\left(1-\cos 2\frac{\pi z_j}{l}\right)\sin\frac{\pi z_j}{l},$$

$$a_{44} = EJ_\omega(2\lambda)^4 - r^2(2\lambda)^2 P +$$
$$+ \frac{2}{l}\sum_{j=1}^{m}\sum_{i=1}^{m_j}\left[k_{\xi i}^{(j)}(h_{yi}^{(j)} - a_y)^2 + k_{\eta i}^{(j)} h_{xi}^{(j)2} + k_{\theta i}^{(j)}\right]\left(1-\cos 2\frac{\pi z_j}{l}\right)^2,$$

$$a_{55} = EJ_\omega 2^4(m+1)^4\lambda^4 - r^2 2^2(m+1)^2\lambda^2,$$

$$a_{12} = a_{21} = a_{13} = a_{31} = a_{23} = a_{32} = a_{35} = a_{53} = a_{45} = a_{54} = 0. \qquad (10)$$
(cont'd)

The critical load of a longitudinally corrugated stiffened cylindrical shell at axial compression will be

$$P_{cr}^{(sh)} = n\, P_{cr}^{(b)}, \qquad (11)$$

where $P_{cr}^{(sh)}$ is the critical load of the bar cut from the shell and n is the number of corrugation waves along the circumference of the cross section.

Obviously, in the case of a stiffened longitudinally corrugated cylindrical shell, the critical value of the load must increase with the number of stiffeners. It has been observed, however, in experimental studies of stiffened shells that, after a certain value has been reached, $P_{cr}^{(sh)}$ does not increase further with the increase of the number of stiffeners. What occurs in this case is not a loss of the general stability of the shell, but the loss of its load-carrying capacity (warping in separate sections). Consider as an example a cylindrical shell with longitudinal rectangular or trapezoidal crimps. Let the maximum value of the critical load be attained for i stiffeners, fastened on the inside of the shell. It has been established that in this case the maximum value of the critical stress of the shell coincides with the minimum value of the critical stress of the plates forming the crimp. Let the plates be simply supported on the two sides parallel to the direction of compression, and have different conditions of support at the other two sides. The plates to which the stiffeners are welded will be $i + 1$ in number, and their length will be $\frac{l}{i+1}$, where l is the shell length. We shall assume the following boundary conditions for the plate sides perpendicular to the direction of compression: the sides supported on the stiffeners are hinged, and the sides coinciding with the shell ends have the same end conditions as the shell. These boundary conditions correspond to an increase in the rigidity of the restraints imposed on the plate, and lead to a fictitious increase in its stability.

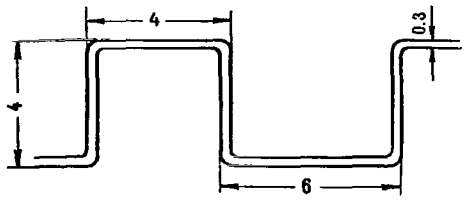

FIGURE 3.

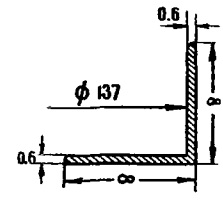

FIGURE 4.

We shall now consider the results of the experimental study of corrugated shells. The form and dimensions of the corrugation are shown in Figure 3. The samples had the following dimensions: l = 200 mm, R_{av} = 70.5 mm, n = 32. The elastic characteristics of the material of the samples are $E = 7 \times 10^5$ kg/cm^2, $\nu = 0.3$. The profile and dimensions of the stiffeners are shown in Figure 4. Unstiffened corrugated cylindrical shells and

corrugated shells stiffened by one, two, and three stiffeners, were tested at axial compression. The results of the experiments and of the theoretical calculation by the theory of bars are shown in Table 1. A satisfactory coincidence is observed between the experimental and theoretical results. The maximum discrepancy for stiffened shells turned out to be 6.4%. The experiment to determine the maximum critical load of a stiffened corrugated shell also agreed well with the theoretical results.

TABLE 1

	$P_{max}^{(exp)}$, kg	$P_{max_{av}}^{(exp)}$, kg	$P_{cr}^{(theor)}$, kg	Discrepancy, %
Unstiffened	1585 1825 1830	1747	1568.2	11.4
One stiffener	2020 1990 2000	2003	2028.4	1.3
Two stiffeners	2450 2325	2387.5	2258.5	5.7
Three stiffeners	2750 2550	2650	2488.7	6.4

TABLE 2

	P_{cr}, kg (structurally orthotropic shell)	P_{cr}, kg (theory of bars)	P_{cr}, kg (other theories)	Experiment
Mushtari's example (sinusoidal corrugation) $k = 945$	4316	2859	6632	
Rectangular corrugation $k = 371$	2312	1568		1747
Trapezoidal corrugation $k = 1372$	6697	4126		4955
Sinusoidal corrugation $k = 15.7$	2543			1590

$$l \leqslant 4R; \quad \frac{h}{R} = \frac{1}{250} \div \frac{1}{300}$$

We shall now compare the results of the numerical calculation of corrugated shells by the theory of bars and by the theory of structurally orthotropic shells (Andreeva's reduction method /1/) with the experimental data. It is seen from Table 2 that the critical load $P_{cr}^{(s.o.)}$ found by the theory of structurally orthotropic shells is higher by 40 to 50% than the experimental value, while $P_{cr}^{(b)}$ found by the theory of bars coincides satis-

factorily with the experimental results. $P_{cr}^{(s.o.)}$ can be considered as the upper boundary of the critical load of a corrugated shell, and $P_{cr}^{(b)}$ as its lower boundary. The true value of the critical load lies between $P_{cr}^{(s.o.)}$ and $P_{cr}^{(b)}$, and is nearer to $P_{cr}^{(b)}$.

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ON THE STABILITY OF SHALLOW CYLINDRICAL SHELLS SUPPORTED ON THE INNER SURFACE BY AN ELASTIC FOUNDATION

B.A. Korbut

(Zaporozh'e)

Some characteristic features of "large scale" buckling are brought to light in the example of an elongated cylindrical panel lying on an elastic Winkler foundation. It is assumed that the panel dimension in the direction of the generator (the length) is considerably larger than the dimension in the direction of the arc (the width); this makes it possible to consider the bent surface as likewise cylindrical. The panel is hinged on fixed supports located along the generator and is subjected from the convex (outer) side to the action of a uniformly distributed pressure (Figure 1 a).

The stresses in the middle surface, acting along the width, are assumed to be constant for all points of the panel and equal to $\sigma_y = \sigma$. The stresses σ_x, τ are considered as equal to zero /1/.

The phenomenon of "snap-through" is studied as a function of the foundation rigidity and the panel curvature. The problem is solved by the method of Bubnov-Galerkin in two successive approximations.

The system of initial nonlinear differential equations for shallow cylindrical shells is

$$\frac{D}{h}\nabla^2\nabla^2 w = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 \Phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2} + \frac{P}{h}, \tag{1}$$

$$\frac{1}{E}\nabla^2\nabla^2\Phi = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}, \tag{2}$$

where w is the deflection; Φ is the stress function; x, y are coordinates; R is the panel radius; h is the panel thickness; D is the flexural rigidity; E is the modulus of elasticity; P is the external load.

On the strength of the assumptions made relative to the pattern of deformation of the panel, and taking into account that

$$P = q - \alpha w, \cdots, \tag{3}$$

equation (1) can be written in the form

$$\frac{D}{h}\frac{d^4 w}{dy^4} + \sigma\frac{d^2 w}{dy^2} + \frac{\alpha w}{h} + \frac{1}{R}\sigma = \frac{q}{h}, \tag{4}$$

where q is the external pressure and α is the foundation modulus, assumed to be constant.

The stress σ is taken with the minus sign, since the panel is compressed.

The continuity equation (2) drops out, since the deflections are independent of x because of the assumption relative to the pattern of deformation.

The problem reduces thus to integrating equation (4) for given boundary conditions.

1. FIRST APPROXIMATION

The following expression is selected as first approximation for the deflections, satisfying the end conditions:

$$w = f \sin \frac{\pi y}{b}, \tag{5}$$

where f is the sag and b is the panel width. Expression (5) presupposes symmetrical deformation of the panel.

The Bubnov-Galerkin equation is

$$\int_0^b X \sin \frac{\pi y}{b}\, dy = 0, \tag{6}$$

where X is determined from (4) and is equal to

$$X = D \frac{d^4 w}{dy^4} + \sigma h \left(\frac{d^2 w}{dy^2} + \frac{1}{R} \right) + \alpha w - q. \tag{7}$$

By substituting (7) and (5) in (6) and integrating we obtain

$$q = D \frac{\pi^5}{4b^4} f - \frac{\pi^3}{4b^2} f \sigma h + \sigma \frac{h}{R} + \frac{\pi \alpha f}{4}. \tag{8}$$

The unknown σ will be determined from the expression for the strain ε_y /1/:

$$\varepsilon_y = \frac{dv}{dy} - \frac{w}{R} + \frac{1}{2} \left(\frac{dw}{dy} \right)^2. \tag{9}$$

Since

$$\varepsilon_y = \frac{\sigma}{E} (1 - \mu^2), \tag{10}$$

(9) can be written in the form

$$v = \int \left[\frac{\sigma}{E} (1 - \mu^2) - \frac{1}{2} \left(\frac{dw}{dy} \right)^2 + \frac{w}{R} \right] dy. \tag{11}$$

The boundary conditions for v will be

$$v|_{y=0} = v|_{y=b} = 0. \tag{12}$$

Integrating (11), and taking these boundary conditions into account, we obtain

$$\sigma = -\frac{E}{1-\mu^2}\left(\frac{\pi^2 f^2}{4b^2} - \frac{2}{\pi}\frac{f}{R}\right). \tag{13}$$

By substituting (13) in (8) and introducing the dimensionless parameters

$$\zeta = \frac{f}{h}; \quad k = \frac{b^2}{Rh}; \quad q^* = \frac{q}{E}\left(\frac{b}{h}\right)^4 (1-\mu^2); \quad \omega = \frac{ah}{E}\left(\frac{b}{h}\right)^4 (1-\mu^2), \tag{14}$$

we obtain

$$q^* = \frac{\pi^5}{16}\zeta^3 - \frac{3}{4}\pi^2 k\zeta^2 + \left(\frac{\pi^5}{48} + \frac{2}{\pi}k^2 + \frac{\pi}{4}\omega\right)\zeta. \tag{15}$$

Curves of q^* as a function of ζ have been plotted in Figure 1 b for $k = \pi^3$ and different parameters of the foundation rigidity ω.

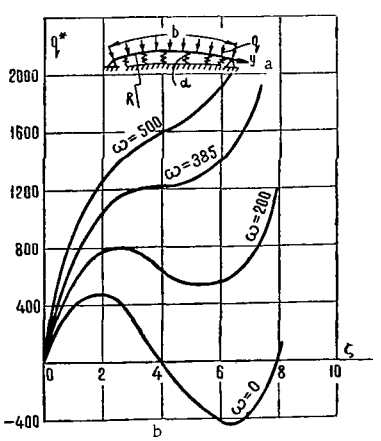

FIGURE 1.

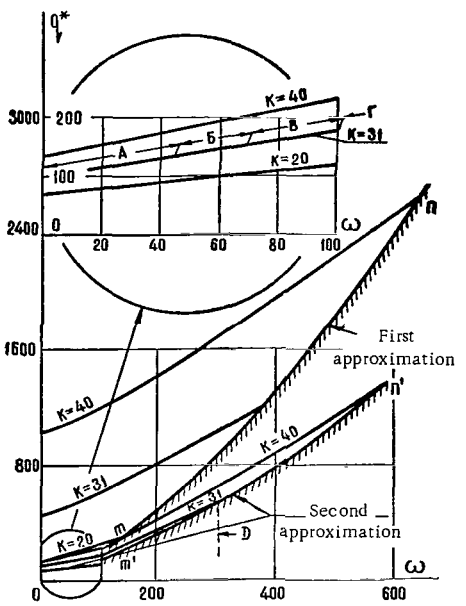

FIGURE 2.

It is seen from these curves that at negligible ω the curves $q^* = f(\zeta)$ have the usual loop shape. With the increase of ω they go higher and become more level. At $\omega \geqslant 385$ the zone of negative rigidity disappears. From this moment the phenomenon of "snap-through" becomes impossible.

The study of the extremal values of (15) established the upper values of the snap-through q_+^* for different ω and k. The relationships obtained are represented in Figure 2.

It is seen that the phenomenon of "snap-through" increases with the increase of the foundation rigidity ω and the panel curvature k. The shaded curve $m-n$ represents the boundary of the values of ω for which a snap-through is still possible.

2. SECOND APPROXIMATION

In the second approximation we consider both symmetrical and non-symmetrical deformations. In the case of symmetrical deformation we represent the deflection in the form

$$w = f_1 \sin\frac{\pi y}{b} + f_3 \sin\frac{3\pi y}{b}, \tag{16}$$

where f_1 is the sag when the panel bends in one half-wave and f_3 is the sag when it bends in three half-waves.

The Bubnov-Galerkin equations will be

$$\int_0^b X \sin\frac{\pi y}{b}\, dy = 0 \quad \text{and} \quad \int_0^b X \sin\frac{3\pi y}{b}\, dy = 0. \tag{17}$$

By integrating (17) after substituting (7) and (16) into it we obtain

$$\left.\begin{aligned} D\beta^4 f_1 - \sigma h\beta^2 f_1 + \frac{4}{\pi}\sigma\frac{h}{R} + \alpha f_1 - \frac{4}{\pi} q = 0, \\ 81 D\beta^4 f_3 - 9\sigma h\beta^2 f_3 + \frac{4}{3\pi}\sigma\frac{h}{R} + \alpha f_3 - \frac{4}{3\pi} q = 0, \end{aligned}\right\} \tag{18}$$

where we have written for the sake of brevity $\beta = \frac{\pi}{b}$.

Introduce the designations

$$\zeta = \frac{f_1}{h}; \quad \psi = \frac{f_3}{f_1}; \quad \eta = \frac{f_3}{h}. \tag{19}$$

We obtain then from (18)

$$\psi = \frac{D\beta^2 - \sigma h + \frac{\alpha}{\beta^2}}{243 D\beta^2 - 27\sigma h + \frac{3\alpha}{\beta^2}}. \tag{20}$$

The stress σ is found as above with the aid of (11) and (12):

$$\sigma = -\frac{E}{1-\mu^2}\left(\frac{\beta^2}{4} f_1^2 + \frac{9}{4}\beta^2 f_3^2 - \frac{2}{\pi}\frac{f_1}{R} - \frac{2}{3\pi}\frac{f_3}{R}\right). \tag{21}$$

Substituting (21) in (20), and taking (14) and (19) into account, we obtain the following cubic equation relative to ψ:

$$729\zeta^2\psi^3 - \left(27\zeta^2 + \frac{216}{\pi^3} k\zeta\right)\psi^2 + \left(\frac{36}{\pi^4}\omega + 81\zeta^2 - \right.$$

$$\left. - \frac{640}{\pi^3} k\zeta + 243\right)\psi - \left(3\zeta^2 - \frac{24}{\pi^3} k\zeta + \frac{12}{\pi^4}\omega + 1\right) = 0. \tag{22}$$

Having determined ψ from here and substituted it in (21), we determine σ^* with the aid of (14) and (19):

$$\sigma^* = -\frac{\pi^2}{4}\zeta^2 - \frac{9}{4}\pi^2\zeta^2\psi + \frac{2}{\pi}k\zeta + \frac{2}{3\pi}k\zeta\psi. \tag{23}$$

From the first equation of (18) we find q^*:

$$q^* = \frac{\pi^5}{48}\zeta - \frac{\pi^2}{4}\sigma^*\zeta + k\sigma^* + \frac{\pi}{4}\omega\zeta. \tag{24}$$

The symbols used in (23) and (24) have the following meaning:

$$\sigma^* = \frac{\sigma}{E}\left(\frac{b}{h}\right)^2(1-\mu^2); \qquad q^* = \frac{q}{E}\left(\frac{b}{h}\right)^4(1-\mu^2). \tag{25}$$

Using equations (22), (23), and (24) it is possible to plot the relationship $q^* = f(\zeta)$ for different ψ, ω, and k, and thus to study the phenomenon of "snap-through," to determine the critical pressures, and to find the parameters of the foundation rigidity ω at which the snap-through disappears.

A different procedure will be followed in the case of nonsymmetrical deformation /2/. The calculations are somewhat simpler in this case.

We take the variation of each term of the equilibrium equation (4) and the strain equation (9), and pass from the state of symmetrical deformation in a first approximation to the neutral state of nonsymmetrical deformation. We start with equation (9):

$$\delta\frac{dv}{dy} = -\frac{(1-\mu^2)}{E}\delta\sigma - \frac{dw}{dy}\delta\frac{dw}{dy} + \frac{\delta w}{R}, \tag{26}$$

where w is the deflection in the symmetrical mode

$$w = f_1 \sin\frac{\pi y}{b}, \tag{27}$$

and δw is the deflection in the nonsymmetrical mode

$$\delta w = \delta f_2 \sin\frac{2\pi y}{b}. \tag{28}$$

By substituting (27) and (28) in (26) and integrating we obtain

$$\delta v = -\frac{(1-\mu^2)}{E}y\delta\sigma - 2\frac{\pi^2}{b^2}f_1\delta f_2\left(\frac{\sin\frac{\pi}{b}y}{2\frac{\pi}{b}} + \frac{\sin\frac{3\pi}{b}y}{6\frac{\pi}{b}}\right) -$$

$$-\frac{b}{2\pi R}\delta f_2\cos\frac{2\pi}{b}y + c. \tag{29}$$

To determine σ and c we use the boundary conditions (12). We obtain

$$c = \frac{b}{4\pi}\delta f_2,$$

$$\delta\sigma = 0. \tag{30}$$

Take now the variation of the equilibrium equation (4):

$$\frac{D}{h}\delta\frac{d^4w}{dy^4} + \sigma\delta\frac{d^2w}{dy^2} + \frac{\alpha}{h}\delta w + \frac{d^2w}{dy^2}\delta\sigma + \frac{\delta\sigma}{R} = 0. \tag{31}$$

By substituting (27) and (28) in (31), and allowing for (14), (25), and (30), we obtain:

$$\sigma^* = \frac{\pi^2}{3} + \frac{\omega}{4\pi^2}. \tag{32}$$

To find the relationship $q^* = f(\zeta)$ we use the equilibrium equation (4). Solving it by the Bubnov-Galerkin method, and taking as deflection the sum of (27) and (28), we obtain

$$q^* = \frac{\pi^5}{48}\zeta - \frac{\pi^3}{4}\sigma^*\zeta + k\sigma^* + \frac{\pi}{4}\omega\zeta. \tag{33}$$

The substitution of (32) in (33) yields finally:

$$q^* = -\frac{(\pi^5 - 3\omega)}{16}\zeta + k\left(\frac{\pi^2}{3} + \frac{\omega}{4\pi^2}\right). \tag{34}$$

Using (34) it is possible to study the phenomenon of "snap-through" to find the critical pressures, and also to determine the boundary of the values of ω for which snap-through is still possible.

We now turn back to the results obtained in the second approximation for the symmetrical modes of equilibrium of the panel. Curves of $q^* = f(\zeta)$ for different ω were plotted with the aid of (22), (23), and (24). Such relationships are given in Figure 3a for $k = \pi^3$. Since equation (22) is cubic relative to ψ, we have three roots; to each root there corresponds a particular equilibrium mode (Figure 3b) and a pressure q^*. For mode I (ψ negative) the pressure curve has the usual loop shape, while II and III (ψ positive) correspond to closed eights /1/. With the increase of ω the curve of mode I goes up, straightening at the same time. At $\sim\omega \geqslant 75$ the zone of negative rigidity disappears, and a snap-through becomes impossible. In mode III snap-through also becomes impossible for the same value $\sim\omega \geqslant 75$ (the dotted branch of the eight becomes horizontal). With the further increase of ω the eight goes up and rotates, occupying a position at which the second branch likewise (the dash-dot line on Figure 3a) becomes horizontal; this happens at $\omega = 385$. Starting with $\omega = 385$, the panel deformation by symmetric modes becomes therefore smooth, without snap-throughs.

Curves of the pressure, corresponding to the nonsymmetrical deformation IV obtained by (34) have also been plotted in Figure 3a. With the increase of ω they go up and rotate. At $\omega = 102$ the straight line occupies a horizontal position. From this moment snap-through by the nonsymmetrical mode becomes impossible, and the deformation increases smoothly.

Since actually snap-through by a given mode is determined by a lower critical pressure, this pressure should be determined from the simultaneous consideration of the curves of symmetrical and nonsymmetrical deformation. It is seen from Figure 3a that at the beginning deformation proceeds by the symmetrical mode I (OA). At point A there occurs a jump to the nonsymmetrical state IV with the simultaneous realization of the symmetrical modes I, II, III (the horizontal line drawn through

point A intersects the descending branches of these modes). At $\omega \geqslant 45$ snap-through by the first mode becomes impossible — the horizontal line does not intersect the corresponding loop curve. Starting with $\omega \approx 65$ to 75 the horizontal line does not intersect the descending branch of mode III, and snap-through by this mode becomes impossible. Finally, at $\omega = 102$ snap-through by the nonsymmetrical mode IV becomes impossible, since the corresponding branch becomes horizontal. With the further increase of ω the deformation develops at first by branch OB, and then passes over smoothly to the nonsymmetrical branch BC. A snap-through by mode II takes place at the point of intersection with the descending branch of this mode (C). Starting with $\omega \approx 300$ the branches do not intersect (the curve of mode II rises more swiftly than the curve of mode IV), the possibility of snap-through by mode II disappears, and the panel deformation can be only smooth. At $\omega > 300$ none of the modes studied gives a snap-through.

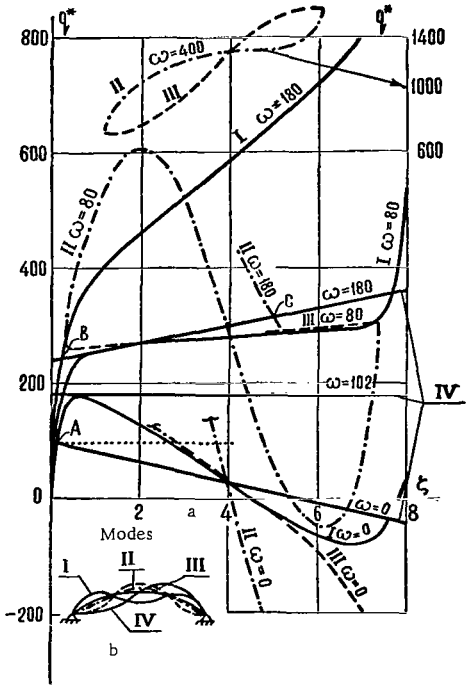

FIGURE 3.

The results obtained are illustrated graphically in Figure 2. In region (A) the snap-through is realized by all four modes. In region (B) the snap-through in the new state is possible by modes II, III, and IV only. To region (C) there correspond modes II and IV. In region (D) a snap-through can be realized by mode II only. Finally, in region (E) the deformation increases smoothly, without snap-through. Naturally, panels with different values of the curvature will have different values of ω at which the possibility of snap-through disappears.

The relationship between the upper critical pressure q_+^* and the foundation rigidity ω is represented graphically in Figure 2 for panels of different curvature k. As expected, the pressure increases with the increase of the foundation rigidity and the panel curvature. The shaded curve $m' - n'$ indicates the boundary to the right of which snap-through is no longer possible.

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BENDING OF AN ELASTIC THIN PLATE SUPPORTED BY RIGID COLUMNS

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(Saratov)

The bending of a finite isotropic plate supported by rigid or yielding columns is considered. The problem of the stressed-strained state of a plate rigidly bound to the columns is treated in detail.

1. Consider an isotropic thin plate of finite dimensions. The plate is supported on rigid columns, on which it is either rigidly bound or simply supported.

Such a plate can be considered as a plate with holes on the contours of which are specified some boundary conditions. It shall be assumed that the plate is either rigidly clamped or simply supported on its outer contour as well.

The plate bends under the action of a lateral load and bending moments applied on the simply supported contours of the plate.

The problem of the distribution of the moments and shearing forces, and the determination of the deflections of such a plate reduces, as known, to integrating the equation

$$\nabla^2\nabla^2 w = \frac{q}{D}. \tag{1.1}$$

The boundary conditions on the contours where the plate is rigidly clamped are

$$w = \frac{dw}{dn} = 0, \tag{1.2}$$

and on the contours where it is simply supported

$$w = M_n = 0. \tag{1.3}$$

The solution of this problem can be reduced to determining two functions of a complex variable $\varphi(z)$ and $\chi(z)$ from the corresponding boundary conditions on the contours of the holes and on the outer contour of the plate /1, 2/. In the case of simply supported contour the boundary conditions reduce to

$$\begin{gathered}\operatorname{Re}\left[\bar{t}\varphi(t) + \chi(t)\right] + w_0 = 0, \\ \operatorname{Re}\left[\left(\frac{4}{1-\nu} + \bar{t}\, t_{ss}\right)\varphi'(t) + \bar{t}_{ss}\,\varphi(t) + t_{ss}\,\chi'(t)\right] = \frac{1}{D(1-\nu)}(M_n^0 - M),\end{gathered} \tag{1.4}$$

and in the case of rigidly clamped contour to

$$\operatorname{Re}\,[\bar{t}\,\varphi(t)+\chi(t)]+w_0=0.$$
$$\operatorname{Re}\,\{i\bar{t}s[\varphi(t)+t\overline{\varphi'(t)}+\overline{\chi'(t)}]\}+\frac{dw_0}{dn}=0. \tag{1.5}$$

Here t is the affix of the point on one of the contours; ν is the Poisson ratio; t_s and t_{ss} are, respectively, the first and second derivatives of t by the arc of the contour considered; w_0 is a particular solution of (1.1); M_n^0 is the moment corresponding to w_0; M represents external moments.

After the functions $\varphi(z)$ and $\chi(z)$ have been determined from the boundary conditions it is easy to find the plate deflections and the moments and shearing forces appearing in the plate. They are expressed as follows through these functions /2/:

$$w=w_0+\operatorname{Re}\,[z\,\varphi(z)+\chi(z)],$$
$$M_x+M_y=M_x^0+M_y^0-2D(1+\nu)\,[\varphi'(z)+\overline{\varphi'(z)}],$$
$$M_y-M_x+2iH_{xy}=M_y^0-M_x^0+2iH_{xy}^0+2D(1-\nu)\,[\bar{z}\varphi''(z)+\chi''(z)], \tag{1.6}$$
$$N_x-iN_y=N_x^0-iN_y^0-4D\,\varphi''(z).$$

2. We shall consider the case of a doubly-connected region. Let the external contour L_0 be circular, and the interior L_1 curvilinear (Figure 1), and let L_1 be rigidly clamped while L_0 is simply supported. The plate bends under the action of moments distributed uniformly over the contour.

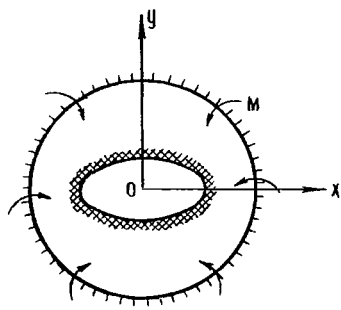

FIGURE 1.

In this case we shall represent the functions $\varphi(z)$ and $\chi(z)$ in the form /3/

$$\varphi(z)=Az\ln\zeta+\sum_{k=1}^{\infty}a_kz^k+\varphi^*(\zeta).$$
$$\chi(z)=B\ln\zeta+\sum_{k=0}^{\infty}C_kz^k+\chi^*(\zeta). \tag{2.1}$$

Here $\varphi^*(\zeta)$ and $\chi^*(\zeta)$ are functions holomorphic outside the curved contour L_1, and

$$z=\omega(\zeta)=\zeta+\sum_{k=1}^{r}\frac{m_k}{\zeta^k} \tag{2.2}$$

is the function which conformally maps the region outside the unit circle γ onto the region outside the contour L_1; m_k and r are constants. If the contour L_1 is elliptical, then

$$z = \omega(\zeta) = \zeta + \frac{m}{\zeta}. \tag{2.3}$$

The functions $\varphi^*(\zeta)$ and $\chi^*(\zeta)$ can be represented in region (ζ) by means of the following series:

$$\varphi^*(\zeta) = \sum_{k=1}^{\infty} \frac{b_k}{\zeta_k}, \quad \chi^*(\zeta) = \sum_{k=1}^{\infty} \frac{d_k}{\zeta_k}. \tag{2.4}$$

The first and third terms of (2.1) will be functions of the small parameter R^{-1} on the contour L_0. We can therefore expand them in series by powers of R^{-1} and neglect all terms of power higher than some power p. Using boundary conditions (1.4), in which we write $w_0 = M_n^0 = 0$ we use series to express the coefficients a_k and C_k through the remaining coefficients. Considering the coefficients a_k and C_k as known magnitudes, we find the functions $\varphi^*(\zeta)$ and $\chi^*(\zeta)$ from the boundary conditions (1.5), in which we write $w_0 = \frac{dw_0}{dn} = 0$ (using Muskhelishvili's method /4/), and obtain an algebraic system for determining the unknown coefficients introduced.

In a first approximation, corresponding to $p = 2$, we obtain

$$\begin{gathered} \varphi^*(\zeta) = -\frac{\zeta m}{1 - m\zeta}\left(b_1 - \frac{2a_1 + A}{\zeta}\right), \\ \chi^*(\zeta) = -\frac{1 + m\zeta^2}{\zeta}\varphi^*(\zeta) + m\, b_1 - \frac{2a_1 m}{\zeta^2}. \end{gathered} \tag{2.5}$$

The coefficients in (2.1) and (2.5) are determined from the system

$$\begin{gathered} AR^2 \ln R + a_1 R^2 + B \ln R + C_0 = 0, \\ 2a_1(1+\nu) + A\,[2(1+\nu)\ln R + 3 + \nu] - \frac{B}{R^2}(1-\nu) = -\frac{M}{D}, \\ 2a_1(1 - m^2) - 2mb_1 + A(1 + m^2) + B = 0, \\ C_0 = -a_1(1 + m^2) - mb_1, \quad b_1 = -m(2a_1 + A). \end{gathered} \tag{2.6}$$

In a second approximation $p = 4$ we obtain

$$\begin{gathered} \varphi^*(\zeta) = -\frac{\zeta}{1 - m\zeta^2}\left\{mb_1 + \frac{1}{\zeta^2}[m(2a_1 + A) + 2C_2 + a_3(3 + 6m^2 - m^4)] - \frac{a_3 m(3 + m^2)}{\zeta^4}\right. \\ \chi^*(\zeta) = -\left(\frac{1}{\zeta} + m\zeta\right)\varphi^*(\zeta) + mb_1 - \frac{1}{\zeta^2}[2a_1 m + a_3(1 + 6m^2 + m^4) + \\ + C_2(1 + m^2)] - \frac{a_3 m(1 + m^2)}{\zeta^4}. \end{gathered} \tag{2.7}$$

The system for determining the unknown coefficients becomes

$$\begin{gathered} AR^2 \ln R + a_1 R^2 + B \ln R + C_0 = 0, \\ -Am + a_3 R^4 + b_1 - \frac{Bm}{R^2} + \frac{d_2}{R^2} + C_2 R^2 = 0, \end{gathered} \tag{2.8}$$

$$2a_1(1+\nu)+A\,[2\,(1+\nu)\ln R+3+\nu]-\frac{B}{R^2}\,(1-\nu)=-\frac{M}{D},$$

$$-\frac{2Am(1+\nu)}{R^2}+4a_3R^2\,(2+\nu)-\frac{4b_1}{R^2}-\frac{2m(1-\nu)\,B}{P^4}+ \qquad (2.8)\ \text{(cont'd)}$$

$$+\frac{2d_2(1-\nu)}{R^4}-2\,(1-\nu)\,C_2=0,$$

$$C_0=-a_1\,(1-m^2)+6a_3\,m^3+Am^2,$$

$$d_2=Am+2a_3\,(1+3m^2)+C_2\,(1-m^2),$$

$$B=-2a_1\,(1+m^2)-A\,(1+3\,m^2)-4\,mC_2-12\,m\,(1+m^2)\,a_3,$$

$$b_1=-\,[(A+2a_1)\,m+2C_2+3a_3\,(1+3\,m^2)].$$

If the column is of circular cross section, this method leads to a rigorous solution of the problem. In this case

$$\varphi\,(z)=Az\ln z+a_1z,\qquad \chi(z)=B\ln z+C_0. \qquad (2.9)$$

The system for determining the constants in (2.9) will be

$$a_1\,(R^2-1)-\ln R\;(2a_1+A)+AR^2\ln R=0,$$

$$2a_1\,(1+\nu)+\frac{1-\nu}{R^2}\,(2a_1+A)+A\,[(3+\nu)+2\,(1+\nu)\ln R]=-\frac{M}{D}, \qquad (2.10)$$

$$B=-\,(2\,a_1+A),\qquad C_0=-\,a_1.$$

We conducted numerical calculations for determining the deflections, moments, and shearing forces for the case $R = 10$, $m = 0.5$ (the large axis of the ellipse is three times larger than the small axis), $\nu = 1/3$. The values of the unknown coefficients are given in Table 1 with an accuracy up to M/D.

The following values of the coefficients were obtained for the case of a circular column:

$$A=-0.33;\qquad a_1=-\,C_0=0.80;\qquad B=-1.27.$$

After these coefficients are determined the functions $\varphi(z)$ and $\chi(z)$ become known, which makes it possible to determine by (1.6) all the magnitudes which characterize the stressed-strained state of the plate.

TABLE 1

Approximations \ Coefficients	A	B	a_1	a_3	C_0	C_2	b_1	b_3	d_2	d_4
First	− 0.34	− 1.44	0.81	—	−0.69	—	− 0.64	—	− 0.17	—
Second	− 0.34	− 1.46	0.81	− 0.00002	−0.69	0.0067	− 0.66	0.00003	− 0.16	− 0.00002

The values of the deflections, moments, and shearing forces at the points of the horizontal and vertical diameters of a plate with elliptical column are given in Tables 2 and 3 with an accuracy up to M. Table 4 is given for comparison: it corresponds to the case of a plate with circular column. The values of the shearing forces are not given in this table, since these are calculated by the elementary formula

$$N_x-iN_y=-\frac{4AD}{z}. \qquad (2.11)$$

TABLE 2

	Approximations \ z	1.5	2	3	4	5	6	7	8	9	10
wD	First	0	0.25	1.28	2.40	3.33	3.86	3.85	3.19	1.78	−0.49
	Second	0	0.26	1.32	2.48	3.46	4.06	4.12	3.52	2.18	−0.002
M_x	First	−4.38	−1.08	−0.28	0.08	0.32	0.51	0.66	0.78	0.90	0.99
	Second	−4.49	−1.11	−0.29	0.07	0.32	0.51	0.66	0.79	0.90	1.007
M_y	First	−1.46	−1.19	−0.63	−0.32	−0.10	0.08	0.22	0.35	0.46	0.55
	Second	−1.50	−1.20	−0.63	−0.31	−0.09	0.09	0.24	0.36	0.47	0.57
N_x	First	22.46	1.82	0.64	0.41	0.30	0.24	0.20	0.18	0.15	0.14
	Second	23.35	1.86	0.65	0.41	0.31	0.25	0.21	0.18	0.16	0.14
N_y	First	0	0	0	0	0	0	0	0	0	0
	Second										

TABLE 3

	Approximations \ z	$0.5i$	i	$2i$	$3i$	$4i$	$5i$	$6i$	$7i$	$8i$	$9i$	$10i$
wD	First	0	0.13	0.94	2.11	3.29	4.24	4.78	4.78	4.12	2.71	0.47
	Second	0	0.12	0.92	2.05	3.18	4.09	4.57	4.50	3.78	2.31	−0.004
M_x	First	−0.36	−0.41	−0.38	−0.25	−0.09	0.05	0.18	0.30	0.41	0.50	0.59
	Second	−0.35	−0.40	−0.38	−0.25	−0.10	0.04	0.17	0.29	0.30	0.49	0.57
M_y	First	−1.08	−0.88	−0.43	−0.08	0.18	0.38	0.55	0.69	0.81	0.91	1.007
	Second	−1.05	−0.86	−0.42	−0.07	0.18	0.38	0.55	0.68	0.80	0.90	0.99
N_x	First	0	0	0	0	0	0	0	0	0	0	0
	Second											
N_y	First	0.16	0.30	0.38	0.34	0.28	0.24	0.21	0.18	0.16	0.14	0.13
	Second	0.13	0.28	0.37	0.33	0.28	0.24	0.20	0.18	0.16	0.14	0.13

TABLE 4

z	1	2	3	4	5	6	7	8	9	10
wD	0	0.60	1.74	2.90	3.84	4.37	4.36	3.69	2.26	0
M_x	−1.88	−0.63	−0.15	0.14	0.36	0.53	0.67	0.80	0.90	1.0
M_y	−0.63	−0.65	−0.41	−0.19	−0.02	0.14	0.27	0.38	0.46	0.65

Figure 2 represents the distribution of the moments M_x and M_y and the deflections along the diameters of a plate supported by an elliptic column.

Tables 2 to 4 show that the pattern of distribution of the moments and the shearing forces along the plate diameters is the same for all types of column cross section considered here. A concentration of these moments and forces is observed only near the column. Of greatest interest at the points of the clamped hole contour is the determination of the moment M_r, since $M_\theta = \nu M_r$, while the moment $H_{r\theta}$ has values considerably smaller than M_r. The moment M_r should be determined by the formula

$$M_r = -4\, D \operatorname{Re} \varphi'(z), \tag{2.12}$$

which is easily obtained from the second formula of (1.6) by taking into account that in the given problem $M_\theta = \nu M_r$ and $M_x + M_y = M_r + M_\theta$. The

shearing forces N_r and N_θ are determined by the formula /3/

$$N_r - iN_\theta = -4De^{i\theta}\varphi''(z). \tag{2.13}$$

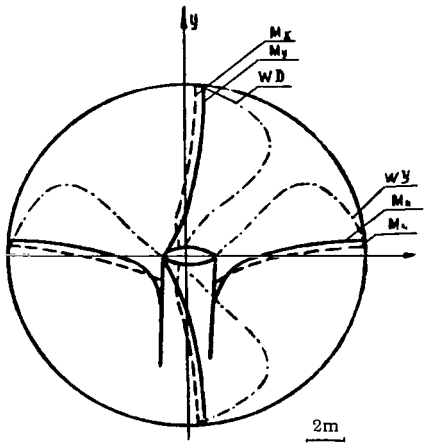

FIGURE 2.

The values of M_r, N_r, and N_θ, calculated for several points of the contour of an elliptical hole with an accuracy up to M, are given in Table 5.

TABLE 5

	Approximations \ θ	0	15	30	45	60	75	90
M_r	First	− 4.38	− 3.08	−1.91	−1.41	−1.20	−1.11	−1.08
	Second	− 4.49	− 3.14	−1.91	−1.39	−1.17	−1.08	−1.05
N_r	First	22.46	− 0.07	−1.96	−0.52	−0.11	0.10	0.16
	Second	23.35	− 0.16	−2.10	−0.58	−0.16	0.07	0.13
N_θ	First	0	11.06	2.68	0.66	0.38	0.16	0
	Second	0	11.52	2.76	0.67	0.38	0.16	0

The analysis of these tables shows that some of the laws in force in this case can be determined from the first approximation. Thus, the stress concentration is much larger in a plate with elliptical column than in a plate with circular column. This is particularly noticeable when comparing the maximum values for the shearing forces.

3. It is of interest to consider for the sake of comparison the case of a yielding column. Such a case can occur, for instance, when the foundation under the column collapses. In this case the principal vector of forces applied to the contour L_1 will be equal to zero. Then, as follows from /3/, the coefficient A in (2.1) will likewise be equal to zero. The first boundary condition of (1.5) becomes

$$\mathrm{Re}[\bar{t}\varphi(t) + \chi(t)] = C, \tag{3.1}$$

where C is a constant. The other boundary conditions remain unchanged.

The equations for determining the unknown coefficients in (2.1) are obtained as in the preceding problem: the first equations of (2.6) and (2.8) are discarded, and the value $A = 0$ is substituted. The values of the remaining coefficients for the same plate as in the first problem are given in Table 6. Here and below the magnitudes corresponding to a yielding column will be designated by an asterisk.

TABLE 6

Approximations \ Coefficient	C_0^*	C_2^*	a_1^*	a_3^*	d_2^*	d_4^*	B^*	b_1^*	b_3^*
First	35.12	0	−0.37	0	0	0	0.93	0.37	0
Second	35.10	−0.0047	−0.37	0.000009	−0.0035	0.000009	0.94	0.38	−0.000015

As explained in Section 2, the most important problem is that of determining the moments and shearing forces near the column. The results of the corresponding calculations for a plate with yielding elliptic column are given in Table 7. The distribution of the moments M_r and M_r^* over the contour of the elliptic hole is represented in Figure 3.

TABLE 7

	Approximations \ θ	0	15	30	45	60	75	90
M_r^*	First	4.47	2.91	1.49	0.89	0.64	0.53	0.50
	Second	4.45	2.95	1.49	0.88	0.62	0.50	0.47
N_r^*	First	−23.85	2.47	3.98	1.63	1.29	0.97	0.88
	Second	−24.45	2.53	4.07	1.67	1.32	0.99	0.91
N_θ^*	First	0	−12.28	−2.30	−0.30	−0.06	−0.01	0
	Second	0	−12.59	−2.35	−0.30	−0.06	−0.01	0

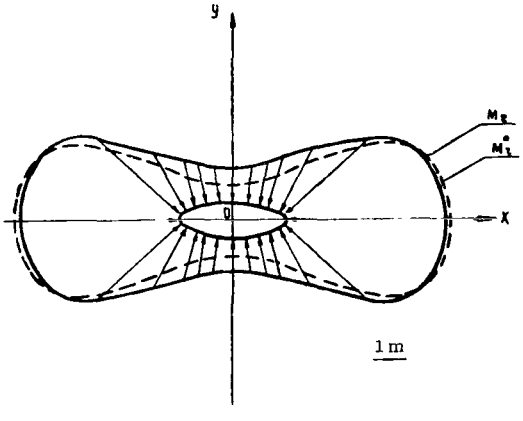

FIGURE 3.

The value obtained for the deflection of the points of a fastened elliptical contour was 34.5 M/D. The following values were obtained for the coefficients different from zero corresponding to a circular column (m = 0):

$$a_1 = -0.37; \quad B = 0.75; \quad C_0 = 35.60.$$

The deflection for the points of the fastened circular contour was 35.2 M/D. The variation of the deflection and moments along the real axis is given in Table 8.

TABLE 8

z	1	2	3	4	5	6	7	8	9	10
$D \cdot w \cdot$	35.22	34.62	33.06	30.66	27.47	23.50	18.76	13.27	7.01	1
M_x^*	1.49	1.12	1.05	1.03	1.01	1.01	1.01	1.00	1.00	1.00
M_y^*	0.50	0.87	0.94	0.96	0.98	0.98	0.98	0.99	0.99	0.99

It is seen from a comparison of these tables that the concentration of bending moments and shearing forces is somewhat larger in a plate with yielding elliptic column than in a plate with rigid column, while in the case of circular column the reverse is true; in addition, the sign of these forces and moments is reversed in the case of yielding column. As to the deflections, they are roughly the same in the cases of circular and elliptic columns.

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CENTRAL IMPACT ON A CIRCULAR PLATE OF UNIFORM THICKNESS

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The problem of determining the external forces in the case of central impact of an axisymmetrical body on a circular plate of uniform thickness with various conditions of support at the edges is treated on the basis of Timoshenko's theory.

The impact of an axisymmetrical body on a circular plate with arbitrary conditions of support (including the case of an unsupported plate) represents a fairly complex phenomenon.

The impact is accompanied by local deformations on the area of contact of the two bodies, and by plate vibrations symmetrical relative to the center; in addition, the points of a free plate start to execute a complex motion, representing the superposition of vibrations on the translatory motion of the system.

In order to study the phenomenon as a whole it is therefore necessary to have a solution of the problem of free and forced vibrations of the plate and to study the processes accompanying impact.

1. VIBRATIONS OF A CIRCULAR PLATE

The differential equation of symmetrical vibrations of a free plate of uniform thickness can be presented in the form

$$\Delta\Delta w + \frac{\rho h R^4}{D}\ddot{w} = \frac{R^4}{D}p, \tag{1}$$

where $w = w(\xi, t)$ is the plate deflection; h, R are the thickness and radius of the plate; ρ is the density per unit surface; r is the polar coordinate; t is the time; $\xi = \frac{r}{R}$ is a dimensionless coordinate; $D = \frac{Eh^3}{12(1-\mu^2)}$ is the flexural rigidity; E is the modulus of normal elasticity; μ is the Poisson ratio; $p = p(\xi, t)$ is the external load; $\Delta\Delta$ is the double Laplacian

$$\Delta\Delta(\ ) = \left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\right)\left(\frac{\partial^2}{\partial\xi^2}(\ ) + \frac{1}{\xi}\frac{\partial}{\partial\xi}(\ )\right).$$

$$\frac{\partial}{\partial t}(\ ) = (\ ).$$

As known /1/, the solution of this equation can be found by Fourier's method:

$$w(\xi, t) = \sum_n f_n(\xi) \cdot q_n(t). \tag{2}$$

Here $f_n(\xi)$ are vibration modes, representing a solution of Bessel's differential equation

$$\frac{d^2 f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} \pm a^2 f = 0, \tag{3}$$

where $a^4 = \frac{\rho h R^4}{D}\lambda^2$, λ^2 are essentially positive numbers and $q_n(t)$ are generalized coordinates, determined from the equation of vibrations of a system with one degree of freedom:

$$q_n + \lambda_n^2 q_n = \frac{Q_n(t)}{a_n}, \tag{4}$$

where

$$Q_n(t) = \int_0^1 p(\xi, t) f_n(\xi)\, \xi \cdot d\xi,$$

$$a_n = \int_0^1 \rho h f_n^2(\xi) \cdot \xi \cdot d\xi.$$

The vibration modes of the plate for various conditions of support were determined on the basis of /3/. The results of the calculation are represented in the table. In the last line of this table are written the first three normal functions $f_n(\xi)$ for the case when the integration constant C_1 is determined from the condition $f_n(0) = 1$.

At zero initial conditions the solution of equation (4) will be, as known,

$$q_n = \frac{1}{a_n \lambda_n}\int_0^t Q_n(t_1) \sin \lambda_n (t - t_1)\, dt_1. \tag{5}$$

2. IMPACT OF AN AXISYMMETRICAL BODY ON A CIRCULAR PLATE

The displacement of the points of the striking body during impact can be represented as the sum of three components: the displacements resulting from local warping, from the plate deflection, and from the displacement of the plate as a rigid body:

$$z = l + w + w_0, \tag{6}$$

where z is the total displacement of the body points; l is the local warping; w is the plate deflection; w_0 is the displacement of the plate as a rigid body.

TABLE

$$\lambda_n = a_n^2 \sqrt{\frac{D}{\rho h R^4}}$$

Conditions of support		Rigidly built-in plate	Simply supported plate	Free plate
Boundary conditions ($\xi=0$)		$f=\frac{\partial t}{\partial \xi}=0$	$f=0$ $\frac{\partial^2 f}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial f}{\partial \xi}=0$	$\frac{\partial^2 f}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial f}{\partial \xi}=0,$ $\frac{\partial}{\partial \xi}\left(\frac{\partial^2 f}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial f}{\partial \xi}\right)=0$
Frequency equation		$J_1 I_0 + J_0 I_1 = 0$	$J_1 I_0 + J_0 I_1 = \frac{2a}{1-\mu} J_0 I_0$	$J_1 I_0 + J_0 I_1 = \frac{2(1-\mu)}{a} J_1 I_0.$
Roots of the frequency equation	a_0	3.20	2.23	3.00
	a_1	6.30	3.45	6.20
	a_2	9.43	8.61	9.37
Deflection mode f_n		$f_n(\xi)=C_1\left[J_0(a_n\xi)-\frac{J_0(a_n)}{I_0(a_n)}I_0(a_n\xi)\right]$	$f_n(\xi)=C_1\left[J_0(a_n\xi)-\frac{J_0(a_n)}{I_0(a_n)}I_0(a_n\xi)\right]$	$f_n(\xi)=C_1\left[J_0(a_n\xi)-\frac{(1-\mu)J_1(a_n)-a_nJ_0(a_n)}{(1-\mu)I_1(a_n)-a_nI_0(a_n)}I_0(a_n\xi)\right]$
First three normal functions f_n ($f_n(0)=1$)	f_0	$0.4224\,[J_0(3.2\xi)+1.3675I_0(3.2\xi)]$	$1.479\,[J_0(2.23\xi)-0.324I_0(2.23\xi)]$	$-1.3862\,[J_0(3\xi)-1.7214\,I_0(3\xi)]$
	f_1	$-2.6364\,[J_0(6.3\xi)-1.3793I_0(6.3\xi)]$	$0.879\,[J_0(5.45\xi)+0.1368\,I_0(5.45\xi)]$	$0.3918\,[J_0(6.2\xi)+1.5523\,I_0(6.2\xi)]$
	f_2	$0.4149\,[J_0(9.43\xi)+1.41033I_0(9.43\xi)]$	$1.093\,[J_0(8.61\xi)-0.0356\,I_0(8.61\xi)]$	$-1.9697[J_0(9.37\xi)-1.5077I_0(9.37\xi)]$

$J_0(a_n\xi)$, $I_0(a_n\xi)$ - are a Bessel function and a modified Bessel function of zero order.

The displacement z satisfies the equation

$$m_1 \frac{d^2 z}{dt^2} = -P(t) \tag{7}$$

and the initial conditions

$$\text{at } t=0 \qquad z=0, \ \frac{dz}{dt} = v_0. \tag{8}$$

Here m_1 is the mass of the striking body; $P(t)$ is the total acting force; and v_0 is the velocity at the beginning of impact.

The integration of (7) with initial conditions (8) yields

$$z = v_0 t - \frac{1}{m_1} \int_0^t \int_0^{t_1} P(t_2)\, dt_2 \cdot dt_1. \tag{9}$$

The displacement of the plate as a rigid body can be found from the equation

$$m_2 \frac{d^2 w_0}{dt^2} = P(t). \tag{10}$$

By integrating (10) with zero initial conditions we obtain

$$w_0 = \frac{1}{m_2} \int_0^t \int_0^{t_1} P(t_2) dt_2 \cdot dt_1. \tag{11}$$

In the case of elastic-plastic impact /2/

$$l = \varkappa P + \varkappa_1 P^{\frac{2n}{2n+1}}, \tag{12}$$

where $\varkappa$, $\varkappa_1$ are experimentally determined coefficients.

The experimental coefficients $\varkappa_i$ have been determined by several authors. On the strength of the studies of high-speed impact phenomena conducted by Valkenburg, Clay, and Huth /3, 4/, the expression for $\varkappa$ can be written in the form

$$\varkappa = \left(\frac{k_1 k_2^2}{2\pi}\right)^{2/3} \frac{1}{\rho_c V_c^{1/3} v_0^2} \left(\frac{\rho_c}{\rho}\right)^{2/3} \left(\frac{v_0}{c}\right)^{4/3}, \tag{13}$$

where ρ_c is the density of the striking body; V_c is its volume; ρ is the density of the obstacle; c is the velocity of sound in the obstacle; k_1, k_2 are empirical coefficients. The following values of k and k_2 were obtained in experiments with $\left(\frac{1}{8}\right)''$ pellets /4/: $k_1 = 25.4$, $k_2 = 1.66$.

The coefficient $\varkappa_1$ can be determined theoretically. Shtaerman /2/ found the following expression for this coefficient by solving the contact

problem under conditions of sufficiently close contact:

$$\varkappa_1 = \left[\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)} A\right]\left[\frac{2n+1}{2n}\,\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\,\frac{n(\vartheta_1+\vartheta_2)}{2A}\right]^{\frac{2n}{2n+1}}, \qquad (14)$$

where

$$A = \frac{1}{2n!}\left(\frac{d^{2n}z_{1,2}}{dr^{2n}}\right)_{r=0}, \quad z_{1,2} = Ar^{2n} + \cdots,$$

$$\vartheta_1 = \frac{1-\mu_1^2}{2\pi G_1}, \quad \vartheta_2 = \frac{1-\mu_2^2}{2\pi G_2}, \quad G_{1,2} = \frac{E_{1,2}}{2(1+\mu_{1,2})}.$$

E_1, μ_1 are the modulus of normal elasticity and the Poisson ratio for the striking body; E_2, μ_2 are the same for the plate; n is the order of the equation of the relative indicatrix of the surfaces of the two bodies.

Taking (2), (6), (9), (11), and (12) into account, the equation of impact of an axisymmetrical body on a free plate will be

$$v_0 t - \frac{1}{m}\int_0^t\int_0^{t_1} P(t_2)\, dt_2\cdot dt_1 = \varkappa P + \varkappa_1 P^{2/3} + \sum_n \frac{1}{a_n\lambda_n}\int_0^t Q_n(t_1)\sin\lambda_n(t-t_1)dt_1, \qquad (15)$$

where $m = \dfrac{m_1 m_2}{m_1+m_2}$; here we have taken $n = 1$, which corresponds to the case of elastic impact according to Hertz /2/.

In the case of impact on a plate with rigidly clamped or simply supported edges, the value $m = m_1$ is obviously to be introduced in (15).

Note further that if the impact area is considerably smaller than the plate area, i.e., $\dfrac{a}{R} \ll 1$ (a is the radius of the region of contact), then $Q_n(t) = \dfrac{P(t)}{2\pi}$.

We obtain then instead of (15) Timoshenko's equation

$$v_0 t - \frac{1}{m}\int_0^t\int_0^{t_1} P(t_2)\, dt_2\cdot dt_1 = \varkappa P + \varkappa_1 P^{2/3} +$$

$$+ \sum_n \frac{1}{a_n\lambda_n}\int_0^t \frac{P(t_1)}{2\pi}\sin\lambda_n(t-t_1)dt_1. \qquad (16)$$

Equation (16) can be solved numerically by iterations with as high an accuracy as desired /2/.

Consider the case at the boundary of applicability of the Timoshenko theory; we shall consider $\varkappa_1 = 0$, and in the case of free plate we shall take approximately $m \approx m_1$; $\delta\tau$ is the integration step.

The initial conditions for the displacement, bending, and velocity of the mass m_1 are: $t = 0$, $z = 0$, $w = 0$, $\dot{z} = v_0$.

First stage:

$$t_1 = \delta\tau, \quad z_1 = v_0\,\delta\tau, \quad w_1 = 0, \quad P_1 = \frac{v_0\,\delta\tau}{\varkappa},$$

acceleration $\ddot{z}_1 = -\dfrac{P_1}{m_1}$, velocity $\dot{z}_1 = v_0$.

Second stage:

$$t_2 = 2\cdot\delta\tau, \quad z_2 = z_1 + \dot{z}_1\,\delta\tau + \frac{\ddot{z}_1(\delta\tau)^2}{2},$$

$$w_2 = \frac{\delta\tau}{2\pi}\sum_n \frac{1}{a_n\lambda_n} P_1 \sin\,(\lambda_n\,\delta\tau),$$

$$P_2 = \frac{z_2 - w_2}{\varkappa}, \quad \ddot{z}_2 = -\frac{P_2}{m_1}, \quad \dot{z}_2 = v_0 + \ddot{z}_1\,\delta\tau.$$

r-th stage:

$$t_r = r\delta\tau, \quad z_r = z_{r-1} + \dot{z}_{r-1}\,\delta\tau + \frac{\ddot{z}_{r-1}(\delta\tau)^2}{2},$$

$$w_r = \frac{\delta\tau}{2\pi}\sum_n \frac{1}{a_n\lambda_n}\sum_i^{r-1} P_i \sin\lambda_n\,(r-i)\,\delta\tau,$$

$$P_r = \frac{z_r - w_r}{\varkappa}, \quad \ddot{z}_r = -\frac{P_r}{m_1},$$

$$\dot{z}_r = \dot{z}_{r-1} + \ddot{z}_{r-1}\,\delta\tau.$$

Calculations by this scheme were conducted by means of a computer for different cases of fastening of the plate edges. The results of the calculations are represented in the figure.

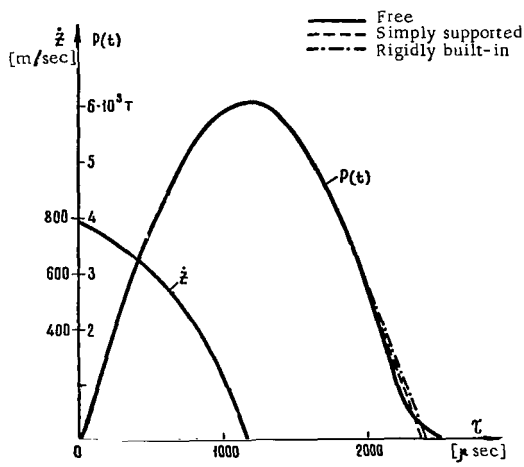

FIGURE.

The following initial values were used in the calculations: $m_1 = m = 5.61$ kg·sec²/m, $v_0 = 800$ m/sec, $\varkappa = 10^{-7}$ m/kg, $\rho = 795$ kg·sec²/m⁴, $h = 0.15$ m, $E = 2\times10^{10}$ kg/m², $\mu = 0.3$, $R = 0.5$ m; only the first three modes of free vibrations of the plate were kept in the calculation. As follows from the figure, the impact time in the case of impact on a free plate is $\tau = 2500\,\mu$ sec, while according to the elementary theory /2/ $\tau = \pi\sqrt{\varkappa m_1} = 2340\,\mu$ sec. The maximum value of the interaction force is $P_{max} = 6.04\times10^3$ t, while the elementary theory yields $P_{max} = v_0\sqrt{\frac{m_1}{\varkappa}} = 5.98\times10^3$ t.

The pattern of increase of the force with the time is likewise represented in the figure. It is seen from the curves that Timoshenko's theory of impact does not lead in the case considered to essential corrections in the determination of the impact time and of P_{max} of the bodies considered, but can be used to determine the pattern of variation of the force with time.

The conditions of edge fastening in the case considered (rigid plate, $\delta\tau = \frac{\tau}{40}$, where τ is the period of natural vibrations of the plate in the lowest mode) do not substantially influence the phenomenon of impact, since w is very small.

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VIBRATIONS OF A PLATE LYING ON AN ELASTIC FOUNDATION WITH AFTEREFFECT, ALLOWING FOR THE MASS OF A MOVING LOAD

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(L'vov)

Consider the vibrations of a system consisting of a rectangular elastic plate and an elastic-viscous foundation lying on a half-space characterized by a foundation modulus. The foundation itself will likewise be considered as a plate made of elastic-viscous material capable of showing resistance in bending. Let the system be acted upon by a moving load, whose velocity of motion will be considered as constant (figure). This system represents the mechanical model of concrete-coated highways, airfields, and special foundations /1/.

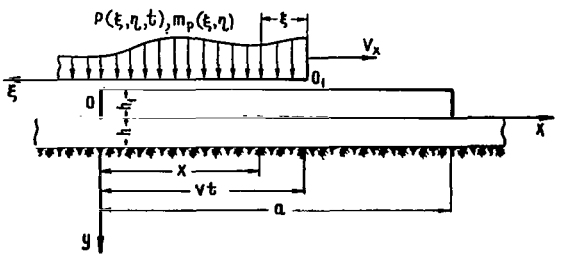

FIGURE.

We shall take as computing model of the foundation an isotropic medium linearly deformed with time, whose state of strain will be represented in an integral form, expressing the properties of creep and relaxation.

The vibrations of an elastic plate are determined from the well-known differential equation

$$m_1 \frac{\partial^2 w_1}{\partial t^2} + D_1 \Delta \Delta w_1 = p(x, y, t) - q(x, y, t), \qquad (1)$$

where m_1 is the plate mass per unit area; $w_1(x, y, t)$ is the dynamic deflection of the middle plane of the plate; t is time; D_1 is the flexural rigidity of the plate; p is the external load; q is the reactive pressure of the foundation; Δ is the Laplacian.

To obtain the equation of vibrations of the elastic-viscous foundation we shall use Volterra's relationships between the stress and strain

components in time. In the two-dimensional case we shall have /3/

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) + \int_0^t [f_1(t-\tau)\,\varepsilon_{11}(\tau) + f_2(t-\tau)\,\varepsilon_{22}(\tau)]\,d\tau,$$

$$\sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}) + \int_0^t [f_1(t-\tau)\,\varepsilon_{22}(\tau) + f_2(t-\tau)\,\varepsilon_{11}(\tau)]\,d\tau, \qquad (2)$$

$$\sigma_{12} = \frac{E}{2(1+\nu)}\,\varepsilon_{12} + \int_0^t f_3(t-\tau)\,\varepsilon_{12}(\tau)\,d\tau,$$

where σ_{11}, σ_{22}, σ_{12}, ε_{11}, ε_{22}, and ε_{12} are the stress and strain components, respectively; E is the modulus of elasticity of the foundation material; ν is the Poisson ratio; $f_1(t-\tau)$, $f_2(t-\tau)$, and $f_3(t-\tau)$ are experimentally determined aftereffect functions; t is the running time coordinate; τ is the time of application of the load.

The following relationship exists between f_1, f_2, and f_3 /3/:

$$f_1 = f_2 + 2f_3.$$

Using the formulas

$$M_x = \int_{-h/2}^{h/2} \sigma_{11} z\,dz, \qquad M_y = \int_{-h/2}^{h/2} \sigma_{22} z\,dz, \qquad M_{xy} = \int_{-h/2}^{h/2} \sigma_{12} z\,dz, \qquad (3)$$

allowing for (2), and writing

$$\varepsilon_{11} = -z\frac{\partial^2 w_2}{\partial x^2}, \qquad \varepsilon_{22} = -z\frac{\partial^2 w_2}{\partial y^2}, \qquad \varepsilon_{12} = -2z\frac{\partial^2 w_2}{\partial x \partial y},$$

we obtain the following relationships for the moments in the foundation:

$$M_x = -\frac{h^3}{12}\left\{\frac{E}{1-\nu^2}\left(\frac{\partial^2 w_2}{\partial x^2} + \nu\frac{\partial^2 w_2}{\partial y^2}\right) + \int_0^t \left[f_1(t-\tau)\frac{\partial^2 w_2(\tau)}{\partial x^2} + f_2(t-\tau)\frac{\partial^2 w_2(\tau)}{\partial y^2}\right]d\tau\right\},$$

$$M_y = -\frac{h^3}{12}\left\{\frac{E}{1-\nu^2}\left(\frac{\partial^2 w_2}{\partial y^2} + \nu\frac{\partial^2 w_2}{\partial x^2}\right) + \int_0^t \left[f_1(t-\tau)\frac{\partial^2 w_2(\tau)}{\partial y^2} + f_2(t-\tau)\frac{\partial^2 w_2(\tau)}{\partial x^2}\right]d\tau\right\}, \qquad (4)$$

$$M_{xy} = -\frac{h^3}{12}\left[\frac{E}{1+\nu}\frac{\partial^2 w_2}{\partial x \partial y} + \int_0^t 2f_3(t-\tau)\frac{\partial^2 w_2(\tau)}{\partial x \partial y}\,d\tau\right],$$

where z is the distance of the layer considered from the middle plane of the foundation; h is the foundation thickness; $w_2(x, y, t)$ is the dynamic deflection of the middle plane of the foundation.

The conditions of equilibrium of an infinitesimal element of the foundation, of height h, yield

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q(x, y, t) + kw_2(x, y, t), \qquad (5)$$

where q is the load acting on the foundation; k is the foundation modulus of the half-space.

Substituting the values of the moments M_x, M_y, and M_{xy} from (4) in (5) and applying the d'Alembert principle, we obtain the law of vibrations of an elastic-viscous foundation in the form of the following integral-differential equation:

$$m_2 \frac{\partial^2 w_2}{\partial t^2} + D\Delta\Delta w_2 + \frac{h^3}{12}\int_0^t f_1(t-\tau)\,\Delta\Delta w_2(\tau)\,d\tau = q(x, y, t) - kw_2(x, y, t), \qquad (6)$$

where m_2 is the foundation mass per unit area; D is the flexural rigidity of the elastic-viscous foundation

$$D = \frac{Eh^3}{12(1-\nu^2)}.$$

Considering that the deflections of the elastic plate and the elastic-viscous foundation are equal ($w_1 = w_2 = w$), and adding equations (1) and (6), we obtain the following integral-differential equation of vibrations of the plate together with the elastic-viscous foundation:

$$m\frac{\partial^2 w}{\partial t^2} + (D_1 + D)\,\Delta\Delta w + kw + \frac{h^3}{12}\int_0^t f_1(t-\tau)\,\Delta\Delta w(\tau)\,d\tau = p(x, y, t), \qquad (7)$$

where $m = m_1 + m_2$ is the mass of the system (plate and foundation) per unit area; $w(x, y, t)$ is the dynamic deflection of the middle plane of the plate when it is deflected together with the foundation.

Equation (7) will be solved by Galerkin's variational method /4/.

Represent the dynamic deflection of the plate $w(x, y, t)$ in the form of a series by the first n modes of natural vibrations of the system $\varphi_{ij}(x, y)$:

$$w(x, y, t) = \sum_{ij=1}^{n} T_{ij}(t)\,\varphi_{ij}(x, y), \qquad (8)$$

where $T_{ij}(t)$ is an unknown time function for the present; $\varphi_{ij}(x, y)$ are the modes of natural vibrations of the system satisfying the boundary conditions.

Substitute (8) in (7) and require, in accordance with Galerkin's method, that the result of the substitution be orthogonal to each of the fundamental functions $\varphi_{ij}(x, y)$. We obtain as a result the following system of ordinary differential equations, solved relative to $T_{ij}(t)$:

$$m\sum_{ij=1}^{n}\frac{d^2 T_{ij}}{dt^2}\iint \varphi_{ij}\varphi_{gl}\,dxdy + (D_1 + D)\sum_{ij=1}^{n} T_{ij}\iint \Delta\Delta\varphi_{ij}\varphi_{gl}\,dxdy +$$

$$+\sum_{ij=1}^{n} kT_{ij}\iint \varphi_{ij}\varphi_{gl}\,dxdy + \sum_{ij=1}^{n}\frac{h^3}{12}\int_0^t f_1(t-\tau)\,T_{ij}(\tau)\,d\tau\iint \Delta\Delta\varphi_{ij}\varphi_{gl}\,dxdy =$$

$$= \iint p(x, y, t)\,\varphi_{gl}\,dxdy. \qquad (9)$$

Here and below it will be assumed that the integration is conducted over the entire area of the plate.

The modes of natural vibrations of the system $\varphi_{ij}(x, y)$ satisfy the differential equation

$$\Delta\Delta\varphi_{ij} - \lambda_{ij}^2 \varphi_{ij} = 0, \tag{10}$$

in which λ_{ij} represent the fundamental numbers (eigenvalues) of the problem, expressing the frequencies of natural vibrations.

Taking (10) into account and using the condition of orthogonality of the function $\varphi_{ij}(x, y)$

$$\iint \varphi_{ij} \varphi_{gl} dxdy = 0 \quad \text{at } i \neq g \text{ and } j \neq l, \tag{11}$$

we obtain from (9) an equation of the type

$$m\frac{d^2T_{gl}}{dt^2} + [(D_1 + D)\lambda_{gl}^2 + k]\, T_{gl} + \frac{\lambda_{gl}^2 h^3}{12}\int_0^t f_1(t-\tau)\, T_{gl}(\tau)\, d\tau =$$
$$= \frac{\iint p(x, y, t)\, \varphi_{gl}\, dxdy}{\iint \varphi_{gl}^2(x, y)\, dxdy} \qquad (g, l = 1, 2, 3, \cdots n). \tag{12}$$

Proceeding to the statement of the problem, we substitute in the right-hand part of equation (12) the value of a moving load, allowing for its mass.

Designating the load in the reference frame linked with the moving system by $P(\xi, \eta, t)$, and the corresponding mass by $m_p(\xi, \eta)$ (see figure), we obtain

$$p(x, y, t) = P(\xi, \eta, t) - m_p(\xi, \eta)\left[\frac{d^2w}{dt^2}\right]_{\substack{\xi = v_x t - x \\ \eta = v_y t - y}}, \tag{13}$$

where v_x and v_y are the x and y components of the velocity of motion.

Assuming that traffic on the road proceeds mainly in a straight line, and that traffic has practically no influence on the surface of the road left behind, we can write $v_y = 0$. Then, using (8) and expressing $\frac{d^2w(x, y, t)}{dt^2}$ through partial derivatives, we obtain

$$p(x, y, t) = P(\xi, \eta, t) -$$
$$- m_p(\xi, \eta)\sum_{ij=1}^{n}\left[\frac{d^2T_{ij}}{dt^2}\varphi_{ij} + 2v_x\frac{dT_{ij}}{dt}\frac{d\varphi_{ij}}{dx} + v_x^2 T_{ij}\frac{d^2\varphi_{ij}}{dx^2}\right]. \tag{14}$$

The substitution of (14) in the right-hand part of (12) yields

$$m\frac{d^2T_{gl}}{dt^2} + \frac{1}{N_{gl}}\sum_{ij=1}^{n}\frac{d^2T_{ij}}{dt^2}\iint m_p(\xi, \eta)\, \varphi_{ij}\, \varphi_{gl} dxdy +$$
$$+ \frac{2v_x}{N_{gl}}\sum_{ij=1}^{n}\frac{dT_{ij}}{dt}\iint m_p(\xi, \eta)\frac{d\varphi_{ij}}{dx}\varphi_{gl} dxdy +$$

$$+\frac{v_x^2}{N_{gl}}\sum_{ij=1}^{n} T_{ij}\iint m_p(\xi,\eta)\frac{d^2\varphi_{ij}}{dx^2}\varphi_{gl}dxdy+\frac{\lambda_{gl}^2h^3}{12}\int_0^t f_1(t-\tau)\,T_{gl}(\tau)\,d\tau+$$

$$+[(D_1+D)\lambda_{gl}^2+k]\,T_{gl}=\frac{1}{N_{gl}}\iint P(\xi,\eta,t)\,\varphi_{gl}\,dxdy \qquad (ij,gl=1,2,3,\cdots,n), \quad (15)$$

where

$$N_{gl}=\iint \varphi_{gl}^2(x,y)\,dxdy.$$

N_{gl} are norms of the fundamental functions.

Since the roads can be used by cars with trailers, or by columns of cars /5/, whose total length exceeds the length of the plate between seams, the moving load can be represented as a continuous load uniformly distributed over a large length, and its mass can be assumed as constant /4, 6/, i.e.,

$$m_p(\xi,\eta)=m_p=\text{const}.$$

Since the external load acting on the system is symmetrical, and since damping is not allowed for directly in the problem, the third term of (15), which determines the Coriolis component of the moving load, can be neglected /7/.

Assuming that the plate is hinged along its contour, writing

$$\varphi_{ij}(x,y)=\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}, \quad (16)$$

where a and b are the dimensions of the plate sides, and substituting this value in (15), we obtain after some transformations the following integral-differential equation of vibrations:

$$\frac{d^2T_{ij}}{dt^2}\left(1+\frac{m_p}{m}\right)+\left\{\left[(D_1+D)\frac{\lambda_{ij}^2}{m}+\frac{k}{m}\right]-\frac{v_x^2i^2\pi^2m_p}{a^2m}\right\}T_{ij}+$$

$$+\frac{\lambda_{ij}^2h^3}{12m}\int_0^t f_1(t-\tau)\,T_{ij}(\tau)\,d\tau=\frac{16P(t)}{\pi\cdot i\cdot j\cdot m} \qquad (ij=1,3,5,\cdots). \quad (17)$$

On the basis of the experimental data on the deformation of natural and stabilized soils with time under a constant load we conclude that the kernel $f_1(t-\tau)$ describing the laws of variation of the strain rates of elastic-viscous bodies with time is satisfactorily approximated by an exponential function of the form $Ae^{-\alpha(t-\tau)}$, where A and α are empirically determined parameters.

Writing

$$f_1(t-\tau)=Ae^{-\alpha(t-\tau)},$$

we can reduce the integral-differential equation (17) to the following differential equation with constant coefficients:

$$T_{ij}'''\left(1+\frac{m_p}{m}\right)+T_{ij}''\alpha_{ij}\left(1+\frac{m_p}{m}\right)+T_{ij}'\left[\frac{(D_1+D)\lambda_{ij}^2+k}{m}-\frac{v_x^2i^2\pi^2m_p}{a^2m}\right]+$$

$$+ T_{ij}\left\{\alpha_{ij}\left[\frac{(D_1+D)\lambda_{ij}^2+k}{m} - \frac{v_x^2 i^2\pi^2 m_p}{a^2 m}\right] + A_{ij}\right\} =$$

$$= \frac{16}{\pi^2 m\cdot i\cdot j}\left[P'(t) + \alpha_{ij}P(t)\right]. \qquad (18)$$

Dividing (18) by $1+\frac{m_p}{m}$, and introducing the designations

$$\frac{(D_1+D)\lambda_{ij}^2+k}{m+m_p} = \psi_{ij}^2; \qquad \frac{v_x\pi}{a} = \beta; \qquad \frac{m_p}{m+m_p} = \mu; \qquad \frac{m}{m+m_p} = \mu_1.$$

we obtain the following differential equation:

$$T_{ij}''' + T_{ij}''\alpha_{ij} + T_{ij}'(\psi_{ij}^2 - i^2\beta^2\mu) + T_{ij}\left[(\psi_{ij}^2 - i^2\beta^2\mu)\alpha_{ij} + A_{ij}\mu_1\right] =$$

$$= \frac{16}{\pi^2 i\, j\,(m+m_p)}\left[P'(t) + \alpha_{ij}P(t)\right] \qquad (ij = 1, 3, 5, \cdots) \qquad (19)$$

(here and in what follows the strokes represent differentiation with respect to the time t).

Let the system be subjected to the action of a periodic moving fluctuating load of the type

$$P(t) = P_0 + P_t \sin\theta t. \qquad (20)$$

The periodic component of the load allows for the influence of the imbalance of the wheels of the cars, and has a frequency equal to

$$\theta = \frac{2v_x}{D_0}, \qquad (21)$$

where D_0 is the diameter of the wheels.

Substitute (20) in the right-hand part of (19):

$$T_{ij}''' + T_{ij}''\alpha_{ij} + T_{ij}'(\psi_{ij}^2 - i^2\beta^2\mu) + T_{ij}\left[(\psi_{ij}^2 - i^2\beta^2\mu)\alpha_{ij} + A_{ij}\mu_1\right] =$$

$$= \frac{16}{\pi^2 i\cdot j\,(m+m_p)}\left[\alpha_{ij}P_0 + \alpha_{ij}P_t\sin\theta t + P_t\theta\cos\theta t\right] \qquad (ij=1, 3, 5, \cdots). \qquad (22)$$

We have obtained a nonhomogeneous differential equation with constant coefficients.

This equation will have a cubic characteristic equation, with either one real and two complex roots, or three real roots.

When the aftereffect kernel is represented through an exponential function, the characteristic equation has always one real negative root and a pair of complex conjugate roots, whose real part is likewise negative /8, 9/.

If all three roots were real the system would have an aperiodic motion, which actually does not take place in the case considered.

We shall look for a particular solution of (22) in the form

$$T_{2ij} = B_{1ij}\cos\theta t + B_{2ij}\sin\theta t + B_{3ij}. \qquad (23)$$

The general solution of (7) will therefore be

$$w = \sum_{ij=1,3,5}^{n}\left[c_{1ij}e^{-r_{ij}t} + e^{-s_{ij}t}(c_{2ij}\cos\omega_{ij}t + c_{3ij}\sin\omega_{ij}t) + \right.$$

$$\left. + B_{1ij}\cos\theta t + B_{2ij}\sin\theta t + B_{3ij}\right]\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}. \qquad (24)$$

The constants c_1, c_2, and c_3 are found from the initial conditions

$$u(x, y, 0) = \rho_1(x, y); \qquad \frac{\partial u(x, y, 0)}{\partial t} = \rho_2(x, y);$$
$$m\frac{\partial^2 u(x, y, 0)}{\partial t^2} + (D_1 + D)\Delta\Delta u(x, y, 0) = 0, \tag{25}$$

where u is a solution of the homogeneous equation corresponding to (7).

By substituting the general solution of the homogeneous equation corresponding to (7) in the initial conditions (25) we obtain the following system of equations for determining the constants c_1, c_2, and c_3:

$$c_{1ij} + c_{2ij} = \frac{4}{ab}\int_0^a\int_0^b \rho_1(x, y)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\,dxdy,$$

$$\omega_{ij}c_{3ij} - s_{ij}c_{2ij} - r_{ij}c_{1ij} = \frac{4}{ab}\int_0^a\int_0^b \rho_2(x, y)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\,dxdy,$$

$$m\sum_{ij=1,3,5}^{n}[r_{ij}^2c_{1ij} + (s_{ij}^2 - \omega_{ij}^2)c_{2ij} - 2s_{ij}\omega_{ij}c_{3ij}] +$$
$$+ (D_1 + D)\sum_{ij=1,3,5}^{n}(c_{1ij} + c_{2ij})\pi^4\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^2 = 0. \tag{26}$$

By substituting (23) in (22) and equating the terms with identical coefficients we obtain the following system of algebraic equations for determing the constants B_1, B_2, and B_3:

$$[\theta^3 - \theta(\psi_{ij}^2 - i^2\beta^2\mu)]B_{1ij} + [\alpha_{ij}(\psi_{ij}^2 - i^2\beta^2\mu - \theta^2) + A_{ij}\mu_1]B_{2ij} = \frac{16\theta P_t}{\pi^2 ij(m+m_p)},$$

$$[\alpha_{ij}(\psi_{ij}^2 - i^2\beta^2\mu - \theta^2) + A_{ij}\mu_1]B_{1ij} + [\theta(\psi_{ij}^2 - i^2\beta^2\mu) - \theta^3]B_{2ij} =$$
$$= \frac{16\alpha_{ij}P_t}{\pi^2 i\cdot j(m+m_p)}. \tag{27}$$

$$[\alpha_{ij}(\psi_{ij}^2 - i^2\beta^2\mu) + A_{ij}\mu_1]B_{3ij} = \frac{16\alpha_{ij}P_0}{\pi^2 i\cdot j(m+m_p)}.$$

The frequencies of natural vibrations of the system are obtained by solving the characteristic equation.

Let the approximate values of the roots of the characteristic equation at $ij = 1$ be, according to Volterra /9/,

$$x_1 = -r_1 = -\frac{(\psi_1^2 - \beta^2\mu)\alpha_1 + A_1\mu_1}{\psi_1^2 - \beta^2\mu},$$

$$x_2 = -s_1 + i\omega_1 = -\frac{1}{2}\frac{A_1\mu_1\alpha_1}{\psi_1^2 - \beta^2\mu} + i\sqrt{\psi_1^2 - \beta\mu},$$

$$x_3 = -s - i\omega = -\frac{1}{2}\frac{A_1\mu_1\alpha_1}{\psi_1^2 - \beta^2\mu} - i\sqrt{\psi_1^2 - \beta^2\mu},$$

where i is the imaginary unit. The natural frequency, equal to the imaginary term, will be

$$\omega_1 \cong \sqrt{\frac{(D_1 + D)\pi^4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 + k}{m + m_p} - \frac{v_x^2\pi^2 m_p}{a^2(m + m_p)}}. \tag{28}$$

It is seen from formula (28) that the moving external load reduces the natural frequencies of the system, and that the larger the mass of the load and the larger its velocity, the larger is the period of natural vibrations.

Since the road is used by heavy vehicles traveling at high speeds, it is impossible to neglect the mass and velocity of the moving load in the calculation of concrete coatings of highways, all the more so since a majority of the vehicles using the road have a weight and mass several times exceeding the weight and mass of the plate itself with the foundation.

The amplitude of the dynamic deflection at the middle of the plate under the action of load (20) is, for the first vibration modes,

$$R = \sqrt{B_1^2 + B_2^2} + \delta. \tag{29}$$

The first term represents the amplitude caused by the fluctuating component of the external load, and the second, the amplitude caused by the constant component of the load.

By substituting in (29) the values of B_1 and B_2 found from (27), and the value

$$\delta = \frac{16P_0}{\pi^2(m+m_p)} \cdot \frac{\alpha_1}{\psi_1^2\alpha_1 + A_1\mu_1 - \theta^2(1+\sigma^2\mu)\alpha_1},$$

we obtain the following expression for the amplitude of the forced vibrations:

$$R = \frac{16}{\pi^2(m+m_p)} \cdot \frac{P_t\sqrt{\theta^2+\alpha_1^2}}{\sqrt{\theta^2[\psi_1^2-\theta^2(1+\sigma^2\mu)]^2+[\psi_1^2\alpha_1-\alpha_1\theta^2(1+\sigma^2\mu)+A_1\mu_1]^2}} + \\ + \frac{16}{\pi^2(m+m_p)} \cdot \frac{P_0\alpha_1}{\psi_1^2\alpha_1 + A_1\mu_1 - \theta^2(1+\sigma^2\mu)\alpha_1}, \tag{30}$$

where

$$\sigma = \frac{\beta}{\theta} = \frac{\pi D_0}{2a}.$$

It is seen from (30) that the phenomenon of resonance, as met in absolutely elastic media, cannot occur here. In fact, for the amplitude R to increase to infinity for some values of the frequency θ it is necessary that the denominator in (30) become equal to zero; this is possible only when the following equalities are simultaneously fulfilled:

$$\psi_1^2 - \theta^2(1+\sigma^2\mu) = 0, \quad \psi_1^2\alpha_1 + A_1\mu_1 - \theta^2(1+\sigma^2\mu)\alpha_1 = 0,$$

These two equalities cannot occur at the same time since $\psi_1^2 \neq \psi_1^2\alpha_1 + A_1\mu$.

Note further that, as seen from (24), the rheological properties of the foundation lead to rapid damping of the free vibrations, so that the system considered actually lies in the region of stable vibrations. Cases of road vibrations with beats are possible when a column of cars moves along the road.

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ON THE FLUTTER OF ELASTIC CYLINDRICAL COAXIAL SHELLS BETWEEN WHICH FLOWS A COMPRESSIBLE FLUID

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The flutter and stability of coaxial elastic circular cylindrical shells of infinite length in the stream of an ideally compressible gas are considered. The gas stream flowing between the cylindrical shells is considered as potential, and the disturbances are considered as small. It is assumed that the equations of the moment shell theory are applicable to the shells.

Consider two coaxial shells (Figure 1). The equations of motion of each of them can be written in the following form /1/ in a cylindrical system of coordinates:

$$\begin{gathered}
\left(\frac{\partial^2}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial s^2}\right)u+\left(\frac{1+\mu}{2}\frac{\partial^2}{\partial x\partial s}\right)v+ \\
+\left[\frac{\mu}{R}\frac{\partial}{\partial x}-a^2R\left(\frac{\partial^3}{\partial x^3}-\frac{1-\mu}{2}\frac{\partial^3}{\partial x\cdot\partial s^2}\right)\right]w=\frac{1-\mu^2}{E}\rho_0\frac{\partial^2 u}{\partial t^2}; \\
\left(\frac{1+\mu}{2}\frac{\partial^2}{\partial x\cdot\partial s}\right)u+\left(\frac{1-\mu}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial s^2}\right)v+ \\
+\left(\frac{1}{R}\frac{\partial}{\partial s}-\frac{3-\mu}{2}a^2R\frac{\partial^3}{\partial x^2\partial s}\right)w=\frac{1-\mu^2}{E}\rho_0\frac{\partial^2 v}{\partial t^2}; \qquad (1) \\
\left[\frac{\mu}{R}\frac{\partial}{\partial x}-a^2R\left(\frac{\partial^3}{\partial x^3}-\frac{1-\mu}{2}\frac{\partial^3}{\partial x\cdot\partial s^2}\right)\right]u+ \\
+\left(\frac{1}{R}\frac{\partial}{\partial s}-\frac{3-\mu}{2}a^2R\frac{\partial^3}{\partial x^2\partial s}\right)v+\left[a^2\left(R^2\nabla^2\nabla^2+2\frac{\partial^2}{\partial s^2}+\right.\right. \\
\left.\left.+\frac{1}{R^2}\right)+\frac{1}{R^2}\right]w=-\frac{1-\mu^2}{Eh}\left(\rho_0 h\frac{\partial^2 w}{\partial t^2}\mp p\right).
\end{gathered}$$

Here u, v, and w are displacements of the middle surface; μ is the Poisson ratio; E is the modulus of elasticity; h is the shell thickness; ρ_0 is the density of the shell material;

$$s=R\theta; \qquad a^2=\frac{h^2}{12R^2}; \qquad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial s^2};$$

the upper sign in the right-hand part of the third equation corresponds to the inner shell, and the lower sign to the outer shell.

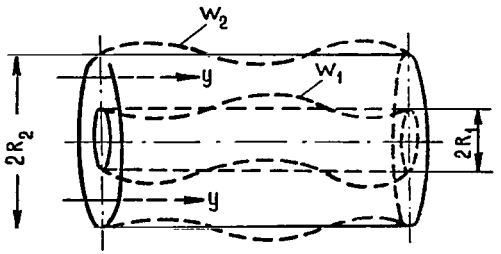

FIGURE 1.

The disturbance potential φ for the gas stream satisfies the well-known linearized equation

$$(1-M^2)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}-\frac{2M}{c_\infty}\frac{\partial^2\varphi}{\partial x\cdot\partial t}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=0, \quad (2)$$

where c_∞ is the velocity of sound in the undisturbed stream; $M=U/c_\infty$; U is the velocity of the gas stream.

The fluttering walls must fulfill the conditions of unpenetrability

$$\frac{\partial\varphi}{\partial r}=U\frac{\partial w_1}{\partial x}+\frac{\partial w_1}{\partial t}\quad (r=R_1+0);\quad \frac{\partial\varphi}{\partial r}=U\frac{\partial w_2}{\partial x}+\frac{\partial w_2}{\partial t}\quad (r=R_2-0), \quad (3)$$

and the pressures p_1 and p_2 can be determined from the formulas following from the Cauchy integral for an unsteady potential flow:

$$\begin{aligned} p_1&=\rho\left(\frac{\partial\varphi}{\partial t}+U\frac{\partial\varphi}{\partial x}\right) \qquad (r=R_1+0),\\ p_2&=\rho\left(\frac{\partial\varphi}{\partial t}+U\frac{\partial\varphi}{\partial x}\right) \qquad (r=R_2-0)\end{aligned} \quad (4)$$

(ρ is the gas density).

Consider a shell of infinite length. We shall look for a solution in the class of waves propagating along the shells. We can write, therefore /1/,

$$\begin{aligned} &\varphi=f(r)\,e^{i(\Omega t-kx)}\cos n\theta,\\ &u_1=i\xi_1e^{i(\Omega t-kx)}\cos n\theta,\quad u_2=i\xi_2e^{i(\Omega t-kx)}\cos n\theta,\\ &v_1=\eta_1e^{i(\Omega t-kx)}\sin n\theta,\quad v_2=\eta_2e^{i(\Omega t-kx)}\sin n\theta,\\ &w_1=\zeta_1e^{i(\Omega t-kx)}\cos n\theta,\quad w_2=\zeta_2e^{i(\Omega t-kx)}\cos n\theta.\end{aligned} \quad (5)$$

The substitution of (5) in the linearized potential equation leads to the equations

$$\begin{aligned} &\frac{d^2f}{dr_0^2}+\frac{1}{r_0}\frac{df}{dr_0}-\left(1+\frac{n_2}{r_0^2}\right)f=0 \qquad (M_1<1),\\ &\frac{d^2f}{dr_0^2}+\frac{1}{r_0}\frac{df}{dr_0}+\left(1-\frac{n^2}{r_0^2}\right)f=0 \qquad (M_1>1),\end{aligned} \quad (6)$$

where

$$M_1=\frac{|U-V|}{c_\infty};\quad V=\frac{\Omega}{k};$$

V is the propagation velocity of the elastic wave in the shell

$$r_0 = \nu r; \qquad \nu = k\sqrt{|M_1^2 - 1|}.$$

The solutions of equations (6) are expressed through Bessel functions of an imaginary ($M_1 < 1$) or real ($M_1 > 1$) argument:

$$\begin{aligned} f(r_0) &= C_1 I_n(r_0) + C_2 K_n(r_0) \qquad (M_1 < 1), \\ f(r_0) &= C_1 J_n(r_0) + C_2 N_n(r_0) \qquad (M_1 > 1). \end{aligned} \tag{7}$$

Consider the conditions at the boundaries of the gas stream. The constants C_1 and C_2 can be determined from the conditions of unpenetrability of the cylinder walls, and as a result the gas pressure on the walls can be expressed through the displacements w_1 and w_2:

$$\begin{aligned} p_1 &= -\rho R_1 \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\alpha_{11} w_1 + \alpha_{12} w_2), \\ p_2 &= \rho R_2 \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\alpha_{21} w_1 + \alpha_{22} w_2). \end{aligned} \tag{8}$$

The coefficients α_{ik} in these formulas are expressed through Bessel functions:

$$\frac{1}{\alpha_{11}} = -n + \nu R_1 \frac{\frac{\nu R_2}{n} A'_n + B'_n}{\frac{\nu R_2}{n} T'_n + S'_n}; \quad \frac{1}{\alpha_{22}} = -n + \nu R_2 \frac{-\frac{\nu R_1}{n} A'_n + T'_n}{\frac{\nu R_1}{n} B'_n - S'_n}$$

$$(M_1 < 1),$$

$$\frac{1}{\alpha_{11}} = -n - \nu R_1 \frac{\frac{\nu R_2}{n} A_n + B_n}{\frac{\nu R_2}{n} T_n + S_n}; \quad \frac{1}{\alpha_{22}} = -n - \nu R_2 \frac{-\frac{\nu R_1}{n} A_n + T_n}{\frac{\nu R_1}{n} B_n + S_n} \tag{9}$$

$$(M_1 > 1),$$

$$\begin{aligned} A_n &= J_{n-1}(\nu R_2) N_{n-1}(\nu R_1) - N_{n-1}(\nu R_2) J_{n-1}(\nu R_1), \\ B_n &= N_n(\nu R_2) J_{n-1}(\nu R_1) - J_n(\nu R_2) N_{n-1}(\nu R_1), \\ T_n &= N_{n-1}(\nu R_2) J_n(\nu R_1) - J_{n-1}(\nu R_2) N_n(\nu R_1), \\ S_n &= J_n(\nu R_2) N_n(\nu R_2) - N_n(\nu R_2) J_n(\nu R_1), \\ A'_n &= I_{n-1}(\nu R_1) K_{n-1}(\nu R_2) - K_{n-1}(\nu R_1) I_{n-1}(\nu R_2), \\ B'_n &= I_{n-1}(\nu R_1) K_n(\nu R_2) + K_{n-1}(\nu R_1) I_n(\nu R_2), \\ T'_n &= I_n(\nu R_1) K_{n-1}(\nu R_2) + K_n(\nu R_1) I_{n-1}(\nu R_2), \\ S'_n &= I_n(\nu R_1) K_n(\nu R_2) - K_n(\nu R_1) I_n(\nu R_2). \end{aligned}$$

We shall consider the case when one of the two shells is absolutely rigid.

The equations of motion of the elastic shell can then be determined by substituting (5) and (8) in (1):

$$\sum_{k=1}^{3} \left[\frac{E}{\rho_0 R^2 (1-\mu^2)} a_{jk} - \Omega^2 \delta_{jk} - \chi(\Omega - kU)^2 \delta_{jk} \right] u_k = 0. \tag{10}$$

Here

$$a_{11} = m^2 + \frac{1-\mu}{2} n^2, \quad a_{22} = \frac{1-\mu}{2} m^2 + n^2,$$

$$a_{33} = a^2 [(m^2 + n^2)^2 - 2n^2 + 1] + 1,$$

$$a_{12} = a_{21} = \frac{1+\mu}{2} mn, \quad a_{13} = a_{31} = m\mu + a^2 m \left(m^2 - \frac{1-\mu}{2} n^2 \right),$$

$$a_{23} = a_{32} = n + \frac{3-\mu}{2} a^2 m^2 n,$$

$$m = kR = \frac{\pi R}{\lambda},$$

$$u_j = (\xi, \eta, \zeta), \quad \delta_{j3} = (0, 0, 1), \quad \chi = \frac{\rho R}{\rho_0 h} \alpha$$

(λ is the half-wave length).

The frequencies of the flutter of the elastic shell in the gas stream are found from the equation

$$\left| \frac{E a_{jk}}{\rho_0 R^2 (1-\mu^2)} - \Omega^2 \delta_{jk} - \chi (\Omega - kU)^2 \delta_{j3} \right| = 0, \tag{11}$$

which can be reduced to the form (cf. /1/)

$$\sum_{\alpha=1}^{3} \frac{\gamma_\alpha}{\Omega_\alpha^2 - \Omega^2} = \frac{1}{\chi (\Omega - kU)^2}, \text{ where } \gamma_\alpha = \frac{1}{\xi_\alpha^2 + \eta_\alpha^2 + 1}. \tag{12}$$

Here ξ_α, η_α, and $\zeta_\alpha = 1$ are elements of a matrix formed from the coefficients for the modes of natural vibrations of the shell in vacuum.

When the wave numbers n are sufficiently large one of the natural frequencies (corresponding to mainly bending vibrations) is considerably smaller than the two others:

$$\Omega_1^2 \ll \Omega_2^2, \quad \Omega_1^2 \ll \Omega_3^2, \quad \gamma_1 \approx 1.$$

In this case equation (12) can be represented in the form

$$\Omega_1^2 - \Omega^2 - \chi (\Omega - kU)^2 = 0. \tag{13}$$

The natural frequency Ω_1 can then be determined by the formula

$$\Omega_1 = \frac{k}{R} \sqrt{\frac{D}{\rho_0 h}} F_1 (m, n),$$

where

$$F_1 (m, n) = \sqrt{\frac{(m^2 + n^2)^2}{m^2} + \frac{1-\mu^2}{a^2} \frac{m^2}{(m^2 + n^2)^2}}. \tag{14}$$

The parameter χ is a complex transcendental function of the stream velocity U and the frequency Ω. The numerical solution of the frequency equation of a shell interacting with a flowing gas can be obtained by the method of successive approximations.

We shall now transform the equations obtained into a form suitable for graphic solution. We take into account that

$$(\Omega - kU)^2 = k^2 c_\infty^2 M_1^2,$$
$$M_1^2 = \frac{(kR)^2 \mp (\nu R)^2}{(kR)^2}. \tag{15}$$

The upper sign corresponds to the case $M_1 < 1$, and the lower sign to $M_1 > 1$.

Introduce the designation $\Omega/\Omega_\alpha = \varphi_\alpha$ ($\alpha = 1, 2, 3$). Equation (12) can now be reduced to the form

$$\alpha(\nu R) = \frac{E}{2(1+\mu)\rho c_\infty^2} \cdot \frac{h}{R} \cdot \frac{1}{(kR)^2 \mp (\nu R)^2} \cdot \frac{1}{\left[\sum_{\alpha=1}^{3} \frac{\gamma_\alpha}{\bar{\Omega}_\alpha^2 (1-\varphi_\alpha^2)}\right]},$$
$$\bar{\Omega}_\alpha \cdot \frac{1}{R}\sqrt{\frac{E}{2(1+\mu)\rho_0}} = \Omega_\alpha, \tag{16}$$

and equation (13), correct at large n, to the form

$$\alpha(\nu R) = \frac{E}{(1-\mu^2)\,12}\left(\frac{h}{R}\right)^3 \frac{1}{\rho c_\infty^2} \cdot \frac{(kR)^2}{(kR) \mp (\nu R)^2}(1-\varphi_1^2) F_1^2. \tag{17}$$

The left-hand parts of equations (16) and (17) are functions of the parameters νR and n, and the right-hand parts of the parameters νR, kR, φ_α. Using equations (16) and (17), we determine in a graphical or tabular manner the frequencies Ω corresponding to any fixed values n, νR, and kR. The relative velocity of the gas stream $\frac{U}{c_\infty}$, corresponding to the problem considered, is determined, as follows from (15), from the relationship

$$\frac{U}{c_\infty} = \left|\frac{\Omega_1 \varphi_1}{k c_\infty} \pm M_1\right|. \tag{18}$$

The upper sign in (18) corresponds to the forward-facing wave, and the lower sign to the backward-facing wave.

Assume some specific values for n and kR. Assuming different values in turn for νR, and plotting the curve $U/c_\infty = f(\Omega)$ with the aid of (16), (17), and (18), we can find the largest velocity of the gas stream (U^{**}) for which the frequencies remain real. The critical velocity can be found by minimizing the solution by the parameters kR and n.

The critical gas velocities were calculated for shells of thickness $0.01R \leqslant h \leqslant 0.03R$ for ratios of the radii of the coaxial shells R_2/R_1 = 1.1, 1.5, 3.

As an example, the results determining the critical velocity at $h = 0.01R$, $R_2/R_1 = 3$, and an absolutely rigid outer shell are represented in Figure 2. The unbroken curves $M^{**} = U^{**}/c_\infty = f_1(kR)$ were plotted by (13). The curve portions for n = 1, 2 at $kR \leqslant 3.14$. are an exception: the values of M^{**} for them were calculated both by (13) (the broken lines) and by (12) (the unbroken lines).

The dotted lines in Figure 2 represent the results of the calculation of M^{**} under the assumption $M_1 = 1$:

$$M^{**} = \frac{U^{**}}{c_\infty} = \frac{\Omega_1}{k c_\infty} + 1.$$

Here $\frac{\Omega_1}{k}$ is the propagation velocity of the elastic wave for a shell in vacuum.

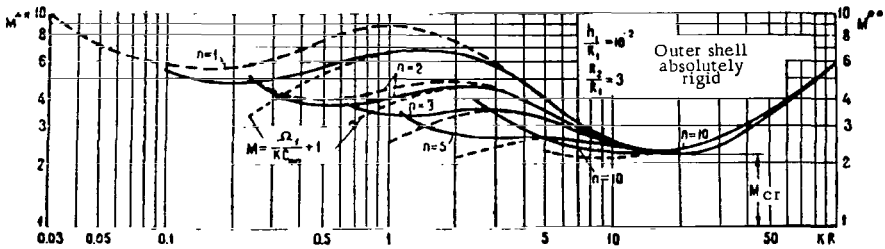

FIGURE 2.

The following conclusions can be drawn from these results:

1. In the range of values R_2/R_1 studied the critical velocity is practically independent of the value of the ratio R_2/R_1.

2. At equal values of h/R the case of rigid inner shell and the case of rigid outer shell are characterized by almost equal critical velocities.

3. In the case of not too slender shells $(h/R \geqslant 0.01)$ the critical velocity can be found with a sufficient degree of accuracy by the Miles formula /3/:

$$M_{cr} = \frac{V_{min}}{c_\infty} + 1,$$

where V_{min} is the minimum propagation velocity of the elastic waves in the shell.

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ON THE GEOMETRY OF HANGING ROOFINGS WITH A RADIAL GUY NET AND MOMENTLESS CONTOUR RING

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1. STATEMENT OF THE PROBLEM

The interest in hanging roofs is increasing among building engineers /1/. This paper treats the question of hanging roofings in which the basic load-carrying element is a guy net with radial disposition of the guys in the plan. The behavior of the radial nets under static strain distinguishes them favorably from the other types of guy nets. An elementary calculation shows that, all other conditions being equal (same contour in the plan, same sag, distributed load of the same intensity), one and a half times less material is necessary for a radial net than for a cross net with guys arranged orthogonally in the plan.

The radial net gives a maximum effect when no moments in the horizontal direction act on the contour ring limiting it, to which the guys are fastened, at least under the basic type of loading. The object of the present paper is to determine the geometrical form of hanging roofings with a radial guy net and a momentless contour ring of arbitrary outline in the plan.

The following assumptions are made:

1) the radial net is considered as a set of flexible filaments unconnected in the peripheral direction and arranged continuously along the angular coordinate;

2) the contour ring is considered as plane and undeformable, and the inner ring of the net is treated as a nodal point (net node).

2. CONDITION OF MOMENTLESSNESS OF A NET OF ARBITRARY SHAPE

Let a plane ring of arbitrary shape be loaded by a system of arbitrarily directed forces per unit length H_s acting in the ring plane. In the absence of bending moments and shearing forces the conditions of equilibrium of a ring element will be, in projections on the tangent and normal (Figure 1):

$$\begin{cases} \dfrac{dN}{ds} + H_s \cos\gamma = 0, \\ N - \rho H_s \sin\gamma = 0. \end{cases} \tag{2.1}$$

Here N is the longitudinal force in the ring, ρ is the radius of curvature of the ring axis, and γ is the angle between the direction of the force H_s and the positive direction of the tangent.

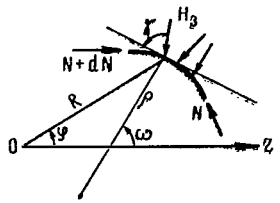

FIGURE 1,

The elimination of N from system (2.1) yields

$$\frac{d}{ds}(\rho H_s \sin\gamma) + H_s \cos\gamma = 0, \tag{2.2}$$

The general integral of this equation is

$$H_s = \frac{Ce^{-\int \operatorname{ctg}\gamma d\omega}}{\rho \sin\gamma}, \tag{2.3}$$

where

$$d\omega = \frac{ds}{\rho}. \tag{2.4}$$

If the forces H_s are everywhere directed along the normal to the ring axis, then by writing in (2.3) $\gamma = \pi/2$, we obtain the following obvious result:

$$\rho H_s = N = C, \tag{2.5}$$

i.e., in the absence of a tangential component of the load the longitudinal force appearing in the ring will be constant.

In the case of a central force field, when the lines of action of all the forces intersect at the same point (pole), we have

$$\omega = \varphi + \gamma - \frac{\pi}{2}. \tag{2.6}$$

On the strength of this equality and of the following relationship, known from differential geometry,

$$\frac{dR}{d\varphi} = R\operatorname{ctg}\gamma \tag{2.7}$$

we can calculate the integral in (2.3); we obtain the following expression for the radial thrust per unit length

$$H_s = \frac{C}{\rho R \sin^2\gamma}. \tag{2.8}$$

Using the formulas of differential geometry it is possible to pass from here to the thrust per unit polar angle:

$$H_\varphi = H_s \frac{ds}{d\varphi} = \frac{C}{\rho \sin^2\gamma} \tag{2.9}$$

or

$$H_\varphi = C\,\frac{R^2 + 2R'^2 - RR''}{R^3}, \tag{2.10}$$

Relationships (2.8) to (2.10) give the law of loading necessary and sufficient for the equilibrium of a momentless ring of arbitrary shape in the case of an arbitrarily located pole of radial forces. In addition, relationship (2.10), considered as a differential equation, makes it possible to solve the inverse problem: *for a given system of radial forces, to find the outline of the axis of a momentless ring* $R = R(\varphi)$; this fact will be used below.

3. SOME PARTICULAR CASES

The results obtained will be used for determining the shape of a momentless contour ring of a uniformly loaded radial net in which the guys in the deformed state horizontally approach the net node (Figure 2). The condition of equilibrium of a sector of unit central angle yields

$$H_\varphi = \frac{qR^3}{6f}, \tag{3.1}$$

where q is the load per unit area of the horizontal projection of the net and f is the sag.

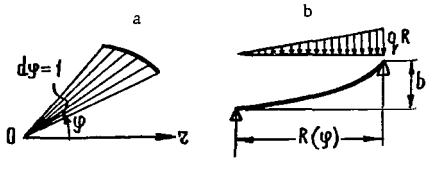

FIGURE 2.

The substitution of (3.1) in (2.10) leads to the following differential equation:

$$\bar{C}R^6 = R^2 + 2R'^2 - RR''. \tag{3.2}$$

This equation is reduced by a number of substitutions and transformations into an Abel differential equation of the second kind /2/, of a particular type which allows separation of the variables. The solution of (3.2) can finally be represented in the following form:

$$R(\varphi) = C_1 \sqrt{\frac{1}{1 - \sqrt{1 - C_2}\,\sin 2(\varphi + \varphi_0)}}. \tag{3.3}$$

Relationship (3.3) describes in polar coordinates a three-parameter family of integral curves of equation (3.2). At $0 < C_2 < 1$ this will be a family of ellipses of arbitrary shape and arbitrary orientation with common center at the origin of coordinates; at $C_2 < 0$ this will be a similar family of hyperbolas.

The conclusion relative to the momentlessness of a contour ring of elliptic outline is of basic interest. Since this result was obtained by

integrating a differential equation describing all shapes of momentless ring, it can be asserted that the ellipse represents the only shape of compressed momentless contour ring of a radial net of the type considered (Figure 3, a).

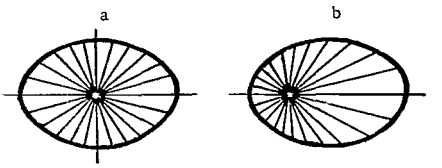

FIGURE 3.

As a second example we shall determine the outline of the momentless contour ring of a radial net with constant thrust per unit angle. By writing H_φ = constant, in (2.10), we obtain

$$\tilde{C}R^3 = R^2 + 2R'^2 - RR''. \tag{3.4}$$

This equation can be integrated just as (3.2); we obtain

$$R(\varphi) = \frac{C_1}{1 - \sqrt{1 - C_2 \sin(\varphi + \varphi_0)}}. \tag{3.5}$$

The integral curves represent a three-parameter family of ellipses ($0 < C_2 < 1$) or hyperbolas ($C_2 < 0$) of arbitrary shape and arbitrary orientation, with common focus at the origin of coordinates.

The ellipse is thus the only shape of compressed momentless contour ring of a radial net in which the thrust per angular unit is constant, and the projection of the node in the plan coincides with one of the foci of the ellipse (Figure 3,b).

4. GEOMETRICAL PARAMETERS OF THE HANGING ROOFING

The law of thrust distribution (2.9) necessary to ensure the momentlessness of a contour ring of arbitrary shape can be realized by means of a special selection of the geometrical parameters of the guys. Consider the equilibrium of a radial guy net in its deformed state under the action of an arbitrary vertical load $q(r, \varphi)$. Removing a sector of central angle $d\varphi = 1$ from the net, we write the condition that the moment of all the forces relative to the axis of the contour ring is equal to zero (Figure 4).

$$H_\varphi f - H_\varphi \beta R - S_\varphi = 0. \tag{4.1}$$

Here $\beta(\varphi) \simeq \operatorname{tg} \beta$ is the slope of the tangent to the line of equilibrium of the guys in the net node and S_φ is the static moment of the external load relative to the axis of the contour ring.

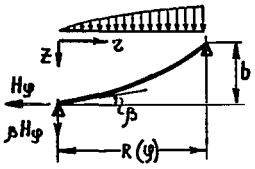

FIGURE 4.

The value of the thrust per unit angle will be

$$H_\varphi = \frac{S_\varphi}{R\vartheta}, \tag{4.2}$$

where

$$\vartheta = \frac{f}{R} - \beta = \eta - \beta. \tag{4.3}$$

The geometrical parameters at our disposal η and β can always be selected in such a way as to obtain the law of thrust distribution required. What is more, these parameters create a surplus of arbitrary elements, which gives a certain freedom in the selection of the roofing contour.

At given outline of the contour ring axis and position of the net node, the parameter $\eta(\varphi) = \frac{f}{R}$ is strictly determined, and we obtain in accordance with (4.2) and (4.3)

$$\beta(\varphi) = \eta - \vartheta = \frac{f}{R} - \frac{S_\varphi}{RH_\varphi}. \tag{4.4}$$

The value of H_φ entering here, found from (2.9), contains an arbitrary constant C, which must be calculated from the condition of net equilibrium in the vertical direction:

$$\int_0^{2\pi} H_\varphi \beta(\varphi)\, d\varphi = F_0. \tag{4.5}$$

Here F_0 is the external concentrated load applied at the central node.

From (4.4) and (4.5) we obtain

$$C = \frac{F}{A}, \tag{4.6}$$

where

$$F = F_0 + \int_0^{2\pi} \frac{S_\varphi}{R}\, d\varphi, \tag{4.7}$$

$$A = \int_0^{2\pi} \frac{\eta}{\rho \sin^3 \gamma}\, d\varphi. \tag{4.8}$$

The parameter $\beta(\varphi)$ is completely determined from the formulas obtained.

As an example we shall calculate the parameter $\beta(\varphi)$ for a uniformly loaded roofing of elliptic outline in the plan with radial net node at the ellipse focus. Formula (4.7) gives under a uniform load

$$F = \frac{qB}{3} = \frac{q\pi ab}{3},$$

where $B = \pi ab$ is the ellipse area.

In a polar system of coordinates with pole at the ellipse focus we can write

$$R = \frac{p}{1 + e\cos\varphi}, \qquad \rho \sin^2 \gamma = p,$$

where $p = b^2/a$ is the latus rectum and $e = \sqrt{a^2 - b^2}/a$ is the eccentricity.

With the aid of these equalities we find by (4.8)

$$A = \int_0^{2\pi} \frac{f}{Rp} = \frac{2\pi f}{p^2},$$

after which we obtain

$$C = \frac{qp^2ab}{6f}.$$

The magnitudes S_φ and H_φ in (4.4) acquire the following values:

$$S_\varphi = \frac{qR^3}{6}, \qquad H_\varphi = \frac{C}{p} = \frac{qpab}{6f}.$$

We finally find by (4.4)

$$\beta(\varphi) = \frac{f}{R}\left(1 - \frac{R^3}{b^3}\right).$$

This is the law according to which the guy slope at the net node must vary in order that the plane elliptic contour ring be momentless in the case of uniform load on the net.

The case when the shape of the contour ring in the plan is given graphically is of considerable practical interest. All the necessary calculations are conducted in this case numerically, and to that end the ring axis is split into small segments, and the values of $R(\varphi_i)$ are measured directly from the drawing for each division point. The values of ρ and $\sin\gamma$ can either be taken from the drawing or determined from the formulas of differential geometry using difference differentiation. The subsequent calculations present no basic difficulties.

Note that the results obtained can be approximately extended to the case of a deplanated contour ring, provided that the magnitude of the deplanation is not large. In this case all the formulas remain in force, and only f will become a function of the angular coordinate φ. If, at given shape of the roofing in the plan, we require that $\beta(\varphi)=0$, then the momentlessness of the contour ring can be achieved, generally speaking, only by its deplanation.

We have in fact for $\beta = 0$ from (3.4)

$$f(\varphi) = \frac{S_\varphi}{H_\varphi}. \tag{4.9}$$

5. ALLOWING FOR THE ELASTIC DISPLACEMENTS AND OPTIMUM STATE OF THE GUY NET

The geometrical parameters of the roofing were determined for the deformed state, and were accordingly based on relationships obtained under the assumption of unextensible guys. In the case considered this is not a simplifying hypothesis, which facilitates the solution while making it less rigorous. In fact, in accordance with the principle of hardening of the theory of flexible filaments, the flexible filament can be considered in its final deformed state as unextensible. The thrust is determined in this case by the conditions of statics only, and there are no displacements or strains. It follows from here, on the one hand, that a complete momentlessness of the contour ring can be achieved in the general case only for one specific load on the roofing; the variation of the law of distribution, or even of the intensity, of the load, and the kinematic and elastic displacements of the guys resulting from this variation alter the pattern of the diagrams of the thrusts transmitted to the contour ring. On the other hand, by applying the hardening principle to the state of the guy net at which there is no bending in the ring (this state will be called the optimum state), we can achieve the momentlessness of the ring at any specified load on the roofing. It is true that the question arises of determining the initial (unloaded) position of the guy net from which it would have passed into the necessary optimum state under the action of the specified load (and allowing for the elastic elongations).

This initial position of the net is realized by fixing the corresponding initial lengths of the guys, which differ from the lengths in the optimum state by the magnitude of the elastic elongations. When the forces and sections of the guys are known, the elastic elongations are easily determined; there remains to calculate the length of the guys in the optimum state from the given geometrical parameters of the net. In the cases interesting us the lines of equilibrium of the guys have small slope, and the following approximate formula of the arc length can therefore be used:

$$L = \int_0^R \sqrt{1 + \left(\frac{dz}{dr}\right)^2}\, dr \simeq \int_0^R \left[1 + \frac{1}{2}\left(\frac{dz}{dr}\right)^2\right] dr, \tag{5.1}$$

where

$$z(r) = f\left(1 - \frac{r}{R}\right) + \frac{M_b}{H}. \tag{5.2}$$

Here M_b is the bending moment under the action of the external load, calculated as for a simply supported beam.

We shall restrict ourselves to the case of a uniform load on the net, for which

$$z(r) = R\left(1 - \frac{r}{R}\right)\left[\eta + \vartheta \frac{r}{R}\left(1 + \frac{r}{R}\right)\right]. \tag{5.3}$$

By substituting (5.3) in (5.1) we obtain the following expression, sufficiently accurate for practical calculations, of the guy length through its geometrical parameters

$$L \simeq R\left(1 + \frac{2}{5}\vartheta^2 + \frac{1}{2}\eta^2\right). \tag{5.4}$$

The case of two-belt prestressed radial guy systems is somewhat more complex. By using the hardening principle it is, however, possible to determine in this case as well, without particular difficulties, the geometrical parameters of the optimum state.

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ON THE BEST INITIAL APPROXIMATIONS IN THE EIGENVALUES PROBLEM OF THE RITZ AND BUBNOV-GALERKIN METHODS

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The problem of finding the best approximations for the minimum eigenvalue in the Ritz and Bubnov-Galerkin methods is considered for some cases when the solution by these methods is considerably complicated by the necessity of taking a large number of terms of expansion (1).

When solving problems by the Ritz or the Bubnov-Galerkin method, the solution is represented in the form of a linear combination of the coordinate functions φ_k:

$$\varphi = \sum_{k=1}^{n} a_k \varphi_k , \tag{1}$$

where a_k are some parameters which must be determined. Usually, some known complete system of linearly independent functions is selected as system of coordinate functions. The summation in expansion (1) is then conducted in the order of increasing index k. If thus $\{\varphi_k\}$ is $\left\{\sin \frac{k\pi x}{l}\right\}$, then

$$\varphi = a_1 \sin \frac{\pi x}{l} + a_2 \sin 2 \frac{\pi x}{l} + \cdots$$

In some cases this standard approach corresponds satisfactorily to the exact solution even in the initial approximations, while in other cases this is not so at all; thus, if the solution represents a large number of waves in the given region, the approximation of such a function by the initial harmonics leads to very unsatisfactory results. It is then necessary to take a large number of terms in (1), as a result of which the solution of the problem frequently becomes very complicated.

Many examples can be given when the function can be approximated with a sufficient accuracy by several harmonics, not necessarily of low indices

Bearing in mind several problems of shell stability under the action of variable loads, it is natural to adopt a method such that the fundamental harmonics are determined in the solution. Problems of this type are met frequently in aircraft engineering and other branches of engineering.

In expansion (1) there are usually principal functions (principal harmonics), i.e., functions nearest the exact solution. If we establish these principal harmonics, the corresponding approximations will naturally be the best ones.

There arises therefore the question how to find these principal functions. The following approach is thus proposed.

In contradistinction to the usual procedure, we do not fix beforehand the indices of the coordinate functions. Let the indices be arranged in some unknown sequence (n_k). The function φ is then represented as follows:

$$\varphi = \sum_{k=1}^{p} a_k \varphi_{n_k}. \tag{2}$$

Next, by using the condition of functional minimum of the Ritz method, or the condition of orthogonality of the Bubnov-Galerkin method, we obtain the following system of algebraic equations:

$$\sum_{k=1}^{p} a_k \left[(A\varphi_{n_k}, \varphi_{n_m}) - \lambda (B\varphi_{n_k}, \varphi_{n_m})\right] = 0 \quad (m=1, \cdots, p), \tag{3}$$

where A and B are some operators of the equation to be solved.

From the condition of nontriviality of (2) (not all $a_k = 0$) we obtain that the determinant of the homogeneous algebraic system (3) must be equal to zero, i.e., $\Delta = 0$.

The expansion of the determinant yields an equation for λ. In a p-th approximation we obtain p roots. The minimum root will be designated by $\lambda^{(p)} = f(n_1, \cdots, n_p)$.

As known, both the Ritz and the Bubnov-Galerkin methods give the eigenvalues with an excess, and in such a manner that

$$\lambda^{(1)} > \lambda^{(2)} > \cdots > \lambda^{(p)} > \cdots \to \lambda. \tag{4}$$

If then we take as first approximation one term of expansion (2), where n_1 is some variable index, we obtain on the strength of (4) (since $\lambda^{(1)} > \lambda$) that the best first approximation will be

$$\min \lambda^{(1)}(n_1) = \lambda^{(1)}(n_1^0),$$

and therefore n_1^0 corresponds to the principal harmonic nearest to the exact value.

If we take two terms in (2), the best second approximation will be (likewise from (4))

$$\min \lambda^{(2)}(n_1, n_2) = \lambda^{(2)}(\bar{n}_1^0, n_2^0),$$

where $\bar{n}_1^0$ and n_2^0 correspond to the pair of principal harmonics nearer the exact solution than all the other possible pairs, etc.

It follows that the determination of the best initial approximations of the eigenvalue and the principal harmonics of expansion (2) reduces to determining the indices n_k making the corresponding approximation minimum at a fixed number of harmonics.

In practical applications the physical pattern is frequently known. If it is known, for instance, that the bar, shell, or any other structure buckles with the formation of a large number of waves, then it is clear that the principal harmonics will be far from the initial harmonics, and the approach proposed will be suitable for obtaining satisfactory initial approximations.

The following approximate method for determining the indices of the principal harmonics can be proposed in this case.

We determine first the index of the first approximation n_1^0

$$\min \lambda^{(1)}(n_1)=\lambda^{(1)}(n_1^0);$$

n_1^0 corresponds to the principal harmonic nearer the exact solution than all the other harmonics; the other principal harmonics must therefore be in the region of this value. In a second approximation n_1 can therefore be considered as approximately equal to n_1^0, and we have to look for a minimum by the second index n_2 only:

$$\min \lambda^{(2)}(n_1^0, n_2)=\lambda^{(2)}(n_1^0, n_2^0).$$

The same procedure is used for the third approximation

$$\min \lambda^{(3)}(n_1^0, n_2^0, n_3)=\lambda^{(3)}(n_1^0, n_2^0, n_3^0),$$

etc.

If the problem solution necessitates taking into account two terms of series (2) in the first step, or if the determination of the minimum by two indices is not difficult, the iterative process begins with the determination of the minimum by two indices. Having determined n_1^0, n_2^0 from the condition of minimum of the corresponding approximation λ, we substitute them in the next approximation and determine the minimum of this approximation by n_3 only, etc.

Such was, for instance, the procedure used for solving the problem of the stability of a shell under the action of torques applied at the end. The unknown function w was represented as a double series

$$w=\sum_{m,k} a_{mn_k}\sin\frac{m\pi R}{l}\xi\sin\frac{n_k}{2}\theta, \tag{5}$$

which satisfies the boundary conditions. The indices m successively take the values 1, 2, 3, ..., while the indices (n_k) represent some unknown subsequence taken out of the natural series of numbers. Since the shell buckles at torsion in a quasisinusoidal mode with a large number of half-waves in the peripheral direction, the harmonics for which the numbers (n_k) will be near to the number of half-waves formed at buckling will naturally be the principal harmonics in expansion (5).

The following terms were considered on the basis of the data of the problem solution:

m	1	2	1	3
n_k	n_1	n_2	n_3	n_4

In the given case the problem solution necessitates allowing for two terms of series (5) in the first step. We obtain therefore the following expression in a first approximation:

$$\lambda^{(1)}=\sqrt{\frac{9}{8}K_{1n_1}K_{2n_2}\varphi(n_1, n_2)},$$

where $\varphi(n_1, n_2)=\frac{(n_2^2-n_1^2)^2}{4n_1n_2}$ and the given function is equal to zero at point $n_2 = n_1$. In addition, $\lambda'_{n_2}(n_1=\text{const}, n_2)>0$ and therefore the minimum value of $\lambda^{(1)}$ is attained at point $n_2=n_1+1$, where $\varphi(n_1, n_2)=1$. Taking this into

account, we find the minimum of $\lambda^{(1)}$ as a function of one index n_1 only.

In a second approximation we restrict ourselves to three terms in expansion (2). We obtain the following expression for λ:

$$\lambda^{(2)}=\sqrt{\frac{9}{8}K_{1n_1}K_{2n_2}\frac{1}{\left(1+\frac{K_{1n_1}}{K_{1n_2}}\right)}}.$$

A similar investigation shows that $\lambda^{(2)}$ becomes minimum at the point $n_3=n_1+2$.

In a third approximation, when we restrict ourselves to four terms:

$$\lambda^{(3)}=\sqrt{\frac{9}{8}K_{1n_1}K_{2n_2}\frac{1}{\left(1+\frac{K_{1n_1}}{K_{1n_2}}+1.08\frac{K_{1n_1}}{K_{3n_4}}\right)}}.$$

It is established in the same manner as above that $\lambda^{(3)}$ becomes minimum at the point $n_4=n_1$.

The calculations show that the critical load in the third approximation differs by 2 to 4% from the results of the known solution, while the value λ at the initial harmonics differs from this solution by a factor larger than two. The comparisons showed that the discrepancy between the value of the exact minimum of λ and the value of the minimum of λ obtained by the above-considered iterative method will be about 1 to 2%.

The main field of application of the method considered lies in the realm of stability problems. Many such problems have been solved by this method, in particular the problem of linearly distributed torques /4/.*

This method can likewise be used for determining the critical loads of shells subjected simultaneously to torsion and a variable pressure, since in such problems an increase in the number of terms of series (1) leads to an increase in the order of the corresponding secular equation, which considerably complicates the determination of the roots. (Another factor which makes the use of a large number of terms of the series necessary is the fact that the principal harmonics are in this case far from the initial ones.)

An additional corroboration of the fact that the principal harmonics can be far from the initial ones can be found in Flügge's problem of the stability of a cylindrical shell under the action of a variable axial load /5/. Flügge calculated the coefficients and determined through them the principal harmonics. In the method proposed there is no need for determining the coefficients of expansion (2): the principal harmonics are directly determined from the condition of minimum of the corresponding approximation of the eigenvalue.

The limits of applicability of the method proposed have not been determined so far; they seem, however, fairly wide both for static and for dynamic problems.

Thus, without pretending to the universality of the approach suggested, it can be shown that in many cases it considerably simplifies the solution, by reducing the number of terms taken into account without impairing the accuracy of the solution.

* Reported to the All-Union Symposium on the Theory of Plates and Shells held at Kazan' in 1960.

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SOME STABILITY PROBLEMS OF SANDWICH SHELLS

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Sandwich shells with a filler are usually treated in the literature as shallow shells. This assumption stems from the fact that interest has been paid mainly to stability problems, for which the theory of shallow shells is usually sufficiently accurate.

The development of the theory of nonshallow shells is of interest both for the study of problems of shell bending and for some special types of stability problems. Such is the problem of the buckling of a sandwich cylindrical shell under the action of an external pressure, under the simultaneous action of pressure and compression: the buckling of a cylindrical shell under compression with the formation of long waves.

The equations of the simple nonshallow shells forming the outer layers of the sandwich shell are derived here with a number of assumptions, which lead to Reissner's elasticity relationships /1/. These relationships are as simple as Love's relationships, and differ from them by the expression for torsion only. Unlike the equations to which Love's relationships lead, we obtain in this case the equations in displacements with a symmetrical matrix of coefficients.

The linear and nonlinear equations of simple nonshallow shells necessary for deriving the equations of a sandwich shell are obtained in the paper from the equations of the tridimensional theory of elasticity, allowing for the Kirchhoff-Love hypothesis and using the same system of approximations as in the derivation of the elasticity equations.

The shell strains are represented as follows:

$$\varepsilon_{11} = e_1 + \frac{1}{2}\psi_1^2 + z\chi_{11}, \qquad \varepsilon_{22} = e_2 + \frac{1}{2}\psi_2^2 + z\chi_{22}, \qquad \varepsilon_{12} = \omega + \psi_1\psi_2 + 2z\chi_{12}, \quad (1)$$

where

$$e_1 = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u_2 + \frac{w}{R_1}, \qquad \chi_{11} = \frac{1}{A_1}\frac{\partial \psi_1}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\psi_2,$$

$$e_2 = \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u_1 + \frac{w}{R_2}, \qquad \chi_{22} = \frac{1}{A_2}\frac{\partial \psi_2}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\psi_1, \quad (2)$$

$$\omega = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right), \qquad 2\chi_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\psi_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\psi_2}{A_2}\right)$$

Here u_1, u_2, w are the components of the displacements in an orthogonal curved system of coordinates; R_1, R_2, A_1, A_2 are the radii of the principal curvatures and Lamé coefficients of the middle surface of the shell;

ψ_1, ψ_2 are the angles of rotation of the normal to the middle surface

$$\psi_1 = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u_1}{R_1}, \qquad \psi_2 = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{u_2}{R_2}.$$

The stability equations of the shell are obtained by linearizing the equations of neutral equilibrium. In the case of a cylindrical shell these equations differ somewhat from the usual equations, due to the fact that here the expression for the torsion in the elasticity relationships is somewhat different from that given in /2/. However, this does not lead to any difference in the results obtained in solving specific problems of the theory of stability. The only exception is the case of longitudinal compression of a very long cylindrical shell buckling as a pole. The following expression is obtained in this case for the critical load:

$$T = T_{\mathrm{Eul}}\left[1 + \frac{1}{24\pi^2(1+\nu)}\frac{t^2}{R^2}\frac{l^2}{R^2}\right],$$

where t, R, l are the thickness, radius, and length of the shell.

The presence of a second term in the formula is a result of the fact that the characteristic equation of the shell in the case considered at $n = 1$ has two zero roots only. This makes it possible to draw some conclusions relative to the order of the error introduced by the simplifying assumptions made.

When deriving the equations of a nonshallow sandwich shell, the outer layers of thickness t, made from the same isotropic material, are considered as thin elastic shells.

It is assumed that the middle layer — the filler of thickness $2h$ — does not take moments and forces parallel to the middle surface, so that in the filler

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0.$$

The filler, which connects the outer layers of the sandwich shell, takes therefore the shearing forces. In the lateral direction the filler is undeformable, so that

$$w = w_{\mathrm{L}} = w_{\mathrm{U}}.$$

The indices U, L here correspond to the upper and lower load-carrying layers, respectively.

The same assumptions are used in the derivation of a sandwich shell as in the derivation of the equations of the outer layers. The magnitude $\frac{h+\frac{t}{2}}{R}$ is considered as small compared with unity.

The load-carrying layers in the sandwich shell behave as thin elastic shells loaded by external forces and distributed loads, applied on the surfaces of contact with the filler and determined by the interaction of the layers during their common strain. These stresses are expressed through the displacements of the middle surfaces of the outer layers by integrating the equations for the filler (the integration is possible under the assumptions used for the filler).

The system of nonlinear equations of the sandwich shell is

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha_1}\{A_2[(1+e_{2\alpha})T_{11\alpha}+\omega_{2\alpha}S_\alpha+\psi_{1\alpha}N_{1\alpha}]\}+\\
&+\frac{\partial}{\partial\alpha_2}\{A_1[(1+e_{1\alpha})S_\alpha+\omega_{2\alpha}T_{22\alpha}+\psi_{1\alpha}N_{2\alpha}]\}+\\
&+\frac{\partial A_1}{\partial\alpha_2}[\omega_{1\alpha}T_{11\alpha}+(1+e_{2\alpha})S_\alpha+\psi_{2\alpha}N_{1\alpha}]-\\
&-\frac{\partial A_2}{\partial\alpha_1}[\omega_{1\alpha}S_\alpha+(1+e_{1\alpha})T_{2\alpha}+\psi_{2\alpha}N_{2\alpha}]+\\
&+\frac{A_1A_2}{R_1}[-\psi_{1\alpha}T_{11\alpha}-\psi_{2\alpha}S_\alpha+(1+e_{2\alpha})N_{1\alpha}]+\\
&+A_1A_2\left[\frac{2h}{R_1}f_1-\psi_{1\alpha}p-\psi_{1\alpha}f_3\right]=0,\\
&\frac{\partial}{\partial\alpha_1}\{A_2[\omega_{12}T_{11\alpha}+(1+e_{2\alpha})S_\alpha+\psi_{2\alpha}N_{1\alpha}]\}+\\
&+\frac{\partial}{\partial\alpha_2}\{A_1[\omega_{1\alpha}S_\alpha+(1+e_{1\alpha})T_{22\alpha}+\psi_{2\alpha}N_{2\alpha}]\}+\\
&+\frac{\partial A_2}{\partial\alpha_1}[(1+e_{1\alpha})S_\alpha+\omega_{2\alpha}T_{22\alpha}+\psi_{1\alpha}N_{2\alpha}]-\\
&-\frac{\partial A_1}{\partial\alpha_2}[(1+e_{2\alpha})T_{11\alpha}+\omega_{2\alpha}S_\alpha+\psi_{1\alpha}N_{1\alpha}]+\\
&+\frac{A_1A_2}{R_2}[-\psi_{1\alpha}S_\alpha-\psi_{2\alpha}T_{22\alpha}+(1+e_{1\alpha})N_{2\alpha}]+ \qquad (4)\\
&+A_1A_2\left[\frac{2h}{R_2}f_2-\psi_{2\alpha}p-\psi_{2\alpha}f_3\right]=0,\\
&\frac{\partial}{\partial\alpha_1}\{A_2[-\psi_{1\alpha}T_{11\alpha}-\psi_{2\alpha}S_\alpha+(1+e_{2\alpha})N_{1\alpha}]\}+\\
&+\frac{\partial}{\partial\alpha_2}\{A_1[-\psi_{1\alpha}S_\alpha-\psi_{2\alpha}T_{22\alpha}+(1+e_{1\alpha})N_{2\alpha}]\}-\\
&-\frac{A_1A_2}{R_1}[(1+e_{2\alpha})T_{11\alpha}+\omega_{2\alpha}S_\alpha+\psi_{1\alpha}N_{1\alpha}]-\\
&-\frac{A_1A_2}{R_2}[\omega_{1\alpha}S_\alpha+(1+e_{1\alpha})T_{22\alpha}+\psi_{2\alpha}N_{2\alpha}]+\\
&+A_1A_2[-(1+e_{1\alpha}+e_{2\alpha})p-(1+e_{1\alpha}+e_{2\alpha})f_3]=0,\\
&\frac{\partial}{\partial\alpha_1}(A_2T_{11\beta})+\frac{\partial}{\partial\alpha_2}(A_1S_\beta)+\frac{\partial A_1}{\partial\alpha_2}S_\beta-\frac{\partial A_2}{\partial\alpha_1}T_{22\beta}-2A_1A_2f_1=0,\\
&\frac{\partial}{\partial\alpha_1}(A_2S_\beta)+\frac{\partial}{\partial\alpha_2}(A_1T_{22\beta})+\frac{\partial A_2}{\partial\alpha_1}S_\beta-\frac{\partial A_1}{\partial\alpha_2}T_{11\beta}-2A_1A_2f_2=0,\\
&\frac{\partial}{\partial\alpha_1}(A_2M_{11\alpha})+\frac{\partial}{\partial\alpha_2}(A_1H_\alpha)+\frac{\partial A_1}{\partial\alpha_2}H_\alpha-\frac{\partial A_2}{\partial\alpha_1}M_{22\alpha}-A_1A_2N_{1\alpha}+A_1A_2tf_1=0,\\
&\frac{\partial}{\partial\alpha_1}(A_2H_\alpha)+\frac{\partial}{\partial\alpha_2}(A_1M_{22\alpha})+\frac{\partial A_2}{\partial\alpha_1}H_\alpha-\frac{\partial A_1}{\partial\alpha_2}M_{11\alpha}-A_1A_2N_{2\alpha}+A_1A_2tf_2=0.
\end{aligned}
$$

Here

$$f_1 = \frac{G}{h}\left[u_{1\beta} - \left(h + \frac{t}{2}\right)\psi_{1\alpha}\right]; \qquad f_2 = \frac{G}{h}\left[u_{2\beta} - \left(h + \frac{t}{2}\right)\psi_{2\alpha}\right],$$

$$f_3 = -\frac{2h}{A_1 A_2}\left[\frac{\partial}{\partial \alpha_1}(A_2 f_1) + \frac{\partial}{\partial \alpha_2}(A_1 f_2)\right].$$

The forces per unit length in the shell are determined by the formulas

$$\begin{gathered}
T_{11\alpha} = T_{11U} + T_{11L} = 2B\left[e_{1\alpha} + \frac{1}{2}\psi_{1\alpha}^2 + \nu\left(e_{2\alpha} + \frac{1}{2}\psi_{2\alpha}^2\right)\right],\\
T_{22\alpha} = T_{22U} + T_{22L} = 2B\left[e_{2\alpha} + \frac{1}{2}\psi_{2\alpha}^2 + \nu\left(e_{1\alpha} + \frac{1}{2}\psi_{1\alpha}^2\right)\right],\\
S_\alpha = S_U + S_L = B(1-\nu)\left[\omega_\alpha + \psi_{1\alpha}\psi_{2\alpha}\right],\\
T_{11\beta} = T_{11U} - T_{11L} = 2B(e_{1\beta} + \nu e_{2\beta}),\\
T_{22\beta} = T_{22U} - T_{22L} = 2B(e_{2\beta} + \nu e_{1\beta}),\\
S_\beta = S_U - S_L = B(1-\nu)\,\omega_\beta.
\end{gathered} \tag{5}$$

The moments in the outer layers are

$$\begin{gathered}
M_{11\alpha} = 2D(\chi_{11\alpha} + \nu\chi_{22\alpha}),\\
M_{22\alpha} = 2D(\chi_{22\alpha} + \nu\chi_{11\alpha}),\\
H_\alpha = 2D(1-\nu)\,\chi_{12}.
\end{gathered} \tag{6}$$

Here

$$\begin{gathered}
e_{1\alpha} = \frac{1}{A_1}\frac{\partial u_{1\alpha}}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_{2\alpha} + \frac{w}{R_1},\\
e_{2\alpha} = \frac{1}{A_2}\frac{\partial u_{2\alpha}}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}u_{1\alpha} + \frac{w}{R_2},\\
\omega_{1\alpha} = \frac{1}{A_1}\frac{\partial u_{2\alpha}}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_{1\alpha},\\
\omega_{2\alpha} = \frac{1}{A_2}\frac{\partial u_{1\alpha}}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}u_{2\alpha},
\end{gathered} \tag{7}$$

$$\begin{gathered}
e_{1\beta} = \frac{1}{A_1}\frac{\partial u_{1\beta}}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_{2\beta}, \qquad 2\chi_{12\alpha} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\psi_{2\alpha}}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\psi_{1\alpha}}{A_1}\right),\\
e_{2\beta} = \frac{1}{A_2}\frac{\partial u_{2\beta}}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}u_{1\beta}, \qquad \chi_{11\alpha} = \frac{1}{A_1}\frac{\partial \psi_{1\alpha}}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\psi_{2\alpha},\\
\omega_\beta = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_{1\beta}}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_{2\beta}}{A_2}\right), \qquad \chi_{22\alpha} = \frac{1}{A_2}\frac{\partial \psi_{2\alpha}}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\psi_{1\alpha},\\
\psi_{1\alpha} = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{1}{R_1}u_{1\alpha}, \qquad \psi_{2\alpha} = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{1}{R_2}u_{2\alpha}.
\end{gathered} \tag{8}$$

The following symbols are used in equations (4) and relationships (5) to (8):

$$u_{1\alpha} = \frac{1}{2}(u_{1U} + u_{1L}), \qquad u_{2\alpha} = \frac{1}{2}(u_{2U} + u_{2L}),$$

$$u_{1\beta} = \frac{1}{2}(u_{1U} - u_{1L}), \qquad u_{2\beta} = \frac{1}{2}(u_{2U} - u_{2L}).$$

B, D designate the tensile and flexural rigidities of the outer layers:

$$B = \frac{Et}{1-\nu^2}, \qquad D = \frac{Et^3}{12(1-\nu^2)}.$$

E, ν are the modulus of elasticity and the Poisson ratio of the load-carrying layers; G is the shear modulus of the filler; p is the distributed normal pressure.

From equations (4) one easily obtains the system of stability equations of a sandwich shell as equations of neutral equilibrium; this system reduces to five equations relative to the five displacements $u_{1\alpha}$, $u_{2\alpha}$, $u_{1\beta}$, $u_{2\beta}$, w.

The system of equations is considerably simplified in the case of a shallow shell. It can then be reduced to three equations relative to three functions. There is no need in this case to introduce the assumption $A_1 \approx A_2 \approx 1$ relative to the Lamé coefficients.

The stability equations of a shallow sandwich shell reduce to

$$\Delta\Delta F + 2Et\left(\frac{\chi_{22\alpha}}{R_1} + \frac{\chi_{11\alpha}}{R_2}\right) = 0,$$

$$\frac{Bh}{G}\Delta\Psi - \Psi - \frac{Bh\left(h+\frac{t}{2}\right)}{G}\Delta\Delta w = 0, \tag{9}$$

$$2D\Delta\Delta w + \frac{T_{11\alpha}}{R_1} + \frac{T_{22\alpha}}{R_2} - \frac{2G\left(h+\frac{t}{2}\right)}{h}\Psi + \chi_{11\alpha}\overset{\circ}{T}_{11\alpha} + \chi_{22\alpha}\overset{\circ}{T}_{22\alpha} + 2\chi_{12\gamma}\overset{\circ}{S}_{\alpha} = 0.$$

Here F is the stress resultant function; Ψ is the shear function

$$\Psi = \frac{1}{A_1A_2}\left[\frac{\partial(A_2u_{1\beta})}{\partial\alpha_1} + \frac{\partial(A_1u_{2\beta})}{\partial\alpha_2}\right] + \left(h+\frac{t}{2}\right)\Delta w;$$

Δ is the operator

$$\Delta = \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\right)\right].$$

The problem of the stability of a sandwich circular cylindrical shell hinged at the ends and subjected to the simultaneous action of longitudinal compression and external pressure was considered in a linear formulation on the basis of the equations of both the theory of shallow shells and the theory of nonshallow shells.

The relationship connecting the values of the critical load parameters m_p and m_t is, by the theory of shallow shells,

$$m^2\lambda^2 m_t + n^2 m_p = \frac{q^2}{\beta[1+rq]} + \frac{k^2q^2}{\beta} + \frac{\beta m^4\lambda^4}{q^2}, \tag{10}$$

and by the theory of nonshallow shells (waves of large length are formed along the shell)

$$\frac{m^2\lambda^2(n^2+1)}{n^2}m_t + (n^2-1)m_p = \frac{q^2+1-2n^2}{\beta[1+rq]} + \frac{k^2q^2}{\beta} + \frac{\beta m^4\lambda^4}{q^2}. \tag{11}$$

Here

$$r = \frac{Bh}{GR^2}, \quad \beta^2 = \frac{(1-\nu^2)R^2}{\left(h+\frac{t}{2}\right)^2}, \quad k^2 = \frac{D}{B\left(h+\frac{t}{2}\right)^2}, \quad \lambda = \frac{\pi R}{l},$$

$$m_t = \frac{T\beta}{2Et}, \quad m_p = \frac{pR\beta}{2Et}, \quad q = m^2\lambda^2 + n^2.$$

T is the force per unit length on the two load-carrying layers; and p is the external pressure.

The values of the integers n and m must be selected in such a way as to obtain minimum m_p and m_t at given ratio m_p/m_t.

In the case of longitudinal compression ($m_p = 0$) it follows from (10) that m_t is minimum in the case of symmetrical buckling ($n = 0$). The bending moment in the outer layers can be neglected ($k = 0$) for shells in which $r\beta < 1.0$, and the calculation can be conducted by the appropriate formula

$$m_t = 2 - r\beta. \tag{12}$$

At $r\beta > 1.0$ it is necessary to take the parameter k into account. The following approximate formula can be used in this case:

$$m_t = \frac{1}{r\beta} + 2k. \tag{13}$$

Formula (12) coincides with earlier obtained results /3, 4, 5/. Formula (13) is somewhat more refined.

The analysis of (11) for the case of compression ($m_p = 0$) shows that, unlike the case of simple shells, in which the buckling with formation of lengthy waves (the Southwell solution) corresponds to a lower value of the critical load than short-wave buckling, in sandwich shells it is the short-wave buckling which corresponds as a rule to a lower value of the critical load. The following approximate formula is obtained in this case for the parameter m_t:

$$m_t = \frac{6}{5}(1 - 2r).$$

It is seen from this formula that in the case of long-wave buckling of a sandwich shell the weakening of the shearing strain in the filler affects only little the critical load (the value of r is usually small).

The study of shell stability under pressure ($m_t = 0$) shows that buckling occurs in this case with the formation of one half-wave along the generator, just as in the case of a simple shell.

It was shown that the bending moment in the outer layers can be neglected here in a majority of cases. It is essential to note that the calculation must generally be conducted in this case by formula (11). In the case of a lengthy cylindrical shell (tube) the computing formula is

$$m_p = \frac{3}{\beta(1+4r)}.$$

The factor $1 + 4r$ in the denominator distinguishes this formula from the well-known formula for simple shells.

Thorough calculations for the case of combined action of compression and pressure were conducted by means of an electronic computer by formulas (10) and (11). The results were plotted on curves suitable for practical utilization.

The analysis of the results makes it possible to form an opinion on the limits of applicability of the formulas of the theory of shallow shells. It may be concluded that the limits of applicability of the theory of shallow shells are generally widened with the increase of the parameter r.

It was established that the allowance for the bending moment in the outer layers is of importance when compression predominates. When the pressure predominates the assumption $k = 0$ is wholly acceptable.

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ON THE INFLUENCE OF COMPRESSIBILITY ON THE BENDING OF ELASTO-PLASTIC PLATES

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(Tartu)

The plate material is usually assumed to be uncompressible when developing the theory of elasto-plastic bending of thin plates. Although such an assumption considerably simplifies the problem solution, it cannot always be accepted. If the zone of plastic deformations in the plate is very restricted, the hypothesis of uncompressibility can lead to errors of the order of 30 to 40% in the determination of the load or stresses (cf. /1/). The accurate allowance for the influence of the compressibility being difficult, a number of methods in which it is approximately allowed for have been developed lately /1-5/.

An algorithm for the rigorous allowance for the influence of compressibility is developed in this paper. The problem of the equilibrium of a circular simply supported plate is solved as an example. The results obtained are compared with the results found by approximate methods /1-5/.

1. BASIC RELATIONSHIPS AND METHOD OF SOLUTION

We start from the Hencky-Il'yushin relationships, which reduce in the case of a thin plate to

$$X_x - \frac{1}{2} Y_y = 3G(1-\omega)(e_{xx} - \epsilon),$$

$$Y_y - \frac{1}{2} X_x = 3G(1-\omega)(e_{yy} - e), \qquad (1.1)$$

$$X_y = G(1-\omega) e_{xy}.$$

Here G is the shear modulus; $\omega = 1 - \sigma_i / 3Ge_i$ is the modulus of softening; and $e = (e_{xx} + e_{yy} + e_{zz})/3$.

It follows from the law of volume compression that

$$e = \frac{1-2\nu}{3E} (x_x + Y_y). \qquad (1.2)$$

Eliminating e from (1.1) and (1.2), we obtain

$$X_x = G(\alpha e_{xx} + \beta e_{yy}),$$
$$Y_y = G(\beta e_{xx} + \alpha e_{yy}), \tag{1.3}$$
$$X_y = G(1-\omega)\, e_{xy},$$

where

$$\alpha = 2(1-\omega)\,[3-\omega(1-2\nu)]\,\Delta^{-1},$$
$$\beta = 2(1-\omega)\,[3\nu+\omega(1-2\nu)]\,\Delta^{-1}, \tag{1.4}$$
$$\Delta = 3(1-\nu) - 2\omega(1-2\nu).$$

The strain intensity e_i is calculated by the formula

$$e_i = \frac{\sqrt{2}}{3}\sqrt{(e_{xx}-e_{yy})^2 + (e_{yy}-e_{zz})^2 + (e_{zz}-e_{xx})^2 + \frac{3}{2}e_{xy}^2}\,. \tag{1.5}$$

We shall represent formula (1.5) in a somewhat different form. Calculating e_{zz} from (1.2), we find

$$e_{zz} = \gamma(e_{xx} + e_{yy}), \tag{1.6}$$

where

$$\gamma = -[3\nu + (1-2\nu)\,\omega]\,\Delta^{-1}. \tag{1.7}$$

In the case of small deflections we have

$$e_{xx} = -z\frac{\partial^2 w}{\partial x^2}, \quad e_{yy} = -z\frac{\partial^2 w}{\partial y^2}, \quad e_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}. \tag{1.8}$$

We shall restrict ourselves in what follows to the case of linear work-hardening of the material. We find then (λ is the modulus of work-hardening, e_s is the strain intensity at the yield point)

$$e_i = \frac{\lambda e_s}{\lambda - \omega}. \tag{1.9}$$

We pass now to dimensionless magnitudes (h is the plate thickness)

$$z^* = \frac{2z}{h}, \quad w^*_{xx} = \frac{h}{e_s}\frac{\partial^2 w}{\partial x^2}, \quad w^*_{yy} = \frac{h}{e_s}\frac{\partial^2 w}{\partial y^2}, \quad w^*_{xy} = \frac{h}{e_s}\frac{\partial^2 w}{\partial x \partial y}, \tag{1.10}$$
$$s = (w^*_{xx} + w^*_{yy})^2, \qquad t = 3(w'^2_{xy} - w^*_{xx}w^*_{yy}).$$

On the strength of formulas (1.5) to (1.10) we obtain the following algebraic equation of order 4 for determining ω:

$$z^{*2}\sum_{i=0}^{4} a_i\,\omega^i = 9\lambda^2\sum_{i=0}^{2} b_i\,\omega^i, \tag{1.11}$$

where

$$a_4 = (1-2\nu)^2(3s+4t),$$
$$a_3 = -(1-2\nu)\,[2\lambda(1-2\nu) + 3(1-\nu)]\,(3s+4t),$$

$$a_2 = (1-2\nu)\lambda\,[6(1-\nu)+\lambda(1-2\nu)]\,(3s+4t)+9\,[(1-\nu+\nu^2)\,s+(1-\nu)^2 t],$$

$$a_1 = -3(1-\nu)(1-2\nu)\lambda^2(3s+4t)-18\lambda\,[(1-\nu+\nu^2)\,s+(1-\nu)^2 t],$$

$$a_0 = 9\lambda^2\,[(1-\nu+\nu^2)\,s+(1-\nu)^2 t],$$

$$b_2 = 4(1-2\nu)^2,$$

$$b_1 = -12(1-\nu)(1-2\nu),$$

$$b_0 = 9(1-\nu)^2.$$

Equation (1.11) can be used to determine the boundary $z^* = z_s$ between the elastic and plastic zones. On this boundary $\omega = 0$, and therefore

$$z_s = \frac{3}{\sqrt{\nu_1 s + t}} \qquad \left(\nu_1 = \frac{1-\nu+\nu^2}{(1-\nu)^2}\right). \tag{1.12}$$

It follows from the definition of z_s that always $0 \leqslant z_s \leqslant 1$; at the extreme values $z_s = 0$ and $z_s = 1$ all the sections along the plate normal are deformed either plastically or elastically.

The bending moments M_1, M_2 and the torsional moment H are calculated by the usual formulas:

$$M_1 = \int_{-h/2}^{+h/2} X_x\, z dz, \qquad M_2 = \int_{-h/2}^{+h/2} Y_y\, z dz, \qquad H = \int_{-h/2}^{+h/2} X_y\, z dz. \tag{1.13}$$

Substituting (1.3) and allowing for the fact that in the elastic zone $0 \leqslant z^* \leqslant z_s$ the magnitudes α and β have the values $\alpha = 2/(1-\nu)$, $\beta = 2\nu/(1-\nu)$, we find that

$$M_1 = -\frac{1}{4} G h^2 e_s \left[\frac{2z_s^3}{4(1-\nu)}(w^*_{xx} + \nu w^*_{yy}) + A w^*_{xx} + B w^*_{yy}\right],$$

$$M_2 = -\frac{1}{4} G h^2 e_s \left[\frac{2z_s^3}{3(1-\nu)}(w^*_{yy} + \nu w^*_{xx}) + B w^*_{xx} + A w^*_{yy}\right], \tag{1.14}$$

$$H = -\frac{1}{2} G h^2 e_s \left(\frac{1}{3} - C\right) w^*_{xy},$$

where

$$A = \int_{z_s}^{1} \alpha z^{*2} dz^*, \qquad B = \int_{z_s}^{1} \beta z^{*2} dz^*, \qquad C = \int_{z_s}^{1} \omega z^{*2} dz^*. \tag{1.15}$$

The expressions for the moments must satisfy the Germain equation

$$\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} = -q. \tag{1.16}$$

The substitution of expressions (1.14) here yields a differential equation for determining the deflection $w(x, y)$. This equation can be integrated in successive approximations by the following scheme:

1) we assume a lateral load $q(x, y)$;

2) we select a certain initial approximation $w_0(x, y)$ for the deflection $w(x, y)$ (as such can be taken the value of the deflection found from (1.6) for the case of an uncompressible material);

3) starting from the expression $w = w_n(x, y)$ we calculate on the basis of formulas (1.11), (1.4), and (1.12) the magnitudes ω, α, β, z_s in a first approximation;

4) we calculate the integrals A, B, C in a first approximation;

5) from (1.16) and (1.14) we determine the first approximation for the deflection $w = w_1(x, y)$.

The process is repeated until the required accuracy is attained for $w(x, y)$.

The same problem can also be solved by the variational method. This method is particularly efficient in the case when the plate deflection is given by the one-term formula $w = cw_*(x, y)$ (c is the coefficient to be varied; $w_*(x, y)$ is the given function). In this case the problem solution can be found directly, i.e., without successive approximations (cf. Section 3 of this paper).

2. APPROXIMATE ALLOWANCE FOR THE INFLUENCE OF COMPRESSIBILITY

We shall consider several variants in which the material compressibility is allowed for approximately.

Variant 1. The coefficient γ in (1.6) has in the elastic case the value $\gamma = -\nu/(1-\nu)$. Let us assume that γ has the same value even beyond the elastic limit /1/. Formula (1.11) simplifies in this case and reduces to

$$\omega = \lambda\left(1 - \frac{3}{z^* \sqrt{\nu_1 s + t}}\right). \tag{2.1}$$

By taking $\omega = 0$ in (2.1) we again obtain relationship (1.12). It follows that in this variant the boundary between the elastic and plastic zones is determined exactly.

Variant 2. We assume that the influence of compressibility on the strain intensity e_i can be neglected /5/. In this case we must write in (1.6) $\gamma = -1$; ω and z_s are determined from formulas (2.1) and (1.12), with ν_1 replaced by 3. Since in the case of an incompressible material $\nu_1 < 3$, it follows from (1.12) that the substitution $\nu_1 \to 3$ widens the plastic zone.

Variant 3. The Poisson ratio in the zone of plastic deformations is taken in many papers (cf., for instance, /3/) as equal to 0.5, while in elastic deformations a value $\nu \neq 0.5$ is used (i.e., the influence of compressibility is allowed for in the region of purely elastic deformations only).

Variant 4. A peculiar method was proposed by Grigor'ev in /4/. According to this method, the sag is calculated for $\nu \neq 0.5$ and $\nu = 0.5$ for the load $q = q_s$ at which plastic deformations start to appear in the plate. We designate the difference between these magnitudes by the symbol $\Delta\delta$, i.e.,

$$\Delta\delta = (\delta_s)_{\nu \neq 0.5} - (\delta_s)_{\nu = 0.5}. \tag{2.2}$$

If $q > q_s$, we add to the sag found for the uncompressible material the

magnitude $\Delta\delta$, i.e.,

$$\delta = \delta_{0.5} + \Delta\delta. \tag{2.3}$$

3. BENDING OF A CIRCULAR PLATE SIMPLY SUPPORTED ALONG THE CONTOUR

The following example will illustrate the relationships given in Section 1 and the degree of accuracy of the approximate solutions given in Section 2.

A circular plate of radius R is loaded by a uniform lateral load of intensity q. The plate edge $r = R$ is simply supported.

In solving this problem we start from Lagrange's variational equation

$$\int_0^R (M_1\delta\varkappa_1 + M_2\delta\varkappa_2 - q\delta w)\, r dr = 0. \tag{3.1}$$

Since in the axisymmetrical problem $\varkappa_1 = -d^2w/dr^2$, $\varkappa_2 = -(1/r)\, dw/dr$, all the formulas of Section 1 remain in force, we make the substitutions

$$w^*_{xx} \to \frac{h}{R^2 e_s}\frac{d^2w}{d\rho^2}, \quad w^*_{yy} \to \frac{h}{R^2 e_s}\frac{1}{\rho}\frac{dw}{d\rho}, \quad w^*_{xy} \to 0 \qquad \left(\rho = \frac{r}{R}\right).$$

We shall look for a solution to the variational problem (3.1) in the form

$$w = f[1 + a\rho^2 - (1 + a)\rho^2], \tag{3.2}$$

which satisfies the boundary condition $w = 0$ at $\rho = 1$. The coefficient a in this formula is determined from the condition $M_1 = 0$ at $\rho = 1$. Satisfying this condition and taking into account the first formula of (1.14), we obtain the equation

$$\frac{2}{3(1-\nu)}[z_s(1)]^3[6 + 2\nu + (5 + \nu)a] + (6 + 5a)A(1) + (2 + a)B(1) = 0. \tag{3.3}$$

If the zone of plastic deformations does not attain the plate edge $\rho = 1$, then $A(1) = B(1) = 0$, and it follows from (3.3) that $a = -2(3 + \nu)(5 + \nu)^{-1}$. If the plastic deformations embrace the contour section $\rho = 1$, equation (3.3) is to be solved by successive approximations. It is best to take the value $a = -2(3 + \nu)(5 + \nu)^{-1}$ as initial approximation for a. Having determined the first approximation of $z_s(1)$, $A(1)$, $B(1)$ on the basis of formulas (1.12) and (1.15), we find from (3.3) the first approximation for a. This process is repeated until the required accuracy is attained.

We introduce the following dimensionless magnitudes:

$$\zeta = \frac{fh}{R^2 e_s}, \quad q^* = \frac{R^2 q}{Gh^2 e_s}, \tag{3.4}$$

$$\rho_1 = a - 6(1 + a)\rho^2, \quad \rho_2 = a - 2(1 + a)\rho^2.$$

The variational equation (3.1) yields, after substituting (3.2):

$$q^* = \frac{12\zeta}{4 + a}\int_0^1 \left[\frac{2z_s^3}{3(1-\nu)}(\rho_1^2 + 2\nu\rho_1\rho_2 + \rho_2^2) + A(\rho_1^2 + \rho_2^2) + 2B\rho_1\rho_2\right]\rho d\rho. \tag{3.5}$$

The calculations were conducted for the case $\lambda = 0.95$, $\nu = 0.3$. The results are represented in the figure; the symbol "0" corresponds to the rigorous solution, and the numbers "1" to "4" to the approximate solutions obtained by variants 1 to 4, respectively. The following conclusions can be drawn from an examination of the figure: a) variants 1 and 2 give unacceptable results; b) Grigor'ev's method gives the best approximation to the rigorous solution; another factor in its favor is its simplicity.

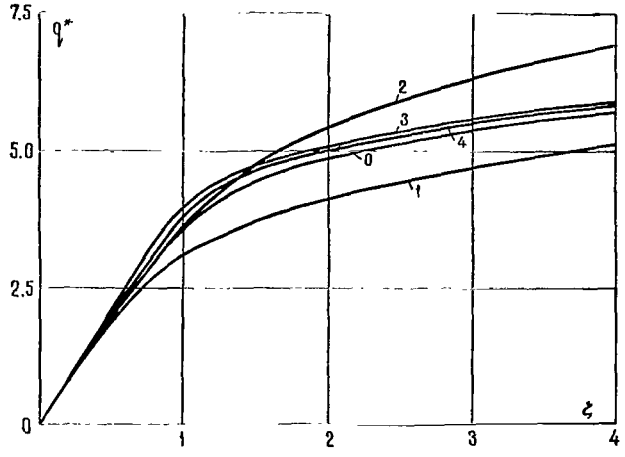

FIGURE.

Similar conclusions apply to the distribution of the bending moments in the plate.

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AXISYMMETRICAL FLUTTER OF CYLINDRICAL SHELLS IN A SUPERSONIC GAS STREAM

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A method is proposed for solving exactly the problem of the axisymmetrical vibrations of a cylindrical shell, allowing for the inertia forces of all the displacements. The results obtained on the "Strela" computer are given.

1. The axisymmetrical vibrations of a cylindrical shell are described by the equations (/1/, p. 257)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\nu}{R}\frac{\partial w}{\partial x} - \frac{h^2}{12R}\frac{\partial^3 w}{\partial x^3} + \frac{1-\nu^2}{Eh}X = 0,$$
$$\frac{\nu}{R}\frac{\partial u}{\partial x} - \frac{h^2}{12R}\frac{\partial^3 u}{\partial x^3} + \frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} + \left(\frac{h^2}{12R^4} + \frac{1}{R^2}\right)w - \frac{1-\nu^2}{Eu}Z = 0. \qquad (1.1)$$

Here u, w are the longitudinal and normal displacements of the middle surface; X, Z are the longitudinal and normal forces acting on a unit area of the middle surface; h, R are the thickness and radius of the cylindrical shell, E, ν are Young's modulus and the Poisson ratio of the shell material; x is the coordinate along the cylinder generator.

Let the shell vibrate in a supersonic gas stream whose undisturbed velocity C is directed parallel to the axis of the cylindrical shell. The forces acting in the middle surface of the shell will be made then of the inertia forces and the aerodynamic forces. If we use the expressions for the latter given by the linear approximation of the "piston theory" /2/, we obtain

$$X = -\mu h\frac{\partial^2 u}{\partial t^2}, \quad Z = -\mu h\frac{\partial^2 w}{\partial t^2} + B\left(C\frac{\partial w}{\partial x} - \frac{\partial w}{\partial t}\right). \qquad (1.2)$$

Here μ is the density of the shell material; $B = p_0\varkappa/C_0$ is the damping coefficient; p_0, $\varkappa$, C_0 are the pressure, polytropic index, and sound velocity in the undisturbed gas.

We introduce the dimensionless coordinate $\alpha = x/a$, were a is the shell length, and consider displacements representable in the following form:

$$u(\alpha, t) = au(\alpha)e^{\omega t}, \quad w(\alpha, t) = aw(\alpha)e^{\omega t}, \qquad (1.3)$$

where ω is the complex frequency. The substitution of (1.2) and (1.3) in (1.1) yields:

$$\left(\frac{d^2}{d\alpha^2} - \Omega^2 c\right)u + \delta\left(\nu\frac{d}{d\alpha} - c\frac{d^3}{d\alpha^3}\right)w = 0, \qquad (1.4)$$

$$\frac{\delta}{c}\left(\nu\frac{d}{d\alpha}-c\frac{d^3}{d\alpha^3}\right)u+\left(\frac{d^4}{d\alpha^4}-A\frac{d}{d\alpha}+\Omega^2+K\Omega+\delta^4+\frac{\delta^2}{c}\right)w=0. \qquad (1.4)$$
(cont'd)

where the dimensionless frequency of vibrations Ω, stream velocity A, damping coefficient K, shell curvature δ, and thickness characteristic c are given by the formulas

$$\Omega^2=\frac{12(1-\nu^2)a^4\mu}{Eh^2}\omega^2, \qquad A=\frac{12(1-\nu^2)a^3B}{Eh^3}C,$$

$$K=\frac{a^2}{h^2}\sqrt{\frac{12(1-\nu^2)}{\mu E}}B, \qquad \delta=\frac{a}{R}, \qquad c=\frac{h^2}{12a^2}.$$

It can be shown that at $\delta\neq 0$ any solution of system (1.4) can be represented in the form

$$u=\delta\,(c\Phi'''-\nu\Phi'), \qquad w=\Phi''-\Omega^2c\Phi, \qquad (1.5)$$

where $\Phi(\alpha)$ is the displacement function.

The substitution of (1.5) in system (1.4) transforms the first equation into an identity, and the second one into

$$\left(1-\delta^2c)\Phi^{VI}+(2\delta^2\nu-\Omega^2c)\Phi^{IV}-A\Phi'''+\left(\Omega^2+K\Omega+\delta^4+\frac{\delta^2(1-\nu^2)}{c}\right)\Phi''+\right.$$
$$+A\Omega^2c\Phi'-\Omega^2c\left(\Omega^2+K\Omega+\delta^4+\frac{\delta^2}{c}\right)\Phi=0. \qquad (1.6)$$

Expressing the forces and moments through the displacements (/1/, p.256), and allowing for (1.5), we obtain

$$N=\frac{\Omega^2Eh^3}{12(1-\nu^2)aR}(c\Phi''-\nu\Phi)\,e^{\omega t},$$
$$M=\frac{Eh^3(1-\delta^2c)}{12a(1-\nu^2)}\left(\Phi^{IV}+\frac{\delta^2\nu-\Omega^2c}{1-\delta^2c}\Phi''\right)e^{\omega t}, \qquad (1.7)$$
$$Q=-\frac{Eh^3(1-\delta^2c)}{12a^2(1-\nu^2)}\left(\Phi^{V}+\frac{\delta^2\nu-\Omega^2c}{1-\delta^2c}\Phi'''\right)e^{\omega t}.$$

Using formulas (1.5) and (1.7), the linear end conditions for the displacements, forces, and moments can be written in the form

$$L_k\Phi=0 \quad \text{at} \quad \alpha=0,$$
$$k=1,2,3,$$
$$R_k\Phi=0 \quad \text{at} \quad \alpha=1, \qquad (1.8)$$

Here L_k, R_k are linear differential operators corresponding to definite boundary conditions.

We shall look for a solution of (1.6) in the form $\Phi=e^{z\alpha}$. We obtain the following equation for Z:

$$(1-\delta^2c)z^6+(2\delta^2\nu-\Omega^2e)z^4-Az^3+\left(\Omega^2+K\Omega+\delta^4+\frac{\delta(1-\nu^2)}{c}\right)z^2+$$
$$+A\Omega^2cz-\Omega^2c\left(\Omega^2+K\Omega+\delta^4+\frac{\delta^2}{c}\right)=0. \qquad (1.9)$$

Assuming that the roots $z_1,\ldots,z_6$ of equation (1.9) are simple, we can represent the general solutions of (1.6) in the form $\Phi=c_1e^{z_1\alpha}+\cdots+c_6e^{z_6\alpha}$.

Using the boundary conditions (1.8), we can form the characteristic determinant, which in the case of simple roots of (1.9) will be of the form

$$\Delta = \begin{vmatrix} L_1(z_1) & \cdots & L_1(z_6) \\ L_2(z_1) & \cdots & L_2(z_6) \\ L_3(z_1) & \cdots & L_3(z_6) \\ R_1(z_1)\, e^{z_1} & \cdots & R_1(z_6)\, e^{z_6} \\ R_2(z_1)\, e^{z_1} & \cdots & R_2(z_6)\, e^{z_6} \\ R_3\, z_1)\, e^{z_1} & \cdots & R_3(z_6)\, e^{z_6} \end{vmatrix} = 0.$$

Here $L_k(z)$ and $R_k(z)$ are polynomials obtained from the operators L_k and R_k by replacing the symbol $\frac{d^n}{d\alpha^n}$ by z^n. The characteristic determinant will be different in the case of multiple roots of (1.9). It has been shown, however /3/, that the study of the numerous cases of different multiplicity can be avoided by studying the equation

$$\frac{\Delta}{\sigma} = 0, \tag{1.10}$$

where σ is the Vandermonde determinant, made of the magnitudes $z_1, \ldots, z_6$.

The set of equations (1.9) and (1.10) will be called the fundamental characteristic system. Equation (1.9) gives the roots of the characteristic polynomial as functions of the problem parameters. By substituting these functions in (1.10) instead of $z_1, \ldots, z_6$ we obtain the equation $\varphi(\Omega, A, K, \delta, c, \nu) = 0$. This equation establishes the connection between the parameters of the shell vibrating in the stream. If the shell vibrates in a vacuum of a gas at rest, this equation can be written in an explicit form, since in this case equation (1.9) becomes bicubic. At $A \neq 0$ the value of the function φ can be calculated for any set of values of the parameters, in the following way: on the strength of the values of the parameters we find the coefficients of (1.9), and we can then calculate its roots; substituting the values of the roots in the left-hand side of (1.10), we obtain the value of the function φ for the given set of values of the parameters. Fixing, in the equation $\varphi = 0$, all the parameters except the first one, we obtain the frequency equation of the given shell in the given gas stream. We shall show how this equation can be solved numerically. Let it be required to know how the lowest frequency varies with the variation of the curvature, i.e., to calculate the values of the function $\Omega(\delta)$ for a number of values of δ. We give specific values to all parameters except Ω and δ. The equation $\varphi = 0$ can then be written in the form $f(\Omega, \delta) = 0$. Note that the values of the function $f(\Omega, \delta)$ can be calculated for any Ω and δ. For $\delta = 0$ (for a strip), the lowest frequency is easily found. This frequency is taken as initial approximation for $\delta = \delta_1$, and the "chord method" is then used:

$$\Omega_{n+1} = \Omega_n - \frac{\Omega_n - \Omega_{n-1}}{f(\Omega_n, \delta_1) - f(\Omega_{n-1}, \delta_1)} f(\Omega_n, \delta_1).$$

Here $\Omega_0 = \Omega(0)$, $\Omega_1 = \Omega(0) + \gamma$, where γ is a small magnitude. $\Omega(\delta_1)$ is found by using the iterative form of the "chord method." The initial

approximation for the frequency at $\delta = \delta_2$ can be found by linear interpolation from the known points $\Omega(0)$ and $\Omega(\delta_1)$. Next, specifying the initial approximation of γ and using the iterative formula of the "chord method," we find $\Omega(\delta_2)$. The table of values of the function $\Omega(\delta)$ can be determined in this way. The same procedure is used for determining the values of the function $\Omega(A)$. Due to the large amount of computations required, the calculations are best conducted on high-speed computers.

2. Consider the shell flutter in a gas stream. Let the shell be supported at the ends on hinges mobile in the axial direction. We form the characteristic determinant for these end conditions. Expanding it by the first three rows, we find that the left-hand part of (1.10) consists of 10 terms. The first term is of the form

$$\frac{(z_1+z_2)(z_1+z_3)(z_2+z_3)(z_4+z_5)(z_4+z_6)\ (z_5+z_6)\left[e^{z_4+z_5+z_6}-e^{z_1+z_2+z_3}\right]}{(z_4-z_1)(z_5-z_1)(z_6-z_1)(z_4-z_2)(z_5-z_2)(z_6-z_2)(z_4-z_3)(z_5-z_3)(z_6-z_3)},$$

and the other ones are obtained by replacing the indices 123456, respectively, by

356124	136245	145236
125346	246135	235146
345126	134256	156234

The representation of the left-hand part of equation (1.10) in the form of a sum of terms of the same type is convenient when programing the problems for computerized solution.

At $A \neq 0$ equation (1.9) is a complete sixth-order equation. Its roots can therefore be determined in the general case by numerical methods only. In actual calculations on the "Strela" computer this equation is solved by the slope method /4/.

The calculations were conducted in the absence of damping ($K=0$). For convenience sake, the calculation of the frequency was replaced by the calculation of $\lambda = -\left(\Omega^2 + \delta^4 + \frac{\delta^2(1-\nu^2)}{c}\right)$. The curves of $\lambda(A)$ resemble in their external shape the loops of the (A, λ) plane drawn in /3/. Designate by A_1 the maximum value of the velocity A on the first loop. The velocity A_1 is some characteristic preflutter velocity; its value was calculated for $c = {}^1/_3 \cdot 10^{-6}$, $\nu = {}^1/_3$ and different δ. The results of the calculation are given in the table. The velocity A_1 was calculated without allowing for the inertial forces caused by the tangential displacement /5/. The dependence of this velocity on the curvature is represented in the figure by a dotted line. It is clear that the allowance for the inertial forces influences the determination of the preflutter velocity to an extent which increases with the increase of the curvature.

It should be noted that the calculation of the frequencies from the exact equation requires a great deal of computer time. The calculation of one loop took half an hour on the Strela computer. In order to speed up the calculations, a program based on the Galerkin method was worked out, using a five-term approximation. The calculation time was reduced as a result to about 30 sec.

We do not examine here the accuracy of the Galerkin method. An idea on the reliability of the method can be formed by examining the table.

TABLE

δ	A_1 (rigorous)	A_1 according to Galerkin)
0.00	343	340
0.10	344	341
0.20	354	351
0.30	400	396
0.35	454	449
0.40	545	540
0.45	718	713

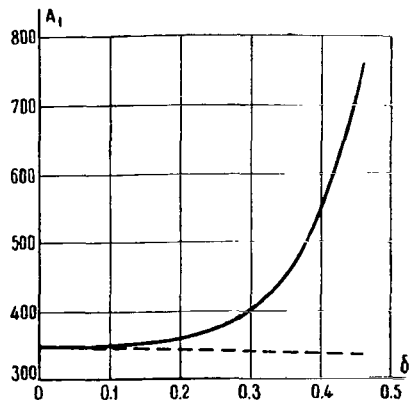

FIGURE.

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BENDING OF FLEXIBLE PLATES BUILT-IN ECCENTRICALLY IN AN ELASTIC CONTOUR

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(Kiev)

The nonlinear problem of calculating a flexible plate is today not only of theoretical interest, but of practical interest as well. Plates whose deflections are of the same order as their thickness are widely used in shipbuilding and in the aircraft industry, and recently also in the building industry.

Such plates generally bend together with the ribs bordering and stiffening them, occupying different positions relative to the latter.

We consider below a rectangular flexible plate rigidly clamped to elastic ribs bordering it in such a way that the neutral surface of the plate is eccentric relative to the rib axes (Figure 1).

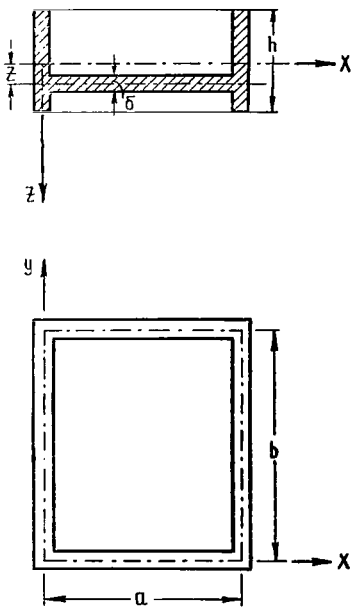

FIGURE 1.

The problem is described by the system of fundamental equations of equilibrium and compatibility (the Karman equations) and by a system of contour equations (the conditions on the contour, written by the author).

The fundamental equations are /1/

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{\delta}{D}\left(\frac{p}{\delta} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right),$$

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]. \tag{2}$$

The contour equations for the edge parallel to the x axis are

$$C\frac{\partial^3 w}{\partial x^2 \partial y} = D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial^2 F}{\partial x^2}\delta Z, \tag{3}$$

$$B_1\frac{\partial^4 w}{\partial x^4} = D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right], \tag{4}$$

$$\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2} = -E\frac{\partial^2 w}{\partial x^2}Z, \tag{5}$$

$$\frac{\partial^3 F}{\partial y^3} + (2+\mu)\frac{\partial^3 F}{\partial x^2 \partial y} = -\frac{E\delta}{B_2}F. \tag{6}$$

Here C is the torsional rigidity of the rib; B_1 is the flexural rigidity of the rib in the vertical plane; B_2 is the flexural rigidity of the rib in the horizontal plane; D is the flexural rigidity of the plate.

Equations (3), (4), (5), and (6) need no explanation. The last was obtained from the equation of the bent rib axis at bending in the plate plane $B_2\frac{\partial^2 v}{\partial x^2} = M$, taking into account that $-E\frac{\partial^2 v}{\partial x^2} = \frac{\partial^3 F}{\partial y^3} + (2+\mu)\frac{\partial^3 F}{\partial x^2 \partial y}$ and using Prof. Varvak's frame analogy /2/, on the basis of which the values of the stress function F on the contour are equal to the values of the bending moments appearing in the frame created by the contour under the load per unit plate thickness acting on the contour.

Either elastic or rigid support can be realized at the corner points; the bordering ribs at the corners can be either rigidly built-in in columns, or rigidly interconnected, forming a frame contour.

In the first case we have at the corner point k

$$\left(\frac{\partial w}{\partial x}\right)_k = 0,$$

and in the second case either

$$B_1\left(\frac{\partial^2 w}{\partial y^2}\right)_k = C\left(\frac{\partial^2 w}{\partial x \partial y}\right)_k, \tag{7}$$

or

$$D\int_0^a\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)dx=2B_1\left(\frac{\partial w}{\partial y^2}\right)_k. \tag{8}$$

Equation (8), which expresses the condition of equilibrium of a rib, is more convenient for use in finite-difference solutions, since it introduces a smaller number of points outside the contour than equation (7).

An additional equation necessary when solving in finite differences is the equation expressing the equality to zero of the area of the moment diagram for the contour, considered as a closed frame:

$$\int_S F dS = 0. \tag{9}$$

Equations (3) to (9) are the most general expressions of the conditions on the contour for both flexible plates and thin slabs. By varying the values of C, B_1, B_2, and Z we can obtain both the "classic" conditions of fastening on the contour (rigid, hinged, and simple support) and various other conditions.

We shall consider several of these.

1) $C=\infty$; $B_2=0$; $B_1\neq 0$ represents a built-in plate whose edges bend freely. The contour equations are

$$\text{a)}\ \frac{\partial w}{\partial y}=0;\qquad \text{b)}\ B_1\frac{\partial^4 w}{\partial x^4}=D\left[\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}\right];$$

$$\text{c)}\ \frac{\partial^2 F}{\partial y^2}=-E\frac{\partial^2 w}{\partial x^2}Z;\qquad \text{d)}\ F=0;$$

2) $C=0$; $B_2=0$; $B_1\neq 0$ represents hinged support on ribs flexible outside their plane. The contour equations are

$$\text{a)}\ \frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}=0;\qquad \text{b)}\ B_1\frac{\partial^4 w}{\partial x^4}=D\left[\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}\right];$$

$$\text{c)}\ \frac{\partial^2 F}{\partial x^2}=0;\qquad \text{d)}\ F=0;$$

3) $C=0$; $B_1=0$; $B_2=0$ represents a flexible plate fastened at four points. The contour equations are

$$\text{a)}\ \frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}=0;\qquad \text{b)}\ \frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}=0;$$

$$\text{c)}\ \frac{\partial^2 F}{\partial x^2}=0;\qquad \text{d)}\ F=0;$$

The following additional equation expresses the immobility of the support points /3/:

$$\int_0^a\left[\frac{\partial^2 F}{\partial y^2}-\frac{1}{2}E\left(\frac{\partial w}{\partial x}\right)^2\right]dx=0.$$

In the case of a relatively large plate rigidity (low values of the parameter

$S=\frac{p}{E}\left(\frac{a}{\delta}\right)^4$), the problem is reduced to that of a thin plate eccentrically built-in in an elastic contour. The fundamental equations are then

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{D}, \tag{10}$$

$$\frac{\partial^4 F}{\partial x^4}+2\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{\partial^4 F}{\partial y^4}=0. \tag{11}$$

The contour equations (3) to (9) remain in force.

It is proposed to solve these problems by the net method, which can be programed for computerized solution.

We shall give the expressions of all the equations in finite differences.

Fundamental equations:

$$\begin{gathered}20\bar{w}_k-8(\bar{w}_a+\bar{w}_b+\bar{w}_c+\bar{w}_d)+2(\bar{w}_e+\bar{w}_f+\bar{w}_g+\bar{w}_h)+\bar{w}_i+\bar{w}_l+\\+\bar{w}_m+\bar{w}_n=\frac{10^4}{n^4}-0.0144(1-\mu^2)^2\frac{p}{E}\left(\frac{a}{\delta}\right)^4[(2\bar{F}_k-\bar{F}_a-\bar{F}_c)\times\\\times(\bar{w}_b-2\bar{w}_k+\bar{w}_d)+(2\bar{F}_k-\bar{F}_b-\bar{F}_d)(\bar{w}_a-2\bar{w}_k+\bar{w}_c)+\\+0.125(\bar{F}_e-\bar{F}_f+\bar{F}_g-\bar{F}_h)(\bar{w}_e-\bar{w}_f+\bar{w}_g-\bar{w}_h),\end{gathered} \tag{12}$$

$$\begin{gathered}20\bar{F}_k-8(\bar{F}_a-\bar{F}_b+\bar{F}_c+\bar{F}_d)+2(\bar{F}_e+\bar{F}_f+\bar{F}_g+\bar{F}_h)+\\+\bar{F}_i+\bar{F}_l+\bar{F}_m+\bar{F}_n=12(1-\mu^2)\frac{p}{E}\left(\frac{a}{\delta}\right)^4\times\\\times\left[\left(\frac{\bar{w}_f-\bar{w}_e+\bar{w}_h-\bar{w}_g}{4}\right)^2-(\bar{w}_b-2\bar{w}_k+\bar{w}_d)(\bar{w}_a-2\bar{w}_k+\bar{w}_c)\right].\end{gathered} \tag{13}$$

In the general case, the contour equations are

$$\begin{gathered}\left(\frac{Cn}{Da}+1\right)\bar{w}_a-\frac{Cn}{2Da}(\bar{w}_e+\bar{w}_f-\bar{w}_g-\bar{w}_h)-\left(\frac{Cn}{Da}-1\right)\bar{w}_c-\\-2(\mu+1)\bar{w}_k+\mu(\bar{w}_b+\bar{w}_d)=6(1-\mu^2)\frac{h-\delta}{\delta}(2\bar{F}_k-\bar{F}_b-\bar{F}_d),\end{gathered} \tag{14}$$

$$\begin{gathered}12\frac{B_1 n}{Da}\bar{w}_k+(6-2\mu)(\bar{w}_a-\bar{w}_c)-8\frac{B_1 n}{Da}(\bar{w}_b+\bar{w}_d)+2\frac{B_1 n}{Da}(\bar{w}_n+\bar{w}_l)-\\-(2-\mu)(\bar{w}_e+\bar{w}_f-\bar{w}_h-\bar{w}_g)-\bar{w}_i-\bar{w}_m=0,\end{gathered} \tag{15}$$

$$(\bar{w}_b-2\bar{w}_k+\bar{w}_d)\frac{z}{\delta}+2\bar{F}_k(\mu-1)+\bar{F}_a+\bar{F}_c-\mu(\bar{F}_b+\bar{F}_d)=0, \tag{16}$$

$$\begin{gathered}2(3+\mu)(\bar{F}_a-\bar{F}_c)-\\-(2+\mu)(\bar{F}_e+\bar{F}_f-\bar{F}_g-\bar{F}_h)+\bar{F}_i-\bar{F}_m-\frac{2E\delta a^3}{B_2 n^3}\bar{F}_k=0.\end{gathered} \tag{17}$$

Here

$$\bar{w}=w\cdot 10^4\frac{D}{pa^4};\qquad \bar{F}=F\,10^4\frac{D}{Epa^4\delta};\qquad n=\frac{a}{\Delta x}.$$

A square net is used; the designations of the nodal points are shown in Figure 2.

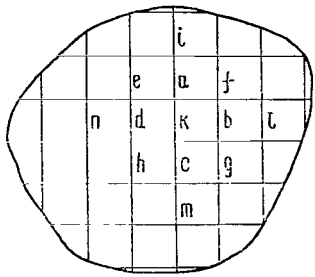

FIGURE 2.

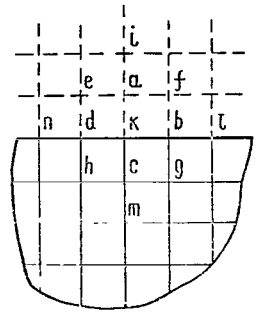

FIGURE 3.

When solving by the net method, the unknowns are the deflections and the stress functions at the points within the contour, at the contour points, and at the first and second points outside the contour (k, a, i, see Figure 3), six unknowns in all at every point of the contour. For each such point it is recommended to write four contour equations and two fundamental ones. As the comparative calculations conducted by the author have shown, such an approach ensures a considerably higher accuracy than the use of various interpolative formulas for eliminating the unknowns in the points outside the contour (instead of writing the fundamental equations for the contour points). Equation (1) for point k of the contour, in finite differences, allowing for the expression of the complete reaction in the Kirchhoff sense and half the length of the element in the direction of the y axis, becomes of the form

$$(19-4\mu)\,\bar{w}_k-(9-2\mu)(\bar{w}_a+\bar{w}_c)-(7-2\mu)(\bar{w}_b+\bar{w}_d)+$$
$$+(2.5-\mu)(\bar{w}_e+\bar{w}_f+\bar{w}_g+\bar{w}_h)+\bar{w}_l+\bar{w}_m+0.5(\bar{w}_l+\bar{w}_n)=$$
$$=\frac{10^3\cdot 5}{n^4}-0.0144(1-\mu^2)\frac{p}{E}\left(\frac{a}{\delta}\right)^4\Big[0.5(2\bar{F}_k-\bar{F}_a-\bar{F}_c)(\bar{w}_b-2\bar{w}_k+\bar{w}_d)+$$
$$+(2\bar{F}_k-\bar{F}_b-\bar{F}_d)(\bar{w}_a-2\bar{w}_k+\bar{w}_c)+$$
$$+0.09375(\bar{F}_e-\bar{F}_f-\bar{F}_g-\bar{F}_h)(\bar{w}_e-\bar{w}_f+\bar{w}_g-\bar{w}_h)\Big]. \qquad (18)$$

It is suggested to conduct the solution in the following sequence.

The linear problem of bending of a thin plate eccentrically built-in in an elastic contour (equations (10) and (11)) is solved first. F_i^0 and $\bar{w}^0$ are values of the stress functions and the deflections at the nodal points. The nets of such a plate are considered as first approximations of the nonlinear problem.

Using the system of contour equations (14) to (16) and the additional equations (8) and (9), we express $\bar{w}$ and $\bar{F}$ in the points outside the contour through $\bar{w}$ and $\bar{F}$ in the points on the contour and inside it. These expressions are substituted in the system of fundamental equations (12), (13), and (18). Considering the right-hand parts of these equations as some free terms, we solve separately system (10)-(16) and (11) in influence numbers. We obtain the matrix of the influence numbers for $\bar{w}-\alpha_{ik}$ and $\bar{F}-\beta_{ik}$.

Having the values of $\overline{w}$ and $\overline{F}$ in a first approximation ($\overline{w}^{\circ}$ and $\overline{F}^{\circ}$) we calculate the right-hand parts of equations (10)-(16) and (11) and then the second approximations of $\overline{w}$ and $\overline{F}$ by means of the matrices of the influence numbers α_{ik} and β_{ik}, respectively. The third approximations of $\overline{w}$ and $\overline{F}$ are then similarly calculated, and so on until the values of $\overline{w}$ and $\overline{F}$ for two consecutive approximations practically coincide. Such an iterative process is convenient for computer programing. In the process of the solution it is necessary to pay attention to the writing of the equations for the corner point, using (8) and (9).

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SOME PROBLEMS OF SHELL DYNAMICS

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1. FREE AND FORCED VIBRATIONS OF SHELLS OF REVOLUTION AND STRUCTURES REPRESENTING COMBINATIONS OF SUCH SHELLS

A large number of original papers have been devoted to the examination of free vibrations of shells of revolution of various shapes, and it is impossible to survey them all within the scope of the present paper. We shall restrict ourselves here to a brief survey of recent studies in this region. All these studies are based as a rule on the classic theory of calculation of moment shells. The case of a cylindrical isotropic shell has been most widely studied. In papers /1/ to /11/ are given rigorous (within the frame of the starting hypotheses) and approximate solutions of the problem of free vibrations. In /7/ the problem is solved without using the kinematic hypotheses which characterize the theory of thin shells. In /3/ and /4/ the influence of the shear and of the inertia of rotation is allowed for. Orthotropic cylindrical shells and shells stiffened by longitudinal ribs and lateral rings are considered in /10/ to /15/.

The determination of the spectrum of frequencies of the free vibrations of spherical shells was considered in /16/ to /27/, while the dynamics of conic shells were considered in /20/ and in /28/ to /36/. The paper /37/ was devoted to the dynamic calculation of a paraboloidal shell of revolution.

Shells of revolution of an arbitrary shape were studied in /38/ to /40/.

Without entering in the details of the methods of solution of the problems mentioned, we only note that the determination of the frequencies of the free vibrations and of the corresponding modes involves a) the formation of the equation, b) the solution of the equation obtained, c) the examination of the boundary conditions, and d) the determination of the frequency equation.

Since we are considering here closed shells of revolution having homogeneous boundary conditions along the corresponding parallels, the solution of the problem of the modes of free vibrations leads to one resolving differential equation of order eight. When only axisymmetrical vibrations are considered the order of the equation is reduced to six; this particular case will not, however, be considered separately below.

In the most general case of a shell of revolution, the determination of its state of strain and stress necessitates the knowledge of three displacements u, v, w along the meridian, parallel, and radius respectively, ϑ, the angle of rotation of the tangent to the meridian, and four force

factors: N_1, the axial force per unit length along the meridian, S, the shearing force per unit length, M_1, the bending moment along the meridian, and Q_1, the lateral force along the meridian. The solution of the initial equations and the existence of known differential relationships between the kinematic and force factors lead to the following system of functions:

$$\begin{aligned}
u_\theta &= A_1 f_{11}(\theta, m, k) + A_2 f_{12}(\theta, m, k) + \cdots + A_8 f_{18}(\theta, m, k);\\
v_\theta &= A_1 f_{21}(\theta, m, k) + A_2 f_{22}(\theta, m, k) + \cdots + A_8 f_{28}(\theta, m, k);\\
w_\theta &= A_1 f_{31}(\theta, m, k) + A_2 f_{32}(\theta, m, k) + \cdots + A_8 f_{38}(\theta, m, k);\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
Q_{1\theta} &= A_1 f_{81}(\theta, m, k) + A_2 f_{82}(\theta, m, k) + \cdots + A_8 f_{88}(\theta, m, k);
\end{aligned} \tag{1}$$

which will be used below in the matrix notation

$$U_\theta = F(\theta, m, k)\, A \tag{2}$$

Here U_θ and A are matrix columns, and F a quadratic matrix

$$U_\theta = \begin{Vmatrix} u_\theta \\ v_\theta \\ w_\theta \\ \vartheta_\theta \\ N_{1\theta} \\ S_\theta \\ M_{1\theta} \\ Q_{1\theta} \end{Vmatrix}; \quad A = \begin{Vmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_8 \end{Vmatrix}; \quad F(\theta, m, k) = \begin{Vmatrix} f_{11}(\theta, m, k); f_{12}(\theta, m, k) \ldots f_{18}(\theta, m, k) \\ f_{21}(\theta, m, k)\ f_{22}(\theta, m, k) \ldots f_{28}(\theta, m, k) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ f_{81}(\theta, m, k)\ f_{82}(\theta, m, k) \ldots f_{88}(\theta, m, k) \end{Vmatrix}$$

Here A_1, A_2, etc. are arbitrary constants, devoid in the general case of clear physical meaning; θ is the polar angle, characterizing the position of the parallel considered in the spherical shell, or the distance from some reference point in the cylindrical or conical shell; m is the number of waves in the direction of the parallel; k is an unknown magnitude, equal up to some constant factor to the corresponding frequency of free vibrations.

It is expedient to pass from expressions (1) to similar formulas represented in the form of the method of initial parameters, when the arbitrary constants are the kinematic and force factors. Write expression (2) relative to the shell edge, defined by the angle α_0:

$$U_{\alpha_0} = F(\alpha_0, m, k)\, A. \tag{3}$$

Multiply both sides of (3) by the matrix $F^{-1}(\alpha_0, m, k)$. We obtain as a result

$$F^{-1}(\alpha_0, m, k)\, U_{\alpha_0} = A. \tag{4}$$

Using the expression obtained it is possible to find the displacements and forces per unit length at any section of the shell, if the frequency of free vibrations and the initial parameters are known:

$$U_\theta = F(\theta, m, k)\, F^{-1}(\alpha_0, m, k)\, U_{\alpha_0}. \tag{5}$$

Expression (5) relative to the right-hand edge of the shell, characterized by the angle α_1, is

$$U_{\alpha_1} = \Psi(k)\, U_{\alpha_0}, \tag{6}$$

where the matrix $\Psi(k)$ represents the product

$$\Psi(k) = F(\alpha, m, k)\, F^{-1}(\alpha_0, m, k). \tag{7}$$

From expression (6) it is easy to obtain the frequency equation at different support conditions. Thus, if the left-hand edge of the shell is free while the right-hand one is clamped, it is necessary to take the first four rows of the matrix $\Psi(k)$, dropping the last four rows. The frequency equation will be, in a determinant form:

$$\begin{vmatrix} \varphi_{11}(k) & \varphi_{12}(k) & \varphi_{13}(k) & \varphi_{14}(k) \\ \varphi_{21}(k) & \varphi_{22}(k) & \varphi_{23}(k) & \varphi_{24}(k) \\ \varphi_{31}(k) & \varphi_{32}(k) & \varphi_{33}(k) & \varphi_{34}(k) \\ \varphi_{41}(k) & \varphi_{42}(k) & \varphi_{43}(k) & \varphi_{44}(k) \end{vmatrix} = 0.$$

Since the expansion of the determinant leads to a fairly complex expression, whose solution in a closed form is impossible in the general case, the method of trial and error seems indicated for solving this equation.

The matrix notation of the equations determining the vibration of a shell of revolution is especially efficient in the case of a shell made of several separate parts, such as a combination of a truncated cone, a cylinder, and a sphere. We first assume that this complex shell is fastened only by the end sections.

We now introduce the concept of transition matrix, which establishes the connection between the displacements and the forces at the ends of two contiguous parts of the shell. Let us consider the junction of part i-1 with part i. The end i-1 of the shell is defined by the angle α_1^{i-1}, and the end i by the angle α_0^i. The forces and displacements will be connected by the condition

$$U_{\alpha_0^i} = S_{\alpha_1^{i-1}}^{\alpha_0^i} U_{\alpha_1^{i-1}}, \tag{8}$$

where

$$U_{\alpha_0^i} = \begin{Vmatrix} u \\ v \\ w \\ \vartheta \\ N_1 \\ S \\ M_1 \\ Q_1 \end{Vmatrix}_{\alpha_0^i}$$

$$S_{\alpha_1^{i-1}}^{\alpha_0^i} = \begin{Vmatrix} \cos(\alpha_0^i - \alpha_1^{i-1}) & 0 & \sin(\alpha_0^i - \alpha_1^{i-1}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin(\alpha_0^i - \alpha_1^{i-1}) & 0 & \cos(\alpha_0^i - \alpha_2^{i-1}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin(\alpha_0^i + \alpha_1^{i-1}) & 0 & 0 & \cos(\alpha_0^i + \alpha_1^{i-1}) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\cos(\alpha_0^i + \alpha_1^{i-1}) & 0 & 0 & \sin(\alpha_0^i + \alpha_1^{i-1}) \end{Vmatrix}$$

$$U_{\alpha_1^{l-1}} = \begin{Vmatrix} u \\ v \\ w \\ \vartheta \\ N_1 \\ S \\ M_1 \\ Q_1 \end{Vmatrix}_{\alpha_1^{l-1}}$$

The forces and displacements at the ends of an arbitrary shell of revolution can now be connected by the expression

$$U_{\alpha_1^n} = \Psi_n \left(\prod_{l=n-1}^{l=1} S_{\alpha_1^l}^{\alpha_0^{l+1}} \Psi_l \right) U_{\alpha_0^1}.$$

n is the number of components of the complex structure. The expression $\Psi_n \left(\prod_{l=n-1}^{l=1} S_{\alpha_1^l}^{\alpha_0^{l+1}} \Psi_l \right)$ is a quadratic matrix of order eight. The problem of determining the determinant corresponding to the frequency equation is solved now in exactly the same way as in the case of a simple shell.

In the case when external constraints are imposed on the shell not only at the ends, but in the intermediate sections as well, the application of the method of initial parameters is not always justified. In separate cases it is possible to use the method of forces, the method of displacements, or a mixed method. The scope of this paper does not allow us to give specific examples.

After the frequencies of the free vibrations have been determined and the corresponding modes have been found, the problem of the calculation of forced vibrations can be solved by elementary means, consisting of expanding the dynamic load by the modes of the natural vibrations.

2. DETERMINING THE SPECTRUM OF FREQUENCIES AND THE MODES OF VIBRATIONS OF A SPHERICAL SHELL ON AN ELECTRONIC COMPUTER

The problem was solved on an electronic computer on the strength of the formulas given in /26/, and the values of the frequencies of natural vibrations were determined for various boundary conditions, various ratios of the shell radius to its thickness, and various shell angles. Below are given the first three frequencies for rigidly clamped and hinged domes at a Poisson ratio $\mu = 0.25$ and $\alpha = 90°$:

$$\omega = \frac{k}{R} \sqrt{\frac{E}{(1-\mu^2)\rho}}.$$

$\frac{R}{\delta}$	Clamped edge k_1	k_2	k_3	Hinged edge k_1	k_2	k_3
20	1.245	1.501	1.729	1.196	1.499	1.629
50	1.094	1.242	1.437	1.071	1.207	1.400
100	1.036	1.105	1.202	1.025	1.088	1.178
200	0.998	1.027	1.067	0.991	1.020	1.057

3. EXPERIMENTAL TEST

As already mentioned earlier /26/, many of the approximate methods used give results differing from those obtained in the exact solution of the equations of dynamics of the general linear theory of shells. Note that the results obtained in /41/ for a shell with free edges coincide with the results obtained in /26/.

A semisphere of radius 12.5 cm and thickness 0.25 cm was clamped by the edge. The vibrations were excited by a symmetrically acting load. They were recorded on a cathode-ray oscillograph by means of a piezoelectric gage. The vibrations bore the pattern of beats of period 1.07×10^{-3} sec; the period of one vibration was 0.132×10^{-3} sec. It was obtained theoretically that the maximum period is equal to 0.139×10^{-3} sec, and the beat period to 1.03×10^{-3}; as can be seen, the theoretical and experimental results are sufficiently close.

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INFLUENCE OF THE RIGIDITY OF TRANSVERSE DIAPHRAGMS ON THE TEMPERATURE STRESSES IN THIN-WALLED SYSTEMS (SHELLS)

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(Khar'kov)

An engineering method of calculating the temperature stresses in thin-walled bars and shells of closed and open profile is proposed. With this method the influence of various structural features of the system (rigidity of the transverse diaphragms, skin thickness, conditions of support, etc.) and of the parameters of a two-dimensional temperature field can be assessed relatively simply and reliably. The longitudinal ribs are allowed for approximately by reducing the system to a structurally orthotropic one. The shell structure can be either homogeneous or made from different materials.

The analysis is conducted for a system free of external forces. The external forces are allowed for independently /2/.

The temperature field for the entire system or for a part of it is considered as given in the form

$$T(x,\ s) = T(x)\,T(s), \tag{1}$$

where $T(x)$ and $T(s)$ are functions defining the law of variation of the temperature along the axis of the thin-walled system, x, and along the middle line of the cross section, s, respectively (Figure 1,a).

The temperature drop through the skin thickness is neglected.

We assume that the longitudinal elongations of the system fibers are absolutely restrained as a result of the introduction of rigid undeformable constraints, i.e., that the total deformation is equal to zero:

$$\varepsilon_n = \varepsilon_t + \varepsilon_y = 0, \tag{2}$$

where ε_n, ε_t, and ε_y are, respectively, the total, temperature, and elastic deformations of the longitudinal fibers of the thin-walled system.

Such a restraint of the total deformation exists in the case of an infinitely long shell.

The system stresses corresponding to such a state will be called primary.

Using the simplest physical law of Hooke and condition (2), we find the longitudinal normal stresses (Figure 1,a)

$$\sigma_{x1} = \varepsilon_y E_t = -\varepsilon_t E_t = -\alpha_t\, T(x,\ s)\, E_t = -\eta_1(s)\, T(x), \tag{3}$$

where α_t is the coefficient of linear expansion of the material,

$$\eta_1(s) = \alpha_t E_t T(s).$$

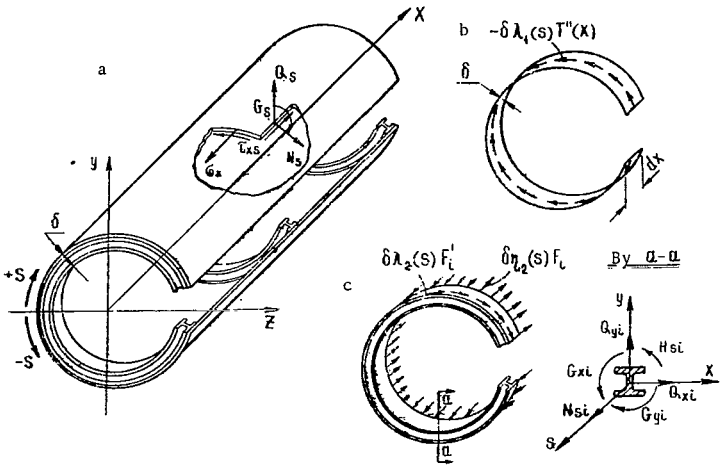

FIGURE 1.

In order to simplify the calculations we shall assume in what follows that E_t and α_t are constant and equal to some averaged value of the modulus of elasticity and the coefficient of linear expansion, respectively, by the temperature. This simplification is of no basic importance, it being possible to allow by the method proposed for the variation of the material characteristics (α_t, E_t, G_t) with the temperature.

From the differential equation of equilibrium of an element of the middle surface of the shell we obtain (Figure 1,a)

$$\tau_{xs1} = -\lambda_1(s)\, T'(x), \tag{4}$$

where

$$\lambda_1(s) = \frac{1}{\delta}\left[\frac{\oint\left(\int_s \eta_1(s)\,\delta\, ds\right)\rho\, ds}{\oint \rho\, ds} - \int_s \eta_1(s)\,\delta\, ds\right]$$

for systems of closed profile, and

$$\lambda_1(s) = \frac{1}{\delta}\left[\int_{\bar{s}_1} \eta_1(s)\,\delta\, ds - \int_s \eta_1(s)\,\delta\, ds\right]$$

for systems of open profile.

Here ρ is the arm of the shearing stresses relative to some points, and $\bar{s}_1$ is the semi-perimeter of the middle line of the cross section.

The transverse forces in the longitudinal sections (G_{s1}, Q_{s1}, N_{s1}) are determined from a calculation of an elementary strip of the system cut out

by two cross sections x = const and $x+dx$ = const and loaded along the contour by the drop of shearing forces $\frac{\partial(\delta\tau_{xsI})}{\partial x} = -\lambda_1(s)\,T''(x)$ (Figure 1,b)

$$G_{sI} = -\gamma_1(s)\,T''(x). \tag{5}$$

In the constraints introduced there appear simultaneously forces whose resultants are

$$N(x)_I = \int_s \sigma_{xI}\delta\,ds, \qquad M_z(x)_I = \int_s \sigma_{xI}\,y\delta\,ds,$$
$$M_y(x)_I = \int_s \sigma_{xI}z\delta\,ds, \qquad M_{cr}(x)_I = \int_s \delta_{xsI}\,\delta\rho\,ds, \tag{6}$$

where $M(x)_I$, $M_z(x)_I$, $M_y(x)_I$, and $M_{cr}(x)_I$ are the resultants of the longitudinal forces, bending moments, and torsional moments; s is the perimeter of the middle line of the cross section; y, z are coordinates of the middle line of the profile in the principal central axes.

In the secondary state we remove the additional rigid constraints, applying to the system external loads equal in magnitude to the forces in the constraints but oppositely directed. The resultants of these forces are

$$N(x)_{II} = -N(x)_I, \quad M_z(x)_{II} = -M_z(x)_I, \quad M_y(x)_{II} = -M_y(x)_I,$$
$$M_{cr}(x)_{II} = -M_{cr}(x)_I. \tag{7}$$

The stresses in the system appearing in isothermal conditions under the action of the external loads (7) will be called secondary. In the determination of these stresses we shall make use of the well-known Papkovich theorem relative to the possibility of splitting the stress tensor into a fundamental one, which satisfies the equilibrium conditions, and an additional tensor.

If we take as fundamental stressed state the primary state with reverse sign, the equilibrium equations will be tranformed into identities. Then

$$\left.\begin{aligned} \sigma_x\,\text{fund} &= -\sigma_{xI} = \eta_1(s)\,T(x), \\ \tau_{xs}\,\text{fund} &= -\tau_{xsI} = \lambda_1(s)\,T'(x), \\ G_s\,\text{fund} &= -G_{sI} = \gamma_1(s)\,T''(x). \end{aligned}\right\} \tag{8}$$

The additional self-equilibrated stress tensor will be determined by the variational method. As independent unknown we select the additional normal stress, which will be represented in the form

$$\sigma_x\,\text{ad} = \eta_2(s)\,F(x), \tag{9}$$

where $\eta_2(s)$ is a function of the arc coordinate s, and $F(x)$ is a function of the coordinate x along the system axis.

We leave $F(x)$ as an unknown function, and specify $\eta_2(s)$, subjecting it to the conditions of self-equilibration of the additional stresses:

for systems of closed profile

$$\eta_2(s) = T(s) - \frac{\int_s T(s)\,\delta\,ds}{F} - \frac{\int_s T(s)\,y\delta\,ds}{J_{yy}}\,y - \frac{\int_s T(s)\,z\,\delta ds}{J_{zz}}\,z; \tag{10}$$

for systems of open profile

$$\eta_2^*(s) = T(s) - \frac{\int\limits_s T(s)\,\delta ds}{F} - \frac{\int\limits_s T(s)\,y\delta ds}{J_{yy}}\,y - \frac{\int\limits_s T(s)\,z\delta ds}{J_{zz}}\,z +$$

$$+ \frac{\beta_t - \beta_0 \frac{\int\limits_s T(s)\,\delta ds}{F} - \beta_y \frac{\int\limits_s T(s)\,y\delta ds}{J_{yy}} - \beta_z \frac{\int\limits_s T(s)\,z\delta ds}{J_{zz}}}{\beta_y \frac{\int\limits_s y\varphi\delta ds}{J_{yy}} + \beta_z \frac{\int\limits_s z\varphi\delta ds}{J_{zz}} - \beta_c} \times$$

$$\times \left(\varphi - \frac{\int\limits_s y\varphi\delta ds}{J_{yy}}\,y - \frac{\int\limits_s z\varphi\delta ds}{J_{zz}}\,z\right). \tag{11}$$

Here

$$\varphi = zy, \quad J_{yy} = \int\limits_s y^2\delta ds, \quad J_{zz} = \int\limits_s z^2\delta ds, \quad F = \int\limits_s \delta ds,$$

$$\beta_0 = \int\limits_s \left[\int\limits_{s_1} \delta ds - \int\limits_s \delta\, ds\right]\rho\, ds, \quad \beta_y = \int\limits_s \left[\int\limits_{s_1} y\delta\, ds - \int\limits_s y\delta\, ds\right]\rho\, ds, \tag{12}$$

$$\beta_z = \int\limits_s \left[\int\limits_{s_1} z\delta\, ds - \int\limits_s z\delta\, ds\right]\rho\, ds, \quad \beta_\varphi = \int\limits_s \left[\int\limits_{s_1} \varphi\delta\, ds - \int\limits_s \varphi\delta\, ds\right]\rho\, ds.$$

The additional shearing stresses and forces in the longitudinal sections are determined similar to the primary stresses and forces (4), (5):

$$\begin{aligned} \tau_{xs\,\mathrm{ad}} &= \lambda_2(s)\,F'(x), \\ G_{s\,\mathrm{ad}} &= \gamma_2(s)\,F''(x). \end{aligned} \tag{13}$$

The unknown function $F(x)$ will be found by the strain energy method, using the Castigliano principle. We include in the expression for the energy the potential energy of the fundamental deformations only

$$U = \int\limits_l \left[\int\limits_s \frac{(\sigma_{x\,\mathrm{fund}} + \sigma_{x\,\mathrm{ad}})^2}{2E_t}\,\delta\, ds + \int\limits_s \frac{(G_{s\,\mathrm{fund}} + G_{s\,\mathrm{ad}})^2}{2D_t}\, ds\right] dx.$$

It follows from Euler's differential equation that

$$F^{\mathrm{IV}}(x) + 4k^4 F(x) = -\frac{a_{21}}{d_{22}}\,T(x) - \frac{d_{21}}{d_{22}}\,T^{\mathrm{IV}}(x), \tag{14}$$

where

$$a_{2i} = \int\limits_s \frac{\eta_2(s)}{E_t}\,\eta_i(s)\,\delta\, ds, \quad d_{2i} = \int\limits_s \frac{\gamma_2(s)}{D_t}\,\gamma_i(s)\, ds \quad (i = 1,\ 2).$$

$$k=\sqrt[4]{\frac{a_{22}}{4d_{22}}}, \qquad D_i=\frac{\delta^3 E_i}{12(1-\mu_i^2)}.$$

For systems of closed profile $a_{21}=\alpha_i E_i a_{22}$.

The solution of differential equation (14) by the method of initial parameters is

$$F(x)=F(0)\,Y_1(x)+F'(0)\frac{Y_2(x)}{k}+F''(0)\frac{Y_3(x)}{k^2}+F'''(0)\frac{Y_4(x)}{k^3}+$$
$$+\frac{1}{k^3}\int_0^x\left[-\frac{a_{21}}{d_{22}}T(t)-\frac{d_{21}}{d_{22}}T^{\mathrm{IV}}(t)\right]Y_4(x-t)\,dt+$$
$$+\sum_{i=1}^{m}\left[F_i\,Y_1(x-x_i)+F_i'\frac{Y_2(x-x_i)}{k}\right], \tag{15}$$

where $F(0)$, $F'(0)$, $F''(0)$, and $F'''(0)$ are initial parameters; $Y_1(x)$, $Y_2(x)$, $Y_3(x)$, and $Y_4(x)$ are Krylov functions; F_i and F_i' are concentrated parameters, allowing for the influence of the intermediate transverse diaphragm elastic in its plane and out of its plane.

The parameters F_i and F_i' are determined using the natural boundary conditions of the mixed variational problem /1/

$$\left[\frac{\partial\Gamma}{\partial F'(x)}-\frac{d}{dx}\frac{\partial\Gamma}{\partial F''(x)}\right]_{x=x_i}-\frac{\partial U_{\partial i}}{\partial F_i}=0, \tag{16}$$

$$\left[\frac{\partial\Gamma}{\partial F''(x)}\right]_{x=x_i}-\frac{\partial U_{\partial i}}{\partial F_i'}=0, \tag{17}$$

where

$$\Gamma=\int_s\frac{\sigma_x^2}{2E_i}\delta\,ds+\int_s\frac{G_s^2}{2D_i}ds,$$

$$U_{\partial i}=\int_s\frac{G_{xi}^2}{2E_iJ_{xi}}ds+\int_s\frac{G_{yi}^2}{2E_iJ_{yi}}ds+\int_s\frac{H_{si}^2}{2G_iJ_{di}}ds.$$

G_{xi}, G_{yi}, and H_{si} are bending moments and torsional moment (Figure 1,c); J_{xi}, J_{yi}, and J_{di} are moments of inertia of the diaphragm cross section at bending and at pure torsion, respectively.

Equations (16) and (17) follow also from the conditions of compatibility of the strains of the skin and diaphragm in the longitudinal and lateral directions.

The forces G_{xi}, G_{yi}, and H_{si} are found from the calculation of the diaphragm as a tridimensional structure, loaded by the drop of the self-equilibrated shearing forces $\delta\lambda_2(s)F_i'$ and normal forces $\delta\eta_2(s)F_i$ (Figure 1,c):

$$G_{xi}=\gamma_{xi}(s)F_i', \quad G_{yi}=\gamma_{yi}(s)F_i, \quad H_{si}=\gamma_{si}(s)F_i. \tag{18}$$

Expanding conditions (16) and (17) with the aid of (18), we obtain

$$F_i=a_i\,J_{yi}F'''(x_i), \qquad F_i'=B_i\,J_{xi}F''(x_i), \tag{19}$$

where

$$a_l = -\frac{d_{22}E_l}{\int\limits_{\bar{s}} \gamma_{yl}^2(s)\,\mathrm{d}s + \frac{E_l J_{yi}}{G_l J_{dl}} \int\limits_{\bar{s}} \gamma_{sl}^2(s)\,\mathrm{d}s}, \qquad B_l = \frac{d_{22}E_l}{\int\limits_{\bar{s}} \gamma_{xl}^2(s)\mathrm{d}s}.$$

The boundary conditions for determining the initial parameters are similarly expanded:

a) at $x_i = 0$

$$F(0) = a_0 J_{y0} F'''(0), \qquad F'(0) = b_0 J_{x0} F''(0), \tag{20}$$

b) at $x_i = l$ /1/

$$F(l) = -a_l J_{yl} F'''(l), \qquad F'(l) = -b_l J_{xl} F''(l). \tag{21}$$

In the two extreme cases of absolutely rigid and absent end diaphragms, the values $J_{xl} \to \infty$, $J_{yi} \to \infty$ and $J_{xl} = 0$, J_{yl}, respectively, must be substituted in (20) and (21). Other variants of the boundary conditions can be similarly considered.

For a system with undeformable cross-sectional contour

$$F''(x) - u^2 F(x) = \frac{a_{21}}{c_{22}} T(x) - \frac{c_{21}}{c_{22}} T''(x), \tag{22}$$

where

$$c_{2i} = \int\limits_{\bar{s}} \frac{\lambda_2(s)}{G_l} \lambda_i(s)\,\delta\,\mathrm{d}s \qquad (i = 1,\ 2), \qquad u = \sqrt{\frac{a_{22}}{c_{22}}}.$$

The solution of differential equation (22) by the method of initial parameters is

$$F(x) = F(0)\,\mathrm{ch}\,ux + F'(0)\frac{\mathrm{sh}\,ux}{u} +$$

$$+ \frac{1}{u}\int\limits_0^x \left[\frac{a_{21}}{c_{22}} T(t) - \frac{c_{21}}{c_{22}} T''(t)\right] \mathrm{sh}\,(ux - ut)\,dt + \sum_{i=1}^{m} F_i\,\mathrm{ch}\,(ux - ux_i). \tag{23}$$

The parameter F_i is found from equation (16)

$$F_i = -\frac{c_{22}d_{22}}{a_i} J_{yi} F'(x_i). \tag{24}$$

The boundary conditions for determining the initial parameters are similarly expanded /1/:

$$F(0) = -\frac{c_{22}d_{22}}{a_0} J_{y0} F'(0), \qquad F(l) = \frac{c_{22}d_{22}}{a_l} J_{yl} F'(l). \tag{25}$$

Other variants of the boundary conditions follow from relationships (25) by passing to the limit $J_{yi} = 0$ (no diaphragm) or $J_{yl} \to \infty$ (absolutely rigid diaphragm).

The total stresses in the thin-walled system are equal to the sum of the stresses of the primary and secondary states, i.e.,

$$\sigma_x = \sigma_{x1} + \sigma_{x\,\mathrm{fund}} + \sigma_{x\,\mathrm{ad}} = \sigma_{x\,\mathrm{ad}} = \eta_2(s)\,F(x), \tag{26}$$

since, according to (8), $\sigma_{x\,\mathrm{fund}} = -\sigma_x$.

Similarly

$$\begin{aligned} \tau_{xs} &= \tau_{xs\,\mathrm{ad}} = \lambda_2(s)\,F'(x), \\ G_s &= G_{s\,\mathrm{ad}} = \gamma_2(s)\,F''(x). \end{aligned} \tag{27}$$

i.e., finally the temperature stresses are equal to the additional stresses.

Figure 2 represents the variation of the maximum values of the temperature normal stresses as a function of the rigidity of the end diaphragm elastic in its plane, and of the length of the system at $T(x) = \mathrm{const}$, while Figure 3 represents the same functional relationships for $T(x) = bx$. Such curves can easily be plotted for different variants of the boundary conditions and the function $T(x)$ as well.

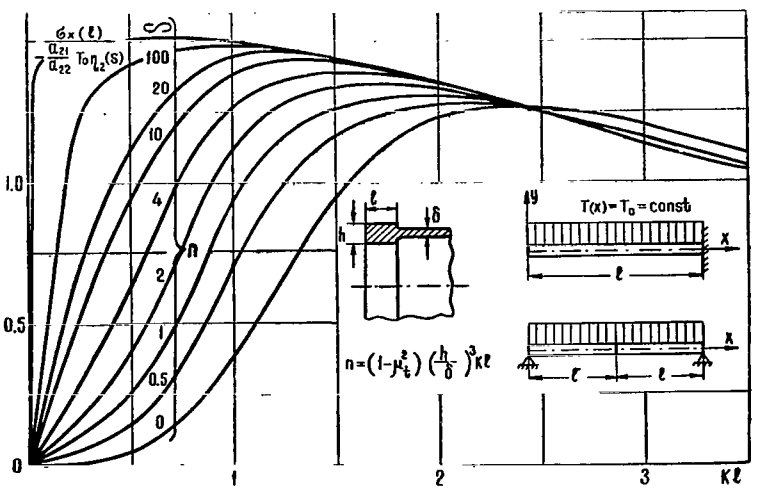

FIGURE 2.

The analysis of the relationships given shows that a considerable increase of the value of $\sigma_{x\max}$ with the increase of the rigidity of the end diaphragm is observed only for $kl < 2.5$ to 3.5. At larger values of kl, $\sigma_{x\max} \cong 1$ and is practically independent of the diaphragm rigidity and the system length. The end diaphragm then noticeably influences only the lateral bending moments $G_{s\max}$.

Example. Let it be required to determine the maximum temperature stresses of a thin-walled system of open profile (Figure 4).

Solution. According to (11)

$$\eta_2(s) = T(s) - 0.5T_0 - 0.54\frac{y}{h}T_0 - 0.033\frac{zy}{h^2}T_0.$$

Having determined the function $\eta_2(s)$ from the calculation of an elementary strip of the system, we find

$$a_{21} = 0.97\,\alpha_t\,E_t\,a_{22}, \qquad k = \frac{0.44}{h}\sqrt{\frac{\delta}{h}}.$$

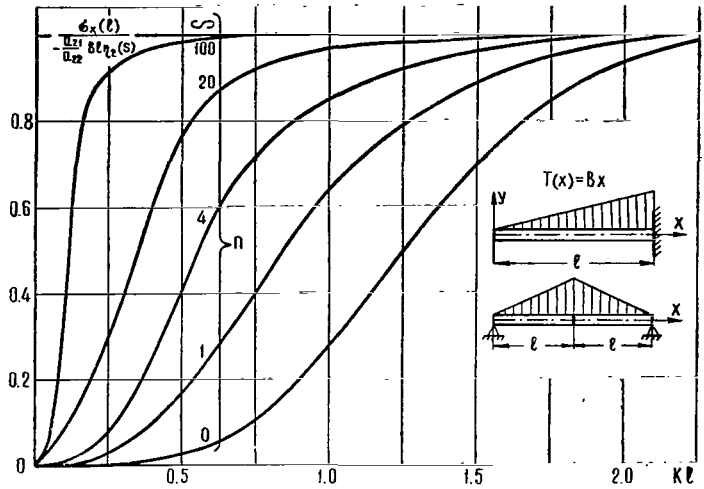

FIGURE 3.

Let $\delta = 8$ mm, $2h = 95$ mm, $l = 816$ mm, $T(x, s) = T_0 = 100°$, $\alpha_t = 11.1\times10^{-6}$, $E_t = 2.1\times10^6$ kg/cm^2.

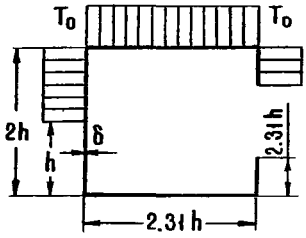

FIGURE 4.

Using Figure 2, we consider the following cases:

a) a simply supported system with rigid diaphragm at the middle

$$\sigma_{x\max} = -0.368\,\alpha_t\,E_t\,\eta_2(s)_{\max} = -430 \text{ kg/cm}^2;$$

b) diaphragms rigid in their plane placed at the ends and in the middle

$$\sigma_{x\max} = -1.455\,\alpha_t\,E_t\,\eta_2(s)_{\max} = -1700 \text{ kg/cm}^2;$$

c) the cross-sectional contour is undeformable

$$\sigma_{x\max} = -0.97\,\alpha_t\,E_t\,\eta_2(s)_{\max} = -1130 \text{ kg/cm}^2.$$

BRIEF CONCLUSIONS

1. The engineering method proposed for calculating the temperature stresses in thin-walled systems is relatively simple. It permits one to allow easily for various structural peculiarities of a system (elasticity of the transverse diaphragms, presence of elements with various coefficients of linear expansion, etc.) subjected to forces /2/ and to a temperature field.

2. The comparison of the results of the calculation by the method considered with the experimental data given in the papers of Kan, Rudakov, and others shows that the computing relationships obtained give reliable results.

3. The particular examples of calculation of the temperature stresses coincide satisfactorily with the results obtained by the methods of Vlasov, Kan, and others. However, unlike the latter, the method proposed is suitable for thin-walled systems of both open and closed profiles of arbitrary cross-sectional shape, with or without transverse diaphragms, under an arbitrary law of variation of the temperature along the system axis x and along the middle line of the profile s.

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APPLICATION OF THE METHOD OF ASYMPTOTIC INTEGRATION TO PROBLEMS OF SHELL VIBRATIONS

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Several problems of free vibrations of shells of positive curvature are treated by the method of asymptotic integration /1, 2, 3/.

It is assumed that the state of stress and strain of the shell varies considerably during the vibrations. In this case the problem of the vibrations of an arbitrary shell can be solved by means of the equation

$$\Delta\Delta\varphi - Eh\Delta_k w = 0,$$
$$\Delta_k\varphi + D\Delta\Delta w + \frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where

$$\Delta = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right],$$
$$\Delta_k = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{1}{R_2}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{1}{R_1}\frac{\partial}{\partial\beta}\right)\right],$$
$$D = \frac{Eh^3}{12(1-\nu^2)}.$$

Here w is the normal deflection; φ is the stress function; α, β are the curvilinear cooordinates of the middle surface, coinciding with the curvature lines; A, B are Lamé parameters; γ/g is the density of the material; t is the time.

The functions φ and w will be represented in the form

$$\varphi(\alpha, \beta, t) = \varphi(\alpha, \beta)\cos\omega t, \quad w(\alpha, \beta, t) = w(\alpha, \beta)\cos\omega t,$$

where ω is the frequency of the natural vibrations.

Consider a spherical shell in a geographical system of coordinates (Figure 1). We can write then in equations (1): $R_1 = R_2 = R$; $A = R$; $B = R\sin\alpha$. By eliminating the function φ from system (1), we obtain the equation

$$D\Delta\Delta\Delta w - \left(\frac{Eh}{R^2} - \bar{p}^2\right)\Delta w = 0, \qquad \left(\bar{p}^2 = \frac{\gamma h\omega^2}{g}\right). \tag{2}$$

Consider a segment of the spherical shell limited by the arbitrary parallels $\alpha = \alpha_1$, $\alpha = \alpha_2$ or a spherical dome. In this case all forces and

displacements will be periodic functions of β of period 2π, and the problem solution can be represented in the form

$$w(\alpha, \beta) = \sum_{m=0,1,2} w_m(\alpha) \cos m\beta. \tag{3}$$

After the substitution of (3) in (2), the problem reduces to integrating the equation

$$h^{*2}\Delta\Delta\Delta w_m(\alpha) - (p^2 - 1)\Delta w_m(\alpha) = 0 \qquad (m = 0, 1, 2, \cdots), \tag{4}$$

where

$$p^2 = \frac{\gamma\omega^2 R^2}{gE},$$

$$h^{*2} = \frac{h^2}{12(1-\nu^2)R^2}, \qquad \Delta = \frac{d^2}{d\alpha^2} + \operatorname{ctg}\alpha \frac{d}{d\alpha} - \frac{m^2}{\sin^2\alpha} \tag{5}$$

We shall restrict ourselves to finding the frequency spectrum satisfying the condition

$$p^2 - 1 > 0. \tag{6}$$

The meaning of this restriction will be explained below.

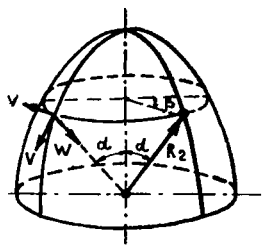

FIGURE 1.

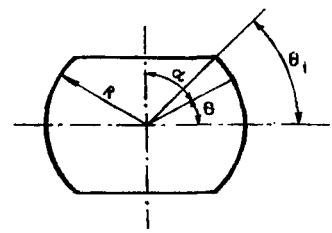

FIGURE 2.

Introduce the designations

$$\Delta w = w^*, \quad b^4 = h^{*-2}(p^2 - 1) \tag{7}$$

and consider first instead of (4) the following equation:

$$\Delta\Delta w^* - b^4 w^* = 0. \tag{8}$$

Assuming that the parameter h^{*2} is sufficiently small, we obtain on the strength of (6) that $b^4 \gg 1$.

Equation (8) splits into two Legendre equations, and its general solution can be represented in the form

$$w^*_m = c_{1m} P^m_{n1} + c_{2m} Q^m_{n1} + c_{3m} P^m_{n2} + c_{4m} Q^m_{n2}, \tag{9}$$

where P_n, Q_n are generalized Legendre functions of the first and second kind, of order determined from the conditions

$$\left(n_1 + \frac{1}{2}\right)^2 \approx b^2, \text{ and} \left(n_2 + \frac{1}{2}\right)^2 \approx -b^2; \qquad c_{im} \ (i = 1, 2, 3, 4)$$

are integration constants.

Having obtained solution (9) of equation (8), it is easily shown that the general solution of the initial equation (4) will be

$$w_m = A_{1m} + A_{2m} Q_0 + A_{3m} P_{n1}^m + A_{4m} Q_{n1}^m + A_{5m} P_{n2}^m + A_{6m} Q_{n2}^m, \tag{10}$$

where $A_i\,(i = 1, 2, \cdots, 6)$ are integration constants.

At large values of α the modulus αb will be very large, and it is possible to use an asymptotic representation for the Legendre functions. Using the corresponding asymptotic formulas /4/, for instance for the case

$$n \gg m, \quad \varepsilon < \alpha < \pi - \varepsilon, \quad 0 < \varepsilon < \frac{\pi}{6}, \tag{11}$$

we obtain

$$w(\alpha) = (\sin\alpha)^{-1/2}(c_1 + c_2\alpha + c_3 \sin b\alpha + c_4 \cos b\alpha + c_5 \operatorname{sh} b\alpha + c_6 \operatorname{ch} b\alpha). \tag{12}$$

We shall show now that this result can be obtained much more simply by using the general method of asymptotic integration /1, 2, 3/. The advantages of this method lie not only in its simplicity, but also in its applicability to a wide circle of problems.

We shall represent the solution of (4) in the form*

$$w = e^{kf(\alpha)}[w_0(\alpha) + w_1(\alpha) k^{-1} + k^{-2} w_2(\alpha)], \tag{13}$$

where $f(\alpha)$ is a so far unknown function of the variability, and the expression in brackets is an unknown intensity function, expanded in an asymptotic series by decreasing powers of the large parameter k.

As usual, k is a constant number, linked with the small parameter h^* by the relationship

$$k = h^{*-\tau} \qquad \left(h^* = \frac{h}{2R\sqrt{3(1-\nu^2)}}\right),$$

where τ is a variability index selected by us. We take $\tau = 0.5$.

Let us write $m = \eta k$, where η is a so far unknown number. Substituting (13) in (4), and restricting ourselves to two terms of the series, we obtain

$$\left(f'^2 - \frac{\eta^2}{\sin^2\alpha}\right)\left\{(1-p^2)\left[w_0 + k^{-1}w_1 + k^0\left(f'^2 - \frac{\eta^2}{\sin^2\alpha}\right)^2 w_0 + \right.\right.$$
$$\left.\left. + k^{-1}\left[\left(f'^2 - \frac{\eta^2}{\sin^2\alpha}\right)w_1 + \xi w_0 + \zeta w_0'\right]\right\} = 0, \tag{14}$$

ξ and ζ are known functions of α and of the derivatives of f by α. Assuming a certain value of η, we require that the coefficients of all powers of k in (14) become equal to zero. Let $\eta = 0$, i.e., $m = 0$. Then, by equating to zero the coefficients of the high powers of k, we obtain

$$f'^2(1 - p^2 + f'^4) = 0, \tag{15}$$

$$w_0(\alpha) = f'^{-5/2}(\sin\alpha)^{-3/2}. \tag{16}$$

* We shall omit in what follows the subscript m.

Equation (15) gives six solutions for the function f:

$$kf_1 = -kf_2 = b(\alpha - \alpha_0), \quad kf_3 = -kf_4 = ib(\alpha - \alpha_0).$$

If we restrict ourselves to the zero approximation in determining the intensity function, the general solution for w will be

$$w = (\sin\alpha)^{-1/2}[c_1 + c_2\alpha + c_3 \sin b(\alpha - \alpha_0) + c_4 \cos b(\alpha - \alpha_0) + \\ + c_5 \operatorname{sh} b(\alpha - \alpha_0) + c_6 \operatorname{ch} b(\alpha - \alpha_0)]. \tag{17}$$

This solution is reduced by elementary transformations to the earlier obtained solution (12).

Let the variability index be $\tau < 0.5$; take for instance $\tau = 0.25$, i.e., $h^{*2} = k^{-8}$.

By substituting under this assumption (13) in (4) and equating to zero the coefficient of the higher power of k, we obtain instead of (15) the equation

$$(p^2 - 1)\left(f'^2 - \frac{\eta^2}{\sin^2\alpha}\right) = 0,$$

whence it follows that

$$p^2 = 1, \quad f'^2 - \frac{\eta^2}{\sin^2\alpha} = 0. \tag{18}$$

We thus obtain $p^2 = 1$ for a spherical shell at $\tau < 0.5$.

It was shown in /3/ that in shells of positive curvature the frequency of vibrations does not increase with the increase of τ as long as $\tau < 0.5$, and remains commensurable with $\left(\frac{h}{R}\right)^0$, while at $\tau > 0.5$ the frequency increases with the increase of τ. If thus we obtain at $\tau < 0.5$ that $p^2 = 1$, at $\tau \geqslant 0.5$ we shall obtain $p^2 > 1$, i.e., condition (6) will be satisfied.

Having determined w, we find the meridional displacement u. We finally obtain

$$w(\theta) = (\cos\theta)^{-1/2}(A_1 \sin b\theta + A_2 \cos b\theta + A_3 \operatorname{sh} b\theta + A_4 \operatorname{ch} b\theta + A_5),$$
$$u(\theta) = (\cos\theta)^{-1/2}(1 + \nu)\left\{\frac{1}{b}[-A_1 \cos b\theta + A_2 \sin b\theta + A_3 \operatorname{ch} b\theta + \right. \\ \left. + A_4 \operatorname{sh} b\theta + (1 - \nu)^{-1} A_5 [2 - p^2(1 - \nu)]\theta\right\} + A_6. \tag{19}$$

Here $\theta = \frac{\pi}{2} - \alpha$. In the formulation of the final solution (19) we have discarded the redundant solutions due to the increase of the order of the differential equation realized in order to find the axisymmetrical vibrations of the spherical shell.

By satisfying with the aid of (19) the end conditions $\theta = \theta_1$, $\theta = \theta_2$, we obtain the frequency equation (Figure 2).

Note that this same result can be obtained directly from system (1) with the aid of the same substitution as before: $R_1 = R_2 = A = R$.

Consider the axisymmetrical vibrations of a spherical shell. We shall represent the solution in the form

$$w(\alpha, t) = k^{\mu} w_0(\alpha) e^{kf(\alpha)} \cos\omega t,$$
$$\varphi(\alpha, t) = k^{\lambda} \varphi_0(\alpha) e^{kf(\alpha)} \cos\omega t. \tag{20}$$

After the substitution of (20) in (1), we find noncontradictory values of the intensity indices $\mu + 2 = \lambda$. Writing $\tau = 0.25$ and equating to zero the coefficient of the higher power of k, we obtain

$$f'^4\varphi_0 - EhRf'^2w_0 = 0,$$
$$f'^2\varphi_0 - \frac{\gamma h\omega^2 R^3}{g} w_0 = 0, \tag{21}$$

whence follows $f'^4(p^2 - 1) = 0$, i.e., $p^2 = 1$.

Taking $\tau = 0.5$, we obtain instead of (21) the following system:

$$f'^4\varphi_0 - EhRf'^2w_0 = 0,$$
$$f'^2\varphi_0 + \left(EhRf'^4 - \frac{\gamma h\omega^2 R^3}{g}\right) w_0 = 0, \tag{22}$$

from which follows $(f'^4 - p^2 + 1)f'^4 = 0$, where $p^2 > 1$.

Consider an arbitrary shell of revolution of positive curvature. Let the shell be closed relative to the axis of revolution and undergo axisymmetrical vibrations. By substituting in equations (1) $A = R_1$, $B = R_2 \sin\alpha$, we obtain the operators

$$\Delta = \frac{1}{R_1^2\rho \sin\alpha} \frac{d}{d\alpha}\left(\rho \sin\alpha \frac{d}{d\alpha}\right), \qquad \rho = \frac{R_2}{R_1},$$
$$\Delta_\alpha = \frac{1}{R_1^2\rho \sin\alpha} \frac{d}{d\alpha}\left(\frac{\sin\alpha}{R_1} \frac{d}{d\alpha}\right). \tag{23}$$

We shall represent the solution corresponding to (1) and (23) in the form (20). Then, writing $\tau = 0.5$, we obtain the following system instead of (22):

$$f'^4\varphi_0 - Eh\frac{R_1^2}{R_2^2} f'w_0^2 = 0,$$
$$f'^2\varphi_0 + \left(EhR_2f'^4 - \frac{R_1^2R_2\gamma h\omega^2}{g}\right) w_0 = 0,$$

from which it follows that

$$\left(f'^4 - \frac{R_1^2\gamma\omega^2}{gE} + \frac{R_1^2}{R_2^2}\right) = 0, \quad f'^4 = 0. \tag{24}$$

The integration of the first equation of (24) yields

$$f = \int_{\alpha_0}^{\alpha} \sqrt[4]{\frac{\gamma\omega^2R_1^2}{gE} - \frac{R_1^2}{R_2^2}}\, d\alpha.$$

In order to calculate this integral it is necessary to specify the shell shape, i.e., the functions $R_1(\alpha)$, $R_2(\alpha)$.

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VIBRATIONS OF A SPHERICAL SHELL UNDER THE ACTION OF A CONCENTRATED FORCE

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The case of a concentrated force P is usually treated as the limiting case of a distributed load of intensity q, equal to zero everywhere except on some portion S of the surface, so that $\lim\limits_{s\to 0} Sq = P$.

Unlike the studies with discontinuous transition to an infinite intensity, we consider here the concentrated force as limiting case of a load continuously distributed over the surface, which tends to zero everywhere except at the zero point, where it tends to infinity.

The initial equations of motion of a shell element are obtained from the equilibrium equations given in /1/, with inertial terms added. After elementary transformations these are reduced to the following form:

$$
\begin{gathered}
(\nabla^2 - 1 - 3\nu)\,\theta - (3c^2\nabla^2 + 1 - \nu)(\nabla^2 + 2)\,\frac{w}{R} = -a^2\left(P_\theta - \rho\frac{\partial^2\theta}{\partial t}\right) \\
(c^2\nabla^2 - 1 - \nu)\,\theta - c^2(\nabla^2 + 1 - \nu)(\nabla^2 + 2)\,\frac{w}{R} = a^2\left(P_z - \rho\frac{\partial^2 w}{\partial t^2}\right) \qquad (1) \\
(\nabla^2 + 2)\,\chi = \frac{2a^2}{1-\nu}\left(-P_\chi + \rho\frac{\partial^2\chi}{\partial t^2}\right)
\end{gathered}
$$

where

$$
a^2 = \frac{R^2(1-\nu^2)}{Eh},
$$

$$
P_\theta = \frac{1}{R\sin\alpha}\left[\frac{\partial}{\partial\alpha}(X\sin\alpha) + \frac{\partial Y}{\partial\beta}\right] + \frac{2Z}{R},
$$

$$
P_z = \frac{Z}{R},
$$

$$
P_\chi = \frac{1}{2R\sin\alpha}\left[-\frac{\partial X}{\partial\beta} + \frac{\partial}{\partial\alpha}(Y\sin\alpha)\right],
$$

$$
\rho = \frac{\gamma h}{g},
$$

where γ is the specific weight of the material. The remaining symbols have the same meaning as in /1/.

Note that the system splits just as in the problems of statics, the first two equations containing only w and θ, and the third one only χ.

We shall determine the solution of (1) for zero initial conditions. The forces are given in the following form (see figure):

$$Z_1 = \begin{cases} -\dfrac{P(t)}{2\pi R^2} k \cos^k \alpha, & 0 \leqslant \alpha \leqslant \dfrac{\pi}{2}, \\ 0, & \dfrac{\pi}{2} \leqslant \alpha \leqslant \pi. \end{cases} \tag{2}$$

The inertia forces acting on the system are equilibrated by the external loads, so that

$$q(t) = \frac{P(t)}{4\pi R^2} \frac{k}{k+2}.$$

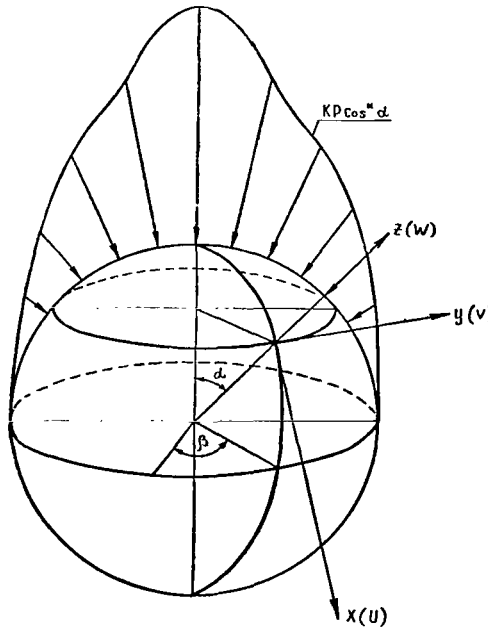

FIGURE.

In the limiting case $k \to \infty$ we obtain the solution corresponding to the case of a concentrated force.

The problem is solved by the Fourier method, i.e., expanding by the eigenfunctions.

The eigenfunctions are in the general case

$$\theta_n = Y_n^{(1)} = \sum_{m=0}^{n} (a_{m,n}^{(1)} \cos m\beta + b_{m,n}^{(1)} \sin m\beta) P_{m,n}(\cos \alpha),$$
$$w_n = Y_n^{(2)} = \sum_{m=0}^{n} (a_{m,n}^{(2)} \cos m\beta + b_{m,n}^{(2)} \sin m\beta) P_{m,n}(\cos \alpha), \tag{3}$$

$$\chi = Y_n^{(3)} = \sum_{m=0}^{n} (a_{m,n}^{(3)} \cos m\beta + b_{m,n}^{(3)} \sin m\beta)\, P_{m,n}(\cos\alpha), \qquad (3)$$
(cont'd)

where

$$P_{m,n}(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m};$$

$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^2$ are Legendre polynomials. Taking into account that $\nabla^2 Y_n = -n(n+1) Y_n$, we easily obtain, after separating the variables from the homogeneous equation corresponding to (1), the following characteristic equation:

$$\begin{vmatrix} a_n + a^2\rho\lambda_n^2 & b_n \\ C_n & a_n + a^2\rho\lambda_n^2 \end{vmatrix} = 0. \qquad (4)$$

where

$$a_n = -n(n+1) - 1 - 3\nu,$$
$$b_n = -[-3c^2 n(n+1) + 1 - \nu][2 - n(n+1)],$$
$$c_n = -[c^2 n(n+1) + 1 + \nu],$$
$$a_n = c^2[n(n+1) - 1 + \nu][2 - n(n+1)].$$

Note that at $c^2 = 0$ we obtain from (4) the frequency of a spherical shell obtained first by Lamb /2/.

The solution of the nonhomogeneous system (1) at the given loads (2) will be represented in the form

$$\theta_n = [k_1(\lambda_{1,n}) M_n^{(1)}(t) + k_1(\lambda_{2,n}) M_n^{(2)}(t)]\, P_n(\cos\alpha),$$
$$\frac{w_n}{R} = [k_2(\lambda_{1,n}) M_n^{(1)}(t) + k_2(\lambda_{2,n}) M_n^{(2)}(t)]\, P_n(\cos\alpha). \qquad (5)$$

Here

$$k_1(\lambda_{i,n}) = -b_n - d_n - a^2\rho\lambda_{i,n}^2,$$
$$k_2(\lambda_{i,n}) = a_n + c_n + a^2\rho\lambda_{i,n}^2.$$

It is easily obtained that at load (2)

$$P_s = \begin{cases} -\frac{P(t)}{2\pi R^3} k\cos^k\alpha + \frac{q(t)}{R}\cos\alpha, & 0 \leqslant \alpha \leqslant \frac{\pi}{2}, \\ \frac{q(t)}{R}\cos\alpha, & \frac{\pi}{2} \leqslant \alpha \leqslant \pi, \end{cases}$$

$$P_\theta = \begin{cases} -\frac{P(t)}{\pi R^3} k\cos^k\alpha, & 0 \leqslant \alpha \leqslant \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} \leqslant \alpha \leqslant \pi. \end{cases}$$

Expanding P_z, P_θ in series of the form of (5), designating the expansion coefficients by $H_n^{(i)}(t)$, and noting that /3/

$$\int_0^1 k\,\mu^k P_n(\mu)\,d\mu = \begin{cases} \dfrac{k^2(k-2)\cdots(k-n+2)}{(k+n+1)(k+n-1)\cdots(k+3)(k+1)} & \text{for even } n, \\ \dfrac{k(k-1)\cdots(k-n+1)}{(k+n+1)(k+n-1)\cdots(k+4)(k+2)} & \text{for odd } n, \end{cases} \tag{6}$$

we find

$$\begin{aligned} H_n^{(1)} &= \frac{2n+1}{2}\,c(n,k)\,\frac{-2k_2(\lambda_{2,n}) + k_1(\lambda_{2,n})}{k_1(\lambda_{1,n})\,k_2(\lambda_{2,n}) - k_2(\lambda_{1,n})\,k_1(\lambda_{2,n})}\,\frac{P(t)}{R}, \\ H_n^{(2)} &= \frac{2n+1}{2}\,c(n,k)\,\frac{-2k_2(\lambda_{1,n}) + k_1(\lambda_{1,n})}{k_1(\lambda_{1,n})\,k_2(\lambda_{2,n}) - k_2(\lambda_{1,n})\,k_1(\lambda_{2,n})}\,\frac{P(t)}{R}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \theta = \frac{1}{2\pi R^2}\Bigg[&-\frac{k}{k+1}\,\frac{1}{R\rho\lambda_{1,0}}\int_0^t P(\tau)\sin\lambda_{1,0}(t-\tau)\,d\tau - \\ &-\frac{3k}{k+2}\,\frac{P_1}{R\rho\lambda_{1,1}}\int_0^t P(\tau)\sin\lambda_{1,1}(t-\tau)\,d\tau + \\ &+\sum_{n=2}^{\infty}\sum_{l=1}^{2}\frac{k_1(\lambda_{l,n})\,P_n}{\rho R\,\lambda_{l,n}}\int_0^t H_n^{(l)}(\tau)\sin\lambda_{l,n}(t-\tau)\,d\tau\Bigg]. \end{aligned}$$

$$\begin{aligned} w = \frac{1}{2\pi R^2}\Bigg[&\frac{-k}{k+1}\,\frac{1}{2\rho\lambda_{1,0}}\int_0^t P(\tau)\sin\lambda_{1,0}(t-\tau)\,d\tau - \\ &-\frac{3k}{k+2}\,\frac{P_1}{\rho\lambda_{1,1}}\int_0^t P(\tau)\sin\lambda_{1,1}(t-\tau)\,d\tau + \\ &+\sum_{n=2}^{\infty}\sum_{l=1}^{2}\frac{P_n\,k_2(\lambda_{l,n})}{\rho\lambda_{l,n}^2}\int_0^t H_n^{(l)}(\tau)\,\lambda_{l,n}\sin\lambda_{l,n}(t-\tau)\,d\tau\Bigg]. \end{aligned} \tag{8}$$

In order to separate the quasistatic solution from (8), we integrate twice by parts and obtain

$$\begin{aligned} \theta = \frac{1}{2\pi R^2}\Bigg\{ &-P(t)\left[\frac{k}{k+1}\,\frac{1}{\rho R\lambda_{1,0}^2} + \frac{3k}{k+2}\,\frac{P_1}{\rho R\lambda_{1,1}^2}\right] + \frac{k}{k+1}\,\frac{1}{\rho R\lambda_{1,0}^2}\times \\ &\times\left[P(0)\cos\lambda_{1,0}t + \frac{\dot P(0)}{\lambda_{1,0}}\sin\lambda_{1,0}t + \frac{1}{\lambda_{1,0}}\int_0^t \ddot P(\tau)\sin\lambda_{1,0}(t-\tau)\,d\tau\right] + \\ &+\frac{3k}{k+2}\,\frac{P_1}{\rho R\lambda_{1,1}^2}\left[P(0)\cos\lambda_{1,1}t + \frac{\dot P(0)}{\lambda_{1,1}}\sin\lambda_{1,1}t + \frac{1}{\lambda_{1,1}}\int_0^t \ddot P(\tau)\sin\lambda_{1,1}(t-\tau)\,d\tau\right] + \\ &+\sum_{n=2}^{\infty}\sum_{l=1}^{2}\frac{P_n\,k_1(\lambda_{l,n})}{\rho\lambda_{l,n}^2}\Bigg[H_n^{(l)}(t) - H_n^{(l)}(0)\cos\lambda_{l,n}t - \\ &-\frac{\dot H_n^{(l)}(0)}{\lambda_{l,n}}\sin\lambda_{l,n}t - \frac{1}{\lambda_{l,n}}\int_0^t \ddot H_n^{(l)}(\tau)\sin\lambda_{l,n}(t-\tau)\,d\tau\Bigg]\Bigg\}; \end{aligned} \tag{9}$$

$$w=\frac{1}{2\pi R^2}\left\{-P(t)\left[\frac{k}{k+1}\frac{1}{2\rho\lambda_{1,0}^2}+\frac{k}{k+2}\frac{P_1}{\rho\lambda_{1,1}^2}\right]+\frac{k}{k+1}\frac{1}{2\rho\lambda_{1,0}^2}\left[P(0)\cos\lambda_{1,0}t+\right.\right.$$

$$\left.+\frac{\dot P(0)}{\lambda_{1,0}}\sin\lambda_{1,0}t+\frac{1}{\lambda_{1,0}}\int_0^t\ddot P(\tau)\sin\lambda_{1,0}(t-\tau)\,d\tau\right]+$$

$$+\frac{k}{k+2}\frac{P_1}{\rho\lambda_{1,1}^2}\left[P(0)\cos\lambda_{1,1}t+\frac{\dot P(0)}{\lambda_{1,1}}\sin\lambda_{1,1}t+\frac{1}{\lambda_{1,1}}\int_0^t\ddot P(\tau)\sin\lambda_{1,1}(t-\tau)\,d\tau\right]+$$

$$+\sum_{n=2}^{\infty}\sum_{i=1}^{2}\frac{P_n k_2(\lambda_{i,n})}{\rho\lambda_{i,n}^2}\left[H_n^{(i)}(t)-H_n^{(i)}(0)\cos\lambda_{i,n}t-\right.$$

$$\left.\left.-\frac{\dot H_n^{(i)}(0)}{\lambda_{i,n}}\sin\lambda_{i,n}t-\frac{1}{\lambda_{i,n}}\int_0^t\ddot H_n^{(i)}(\tau)\sin\lambda_{i,n}(t-\tau)\,d\tau\right]\right\}. \qquad (9)\ \text{(cont'd)}$$

The terms in (9) which do not contain trigonometric functions of the time represent the quasistatic solution of the problem. In order to improve the convergence of these series it is possible to apply the existing methods of static solution.

If $P(0)=0$, the terms containing $\cos\lambda_{i,n}t$ and having the same order of convergence as the static solution become equal to zero. The remaining terms converge more rapidly than the static solution, having a convergence order $\lambda_{i,n}\sim n^2$. In practical calculations the first three terms of the series are sufficient.

Taking into account that

$$\lim_{k\to\infty} k\cos^k\alpha=\begin{cases}\infty & \text{for } \alpha=0,\\ 0 & \text{for } \alpha\neq 0,\end{cases}$$

we obtain from solution (9) the solution of the problem under the action of the concentrated force $P(t)$:

$$\theta=\frac{1}{2\pi R^2}\left\{-P(t)\frac{1}{\rho R}\left[\frac{1}{\lambda_{1,0}^2}+\frac{3P_1}{\lambda_{1,1}^2}\right]+\frac{1}{\rho R\lambda_{1,0}^2}\left[\frac{\dot P(0)}{\lambda_{1,0}}\sin\lambda_{1,0}t+\right.\right.$$

$$\left.+\frac{1}{\lambda_{1,0}}\int_0^t\ddot P(\tau)\sin\lambda_{1,0}(t-\tau)\,d\tau\right]+\frac{3P_1}{\rho R\lambda_{1,1}^2}\left[\frac{\dot P(0)}{\lambda_{1,1}}\sin\lambda_{1,1}t+\right.$$

$$\left.+\frac{1}{\lambda_{1,1}}\int_0^t\ddot P(\tau)\sin\lambda_{1,1}(t-\tau)\,d\tau\right]+\sum_{n=2}^{\infty}\sum_{i=1}^{2}\frac{P_n k_1(\lambda_{i,n})}{\rho R\lambda_{i,n}^2}\times$$

$$\left.\times\left[\tilde H_n^{(i)}(t)-\frac{\dot{\tilde H}_n^{(i)}(0)}{\lambda_{i,n}}\sin\lambda_{i,n}t-\frac{1}{\lambda_{i,n}}\int_0^t\ddot{\tilde H}_n^{(i)}(\tau)\sin\lambda_{i,n}(t-\tau)\,d\tau\right]\right\}, \qquad (10)$$

$$w=\frac{1}{2\pi R^2}\left\{-P(t)\frac{1}{\rho}\left[\frac{1}{2\lambda_{1,0}^2}+\frac{P}{\lambda_{1,1}^2}\right]+\frac{1}{2\rho\lambda_{1,0}^2}\left[\frac{\dot P(0)}{\lambda_{1,0}}\sin\lambda_{1,0}t+\right.\right.$$

$$\left.+\frac{1}{\lambda_{1,0}}\int_0^t\ddot P(\tau)\sin\lambda_{1,0}(t-\tau)\,d\tau\right]+\frac{P_1}{\rho\lambda_{1,1}^2}\left[\frac{\dot P(0)}{\lambda_{1,1}}\sin\lambda_{1,1}t+\right.$$

$$+\frac{1}{\lambda_{1,1}}\int_0^t \ddot{P}(\tau)\sin\lambda_{1,1}(t-\tau)\,d\tau\Bigg]+\sum_{n=2}^{\infty}\sum_{i=1}^{2}\frac{P_n k_2(\lambda_{i,n})}{\rho\lambda_{i,n}^2}\times$$

$$\times\left[\widetilde{H}_n^{(i)}(t)-\frac{\dot{\widetilde{H}}_n^{(i)}(0)}{\lambda_{i,n}}\sin\lambda_{i,n}t-\frac{1}{\lambda_{i,n}}\int_0^t \ddot{\widetilde{H}}_n^{(i)}(\tau)\sin\lambda_{i,n}(t-\tau)\,d\tau\right]\Bigg\}, \qquad (10)\ (\text{cont'd})$$

where $\widehat{H}_n^{(i)}$, $\widetilde{H}_n^{(i)}$ are obtained from $H_n^{(i)}(7)$ by writing $(n, k) = 1$.

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AXISYMMETRICAL BENDING OF A THICK CIRCULAR SANDWICH PLATE

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The problem of the bending of a thick circular sandwich plate, made of layers having different moduli of elasticity, under an arbitrary axisymmetrical load applied to the upper base is treated. The modulus of elasticity is considered as constant within the limits of each layer. It is assumed that the materials of the three layers have the same Poisson ratio.

Let a thick circular plate of radius a and of layer thicknesses δ_1, δ_2, and δ_3 be loaded by a lateral axisymmetrical load of intensity $p(r)$ applied to the upper base of the plate, and let each of the three layers be likewise thick. Let the moduli of elasticity of the three layers be E_1, E_2, and E_3, and let their Poisson ratios be equal to the same value ν. The subscript 1 shall refer to all magnitudes corresponding to the upper layer, the subscript 2 to the magnitudes corresponding to the middle layer, and the subscript 3 to those corresponding to the lower layer.

We select the cylindrical system of coordinates as shown in the figure.

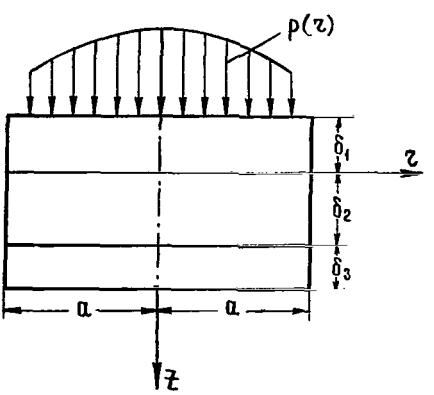

FIGURE.

The solution of problems of axisymmetrical deformation of bodies of revolution reduces as known /1/ to determining a stress function $\varphi(r, z)$ which satisfies the equation

$$\Delta\Delta\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2}\right) = 0. \qquad (1)$$

The stresses and displacements are expressed by the following relationships through $\varphi(r, z)$ /1/:

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\Delta\varphi - \frac{\partial^2\varphi}{\partial r^2}\right), \qquad \tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right],$$

$$\sigma_\theta = \frac{\partial}{dz}\left(\nu\Delta\varphi - \frac{1}{r}\frac{\partial\varphi}{\partial r}\right), \qquad \sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right], \tag{2}$$

$$u = -\frac{1+\nu}{E}\frac{\partial^2\varphi}{\partial r\partial z}, \qquad w = \frac{1+\nu}{E}\left[2(1-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right].$$

We select the stress function $\varphi(r, z)$ corresponding to each layer in the form

$$\varphi_i(r, z) = -\frac{E_i}{4}\frac{1-\nu}{1+\nu}\left\{\left[(z - m\delta_2)r^2 - \frac{2(2-\nu)}{3(1-\nu)}(z - m\delta_2)^3\right]F + \right.$$

$$+\left[\frac{(z - m\delta_2)^2}{2}r^2 - \frac{a^2}{8(1-\nu)}r^2 + \frac{\nu}{16(1-\nu)}r^4 - \right.$$

$$\left.\left.\frac{2-\nu}{6(1-\nu)}(z - m\delta_2)^4\right]H\right\} +$$

$$+\sum_{k=1} J_0(\omega_k r)\,[A_k^i \operatorname{sh}\omega_k z + B_k^i \operatorname{ch}\omega_k z + C_k^i z \operatorname{ch}\omega_k z + D_k^i z \operatorname{sh}\omega_k z], \tag{3}$$

where $J_0(\omega_k r)$ is a zero-order Bessel function of the first kind; $\omega_k = \frac{\mu_k}{a}$; μ_k are roots of the equation $J_0(\mu_k) = 0$; $i = 1, 2, 3$ for the upper, middle and lower layers, respectively; m is a constant, whose value lies in the range $\frac{\delta_1}{\delta_2} \leqslant m \leqslant \frac{\delta_2 + \delta_3}{\delta_2}$; the value of m in each particular case is determined by the position of the circle on the lateral surface along which the plate is fastened.

The substitution of (3) in (2) yields:

$$\sigma_r^i = \frac{E_i}{2}[F + H(z - m\delta_2)] + \sum_{k=1}^{\infty}\omega_k^2 J_0(\omega_k r)\,\{A_k^i\omega_k \operatorname{ch}\omega_k z + B_k^i\omega_k \operatorname{sh}\omega_k z +$$

$$+ C_k^i[(1+2\nu)\operatorname{ch}\omega_k z + \omega_k z \operatorname{sh}\omega_k z] + D_k^i[(1+2\nu)\operatorname{sh}\omega_k z + \omega_k z \operatorname{ch}\omega_k z]\} -$$

$$-\sum_{k=1}^{\infty}\omega_k\frac{J_1(\omega_k r)}{r}[A_k^i\omega_k \operatorname{ch}\omega_k z + B_k^i\omega_k \operatorname{sh}\omega_k z + C_k^i(\operatorname{ch}\omega_k z + \omega_k z \operatorname{sh}\omega_k z) +$$

$$+ D_k^i(\operatorname{sh}\omega_k z + \omega_k z \operatorname{ch}\omega_k z)]; \tag{4}$$

$$\sigma_\theta^i = \frac{E_i}{2}[F + H(z - m\delta_2)] + 2\nu\sum_{k=1}^{\infty}\omega_k^2 J_0(\omega_k r)\,[C_k^i \operatorname{ch}\omega_k z + D_k^i \operatorname{sh}\omega_k z] +$$

$$+\sum_{k=1}^{\infty}\omega_k\frac{J_1(\omega_k r)}{r}[A_k^i\omega_k \operatorname{ch}\omega_k z + B_k^i\omega_k \operatorname{sh}\omega_k z + C_k^i(\operatorname{ch}\omega_k z +$$

$$+ \omega_k z \operatorname{sh}\omega_k z) + D_k^i(\operatorname{sh}\omega_k z + \omega_k z \operatorname{ch}\omega_k z)]; \tag{5}$$

$$\sigma_z^i = \sum_{k=1}^{\infty}\omega_k^2 J_0(\omega_k r)\,\{A_k^i\omega_k \operatorname{ch}\omega_k z + B_k^i\omega_k \operatorname{sh}\omega_k z + C_k^i[\omega_k z \operatorname{sh}\omega_k z -$$

$$-(1-2\nu)\operatorname{ch}\omega_k z] + D_k^i[\omega_k z \operatorname{ch}\omega_k z - (1-2\nu)\operatorname{sh}\omega_k z]\}; \tag{6}$$

$$\tau_{rz}^i = \sum_{k=1}^{\infty}\omega_k^2 J_1(\omega_k r)\,[A_k^i\omega_k \operatorname{sh}\omega_k z + B_k^i\omega_k \operatorname{ch}\omega_k z + C_k^i(2\nu \operatorname{sh}\omega_k z +$$

$$+ \omega_k z \operatorname{ch}\omega_k z) + D_k^i(2\nu \operatorname{ch}\omega_k z + \omega_k z \operatorname{sh}\omega_k z)]; \tag{7}$$

$$u^i = \frac{1-\nu}{2}[Fr + Hr(z - m\delta_2)] + \frac{1+\nu}{E_i}\sum_{k=1}^{\infty}\omega_k J_1(\omega_k r)\,[A_k^i\omega_k \operatorname{ch}\omega_k z +$$

$$+ B_k^i\omega_k \operatorname{sh}\omega_k z + C_k^i(\operatorname{ch}\omega_k z - \omega_k z \operatorname{sh}\omega_k z) + D_k^i(\operatorname{sh}\omega_k z + \omega_k z \operatorname{ch}\omega_k z)]; \qquad (8)$$

$$w^i = \frac{1-\nu}{4}(a^2 - r^2)H - \frac{\nu}{2}(z - m\delta_2)^2 H - \nu(z - m\delta_2)F -$$

$$- \frac{1+\nu}{E_i}\sum_{k=1}^{\infty}\omega_k J_0(\omega_k r)\,\{A_k^i\omega_k \operatorname{sh}\omega_k z + B_k^i\omega_k \operatorname{ch}\omega_k z + C_k^i[\omega_k z \operatorname{ch}\omega_k z -$$

$$- 2(1-2\nu)\operatorname{sh}\omega_k z + D_k^i[\omega_k z \operatorname{sh}\omega_k z - 2(1-2\nu)\operatorname{ch}\omega_k z]\}. \qquad (9)$$

The fourteen coefficients A_k^i, B_k^i, C_k^i, D_k^i, F, and H are determined from the boundary conditions and the conditions of connection at the planes of contact of the layers.

In view of the absence of shearing stresses on the upper and lower boundary planes of the plate, the boundary conditions on these planes will be

$$\begin{aligned} z &= -\delta_1 \qquad \sigma_z^{(1)} = -p(r), \qquad \tau_{rz}^{(1)} = 0; \\ z &= \delta_2 + \delta_3 \qquad \sigma_z^{(3)} = 0, \qquad \tau_{rz}^{(3)} = 0. \end{aligned} \qquad (10)$$

We require that the load $p(r)$ be representable in a Fourier-Bessel series:

$$p(r) = \sum_{k=1}^{\infty} K_k J_0(\omega_k r), \text{ where } K_k = \frac{2}{a^2 J_1^2(\mu_k)}\int_0^a p(r) J_0 \omega_k r)\,dr,$$

J_1 is a first-order Bessel function of the first kind.

Assuming that the stress and displacement vectors vary continuously across the plane of contact of the layers, we obtain the following conditions on the planes of contact:

$$z = 0 \quad \sigma_z^{(1)} = \sigma_z^{(2)}, \quad \tau_{rz}^{(1)} = \tau_{rz}^{(2)}, \quad u^{(1)} = u^{(2)}, \quad w^{(1)} = w^{(2)}; \qquad (11)$$

$$z = \delta_2 \quad \sigma_z^{(2)} = \sigma_z^{(3)}, \quad \tau_{rz}^{(2)} = \tau_{rz}^{(3)}, \quad u^{(2)} = u^{(3)}, \quad w^{(2)} = w^{(3)}. \qquad (12)$$

It follows from the compatibility conditions (11) and (12) that the coefficients of the layers are linked by the following relationships:

$$\left.\begin{aligned} A_k^{(1)} &= A_k^{(2)} + \frac{1-2\nu}{\omega_k} R(\omega_k A_k^{(2)} + C_k^{(2)}), \\ B_k^{(1)} &= B_k^{(2)} + \frac{2\nu}{\omega_k} R[\omega_k B_k^{(2)} - 2(1-2\nu) D_k^{(2)}], \\ C_k^{(1)} &= C_k^{(2)} + R(\omega_k A_k^{(2)} + C_k^{(2)}), \\ D_k^{(1)} &= D_k^{(2)} - R[\omega_k B_k^{(2)} - 2(1-2\nu) D_k^{(2)}], \end{aligned}\right\} \qquad (13)$$

$$A_k^{(3)} = A_k^{(2)} + Q\Big\{A_k^{(2)}[\beta_k \operatorname{sh}\beta_k - 2\nu \operatorname{ch} 2\beta_k + \operatorname{ch}^2\beta_k] +$$

$$+ B_k^{(2)}\Big[\beta_k \operatorname{ch} 2\beta_k + \frac{1}{2}(1-4\nu)\operatorname{sh} 2\beta_k\Big] + C_k^{(2)}\frac{1}{\omega_k}[\beta_k^2 \operatorname{ch} 2\beta_k +$$

$$+ (1-2\nu)(1+4\nu)\operatorname{sh}^2\beta_k + (1-2\nu)] +$$

$$+ D_k^{(2)}\frac{1}{\omega_k}\Big[\beta_k^2 \operatorname{sh} 2\beta_k + \frac{1}{2}(1-2\nu)(1+4\nu)\operatorname{sh} 2\beta_k - (1-2\nu)\beta_k\Big]\Big\}, \qquad (14)$$

$$B_k^{(3)} = B_k^{(2)} - Q\Big\{ A_k^{(2)} \Big[\frac{1}{2}(1-4\nu)\,\mathrm{sh}\,2\beta_k + \beta_k\,\mathrm{ch}\,2\beta_k \Big] + B_k^{(2)} [\beta_k\,\mathrm{sh}\,2\beta_k -$$

$$- 2\nu\,\mathrm{ch}\,2\beta_k + \mathrm{sh}^2\beta_k] + C_k^{(2)} \frac{1}{\omega_k} \Big[\frac{1}{2}(1-2\nu)(1+4\nu)\,\mathrm{sh}\,2\beta_k + \beta_k^2\,\mathrm{sh}\,2\beta_k +$$

$$+ (1-2\nu)\beta_k \Big] + D_k^{(2)} \frac{1}{\omega_k} [\beta_k^2\,\mathrm{ch}\,2\beta_k + (1-2\nu)(1+4\nu)\,\mathrm{ch}^2\beta_k - (1-2\nu)] \Big\},$$

$$C_k^{(3)} = C_k^{(2)} + Q\Big\{ A_k^{(2)}\omega_k\,\mathrm{ch}\,2\beta_k + B_k^{(2)}\omega_k\,\mathrm{sh}\,2\beta_k + C_k^{(2)} [\beta_k\,\mathrm{sh}\,2\beta_k - \tag{14 (cont'd)}$$

$$- (1-4\nu)\,\mathrm{sh}^2\beta_k + 1] + D_k^{(2)} \Big[\beta_k\,\mathrm{ch}\,2\beta_k - \frac{1}{2}(1-4\nu)\,\mathrm{sh}\,2\beta_k \Big] \Big\},$$

$$D_k^{(3)} = D_k^{(2)} - Q\Big\{ A_k^{(2)}\omega_k\,\mathrm{sh}\,2\beta_k + B_k^{(2)}\omega_k\,\mathrm{ch}\,2\beta_k +$$

$$+ C_k^{(2)} \Big[\beta_k\,\mathrm{ch}\,2\beta_k - \frac{1}{2}(1-4\nu)\,\mathrm{sh}\,2\beta_k \Big] +$$

$$+ D_k^{(2)} [\beta_k\,\mathrm{sh}\,2\beta_k - (1-4\nu)\,\mathrm{ch}^2\beta_k - 1] \Big\},$$

where

$$R = \frac{1}{2(1-\nu)} \left(\frac{E_1}{E_2} - 1 \right), \qquad Q = \frac{1}{2(1-\nu)} \left(\frac{E_3}{E_2} - 1 \right), \quad \beta_k = \omega_k \delta_2.$$

By using the conditions on the planes of contact we have thus reduced the number of unknown constants from fourteen to six: $A_k^{(2)}$, $B_k^{(2)}$, $C_k^{(2)}$, $D_k^{(2)}$, F, and H.

The boundary conditions (10) and relationships (13) and (14) lead to a system of four algebraic equations relative to the coefficients of the middle layer $A_k^{(2)}$, $B_k^{(2)}$, $C_k^{(2)}$, $D_k^{(2)}$. The solution of this system yields

$$A_k^{(2)} = \frac{\Delta_A}{\Delta}, \quad B_k^{(2)} = \frac{\Delta_B}{\Delta}, \quad C_k^{(2)} = \frac{\Delta_C}{\Delta}, \quad D_k^{(2)} = \frac{\Delta_D}{\Delta}, \tag{15}$$

where

$$\frac{\Delta}{\omega_k^2} = [(\alpha_k + \gamma_k)^2 - \mathrm{sh}^2(\alpha_k + \gamma_k)] + R\{-4(1-\nu)[\mathrm{sh}\,\alpha_k\,\mathrm{ch}\,\gamma_k\,\mathrm{sh}(\alpha_k+\gamma_k) -$$

$$- \alpha_k(\alpha_k + \gamma_k)] + 2(\alpha_k^2\,\mathrm{sh}\,\gamma_k - \gamma_k\,\mathrm{sh}^2\alpha_k)\} + R^2\{-[\gamma_k^2 + (1-2\nu)^2 +$$

$$+ (3-4\nu)\,\mathrm{ch}^2\gamma_k](\mathrm{sh}^2\alpha_k - \alpha_k^2)\} +$$

$$+ Q\{-4(1-\nu)[\mathrm{sh}\,\varphi_k\,\mathrm{ch}(\alpha_k+\beta_k)\,\mathrm{sh}(\alpha_k+\gamma_k) - (\alpha_k+\gamma_k)\varphi_k] +$$

$$+ 2[\varphi_k^2\,\mathrm{sh}^2(\alpha_k+\beta_k) - (\alpha_k+\beta_k)^2\,\mathrm{sh}^2\varphi_k]\} +$$

$$+ Q^2\{-[(1-2\nu)^2 + (\alpha_k+\beta_k)^2 + (3-4\nu)\,\mathrm{ch}^2(\alpha_k+\beta_k)](\mathrm{sh}^2\varphi_k - \varphi_k^2)\} +$$

$$+ RQ\{2(3-4\nu)\,\mathrm{sh}^2\alpha_k\,\mathrm{sh}^2\varphi_k + 4\beta_k^2\,\mathrm{sh}^2\alpha_k\,\mathrm{sh}^2\varphi_k -$$

$$- 8(1-\nu)\beta_k(\varphi_k\,\mathrm{sh}^2\alpha_k + \alpha_k\,\mathrm{sh}^2\varphi_k) - 2(3-4\nu)(\varphi_k^2\,\mathrm{sh}^2\alpha_k + \alpha_k^2\,\mathrm{sh}^2\varphi_k) +$$

$$+ 8(1-\nu)^2\alpha_k\varphi_k + 4(1-\nu)[\varphi_k^2\,\mathrm{sh}\,\alpha_k\,\mathrm{sh}(\alpha_k+2\beta_k) + \alpha_k^2\,\mathrm{sh}\,\varphi_k\,\mathrm{sh}(\varphi_k+2\beta_k)] -$$

$$
\begin{gathered}
-2\alpha_k^2\varphi_k^2 \operatorname{ch} 2\beta_k - 8(1-\nu)^2 \operatorname{sh}\alpha_k \operatorname{sh}\varphi_k \operatorname{ch}(\alpha_k+\beta_k+\gamma_k)\} + \\
+ R^2 Q\{2(\beta_k^2 \operatorname{sh}^2\varphi_k + \varphi_k^2 \operatorname{sh}^2\beta_k) + 4(1-\nu)[\varphi_k^2 \operatorname{sh}^2\beta_k - (3-4\nu)\operatorname{sh}\varphi_k \operatorname{sh}\beta_k \operatorname{ch}\gamma_k - \\
-\beta_k\varphi_k](\operatorname{sh}^2\alpha_k - \alpha_k^2)\} + RQ^2\{2(\beta_k^2 \operatorname{sh}^2\alpha_k + \alpha_k^2 \operatorname{sh}^2\beta_k) + 4(1-\nu)\times \\
\times[\alpha_k^2 \operatorname{sh}^2\beta_k - (3-4\nu)\operatorname{sh}\alpha_k \operatorname{sh}\beta_k \operatorname{ch}(\alpha_k+\beta_k) - \alpha_k\beta_k]\}(\operatorname{sh}^2\varphi_k - \varphi_k^2) + \\
+ R^2Q^2\{[\beta_k^2 - (3-4\nu)^2 \operatorname{sh}^2\beta_k](\operatorname{sh}^2\alpha_k - \alpha_k^2)(\operatorname{sh}^2\varphi_k - \varphi_k^2)\} \qquad (16)
\end{gathered}
$$

$$
\begin{gathered}
\frac{\omega_k}{K_k}\Delta_A = [(\beta_k+\varphi_k)\operatorname{ch}\alpha_k + 2\nu \operatorname{ch}\alpha_k](\alpha_k+\gamma_k) - \\
-[\alpha_k \operatorname{ch}\gamma_k + 2\nu \operatorname{sh}\gamma_k]\operatorname{sh}(\alpha_k+\gamma_k) - \\
-R\left\{\frac{1}{2}[\operatorname{sh}2\gamma_k - 2\gamma_k](\operatorname{sh}\alpha_k + \alpha_k \operatorname{ch}\alpha_k) + \right. \\
\left. + [\gamma_k^2 + 2(1-2\nu)(\operatorname{ch}^2\gamma_k - \nu)\alpha_k \operatorname{sh}\alpha_k\right\} - Q\{(1+4\nu-4\nu^2)\operatorname{ch}\alpha_k \operatorname{sh}^2\varphi_k + \\
+ 2\beta_k(\alpha_k+\beta_k)\operatorname{ch}\alpha_k \operatorname{sh}^2\varphi_k - \varphi_k^2[\alpha_k \operatorname{sh}(\alpha_k+2\beta_k) + 2\nu \operatorname{ch}(\alpha_k+2\beta_k)] + \\
+ 2(1-\nu)\operatorname{sh}\varphi_k \operatorname{ch}\gamma_k[\alpha_k \operatorname{ch}(\alpha_k+\beta_k) + 2\nu \operatorname{sh}(\alpha_k+\beta_k)] + \\
+ 2(1-\nu)\operatorname{sh}\varphi_k \operatorname{sh}\beta_k[\alpha_k \operatorname{sh}(\alpha_k+\gamma_k) + 2\nu \operatorname{ch}(\alpha_k+\gamma_k)] + 4\nu\beta_k \operatorname{sh}\alpha_k \operatorname{sh}^2\varphi_k - \\
-4\nu(1-\nu)\varphi_k \operatorname{sh}\alpha_k + (1+2\nu)\alpha_k \operatorname{sh}\alpha_k \operatorname{sh}^2\varphi_k - 2(1-\nu)\alpha_k\varphi_k \operatorname{ch}\alpha_k - \\
-[4(1-\nu)\beta_k + (3-4\nu)\varphi_k]\varphi_k \operatorname{ch}\alpha_k\} - Q^2\{2(1-\nu)\operatorname{ch}\alpha_k + \\
+\beta_k(\alpha_k+\beta_k)\operatorname{ch}\alpha_k + (\alpha_k+2\nu\beta_k)\operatorname{sh}\alpha_k + (3-4\nu)\operatorname{sh}\beta_k[\alpha_k \operatorname{ch}(\alpha_k+\beta_k) + \\
+2\nu \operatorname{sh}(\alpha_k+\beta_k)]\}(\operatorname{sh}^2\varphi_k - \varphi_k^2) + RQ\{[[2\beta_k^2 - (1-2\nu)]\operatorname{sh}^2\varphi_k - \\
-[4(1-\nu)\beta_k + (3-4\nu)\varphi_k + 2(1-2\nu)\varphi_k^2 \operatorname{ch}2\beta_k - \\
-8(1-\nu)(1-2\nu)\operatorname{sh}\beta_k \operatorname{sh}\varphi_k \operatorname{ch}\gamma_k]\alpha_k \operatorname{sh}\alpha_k - [2\beta_k \operatorname{sh}^2\varphi_k - \varphi_k^2 \operatorname{sh}2\beta_k - \\
-2(1-\nu)\varphi_k + 2(1-\nu)\operatorname{sh}\varphi_k \operatorname{ch}(\gamma_k+\beta_k)](\operatorname{sh}\alpha_k + \alpha_k \operatorname{ch}\alpha_k)\} + \\
+ RQ^2\left\{[\beta_k^2 - 2(1-2\nu)(3-4\nu)\operatorname{sh}^2\beta_k]\alpha_k \operatorname{sh}\alpha_k - \left(\beta_k + \frac{3-4\nu}{2}\operatorname{sh}2\beta_k\right)\times\right. \\
\left.\times(\operatorname{sh}\alpha_k + \alpha_k \operatorname{ch}\alpha_k)(\operatorname{sh}^2\varphi_k - \varphi_k^2)\right\}. \qquad (17)
\end{gathered}
$$

$$
\begin{gathered}
\frac{\omega_k}{K_k}\Delta_B = [\gamma_k \operatorname{sh}\alpha_k + 2\nu \operatorname{ch}\alpha_k](\alpha_k+\gamma_k) + (\alpha_k \operatorname{sh}\gamma_k + 2\nu \operatorname{ch}\gamma_k)\operatorname{sh}(\alpha_k+\gamma_k) + \\
+ R\{[\gamma_k^2 + \operatorname{ch}^2\gamma_k - (1-2\nu)^2](\operatorname{sh}\alpha_k + \alpha_k \operatorname{ch}\alpha_k) + \\
+(1-2\nu)(\operatorname{sh}2\gamma_k + 2\gamma_k)\alpha_k \operatorname{sh}\alpha_k\} - Q\{(1+4\nu-4\nu^2)\operatorname{sh}\alpha_k \operatorname{sh}^2\varphi_k + \\
+2\beta_k(\alpha_k+\beta_k)\operatorname{sh}\alpha_k \operatorname{sh}^2\varphi_k + \varphi_k^2[\alpha_k \operatorname{ch}(\alpha_k+2\beta_k) + 2\nu \operatorname{sh}(\alpha_k+2\beta_k)] - \\
-2(1-\nu)\operatorname{sh}\varphi_k \operatorname{sh}\gamma_k[\alpha_k \operatorname{ch}(\alpha_k+\beta_k) + 2\nu \operatorname{sh}(\alpha_k+\beta_k)] - \\
-2(1-\nu)\operatorname{sh}\varphi_k \operatorname{ch}\beta_k[\alpha_k \operatorname{sh}(\alpha_k+\gamma_k) + 2\nu \operatorname{ch}(\alpha_k+\gamma_k)] + 4\nu\beta_k \operatorname{ch}\alpha_k \operatorname{sh}^2\varphi_k - \\
-4\nu(1-\nu)\varphi_k \operatorname{ch}\alpha_k + (1+2\nu)\alpha_k \operatorname{ch}\alpha_k \operatorname{sh}^2\varphi_k - 2(1-\nu)\alpha_k\varphi_k \operatorname{sh}\alpha_k - \\
-[4(1-\nu)\beta_k + (3-4\nu)\varphi_k]\varphi_k \operatorname{sh}\alpha_k\} - Q^2\{2(1-\nu)\operatorname{sh}\alpha_k + \\
+\beta_k(\alpha_k+\beta_k)\operatorname{sh}\alpha_k + (\alpha_k+2\nu\beta_k)\operatorname{ch}\alpha_k - (3-4\nu)\operatorname{ch}\beta_k[\alpha_k \operatorname{ch}(\alpha_k+\beta_k) + \\
+2\nu \operatorname{sh}(\alpha_k+\beta_k)]\}(\operatorname{sh}^2\varphi_k - \varphi_k^2) + RQ\{[4(1-\nu)\operatorname{ch}\beta_k \operatorname{sh}\varphi_k \operatorname{sh}\gamma_k + \\
+[4(1-\nu)\beta_k + (3-4\nu)\varphi_k]\varphi_k - \varphi_k^2 \operatorname{ch}2\beta_k - 2\beta_k^2 \operatorname{sh}^2\varphi_k -
\end{gathered}
$$

$$
\begin{aligned}
&- 2(1+2\nu-4\nu^2)\operatorname{sh}^2\varphi_k]\,(\operatorname{sh}\alpha_k+\alpha_k\operatorname{ch}\alpha_k)+[2(1-\nu)\operatorname{sh}\varphi_k\operatorname{ch}(\beta_k+\gamma_k)+\\
&+2(1-\nu)\varphi_k-\varphi_k^2\operatorname{sh}2\beta_k-2\beta_k\operatorname{sh}^2\varphi_k]\,2(1-2\nu)\alpha_k\operatorname{sh}\alpha_k\}+\\
&+RQ^2\{[(3-4\nu)\operatorname{sh}^2\beta_k-\beta_k^2]\,(\operatorname{sh}\alpha_k+\alpha_k\operatorname{ch}\alpha_k)+\\
&+(1-2\nu)\,[(3-4\nu)\operatorname{sh}2\beta_k-2\beta_k]\,\alpha_k\operatorname{sh}\alpha_k\}\,(\operatorname{sh}^2\varphi_k-\varphi_k^2),
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\frac{\Delta_C}{K_k}&=\operatorname{sh}\gamma_k\operatorname{sh}(\alpha_k+\gamma_k)-(\alpha_k+\gamma_k)\operatorname{sh}\alpha_k+R\left\{\frac{1}{2}(\operatorname{sh}2\gamma_k-2\gamma_k)\times\right.\\
&\left.\times(\operatorname{sh}\alpha_k+\alpha_k\operatorname{ch}\alpha_k)-(1-2\nu+\operatorname{ch}^2\gamma_k)\alpha_k\operatorname{sh}\alpha_k\right\}+Q\,\{2(\alpha_k+\beta_k)\times\\
&\times\operatorname{sh}\alpha_k\operatorname{sh}^2\varphi_k-\varphi_k\operatorname{ch}(\alpha_k+2\beta_k)-(1-2\nu)\operatorname{ch}\alpha_k\operatorname{sh}^2\varphi_k-2(1-\nu)\varphi_k\operatorname{sh}\alpha_k+\\
&+2(1-\nu)\operatorname{sh}\varphi_k[\operatorname{ch}\beta_k\operatorname{sh}(\alpha_k+\gamma_k)+\operatorname{sh}\gamma_k\operatorname{ch}(\alpha_k+\gamma_k)]\}+\\
&+Q^2\{(3-4\nu)\operatorname{ch}\beta_k\operatorname{ch}(\alpha_k+\beta_k)+(\alpha_k+\beta_k)\operatorname{sh}\alpha_k-(1-2\nu)\operatorname{ch}\alpha_k\}(\operatorname{sh}^2\varphi_k-\varphi_k^2)+\\
&+RQ\,\{[2\beta_k\operatorname{sh}^2\varphi_k-\varphi_k^2\operatorname{sh}2\beta_k-2(1-\nu)\varphi_k+\\
&+2(1-\nu)\operatorname{sh}\varphi_k\operatorname{ch}(\beta_k+\gamma_k)]\,(\operatorname{sh}\alpha_k+\alpha_k\operatorname{ch}\alpha_k)+\\
&+[\varphi_k^2\operatorname{ch}2\beta_k-4(1-\nu)\operatorname{ch}\beta_k\operatorname{sh}\varphi_k\operatorname{sh}\gamma_k+(5-6\nu)\operatorname{sh}^2\varphi_k]\,\alpha_k\operatorname{sh}\alpha_k\}+\\
&+RQ^2\left\{\left(\frac{3-4\nu}{2}\operatorname{sh}2\beta_k+\beta_k\right)(\operatorname{sh}\alpha_k+\alpha_k\operatorname{ch}\alpha_k)-\right.\\
&\left.-(3-4\nu)\,\alpha_k\operatorname{sh}\alpha_k\operatorname{sh}^2\beta_k\right\}(\operatorname{sh}^2\varphi_k-\varphi_k^2)\,.
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\frac{\Delta_D}{K_k}&=-\operatorname{ch}\gamma_k\operatorname{sh}(\alpha_k+\gamma_k)-(\alpha_k+\gamma_k)\operatorname{ch}\alpha_k-R\left\{(1-2\nu+\operatorname{ch}^2\gamma_k)(\operatorname{sh}^2\alpha_k+\right.\\
&\left.+\alpha_k\operatorname{ch}\alpha_k)-\frac{1}{2}(\operatorname{sh}2\gamma_k+2\gamma_k)\,\alpha_k\operatorname{sh}\alpha_k\right\}+Q\,\{2(\alpha_k+\beta_k)\operatorname{ch}\alpha_k\operatorname{sh}^2\varphi_k+\\
&+\varphi_k^2\operatorname{sh}(\alpha_k+2\beta_k)-(1-2\nu)\operatorname{sh}\alpha_k\operatorname{sh}^2\varphi_k-2(1-\nu)\varphi_k\operatorname{ch}\alpha_k-2(1-\nu)\operatorname{sh}\varphi_k\times\\
&\times[\operatorname{ch}\gamma_k\operatorname{ch}(\alpha_k+\beta_k)+\operatorname{sh}\beta_k\operatorname{sh}(\alpha_k+\gamma_k)]\}+Q^2\{(\alpha_k+\beta_k)\operatorname{ch}\alpha_k-\\
&-(1-2\nu)\operatorname{sh}\alpha_k-(3-4\nu)\operatorname{sh}\beta_k\operatorname{ch}(\alpha_k+\beta_k)\}\,(\operatorname{sh}^2\varphi_k-\varphi_k^2)+\\
&+RQ\,\{[2\operatorname{sh}^2\varphi_k+\varphi_k^2\operatorname{ch}2\beta_k-4(1-\nu)\operatorname{sh}\beta_k\operatorname{sh}\varphi_k\operatorname{ch}\gamma_k-\\
&-(1+2\nu)\operatorname{sh}^2\varphi_k]\,(\operatorname{sh}\alpha_k+\alpha_k\operatorname{ch}\alpha_k)+[2(1-\nu)\varphi_k+\\
&+2(1-\nu)\operatorname{sh}\varphi_k\operatorname{ch}(\beta_k+\gamma_k)-2\beta_k\operatorname{sh}^2\varphi_k-\varphi_k^2\operatorname{sh}2\beta_k]\,\alpha_k\operatorname{sh}\alpha_k\}+\\
&+RQ^2\left\{\left(\frac{3-4\nu}{2}\operatorname{sh}2\beta_k-\beta_k\right)\alpha_k\operatorname{sh}\alpha_k-\right.\\
&\left.-(3-4\nu)\operatorname{sh}^2\beta_k(\operatorname{sh}\alpha_k+\alpha_k\operatorname{ch}\alpha_k)\right\}(\operatorname{sh}^2\varphi_k-\varphi_k^2),
\end{aligned}
\tag{20}
$$

where

$$
\alpha_k=\omega_k\delta_1,\quad \beta_k=\omega_k\delta_2,\quad \varphi_k=\omega_k\delta_3,\quad \gamma_k=\beta_k+\varphi_k\,.
$$

Having the values of the coefficients $A_k^{(2)}$, $B_k^{(2)}$, $C_k^{(2)}$, $D_k^{(2)}$, we determine by (13) and (14) the values of the coefficients for the first and third layers, respectively.

Consider the case when the plate is built-in along the contour.* We require that the lateral conditions be satisfied not along the entire cylindrical surface restricting the plate, but only along some arbitrary circumference, i.e.,

$$\text{at } r = a, \quad z = m\delta_2 \text{ we have: } u^{(i)} = 0, \quad w^{(i)} = 0, \quad \frac{\partial w^{(i)}}{\partial r} = 0. \tag{21}$$

Here $i = 1, 2, 3$ depending on the value of m.

The constants F and H are determined from conditions (21) (the second condition of (21) is satisfied identically) and have the form

$$F = -\frac{2}{a}\frac{1+\nu}{1-\nu}\frac{1}{E_i}\sum_{k=1}^{\infty}\omega_k J_1(\mu_k)\,[A_k^{(i)}\omega_k \operatorname{ch} m\beta_k + B_k^{(i)}\omega_k \operatorname{sh} m\beta_k +$$

$$+ C_k^{(i)}(\operatorname{ch} m\beta_k + m\beta_k \operatorname{sh} m\beta_k) + D_k^{(i)}(\operatorname{sh} m\beta_k + m\beta_k \operatorname{ch} m\beta_k)], \tag{22}$$

$$H = \frac{2}{a}\cdot\frac{1+\nu}{1-\nu}\cdot\frac{1}{E_i}\sum_{k=1}^{\infty}\omega_k^2 J_1(\mu_k)\,\{A_k^{(i)}\omega_k \operatorname{sh} m\beta_k + B_k^{(i)}\omega_k \operatorname{ch} m\beta_k +$$

$$+ C_k^{(i)}[m\beta_k \operatorname{ch} m\beta_k - 2(1-2\nu)\operatorname{sh} m\beta_k] + D_k^{(i)}[m\beta_k \operatorname{sh} m\beta_k - 2(1-2\nu)\operatorname{ch} m\beta_k]\}. \tag{23}$$

Solutions for the cases of two-layer and one-layer circular plates can be obtained as particular cases of the solution obtained.**

1. Timoshenko, S.P. Theory of Elasticity.— ONTI. 1937. [Russian edition.]
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3. Melkonyan, A.P. Izgib trekhsloinoi tolstoi plity (Bending of a Sandwich Thick Plate).— Izvestiya AN ArmSSR, seriya fiziko-matematicheskikh nauk, No. 5. 1962.

* The case of simply supported plates is not treated here in view of the limited scope of the paper.

** In view of the limited scope of the paper, the results of the corresponding calculations are not given here. For the case of a supported plate, they are similar to the results obtained for simply supported rectangular thick sandwich plates /3/.

STABILITY OF A CYLINDRICAL SHELL IMMERSED IN A RESTRICTED FLUID MEDIUM UNDER DYNAMIC LOADING

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Consider the process of buckling of a thin-walled cylindrical shell immersed in a restricted fluid nonviscous medium under dynamic loading. Let the shell ends be hinged on rigid diaphragms. The fluid flow is considered as potential. The space occupied by the fluid has the form of the hollow between two coaxial circular cylinders. One wall of the hollow is formed by the thin-walled shell, while the other one is absolutely rigid. The upper end of the shell is situated at a distance l_1 from the free surface (figure), and the lower one coincides with the bottom of the hollow. A pressure p_0 variable with time acts on the shell and liquid from the side of the free surface.

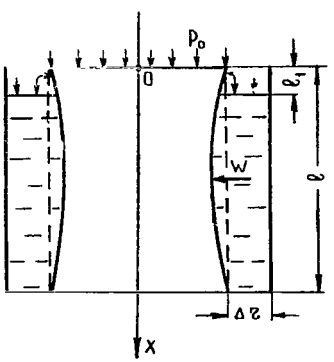

FIGURE.

The liquid disturbances will be considered as small, so that the liquid motion can be studied in a linear approximation. As to the shell, we assume that the nonlinear theory of shallow shells /1/ is applicable to it. We assume further that the tangential accelerations in the shell are small and can be neglected; the liquid is incompressible. The method described below and the results obtained are therefore applicable to shells of medium length at moderate radial dimensions of the annular clearance /3/ and under the condition that the load variation is not too abrupt.

We introduce a cylindrical system of coordinates (r, θ, x), with x axis coinciding with the shell axis. Let w be the normal displacement of the points of the middle surface of the shell, l the length, r_1 the radius of the middle surface, h the shell thickness, r_2 the outer radius of the annular hollow, $k = \frac{\pi}{l}$, $\bar{r} = kr$ a dimensionless coordinate, $\Delta r = r_2 - r_1$, $\Delta r = k\Delta r$, E the modulus of elasticity, μ the Poisson ratio, $h = \frac{h}{r_1}$, $\eta = r_1^2 h$, n the number of waves into which the shell buckles, $n = \frac{n}{\bar{r}}$, t the time, ρ the liquid density, φ the velocity potential, m the shell mass per unit

area, and $p_0 = \frac{p_0}{Eh^2}$ the dimensionless pressure. The positive directions and the coordinate origin are shown in the figure.

We represent the initial deflection of the shell w_{in} in the form

$$w_{in} = q_{in} \sin kx \cos n\theta \tag{1}$$

and the deflection w in the form

$$w = \sum_{j=1}^{3} q_j f_j. \tag{2}$$

Here $q_j = q_j(t)$ are deflection parameters, used in what follows as generalized coordinates, and

$$f_1 = \sin kx, \qquad f_2 = \frac{8}{3\pi - 8}\left(\sin kx - \frac{3\pi}{8}\sin^2 kx\right), \qquad f_3 = \sin kx \cos n\theta \tag{3}$$

are mutually orthogonal functions of the coordinates. The boundary conditions of hinged support are not fully satisfied.

The approximation (2) used for the shell deflection satisfactorily reflects both the precritical and the post-buckling shell deformation. f_1 and f_2 are near to the modes of the first and third tones of the fundamental vibrations of the shell in the liquid; the static lower critical pressure calculated by (3) coincides with the well-known solution /2/.

We proceed to the determination of the hydrodynamic forces caused by the shell deformation.

Using the Cauchy-Lagrange integral, we connect the radiation pressure p_u (the pressure caused by the shell vibration) at $r = r_1$ with the velocity potential φ.

In the linear approximation

$$[p_u]_{r=r_1} = -\rho \left[\frac{\partial \varphi}{\partial t}\right]_{r=r_1} \tag{4}$$

The function φ must satisfy the Laplace equation and the boundary conditions

$$\left[\frac{\partial \varphi}{\partial r}\right]_{r=r_1} = -\frac{dw}{dt}, \tag{5}$$

$$\left[\frac{\partial \varphi}{\partial r}\right]_{r=r_2} = 0, \tag{6}$$

$$\left[\frac{\partial \varphi}{\partial x}\right]_{x=l} = 0. \tag{7}$$

Neglecting the wave motion on the free surface of the liquid (the energy of the wave motion is small compared with the total energy of the system) we write

$$[\varphi]_{x=l_1} = 0. \tag{8}$$

We represent φ as sum of harmonic functions:

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3.$$

Let φ_1 satisfy conditions (5) and (6), and let it satisfy conditions (7) "on the average":

$$\int_{r_1}^{r_2} \left[\frac{\partial \varphi}{\partial x}\right]_{x=l} r dr = 0. \tag{9}$$

This last condition corresponds to the equality to zero of the rate of discharge of liquid through section $x = l$.

We subject φ_2 to the conditions

$$[\varphi_2]_{x=l_1} = \left[\frac{\partial \varphi_2}{\partial r}\right]_{r=r_2} = \left[\frac{\partial \varphi_2}{\partial r}\right]_{r=r_1} = 0, \quad \left[\frac{\partial \varphi_2}{\partial x}\right]_{x=l} = -\left[\frac{\partial \varphi_1}{\partial x}\right]_{x=l} \tag{10}$$

It follows that φ_3 must be a solution of the boundary-value problem

$$\nabla^2 \varphi_3 = 0, \quad \left[\frac{\partial \varphi_3}{\partial r}\right]_{r=r_1} = \left[\frac{\partial \varphi_3}{\partial r}\right]_{r=r_2} = \left[\frac{\partial \varphi_3}{\partial x}\right]_{x=l} = 0, \quad [\varphi_3]_{x=l_1} = -[\varphi_1]_{x=l_1}, \tag{11}$$

where ∇^2 is the Laplacian.

Let us write

$$\varphi_1 = -\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{d}{dt} a_{mn}(t) \sin mkx \cos n\theta \, F_n(\bar{r}m) + \\ + \sum_{m=1}^{\infty} \frac{d}{dt} a_{m0}(t) \frac{2xr_1(-1)^{m+1}}{km(r_2^2 - r_1^2)}. \tag{12}$$

Here are expansion coefficients of the function $w(x, \theta, t)$ in a Fourier series by $\sin mkx \cos n\theta$. We represent φ_2 and φ_3 as series

$$\varphi_2 = \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} b_{ln} A_{ln}, \tag{13}$$

$$\varphi_3 = \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} c_{ln} B_{ln}; \tag{14}$$

$b_{ln}(t)$, $c_{ln}(t)$ are coefficients which remain to be determined, and

$$A_{ln} = \frac{\operatorname{sh} \omega_{ln}(x - l_1)}{\omega_{ln} \operatorname{ch} \omega_{ln}(l - l_1)} \xi_{ln} \cos n\theta, \tag{15}$$

$$B_{ln} = \frac{\operatorname{ch} \omega_{ln}(x - l)}{\operatorname{ch} \omega_{ln}(l_1 - l)} \xi_{ln} \cos n\theta. \tag{16}$$

Here

$$F_n(\bar{r}m) = \frac{I_n'(\bar{r}_2 m) K_n'(\bar{r}m) - K_n'(\bar{r}_2 m) I_n'(\bar{r}m)}{K_n'(\bar{r}_1 m) I_n'(\bar{r}_2 m) - K_n'(r_2 m) I_n'(\bar{r}_1 m)}, \tag{17}$$

$\xi_{ln}(r)$ and ω_{ln}^2 are normalized eigenfunctions and eigenvalues of the

boundary-value problem

$$\xi''_{ln} + \frac{1}{r}\xi'_{ln} - \left(\frac{n^2}{r^2} - \omega^2_{ln}\right)\xi_{ln} = 0, \qquad [\xi'_{ln}]_{r=r_1, r_2} = 0. \tag{18}$$

I_n, K_n are Bessel functions of a purely imaginary argument. The prime stands for differentiation by r. ξ_{ln} form a complete and orthogonal system of functions in the region limited by circumferences of radii r_1 and r_2. Equation (18) is solved in Bessel functions of a real argument. The harmonic functions A_{ln} satisfy the conditions for φ_2 at $r = r_1$, $r = r_2$, and $x = l_1$. It follows that the coefficients b_{ln} must be selected in such a way as to make series (13) satisfy the conditions for φ_2 at $x = l$. Calculating b_{ln}, we obtain

$$b_{ln} = -\int_0^{2\pi}\int_{r_1}^{r_2}\left[\frac{\partial\varphi_1}{\partial x}\right]_{x=l}\xi_{ln}\cos n\theta r dr d\theta. \tag{19}$$

The coefficients c_{ln} are found from the boundary conditions for φ_3 at $x = l_1$ (the remaining conditions for φ_3 are satisfied by selecting the functions B_{ln} in the form (16)). We find

$$c_{ln} = -\int_0^{2\pi}\int_{r_1}^{r_2}[\varphi_1]_{x=l_1}\xi_{ln}\cos n\theta r dr d\theta. \tag{20}$$

If the free surface of the liquid coincides with the upper end of the shell ($l_1 = 0$), then $[\varphi_1]_{x=l_1} = 0$ and therefore $\varphi_3 \equiv 0$. Only this case will be considered in what follows. It follows from (12) and (13), when the deflection w is given by (2):

$$\varphi_1 = \sum_{j=1}^{3}\frac{dq_j}{dt}\psi_{1j}, \qquad \varphi_2 = \sum_{j=1}^{3}\frac{dq_j}{dt}\psi_{2j}, \tag{21}$$

where

$$\left.\begin{aligned}
\psi_{11} &= -\sin kx F_0(\bar{r}) + \frac{2r_1 x}{k(r_2^2 - r_1^2)},\\
\psi_{12} &= -\sum_{m=3}^{\infty} b_m\left[\sin mkx F_0(\bar{r}m) - \frac{2r_1 x}{km(r_2^2 - r_1^2)}\right],\\
\psi_{13} &= -\sin kx\cos n\theta F_n(\bar{r}),
\end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned}
\psi_{21} &= -k\sum_{l=1}^{\infty}\frac{\operatorname{sh}\omega_{0l}x\xi_{0l}}{\omega_{0l}\operatorname{ch}\omega_{0l}l}\int_0^{2\pi}\int_{r_1}^{r_2}F_0(\bar{r})\xi_{0l}r dr d\theta,\\
\psi_{22} &= -k\sum_{m=3}^{\infty}mb_m\sum_{l=1}^{\infty}\frac{\operatorname{sh}\omega_{0l}x\xi_{0l}}{\omega_{0l}\operatorname{ch}\omega_{0l}l}\int_0^{2\pi}\int_{r_1}^{r_2}F_0(\bar{r}m)\xi_{0l}r dr d\theta,\\
\psi_{23} &= -k\sum_{l=1}^{\infty}\frac{\operatorname{sh}\omega_{nl}x\,\xi_{nl}\cos n\theta}{\omega_{nl}\operatorname{ch}\omega_{nl}l}\int_0^{2\pi}\int_{r_1}^{r_2}F_n(\bar{r})\xi_{nl}\cos^2 n\theta r dr d\theta.
\end{aligned}\right\} \tag{23}$$

Here

$$b_m = \begin{cases} -\dfrac{24}{(m-2)\,m\,(m+2)\,(3\pi-8)} & \text{at} \quad m = 3,\ 5,\ 7\cdots \\ 0 & \text{at even } m \end{cases} \tag{24}$$

are expansion coefficients of the function f_2 in a Fourier series by $\sin mkx$.

The work of the radiation pressure and the inertia forces on the displacement w is equal to

$$A_u = \int_0^{2\pi}\int_0^{l} (p_u - m\ddot{w})\, r_1 dx d\theta. \tag{25}$$

Using the relationships given we obtain

$$A_u = -\sum_{j=1}^{3}\sum_{i=1}^{3} \ddot{q}_i q_j (\alpha_{1ij} + \alpha_{2ij} + \gamma_{ij}), \tag{26}$$

where

$$\alpha_{sij} = \rho \int_0^{2\pi}\int_0^{l} [\psi_{si}]_{r=r_1} f_j r_1 dx d\theta \qquad (s = 1,\ 2), \tag{27}$$

$$\gamma_{ij} = \begin{cases} 0 & \text{at} \quad i \neq j, \\ m\displaystyle\int_0^{2\pi}\int_0^{l} f_i^2 r_1 dx d\theta & \text{at} \quad i = j. \end{cases}$$

It is easily shown that

$$\lim_{\Delta\bar{r}\to 0} \alpha_{1ij} = \infty \qquad \text{and} \qquad \lim_{\Delta\bar{r}\to 0} \alpha_{2ij} = 0.$$

Since γ_{ij} is independent of the gap magnitude Δr, we can write the following asymptotic equality:

$$A_u \sim -\sum_{i=1}^{3}\sum_{j=1}^{3} \ddot{q}_i q_j \alpha_{1ij} \qquad (\Delta\bar{r} \to 0). \tag{28}$$

The error in the solution due to replacing the exact boundary conditions at $x = l$ by the approximate conditions (9) and by neglecting the inertial properties of the shell must therefore decrease with the decrease of the radial dimension of the annular hollow and with the increase of the shell length.

We shall use in what follows the assumption that the radial clearance is small ($\Delta\bar{r} \ll 1$). It is then possible to base the calculations on the "inertia-free" shell model and on boundary conditions (9) (in this case $\psi_{2j} \equiv 0$). The calculations of $[\psi_{1j}]_{r=r_1}$ are simplified by the use of the asymptotic formulas

$$[\psi_{11}]_{r=r_1} \sim \frac{1}{k\Delta\bar{r}} \sin kx + \frac{x}{\Delta\bar{r}} \qquad (\Delta\bar{r} \to 0), \tag{29}$$

$$[\psi_{12}]_{r=r_1} \sim \sum_{m=3}^{\infty} b_m \left(\frac{\sin mkx}{m^2 k \Delta \bar{r}} + \frac{x}{m \Delta \bar{r}} \right) \qquad (\Delta \bar{r} \to 0), \tag{30}$$

$$[\psi_{13}]_{r=r_1} \sim \frac{\sin kx \cos n\theta}{k \Delta \bar{r} (1 + \bar{n}^2)} \qquad (\Delta \bar{r} \to 0). \tag{31}$$

In the case of small radial clearance (in the sense explained above) the equations of motion of the shell in dimensionless parameters, obtained by the Ritz method, are

$$\begin{aligned} &\frac{d^2\zeta_1}{d\tau^2} + \Phi_{11}\zeta_1 = \Phi_{12}, \\ &\frac{d^2\zeta_2}{d\tau^2} + \Phi_{21}\zeta_2 = \Phi_{22}, \\ &\frac{d^2\zeta_3}{d\tau^2} + \Phi_{31}\zeta_3 = \Phi_{32}. \end{aligned} \tag{32}$$

Here $\zeta_i = \frac{q_i}{h}$, $\tau = \lambda t$ is dimensionless time,

$$\lambda^2 = 0.226 \frac{E}{\rho r_1^2} \frac{\pi^2 h}{l^2} \Delta r, \tag{33}$$

$$\begin{aligned} \Phi_{11} &= a_{11} + a_{12}\zeta_3^2 - \frac{\bar{p}_0 \eta}{l^2}, \\ \Phi_{12} &= (a_{13} - 0.844\bar{p}_0\eta)\zeta_2 + (a_{14} + a_{15})\zeta_3^2 + a_k\zeta_2\zeta_3^2 - \\ &\quad - a_{14}\zeta_{in}^2 - a_{15}\zeta_3\zeta_{in} + 0.36\bar{p}_0\left(1 - \frac{\mu}{2}\right), \\ \Phi_{21} &= a_{21} + a_{22}\zeta_3^2 - 19\bar{p}_2\eta, \\ \Phi_{22} &= (a_{23} - 32.1\bar{p}_0\eta)\zeta_1 + (a_{24} + a_{25})\zeta_3^2 + a_{26}\zeta_2\zeta_3^2 - \\ &\quad - a_{24}\zeta_{in}^2 - a_{25}\zeta_3\zeta_{in} + 6.92\bar{p}_0\left(1 - \frac{\mu}{2}\right), \\ \Phi_{31} &= a_{31} + a_{32}\zeta_1 + a_{33}\zeta_2 + a_{34}\zeta_3^2 + a_{35}\zeta_1^2 + a_{36}\zeta_1\zeta_2 + a_{37}\zeta_2^2 + a_{38}\bar{p}_0, \\ \Phi_{32} &= a_{31}\zeta_{in} + a_{32}\zeta_1\zeta_{in} + a_{33}\zeta_2\zeta_{in} + a_{34}\zeta_3\zeta_{in}^2. \end{aligned} \tag{34}$$

The coefficients a_{ks} in (34) depend only on the value of the dimensionless parameters η and $\bar{n}$.

An analysis of the structure of system (32) shows that its solution is independent of the radial dimension of the annular hollow $\Delta\bar{r}$. At fixed shell parameters only the time scale m_t, which is proportional to $\frac{1}{\sqrt{\Delta r}}$, will vary with the variation of the clearance.

If follows in particular from here that the frequencies of small natural vibrations of the system are proportional to $\sqrt{\Delta r}$, while the impulses of impact load leading to buckling are inversely proportional to it. This conclusion has been corroborated experimentally in experiments with steel shells $\left(\bar{r}_1 = 0.785,\ \bar{h} = \frac{1}{150}\right)$ with $\Delta\bar{r}$ varying from 0.044 to 0.350. The value of the impulse leading to buckling was determined in the experiments by gradually increasing its value up to the moment of appearance of considerable elastic deformations in the shell.

Equations (32) were solved for the case of action on the system of an impact load of pressure p_0.

The results of the calculations satisfactorily agree with experimental results.

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VIBRATION MODES IN THE PROBLEM OF THE SUPERSONIC FLUTTER OF PANELS

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The characteristics of the natural modes of vibrations of panels in the flow of a supersonic gas stream can now be determined, thanks to the use of high-speed digital electronic computers. Some of these characteristics, common to panels of different shape in the plan, will be represented here for simplicity relative to the simplest equation

$$D\frac{\partial^4 w}{\partial x^4}+\mu\frac{\partial^2 w}{\partial t^2}=\frac{p_0\varkappa}{c_0}\left(c\frac{\partial w}{\partial x}-\varepsilon\frac{\partial w}{\partial t}\right), \tag{1}$$

where $w(x, t)$ is the deflection of the panel occupying a region in the form of a strip; $0\leqslant x\leqslant a$, ε is a dimensionless number indicating the factor by which the external damping exceeds the aerodynamic damping, calculated by the well-known formula of the piston theory /1/; all the other symbols coincide with those used in /2/ (the panel moves in the direction of the positive x axis). The solutions of (1) satisfying the boundary conditions at the ends $x=0$, $x=a$ are compared below with monochromatic waves /3/ of length a

$$w_{mon}(x, t)=\sin\left[\frac{2\pi}{a}x+\left(\frac{2\pi}{a}\right)^2\sqrt{\frac{D}{\mu}}\,t\right], \tag{2}$$

which satisfy the homogeneous equation corresponding to (1), but not the boundary conditions at ends $x=0$, $x=a$. The velocity of propagation of waves (2) is obviously given by the formula

$$v_{mon}=-\frac{2\pi}{a}\sqrt{\frac{D}{\mu}}. \tag{3}$$

We introduce the dimensionless magnitudes:

$$\frac{x}{a},\ \frac{w}{h},\ t:\sqrt{\frac{\mu a^4}{D}}. \tag{4}$$

We shall represent them by the same symbols x, w, t as the corresponding dimensional magnitudes. Consider disturbed panel motions of the form

$$w(x, t)=X(x)\,e^{\omega t}. \tag{5}$$

The right-hand part of (5) can, generally speaking, be a complex function of the real variables x, t, since the frequency ω is complex:

$$\omega = p + iq. \tag{6}$$

Separating the real and imaginary parts in (5), we obtain the real disturbed motions

$$w(x, t) = \mathrm{Re}\,[X(x)e^{\omega t}],\ \mathrm{Im}\,[X(x)\,e^{\omega t}]. \tag{7}$$

At $q \neq 0$ we shall call these motions the natural vibrations of the panel in the gas stream. The function $X(x)$ in (5) is an eigenfunction (nontrivial solution) of the boundary-value problem

$$\frac{d^4X}{dx^4} - A\frac{dX}{dx} = \lambda X, \qquad L_i(X) = 0, \quad i = 1, 2, 3, 4, \tag{8}$$

where $L_i(X)$ represent linear homogeneous boundary conditions at the ends $x = 0$, $x = 1$, and the dimensionless parameters A, λ are linked with the problem parameters by the formulas

$$A = \frac{a^3 p_0 \varkappa}{Dc_0}\,c, \qquad B = \frac{a^2}{\sqrt{D\mu}}\frac{p_0\varkappa}{c_0}\,\varepsilon, \qquad \lambda = -\omega^2 - B\omega. \tag{9}$$

By forming the combination

$$X(x) = \sum_{j=1}^{4} C_j X^j(x, A, \lambda) \tag{10}$$

of four linearly independent solutions $X^j(x, A, \lambda)$ of equation (8) and satisfying the boundary conditions

$$\sum_{j=1}^{4} C_j X^j(x, A, \lambda), \tag{11}$$

we obtain the following equation for determining the eigenvalues λ:

$$F(A, \lambda) \equiv \begin{vmatrix} L_1(X^1) & L_1(X^2) & L_1(X^3) & L_1(X^4) \\ L_2(X^1) & L_2(X^2) & L_2(X^3) & L_2(X^4) \\ L_3(X^1) & L_3(X^2) & L_3(X^3) & L_3(X^4) \\ L_4(X^1) & L_4(X^2) & L_4(X^3) & L_4(X^4) \end{vmatrix} = 0. \tag{12}$$

The left-hand part of this equation is a whole analytical function of the parameters A, λ. To each root λ of (12) there corresponds at least one solution C_j of system (11) and one eigenfunction (10). The frequencies ω corresponding to this function are found from the equation $\omega^2 + B\omega + \lambda = 0$:

$$\omega = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - \lambda}. \tag{13}$$

If the eigenvalue λ is a simple root of (12), there corresponds to it only one (up to a constant factor) eigenfunction $X(x)$. If the eigenvalue is a double root of equation (12) and the rank of matrix (12) is equal to

two, then two linearly independent eigenfunctions will correspond to it; if the rank of the matrix is three, only one eigenfunction will correspond to it. In this case the completeness of the solution necessitates a consideration of the solution

$$w\ (x,\ t) = [X_{(1)}\ (x) + t\ X\ (x)]\ e^{\omega t}. \tag{14}$$

where $X_{(1)}\ (x)$ is a function attached to the eigenfunction $X\ (x)$ /4, 5/.

Every eigenfunction $X\ (x)$ of the boundary-value problem (8) can be represented in the form

$$X\ (x) = X_1\ (x) + i\ X_2\ (x). \tag{15}$$

where $X_1\ (x)$, $X_2\ (x)$ are real functions. The zeros of the eigenfunction (15) divide the segment $0 \leqslant x \leqslant 1$ into a finite number of intervals, in each of which the function (15) can be represented in trigonometric form:

$$X\ (x) = |X\ (x)|\, e^{i\psi(x)}\ . \tag{16}$$

The argument $\psi\ (x)$ in each of these intervals is a continuously differentiable function of x; at the end points of the intervals $\psi\ (x)$ is not defined and may be discontinuous.

Let the functions $X_1\ (x)$, $X_2\ (x)$ of (15) be linearly dependent. We shall call the eigenfunction (15) in this case (conventionally) real. It is easily seen that for a function real in this sense the condition $\psi\ (x)$ = const is satisfied on each of the intervals of continuity of $\psi\ (x)$.

If, on the other hand, the functions $X_1\ (x)$, $X_2\ (x)$ are linearly independent, we shall call the corresponding eigenfunction a complex eigenfunction. It is easily seen that in this case the condition $\psi\ (x) \neq$ const is satisfied in each of the continuity intervals of $\psi\ (x)$.

By substituting (6) and (16) in (5) and separating the real and imaginary parts we obtain that the natural vibrations (7) are given by the equalities

$$\begin{aligned} w\ (x,\ t) &= |X\ (x)| \cos\ [\psi\ (x) + qt]\ e^{pt}, \\ w\ (x,\ t) &= |X\ (x)| \sin\ [\psi\ (x) + qt]\ e^{pt}. \end{aligned} \tag{17}$$

In the case of a real eigenfunction $X\ (x)$ the condition $\psi\ (x)$ = const is satisfied in the intervals of continuity of $\psi\ (x)$, so that the natural vibrations (17) can be called in that case standing waves.

In the case of a complex eigenfunction $X\ (x)$ the condition $\psi\ (x) \neq$ const is fulfilled in the intervals of continuity. On the strength of the analogy with functions (2), the natural vibrations (17) can be called in this case traveling waves. The dimensionless velocity of propagation of the traveling waves (17) is determined by the formula

$$-q\ /(d\psi\ (x)/dx).$$

Taking symbols (4) into account, we obtain the dimensional velocity of propagation of the traveling waves (17):

$$v = -\frac{2\pi}{a}\sqrt{\frac{D}{\mu}}\,\frac{q/2\pi}{d\psi\,(x)\,/dx} = v_{mon}\ \ \frac{q/2\pi}{d\psi\,(x)\,/dx}\ , \tag{18}$$

which is exactly equal to the velocity of propagation of the nodal lines along the panel (the derivative $d\psi\ (x)/dx$ in the right-hand part of (17) is taken by

the dimensionless parameter x). The velocity (18) of the waves traveling along a panel fastened by the edges is generally speaking different in the different points of the panel.

Panel flutter (natural vibrations (17) in the case $p>0$), observed on the basis of (1), is possible only for those complex eigenvalues λ of the boundary-value problem (8) which actually exist at the corresponding values of the velocity A /2/. To the complex eigenvalue λ of the boundary-value problem (8) there corresponds a complex eigenfunction (15) and natural vibrations (17) in the form of waves traveling along the panel. It follows that natural vibrations in the form of traveling waves are a typical phenomenon for the panel flutter; this conclusion has been corroborated by the experiments /6, 7/. The theoretical quantitative study of the pattern of the traveling waves (17) at flutter, linked as a rule with much calculation, has become possible as a result of the introduction of high-speed computers.

With the exception of the case when the rank of matrix (12) is less than three, the eigenfunction (10) can be found by the formula

$$X(x) = C_1 e^{-\alpha x} \sin \beta x + C_2 e^{-\alpha x} \cos \beta x + \\ + C_3 e^{\alpha x} \operatorname{sh} \gamma x + C_4 e^{\alpha x} \operatorname{ch} \gamma x, \tag{19}$$

where α, β satisfy the corresponding transcendental equations /2/, and the constants C_i are determined for hinged edges $x = 0$, $x = 1$ by the equalities

$$\begin{gathered} C_1 = e^{\alpha} [\alpha \gamma \operatorname{ch} \gamma + (\alpha^2 - \beta^2 - k\pi^2) \operatorname{sh} \gamma] - \alpha \gamma e^{-\alpha} \cos \beta, \\ C_2 = \alpha (e^{-\alpha} \gamma \sin \beta - \operatorname{sh} \gamma e^{\alpha} \beta), \\ C_3 = e^{-\alpha} [(\beta^2 - \alpha^2 + k\pi^2) \sin \beta - \alpha\beta \cos \beta] + \alpha \beta e^{\alpha} \operatorname{ch} \gamma, \\ C_4 = -C_2. \end{gathered} \tag{20}$$

and for built-in edges $x = 0$, $x = 1$ by the equalities

$$\begin{gathered} C_1 = (e^{2\alpha} \operatorname{ch} \gamma - \cos \beta) \gamma - 2\alpha e^{2\alpha} \operatorname{sh} \gamma, \\ C_2 = \gamma \sin \beta - \beta e^{2\alpha} \operatorname{sh} \gamma, \\ C_3 = \beta \cos \beta + 2\alpha \sin \beta - \beta e^{2\alpha} \operatorname{sh} \gamma, \\ C_4 = -C_2. \end{gathered} \tag{21}$$

Here $\gamma = \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}$ (in the case of equation (1) we must write $k = 0$, cf. the symbols in /2/).

The process of finding the eigenfunctions $X(x)$ consists first in solving fairly complex transcendental equations relative to the parameters α, β (which are complex numbers in the case of complex λ), and then in using formulas of the type of (19), (20), and (21). In order to preserve the accuracy, which drops sharply when numbers of large modulus and similar value are subtracted from each other, the terms in these equations should first be regrouped.

We give below the results of the calculation of the vibrations of a plate built-in at the edges $x = 0$, $x = 1$, performed on a "Strela" computer.

The real eigenfunctions, corresponding at $A = 0$ to the first two single eigenvalues $\lambda_1 = 500.6$, $\lambda_2 = 3804$, are represented in Figure 1. The corresponding natural vibrations have the form of standing waves. At $A = 636.6$ there corresponds to the coinciding eigenvalues $\lambda_1 = \lambda_2 = 2741$ only one eigenfunction $X(x)$, represented in Figure 1; the corresponding

natural vibrations have the form of standing waves. To the double eigenvalue $\lambda_1 = \lambda_2 = 2741$ there corresponds also the attached function $X_{(1)}(x)$ and solution (14); the form of this function was not determined by us.

At $A > 636.6$ the first two eigenvalues λ_1, λ_2 become complex conjugate numbers. Figure 2 represents the real (to the left) and imaginary (to the right) parts of the complex eigenfunction corresponding to the first eigenvalue $\lambda_1 = 2866 - i772.7$ at $A = 700$, to the first eigenvalue $\lambda_1 = 56{,}849 - i78{,}367$ at $A = 10{,}000$, and to the first eigenvalue $\lambda_1 = 137{,}511 - i206{,}656$ at $A = 20{,}000$. Figures 3 and 4 represent the functions $|X(x)|$ and $\psi(x)$. Knowing the eigenvalues and the functions $|X(x)|$, $\psi(x)$ corresponding to them it is possible to calculate the form of $w(x, t)$ which the traveling waves (17) adopt at different time moments. By successively photographing these forms on a film we obtain a multiplication film from which a clear opinion can be formed of the pattern of the traveling waves (17).

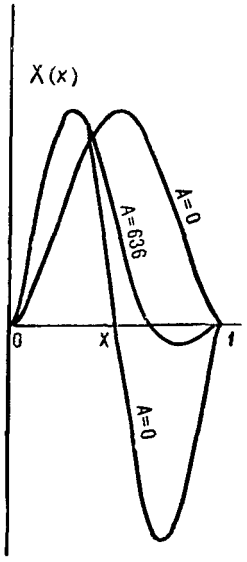

FIGURE 1.

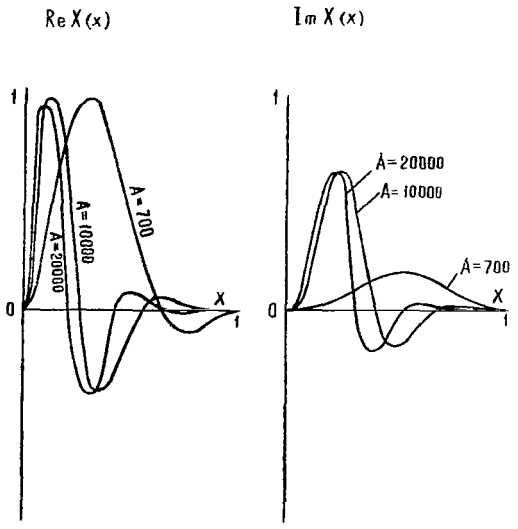

FIGURE 2.

The waves traveling against the stream are damped. The panel flutter is formed of waves traveling downstream. With the increase of the velocity A the maximum amplitude of the waves (17) is shifted sharply toward the back end of the panel; this conclusion is corroborated by the experiments /7/, it having been established that panel collapse as a result of the flutter always starts from the back edge. Another interesting conclusion is the difference between the amplitude distribution of the traveling waves (17) investigated, which satisfy the support conditions at the ends, and that of monochromatic waves of the type of (2). The calculations show that for the panels of practical interest the minimum velocity of the waves (17)

$$v_{\min} = v_{\text{mon}} \max_{x} \left(d\psi(x)/dx\right)^{q/2\pi}$$

is small compared with the velocity c_0 of sound in the gas.

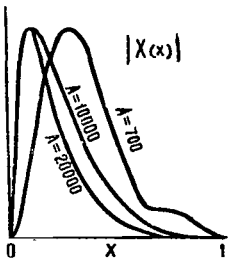

FIGURE 3.

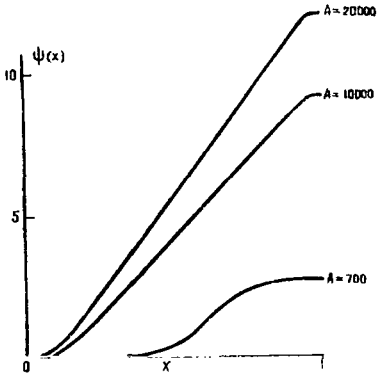

FIGURE 4.

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QUALITATIVE ANALYSIS OF THE MEMBRANE SOLUTION

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The so-called membrane equation of a circular thin plate,

$$32p^2(xp)'' - 1 = 0, \qquad (1)^*$$

is studied; here $0 \leqslant x \leqslant 1$, and p is proportional to the radial force per unit length and of opposite sign.

Three possible boundary conditions exist at $x = 1$:

$$-p(1) = T > 0 \ (a); \quad p(1) = 0 \ (b); \quad 2p'(1) + (1-\sigma)p(1) = 0 \ (c),$$

corresponding respectively to extension on the contour, absence of chain forces on the contour, and absence of radial contour displacements.

Following Hencky /2/, we look for a formal solution of the equation in the form of the exponential series

$$p = \frac{1}{4}\sum_{n=0}^{\infty} a_n x^n. \qquad (2)$$

Substitute (2) in (1). All the coefficients a_n are then determined through a_0 by the recurrent formulas

$$a_1 = \frac{1}{a_0^2}; \quad a_2 = -\frac{2}{3a_0^5} \cdots \text{ etc.} \qquad (3)$$

It is required to establish the possibility of a selection of a_0 satisfying the boundary condition at $x = 1$, and to prove the convergence of the series obtained.

1. STUDY OF PROBLEMS (1)—(a) and (1)—(c)

It was obtained in /3/ that the solution of problems (1)—(a), (1)—(b), and (1)—(c) is nonpositive. The uniqueness of the solution of these problems was proved in /4/; this paper also proved that there exist solutions of problems (1)—(a) and (1)—(c) in the class of twice continuously differentiable functions.

* The equation in this form is taken from Bromberg /1/.

We proceed for problems (1)—(a) and (1)—(c) to the integral equations

$$p(x) = \frac{1}{x}\int_0^x\int_0^{x_0} \frac{d\bar{x}dx_0}{32p^2} - \int_0^1\int_0^{x_0} \frac{d\bar{x}dx_0}{32p^2} - T, \tag{4a}$$

$$p(x) = \frac{1}{x}\int_0^x\int_0^{x_0} \frac{d\bar{x}dx_0}{32p^2} + \frac{1+\sigma}{1-\sigma}\int_0^1\int_0^{x_0} \frac{d\bar{x}dx_0}{32\sigma^2} - \frac{2}{1-\sigma}\int_0^1 \frac{dx}{32p^2}. \tag{4c}$$

We shall consider equation (4a) only (the results obtained for (4c) are similar). Following /4/, we make the substitution of variable $p=p+T$; we obtain

$$\bar{p} = \frac{1}{x}\int_0^x\int_0^{x_0} \frac{d\bar{x}dx_0}{32(T-\bar{p})^2} - \int_0^1\int_0^{x_0} \frac{d\bar{x}dx_0}{32(T-\bar{p})^2} \equiv \Phi(\bar{p}). \tag{5}$$

It was proved in /4/ that there exists a continuous nonpositive $\bar{p}_0(x)$ which satisfies equation (5).

Consider the space of l exponential series of the form $v = \sum_{n=0}^{\infty} C_n x^n$, converging on segment [0.1]. We introduce a norm by the rule

$$\|v\|_l = \sum_{n=0}^{\infty} |C_n|.$$

The space obtained is a Banach space.

Consider series satisfying the conditions:

1) $C_0 \leqslant 0$, 2) $C_0 C_k \leqslant 0$ for all k from one to infinity, 3) $|C_0| \geqslant \sum_{k=1}^{\infty} C_n$.

These series form a cone K in space l.

Let S_R be a sphere in space l. $K_R = S_R \cap K$. K_R is convex, being the intersection of two convex sets. We can select $R = R_0$ so that the operator Φ convert the set K_{R_0} into itself. The operator Φ is completely continuous in space l. We obtain then by Shauder's principle that equation (5) has a solution from set K_{R_0}:

$$\bar{p}_0(x) = \sum_{n=0}^{\infty} C_n x^n. \tag{6}$$

The solution of problem (1)—(a) (and of problem (1)—(c) as well) can therefore be always represented as an exponential series converging on segment [0, 1] and satisfying the boundary condition at $x = 1$. Substitute (6) in (1) and select $a_0 = 4C_0$. The coefficients of series (2) and (6) coincide then. This ends the proof of the solution.

2. EXISTENCE OF A SOLUTION OF PROBLEM (1)—(b)

Let $xp = u$, then equation (1) is transformed to

$$u'' = \frac{x^2}{32u^2}, \tag{7}$$

with boundary conditions $u(0) = u(1) = 0$. Problem (1)—(b) reduces to the integral equation

$$u(x) = \int_0^x dt \int_0^t \frac{\xi^2 d\xi}{32u^2} - x \int_0^1 dt \int_0^t \frac{\xi^2 d\xi}{32u^2}. \tag{8}$$

Take $u_1 = -\frac{\sqrt[3]{9}}{4} x(1-x)^{\frac{2}{3}}$ and consider the process of successive approximations

$$u''_{n+1} = \frac{x^2}{32u_n^2}, \qquad u_{n+1}(0) = u_{n+1}(1) = 0. \tag{9}$$

It can be proved, in the same way as in /5/, that $u_1(x) \leqslant u_2(x) \leqslant u_3(x) \cdots \leqslant 0$ and u_n converges uniformly to $u_0(x)$ on segment $[0, 1]$, with $u_0(x) < 0$ for $x \in (0, 1)$.

3. STUDY OF THE PROBLEM (1)—(b)

Consider an arbitrary x_1 $(0 < x_1 < 1)$; since $u_0(x_1) < 0$, then $u_0(x)$ as a solution of problem (1)—(a) can be represented on segment $[0, x_1]$ in the form of an absolutely convergent power series of type (6).

We shall now examine the solution $u_0(x)$ in the vicinity of the point $x = 1$. It can be obtained from equality (8) that for all "n"

$$u_n(x) = -\frac{\sqrt[3]{9}}{4} x(1-x)^{\frac{2}{3}} Y_n(z),$$

where $Y_n(x)$ is continuous on $[0,1]$, then $Y_n(x)$ converges uniformly to $Y_0(x)$ and

$$u_0(x) = -\frac{\sqrt[3]{9}}{4} x(1-x)^{\frac{2}{3}} Y_0(x). \tag{10}$$

Continuing our study of integral equality (8), we obtain

$$Y_0(x) \underset{x\to 1}{\to} 1; \qquad Y_0'(x) \underset{x\to 1}{\to} -\frac{2}{3}; \qquad Y_0''(x) \underset{x\to 1}{\to} \frac{5}{11}.$$

We can finally write for $p(x)$ formulas of asymptotic character at x tending to unity.

$$\begin{aligned} p(x) &= \frac{\sqrt[3]{9}}{4}(1-x)^{\frac{2}{3}}\left[-1 - \frac{2}{3}(1-x) + \frac{5}{22}(1-x)^2 + 0(1-x)^3\right], \\ p'(x) &= \frac{\sqrt[3]{9}}{4}(1-x)^{\frac{1}{3}}\left[\frac{2}{3} + \frac{10}{9}(1-x) - \frac{20}{33}(1-x)^2 + 0(1-x)^3\right]. \end{aligned} \tag{11}$$

The results obtained justify the form proposed by Bromberg /1/ for looking for the solution of problem (1)—(b).

4. MECHANICAL MEANING OF THE RESULTS OBTAINED

Several qualitative conclusions relative to the deflection functions and the stress function of the membrane can be drawn from the symmetrical solution (6).

In the case of a constant positive load the function of symmetrical deflection is positive and convex.

The radial and circumferential stresses in the case of symmetrical deflection and boundary conditions (c) are positive for all points of the membrane and the membrane is in a state of all-round extension.

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NATURAL VIBRATIONS OF SANDWICH PLATES RECTANGULAR IN THE PLAN

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This report deals with the problem of natural vibrations of elastic sandwich plates rectangular in the plan. An exact solution of the dynamic equations of the theory of elasticity, and solutions based on various theories of sandwich plates, were established for one class of end conditions. The exact solution obtained was compared with the solution obtained on the strength of approximate theories, and conclusions relative to the region of application of the approximate theories considered were drawn on the basis of this comparison. The results obtained were published in /1/. We present here results corresponding to the second part of the report, dealing with the application of Bolotin's asymptotic method /2/ to the problem of the vibrations of sandwich plates.

1. INITIAL EQUATIONS

From among the large number of known approximate equations we shall consider only those of Reissner /3/ and Grigolyuk /4, 5/.

Reissner's equations are

$$\Delta\varphi - \Delta w = \frac{p}{\mu(2h+H)} - \frac{2(\rho h + \rho_1 H)}{\mu(2h+H)}\frac{\partial^2 w}{\partial t^2},$$

$$\Delta\varphi = \frac{\mu(1-\nu_1)}{\mu_1(2h+H)H}(\varphi - w) + \frac{\beta(1-\nu_1)}{\mu_1 H(2h+H)^2}\frac{\partial^2\varphi}{\partial t^2}, \tag{1.1}$$

$$\Delta\theta = \frac{2\mu}{\mu_1 H(2h+H)}\theta - \frac{2\beta}{\mu_1 H(2h+H)^2}\frac{\partial^2\theta}{\partial t^2}.$$

Grigolyuk's equations are

$$\frac{2E_1H^3}{3(1-\nu_1^2)}\Delta\Delta w + \frac{E_1hH^2}{1-\nu_1^2}\Delta\Delta\varphi + \frac{Eh}{1+\nu}\Delta(\varphi - w) =$$

$$= p + \rho_1 hH^2\frac{\partial^2}{\partial t^2}\Delta\varphi + \frac{2}{3}\rho_1 H^3\frac{\partial^2}{\partial t^2}\Delta w - 2(\rho h + \rho_1 H)\frac{\partial^2 w}{\partial t^2}, \tag{1.2}$$

$$\left[\frac{2E_1h^2H}{1-\nu_1^2} + g\frac{2Eh^3}{3(1-\nu^2)}\right]\Delta\varphi + \frac{E_1h^2H}{1-\nu_1^2}\Delta w = \frac{Eh}{1+\nu}(\varphi - w) +$$

$$+ \left(\frac{2}{3}\rho h^3 + 2\rho_1 h^2 H\right)\frac{\partial^2\varphi}{\partial t^2} + \rho_1 hH^2\frac{\partial^2 w}{\partial t^2},$$

$$\left[\frac{E_1h^2H}{1+\nu_1}+g\frac{Eh^3}{3(1+\nu)}\right]\Delta\theta=\frac{Eh}{1+\nu}\theta+\left(\frac{2}{3}\rho h^3+2\rho_1h^2H\right)\frac{\partial^2\theta}{\partial t^2}. \qquad (1.2)\ \text{(cont'd)}$$

Here g is a factor equal to zero or unity. The case $g = 0$ corresponds to a filler of negligible longitudinal rigidity (soft filler), and the case $g = 1$ to a filler possessing a longitudinal rigidity (rigid filler).

2. APPLICATION OF THE ASYMPTOTIC METHOD

The problem of the natural vibrations of a rectangular sandwich plate under different boundary conditions will be solved by Bolotin's approximate method /2/, which consists of representing approximately the mode of natural vibrations as sum of the generating solution and of a set of correcting solutions possessing the properties of an edge effect. If these solutions do not satisfy arbitrary boundary conditions, the dynamic edge effect is called degenerate.

Let a sandwich plate (Figure 1) vibrate with a frequency ω. The generating solution (dropping the factor $\exp(i\omega t)$ will be represented in the form

$$\begin{aligned} w&=w_0\sin k_1(x-x_0)\sin k_2(y-y_0),\\ \varphi&=\varphi_0\sin k_1(x-x_0)\sin k_2(y-y_0),\\ \theta&=\theta_0\cos k_1(x-x_0)\cos k_2(y-y_0), \end{aligned} \qquad (2.1)$$

with the frequency ω linked with the wave numbers k_1, k_2 by one of the formulas

$$\begin{aligned} \alpha_{1,3}&=\frac{\rho\omega_{1,3}^2}{\mu k^2}=\\ &=\frac{1}{a_2}\left(a_1\mp\sqrt{a_1^2-a_0a_2}\right).\\ \alpha_2&=\frac{\rho\omega_2^2}{\mu k^2}=\frac{1}{b_2}\left(b_1+\frac{1}{T}\right). \end{aligned} \qquad (2.2)$$

In the case of Reissner's equations (1.1) we must write

$$\begin{aligned} a_0&=\frac{s(2-s)^2}{\xi(1-\nu_1)[(1-s)+us]},\\ a_1&=\frac{1}{T}+\frac{s(2-s)}{\xi(1-\nu_1)}+\frac{\frac{2}{3}(1-s)^3+\frac{1}{2}us(2-s)^2}{2[(1-s)+us]},\\ a_2&=2\frac{\frac{2}{3}(1-s)^3+\frac{1}{2}us(2-s)^2}{2-s},\qquad b_1=\frac{s(2-s)}{2\xi},\qquad b_2=\frac{1}{2}a_2. \end{aligned} \qquad (2.3)$$

In the case of Grigolyuk's equations (1.2) we must write

$$\begin{aligned} &a_0=a_{00}+\frac{1}{T}a_{01},\ a_1=a_{10}+\frac{1}{T}a_{11}+\frac{1}{T_2}a_{12},\ a_2=a_{20}+\frac{1}{T}a_{21},\\ &b_1=(1-s)\left(\frac{1-s}{3}g+\frac{s}{\xi}\right),\quad b_2=(1-s)\left(\frac{1-s}{3}+us\right), \end{aligned} \qquad (2.4)$$

$$a_{00}=\frac{4\gamma s^3(1-s)}{3(1-\nu)^2}\left(\frac{1-s}{3}g+\frac{\gamma s}{4}\right),\ a_{01}=\frac{2}{3(1-\nu)}[\gamma+(g-\gamma)(1-s)^3],$$

$$a_{10}=\frac{s^3(1-s)}{3(1-\nu)}\left[\frac{1-s}{3}(\gamma+gu)+\frac{1}{2}us\gamma\right],$$

$$a_{11}=\frac{1-s}{1-\nu}\left[(1-s)+us\right]\left(\frac{1-s}{3}g+\gamma s\right)+\frac{1}{6}\left[u+(1-u)(1-s)^3\right],$$

$$a_{12}=\frac{1}{2}\left[(1-s)+us\right], \qquad (2.4)$$

(cont'd)

$$a_{20}=\frac{1}{3}us^3(1-s)\left(\frac{1-s}{3}+\frac{us}{4}\right),\quad a_{21}=(1-s)\left(\frac{1-s}{3}+us\right)\left[(1-s)+us\right]$$

$$\left(\xi=\frac{\mu}{\mu_1},\ \gamma=\frac{1-\nu}{\xi(1-\nu_1)},\ u=\frac{\rho_1}{\rho},\ s=\frac{H}{h+H},\ k^2=k_1^2+k_2^2,\ T=k^2(h+H)^2\right).$$

We proceed to the determination of the correcting solutions. We shall represent the solution expressing the edge effect along the edge $x = 0$ in the form (omitting the factor $\exp(i\omega t)$):

$$w=W(x)\sin k_2(y-y_0),\quad \varphi=\Phi(x)\sin k_2(y-y_0),\quad \theta=\Theta(x)\cos k_2(y-y_0). \qquad (2.5)$$

Substituting (2.5) in Reissner's equation (1.1), we find that the functions W, Φ, Θ must satisfy the system of differential equations

$$W''-k_2^2W-\Phi''+k_2^2\Phi+\frac{2(\rho h+\rho_1H)}{\mu(2h+H)}\omega^2W=0,$$

$$\Phi''-k_2^2\Phi=\frac{\mu(1-\nu_1)}{\mu_1H(2h+H)}(\Phi-W)-\frac{\beta(1-\nu_1)\omega^2}{\mu_1H(2h+H)^2}\Phi, \qquad (2.6)$$

$$\Theta''-k_2^2\Theta=\frac{2\mu}{\mu_1H(2h+H)}\Theta-\frac{2\beta\omega^2}{\mu_1H(2h+H)^2}\Theta.$$

The corresponding characteristic equation has six roots, two of which are deliberately imaginary ($z_{1,2}=\pm ik_1$). In order to satisfy arbitrary boundary conditions at the edge $x = 0$ it is necessary to have two roots with negative real part. Similar considerations relative to the end $x = a$ lead to the requirement of existence of two roots with positive real part.

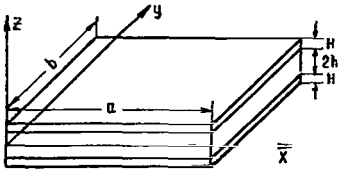

FIGURE 1.

The calculations show that the dynamic edge effect is always degenerate for the second and third frequency series ($\omega=\omega_{2,3}$). The roots corresponding to the first series of frequencies are given by the formulas

$$r_{1,2}^2=-k_1^2,\qquad r_{3,4}^2=p_1^2,\qquad r_{5,6}^2=q_1^2, \qquad (2.7)$$

where

$$p_1^2=k_1^2+2k_2^2-\frac{2}{a_0}\left(a_1-\frac{1}{T}\right)a_1k^2,\qquad q_1^2=k_2^2+\frac{k^2}{b_1T}-\frac{b_2}{b_1}k^2a_1.$$

The conditions that the dynamic edge effect is not degenerate are

$$p_1^2 > 0, \qquad q_1^2 > 0.$$

Only the second of these inequalities is essential. Similar considerations for the ends $y=0$, $y=b$ lead to the requirement $q_2^2>0$, and the region of absence of a dynamic edge effect is determined by the inequalities

$$q_1^2 > 0, \quad q_2^2 > 0. \tag{2.8}$$

The boundary of the region of degeneracy is represented in Figure 2 by a dotted line.

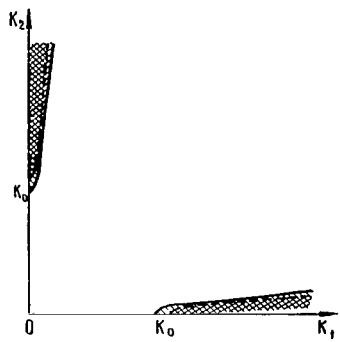

FIGURE 2.

Consider now Grigolyuk's equations (1.2). In this case the functions W, Φ, Θ must satisfy the following system of differential equations:

$$\begin{gathered}
\frac{2E_1H^3}{3(1-\nu_1^2)}(W^{\mathrm{IV}} - 2k_2^2 W'' + k_2^4 W) + \\
+ \frac{E_1 hH^2}{1-\nu_1^2}(\Phi^{\mathrm{IV}} - 2k_2^2\Phi'' + k_2^4\Phi) + \\
+ \frac{Eh}{1+\nu}(\Phi'' - k_2^2\Phi - W'' + k_2^2 W) = -\rho_1 h H^2\omega^2(\Phi'' - k_2^2\Phi) - \\
- \frac{2}{3}\rho_1 H^3\omega^2(W'' - k_2^2 W) + 2(\rho h + \rho_1 H)\omega^2 W, \qquad (2.9) \\
\left[\frac{2E_1 h^2 H}{1-\nu_1^2} + g\frac{2Eh^3}{3(1-\nu^2)}\right](\Phi'' - k_2^2\Phi) + \frac{E_1 h^2 H}{1-\nu_1^2}(W'' - k_2^2 W) = \\
= \frac{Eh}{1+\nu}(\Phi - W) - \left(\frac{2}{3}\rho h^3 + 2\rho_1 h^2 H\right)\omega^2\Phi - \rho_1 h H^2\omega^2 W, \\
\left[\frac{E_1 h^2 H}{1+\nu_1} + g\frac{Eh^3}{3(1+\nu)}\right](\Theta'' - k_2^2\Theta) = \frac{Eh}{1+\nu}\Theta - \left(\frac{2}{3}\rho h^3 + 2\rho_1 h^2 H\right)\omega^2\Theta.
\end{gathered}$$

The corresponding characteristic equation has eight roots, two of which are deliberately imaginary ($r_{1,2} = \pm ik_1$). In order to satisfy arbitrary boundary conditions at $x=0$ and $x=a$ we must have three roots with negative real

part and three roots with positive real part; otherwise the dynamic edge effect is degenerate.

The calculations show that for the second and third series of frequencies the dynamic edge effect is always degenerate. The roots corresponding to the first series of frequencies are given by the formulas

$$r_{1,2}^2 = -k_1^2, \quad r_{3,4}^2 = p_1^2, \quad r_{5,6}^2 = q_1^2, \quad r_{7,8}^2 = s_1^2. \tag{2.10}$$

Here

$$p_1^2 = -k_1^2 + k^2 x_1, \quad q_1^2 = -k_1^2 + k^2 x_2, \quad s_1^2 = -k_1^2 + k^2 \frac{b_2}{b_1}(\alpha_2 - \alpha_1),$$

and x_1, x_2 are roots of the equation

$$a_{00} x^2 + \left(2\alpha_1 a_{10} - 3a_{00} - \frac{1}{T} a_{01}\right) x + \left(3a_{00} + \frac{2}{T} a_{01} - 4\alpha_1 a_{10} + \alpha_1^2 a_{20} - \right.$$
$$\left. - \frac{2}{T} \alpha_1 a_{11}\right) = 0.$$

The conditions of absence of degeneracy of the dynamic edge effect will be

$$\mathrm{Re}\, p_1 \neq 0, \quad \mathrm{Re}\, q_1 \neq 0, \quad \mathrm{Re}\, s_1 \neq 0.$$

Only the last inequality is essential. Similar considerations for the ends $y = 0$, $y = b$ lead to the inequality $\mathrm{Re}\, s_2 \neq 0$, with the expression for s_2 obtained from the expression for s_1 by interchanging k_1, k_2. These inequalities can be written in the following form:

$$s_1^2 > 0, \quad s_2^2 > 0. \tag{2.11}$$

The boundary of the degeneracy region is represented in Figure 2 by the unbroken line. The parameter k_0 is near in magnitude to $\pi/2(h + H)$. This value corresponds to the length of the half-wave of the generating solution of the order of the plate thickness.

3. NATURAL VIBRATIONS OF RECTANGULAR PLATES AT DIFFERENT END CONDITIONS

Consider the natural vibrations of a rectangular sandwich plate of dimensions a and b (Figure 1). We shall first apply the Reissner equations and consider the following three types of boundary conditions

1. $t_n = t_s = w = 0$ — built-in edge;
2. $M_n = t = w = 0$ — supported edge;
3. $M_n = H_s = Q = 0$ — free edge.

Here in the case x = const:

$$t_n = t_x, \quad t_s = t_y, \quad M_n = M_x, \quad H_{ns} = H_{xy},$$

where

$$t_x = \frac{\partial \varphi}{\partial x} + \frac{\partial \Theta}{\partial y}, \qquad t_y = \frac{\partial \varphi}{\partial y} - \frac{\partial \Theta}{\partial x},$$

$$M_x = D\left(\frac{\partial t_x}{\partial x} + \nu_1 \frac{\partial t_y}{\partial y}\right), \qquad H_{xy} = \frac{1-\nu_1}{2} D\left(\frac{\partial t_x}{\partial y} + \frac{\partial t_y}{\partial x}\right),$$

$$Q = \frac{E(2h+H)}{2(1+\nu)}\left(t_x + \frac{\partial w}{\partial x}\right), \qquad D = \frac{E_1 H (2h+H)^2}{2(1-\nu_1^2)}.$$

The functions W, Φ, Θ have near the end $x = 0$ the form

$$W(x) = w_0 \sin k_1 (x - x_0) + A_0 \exp(-p_1 x),$$
$$\Phi(x) = w_0 f \sin k_1 (x - x_0) + A_0 \varkappa_1 \exp(-p_1 x),$$
$$\Theta(x) = B_0 \exp(-q_1 x),$$

where

$$f = 1 - 2\frac{(1-s) + us}{2-s}\alpha_1, \qquad \varkappa_1 = 1 - 2\frac{(1-s)+us}{2-s}\alpha_1 \frac{k_1^2 + k_2^2}{-p_1^2 + k_2^2}.$$

Let conditions (i) exist at $x = 0$, and conditions (j) at $x = a$. It is then possible to obtain from conditions (i) expressions for A_0 B_0 through w_0, and for $k_1 x_0$ through the wave numbers, and from conditions (j) an equation for determining $k_1(a - x_0)$. Eliminating $k_1 x_0$, we obtain the equation

$$m_1 = \frac{k_1 a}{\pi} = \frac{1}{\pi}(\operatorname{arctg} s_i + \operatorname{arctg} s_j) + m_1, \tag{3.1}$$

where

$$s_1 = \frac{k_1 f q_1}{p_1 q_1 \varkappa_1 + k_2^2 (f - \varkappa_1)}, \qquad s_2 = 0,$$

$$s_3 = \frac{2k_1^2 k_2^2 p_1 q_1 (1-\nu_1)(\varkappa_1 - f) - k_1 \varkappa_1 (p_1^2 - \nu_1 k_2^2)[(1-f)q_1^2 + (1+f)k_2^2]}{f p_1 (k_1^2 + \nu_1 k_2)[(1-\varkappa_1) q_1^2 + (1-\varkappa_1) k_2^2]}.$$

In the same way we obtain a second equation

$$m_2 = \frac{k_2 b}{\pi} = \frac{1}{\pi}(\operatorname{arctg} t_l + \operatorname{arctg} t_m) + m_2, \tag{3.2}$$

if we have conditions (l) at $y = 0$, and conditions (m) at $y = b$. The points of intersection of the two families of curves defined by equations (3.1) and (3.2) give the wave numbers k_1 and k_2, from which it is possible to determine the frequency ω, the displacements, and the stresses.

The integers m_1, m_2 in (3.1), (3.2) start from zero or one. The starting value must be determined in each particular case on the strength of some auxiliary considerations.

Figures 3 and 4 depict the families of curves corresponding to $a = b = 80\ (h+H)$, $s = 0.1$, $\nu = 0$, $\nu_1 = 0.3$, $u = 10$, $\xi = 0.001$] and to various edge conditions. The dotted line in the figures represents corresponding curves calculated by equations based on the Kirchhoff-Love hypothesis for the stack as a whole. The comparison of these curves leads to the conclusion that the allowance for the lateral shear in the filler leads to an abrupt variation in the wave numbers k_1 and k_2. This is one of the differences between the influence of the lateral shear in single and sandwich plates.

We shall now consider the problem of the vibrations of a rectangular plate on the basis of Grigolyuk's equations. The following two types

of boundary conditions will be considered:

1) $t_n = t_s = \frac{\partial w}{\partial n} = w = 0$ for a built-in end;

2) $M_{1n} = t_s = M_{2n} = w = 0$ for a supported end.

In the case x = const we have

$$M_{1n} = M_{1x} = \frac{2E_1 h^2 H}{1-\nu_1^2}\left(\frac{\partial t_x}{\partial x} + \nu_1 \frac{\partial t_y}{\partial y}\right) - \frac{E_1 h H^2}{1-\nu_1^2}\left(\frac{\partial^2 w}{\partial x} + \nu_1 \frac{\partial^2 w}{\partial y^2}\right) +$$
$$+ g\frac{2Eh^3}{3(1-\nu^2)}\left(\frac{\partial t_x}{\partial x} + \nu \frac{\partial t_y}{\partial y}\right),$$

$$M_{2n} = M_{2x} = \frac{E_1 h H^2}{1-\nu_1^2}\left(\frac{\partial t_x}{\partial x} + \nu_1 \frac{\partial t_y}{\partial y}\right) - \frac{2E_1 H^3}{3(1-\nu_1^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu_1 \frac{\partial^2 w}{\partial y^2}\right).$$

The functions W, Φ, Θ have in the vicinity of the end x = 0 the form

$$W(x) = w_0 \sin k_1 (x - x_0) + A_0 \exp(-p_1 x) + B_0 \exp(-q_1 x),$$
$$\Phi(x) = w_0 f \sin k_1 (x - x_0) + A_0 \varkappa_1 \exp(-p_1 x) + B_0 \zeta_1 \exp(-q_1 x),$$
$$\Theta(x) = C_0 \exp(-s_1 x).$$

Here

$$f = \frac{\frac{1}{T} + \frac{1}{2} u s^2 \alpha_1 - \frac{s^2}{1-\nu}\gamma}{\frac{1}{T} - (1-s)\left(\frac{1-s}{3} + us\right)\alpha_1 + \frac{2(1-s)}{1-\nu}\left(\frac{1-s}{3} g + s\gamma\right)},$$

$$\varkappa_1 = \frac{\frac{1}{T} + \frac{1}{2} u s^2 \alpha_1 + \frac{s^2\gamma}{1-\nu}\frac{p_1^2 - k_2^2}{k_1^2 + k_2^2}}{\frac{1}{T} - (1-s)\left(\frac{1-s}{3} + us\right)\alpha_1 - \frac{2(1-s)}{1-\nu}\left(\frac{1-s}{3} g + s\gamma\right)\frac{p_1^2 - k_2^2}{k_1^2 + k_2^2}},$$

$$\zeta_1 = \frac{\frac{1}{T} + \frac{1}{2} u s^2 \alpha_1 + \frac{s^2\gamma}{1-\nu}\frac{q_1^2 - k_2^2}{k_1^2 + k_2^2}}{\frac{1}{T} - (1-s)\left(\frac{1-s}{3} + us\right)\alpha_1 - \frac{2(1-s)}{1-\nu}\left(\frac{1-s}{3} g + s\gamma\right)\frac{q_1^2 - k_2^2}{k_1^2 + k_2^2}}.$$

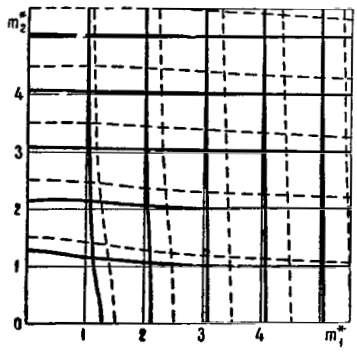

FIGURE 3. Built-in plate.

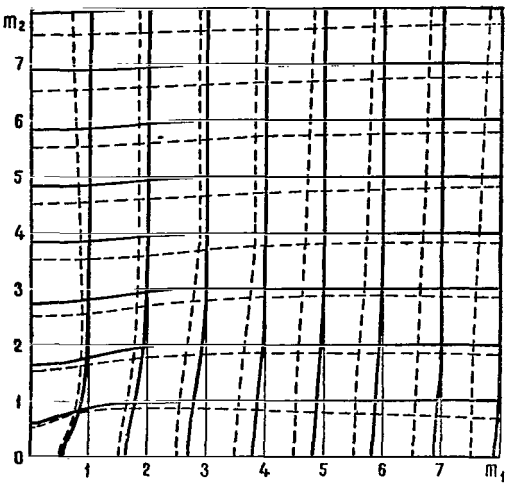

FIGURE 4. Free plate.

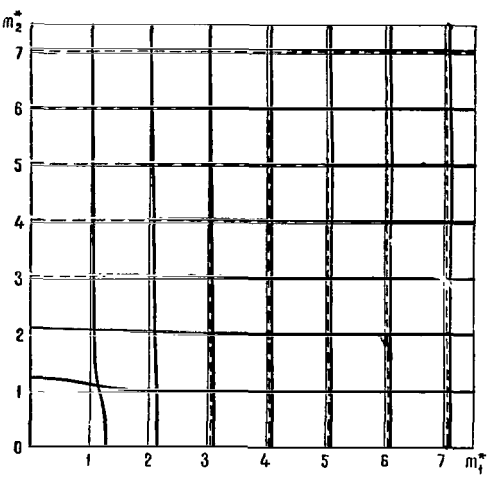

FIGURE 5. Built-in plate.

Let conditions (i), (j), (l), (m) take place at ends $x=0$, $x=a$, $y=0$, $y=b$, respectively. The equations for determining the wave numbers will then be

$$m_1^* = \frac{k_1 a}{\pi} = \frac{1}{\pi}\,(\operatorname{arctg} u_i + \operatorname{arctg} u_j) + m_1, \tag{3.3}$$

$$m_2^* = \frac{k_2 b}{\pi} = \frac{1}{\pi}\,(\operatorname{arctg} v_l + \operatorname{arctg} v_m) + m_2. \qquad (3.4)$$

Here

$$u_1 = \frac{k_1\,[(f-\zeta_1)(k_2^2 - s_1 q_1) + (f-\varkappa_1)(p_1 s_1 - k_2^2)]}{p_1\,(f-\zeta_1)(k_2^2 - s_1 q_1) + q_1 (f-\varkappa_1)(p_1 s_1 - k_2^2)}, \qquad u_2 = 0.$$

The points of intersection of the families of lines defined by equations (3.3) and (3.4), give the wave numbers k_1 and k_2, whose knowledge makes possible the determination of the frequency ω, the displacements, and stresses.

The families of curves corresponding to the natural vibrations of a built-in square plate with soft filler $a = b = 50(h + H)$, $u = 10$, $\xi = 0.001$, $\nu = 0$, $\nu_1 = 0.3$, $s = 0.1$ are represented in Figure 5. The corresponding lines, calculated by Reissner's equations, have been plotted by a dotted line in the same figure. It is seen that Grigolyuk's and Reissner's equations give similar values for k_1, k_2. It follows that the values of the wave numbers found by the Reissner equations can be used as a first approximation in calculating the frequencies and modes of natural vibrations by the formulas based on Grigolyuk's equations.

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ELASTIC-PLASTIC STRESSED STATE OF PLATES IN A THREE-DIMENSIONAL TEMPERATURE FIELD

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(Kiev)

Consider the elastic-plastic state of stress of a rapidly revolving disk of arbitrary profile placed in a nonstationary temperature field (Figure 1).

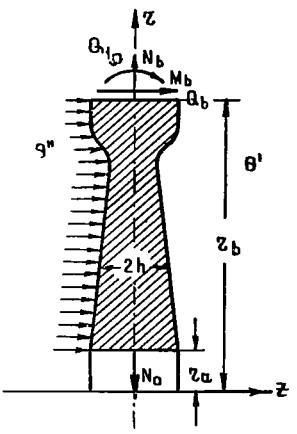

FIGURE 1.

This problem has been treated by numerous authors, and there is extensive literature on the subject. The present study differs from the existing ones in that it treats the simultaneous action of tensile and bending loads at plastic deformation of the disk, and determines the temperature field by solving the corresponding problem of heat conduction. In addition, the variation of the temperature through the disk thickness is allowed for in addition to the allowance for its variation along the radius.

1. DETERMINING THE TEMPERATURE FIELD OF THE DISK

When the difference between the temperatures of the media washing the lateral surfaces of the disk is large, considerable temperature gradients can appear through the disk thickness.

The determination of the nonstationary axisymmetrical distribution of the disk temperature reduces to solving the equation

$$\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{\partial^2 t}{\partial z^2}=\frac{1}{a}\frac{\partial t}{\partial \tau}, \tag{1.1}$$

with conditions

$$\begin{aligned}\frac{\partial t}{\partial n_1}&=-\frac{\alpha}{\lambda}(t-\theta') \quad \text{at} \quad z=h,\\ \frac{\partial t}{\partial n_2}&=-\frac{\alpha}{\lambda}(t-\theta'') \quad \text{at} \quad z=-h,\\ \frac{\partial t}{\partial r}&=-\frac{\alpha_1}{\lambda}(t-\theta_1) \quad \text{at} \quad r=r_b,\end{aligned} \tag{1.2}$$

$$t=\theta_0 \quad \text{at} \quad \tau=0, \tag{1.3}$$

where t is the disk temperature; θ_1, θ', θ'' are temperatures of the washing medium; θ_0 is the initial temperature of the disk; λ, α_1, α are the conductivity and heat-transfer coefficients; a is the thermal diffusivity; n is the external normal to the disk surface; τ is the time.

Conditions (1.2) can be written in the form

$$\begin{aligned}\frac{\partial t}{\partial z}&=-\frac{\alpha}{\lambda}(t-\theta')\frac{dn_1}{dz}-\frac{\partial t}{\partial r}\frac{dr}{dz} \quad \text{at} \quad z=h,\\ \frac{\partial t}{\partial z}&=-\frac{\alpha}{\lambda}(t-\theta'')\frac{dn_2}{dz}-\frac{\partial t}{\partial r}\frac{dr}{dz} \quad \text{at} \quad z=-h.\end{aligned} \tag{1.4}$$

Since

$$\frac{dr}{dz}=-\frac{dh}{dr} \quad \text{at} \quad z=h \text{ and } \frac{dr}{\partial z}=\frac{dh}{dz} \quad \text{at} \quad z=-h,$$

by writing $\frac{dn_1}{dz}\approx 1$, $\frac{dn_2}{dz}\approx -1$ we obtain from (1.4)

$$\begin{aligned}\frac{\partial t}{\partial z}&=-\frac{\alpha}{\lambda}(t-\theta')+\frac{dh}{dr}\frac{\partial t}{\partial r} \quad \text{at} \quad z=h,\\ \frac{\partial t}{\partial z}&=\frac{\alpha}{\lambda}(t-\theta'')-\frac{dh}{dr}\frac{\partial t}{\partial r} \quad \text{at} \quad z=-h.\end{aligned} \tag{1.5}$$

Let the temperature vary through the disk thickness according to the exponential law

$$t=\sum_{j=0}^{l} t_j(r,\tau)z^j. \tag{1.6}$$

To determine the equations satisfied by the functions entering in (1.6), multiply equation (1.1) by z^p $(p=0, 1,\cdots, l)$ and integrate by z from $-h$ to h. We obtain as a result the equation

$$\frac{\partial^2\vartheta_p}{\partial r^2}+\frac{1}{r}\frac{\partial\vartheta_p}{\partial r}-\frac{1}{r}h^p\Theta\frac{dh}{dr}-\frac{\partial}{\partial r}\left(h^p\frac{dh}{dr}\Theta\right)-h^{p-1}\{(\gamma+p)\Theta-$$
$$-\gamma[\theta'+(-1)^p\theta'']\}+p(p-1)\vartheta_{p-1}=\frac{1}{a}\frac{\partial\vartheta_p}{\partial\tau}, \tag{1.7}$$

where

$$\vartheta_p = \int_{-h}^{+h} z^p t dz, \qquad \gamma = \frac{\alpha h}{\lambda}, \qquad \Theta = t(h) + (-1)^p t(-h).$$

In the case of linear variation of the temperature through the disk thickness $t = t_0 + zt_1$ we obtain from (1.7) for $p = 0, 1$, respectively,

$$\begin{gathered}\frac{\partial^2 t_0}{\partial r^2} + \left(\frac{1}{r} + \frac{1}{h}\frac{dh}{dr}\right)\frac{\partial t_0}{\partial r} - m^2(t_0 - \theta_c) = \frac{1}{a}\frac{\partial t_0}{\partial \tau}, \\ \frac{\partial^2 t_1}{\partial r^2} + \left(\frac{1}{r} + \frac{3}{h}\frac{dh}{dr}\right)\frac{\partial t_1}{\partial r} - \varkappa^2\left(t_1 - \frac{\gamma}{1+\gamma}\vartheta\right) = \frac{1}{a}\frac{\partial t_1}{\partial \tau},\end{gathered} \tag{1.8}$$

where

$$\theta_c = \frac{\theta' + \theta''}{2}, \qquad m^2 = \frac{\gamma}{h^2}, \qquad \vartheta = \frac{\theta' - \theta''}{2h}, \qquad \varkappa^2 = \frac{3(1+\gamma)}{h^2}.$$

The solutions of equations (1.8) in the case when the temperatures θ_1 and θ_0 are independent of the coordinate z must satisfy the conditions

$$\begin{gathered}\frac{\partial t_0}{\partial r} = -\frac{\alpha_1}{\lambda}(t_0 - \theta_1), \qquad \frac{\partial t_1}{\partial r} = -\frac{\alpha_1}{\lambda} t_1 \quad \text{at} \quad r = r_b, \\ t_0 = \theta_0, \quad t_1 = 0 \quad \text{at} \quad \tau = 0.\end{gathered} \tag{1.9}$$

Equations (1.8) will be solved by the method of finite differences. In the case of a nonuniform net we find, in accordance with Figure 2, the following computing formulas:

$$\begin{gathered}t_{0,n}(\Delta\tau) = \beta_{0,n} t_{0,n} + \beta_{0,n-1} t_{0,n-1} + \beta_{0,n+1} t_{0,n+1} + a\Delta\tau m_n^2 \theta_c, \\ t_{1,n}(\Delta\tau) = \beta_{1,n} t_{1,n} + \beta_{1,n-1} t_{1,n-1} + \beta_{1,n+1} t_{1,n+1} + a\Delta\tau\varkappa_n^2 \frac{\gamma_n}{1+\gamma_n}\vartheta_n,\end{gathered} \tag{1.10}$$

where

$$\begin{gathered}\beta_{0,n} = 1 - a\Delta\tau\left(\frac{2}{\Delta r_{n,n+1}\Delta r_{n-1,n}} + m_n^2\right), \\ \beta_{0,n+1} = \frac{a\Delta\tau}{\Delta r_{n-1,n+1}}\left(\frac{2}{\Delta r_{n,n+1}} + \frac{1}{r_n} + \frac{h_{n+1} - h_{n-1}}{h_n \Delta r_{n-1,n+1}}\right), \\ \beta_{0,n-1} = \frac{a\Delta\tau}{\Delta r_{n-1,n+1}}\left(\frac{2}{\Delta r_{n-1,n}} - \frac{1}{r_n} - \frac{h_{n+1} - h_{n-1}}{h_n \Delta r_{n-1,n+1}}\right), \\ \beta_{1,n} = 1 - a\Delta\tau\left(\frac{2}{\Delta r_{n,n+1}\Delta r_{n-1,n}} + \varkappa_n^2\right), \\ \beta_{1,n+1} = \frac{a\Delta\tau}{\Delta r_{n-1,n+1}}\left[\frac{2}{\Delta r_{n,n+1}} + \frac{1}{r_n} + \frac{3(h_{n+1} - h_{n-1})}{h_n \Delta r_{n-1,n+1}}\right], \\ \beta_{1,n-1} = \frac{a\Delta\tau}{\Delta r_{n-1,n+1}}\left[\frac{2}{\Delta r_{n-1,n}} - \frac{1}{r_n} - \frac{3(h_{n+1} - h_{n-1})}{h_n \Delta r_{n-1,n+1}}\right].\end{gathered} \tag{1.11}$$

Equations (1.8) for the points on the outer contour are written in finite differences with the aid of the boundary conditions (1.9). We obtain as

a result that the temperature on the disk rim is determined by the formulas:

$$t_{0,n}(\Delta\tau) = \beta_{0,n} t_{0,n} + \beta_{0,n-1} t_{0,n-1} +$$

$$+ a\Delta\tau \left[m_n^2 \theta_c + \frac{\alpha_1}{\lambda} \theta_1 \left(\frac{1}{\Delta r_{n,n-1}} + \frac{1}{r_n} + \frac{h_n - h_{n-1}}{h_n \Delta r_{n-1,n}} \right) \right], \tag{1.12}$$

$$t_{1,n}(\Delta\tau) = \beta_{1,n} t_{1,n} + \beta_{1,n-1} t_{1,n-1} + a\Delta\tau \varkappa_n^2 \frac{\gamma_n}{1+\gamma_n} \vartheta_n,$$

where

$$\beta_{0,n} = 1 - a\Delta\tau \left[\frac{\alpha_1}{\lambda} \left(\frac{1}{\Delta r_{n,n-1}} + \frac{1}{r_n} + \frac{h_n - h_{n-1}}{h_n \Delta r_{n,n-1}} \right) + \frac{1}{\Delta r_{n,n-1}^2} + m_n^2 \right],$$

$$\beta_{1,n} = 1 - a\Delta\tau \left[\frac{\alpha_1}{\lambda} \left(\frac{1}{\Delta r_{n,n-1}} + \frac{1}{r_n} + \frac{3(h_n - h_{n-1})}{h_n \Delta r_{n,n-1}} \right) + \frac{1}{\Delta r_{n,n-1}^2} + \varkappa_n^2 \right], \tag{1.13}$$

$$\beta_{0,n-1} = \frac{a\Delta\tau}{\Delta r_{n,n-1}^2}, \qquad \beta_{1,n-1} = \frac{a\Delta\tau}{\Delta r_{n,n-1}^2}.$$

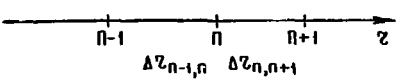

FIGURE 2.

The condition $\frac{\partial t}{\partial r} = 0$ or $t_1 = t_2$ in finite differences, is satisfied on the axis of a disk without hole. If the disk has an opening, we assume that there is no heat exchange on the inner contour, so that the same condition will be satisfied on its axis as well. The time increment $\Delta\tau$ is selected from the condition that the coefficient $\beta_{1,n}$ corresponding to it be equal to zero:

$$\Delta\tau = \frac{1}{a\left(\frac{2}{\Delta r_{n,n+1} \Delta r_{n-1,n}} + \varkappa_n^2 \right)}. \tag{1.14}$$

The smallest value of $\Delta\tau$ determined by this relationships should be used in the calculations.

2. DETERMINING THE ELASTIC-PLASTIC STATE OF STRESS OF THE DISK

The equations of equilibrium of an element of the revolving disk can be written in forces and moments in the following form:

$$\frac{d}{dr}(rN_r) - N_\theta + \Omega r^2 h = 0, \tag{2.1}$$

$$\frac{d}{dr}(rQ) + q_z r = 0, \tag{2.2}$$

$$\frac{d}{dr}(rM_r) - M_\theta - \varphi r N_r - Qr - \Omega z_0 h r^2 = 0, \tag{2.3}$$

where

$$N_r = \int_{-h}^{+h} \sigma_r dz, \qquad N_\theta = \int_{-h}^{+h} \sigma_\theta dz, \qquad Q = \int_{-h}^{+h} \tau_{rz} dz, \tag{2.4}$$

$$M_r = \int_{-h}^{+h} \sigma_r (z - z_0)\, dz, \qquad M_\theta = \int_{-h}^{+h} \sigma_\theta (z - z_0)\, dz. \tag{2.5}$$

It is assumed here that the modulus of elasticity of the material E varies both through the thickness and along the radius of the disk, and therefore the resolving equations of the moments (2.5) are calculated for simplicity relative to the so-called initial surface, whose distance $z_0 = f(r)$ from the symmetry plane $z = 0$ is determined from the equation

$$\int_{-h}^{+h} E(z - z_0)\, dz = 0; \tag{2.6}$$

$\Omega = 2\frac{\gamma}{g}\omega^2$; $\frac{\gamma}{g}$ is the density of the material; ω is the angular velocity of rotation of the disk; $\varphi = -\frac{dz_0}{dr}$ is the angle formed by the normal to the initial surface and the z axis; q_z is the difference between the surface forces per unit area of the symmetry plane acting on the lateral surfaces of the disk along its axis.

In the theory of small elastic-plastic deformations, at simple loading the components of the stress and strain tensors are connected by Hencky's equations, which can be written in the case considered in the form

$$\varepsilon_r = \frac{\sigma_r - \nu^* \sigma_\theta}{E^*} + \alpha_T t, \quad \varepsilon_\theta = \frac{\sigma_\theta - \nu^* \sigma_r}{E^*} + \alpha_T t, \quad \varepsilon_z = -\frac{\nu^* (\sigma_r + \sigma_\theta)}{E^*} + \alpha_T t, \tag{2.7}$$

where

$$E^* = E + E_1 = \frac{3}{2\psi + k}, \qquad \nu^* = \frac{\psi - k}{2\psi + k}, \qquad k = \frac{1 - 2\nu}{2G(1 + \nu)}, \tag{2.8}$$

$$E_1 = \frac{3}{2\psi + k} - E, \tag{2.9}$$

$$\psi = \frac{E}{S}, \tag{2.10}$$

$$S = \sqrt{\frac{1}{3}(\sigma_r^2 + \sigma_\theta^2 - \sigma_r \sigma_\theta)}, \tag{2.11}$$

$$E = \sqrt{\frac{1}{6}[(\varepsilon_r - \varepsilon_\theta)^2 + (\varepsilon_\theta - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_r)^2]}. \tag{2.12}$$

Here ε_r, ε_θ, ε_z are the strain tensor components; α_T is the coefficient of linear expansion of the material; t is the temperature; ν is the Poisson ratio; G is the shear modulus; E_1 is a function characterizing the non-linearity of the relationship between the stress and strain components;

ψ is the plasticity parameter; S is the intensity of the tangential stresses; E is the intensity of the shearing strains.

In the case of incompressible material $\nu = 0.5$, $k = 0$, and

$$E^* = E + E_1 = \frac{3}{2\psi}, \qquad \nu^* = \frac{1}{2}, \qquad \varepsilon_r + \varepsilon_\theta + \varepsilon_z = 3\alpha_T t, \tag{2.13}$$

i.e., ν^* is a constant magnitude. The substitution of (2.13) in (2.12) yields

$$E = \sqrt{\varepsilon_r^2 + \varepsilon_\theta^2 + \varepsilon_r \varepsilon_\theta - 3\alpha_T t\,(\varepsilon_r + \varepsilon_\theta - \alpha_T t)}\,. \tag{2.14}$$

We shall assume in what follows that the disk material is incompressible, $\nu = 0.5$. This assumption is justified to a certain extent, since the increase of the temperature of the material is accompanied by an increase in the value of the Poisson ratio; in addition, the variation of this coefficient only slightly influences the state of stress.

Using the Kirchhoff-Love hypothesis on the invariability of the normal element

$$\varepsilon_r = \varepsilon_{r0} + (z - z_0)\,\varkappa_r, \quad \varepsilon_\theta = \varepsilon_{\theta 0} + (z - z_0)\,\varkappa_\theta, \tag{2.15}$$

we can write the compatibility equations for the strain components ε_{r0}, $\varepsilon_{\theta 0}$ of the initial surface (2.6) and the curvature parameters $\varkappa_r$, $\varkappa_\theta$ of this same surface in the form

$$\frac{d(r\varepsilon_{\theta 0})}{dr} - \varepsilon_{r0} + \theta\varphi = 0, \qquad \frac{d(r\varkappa_\theta)}{dr} - \varkappa_r = 0. \tag{2.16}$$

The parameters $\varkappa_r$, $\varkappa_\theta$ are expressed by the following formulas through the angle of rotation θ of the normal to the initial surface during its deformation:

$$\varkappa_r = \frac{d\theta}{dr}, \qquad \varkappa_\theta = \frac{\theta}{r}. \tag{2.17}$$

Using formulas (2.4) to (2.17), Hencky's equations (2.7) can be written as follows:

$$\begin{aligned} \varepsilon_{r0} &= (\xi - \xi_1)(N_r - \nu^* N_\theta) + \varepsilon_T - \mu_1 \varkappa_r, \\ \varepsilon_{\theta 0} &= (\xi - \xi_1)(N_\theta - \nu^* N_r) + \varepsilon_T - \mu_1 \varkappa_\theta, \end{aligned} \tag{2.18}$$

$$\begin{aligned} M_r &= (\chi - \lambda_1)(\varkappa_r + \nu^* \varkappa_\theta) + \mu_1 N_r + (1 + \nu^*)\,\omega_1 \varepsilon_T - M_T, \\ M_\theta &= (\chi - \lambda_1)(\varkappa_\theta + \nu^* \varkappa_2) + \mu_1 N_\theta + (1 + \nu^*)\,\omega_1 \varepsilon_T - M_T, \end{aligned} \tag{2.19}$$

where

$$\xi^* = \xi - \xi_1 = \frac{1}{\int_{-h}^{+h} E^* dz}, \qquad \xi = \frac{1}{\int_{-h}^{+h} E dz}, \qquad \xi_1 = \frac{\int_{-h}^{+h} E_1 dz}{\int_{-h}^{+h} E dz \int_{-h}^{+h} E^* dz}, \tag{2.20}$$

$$\chi^* = \chi + \chi_1 = \frac{1}{1 - \nu^{*2}} \int_{-h}^{+h} E^* (z - z_0)^2 dz, \qquad \chi = \frac{1}{1 - \nu^{*2}} \int_{-h}^{+h} E (z - z_0)\, dz,$$

$$\varepsilon_T = \xi^* \int_{-h}^{+h} E^* \alpha_T t dz, \qquad M_T = \frac{1}{1-\nu^*} \int_{-h}^{+h} E^* (z - z_0) \alpha_T t dz, \tag{2.20 (cont'd)}$$

$$\omega_1 = \frac{1}{1-\nu^{*2}} \int_{-h}^{+h} E_1 (z - z_0) dz, \qquad \mu_1 = \omega_1 \xi^* (1 - \nu^{*2}), \quad \lambda_1 = \omega_1 \mu_1 - \chi_1.$$

By substituting the strain components (2.18) in the compatibility equation (2.16), eliminating N_θ with the aid of equilibrium equation (2.1), and taking (2.17) into account, we obtain the first resolving equation in the following form:

$$\frac{d}{dr}\left[r^3 \xi \frac{dN_r}{dr}\right] + (1 - \nu^*) r^2 \frac{d\xi}{dr} N_r + \varphi r \theta = f_1 (r, E_1), \tag{2.21}$$

where

$$f_1 (r, E_1) = \frac{d}{dr}\left[r^3 \xi_1 \frac{dN_r}{dr}\right] + (1 - \nu^*) r^2 \frac{d\xi_1}{dr} N_r -$$

$$- \Omega r^2 \left[\frac{d(r^2 h \xi^*)}{dr} + (1 + \nu^*) \xi^* r h\right] - r^2 \frac{d\varepsilon_T}{dr} + r \frac{d\mu_1}{dr} \theta. \tag{2.22}$$

To obtain the second resolving equations we eliminate from the equilibrium equations (2.2), (2.3) the shearing force Q and replace the bending moments M_r and M_θ by their expressions as a function of θ, as given by formulas (2.19) with the aid of (2.1) and (2.17). We obtain

$$\frac{d}{dr}\left[r \chi \frac{d\theta}{dr}\right] + \left(\nu^* \frac{d\chi}{dr} - \frac{\chi}{r}\right) \theta - \varphi r N_r = f_2 (r, E_1), \tag{2.23}$$

where

$$f_2 (r, E_1) = \frac{d}{dr}\left[r \lambda_1 \frac{d\theta}{dr}\right] + \left(\nu^* \frac{d\lambda_1}{dr} - \frac{\lambda_1}{r}\right) \theta - r N_r \frac{d\mu_1}{dr} + \Omega r^2 h (\mu_1 + z_0) -$$

$$- (1 + \nu^*) r \frac{d(\omega_1 \varepsilon_T)}{dr} + r \frac{dM_T}{dr} + C - \int_{r_1}^{r} q_z r dr. \tag{2.24}$$

Here C is an integration constant.

The system of equations (2.21), (2.23) is nonlinear. It will be solved by the method of successive approximations, with E_1 calculated in each step by (2.9) from the preceding approximation. In each approximation it is thus necessary to solve the linear system of differential equations (2.21), (2.23) with new right-hand parts (2.22), (2.24); this is a modification of Il'yushin's method of elastic solutions.

The general solution of system (2.21), (2.23) will be written in the form

$$N_r = C_1 N_r^{(1)} + C_2 N_r^{(2)} + C_3 N_r^{(3)} + C_4 N_r^{(4)} + N_r^{(0)}, \tag{2.25}$$

$$\theta = C_1 \theta_1 + C_2 \theta_2 + C_3 \theta_3 + C_4 \theta_4 + \theta_0. \tag{2.26}$$

The differentiation of these expressions yields

$$\varphi_1(r) = \frac{dN_r}{dr} = C_1\varphi_1^{(1)} + C_2\varphi_1^{(2)} + C_3\varphi_1^{(3)} + C_4\varphi_1^{(4)} + \varphi_1^{(0)},$$
$$\varphi_2(r) = \frac{d\theta}{dr} = C_1\varphi_2^{(1)} + C_2\varphi_2^{(2)} + C_3\varphi_2^{(3)} + C_4\varphi_2^{(4)} + \varphi_2^{(0)}, \tag{2.27}$$

where $N_r^{(l)}$, θ_l, $\varphi_1^{(l)}$, $\varphi_2^{(l)}$ are linearly independent particular solutions of the homogeneous system of differential equations and their derivatives, $N_r^{(0)}$, θ_0, $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ are particular solutions of the nonhomogeneous system of differential equations and their derivatives, and $C_1 \cdots C_4$ are integration constants.

The particular solutions of the homogeneous system of differential equations corresponding to (2.21), (2.23) will be determined by the Adams method (although different methods of numerical integration are likewise possible), with the following boundary conditions on the inner contour of the disk:

At $r = r_a$

$$N_r^{(1)} = 1, \quad \varphi_1^{(1)} = \theta_1 = \varphi_2^{(1)} = 0, \quad \theta_2 = 1, \quad N_r^{(2)} = \varphi_1^{(2)} = \varphi_2^{(2)} = 0,$$
$$\varphi_1^{(3)} = 1, \quad N_r^{(3)} = \theta_3 = \varphi_2^{(3)} = 0, \quad \varphi_2^{(4)} = 1, \quad N_r^{(4)} = \theta_4 = \varphi_1^{(4)} = 0$$

and

$$f_1 = f_2 = 0. \tag{2.28}$$

The particular solutions of the nonhomogeneous system of differential equations (2.21), (2.23) will be determined for boundary conditions

at $r = r_a$

$$N_r^{(0)} = 1, \quad \varphi_1^{(0)} = \theta_0 = \varphi_2^{(0)} = 0 \tag{2.29}$$

and right-hand parts $f_1(r, E_1)$, $f_2(r, E_1)$, where E_1 is calculated by the preceding approximation.

The integration constants $C_1, \cdots, C_4$ are determined from the boundary conditions on the outer and inner contours of the disk. In the case considered (Figure 1) these conditions are

at $r = r_a$

$$N_r = N_a, \quad \theta = 0,$$

at $r = r_b$ $N_r = N_b$,

$$(\chi - \lambda_1)\left(\frac{d\theta}{dr} + \nu^* \frac{\theta}{r}\right) = M_b + M_T - \mu_1 N_b - (1 + \nu^*)\,\omega_1 \varepsilon_T. \tag{2.30}$$

It should be noted in conclusion that the elastic-plastic state of stress of the disk found by the above-described method exists in the case of simple (or quasisimple) loading, i.e., in the case of proportionality of the variation of the centrifugal and surface forces and the heat load, or in the case of the physically nonlinear problem, when loading and unloading proceed according to the same curve of the extension diagram.

The method described for calculating a nonstationary temperature field and the elastic-plastic state of stress of the disk has been programed for use with a digital computer.

BENDING AND STABILITY OF RECTANGULAR PLATES WEAKENED BY RECTANGULAR CUTS

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The method of nets is being wiedely used in solving problems of bending and buckling of plates. By this method, the differential equation of bending and the boundary conditions have to be satisfied at the nodal points of the net only.

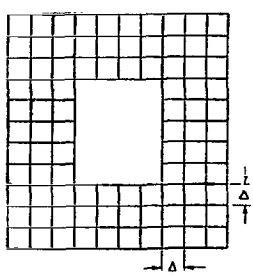

FIGURE 1.

If the plate region contains rectangular cuts (Figure 1), the introduction of a large number of nodal points is necessary in order to satisfy the boundary conditions on the contour. This leads to a very bulky system of equations. The solution obtained as a result is approximate, the degree of accuracy depending on the mesh size.

The inconveniences involved in the use of the net method for solving problems of bending and buckling of rectangular plates weakened by cuts can be overcome by using the net method in a refined formulation. In this formulation the accuracy with which the boundary conditions are satisfied becomes independent of the mesh size.

Instead of the standard equations of the net method, in which the values of the unknown functions at the nodal points appear as unknowns, we use equations in which the unknowns are the values of the function w and of its first derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. The boundary conditions (including the conditions on the contour of the cut) and the differential equation of bending can then be satisfied at every point of the plate (and not only at the nodal points of the net), provided the load acting on the plate varies according to a linear law within the limits of each square of the net, or that it is a momentary load varying linearly on the contour.

It will be assumed in the derivation of the formulas that the plate region is divided by a square net of mesh size Δ.

CONDITIONS OF EQUILIBRIUM

Consider the net cell *abcd* (Figure 2). Let the square *abcd* of the plate occupy a new position *a'b'c'd'* under the action of the deformation.

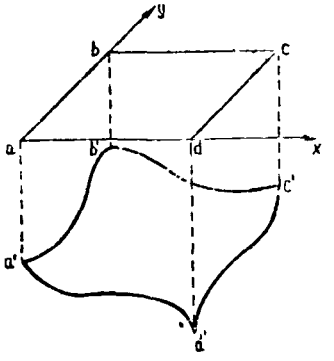

FIGURE 2.

The deflection function of the plate in region $abcd$ can be represented in the following form:

$$w = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + \\ + a_6 y^2 + a_7 x^2 y + a_8 xy^2 + a_9 x^3 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3. \tag{1}$$

This function satisfies the differential equation of bending of the plate under zero load ($q=0$) within the limits of the region

$$\nabla^2 \nabla^2 w = 0. \tag{2}$$

Introduce the following symbols:

w_i will be the displacement of the i-th nodal point of the plate,

$\varphi_i^x = \dfrac{\partial w}{\partial y}$ the angle of rotation of the nodal point about the x axis, and

$\varphi_i^y = \dfrac{\partial w}{\partial x}$ the angle of rotation of the nodal point about the y axis.

The coefficients a_i of equation (1) can be represented through the deflections and angles of rotation of the nodal points of the square. They have the following values:

$$\left.\begin{aligned}
a_1 &= w_a, \quad a_2 = \varphi_a^y \Delta, \quad a_3 = \varphi_a^x \Delta, \\
a_4 &= \frac{-w_a + w_b - w_c + w_d - \varphi_a^y \Delta - \varphi_a^x \Delta - \varphi_b^y \Delta + \varphi_d^x \Delta}{\Delta^2}, \\
a_5 &= \frac{-3w_a + 3w_d - 2\varphi_a^y \Delta - \varphi_d^y \Delta}{\Delta^2}, \\
a_6 &= \frac{-3w_a + 3w_b - 2\varphi_a^x \Delta - \varphi_b^x \Delta}{\Delta^2}, \\
a_7 &= \frac{3w_a - 3w_b + 3w_c - 3w_d + 2\varphi_a^x \Delta + \varphi_b^x \Delta - \varphi_c^x \Delta - 2\varphi_d^x \Delta}{\Delta^3}, \\
a_8 &= \frac{3w_a - 3w_b + 3w_c - 3w_d + 2\varphi_a^y \Delta - 2\varphi_b^y \Delta - \varphi_c^y \Delta + \varphi_d^y \Delta}{\Delta^3}, \\
a_9 &= \frac{2w_a - 2w_d + \varphi_a^y \Delta + \varphi_d^y \Delta}{\Delta^3},
\end{aligned}\right\} \tag{3}$$

$$
\left.
\begin{aligned}
a_{10} &= \frac{2w_a - 2w_b + \varphi_a^x\Delta + \varphi_b^y\Delta}{\Delta^3}, \\
a_{11} &= \frac{-\varphi_a^y\Delta + \varphi_b^y\Delta + \varphi_c^y\Delta - \varphi_d^y\Delta - 2w_a + 2w_b - 2w_c + 2w_d}{\Delta^4}, \\
a_{12} &= \frac{-\varphi_a^x\Delta - \varphi_b^x\Delta + \varphi_c^x\Delta - \varphi_d^x\Delta - 2w_a + 2w_b - 2w_c + 2w_d}{\Delta^4}.
\end{aligned}
\right\} \quad (3)\ \text{(cont'd)}
$$

It is easily seen that equation (1) corresponds to a linear distribution of the intensity of the lateral forces on the square contour.

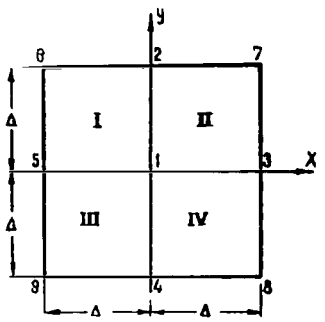

FIGURE 3.

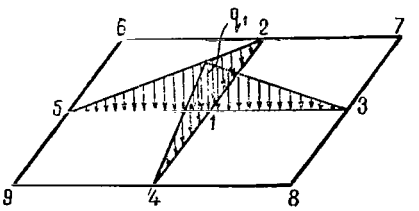

FIGURE 4.

Consider four adjoining squares (Figure 3). Let a linearly distributed load of maximum value q_1 at nodal point 1 act along the lines separating these squares (Figure 4). The following equations are obtained from the fact that regions I, II, III, and IV adjoin each other:

$$
36w_1 - 12\,(w_2 + w_3 + w_4 + w_5) + 3\,(w_6 + w_7 + w_8 + w_9) +
$$

$$
+4\,(\varphi_3^y - \varphi_5^y + \varphi_2^y - \varphi_4^x)\,\Delta + 0.5\,(\varphi_6^y + \varphi_9^y - \varphi_7^y - \varphi_5^y + \varphi_9^y - \varphi_6^x - \varphi_7^x + \varphi_8^x)\Delta = \frac{q_1\Delta^4}{2D}; \quad (4)
$$

$$
(1-\mu)\,(2w_3 - 2w_5 + w_6 - w_7 - w_8 + w_9 - 4\varphi_1^y\Delta + 2\varphi_2^y\Delta + 2\varphi_4^y\Delta) +
$$

$$
+ (12w_5 - 12w_3 + 16\varphi_1^y\Delta + 4\varphi_5^y\Delta + 4\varphi_3^y\Delta) = 0; \quad (5)
$$

$$
(1-\mu)\,(2w_2 - 2w_4 + w_3 - w_6 - w_7 + w_8 - 4\varphi_1^x\Delta + 2\varphi_3^x\Delta + 2\varphi_5^x\Delta) +
$$

$$
+ (12w_4 - 12w_2 + 16\varphi_1^x\Delta + 4\varphi_4^x\Delta + 4\varphi_2^x\Delta) = 0. \quad (6)
$$

Equation (4) corresponds to the differential equation (1) of bending of the plate for point 1.

Three equations (4), (5), and (6) are formed for each nodal point of the net, in the same way as three equilibrium equations are formed for each nodal point of plane frame systems. As a result, the increase of the mesh size has no effect on the accuracy with which the boundary conditions are satisfied.

BOUNDARY CONDITIONS

Consider the plate region *abcd* (Figure 2). We shall formulate the boundary conditions corresponding to the most widely used methods of support of the plate.

THE EDGE *ab* DOES NOT HAVE VERTICAL DISPLACEMENTS

$$|w(x, y)|_{x=0}=0. \tag{7}$$

The displacement function for this edge will be

$$|w|_{x=0}=a_1+a_3y+a_6y^2+a_{10}y^3.$$

In order to satisfy condition (7) (for the entire edge *ab*) it is necessary that

$$a_1=0;\ a_3=0;\ a_6=0;\ a_{10}=0.$$

The last equalities are satisfied if we write

$$w_a=0;\ w_b=0;\ \varphi_a^x=0;\ \ \varphi_b^x=0. \tag{8}$$

THE EDGE *ab* IS RIGIDLY CLAMPED

$$\left|\frac{\partial w}{\partial x}\right|_{x=0}=0,$$
$$\left|\frac{\partial w}{\partial x}\right|_{x=0}=a_2+a_4y+a_7y^2+a_{12}y^3. \tag{9}$$

To satisfy conditions (7) and (9), the following conditions must be added to (8):

$$\left.\begin{aligned}\varphi_a^y=0;\quad \varphi_b^y=0,\\ \varphi_d^x=w_c-w_d;\quad \varphi_c^x=w_c-w_d.\end{aligned}\right\} \tag{10}$$

THE EDGE *ab* IS HINGED

$$\left|\frac{\partial^2 w}{\partial x^2}\right|=0\ (\text{for } w=0). \tag{11}$$

The following conditions are added to conditions (8):

$$\begin{aligned}3w_d-2\varphi_a^y\Delta-\varphi_d^y\Delta=0,\\ 3w_c-2\varphi_b^y\Delta-\varphi_c^y\Delta=0.\end{aligned} \tag{12}$$

THE EDGE *cd* IS FREE

The conditions for the free edge

$$\left.\begin{aligned} &\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} = 0, \\ &\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2} = 0 \ (\text{for } x = b) \end{aligned}\right\} \tag{13}$$

are realized in equations of the following form:

$$(6w_a - 5w_d + 2\varphi_a^y\Delta + 4\varphi_d^y\Delta) + \mu(6w_c - 6w_d - 4\varphi_d^y\Delta) = 0; \tag{14}$$

$$(-6w_a + 6w_b - 6w_c + 6w_d - 2\varphi_a^y\Delta + 2\varphi_b^y\Delta + 4\varphi_c^y\Delta - 4\varphi_d^y\Delta) + \\ + \mu(12w_d - 12w_c + 9\varphi_d^y\Delta) = 0; \tag{15}$$

$$(24w_a - 24w_d - 12w_b + 12w_c + 6\varphi_a^y\Delta + 6\varphi_d^y\Delta + 8\varphi_a^x\Delta - 8\varphi_d^x\Delta) - \\ - \mu(6w_a - 6w_b + 6w_c - 6w_d + 4\varphi_a^x\Delta - 4\varphi_d^x\Delta) = 0; \tag{16}$$

$$36(-w_a + w_b - w_c + w_d) - 6(\varphi_a^y\Delta - \varphi_b^y\Delta + \varphi_c^y\Delta + \varphi_d^y\Delta) - \\ -12(\varphi_a^x\Delta - \varphi_a^y\Delta) - \mu[12(-w_a + w_b - w_c + w_d) - 6(\varphi_a^x\Delta - \varphi_d^y\Delta)] = 0. \tag{17}$$

When these equations are satisfied the boundary conditions are fulfilled exactly.

EXAMPLE 1. BENDING OF SQUARE SIMPLY SUPPORTED PLATE, WEAKENED BY A SQUARE CUT, UNDER THE ACTION OF MOMENTS DISTRIBUTED UNIFORMLY ALONG THE PLATE CONTOUR (FIGURE 5)

The ratio of the hole width to the plate width is equal to 1/3.

We split the plate region by means of a square net (Figure 6).

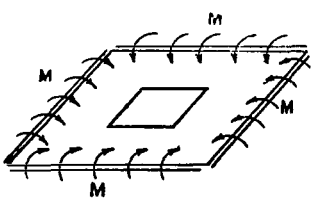

FIGURE 5.

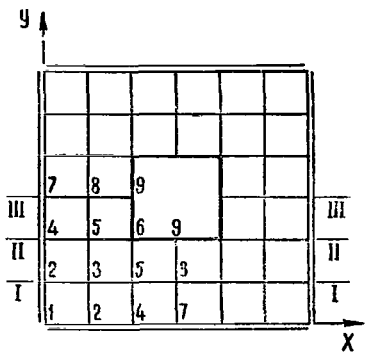

FIGURE 6.

Form equations (4), (5), and (6) for the nodal points of the net. To satisfy the boundary conditions on the plate contour we use equations (8) and (12), equating the moments on the supporting contour to the external distributed moments of intensity M.

To satisfy the boundary conditions in the region of the cut we use equations (14), (15), (16), and (17).

The system of equations with fourteen unknowns is written in Table 1. Its solution yields the following values for the unknowns:

$$w_3=0.27037\,\frac{M\Delta^2}{D};\quad \varphi_2^y\,\Delta=0.66369\,\frac{M\Delta^2}{D};\quad \varphi_7^y\,\Delta=0.30825\,\frac{M\Delta^2}{D};$$

$$w_5=0.13433\,\frac{M\Delta^2}{D};\quad \varphi_4^y\,\Delta=0.50839\,\frac{M\Delta^2}{D};\quad \varphi_8^y\Delta=-0.16390\,\frac{M\Delta^2}{D};$$

$$w_6=-0.04334\,\frac{M\Delta^2}{D};\quad \varphi_5^x\,\Delta=-0.22189\,\frac{M\Delta^2}{D};\quad \varphi_9^y\,\Delta=-0.24124\,\frac{M\Delta^2}{D};$$

$$w_8=-0.01580\,\frac{M\Delta^2}{D};\quad \varphi_5^y\,\Delta=-0.11379\,\frac{M\Delta^2}{D};\quad \varphi_6^y\,\Delta=-0.01627\,\frac{M\Delta^2}{D};$$

$$w_9=-0.20309\,\frac{M\Delta^2}{D};\quad \varphi_6^y\,\Delta=-0.19678\,\frac{M\Delta^2}{D}.$$

The sections of the bent plate surface by cross sections I-I, II-II, and III-III are shown in Figure 7.

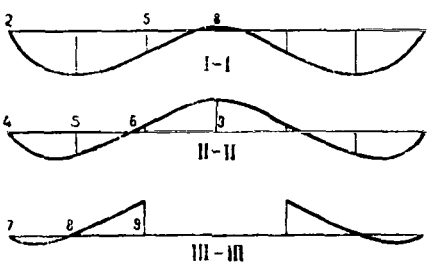

FIGURE 7.

EXAMPLE 2. BENDING OF A SQUARE PLATE WITH CUT, RIGIDLY BUILT-IN BY THE ENDS, UNDER A LOAD q DISTRIBUTED UNIFORMLY ALONG THE HOLE CONTOUR (FIGURE 8)

The plate is divided by a net in the same way as in the preceding example. When satisfying the boundary conditions on the contour of the cut we take into account the action of the uniformly distributed load. The system of equations contains eleven unknowns. Its solution leads to the following results:

$$w_3=0.0998\,\frac{q\Delta^3}{D};\quad \varphi_3^y\Delta=0.02777\,\frac{q\Delta^3}{D};$$

$$w_5=0.0996\,\frac{q\Delta^3}{D};\quad \varphi_5^x\,\Delta=-0.05627\,\frac{q\Delta^3}{D};$$

$$w_6=0.21019\,\frac{q\Delta^3}{D};\quad \varphi_5^y\Delta=0.07956\,\frac{q\Delta^3}{D};$$

$$w_8=-0.03049\,\frac{q\Delta^3}{D};\quad \varphi_6^y\Delta=0.30065\,\frac{q\Delta^3}{D};$$

$$w_9 = 0.32155 \frac{q\Delta^3}{D}; \quad \varphi_8^y \Delta = 0.12031 \frac{q\Delta^3}{D};$$

$$\varphi_9^y \Delta = 0.47286 \frac{q\Delta^3}{D}.$$

The sections of the bent plate surface by cross sections I-I, II-II, III-III are shown in Figure 9.

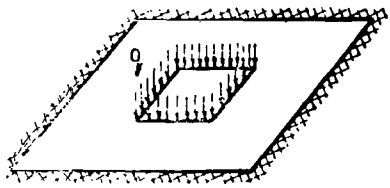

FIGURE 8.

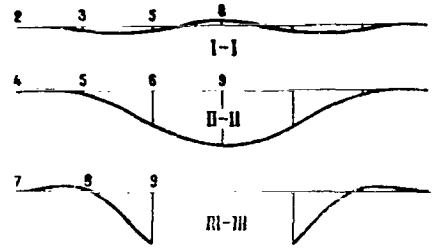

FIGURE 9.

Another problem which was solved was that of the bending of the same plate under a uniformly distributed load acting along the net lines (Figure 10). The pattern of the plate bending is represented in Figure 11.

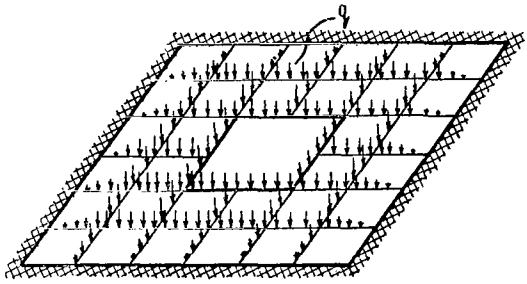

FIGURE 10.

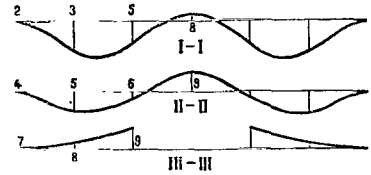

FIGURE 11.

Equations (4), (5), and (6) can also be used in solving buckling problems. In this case the load q in the right-hand part of (4) is replaced by longitudinal compressive forces.

Consider the stability of a hinged square plate weakened by a square cut and loaded by uniformly distributed compressive loads acting on the plate contour (Figure 12).

The determinant D formed of the coefficients of the unknowns is represented in Table 2.

The following symbol has been used in writing this determinant:

$$k = \frac{N\Delta^2}{D}.$$

TABLE 1

w_3	w_5	w_6	w_8	w_9	$\varphi_2^y\Delta$	$\varphi_3^y\Delta$	$\varphi_4^y\Delta$	$\varphi_5^y\Delta$	$\varphi_5^x\Delta$	$\varphi_6^y\Delta$	$\varphi_7^y\Delta$	$\varphi_8^y\Delta$	$\varphi_9^y\nabla$	a_{ik}
	−24		36	−12		6	1		−8	−2	−4		4	0
−12	39	−12	−12	3	0.5	−4	−4	0.5	−0.5	4	0.5		−0.5	0
36	−24	3			−8		1		8	−1				0
		−1.4		−10.6				2.8			4	13.2	4	0
	−0.7	−10.6		−0.7		1.4	4	13.2		4		1.4		0
	−10.6	−0.7			4	13.2		1.4	4					0
10.6	0.7		−10.6	−0.7		4			13.2	1.4				0
	6	−7.8		1.8				2		2.8				0
	−6	9.6	6	−9.6				−2		−2.2		2	4	0
	22.2	−22.2	−10.2	10.2				6	6.8	−0.8				0
	−32.4	32.4	32.4	−32.4				−6		4.2		6	6	0
6					−4	−2								$M\Delta^2/D$
	6						−4	−2						$M\Delta^2/D$
			6								−4	−2		$M\Delta^2/D$

TABLE 2

	10.6	0.7	−10.6	0	−0.7	4	0	0	1.4	0	13.2
	6	−10.6	0	−0.7	0	11.2	11.4	0	0	0	4
	0	0	6	−1.4	−10.6	0	2.8	11.2	0	4	0
	0	5.3	0	−10.6	−0.7	1.4	11.2	1.4	4	0	0
	0	6	0	−7.8	1.8	0	2	0	2.8	0	0
$D =$	0	−6	6	9.6	−9.6	0	−2	2	−2.2	4	0
	0	22.2	−10.2	−22.2	10.2	0	6	0	−0.8	0	6.8
	0	−32.4	32.4	32.4	-32.4	0	−6	6	4.2	6	−10.2
	0	$(-22.5-2k)$	$(30+3.625k)$	6	$(-12-k)$	0	−0.5	$(2+0.125k)$	−2	$(4+0.25k)$	$(-8-0.5k)$
	$(-11.25\ k)$	$(33+3.625k)$	$(-11.25-k)$	$(-12-k)$	3	$(-4.25-0.25k)$	$(2.5+0.125k)$	−0.25	$(4+0.25k)$	−0.5	−0.5
	$(24+3.25k)$	$(-22.5-2k)$	0	3	0	$(4+0.25k)$	−0.5	0	−1	0	$(8+0.5k)$

Calculate the minimum value of k:

$$k_{\min} = 0.230,$$

then

$$N_{cr} = \frac{8.28D}{a^2}.$$

Sections of the bent surface of the plate are represented in Figure 13.

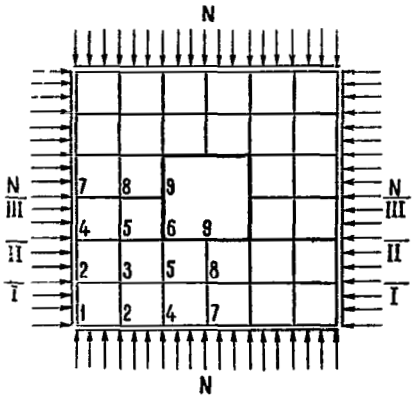

FIGURE 12.

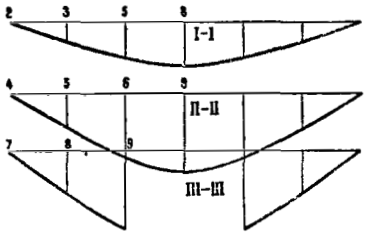

FIGURE 13.

Equations (4), (5), and (6) can also be used in solving problems of plane stress of rectangular plates weakened by rectangular cuts.

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ON THE DETERMINATION OF THE CRITICAL LOAD OF CONICAL AND CYLINDRICAL SHELLS

P. V. Myursepp

(Tartu)

Formulas are given for determining the critical load of a conical shell under the action of a uniformly distributed external pressure. The results obtained by these formulas are compared with the results obtained by Seide's formulas and by other formulas. Expressions are derived for the critical load of short cylindrical and conical shells with built-in ends. In conclusion a formula is given for determining the critical load of a truncated conical shell under an external pressure linearly distributed along the cone generator.

The following formula was derived in Seide's paper /1/ for the critical load of a conical shell under the action of a uniformly distributed external pressure (the formula appears in /1/ with a printing error, which has been corrected here):

$$p = \frac{\dfrac{\pi\sqrt{6}}{9(1-\nu^2)^{3/4}} E \sin\varphi \cos^{3/2}\varphi}{\left(1-\dfrac{\rho_1}{\rho_2}\right)\left(\dfrac{1+\rho_1/\rho_2}{2}\right)^{3/2}\left(\dfrac{\rho_2}{h}\right)^{5/2}} f\left(1-\frac{\rho_1}{\rho_2}\right), \tag{1}$$

where ρ_1, ρ_2 are the radii of the small and large bottoms, respectively, of the truncated cone; h is the shell thickness, φ is the cone angle; E is Young's modulus; ν is the Poisson ratio; $f\left(1-\frac{\rho_1}{\rho_2}\right)$ is a function defining the ratio of the critical load of the conical shell to the critical load of the equivalent cylindrical shell. The equivalent cylindrical shell is, by definition, the cylindrical shell of radius R equal to the principal radius of curvature of the middle section of the cone: $R = \frac{\rho_1+\rho_2}{2\cos\varphi}$.

A formula for determining the critical load of a truncated cone by the method of disturbances was derived in /2/. This formula can be represented in the following form:

$$p = \frac{\varkappa_0\sqrt{6}}{9(1-\nu^2)^{3/4}} \frac{Eh^{5/2}}{L} \cdot \left(\frac{\cos\varphi}{\rho_0}\right)^{3/2} (1-\gamma a^2), \tag{2}$$

where $\varkappa_0$ is a constant which depends on the type of support of the shell ends (at simply supported ends $\varkappa_0 = \pi$, at clamped ends $\varkappa_0 = 4.73$); L is the shell length along the generator; $\rho_0 = \sqrt{\rho_1\rho_2}$; and γ is a constant. The

calculation shows that in the case of simply supported ends $\gamma = 0.25$, and at clamped ends $\gamma = 0.22$; $a = \frac{l}{2}\sin\varphi$, where $l = \frac{L}{\rho_0}$.

In order to compare formulas (1) and (2), we pass in (2) from the geometrical mean ρ_0 to the arithmetical mean ρ_a, related to it by the formula $\rho_a = \rho_0\sqrt{1+a^2}$. We assume that the constant γ is roughly equal to $\frac{1}{4}$ for both types of support. Formula (2) reduces then to

$$p = \frac{\varkappa_0\sqrt{6}}{9(1-\nu^2)^{3/4}}\frac{Eh^{5/2}}{L}\left(\frac{\cos\varphi}{\rho_a}\right)^{1/2}\left(1+\frac{a^2}{2}-\frac{9}{32}a^4\right). \qquad (3)$$

The main part of (3) coincides with the main part of (1). To the function $f\left(1-\frac{\rho_1}{\rho_2}\right)$ there corresponds the expression $\left(1+\frac{a^2}{2}-\frac{9}{32}a^4\right)$. Seide's results and the author's have been plotted in Figure 1 (curves 2 and 3, respectively), together with the results of Niordson /3/ (curve 4), Bijlaard /4/ (curve 5), and Mushtari-Sachenkov /5/ (curve 1). The curve plotted by formula (3) has been drawn up to $1-\frac{\rho_1}{\rho_2} = 0.8$, since to this value of the argument there corresponds a value of a close to unit ($a = 0.896$) (the disturbance parameter must be smaller than 1 according to our assumption).

Seide's results agree closely with the experimental results. Either formula (2) or formula (3) can be recommended for practical use, provided that $\frac{\rho_1}{\rho_2} \geqslant 0.2$. If formula (1) is used, the value of the function $f\left(1-\frac{\rho_1}{\rho_2}\right)$ must be taken from Figure 1.

The number of waves s along the shell circumference /2/ is determined by the formula

$$s^2 = \frac{\varkappa_0}{L}\sqrt{\frac{6\rho_0^3\cos\varphi}{h}}\sqrt[4]{1-\nu^2}\ . \qquad (4)$$

The corresponding formulas for a cylindrical shell of medium length are obtained by substituting $\varphi = 0$ in formulas (2) to (4).

It should be remembered that the difference between the critical loads of a short (cylindrical or conical) shell and of a shell of medium length is not considerable in the case of simply supported ends, the critical load of the short shell being always somewhat larger. It follows that in practical engineering calculations the determination of the critical load of a short shell by the formula for a shell of medium length is wholly admissible.

The situation is totally different in the case of a short (cylindrical or conical) shell of built-in ends. Calculations show that in this case the critical load of a short shell is considerably lower than the value obtained by the formulas for a shell of medium length.

The following simple approximate formula was obtained for determining the critical load of a short cylindrical shell with built-in ends /6/:

$$p = \frac{4.73\sqrt{6}}{9(1-\nu^2)^{3/4}}\frac{Eh}{L}\left(\frac{h}{R}\right)^{1/2}\left(1-2.9\frac{R\sqrt{\lambda}}{L}\right) \qquad (5)$$

$\left(\lambda = \frac{h}{R\sqrt{12(1-\nu^2)}}\right)$. This formula satisfactorily agrees with the results of the exact solution.

The following formula was obtained for determining the critical load of a short truncated cone with built-in ends /2/:

$$p = \frac{4.73\sqrt{6}}{9(1-\nu^2)^{3/4}} \frac{Eh^{5/2}}{L} \left(\frac{\cos\varphi}{\rho_0}\right)^{3/2} (1 + 0.5a^2)\psi. \tag{6}$$

The function ψ is determined from the relationship

$$\psi = \frac{6}{5}\sqrt[4]{\frac{5}{9}}\frac{\varkappa}{\varkappa_0}, \tag{7}$$

with $\varkappa$ found from the equation

$$\varepsilon = \frac{\sqrt{\cos\varphi}}{2(1+\nu)\varkappa}\sqrt[4]{\frac{5}{3}}\frac{\operatorname{tg}\frac{\varkappa}{2} + \operatorname{th}\frac{\varkappa}{2}}{\operatorname{tg}\frac{\varkappa}{2} - \operatorname{th}\frac{\varkappa}{2}}, \tag{8}$$

where

$$\varepsilon = \frac{\sqrt{\lambda}}{l}, \qquad \lambda = \frac{h}{\rho_0\sqrt{12(1-\nu^2)}}, \qquad l = \frac{L}{\rho_0}.$$

The relationship between ψ and ε defined by (7) and (8) has been plotted in Figure 2.

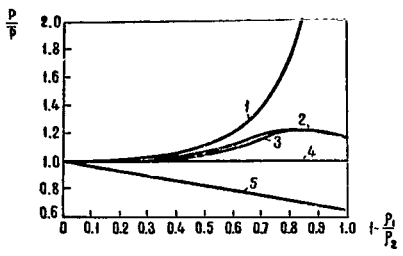

FIGURE 1.

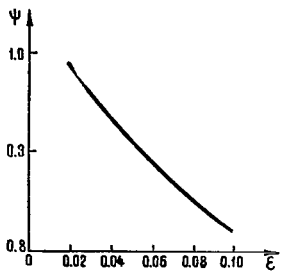

FIGURE 2.

A simple formula for determining the critical load of a truncated cone of medium length under a lateral external pressure distributed linearly along the cone generator (Figure 3) was derived in /7/ by the method of disturbances. This formula is

$$p_1 = \frac{\sqrt{6}}{9(c-d)} \frac{\varkappa_0 E h^{5/2}}{L(1-\nu^2)^{3/4}} \left(\frac{\cos\varphi}{\rho_0}\right)^{3/2} (1 + \gamma a^2), \tag{9}$$

where

$$c = \frac{\rho_2}{\rho_2 - \rho_1} + \frac{p_2}{p_1}, \qquad d = \frac{\rho_0}{\rho_2 - \rho_1}.$$

In the case of simply supported ends

$$x_0 = \pi, \qquad \gamma = -0.034\left(\frac{c-2d}{c-d}\right)^2 + \frac{-0.25c+0.62d}{c-d} + 0.030;$$

in the case of built-in ends

$$x_0 = 4.73, \qquad \gamma = -0.054\left(\frac{c-2d}{c-d}\right)^2 + \frac{-0.20c+0.50d}{c-d} + 0.035.$$

The corresponding formula for a cylinder is obtained from here by substituting $\varphi = 0$.

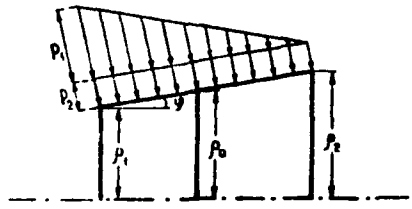

FIGURE 3.

Such a problem was recently solved in the USA /8/ for a cylindrical shell.

The problem for the purely plastic state was solved in /9/, where a formula was derived for determining the minimum thickness of the shell under a given load.

All the formulas given here are simple and convenient for use in engineering practice. At the same time, they ensure the necessary degree of accuracy.

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ASYMPTOTIC THEORY OF STATICS AND DYNAMICS OF ELASTIC CIRCULAR CYLINDRICAL SHELLS AND ANALYSIS OF THE ACCURACY OF THE DIFFERENT VARIANTS OF THE KIRCHHOFF-LOVE THEORY

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Until recently it was widely accepted that different formulations of the elasticity relationships lead to variants of the Kirchhoff-Love theory differing by secondary terms only and having the same error estimate /1/ $\vartheta < a$, where a designates the relative thickness of the shell. In 1961, however, Darevskii /2/ treated problems involving circular cylindrical shells in which different variants of the Kirchhoff-Love theory give different expressions for the principal (independent of a) terms. This result aroused interest in a more thorough analysis of the error of the different variants of the Kirchhoff-Love theory.

To study this problem, an asymptotic (at $a \to 0$) theory of circular cylindrical shells was developed in /3/. Use was made of the results obtained in /4/ on the basis of the three-dimensional theory of elasticity by using the method of power series /1/ in conjunction with the method of asymptotic integration /5/. This asymptotic theory revealed /3/ the numerous essential advantages of Novozhilov's theory /6/ over the other variants of the Kirchhoff-Love theory, and in particular the possibility of using it in solving the Darevskii problems /2/.

The expanded asymptotic formulas, representing the unknown magnitudes in the inner region of the shell through one single function Φ, are presented in this paper. Using these formulas it is possible to estimate the accuracy of any variant of the Kirchhoff-Love theory. It is also possible to derive from them simpler formulas corresponding to various elementary states of stress, including states which were studied on the basis of the three-dimensional theory in /7, 8, 9/.

The asymptotic theory considered exhibits a certain kinship to the theory developed in /10/ for arbitrary shells.

1. MAIN SYMBOLS AND INITIAL PRINCIPLES

We introduce the following designations: E is the modulus of elasticity; μ is the Poisson ratio; ρ is the density of the material; R_0 is the radius of the middle surface of the shell; δ is the shell thickness; x, φ, R are the coordinates in the direction of the length, the transverse circle, and the

radius of the shell, respectively; t is the time; u_j ($j=1, 2, 3$) are displacements; ε_{jk} and σ_{jk} ($j, k=1, 2, 3$) are unit strains and stresses;

$$\zeta=\frac{R}{R_0},\quad \zeta_*=\zeta-1,\quad b=\frac{\delta}{2R_0},\quad a^2=\frac{1}{3}b^2. \tag{1}$$

The following simplifying assumptions are made for the sake of brevity: a) the shell surfaces $\zeta_*=\pm b$ are not loaded; b) pure extension and torsion of the shell are excluded from consideration; c) the states of stress considered depend on x, φ, t through the function

$$\Phi=\exp\left[\lambda\frac{x}{R_0}-im\varphi-i\Omega\frac{t}{R_0}\sqrt{\frac{E}{\rho(1-\mu^2)}}\right], \tag{2}$$

where the coefficients λ, m, Ω can be complex in the general case.

It is assumed further that the following conditions are fulfilled:

$$\vartheta_0=a^2+a^2g^2\ll 1. \tag{3}$$

$$a^2g^2\to 0 \text{ at } a\to 0, \tag{4}$$

where

$$g^2=\max\,[|\lambda^2|,\ |m^2|,\ |\Omega^2|]. \tag{5}$$

Using these assumptions, asymptotic expressions for the unknown magnitudes are easily formed on the basis of /3, 4/. We consider here the dimensionless stress resultants T^*_{ks}, moments M^*_{ks}, lateral forces Q^*_k, displacements u^*_j, and unit strains ε^*_{ks}, defined by the following relationships:

$$T^*_{ks}=N^0_{ks},\quad M^*_{ks}=N^1_{ks},\quad Q^*_k=N^0_{ks}, \tag{6}$$

$$N^n_{kj}=\frac{1-\mu^2}{2bE}\int_{-b}^{b}\alpha_k\zeta_*^n\,\sigma_{kj}\,r_{kj}\,d\zeta, \tag{7}$$

$$u^*_j=\frac{1}{R_0}u_jr_j,\quad \varepsilon^*_{ks}=\varepsilon_{ks}r_{ks}, \tag{8}$$

$$\begin{gathered}(j=1,\ 2,\ 3;\ k,\ s=1,\ 2;\ n=0.1;\ \alpha_1=\zeta,\ \alpha_2=1),\\ r_{11}=r_{22}=r_{13}=r_1=r_3=1,\ r_{12}=r_{23}=r_2=-i.\end{gathered} \tag{9}$$

We introduce the following additional symbols for the sake of brevity:

$$\nu=\frac{1}{1-\mu},\quad k=\frac{1-\mu}{2},\quad \Delta=\lambda-m^2, \tag{10}$$

$$P=\Omega^4+\frac{1}{2}(3-\mu)\,\Omega^2\Delta+k\,\Delta\Delta. \tag{11}$$

2. FORMULAS OF THE ASYMPTOTIC THEORY

The following formulas, derived with an asymptotic error of order ϑ_0, are obtained on the basis of /3, 4/:

1) Stress resultants

$$\begin{gathered}T^*_{11}=\{\mu\Omega^4+\mu k\Omega^2\Delta-(1-\mu^2)\,k\lambda^2m^2-a^2\mu k\,(m^6-m^4)\}\,\Phi,\\ T^*_{12}=k\lambda m\,\{(1+\mu)\,\Omega^2+(1-\mu^2)\,\lambda^2+a^2\,(m^4-m^2)\}\,\Phi,\end{gathered} \tag{12}$$

$$T_{21}^{*} = k\lambda m\ \{(1+\mu)\ \Omega^2 + (1-\mu^2)\ \lambda^2 + a^2\mu\ (m^4 - m^2)\}\ \Phi, \tag{12 (cont'd)}$$

$$T_{22}^{*} = \{\Omega^4 + [(3+2\mu)\ \lambda^2 - m^2]\ k\Omega^2 + (1-\mu^2)\ k\lambda^4 + a^2k\ (m^4 - m^2)(4\lambda^2 - m^2)\}\ \Phi.$$

2) Moments

$$M_{11}^{*} = \{(-\mu\nu\Omega^2 - \lambda^2 + \mu m^2)\ P + \mu\Omega^4 + \mu k\Omega^2\lambda^2 + \frac{1}{2}\mu\ (1+\mu)\ \Omega^2 m^2 - (1-2\mu-2\mu^2)\ k\lambda^2 m^2 - \mu k m^4\}\ a^2\Phi,$$

$$M_{12}^{*} = 2k\lambda m\ \{P + \Omega^2 + (2+\mu)\ k\lambda^2 - km^2\}\ a^2\Phi, \tag{13}$$

$$M_{21}^{*} = 2k\lambda m\ \{P + k\ (\Omega^2 + \Delta)\}\ a^2\ \Phi,$$

$$M_{22}^{*} = \{(-\mu\nu\Omega^2 - \mu\lambda^2 + m^2)\ P - \Omega^4 - (3+2\mu)\ k\Omega^2\lambda^2 + \frac{1}{2}\ (3-\mu)\ \Omega^2 m^2 - (1-\mu^2)\ k\lambda^4 + (2+\mu)\ k\lambda^2 m^2 - km^4\}\ a^2\ \Phi.$$

3) Transverse forces

$$Q_1^{*} = \lambda\ \{-(\nu\Omega^2 + \Delta)\ P + \mu^2\Omega^2 m^2 + \mu\ (1+2\mu)\ k\lambda^2 m^2 - km^4\}\ a^2\Phi,$$

$$Q_2^{*} = m\ \{(\nu\Omega^2 + \Delta)P + 3\Omega^4 + (9+4\mu)\ k\Omega^2\lambda^2 - \frac{1}{2}\ (5-3\ \mu)\ \Omega^2 m^2 + (6+4\mu)\ k^2\lambda^4 - 3k\lambda^2 m^2 + km^4\}\ a^2\ \Phi. \tag{14}$$

4) Displacements

$$u_j^{*} = v_j + \zeta_*\ \psi_j \qquad (j=1,\ 2,\ 3), \tag{15}$$

where

$$v_1 = -\lambda\ [\mu\Omega^2 + \mu k\lambda^2 + km^2]\ \Phi,\ \ v_2 = m\ [\Omega^2 + (2+\mu)\ k\lambda^2 - km^2]\ \Phi,$$

$$v_3 = \left[\Omega^4 + \frac{1}{2}\ (3-\mu)\ \Omega^2\Delta + k\Delta\Delta\right]\Phi, \tag{16}$$

$$\psi_1 = -\lambda v_3,\ \ \psi_2 = v_2 + mv_3,\ \ \psi_3 = -\mu\nu\ (\lambda v_1 + mv_2 + v_3). \tag{17}$$

5) Strain components of the middle surface ($\zeta_* = 0$) of the shell

$$\varepsilon_{11}^{*0} = \lambda v_1,$$

$$\varepsilon_{12}^{*0} = \varepsilon_{21}^{*0} = (1+\mu)\ \lambda m\left[\Omega^2 + (1-\mu)\ \lambda^2 + a^2\frac{1}{2}\ (m^4 - m^2)\right]\Phi.$$

$$\varepsilon_{22}^{*0} = \left[\Omega^4 + \frac{1}{2}(3-\mu)\ \Omega^2\lambda^2 - k\Omega^2 m^2 + k\lambda^4 + \mu k\lambda^2 m^2 + a^2\frac{3}{4}\mu(m^6 - m^4)\right]\Phi. \tag{18}$$

6) Characteristic equation

$$d_0 + a^2\ (d_1 + d_2) = 0, \tag{19}$$

where

$$d_0 = -\Omega^6 - \frac{1}{2}(3-\mu)\ \Omega^4\Delta + \Omega^4 - k\Omega^2\ [\Delta\Delta - (3+2\mu)\ \lambda^2 + m^2] + (1-\mu^2)\ k\lambda^4,$$

$$d_1 = k\ (\Delta\Delta - 2m^6 + m^4),\ \ d_2 = 4k\lambda^2 m^2\ (2m^2 - 1). \tag{20}$$

If the terms in these formulas are not mutually canceled, they define the unknown magnitudes with an asymptotic error θ_0. If the terms of a given expression are mutually canceled, then the magnitude defined by

this expression is much smaller than the other magnitudes, and is practically of no interest. Such a magnitude must be calculated by more accurate formulas; the expressions derived in /3, 4/ can be useful to that end.

These formulas can be simplified at the cost of an increase in the error. With an asymptotic error of order

$$\vartheta_1 = a + a^2 g^2 \tag{21}$$

we obtain

$$\overset{*}{v}_3 = v_3. \tag{22}$$

3. ANALYSIS OF THE ACCURACY OF THE KIRCHHOFF-LOVE THEORY AND THE FORMULAS OF THE ELEMENTARY STATES OF STRESS

A comparison of the formulas of the asymptotic theory with the corresponding formulas obtained on the basis of the different variants of the Kirchhoff-Love theory leads to the conclusion that Novozhilov's theory /6/ is the most reliable variant of the Kirchhoff-Love theory. In Novozhilov's formulas all the terms of the characteristic equation (19) and the expressions for

$$\overset{*}{T}_s \ (k, s = 1, 2), \ \overset{*}{M}_{12}, \ v_j \ (j = 1, 2, 3), \ \psi_k \ (k = 1, 2), \ \overset{*0}{\varepsilon}_{11}$$

appear with correct coefficients.

The other variants of the Kirchhoff-Love theory differ more considerably from the asymptotic theory, among other things by the coefficients of expression d_2 in (19). There exist, as a result, problems for which Novozhilov's theory yields a solution with an asymptotic error $\vartheta_2 \leqslant a^{1/2}$, while other variants of the Kirchhoff-Love theory have an error of the order of a^0. One such example is the problem treated by Darevskii in /2/.

The formulas of the preceding section can also be used for deriving and analyzing the formulas of the elementary states of stress.

If we assume that

$$\Omega \equiv m \equiv 0, \quad |\lambda| \sim a^{-1/2} \tag{23}$$

and admit an asymptotic error of order $\vartheta_3 = a$, we obtain the formulas of the simple edge effect.

If we assume that

$$\Omega \cong 0, \ |\lambda| \sim |m| \sim a^{-1/2} \tag{24}$$

and admit an asymptotic error ϑ_3, we obtain the formulas of a state of stress of large variability index /5/ (Vlasov's technical theory).

Other elementary states of stress can be similarly analyzed.

The particular cases (23), (24) were studied in detail on the basis of the three-dimensional theory in /7, 8, 9/.

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SEPARATE APPLICATION OF THE MEMBRANE THEORY AND THE BENDING-MOMENT THEORY TO THE CALCULATION OF SHALLOW SHELLS

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The fundamental equations of the linear theory of shallow shells with constant principal curvatures subjected to a vertical load of intensity Z are /1/

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}-(k_1+\nu k_2)\frac{\partial w}{\partial x}=0,\\ \frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}-(k_2+\nu k_1)\frac{\partial w}{\partial y}=0,\\ D\nabla^2\nabla^2 w+\frac{Eh}{1-\nu^2}(k_1^2+k_2^2+2\nu k_1k_2)w-\\ -\frac{Eh}{1-\nu^2}(k_1+\nu k_2)\frac{\partial u}{\partial x}-\frac{Eh}{1-\nu^2}(k_2+\nu k_1)\frac{\partial v}{\partial y}=Z.\end{gathered}\tag{1}$$

The exact solution of this system of differential equations is very difficult.

This system of equations can be solved approximately with satisfactory accuracy if the state of stress in the shell can be split into membrane and bending states of stress. This is usually possible when the membrane state of stress is fundamental, and the bending moments result in an edge effect only.

In the case of shallow shells of positive Gaussian curvature rectangular in the plan this splitting was realized as follows under specific conditions /2/.

The term $D\nabla^2\nabla^2 w$ in the third equation of system (1), which represents the part of the vertical load taken by the bending stresses, was neglected, the resulting system of three equations was solved allowing for the tangential boundary conditions, and the term $D\nabla^2\nabla^2 w$ was then allowed for by superposing on the membrane stressed state the state of bending stress in the form of an edge effect.

Such an approach is usually unacceptable in shells of negative Gaussian curvature and in cylindrical shells (in the direction of the curvilinear directrix) in which the bending moments cannot be allowed for through some edge effect. In such shells the calculation should be based not on the total acting load Z, but on the load $Z-D\nabla^2\nabla^2 w$; this reduces to determining the load taken by the membrane stresses.

Represent the system of equations (1) in the following form (the Poisson ratio is taken as zero for simplicity):

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)-k_1\frac{\partial w}{\partial x}=0,\\
&\frac{\partial^2 v}{\partial y^2}+\frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}\right)-k_2\frac{\partial w}{\partial y}=0,\\
&Eh\,(k_1^2+k_2^2)\,w-Eh\left(k_1\frac{\partial u}{\partial x}+k_2\frac{\partial v}{\partial y}\right)=Z\delta;
\end{aligned}\right\}\qquad(2)$$

$$D\nabla^2\nabla^2 w=Z-Z\delta.\qquad(3)$$

Consider the load

$$Z=L\,(y)\,Q\,(x),$$

where $L\,(y)$ is a linear function by means of which a particular solution of (2) or (3) is simply found.

Expand the load in a trigonometric series by x* (the origin of coordinates is placed at the middle of the hinged edge of the shell, lying in the yoz plane)

$$Z=L\,(y)\sum_{n}^{\infty}b_n\sin\alpha_n x.$$

Represent the load Z_m in a certain interval $(-y_1,\ +y_2)$ in the form of the series

$$Z_m=\sum_{n}^{\infty}(1+c_n y)\,b_{n(m)}\sin\alpha_n x.\qquad(4)$$

Solve system (2) for given tangential boundary conditions for the load component $Z_{n(m)}=b_{n(m)}(1+c_n y)\sin\alpha_n x$, and equation (3) for the load component $Z_n-Z_{n(m)}=[b_n L\,(y)-b_{n(m)}(1+c_n y)]\sin\alpha_n x$, allowing for the nontangential conditions with an accuracy up to the unknown parameters $b_{n(m)}$ and c_n.

We obtain the following expressions for the deflections:

$$\begin{aligned}
w_{n(m)}&=\varphi_n\,(y,\ b_{n(m)},\ c_n)\sin\alpha_n x,\\
w_{n(b)}&=f_n(y,\ b_{n(m)},\ c_n)\sin\alpha_n x.
\end{aligned}\qquad(5)$$

First method

The unknown coefficients $b_{n(m)}$ and c_n can be determined from the minimum condition of the quadratic deviation of the function $\varphi_n\,(y,\ b_{n(m)},\ c_n)$ from the function $f_n\,(y,\ b_{n(m)},\ c_n)$ in the interval $(-y_1,\ +y_2)$:

$$I=\int_{-y_1}^{y_2}(\varphi_n-f_n)^2\,dy=\min,\qquad(6)$$

which leads to the solution of two linear equations:

$$\frac{\partial I}{\partial b_{n(m)}}=0,\qquad\frac{\partial I}{\partial c_n}=0.\qquad(7)$$

* It is assumed that the function $Q(x)$ satisfies the Dirichlet conditions.

In the case when the acting load is $Z = Z(x) = \sum_n^\infty b_n \sin \alpha_n x$, it can be assumed that in a certain interval $(-y_1, +y_2)$ $Z_m(x) = \sum_n^\infty b_{n(m)} \sin \alpha_n x$, then $w_{n(m)} = b_{n(m)} \varphi_n(y) \sin \alpha_n x$, $w_{n(b)} = (b_n - b_{n(m)}) f_n(y) \sin \alpha_n x$ and the first equation of system (7) becomes

$$\frac{\partial I}{\partial b_{n(m)}} = b_{n(m)} \int_{-y_1}^{y_2} (\varphi_n + f_n)^2 dy - b_n \int_{-y_1}^{y_2} f_n (\varphi_n + f_n) dy = 0,$$

from which we find

$$b_{n(m)} = b_n \frac{\int_{-y_1}^{y_2} f_n (\varphi_n + f_n) dy}{\int_{-y_1}^{y_2} (\varphi_n + f_n)^2 dy}.$$

Second method

It is possible to determine $b_{n(m)}$ from the condition that

$$b_{n(m)} \int_{-y_1}^{y_2} \varphi_n dy = (b_n - b_{n(m)}) \int_{-y_1}^{y_2} f_n dy, \tag{8}$$

whence

$$b_{n(m)} = b_n \frac{\int_{-y_1}^{y_2} f_n dy}{\int_{-y_2}^{y_2} (\varphi_n + f_n) dy}. \tag{9}$$

The coefficient $b_{n(m)}$ can be determined even more simply by equating the deflections $w_{n(m)}$ and $w_{n(b)}$ along the line $y = 0$; then

$$b_{n,m)} = b_n \frac{f_n(0)}{f_n(0) + \varphi_n(0)}. \tag{10}$$

Third method

In the case of other boundary conditions, when the determination of w from equation (3) is difficult, the splitting of the acting load $Z = L(y) \sum_n^\infty b_n \psi_n(x)$* can be realized in the following way.

By writing $Z_m = \sum_n^\infty b_{n(m)} (1 + c_n y) \psi_n(x)$ and solving system (2) for given tangential boundary conditions for the action of the vertical load component $Z_{n(m)} = b_{n(m)} (1 + c_n y) \psi_n(x)$, we obtain with an accuracy up to the unknown

* The functions $\psi_n(x)$ and $F_n(x)$ form a closed orthogonal system in the interval $(0, l)$.

constants $b_{n(m)}$ and c_n the deflection $w_{n(m)} = \varphi_n(y, b_{n(m)}, c_n) F_n(x)$.* By substituting the function $w_{n(m)}$ in the equation $D\nabla^2\nabla^2 w = Z_n - Z_{n(m)}$ and solving it by some variational method (such as the Bubnov-Galerkin method) we obtain the values of the coefficients $b_{n(m)}$ and c_n.

By one of these methods it is also possible to determine the coefficients $b_{n(m)}$ and c_n when the shell is acted upon by a horizontal load of intensity X or Y only /1/. In this case, the term $-\frac{X}{Eh}$ and $-\frac{Y}{Eh}$ appears in the right-hand parts of the first two equations of system (2), and the substitution $Z = 0$ is to be made in equation (3).

The bending stresses and corresponding normal displacements in the zone $0 \leqslant x \leqslant l$ $-y_1 \leqslant y \leqslant y_2$ are determined from the equation

$$D\nabla^2\nabla^2 w + Ehk_2^2 w = Z + k_1 N_1 + Ehk_2 \frac{\partial v}{\partial y} = Z_{np}(x, y), \tag{11}$$

where N_1 and $\frac{\partial v}{\partial y}$ are the earlier found membrane stresses and corresponding displacements.

By means of the above-described method the solution of equation (11) can be reduced to the solution of two equations:

$$D\frac{\partial^4 w}{\partial x^4} + Ehk_2^2 w = Z_{1np}(x, y), \tag{12}$$

$$D\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 w}{\partial x^2}\right) = Z_{2np}(x, y), \tag{13}$$

where

$$Z_{1np} + Z_{2np} = Z_{np}.$$

The deflection, the bending moments, and the torsional moment can also be determined from the equation

$$D\nabla^2\nabla^2 w + Ehk_1^2 w = Z + k_2 N_2 + Ehk_1 \frac{\partial u}{\partial x} = Z_{np}(x, y). \tag{14}$$

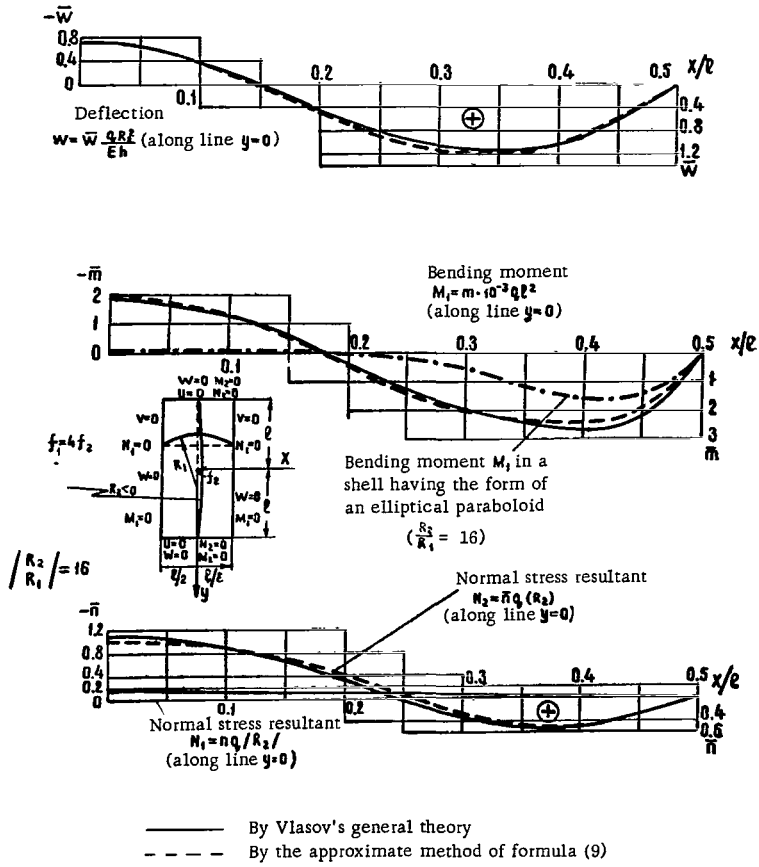
When the extent of the interval $(-y_1, y_2)$ is negligible, the bending factors in this interval can be determined from equation (3).

Two shallow shells rectangular in the plan were calculated by the method described for the action of a load $Z = q = \text{const}$: a shell having the form of a hyperbolic paraboloid with ends oriented along the direction of the principal parabolas, and a cylindrical shell. The shell parameters and the end conditions are given in Figures 1 and 3. Diagrams of the deflections, lateral moments, and shearing stress resultants determined by the approximate method and by Vlasov's general theory of shallow shells using double trigonometric series have been plotted in Figures 1, 2, 3, and 4.

The load was split by the simpler (and therefore less accurate) second method. When determining $b_{n(m)}$ by formula (9) the length of the interval $(-y_1, +y_2)$ in the hyperbolic paraboloid shell and in the cylindrical shell was taken as equal respectively to $1.74\,l$ and $1.5\,l$.

The deflection and the lateral bending moments along line $y = 0$ in the hyperbolic paraboloid shell was determined from (12) for $Z_{1np} = Z_{np}$.

* See foonote on previous page.



Shell in the form of a hyperbolic paraboloid

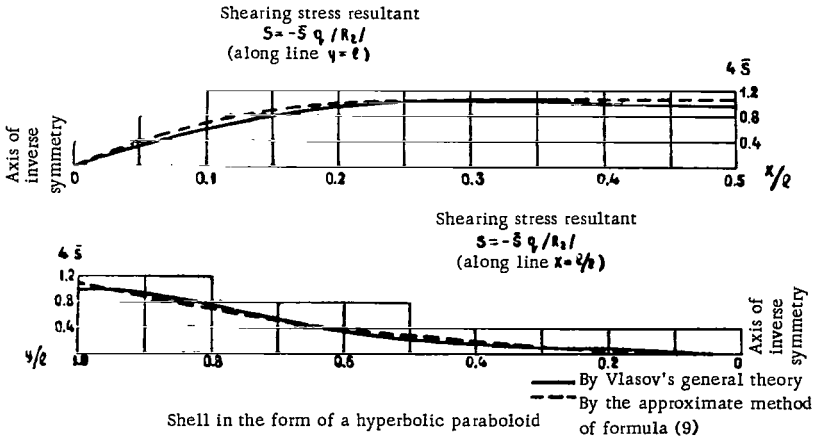
($\frac{f_1}{h}$ = 115.6)

FIGURE 1.

In the case of the cylindrical shell, when splitting the load by formula (9), w and M_1 (Figure 3) were determined according to the equations of the membrane theory for a load Z_m; when using formula (10) M_1 (Figure 3) was determined from (12) for $k_2 = 0$ and $Z_{2np} = Z_{np}$.

The diagrams of the deflections and stress resultants given in Figures 1, 2, 3, 4 show that the approximate solution obtained on the basis of the method proposed has sufficient accuracy.

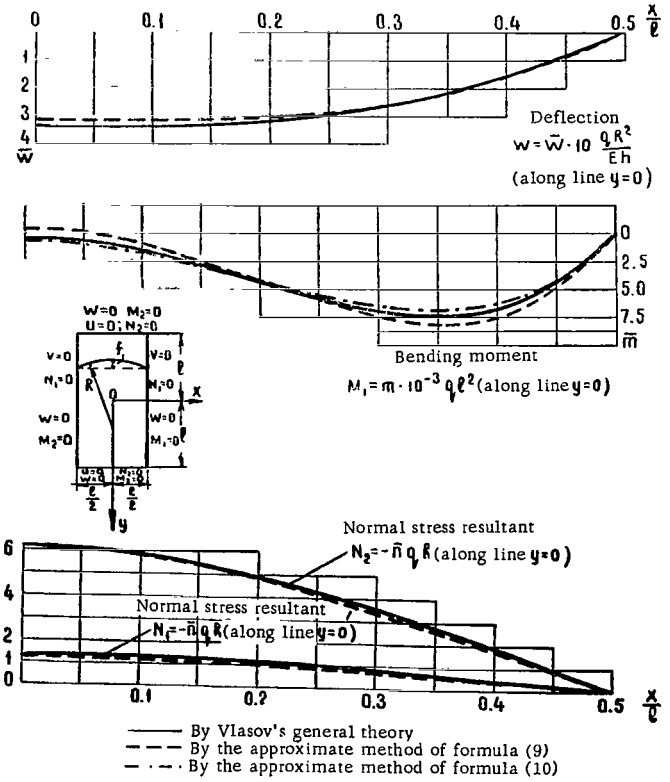
The method of solving system (1) described here can also be applied to the solution of other systems of linear differential equations, and of separate linear differential equations as well.



Shell in the form of a hyperbolic paraboloid

($\frac{f_1}{h}$ = 115.6)

FIGURE 2.



Cylindrical shell (f/h = 10.4)

FIGURE 3.

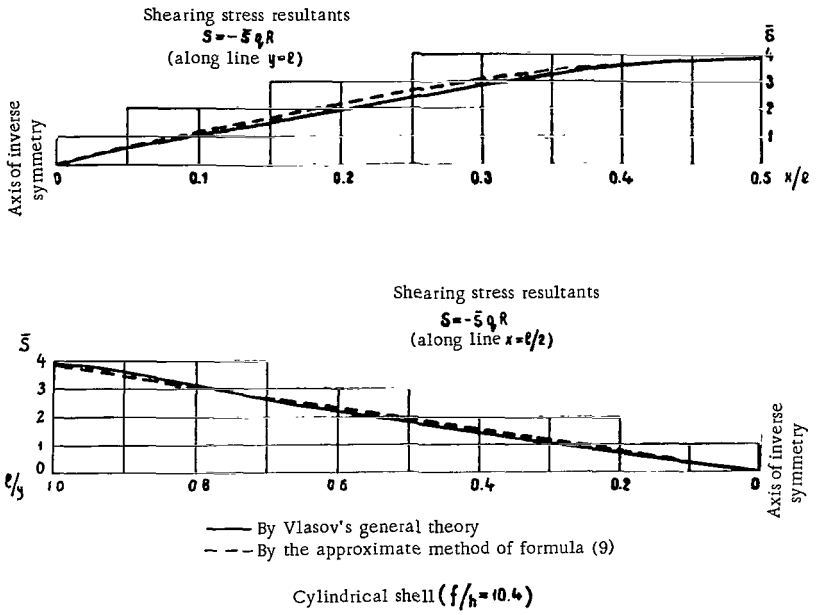


FIGURE 4.

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NONSTATIONARY FLUTTER OF CYLINDRICAL PANELS

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Much literature has already been devoted to the nonlinear problems of panel flutter. Most of the papers dealing with the subject establish stationary solutions of the equations describing the behavior of panels in a gas stream. The nonstationary solutions are, however, also of interest, since through them the behavior of the panel during the simultaneous variation of the velocity and the temperature can be studied. A recent paper by Bolotin /1/ was devoted to the determination of the nonstationary solutions corresponding to nonstationary flutter. It was proposed to describe the nonstationary phenomena at flutter by Krylov-Bogolyubov-Mitropol'skii's method of single-frequency vibrations; the technique for obtaining the first approximation by this method was also outlined there, and the nonstationary flutter of a plane unheated panel was considered as an example.

This paper is devoted to the approximate solution of the problem of nonstationary vibrations of shallow cylindrical panels in a supersonic gas stream. The nonstationary solutions will be determined by the method of single-frequency vibrations. The influence of various parameters on the nonstationary flutter will be illustrated in a particular example.

The panel behavior in a gas stream can be described as known by a Karman-type system of equations. The aerodynamic load is allowed for according to the piston theory in a linear approximation, and the aerodynamic damping is treated together with the structural damping.

Approximating the normal deflection of a cylindrical panel simply supported on all edges by the expression

$$w(x, y, t) = \varphi_1(t)\, h \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \varphi_2(t)\, h \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} \tag{1}$$

and using the Galerkin-Pankovich procedure, the problem is easily reduced to solving the system

$$\left.\begin{aligned} &\varphi_1'' + g\varphi_1' + (d_{11} - s_{11}\vartheta)\varphi_1 - \mu\varphi_2 + c_{11}\varphi_1^3 + c_{12}\varphi_1\varphi_2^2 + d_{12}\varphi_1^2 + \\ &\qquad + d_{13}\varphi_2^2 + d_{14}\vartheta = 0, \\ &\varphi_2'' + g\varphi_2' + (d_{21} - s_{21}\vartheta)\varphi_2 + \mu\varphi_1 + c_{21}\varphi_1^2\varphi_2 + c_{22}\varphi_2^3 + d_{22}\varphi_1\varphi_2 = 0. \end{aligned}\right\} \tag{2}$$

The symbols used here are the same as in /4/. The primes designate differentiation with respect to the dimensionless time $\tau = \Omega t$.

The stationary solutions of this system (the singular points and the limiting cycles) were considered by us in /3, 4/. The nonstationary solutions will be determined, as already mentioned, by the method of single-frequency vibrations. Applications of the method of single-frequency vibrations to nonautonomous systems and to autonomous conservative systems are widely known /5, 6/. The system considered is a peculiar nonconservative autonomous system. Of great importance here is the paired interaction which characterizes flutter problems.

We shall determine the solution near the flutter line /4/. Using the solution corresponding to this line

$$\varphi_j(\tau) = \varphi_j^2 + \zeta_j(\tau) \quad (j = 1,\ 2),$$

we obtain the system of equations

$$\left.\begin{aligned} &\zeta_1'' + g\zeta_1' + (F_{11} - s_{11}\vartheta_*)\zeta_1 + (-\mu_* + F_{12})\zeta_2 = \eta\,[s_{11}(\vartheta - \vartheta_*)\zeta_1 + \\ &\quad + (\mu - \mu_*)\zeta_2 - (F_{111}\zeta_1^2 + 2F_{112}\zeta_1\zeta_2 + F_{122}\zeta_2^2 + c_{11}\zeta_1^3 + c_{12}\zeta_1\zeta_2^2)], \\ &\zeta_2'' + g\zeta_2' + (F_{22} - s_{21}\vartheta_*)\zeta_2 + (\mu_* + F_{21})\zeta_1 = \eta\,[s_{21}(\vartheta - \vartheta_*)\zeta_2 - \\ &\quad - (\mu - \mu_*)\zeta_1 - (F_{211}\zeta_1^2 + 2F_{212}\zeta_1\zeta_2 + F_{222}\zeta_2^2 + c_{21}\zeta_1^2\zeta_2 + c_{22}\zeta_2^3)], \end{aligned}\right\} \tag{3}$$

where

$$F_{jk} = \frac{\partial F_j}{\partial \varphi_k}\bigg|_{\varphi_1=\varphi_1^0,\ \varphi_2=\varphi_2^0} + \delta_{jk}s_{j1}\vartheta_*, \qquad F_{jkl} = \frac{1}{2}\,\frac{\partial^2 F_j}{\partial\varphi_k\partial\varphi_l}\bigg|_{\varphi_1=\varphi_1^0,\ \varphi_2=\varphi_2^0}.$$

The asterisks designate the values of the parameters of the temperature ϑ and the velocity μ corresponding to the flutter line. These parameters are a function of the slow time $\eta\tau$ (η is a small parameter).

All the conditions for applying the method of single-frequency vibrations to system (3) are fulfilled /1/.

We shall look for a solution of (3) in the form

$$\zeta_j = Au_{j1}e^{i\psi} + Au_{j2}e^{-i\psi}, \tag{4}$$

where

$$u_{jk} = u_{jk}^0 + \sum_{s=1}^{\infty}\eta^s u_{jk}^{(s)}(A,\ \psi) \qquad (j = 1,\ 2;\ \ k = 1,\ 2), \tag{5}$$

and the real functions A and ψ are determined from the equations

$$\left.\begin{aligned} \frac{dA}{d\tau} &= \sum_{s=1}^{\infty}\eta^s A_s(A), \\ \frac{d\psi}{d\tau} &= \omega_* + \sum_{s=1}^{\infty}\eta^s B_s(A) \end{aligned}\right\} \tag{6}$$

(ω_* is the frequency of vibrations on the flutter line).

The problem of finding an approximate solution of system (3) reduces to finding the functions u_{jk}^0, $u_{jk}^{(s)}$, A_s, $B_s\,(s = 1,\ 2\cdots)$ and integrating system (6).

These functions must be determined in such a way that expression (4), with (5) substituted in it, will be a solution of (3) in every case when A and ψ satisfy system (6).

Substituting (4) in (3) (allowing for (5) and (6)), and comparing the coefficients of equal powers of η, we obtain complex systems of equations for the determination of u_{jk}^0 and $u_{jk}^{(s)}$. The generating system splits into two conjugate systems with zero determinants, which ensure the existence of a nontrivial solution

$$u_{11}^0 = \frac{1}{2}, \qquad u_{21}^0 = -\frac{1}{2}\sqrt{\chi_3}\, e^{i\gamma}, \tag{7}$$

where the phase shift γ is determined from the relationships

$$\sin\gamma = \frac{g\chi_2}{\sqrt{\chi_3}}, \qquad \cos\gamma = \frac{\chi_1}{\sqrt{\chi_3}}, \qquad \chi_1^2 = \chi_3 - g^2\chi_2^2,$$

$$\chi_2 = \frac{\omega_*}{\mu_* - F_{12}}, \qquad \chi_3 = \frac{\mu_* + F_{21}}{\mu_* - F_{12}}$$

(u_{12}^0 and u_{22}^0 are magnitudes conjugate with u_{11}^0, u_{21}^0).

The solutions of the other systems will be represented in series expansions:

$$u_{j1}^{(s)} = \sum_{n=-1}^{\infty} u_{j1n+1}^{(s)} e^{ni\psi}, \qquad u_{j2}^{(s)} = \sum_{n=-1}^{\infty} u_{j2n+1}^{(s)} e^{-ni\psi}. \tag{8}$$

The coefficients of these series are determined from the equations

$$\left.\begin{aligned} &A(F_{11} - s_{11}\vartheta_*)(u_{110}^{(s)} + u_{120}^{(s)}) + A(-\mu_* + F_{12})(u_{210}^{(s)} + u_{220}^{(s)}) = \Psi_{10}^{(s)};\\ &A(F_{22} - s_{21}\vartheta_*)(u_{210}^{(s)} + u_{220}^{(s)}) + A(\mu_* + F_{21})(u_{110}^{(s)} + u_{120}^{(s)}) = \Psi_{20}^{(s)}; \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} &A[-\omega_*^2(n+1)^2 + ig\omega_*(-1)^{k+1}(n+1) + (F_{11} - s_{11}\vartheta_*)]\, u_{1kn+1}^{(s)} +\\ &\qquad + A(-\mu_* + F_{12})\, u_{2kn+1}^{(s)} = \Psi_{1kn+1}^{(s)};\\ &A[-\omega_*^2(n+1)^2 + ig\omega_*(-1)^{k+1}(n+1) + (F_{22} - s_{21}\vartheta_*)]\, u_{2kn+1}^{(s)} +\\ &\qquad + A(\mu_* + F_{21})\, u_{1kn+1}^{(s)} = \Psi_{2kn+1}^{(s)}; \end{aligned}\right\} \tag{10}$$

$$k = 1, 2, \quad n = 0, 1, 2, 3 \cdots$$

The right-hand parts of these systems represent the corresponding terms of the expansion in the right-hand parts of the equations for determining $u_{jk}^{(s)}$ in a Fourier series. System (9) serves for determining the constant component of the solution. As seen from (4), (5), and (8), there is no need to determine $u_{j10}^{(s)}$ and $u_{j20}^{(s)}$ separately; it suffices to determine their sum. System (10) has determinants different from zero and therefore a unique solution for all n except $n = 0$. At $n = 0$ the determinant of system (10) coincides for any s with the determinant of the generating system and is equal to zero. A solution will, however, exist in this case also, due to the fulfillment of the periodicity conditions

$$\Psi_{1k1}^{(s)}\sqrt{\chi_3}\, e^{i\gamma} + \Psi_{2k1}^{(s)} = 0. \tag{11}$$

These conditions serve for determining the functions A_s and B_s. It is

easily shown that at $n = 0$ the right-hand parts of system (10) have the form

$$\Psi_{k1}^{(s)} = \{[2i\omega_* (-1)^k - g] A_s + [2\omega_* - ig(-1)^k] AB_s\} u_{jk}^0 + \bar{\Psi}_{jk1}^{(s)}.$$

Using these relationships and conditions (11), we obtain

$$\left.\begin{aligned} A_s &= -\frac{2\omega_* H_s + gG_s}{4\omega_*^2 + g^2}, \\ B_s &= \frac{1}{A}\,\frac{2\omega_* G_s - gH_s}{4\omega_*^2 - g^2}, \end{aligned}\right\} \tag{12}$$

where G_s and H_s are, respectively, the real and imaginary parts of a function having at $k = 1$ the form

$$-\frac{\bar{\Psi}_{111}^{(s)}\sqrt{\chi_s}\, e^{i\gamma} + \bar{\Psi}_{211}^{(s)}}{i\sqrt{\chi_s}\sin\gamma} \tag{13}$$

(at $k = 2$ we obtain a function conjugate to (13), so that the same values for G_s and H_s will be obtained in this case as well).

The determination of any approximation will consist thus in determining the functions $(u_{j10}^{(s)} + u_{j20}^{(s)})$, $u_{jkn+1}^{(s)}$ $(k = 1, 2;\ n = 0, 1, 2\cdots)$ from systems (9) and (10) and integrating system (6) after the substitution of (12).

We shall now consider the first approximation in more detail. In this case system (6) reduces to

$$\left.\begin{aligned} \frac{dA}{d\tau} &= \frac{K}{g} A(A_0^2 - A^2), \\ \frac{d\psi}{d\tau} &= \omega + \frac{K}{2\omega_*}(A_0^2 - A^2), \end{aligned}\right\} \tag{14}$$

where

$$\begin{gathered} K = \frac{2\omega_* g}{(4\omega_*^2 + g^2)\sin\gamma}\left(\frac{3}{8}c_0\cos\gamma + \frac{1}{8}c_1\frac{g}{2\omega_*}\sin\gamma\right), \\ A_0^2 = \left[(\mu - \mu_*)\frac{\chi_3 + 1}{2\sqrt{\chi_3}} + (\vartheta - \vartheta_*)\left(s_1\cos\gamma + \frac{gs_2}{2\omega_*}\sin\gamma\right)\right] \times \\ \times \left[\frac{3}{8}c_0\cos\gamma + \frac{1}{8}c_1\frac{g}{2\omega_*}\sin\gamma\right]^{-1}, \\ \omega = \omega_* + \frac{1}{16\omega_*^2}c_1 A_*^2 - \frac{s_2(\vartheta - \vartheta_*)}{2\omega_*}, \qquad s_1 = \frac{s_{21} - s_{11}}{2}, \qquad s_2 = \frac{s_{11} + s_{21}}{2}. \end{gathered} \tag{15}$$

The stationary solutions $\left(\frac{dA}{d\tau} = 0,\ \frac{d\psi}{d\tau} = \omega = \text{const}\right)$ coincide with the solutions obtained by the small-parameter method in a first approximation.

The first equation of (14) was integrated numerically for different laws of variation of the temperature and velocity for a square panel characterized by the parameters $\frac{a}{b} = 1$, $\rho = -0.125$ (curvature parameter /4/), $\beta_x = \beta_y = 1$ (absolutely rigid bar set), $\nu = 0.3$. The results, represented in Figures 1 through 3, show the influence of the rate of variation of the temperature and velocity parameters (Figure 1), the considerable influence of the damping, noted already in /1/ (Figure 2), and the influence of initial deviations on the amplitude of unsteady flutter of cylindrical panels

(Figure 3). Note that at low values of the damping coefficient the nonstationary vibrations are sufficiently near to stationary behavior, and the initial deviations, as expected, influence the amplitude of nonstationary vibrations in the initial portion only.

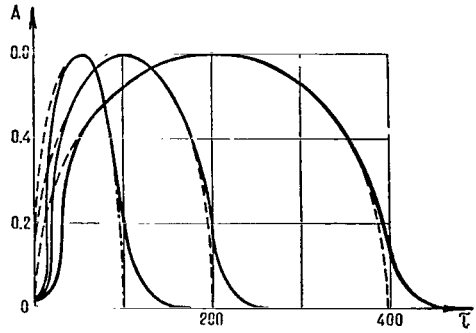

FIGURE 1.

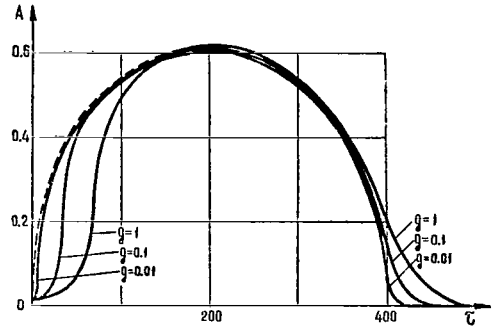

FIGURE 2.

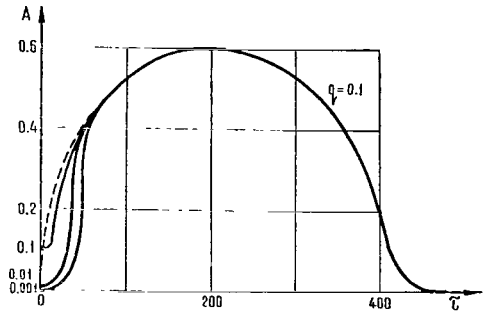

FIGURE 3.

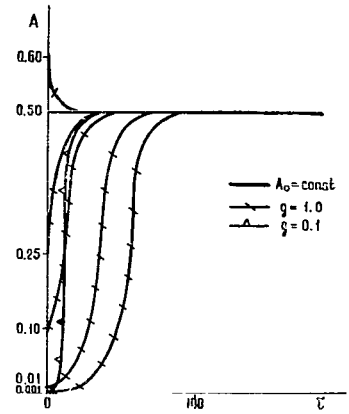

FIGURE 4.

Considering the nonstationary solutions corresponding to unsteady flutter, it is easy to prove the asymptotic stability of the stationary conditions /7/. It was established that on the flutter line there takes place an exchange of the stability of the trivial and nontrivial values for the amplitude.

Using the result of the integration of (14) for $A_0 = \text{const}$ and $A(\tau_0) \neq 0$, it is possible to study the problem of the establishment of stationary conditions in the flutter region. Curves of A have been plotted for these conditions in Figure 4 for different values of the initial deviation and the damping parameter g. It is easily seen that the stationary conditions are rapidly established, the rate at which they are established increasing with the increase of the initial deviation $(A(\tau_0) < A_0)$ and the decrease of the damping coefficient.

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ENGINEERING METHOD OF CALCULATING LAMINATED ANISOTROPIC PLATES

E.S. Osternik and Ya.A. Barg

(Khar'kov)

Problems of bending, buckling, and vibrations of anisotropic laminated plates are considered allowing for the transverse shear.

The plate, made of $2k+1$ layers, is shown in the figure. In the general case all layers are orthotropic, the planes of elastic symmetry of the layers being parallel to the plate edges. If the layers are transversally isotropic, the isotropy planes are parallel to $x0y$. The following assumptions are made.

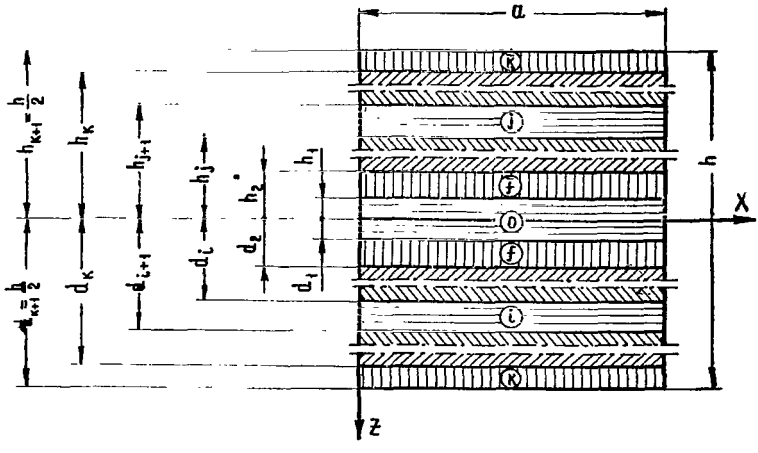

FIGURE.

The vertical displacements w are small, and $\varepsilon_z = w'_z = 0$. The shearing stresses vary according to some given law along the z axis, so that in the i-th layer

$$\tau_{xz}^{(i)} = f(x, y, t)\,[G_{13}^{(i)}\alpha'(z) + A_i]; \quad \tau_{yz}^{(i)} = \varphi(x, y, t)\,[G_{23}^{(i)}\alpha'(z) + B_i],$$
$$i = 0, 1 \cdots k. \tag{1}$$

The function $\alpha(z)$ is odd, $\alpha'(z)$ is even, and

$$\alpha(0) = \alpha'\left(\frac{h}{2}\right) = 0; \quad u(x, y, 0) = v(x, y, 0) \equiv 0.$$

The constants A_i and B_i are determined from the conditions of compatibility of the strains of the layers (without layer sliding) and from the method of loading of the plate:

$$z = d_i \qquad (i = 1, 2 \cdots k-1)$$

at

$$[u_i,\ v_i,\ \tau_{xz}^{(i)},\ \tau_{yz}^{(i)}] = [u_{i-1},\ v_{i-1},\ \tau_{xz}^{(i-1)},\ \tau_{yz}^{(i-1)}]. \tag{2}$$

The plate possesses a symmetrical structure, so that at $i = j$

$$[d_i,\ E_1^{(i)},\ E_2^{(i)},\ E_3^{(i)},\ \nu_{12}^{(i)},\ \nu_{13}^{(i)},\ \nu_{23}^{(i)},\ G_{12}^{(i)},\ G_{13}^{(i)},\ G_{23}^{(i)},\ \mu_i] =$$
$$= [-h_j,\ \bar{E}_1^{(j)},\ \bar{E}_2^{(j)},\ \bar{E}_3^{(j)},\ \bar{\nu}_{12}^{(j)},\ \bar{\nu}_{13}^{(j)},\ \bar{\nu}_{23}^{(j)},\ \bar{G}_{12}^{(j)},\ \bar{G}_{13}^{(j)},\ \bar{G}_{23}^{(j)},\ \bar{\mu}_j],$$

and also

$$\tau_{xz}(z) = \tau_{xz}(-z);\ \tau_{yz}(z) = \tau_{yz}(-z).$$

Here $E^{(i)}$, $\nu^{(i)}$, $G^{(i)}$ are elastic constants, and μ_i is the mass per unit area of the i-th layer. The plate is subjected to a lateral load $q(x, y, t)$ and to longitudinal static stress resultants P_x, P_y, P_{xy} in the middle plane, uniformly distributed along the edges and normal to them. Then at $z = \frac{h}{2}$ $\tau_{xy}^{(k)} = \tau_{yz}^{(k)} = 0$; at $z = -\frac{h}{2}$ $\tau_{xz}^{(k)} = \tau_{yz}^{(k)} = 0$. The volume static forces X, Y, Z of inertia of the axial motion and the damping in the plate material are neglected. The corresponding theory for single-layer plates was developed by Ambartsumyan /1/.

The equations of motion of an element of dimensions $dx \cdot dy \cdot h$ will be

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -g(x, y, t);\ Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y};\ Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}. \tag{3}$$

Here

$$g(x, y, t) = q(x, y, t) + \left[P_x \frac{\partial^2}{\partial x^2} + P_y \frac{\partial^2}{\partial y^2} + 2P_{xy} \frac{\partial^2}{\partial x \partial y} - \mu \frac{\partial^2}{\partial t^2}\right] w;$$

μ is the plate mass per unit area. According to the conditions of equivalence

$$[M_x, M_y, M_{xy}] = \int_{-0.5h}^{0.5h} [\sigma_x, \sigma_y, \tau_{xy}]\, z dz;\quad [Q_x, Q_y] = \int_{-0.5h}^{0.5h} [\tau_{xz}, \tau_{yz}]\, dz. \tag{4}$$

We transform equations (3) in such a way that the functions f, φ, w of x, y, t appear in them. Substitute to that end (1) in the Cauchy equations and integrate the latter, determining the arbitrary integration functions from (2). We obtain

$$\begin{gathered} u_i(x, y, z, t) = fF_i(z) - \frac{\partial w}{\partial x} z;\qquad v_i = \varphi F_i'(z) - \frac{\partial w}{\partial y} z; \\ F_i(z) = \alpha(z) + C_i z + D_i;\qquad F_i'(z) = \alpha(z) + C_i' z + D_i'. \end{gathered} \tag{5}$$

The constants C, D, etc. are determined by the plate parameters and the form of $\alpha(z)$.

Assuming $\sigma_z = 0$, we write for the i-th layer

$$\sigma_x^{(i)} = L_1^{(i)}\left[\frac{\partial u_i}{\partial x} + \nu_{21}^{(i)}\frac{\partial v_i}{\partial y}\right]; \qquad \sigma_y^{(i)} = L_2^{(i)}\left[\frac{\partial v_i}{\partial y} + \nu_{12}^{(i)}\frac{\partial u_i}{\partial x}\right];$$
$$\tau_{xy}^{(i)} = G_{12}^{(i)}\left[\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\right]. \tag{6}$$

The substitution of (5) in (6) yields

$$\sigma_x^{(i)} = \frac{\partial f}{\partial x}L_1^{(i)}F_i(z) + \frac{\partial\varphi}{\partial y}k_iF_i'(z) - \frac{\partial^2 w}{\partial x^2}L_1^{(i)}z - \frac{\partial^2 w}{\partial y^2}k_iz;$$
$$\sigma_y^{(i)} = \frac{\partial f}{\partial x}k_iF_i(z) + \frac{\partial\varphi}{\partial y}L_2^{(i)}F_i'(z) - \frac{\partial^2 w}{\partial x^2}k_i(z) - \frac{\partial^2 w}{\partial y^2}L_2^{(i)}z; \tag{7}$$
$$\tau_{xy}^{(i)} = G_{12}^{(i)}\left[\frac{\partial f}{\partial y}F_i(z) + \frac{\partial\varphi}{\partial x}F_i'(z) - 2\frac{\partial^2 w}{\partial x\partial y}z\right].$$

The substitution of (7) in (4) yields

$$M_x = 2\left(\rho_1\frac{\partial f}{\partial x} + \rho_2\frac{\partial\varphi}{\partial y} - \rho_3\frac{\partial^2 w}{\partial x^2} - \rho_4\frac{\partial^2 w}{\partial y^2}\right);$$
$$M_{xy} = 2\left(\rho_8\frac{\partial f}{\partial y} + \rho_9\frac{\partial\varphi}{\partial x} - \rho_{10}\frac{\partial^2 w}{\partial x\partial y}\right); \tag{8}$$
$$M_y = 2\left(\rho_5\frac{\partial f}{\partial x} + \rho_6\frac{\partial\varphi}{\partial y} - \rho_4\frac{\partial^2 w}{\partial x^2} - \rho_7\frac{\partial^2 w}{\partial y^2}\right); \quad Q_x = 2\rho_{11}f; \quad Q_y = 2\rho_{12}\varphi.$$

The constants ρ are calculated by formulas of the type $\rho_1 = \sum_{i=0}^{k} L_1^{(i)}\Omega_i$. Here

$$L_1^{(i)} = \frac{E_1^{(i)}}{1 - \nu_{12}^{(i)}\nu_{21}^{(i)}}; \qquad \Omega_i = \int_{d_i}^{d_{i+1}} F_i(z)\,z\,dz.$$

The substitution of (8) in (3) yields the following system of three equations:

$$\rho_{11}\frac{\partial f}{\partial x} + \rho_{12}\frac{\partial\varphi}{\partial y} = \frac{1}{2}g(x, y, t);$$
$$\rho_{12}\varphi = (\rho_5 + \rho_8)\frac{\partial^2 f}{\partial x\partial y} + \rho_9\frac{\partial^2\varphi}{\partial x^2} + \rho_6\frac{\partial^2\varphi}{\partial y^2} - (\rho_4 + \rho_{10})\frac{\partial^3 w}{\partial x^2\partial y} - \rho_7\frac{\partial^3 w}{\partial y^3}; \tag{9}$$
$$\rho_{11}f = \rho_1\frac{\partial^2 f}{\partial x^2} + \rho_8\frac{\partial^2 f}{\partial y^2} + (\rho_2 + \rho_9)\frac{\partial^2\varphi}{\partial x\partial y} - \rho_3\frac{\partial^3 w}{\partial x^3} - (\rho_4 + \rho_{10})\frac{\partial^3 w}{\partial x\partial y^2}.$$

If the layers are transversally isotropic, system (9) reduces to

$$\frac{\partial f}{\partial x} + \frac{\partial\varphi}{\partial y} = -\frac{1}{2\gamma_5}g(x, y, t);$$
$$\gamma_5\varphi = (\gamma_2 + \gamma_4)\frac{\partial^2 f}{\partial x\partial y} + \gamma_4\frac{\partial^2\varphi}{\partial x^2} + \gamma_1\frac{\partial^2\varphi}{\partial y^2} - \gamma_3\frac{\partial}{\partial y}\nabla^2 w; \tag{10}$$
$$\gamma_5 f = \gamma_1\frac{\partial^2 f}{\partial x^2} + \gamma_4\frac{\partial^2 f}{\partial y^2} + (\gamma_2 + \gamma_4)\frac{\partial^2 f}{\partial x\partial y} - \gamma_3\frac{\partial}{\partial x}\nabla^2 w.$$

The following equation is easily obtained from here

$$\nabla^4 w = \frac{1-\delta_2\nabla^2}{D^*} g(x, y, t). \tag{11}$$

Here D^* is the reduced flexural rigidity; δ_2 is a constant. Using equation (11) it is possible to calculate the deflections w, the natural lateral vibrations, and the critical states of the plate. The complete solution of the problem, which must include the determination of the stresses, necessitates, however, the knowledge of the functions f and φ. We assume additionally that the rotation $\omega_z = 0$. Following Khachaturyan /2/, we introduce the auxiliary function $\Phi(x, y, t)$, defined by the formulas

$$w = (1-\delta_2\nabla^2)\Phi; \qquad f = -\frac{\gamma_3}{\gamma_5}\frac{\partial}{\partial x}\nabla^2\Phi; \qquad \varphi = -\frac{\gamma_3}{\gamma_5}\frac{\partial}{\partial y}\nabla^2\Phi.$$

The substitution of these formulas in (10) yields

$$\nabla^4\Phi = \frac{1}{D^*}\left[q + \left(P_x\frac{\partial^2}{\partial x^2} + 2P_{xy}\frac{\partial^2}{\partial x\partial y} + P_y\frac{\partial^2}{\partial y^2} - \mu\frac{\partial^2}{\partial t^2}\right)(\Phi - \delta_2\nabla^2\Phi)\right].$$

The standard Sophie Germain equation $\nabla^4\Phi = \frac{q}{D^*}$ is obtained for the lateral bending. The applicability of this equation to the case of a single isotropic plate was studied in /2/. System (10) coincides in this case with system (1.7) in /2/.

Bending problems. Examples of calculation. Write in (11) $P_x = P_y = P_{xy} = \frac{\partial w}{\partial t} = 0$. We shall consider sandwich square hinged plates subjected to the load $q = q_m \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$.

1. The layers are transversally isotropic: $\nu_0 = \nu_1 = \nu$; $d_1 = \frac{h}{4}$; $\frac{E_1}{E_0} = n$; $\frac{E_0}{G_0'} = \frac{E_1}{G_1'} = s$. Write $\alpha'(z) = \frac{h^2}{4} - z^2$. The exact solution of the equation yields

$$w = \frac{24 q_m a^4 (1-\nu^2)}{\pi^4 E_0}\left[1 + \frac{\pi^2 s h^2}{160(1-\nu^2)a}\cdot\frac{8+113n+135n^2}{1+7n}\right]\times$$

$$\times \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}.$$

Writing $\nu = 0$, we obtain a complete coincidence with Khachatryan's solution /3/.

2. The layers are isotropic and of the same thickness, $\nu_0 = \nu_1 = \nu$; $\frac{E_1}{E_0} = n$ Then, writing $\alpha'(z) = \frac{h^2}{4} - z^2$, we obtain

$$\eta^{(2)} = 1 + \frac{2\pi^2}{45(1-\nu)}\left(\frac{h}{a}\right)^2\frac{1+122n+120n^2}{1+26n}; \tag{12}$$

writing $\alpha'(z) = \frac{h^4}{16} - z^4$ we obtain

$$\eta^{(4)} = 1 + \frac{8\pi^2}{189(1-\nu)}\left(\frac{h}{a}\right)^2\frac{1+1136n+1050n^2}{1+242n}. \tag{13}$$

Here $\eta = \frac{w}{w_0}$ is a correction to the classical theory. Take $\nu = 0.25$, and compare the results with those obtained by Melkonyan /4/ on the basis of the three-dimensional theory of thick plates. The results are given in the table.

TABLE

n	by /4/			by (12)		by (13)		$\frac{h}{a}$
	η_{max}	η_{min}	η_{av}	$\eta^{(2)}$	discrep. with η_{av}	$\eta^{(4)}$	discrep. with η_{av}	
10	2.117	2.011	2.071	2.185	5.5%	2.071	0	$\frac{1}{5}$
2	1.302	1.254	1.286	1.320	2.6%	1.297	0.9%	
0.5	1.132	1.096	1.121	1.154	2.7%	1.152	2.8%	
0.1	1.069	1.035	1.059	1.094	3.3%	1.111	4.9%	
10	1.286	1.274	1.282	1.296	1.1%	1.268	1.1%	$\frac{1}{10}$
2	1.073	1.064	1.069	1.080	1%	1.074	0.5%	
0.5	1.040	1.030	1.037	1.038	0.1%	1.038	0.1%	
0.1	1.042	1.033	1.038	1.023	1.4%	1.028	1%	

It is seen that, even for a plate of medium thickness, the discrepancy does not exceed 5.5% by formula (12) and 4.9% by formula (13). Formula (13) generally gives better results than (12).

The problems of the natural frequencies and critical forces were solved for plates of all possible combinations of rigidly supported and hinged edges (9 cases in all). The solution was conducted according to /5/ by means of trigonometric functions.

Example. The layers are transversally isotropic, the plate is hinged. The natural circular frequency is expressed by the formula

$$\omega_{mn}^2 = \frac{\pi^4 D^*}{a^2 b^2 \mu} \frac{\frac{m^4}{r^2} + 2m^2n^2 + n^4r^2}{1 + \delta_2 \frac{\pi^2}{b^2}\left(\frac{m^2}{r^2} + n^2\right)}. \tag{14}$$

Here $r = \frac{a}{b}$. In the case of a single-layer plate formula (14) coincides with the solution of Ambartsumyan and Khachatryan /6/. It is seen from (14) that the classical theory gives an overestimated value for the frequencies.

An experimental test of the theory was conducted on a three-layered beam. The middle layer was made of plexiglass, and the outer layers of steel, fastened by a cold-hardening glue. The beam data were: $a = 52$ cm; $h = 4.8$ cm; $d_1 = 1.2$ cm; $E_0 = 2.5\times10^4$ kg/cm^2; $\nu_0 = 0.36$; $E_1 = 2\times10^6$ kg/cm^2; $\nu_1 = 0.28$. The beam was hinged and loaded by a uniform pressure through a pressure chamber. The deflections were measured at 9 points of the beam by means of micrometers with an electronic contact indicator. The theoretical correction to the classical theory at the middle of the beam was $\eta = 2.08$. At a pressure $p_0 = 6$ kg/cm^2 the calculated deflection was 0.73 mm for a parabolic law for $\alpha'(z)$, while the measured deflection was 0.71 mm (a discrepancy of 3%). Similar results were obtained at the other points as well.

The creation and running of the installation made it possible to refine the testing technique and the design of a universal stand for plate testing, in particular in what concerns the end conditions.

The theory proposed here serves for calculating important parts and subassemblies of high-power turbogenerators and makes it possible to calculate laminated plates allowing for the lateral shear without conducting a preliminary calculation according to the classical theory.

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CALCULATING A CEILING IN THE FORM OF A SHELL RECTANGULAR IN THE PLAN

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This paper treats the problem of calculating the stresses in the structure represented in the figure. Shell 1 represents a part of an elliptical paraboloid, cut out from it by four vertical planes symmetrical relative to the paraboloid apex. The shell ends are connected with the side element 2, which rests on the vertical columns 3. It is also necessary to take into account the presence of the horizontal tie pieces 4, whose tension can be arbitrarily specified within certain limits.

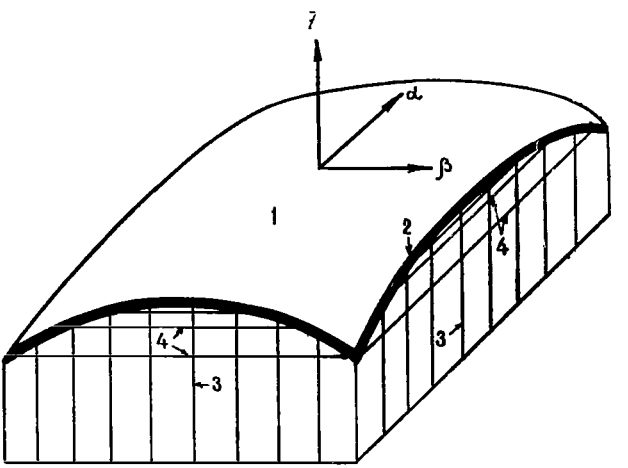

FIGURE.

The columns prevent the vertical displacement of the side element and, therefore, of the shell ends as well. There follows from here one of the boundary conditions for the shell: the absence of vertical displacements of the end. The other end conditions must be obtained as conditions of contact between the shell and the side element. In order to simplify the problem, we assume that the side element is in a state of compressive-tensile strain only, and opposes no resistance to bending and torsion. This is an assumption which is usually made in the practice of ceiling calculation (cf. /1/). It then follows that the bending moment and the

horizontal force perpendicular to the shell boundary vanish at the shell edge. In the direction of the boundary the shell is in a state of strain together with the side element, so that the deformations of the shell and the side element along the edge are equal, and the shearing forces with which the side element acts on the shell are equal and opposite to the forces with which the shell acts on the side element.

Draw a system of coordinate axes α, β on the shell surface. Let the α axis be the line of intersection of the middle surface of the shell with the plane perpendicular to the β axis and drawn through the origin, and let the β axis be the line of intersection of the middle surface with the corresponding plane perpendicular to the α axis (see figure). The advantage of this system of coordinate axes is that the shell ends coincide with the coordinate lines $\alpha = \pm \alpha_0$; $\beta = \pm \beta_0$. Its shortcoming is that it is a system of nonorthogonal curvilinear coordinates.

Using Gol'denveizer's symbols /2/, we write the boundary conditions on the line $\beta = +\beta_0$ in the form

$$uf_1' + vf_2 - \frac{w}{\Delta} = 0,$$

$$G_2 - H_2 \cos\chi = 0,$$

$$T_2 + \frac{\operatorname{ctg}\chi}{R_1'} H_2 + \frac{f_2 B}{\Delta}\left[N_2 + \frac{1}{A}\frac{\partial}{\partial\alpha}(H_2 \sin\chi)\right] = 0. \tag{1}$$

$$\frac{d}{d\alpha}\left(\frac{Q}{A}\right) + S_2 + \frac{H_2}{R_1' \sin\chi} - \frac{f_1' B}{\Delta}\left[N_2 + \frac{1}{A}\frac{\partial}{\partial\alpha}(H_2 \sin\chi)\right] = \Phi.$$

Here

$$Q = E\Omega\varepsilon_1. \tag{2}$$

The boundary conditions are written in the order in which they were discussed above.

Here T_2, S_2 are tangential stress resultants; N_2 is the shearing force; G_2, H_2 are the bending and torsional moments, respectively; u, v, w are components of the displacement vector; ε_1 is the shell strain along the α axis; Ω is the area of the side element; Q is the stress resultant acting in it; Φ is the distributed force of action of the tie pieces on the side element.

In addition, A, B, χ are, in Gol'denveizer's terminology, coefficients of the first quadratic form, and $\frac{A^2}{R_1'}, \frac{B^2}{R_2'}$ are coefficients of the second quadratic form of the middle surface of the shell.

The functions $f_1(\alpha)$, $f_2(\beta)$ enter in the shell equation

$$z = f_1(\alpha) + f_2(\beta).$$

The values of the other magnitudes are then

$$f_1 = \frac{\partial f_1}{\partial\alpha}; \quad f_2 = \frac{\partial f_2}{\partial\beta}; \quad \Delta^2 = 1 + (f_1')^2 + (f_2')^2.$$

The equations of the theory of shells, written in the system of curvilinear nonorthogonal coordinates defined above, are given in /2/. They are very bulky and their integration is complicated by the complexity of the boundary conditions (1).

The problem will be solved by the method of asymptotic integration /2, 3/, which uses efficiently the shell thinness. In order to describe the basic principles of this method as applied to the problem considered, we write the equations of the theory of shells in a special form. We introduce to that end matrix columns: T represents tangential stress resultants, N, shearing stress resultants, M, bending and torsional moments, Q, external forces, and v, displacements:

$$T=\begin{Vmatrix} T_1 \\ T_2 \\ S_2 \end{Vmatrix}; \quad N=\begin{Vmatrix} N_1 \\ N_2 \end{Vmatrix}; \quad M=\begin{Vmatrix} G_1 \\ G_2 \\ H_1 \\ H_2 \end{Vmatrix}; \quad Q=\begin{Vmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Vmatrix}; \quad v=\begin{Vmatrix} u \\ v \\ w \end{Vmatrix}. \tag{3}$$

The equations of equilibrium can then be written as follows:

$$HT+RN=Q; \tag{4}$$

$$N=CM. \tag{5}$$

Here H, R, C are matrices of differential operators. The form of these matrices will not be given here for lack of space. The elasticity relationships will be written as follows:

$$T=\frac{1}{q^2}IU; \tag{6}$$

$$M=\frac{1}{q^6}FU. \tag{7}$$

Here I and F are likewise matrices of differential operators, and q is a "large parameter," i.e., a magnitude inversely proportional to the square root of the ratio of the shell thickness to its characteristic dimension in the plan.

Following /2, 3/, we shall consider that the state of stress of the shell is made of the sum of three elementary states of stress: a membrane state of stress, a bending state of stress, and an end effect. All the magnitudes introduced are magnitudes of a certain order relative to q. If the derivative of any magnitude by the coordinate is of different order than the corresponding magnitude itself, we shall say that it varies substantially when differentiated by this coordinate; in the opposite case it does not vary substantially.

The membrane and bending states of stress are defined as states in which the components of the state of stress and the displacements do not vary substantially when differentiated by the coordinates α and β.

It follows from the elasticity equations (6), (7) and the equilibrium equations (4), (5) that in this case, if $T\neq 0$, then T and N are of different order of magnitude; the equilibrium equation (4) then reduces to

$$HT=Q. \tag{8}$$

In order to form the membrane state of stress, take the general solution of the homogeneous equation corresponding to (8):

$$HT=0.$$

At known T ($T\neq 0$) the displacements of the membrane state are determined as a particular solution of (6). Using the expression found, we

determine the moments by (7), and then the shearing forces by (5). By the same scheme, using a particular solution of (8), we form the solution corresponding to the external distributed load Q.

When forming the bending state of stress it is assumed that there are no extensions and shears in the middle surface, i.e.,

$$IU = 0. \tag{9}$$

The displacements of the bending state of stress are determined as a general solution of (9). We then find the moments by (7), the shearing forces by (5), and the tangential stress resultants by the equation

$$HT + RN = 0$$

We shall now investigate the asymptotic properties of these states of stress. Let the stress resultants of the membrane state of stress be of order q^λ:

$$T = T^{(m)} q^\lambda \tag{10}$$

($T^{(m)}$ are magnitudes independent of q).

It follows from equation (6) that the displacements are of order $q^{\lambda+2}$:

$$U = U^{(m)} q^{\lambda+2}, \tag{11}$$

from equation (7) that the moments are of order $q^{\lambda-4}$:

$$M = M^{(m)} q^{\lambda-4}, \tag{12}$$

and from equation (5) that the shearing forces are likewise of order $q^{\lambda-4}$:

$$N = N^{(m)} q^{\lambda-4}. \tag{13}$$

The particular solution corresponding to an external distributed load has the following asymptotic properties:

$$T = T^{(0)} q^0; \qquad v = v^{(0)} q^2; \qquad M = M^{(0)} q^{-4}; \quad N = N^{(0)} q^{-4}. \tag{14}$$

The properties of the bending state of stress are similarly found:

$$U = U^{(b)} q^{\mu+6}; \qquad M = M^{(b)} q^{\mu}; \qquad N = N^{(b)} q^{\mu}; \qquad T = T^{(b)} q^{\mu}. \tag{15}$$

The coefficients of q^l in (11) to (15) are independent of q.

The end effect is defined as a state of stress whose components vary substantially when differentiated by the coordinates.

We shall restrict ourselves to considering the end effect on the line $\beta = +\beta_0$. In this case the state of stress varies substantially when differentiated by the coordinate β. Passing over the details of the calculation, we note that in this case the normal displacements are of the form

$$w = q^4 e^{q(\beta-\beta_0)} \{q^{\nu_1} \varphi_1(\alpha) [\cos q(\beta-\beta_0) + \sin q(\beta-\beta_0)] - \\ - q^{\nu_2} \varphi_2(\alpha) \cos q(\beta-\beta_0)\}, \tag{16}$$

where $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$ are functions which do not vary substantially when differentiated by the argument; ν_1 and ν_2 are as yet undetermined numbers.

The components of the state of stress, the strains, and displacements on the end $\beta = \beta_0$ possess the following asymptotic properties /3/:

$$u = u^{(k)} \varphi_2 q^{\nu_2+3}; \qquad v = v^{(k)} \varphi_2 q^{\nu_2+3}; \qquad w = w^{(k)}(\varphi_1 q^{\nu_1+4} - \varphi_2 q^{\nu_2+4}); \tag{17}$$

$$\varepsilon_1 = -\frac{w^{(k)}}{R_1}(\varphi_1 q^{\nu_1+4} - \varphi_2 q^{\nu_2+4}); \qquad N_2 = N^{(k)}\varphi_2 q^{\nu_2+1};$$

$$H_2 = H^{(k)}\varphi_1 q^{\nu_1-1}; \quad G_2 = G^{(k)}\varphi_1 q^{\nu_1}; \tag{17 (cont'd)}$$

$$T_2 = 0\, q^{\nu_1+2} + 0\, q^{\nu_2+2}; \quad S_2 = 0 q^{\nu_1+2} + 0\, q^{\nu_2+2}.$$

The end effect on lines $\beta = -\beta_0$; $\alpha = \pm\alpha_0$ possesses similar properties.

We shall now establish the boundary conditions for each of the elementary states of stress. It was mentioned above that the complete solution of the equations of the theory of shells can be represented as the sum of the three states considered and of the solution corresponding to an external distributed load. We substitute these sums in the boundary conditions and keep only the highest-order terms in each of the states of stress. Terms of different order will then appear in the boundary conditions:

in the first condition:

$$0,\ \lambda,\ \mu+4,\ \nu_1+2,\ \nu_2+2;$$

in the second condition:

$$-4,\ \lambda-4,\ \mu,\ \nu_1,\ \nu_2-1;$$

in the third condition:

$$0,\ \lambda,\ \mu,\ \nu_1-4,\ \nu_2+1;$$

in the fourth condition:

$$0,\ \lambda,\ \mu,\ \nu_1+2,\ \nu_2+2.$$

It has been assumed here that the cross-sectional area of the side element is proportional to the shell thickness ($\Omega = \bar{\Omega}\, q^{-2}$). The magnitudes λ, μ, ν_1, ν_2 must be selected under the following considerations.

If, at given values of λ, $\mu \cdots$, we keep in each of the boundary conditions the terms of highest order only, we obtain certain boundary conditions. We require that λ, $\mu \cdots\cdots$ be such that these boundary conditions contain elements of all the states of stress, that the components of the particular solution $T_2^{(0)}$, $S_2^{(0)}$ appear in one of the boundary conditions at least, etc. Each of the states of stress must be determined uniquely with the aid of these conditions; no state of stress must vanish under any external load whatsoever; the boundary conditions must be free of contradictions. The following combination of values of the numbers λ, $\mu \cdots\cdots$ satisfies these requirements:

$$\lambda = 0;\ \mu = -4;\ \nu_1 = -2;\ \nu_2 = -2. \tag{18}$$

The boundary conditions reduce then to the following, when the magnitudes of the highest order only are kept in them:

$$\begin{aligned}
&1.\ f_1'\,(u^{(0)} + u^{(m)} + u^{(b)}) + f_2'\,(v^{(0)} + v^{(m)} + v^{(b)}) - \\
&\quad - \frac{1}{\Delta}(w^{(b)} + w^{(m)} + w^{(0)} - w^{(k)}\varphi_2) = 0. \\
&2.\ G^{(k)}\varphi_1 = 0. \\
&3.\ T_2^{(0)} + T_2^{(m)} = 0. \\
&4.\ \frac{d}{d\alpha}\left\{\frac{E}{A}\bar{\Omega}\left[\varepsilon_1^{(0)} + \varepsilon_1^{(m)} + \varphi_2\frac{w^{(k)}}{R_1'}\right] + S_2^{(0)} + S_2^{(m)}\right\} = 0.
\end{aligned} \tag{19}$$

The boundary conditions on the other ends of the shell are similarly formed.

The membrane state of stress is determined with the aid of the third boundary condition of this system, the end effect with the aid of the second and fourth conditions, and the bending state with the aid of the first condition.

The investigation of the asymptotic properties of the stresses in each of the elementary states of stress shows that at the values selected for λ, $\mu \cdots$ the stresses caused by the tangential stress resultants of the membrane state and by the moments in the end effect are of the highest order of magnitude. These stresses are the design stresses.

The solution of the corresponding membrane problem is given in /4/. Knowing it, the solution of the problem of the end effect can be found from the second and fourth boundary conditions of (19) for the end $\beta = \beta_0$, and from the corresponding boundary conditions on the other ends of the shell. If only the stresses have to be calculated, there is no need to solve the problem of the bending state of stress.

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STUDY OF THE MEMBRANE STRESSED STATE OF ELLIPSOIDAL SHELLS OF REVOLUTION WITH A REGULAR ROW OF HOLES

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The complexity of the problem of stress concentration in shells having several cuts makes it necessary to introduce various simplifying assumptions before solving it. One such simplification consists in splitting the stressed state into a membrane state and a state of the edge-effect type. The advantage of such a splitting lies in the relatively simple solution of the second problem for shells of positive curvature and in the possibility of a clear mathematical formulation of the first problem.

The equations of equilibrium of the membrane state are, in tensor notation /1/:

$$\begin{gathered} \nabla_{x^i} T^{ik} + X^k = 0, \\ b_{ik} T^{ik} + Z = 0, \\ T^{ik} = T^{ki}, \quad (i, k = 1, 2). \end{gathered} \tag{1}$$

The introduction of the complex stress function

$$\hat{W} = \sqrt{g}\,(T^{11} - iT^{12}) + \frac{Z}{2\sqrt{k}}$$

reduces the first two equations of system (1) to one:

$$\frac{\partial \hat{W}}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}}\left(\ln \sqrt{g\sqrt{k}}\right)\hat{W} = F,$$

$$F = \frac{1}{2}\sqrt{k}\,\frac{\partial}{\partial z}\left(\frac{Z}{k}\right) - \frac{\sqrt{g}}{2}(X^1 - iX^2),$$

$$z = x^1 + ix^2.$$

By selecting as coordinate lines the principal curvatures of the surface (orthogonal and conjugated-isometric) we reduce the third equation of (1) to

$$T^{11} + T^{12} = -\frac{Z}{\sqrt{gk}}.$$

The introduction of the function

$$\hat{U} = \sqrt{g\sqrt{k}} \cdot \hat{W}$$

leads to

$$\frac{\partial \hat{U}}{\partial \bar{z}} = \sqrt{g\sqrt{k}}\cdot F, \qquad (1^*)$$

whose general solution is

$$\hat{U} = U^*(z, \bar{z}) + U(z) + \varphi(z).$$

The circumferential force vector $\bar{T}_l$ is of the form

$$\bar{T}_l(s) = 2\,\mathrm{Re}\left[\frac{1}{i}\,\frac{U + U^* + \varphi}{\sqrt{g\sqrt{k}}}\left(\frac{dz}{ds}\right)^2 \bar{r}_{\bar{z}}\right] - \frac{Z}{\sqrt{k}}\,\mathrm{Re}\left[\frac{1}{i}\left(\frac{dz}{ds}\right)^2 \bar{r}_z\right].$$

By specifying the normal load component on the contour $T_{ll} = \bar{T}_l \cdot \bar{l} = f(s)$ we obtain Hilbert's boundary-value problem for the analytical function $U(z)$:

$$\mathrm{Re}\left[\frac{1}{i}\,\frac{U(z)}{\sqrt{g\sqrt{k}}}\left(\frac{dz}{ds}\right)^2 \bar{r}_{\bar{z}}\cdot\bar{l}\right] = \frac{f(s)}{2} + \frac{Z}{2\sqrt{k}}\,\mathrm{Re}\left[\frac{1}{i}\left(\frac{dz}{ds}\right)^2 \bar{r}_z\bar{l}\right] -$$

$$-\,\mathrm{Re}\left[\frac{1}{i}\,\frac{U^*}{\sqrt{g\sqrt{k}}}\left(\frac{dz}{ds}\right)^2 \bar{r}_{\bar{z}}\bar{l}\right] - \mathrm{Re}\left[\frac{1}{i}\,\frac{\varphi(z)}{\sqrt{g\sqrt{k}}}\left(\frac{dz}{ds}\right)^2 \bar{r}_{\bar{z}}\bar{l}\right]. \qquad (2)$$

A similar boundary-value problem is obtained if it is the tangential force component T_{ls} (or a combination of the two) that is specified.

The problem formulated will be solved for the case of ellipsoidal shells of revolution with a regular row of holes, i.e., with m identical holes having equidistant centers lying on the same latitude. We assume that the shell is cut along the equator.

If the holes have a plane contour, it is easily shown that the surface of the shell is homeomorphically mapped on region G^* of the η plane by means of the stereographic projection $\eta = l^2$. The contours of the holes are transformed as a result into the circles $\Gamma_k^* \,|\eta - \Delta e^{i\frac{2\pi}{m}(k-1)}| = \rho_0$, where $k = 1, 2, \cdots, m$. In addition to G^* we shall also make use of the region G, which is the canonical region of the concentric ring on which the region $\sum_{k=1}^{m} G_k^* + G^*$ is mapped by the fractional-linear transformation

$$w = \frac{\eta - c}{1 - \eta\bar{c}}.$$

Here

$$q = \frac{\rho_0}{1 - c\Delta}, \qquad \Delta + \rho_0 > c > \Delta - \rho_0.$$

The Hilbert problem (2) reduces for the case of loading by a normal pressure $Z_1 = P_n$ $X^1 = X^2 \equiv 0$ on plane η to

$$-\,\mathrm{Re}\left[\frac{U(\eta)}{\eta^2}\,\overline{e^{-2i\vartheta}}\right]_{\Gamma_0^* + \Gamma_1^*} = \frac{\sqrt{g}}{k^{1/4}}\,\frac{f(s)}{k_s} + U^*\mathrm{Re}\left[\frac{e^{2i\vartheta}}{\eta^2}\right] +$$

$$+\,\mathrm{Re}\left[\frac{\varphi(z)}{\eta^2}\,e^{2i\vartheta}\right] + \frac{P_n}{2}\,\frac{\sqrt{g}}{k^{1/4}}, \qquad (3)$$

$\vartheta = \frac{d\eta}{d\sigma}$ is the contour slope; σ is the arc along the contours on the plane; and

$$U^* = P_n \frac{a}{2\sqrt{b}} (b^2 - a^2) \operatorname{ch}^{-4} \frac{z + \bar{z}}{2}.$$

The function $U(z)$ contains as a separate term the analytical component $\varphi(z)$, which is very convenient for solving problems with equatorial cut.

It has been shown by Gol'denveizer /2/ that the function $U(z)$ which satisfies (3) must have poles of second order at least at points $\eta = 0$, $\eta = \infty$ in order that the stresses be finite.

The boundary-value problem (3) will be solved for cyclical boundary conditions in order to make the greatest possible use of the cyclical geometry of the region. To that end the solutions obtained for one cut (with region characteristics reflecting the presence of m cuts) will be extended out of the sector $-\frac{\pi}{m} \leqslant l^* (\varphi) \leqslant +\frac{\pi}{m}$ by rotating through the cycle angle $\varphi_0 = \frac{2\pi}{m}$. The region characteristics are the index $\varkappa$ and the corresponding regulating multiplier of the Hilbert problem $P(6)$ /3/.

The problem index $\varkappa = 2(m-1)$ and the corresponding regulating multiplier $P(6)$ are found from the equality on the contour

$$[\eta^\varkappa e^{i\gamma(\eta)}]_{\Gamma^*} = P(6)\, e^{-2i\vartheta}. \tag{4}$$

The separation of the argument in (4) leads to the Dirichlet problem for the function $\gamma(\eta)$ regular in G^*:

$$\operatorname{Re} \gamma(\eta) = -\varkappa \arg \eta - 2\vartheta. \tag{5}$$

By expanding the function $\gamma(\eta)$ in a series in the concentric region $G(w)$ we obtain

$$\gamma(w) = -\pi + i \sum_{k=1}^{\infty} B_k (w^k + w^{-k}) + \frac{2m}{i} \ln \frac{(1 + wc)^2}{w},$$

where

$$B_k = 2(m-1) \frac{(-1)^{k-1}}{k} q^k \left[\left(\frac{1}{q^k} - \frac{1}{c^k}\right) + \left(\frac{1}{q^k} - c^k\right)\right].$$

This function satisfies the condition of regularity

$$P(6) = |\eta^\varkappa e^{i\gamma(\eta)}| = e^{-\sum_{k=1}^{\infty} B_k \operatorname{Re}[w^k + w^{-k}]} \left|\frac{1}{\eta}\right|^2 \left|\frac{(w+c)(1+wc)}{w}\right|^{2m}.$$

It is easily shown that the continuation of $\eta^\varkappa e^{i\gamma(\eta)}$ by rotating through the cycle angle is analytical. The division of both sides of (3) by $P(6)$ yields

$$-\operatorname{Re}\left[\frac{U(\eta)}{\Phi(\eta)}\right]_\Gamma = \frac{\sqrt{g}}{k^{1/4}} \frac{f(s)}{P(6) \cdot k_s} + \operatorname{Re} U^* \left[\frac{1}{\Phi(\eta)}\right] + \operatorname{Re}\left[\frac{\varphi(\eta)}{\Phi(\eta)}\right] + \frac{P_n \sqrt{g}}{2k^{1/4} \cdot P(6)},$$

$$\Phi(\eta) = \eta^{\varkappa+2} e^{i\gamma(\eta)}, \quad \psi(\eta) = \frac{U(\eta)}{\Phi(\eta)}.$$

The function $\psi(\eta)$ can be an ordinary regular function, since $U(\eta) = \psi(\eta)\cdot\Phi(\eta)$ will then have at point $\eta = 0$ a zero of second order at least. We shall select the function $\varphi(\eta)$ in such a way as to make the right-hand part of (5) identically equal to zero on the shell equator Γ_0. By striking out the value of the shell parameters on Γ_0 we obtain

$$\varphi(\eta) = \sum_{mk\geqslant 2} \tilde{D}_{mk}\eta^{mk}.$$

It follows from the method itself of selection of $\varphi(\eta)$ that $U^* + \varphi(\eta)$ is the function of the stressed state of the shell in the absence of internal cuts, so that $U(\eta)$ is a purely concentration function.

The load on the inner cuts will be represented by the series

$$\frac{f(s)}{\sqrt{a^2(\eta\bar{\eta}-1)^2+4\eta\bar{\eta}}} = \tilde{E}_0 + \sum_l \mathrm{Re}\left[\left(\frac{\eta_1}{\rho_0}\right)^l \cdot \tilde{E}_l\right],$$

which is likewise equivalent to a Fourier series representation.

The initial problem has now been reduced to the Dirichlet problem (5) for the twice-connected region of the concentric ring G.

$$-\mathrm{Re}\,[\psi(\tau)]_\Gamma = \begin{cases} 0 & \Gamma_0 \\ \widetilde{f(\tau)} & \Gamma_1 \end{cases}.$$

We write the solution of this problem in the form of a Schwarz integral:

$$\Psi(w) = \frac{1}{2\pi}\int_\Gamma \mathrm{Re}\,[\Psi(\tau)]\,T(\tau, w)\,|d\tau| + iB.$$

The Schwarz operator for the region G is well known and is equal to /4/

$$T(\tau, w)\,|d\tau| = \frac{i\omega}{\pi^2}\cdot\zeta\left(\frac{\omega}{\pi i}\ln\frac{w}{\tau}\right)\frac{d\tau}{\tau}.$$

The condition of uniqueness of the solution is the integral equality

$$\int_{\Gamma_0} \tilde{f}(s)\,ds = \int_{\Gamma_1} \widetilde{f(s')}\,ds',$$

which reduces in our case to

$$\int_{\Gamma_1} \widetilde{f(s')}\,ds' = 0.$$

The following simplified value of the Schwarz operator was obtained by Mikhlin /5/

$$T(\tau, w)\,|d\tau| = i\left[-\frac{2}{\tau - w} + \frac{q^2}{w} - 2q^2\frac{w}{\tau^2}\right]d\tau + \frac{\ln w}{\ln q}\,i\,\frac{d\tau}{\tau}.$$

The uniqueness condition requires that the term containing $\ln w$ be absent. The Mikhlin simplification is based on the smallness of q^2

compared with unity. Since $q = \frac{p_0}{1 - c\Delta}$, it is obvious that the hole in the shell must be sufficiently small and not too close to the edge.

The same simplification leads to the following expression for $P(6)$ on Γ_1:

$$P(6) = \frac{1}{\eta\bar{\eta}}\left[\frac{\prod_{k=1}^{4}(w+\beta_k)}{q^2w^2}\right]^m\left[\frac{q^4w^4}{\prod_{k=1}^{4}(w+\alpha_k)}\right]^{m-1},$$

where α_k and β_k are known constants, determined as a result of the summation of the series for $\gamma(w)$.

Considering the right-hand side of the Dirichlet problem on Γ_1, we find that $f(\tau)$ has the following singularities inside the circle $|w| \leqslant q$:

1) a fourth-order pole at point τ^{**} with main Laurent part $G_4^{(1)}\left(\frac{1}{\tau - \tau^{**}}\right)$,

2) a pole of order $s = \max\{l + m + 1,\ m(k+2)\}$ at point q^2c with $G_s^{(2)}\left(\frac{1}{\tau + cq^2}\right)$,

3) a pole of order $2(m-1)$ at point q^2/c with $G_{2(m-1)}^{(3)}\left(\frac{1}{c\tau + q^2}\right)$,

4) a pole of order $2(m-2)$ at point $\tau = 0$ with $G_{2(m-2)}^{(4)}\left(\frac{1}{\tau}\right)$.

We need, in addition, the Laurent parts of the expressions at the singular points τ_p:

$$-\frac{\operatorname{Re}[\psi(\tau)]}{\tau^2} \div \bar{G}_{r_p}^{(p)}\left(\frac{1}{\tau - \tau_p}\right); \qquad -\frac{\operatorname{Re}[\psi(\tau)]}{\tau} \div \bar{G}_{r_p}^{(p)}\left(\frac{1}{\tau - \tau_p}\right).$$

Here

$$G_{r_p}^{(p)}\left(\frac{1}{\tau - \tau_p}\right) = \sum_{k=1}^{r_p}\frac{a_k^{(p)}}{(\tau - \tau_p)^k}.$$

The knowledge of the principal parts $G_{r_p}^{(p)}$ automatically gives us the value of the constants $a_k^{(p)}$. The problem solution can now be represented as follows:

$$\psi(w) = -\frac{i}{2\pi}\int_{\Gamma_1}\left\{\sum_{p=1}^{4}G_{r_p}^{(p)}\left(\frac{1}{\tau - \tau_p}\right) + \text{analytical function}\right\}\times$$
$$\times\left[-\frac{2}{\tau - w} + \frac{q^2}{w} - 2q^2\frac{w}{\tau^2}\right]d\tau + iB.$$

Using Cauchy's residue theorem we obtain

$$\psi(w) = -\left\{2\sum_{p=1}^{4}G_{r_p}^{(p)}\left(\frac{1}{w - w_p}\right) + \frac{q^2}{w}A - 2q^2w\bar{A}\right\} + iB,$$

$$A = a_1^{(1)} + a_1^{(2)} + a_1^{(3)} + a_1^{(4)}.$$

$$\bar{A} = \bar{a}_1^{(1)} + \bar{a}_1^{(2)} + \bar{a}_1^{(3)} + \bar{a}_1^{(4)}.$$

The uniqueness condition gives after integration

$$\bar{A} = \bar{a}_1^{(1)} + \bar{a}_1^{(2)} + \bar{a}_1^{(3)} + \bar{a}_1^{(4)}.$$

The approximation $q^2 \ll 1$ reduces the stress function to

$$U(\eta) = \left[-iB - 2q^2 w\bar{A} + \frac{q^2}{w} A + 2\sum_{p=1}^{4} G_{r_p}^{(p)}\left(\frac{1}{w - w_p}\right)\right] \times$$

$$\times \left[\frac{(w+c)(1+wc)}{w}\right]^{2m} \left[\frac{w^2}{\prod_{k=1}^{4}(w+\alpha_k)}\right]^{2(m-1)}$$

The solution is easily obtained after the separation in $\mathrm{Re}\,[\psi(\tau)]$ of the principal Laurent parts at the singular points indicated.

The examination of $U(\eta)$ leads to the conclusion that it describes a stress field decreasing away from the contour $|w| = q$, so that the stress concentration will be maximum on the contour; it is also seen that the concentration increases with the increase of q and m.

Another solution of the problem formulated is possible in the particular case of a large number of holes m.

The boundary condition of the Dirichlet problem can be written in the form

$$-\mathrm{Re}\,[\psi(\tau)] = \begin{cases} 0 & \text{on } \Gamma_0 \\ \hat{f}(qe^{i\theta})\dfrac{1}{p^*(\tau)} e^{-4(m-1)u(\theta)} & \text{on } \Gamma_1. \end{cases}$$

Using the Schwarz operator, we obtain the following solution of the problem:

$$\psi(w) = \frac{1}{2\pi} \cdot \frac{i\omega}{\pi^2} \oint_{|\tau|=q} \frac{\hat{f}(\tau)}{p^*(\tau)} e^{-4(m-1)u(\theta)} \zeta\left(\frac{\omega}{\pi i} \ln \frac{w}{\tau}\right) d\theta, \tag{6}$$

where $\hat{f}(\tau)$ depends on the supposedly large number m. We represent $\hat{f}(\tau)$ in the form (on Γ_1)

$$\hat{f}(\tau) = \hat{f}_2(\tau)\, e^{2(m-1)\tilde{u}(\theta)} + \hat{f}_1(\tau),$$

after which the integral (6) is easily calculated by Laplace's asymptotic method:

$$\psi(w) = \frac{1}{2\pi} \cdot \frac{i\omega}{\pi^2} \sqrt{\frac{\pi}{m-1}} \left\{ \frac{\hat{f}_1(q)}{p^*(q)} \frac{e^{-4(m-1)u(\theta)}}{\sqrt{-2u''(0)}} \zeta\left(\frac{\omega}{\pi i} \ln \frac{w}{q}\right) + \right.$$

$$+ \frac{e^{2(m-1)\tilde{u}(\theta_1)}}{p^*(qe^{i\theta_1})} \sqrt{\frac{1}{-\tilde{u}''(\theta_1)}} \left[\hat{f}_1(qe^{i\theta_1}) \zeta\left(\frac{\omega}{\pi i} \ln \frac{w}{qe^{i\theta_1}}\right) + \right.$$

$$\left.\left. + \hat{f}_1(qe^{-i\theta_1})\, \zeta\left(\frac{\omega}{\pi i} \ln \frac{w}{qe^{-i\theta_1}}\right)\right]\right\}. \tag{7}$$

The uniqueness condition is easily written. Since

$$\zeta(u) = cU + f\left(e^{\frac{\pi i}{\omega} u}\right),$$

only the first term can lead to a nonuniqueness. $\delta\psi(w)_L = 2\pi i c\,[\psi(w)]_{\zeta=1}$ after which the solution simplifies and reduces to

$$\psi(w) = \frac{1}{2\pi} \cdot \frac{i\omega}{\pi^2} \sqrt{\frac{\pi}{m-1}} \sqrt{\frac{1}{-\tilde{u}''(\theta_1)}} \hat{f}_1(qe^{i\theta_1}) \frac{e^{2(m-1)\tilde{u}(\theta_1)}}{p^{\cdot}(qe^{i\theta_1})} \times$$

$$\times \left[\zeta\left(\frac{\omega}{\pi i} \ln \frac{w}{q}\right) - \zeta\left(\frac{\omega}{\pi i} \ln \frac{w}{qe^{i\theta_1}}\right) \right].$$

The uniqueness condition leads thus in both solutions to the necessity of having one arbitrariness in the contour loading.

The geometrical formulation of the conditions of existence of a membrane state leads to six more equilibrium conditions of the shell as a rigid whole, which must always be fulfilled.

Geometrical conditions which ensure trivial infinitesimal bendings are the rigid edging $\delta k_s = 0$ of the inner holes and the rigid clamping of the large contour. The triviality of the bendings for the case $\delta k_s = 0$ at $m \geqslant 2$ has been proved by Vekua.

By completing the half-shell by a second symmetrical one we obtain the fulfillment of all conditions of Vekua's theorem. The pattern symmetry gives in this case a zero displacement along the meridian of Γ_0^*, which represents the tangential boundary condition at rigid clamping.

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STABILITY OF AN ELASTIC-PLASTIC CYLINDRICAL SHELL UNDER DYNAMIC LOADING

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When a shell is subjected to impact loading, the appearance of residual dents caused by buckling occurs under considerably higher stresses than in the static case. It is expedient to classify dynamic loads according to the extent to which the residual dents lower the static stability of the shell.

We shall distinguish between three characteristic dynamic loads:

1) Safe load, whose action does not lead to the appearance of residual dents.

2) Dangerous load, under whose action there appear visually noticeable residual dents, but so shallow that the value of the critical load is only negligibly reduced. By definition, a load will be called dangerous if it reduces the static critical load by 10%.

3) Collapse load, which causes the appearance of dents so deep that the shell becomes practically incapable of taking a static load.

This paper treats the problem of determining the dangerous dynamic load. This load characterizes the carrying capacity of the shell better than the safe load, and it has the added advantage of being easier to determine experimentally (by measuring the depth of the residual dents).

The formation of residual dents is linked with the appearance of plastic deformations. It follows that the plastic properties of the materials must be taken into account when studying the dangerous loads. On the other hand, the shell deflections under a dangerous load are small, so that the problem can be solved within the frame of the geometrically linear problem.

Consider the buckling of a circular cylindrical shell immersed in water and subjected to, besides hydrostatic pressure, an impact load applied uniformly to the lateral surface of the shell. The shell is supported by equidistant lateral stiffening ribs; it is assumed that they do not oppose resistance to the pressure applied on the shell and do not lose their circular shape. Buckling will therefore occur only between the ribs.

We shall study the stability of the middle portion of a fairly lengthy shell; the edge effect will accordingly be neglected, and it will also be assumed that the distance between the stiffening ribs does not decrease under dynamic loading.

The shell has an initial deflection coinciding with the form of the buckling.

The system of equations of motion and continuity of a circular cylindrical

shell is, in a geometrically linear formation:

$$M_{1xx} + \frac{2}{r} H_{x\varphi} + \frac{1}{r^2} M_{2\varphi\varphi} + \frac{1}{r} T_2 - T_1 (\varkappa_1 + \varkappa_1^{(0)}) -$$

$$- T_2 (\varkappa_2 + \varkappa_2^{(0)}) - 2S (\gamma + \gamma^{(0)}) = p(t) - \rho \delta \ddot{w}, \tag{1}$$

$$\frac{1}{r^2} \varepsilon_{1\varphi\varphi} - \frac{1}{r} \omega_{x\varphi} + \varepsilon_{2xx} = - \frac{1}{r} w_{xx} . \tag{2}$$

Here r and δ are the radius and thickness of the shell; T_1, T_2, S, M_1, M_2, and H are forces and moments per unit length; ε_1, ε_2, and ω are strains of the middle surface; $\varkappa_1$, $\varkappa_2$, and γ are variations of the curvature and torsion; the superscript "0" designates the values corresponding to the initial deflection; w is the normal component of the shell deflection; ρ is the density of the shell material; $p(t)$ is the external pressure; x and φ are cylindrical coordinates.

The subscripts "x" and "φ" designate differentiation by the coordinates; the dot designates differentiation by time. The subscripts "1" and "2" designate the longitudinal and lateral directions of the magnitudes, respectively.

The shell material will be considered as elastic-plastic. It will be assumed that the stress components remain constant in the plasticity zones. The forces and moments will then be expressed by the following formulas:

$$T_1 = \frac{E\delta}{1-\mu^2} [\varepsilon_1 + \mu\varepsilon_2 - s\delta (\varkappa_1 + \mu\varkappa_2)] , \tag{3}$$

. . . . . .

$$M_1 = Di (\varkappa_1 + \mu\varkappa_2), \tag{4}$$

. . . . . .

where

$$s = \frac{1}{2} [(1-\bar{a})^2 - 2c(1-\bar{a})], \tag{5}$$

$$i = \bar{a} [6\bar{c}(1-\bar{c}) + \bar{a}(3-6\bar{c}-2\bar{a})], \tag{6}$$

$$D = \frac{E\delta^3}{12(1-\mu^2)}, \quad \bar{a} = \frac{a}{\delta}, \quad \bar{c} = \frac{c}{\delta},$$

a is the height of the elastic core of the section; c is the height of the plastic zone of extension; E and μ are the modulus of elasticity and the Poisson ratio.

By using the following expressions for the curvature and the torsion

$$\varkappa_1 = - w_{xx}, \quad \varkappa_2 = - \frac{1}{r^2} w_{\varphi\varphi}, \quad \gamma = - \frac{1}{r} w_{x\varphi} \tag{7}$$

and introducing a stress function by the formulas

$$T_1 = \frac{1}{r^2} \Phi_{\varphi\varphi}, \; T_2 = \Phi_{xx}, \; S = - \frac{1}{r} \Phi_{x\varphi}, \tag{8}$$

we can transform equations (1) and (2) to the form

$$DL\ (i,\ w) + \frac{1}{r}\ \Phi_{xx} + \frac{1}{r^2}\ \Phi_{\varphi\varphi}\ (w_{xx} + w^0_{xx}) + \frac{1}{r^2}\Phi_{xx}(w_{\varphi\varphi} + w^0_{\varphi\varphi}) -$$

$$- \frac{2}{r^2}\ \Phi_{x\varphi}\ (w_{x\varphi} + w^0_{x\varphi}) = p(t) - \rho\delta\ \ddot{w}, \tag{9}$$

$$\frac{r^2}{E}\ \nabla^2\nabla^2\Phi = -\ r w_{xx} + \delta\ (s_{xx}\ w_{\varphi\varphi} - 2 s_{x\varphi}\ w_{x\varphi} + s_{\varphi\varphi}\ w_{xx}\ ), \tag{10}$$

where

$$L\ (i,\ w) = (i \cdot w_{xx}\ )_{xx} + \frac{2}{r^2}\ (i \cdot w_{x\varphi}\ )_{x\varphi} + \frac{1}{r^4}\ (i \cdot w_{\varphi\varphi})_{\varphi\varphi} +$$

$$+ \frac{\mu}{r^2}\ [(i \cdot w_{\varphi\varphi})_{xx} + (i \cdot w_{xx}\ )_{\varphi\varphi} - 2\ (i \cdot w_{x\varphi})_{x\varphi}]. \tag{11}$$

Palii /1/ obtained similar equations for the case of a static load and for plasticity zones by the compressed fibers of the shell only.

The following assumptions will be used in solving system (9), (10).

Following /1/, we shall neglect the terms in (10) caused by the presence of plasticity zones, since this does not introduce a substantial error in the determination of the stress function.

The shell deflection will be described, just as for the case of an elastic shell, by the following expressions:

$$w = f_0 + f_1 \cos\frac{\pi x}{l}\ \cos\ n\varphi, \tag{12}$$

$$w^0 = f_{\text{in}} \cos\frac{\pi x}{l}\ \cos\ n\ \varphi, \tag{13}$$

where l is the distance between the stiffening ribs and n is the number of waves into which the shell buckles.

We finally take the function i as equal to its minimum value

$$i = i_0, \tag{14}$$

where i_0 is the value of the function i at the point with maximum variation of the curvature.

Then

$$L\ (i,\ w) = i_0\ \nabla^2\nabla^2 w \tag{15}$$

and equation (9) will be the same as for an elastic shell, but with a variable flexural rigidity.

It should be noted that the last two assumptions compensate for each other to a certain extent, since the first one leads to an increase of the shell resistance to bending while the second one decreases it. Similar assumptions have been used in the analysis of the static stability of elastic-plastic bars /2/.

The influence of the liquid on the bending motion of the shell will be allowed for by means of the coefficient of attached mass, which can be calculated either by the formula of /3/ or by the approximate formula obtained by Novozhilov:

$$k_{\text{at}} = 1 + \frac{\rho_0\ r}{\rho\delta\ \sqrt{n^2 + \alpha^2}}, \tag{16}$$

where $\alpha = \frac{\pi r}{l}$; ρ_0 is the water density.

In the case of axisymmetrical compression the liquid resistance is determined by means of the hypothesis of plane reflection, i.e., we shall consider it as proportional to the velocity of this motion.

The pressure acting on the lateral surface of the shell will then be equal to

$$p(t) = p_0 + p_d(t) - \rho_0 a_0 \dot{f}_0, \tag{17}$$

where p_0 is the hydrostatic pressure; $p_d(t)$ is the dynamic pressure; a_0 is the velocity of sound in water.

Equation (9) will be integrated by the Bubnov-Galerkin method (equation (10) is integrated rigorously). We obtain as a result two equations, which are written as follows in dimensionless form:*

$$\ddot{\zeta}_1 = -\omega_0^2 i_0 \zeta_1 - \omega_1^2 \zeta_1 + B(\bar{\sigma}_d + \bar{\sigma}_0)(\zeta_1 + \zeta_{in}), \tag{18}$$

$$\ddot{\zeta}_0 = -\frac{1}{1-\mu^2}\zeta_0 - \beta\dot{\zeta}_0 + \bar{p}_d(\tau), \tag{19}$$

where

$$\omega_0^2 = \frac{(1+\lambda^2)^2 \eta^2}{12(1-\mu^2)k_{at}}, \quad \omega_1^2 = \frac{\lambda^4}{(1+\lambda^2)^2 k_{at}}, \quad B = \frac{1+\mu\lambda^2}{k_{at}}.$$

$$\lambda = \frac{\pi r}{nl}, \quad \eta = \frac{n^2\delta}{r}, \quad \zeta = \frac{f}{\delta}, \quad \beta = \frac{\rho_0 a_0}{\rho a_b}\frac{r}{\delta},$$

$$\bar{\sigma}_d = \frac{1}{1-\mu^2}\zeta_0, \quad \bar{\sigma}_0 = \frac{p_0 r^2}{E\delta^2}, \quad \bar{p}_d = \frac{p_d r^2}{E\delta^2}, \quad \tau = \frac{t a_b}{r}, \quad a_b = \sqrt{\frac{E}{\rho}}.$$

Equations (18) and (19) will be solved under the following initial assumptions:**

$$\zeta_1 = \zeta_{in}\frac{p_0}{\bar{p}_* - \bar{p}_0}, \quad \dot{\zeta}_1 = 0, \quad \zeta_0 = \dot{\zeta}_0 = 0, \tag{20}$$

where

$$\bar{p}_* = \frac{1}{1+{}^1/_2\lambda^2}\left[\frac{(1+\lambda^2)^2\eta}{12(1-\mu^2)} + \frac{\lambda^4}{(1+\lambda^2)^2\eta}\right]$$

is the upper critical pressure on the shell.

The height $\bar{a}$ of the elastic core of the section and the height $\bar{c}$ of the plastic extension zone will be determined with the aid of Mises' yield condition

$$\sigma_i = \sigma_y, \tag{21}$$

where σ_i is the stress intensity and σ_y is the yield point of the material.

The calculations have established that condition (21) can be replaced in our case by the simpler approximate expression

$$\sigma_2 = 1.08\,\sigma_y. \tag{22}$$

* It is assumed that the maximum circumferential stresses in the shell are below the yield point.

** It is assumed that the static pressure does not lead to plastic deformations. The function ζ_0 in (19) and (20) describes the deflection due to the dynamic load only.

The following expressions are then obtained with the aid of the hypothesis of straight normals and the condition of force equilibrium in the shell section:

$$A \leqslant 2\,(1-\sigma_2^*)\,\text{(elastic shell)}$$

$$A = \frac{\zeta_1 \eta\,(1+\mu\lambda^2)}{\bar{\sigma}_y}, \tag{23}$$

$$\bar{a} = 1, \quad \bar{c} = 0, \tag{24}$$

at

$$2\,(1-\sigma_2^*) \leqslant A \leqslant \frac{2}{1-\sigma_2^*} \quad \text{(one-sided plasticity)}$$

$$\bar{a} = \left[\frac{2\,(1-\sigma_2^*)}{A}\right]^{1/2}, \quad \bar{c} = 0, \tag{25}$$

at

$$A \geqslant \frac{2}{1-\sigma_2^*} \quad \text{(two-sided plasticity)}$$

$$\bar{a} = \frac{2}{A}, \quad \bar{c} = \frac{1-\sigma_2^*}{2} - \frac{\bar{a}}{2}. \tag{26}$$

Here

$$\bar{\sigma}_y = \frac{1.03\,\sigma_y r}{E\delta}\,(1+\mu^2), \qquad \sigma_2^* = \frac{\sigma_{02}}{1.08\,\sigma_y}, \tag{27}$$

$\sigma_{02} = \frac{T_2}{\delta}$ is the mean lateral stress in the shell section.

Equation (18) with formulas (6), (25), and (26) is correct as long as the plastic deformations continue to increase.

The material will behave elastically during unloading. We assume arbitrarily that unloading starts simultaneously at all fibers of the section, beginning with the moment when the bending deflections reach the maximum.

During unloading the term $\omega_0^2\, i_0\, \zeta_1$ in (18) must be replaced by the expression

$$\omega_0^2\ [i_0^*\, \zeta_{1\max} - (\zeta_{1\max} - \zeta_1)], \tag{28}$$

where $\zeta_{1\max}$ is the maximum deflection, and i_0^* is the value of the function i_0 at the maximum deflection.

In order to determine the dangerous load it is first necessary to find the dependence of the residual deflections of the shell on the load magnitude. We therefore plot the functions $\zeta_0(\tau)$ and $\zeta_1(\tau)$, either by a numerical method or by means of an electronic computer. The typical shape of these functions is represented in Figure 1.

It is seen from the figure that only an axisymmetrical compression of the shell takes place at the beginning of the loading, and that the bending deflections develop after a certain delay. A certain time after the dynamic load has been removed there start free vibrations of the shell, which are described by the equation

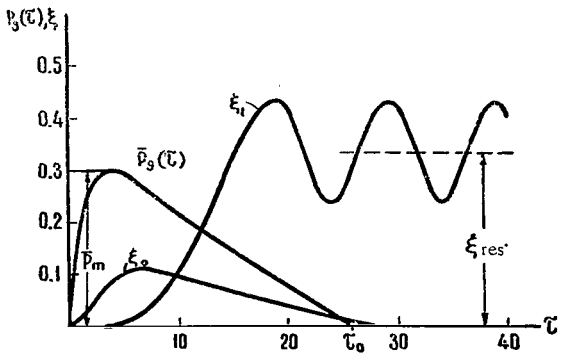
$$\ddot{\zeta}_1 = -\,\omega_0^2\ [i_0^*\, \zeta_{1\max} - (\zeta_{1\max} - \zeta_1)] - \omega_1^2\, \zeta_1 \tag{29}$$

(in the given case we assume for simplicity that $p_0 = 0$).

Since these vibrations must be dampened due to the internal friction in the metal, the residual deflection is found by equating the inertial term to zero:

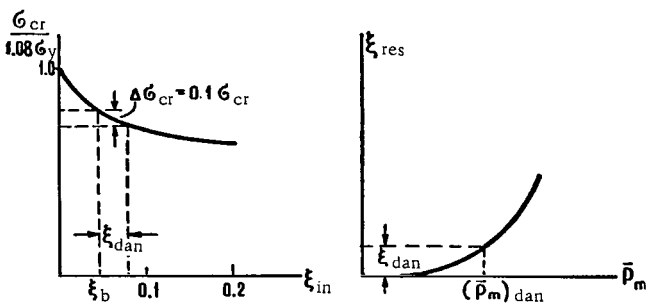
$$\zeta_{res} = \frac{\omega_0^2}{\omega_0^2 + \omega_1^2}(1 - i_0^*)\,\zeta_{1max}. \qquad (30)$$

We thus obtain the relationship between the residual deflection and the maximum magnitude of the load $\zeta_{res} = f(\bar{p}_m)$ (Figure 2).



Dynamic shell deflections (ℓ = 15.5 cm; δ = 0.23 cm; ℓ = 13.0 cm; σ_y = 4500 kg/cm^2).

FIGURE 1.



Scheme of determination of the dangerous dynamic load.

FIGURE 2.

It is further necessary to obtain the dependence of the static critical load on the size of the initial dents.

Such a problem was solved in /1/. The desired relationship can also be obtained with the aid of equations (18) and (19). The shape of the function $\sigma_{cr}(\zeta_{in})$ is represented in Figure 2. Using this curve it is easy to determine the magnitude of the deflection which decreases the static critical load by 10% (this deflection corresponds by hypothesis to a dangerous dynamic load). With this deflection known, the dangerous deflection is determined from the curve $\zeta_{res}(p_m)$.

The laws of variation of the dangerous load were investigated by calculating several particular cases.

The initial irregularities of the shells were selected by the formula /4/

$$\zeta_{in} = \frac{0.2}{n}.$$

The number n of waves into which the shell buckles was selected on the basis of the experimental results.

The dynamic load had the shape represented in Figure 1. Its time of application was varied within the limits $5 \leqslant \tau_0 \leqslant 50$.

It is convenient to characterize the dangerous load by the overload coefficient

$$\varkappa = \frac{\sigma_{d.max}}{\sigma_{cr} - \sigma_0}, \tag{31}$$

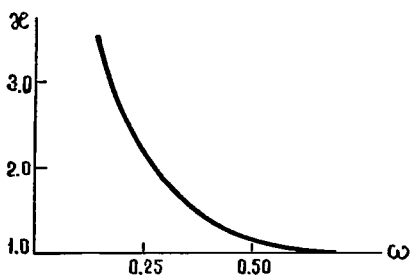
where $\sigma_{d|max}$ are the maximum circumferential stresses at dangerous load under the assumption that the shell does not buckle but is only compressed; σ_{cr} are critical stresses; σ_0 are stresses due to the hydrostatic pressure.

The calculations established that the main factors influencing the magnitude of the overload coefficient are the frequency of the natural vibrations of the shell ω, the value of the yield point σ_y, and the time of action of the load τ_0.

The lower the frequency of natural vibrations, determined by the formula

$$\omega^2 = \frac{\overline{p}_0 \eta}{k_{at}}\left(1 - \frac{\overline{p}_0}{p_*}\right)\left(1 + \frac{1}{2}\lambda^2\right), \tag{32}$$

the slower is the development of buckling deflections, and the higher is the overload coefficient (Figure 3). There results the seemingly paradoxical fact that the dangerous load is the same for a shell with small critical pressure and low frequency and a shell with considerably higher critical stresses and natural frequency.



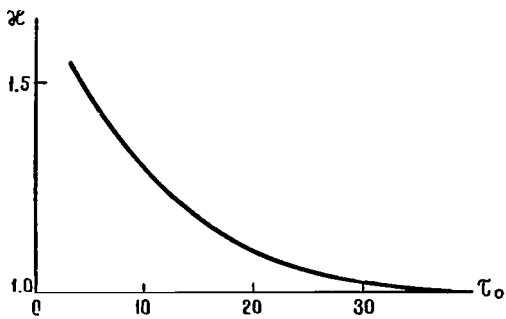
The overload coefficient as a function of the time of application of the load for the shell $r = 15.5$ cm, $\delta = 0.23$ cm, $\ell = 13.0$ cm, $\sigma_y = 4500$ kg/cm².

FIGURE 3.

The existence of a static load influences in the first place the frequency of the natural vibrations, which is decreased as a result. This leads to an increase in the overload coefficient.

The overload coefficient decreases with the increase of the time of action of the load (Figure 4). When the time of action $\tau_0 \geqslant 3T$ (where $T = \frac{2\pi}{\omega}$ is the period of natural vibrations of the shell), the overload coefficient approaches unity, i.e., buckling occurs under such a load at the same stresses as in the static case (if the yield point σ_y is the same in the static and dynamics cases).

The experiments corroborated qualitatively these laws.



The overload coefficient as a function of the frequency of natural vibrations of the shell (τ_0 = 25, ℓ = 15.5 cm, δ = 0.23 cm, σ_y = 4500 kg/cm^2.

FIGURE 4.

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ON THE VIBRATIONS OF HOMOGENEOUS AND SANDWICH PLATES

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This communication is devoted to a logically substantiated derivation of the equations of flexural vibrations of thin homogeneous and unlimited sandwich plates with plane-parallel boundaries of separation. The reasoning is based on the example of a homogeneous elastic plate $|z| \leqslant h$ defined in the Cartesian coordinate system x, y, z.

1. We start by formulating the basic propositions to be taken into account in any derivation of the equations interesting us.

The following equation is used in the engineering theory of vibrations for the displacement w of the points of the middle plane $z = 0$ of a thin plate in the direction of the normal (i.e., the z-axis):

$$L[w(x, y, t)] = M(F). \tag{1}$$

It is assumed that the linear operator L here contains only a finite number of components with derivatives of different orders by the variables x, y, and t. The right-hand part of (1) is determined by the totality of forces applied to the plate.

It is natural to require that the form of the operator L be independent of the forces actually applied to the plate and of the spectrum of vibrations produced in the plate by these forces (proposition 1).

Since the middle plane of the plate is inaccessible to direct observation, and since, on the other hand, it is the displacements of this plane which appear in (1), there must exist an exact algorithm by means of which the displacements $w(x, y, t)$ of the middle plane can be determined from the results of observations of the displacement of the plate boundaries $z = \pm h$. It is assumed in engineering that the displacement of the middle plane coincide sufficiently accurately with the displacements of the corresponding points of the boundary planes $z = \pm h$ (proposition 2).

The classical Kirchhoff equations have successfully been used for a long time in the case of low-frequency flexural vibrations of the plates. It is therefore necessary to require that any "refined" engineering equations differ at low frequencies from the Kirchhoff equations by small corrective terms only (proposition 3).

Finally, in the derivation of the equations of the engineering theory of vibrations it is necessary to strive to achieve the best possible correspondence between equations (1) and the true process of vibrations of the plate as a three-dimensional elastic body (proposition 4).

Note that the basic difference between our reasoning and that of the other authors of "refined" equations in the engineering theory of thin plates lies in the interpretation of "proposition 4." The authors known to us allow for this proposition by formulating some hypotheses relative to either the pattern of deformation of the vibrating plate /1, 2/ or the order of magnitude of different terms in the exact equations of motion of the theory of elasticity; the latter procedure enables them to expand the displacements and stresses in series by a small (dimensional?) parameter /3, 4/. Such an approach is not always justified and does not always lead to a correct result.

It is, however, possible to solve exactly the problems of the dynamic theory of elasticity for an infinite layer $|z| \leqslant h$ with a subsequent thorough investigation of the displacement fields excited in it. Using the results obtained it is possible to elucidate completely the problem interesting us.

2. If any bending forces* varying arbitrarily with time are applied on an infinite layer $|z| \leqslant h$ (either on the lateral surfaces or distributed through the layer thickness), a strict solution of the corresponding problem of the dynamic theory of elasticity is easily obtained in a form convenient for analysis /5-8/.

Formulas of the following form can be written for the components $u_x = u$, $u_y = v$, and $u_z = w$ of the field of elastic displacements:

$$w(x, y, z, t) = w_0 + w_1 + w_2 + \sum_{n=3}^{\infty} w_n; \tag{2}$$

the terms in these formulas have quite different properties, in particular in what concerns their spectral composition.

The displacements u_0, v_0, w_0 describe a purely forced process of vibrations and become equal to zero after the external force is removed. They can therefore be neglected if we are interested in the free vibrations of the plate produced by arbitrary bending forces acting during a time interval $0 \leqslant t \leqslant T$.

Each group (u_m, v_m, w_m) of the remaining terms of the right-hand parts of (2) strictly satisfies the equations of motion of the theory of elasticity of a free layer. These terms differ from each other by the behavior of the phase velocities and the spectral composition.

The terms (u_1, v_1, w_1) can contain any frequencies in the interval $0 < v < \infty$. The following formulas are true here:

$$\begin{aligned} u_1 &= \operatorname{Re}\int_0^{2\pi} d\varphi \int_0^{\infty} [A_1(k, \varphi) \operatorname{sh} kz\sqrt{1-\gamma^2\tau_1^2} + A_2(k, \varphi) \operatorname{sh} kz\sqrt{1-\tau_1^2}] \times \\ &\qquad \times \exp[ik(x\cos\varphi + y\sin\varphi + v_s\tau_1 t)]\, dk, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_1 &= \operatorname{Re}\int_0^{2\pi} d\varphi \int_0^{\infty} [C_1(k, \varphi) \operatorname{ch} kz\sqrt{1-\gamma^2\tau_1^2} + C_2(k, \varphi) \operatorname{ch} kz\sqrt{1-\tau_1^2}] \times \\ &\qquad \times \exp[ik(x\cos\varphi + y\sin\varphi + v_s\tau_1 t)]\, dk. \end{aligned} \tag{3}$$

* I. e., forces antisymmetrical relative to the middle plane of the plate. Symmetrical forces excite longitudinal vibrations only in a homogeneous plate.

In these formuls the coefficients $A_1(k, \varphi), \cdots C_2(k, \varphi)$ are determined by the form of the forces acting on the plate; $\tau_1(k)$ is the smallest root of the equation

$$g^2 \operatorname{th} kh\alpha - 4\tau\beta \operatorname{th} kh\beta = 0, \tag{4}$$

where

$$\beta = \sqrt{1-\tau^2}, \quad \alpha = \sqrt{1-\gamma^2\tau^2}, \quad g = 2-\tau^2, \quad \gamma^2 = \frac{\mu}{\lambda+2\mu}.$$

If $0 < kh\alpha < \frac{\pi}{2}$, then

$$\tau_1 = 2\sqrt{\frac{1-\gamma^2}{3}}\, kh\left[1 - \frac{27-20\gamma^2}{30} k^2h^2 + \cdots\right]. \tag{5}$$

The other terms on the right-hand side of (2) will now be briefly described.

The terms under the summation sign represent the components of the eigenvectors $\vec{u}_n$ of the vibrating layer. They are characterized by phase velocities differing sharply from the phase velocities of the terms u_1, v_1, w_1, and have each several nodal surfaces z = const inside the plate. The spectral expansions in Fourier integrals (by $\exp(i2\pi\nu t)$) of the expressions $\vec{u}_n$ contain only frequencies ν higher than

$$\tilde{\nu}_0 = \min\left(\frac{v_p}{2h}, \frac{3}{4}\frac{v_s}{h}\right) > 2\sqrt{2}\,\nu_0, \text{ where } \nu_0 = \frac{v_s{}^*}{4h}.$$

Therefore, if we are considering nonstationary vibrations of the plate with frequencies $\nu < \tilde{\nu}_0$, there is no need to consider these sums in (2).

As to the terms (u_2, v_2, w_2), finally, the frequencies in them exceed the value ν_0. It will suffice for our needs to give the expression for w_2 in the frequency range $\nu_0 \leqslant \nu \leqslant \nu_0\left(1 + \frac{1}{2\pi^2}\right)$, which adjoins the interval $0 \leqslant \nu \leqslant \nu_0$. This expression is represented with sufficient accuracy by the formula

$$w_2 = \operatorname{Re}\int_0^{2\pi} d\varphi \int_0^{\tilde{k}_0}\left[\tilde{C}_1(k, \varphi)\cos\frac{\pi\gamma z}{2h} + \tilde{C}_2(k, \varphi)\cos\frac{\pi z}{2h}\right] \times$$
$$\times \exp\left\{ik\left[x\cos\varphi + y\sin\varphi + v_s\sqrt{1+\left(\frac{\pi}{2kh}\right)^2}\, t\right]\right\}, \tag{6}$$

in which $\tilde{C}_1$ and $\tilde{C}_2$ are determined by the pattern of the external force, and the number $\tilde{k}_0$ is to be taken as smaller than $(2h)^{-1}$.

The information given about the displacement field in the elastic layer is absolutely sufficient for determining the form of the operator L in (1).

3. Since L must be independent of the forces applied to the plate (proposition 1), it is easiest to determine it under the assumption that the forces applied to the plate have already been removed. The points of the middle plane of the plate must then satisfy the equation

$$L[w(x, y, t)] = 0. \tag{1_1}$$

* If $v_s = 10^3$ cm/sec and $2h = 20$ cm, then $\nu_0 = 2.5 \times 10^3$ cycles/sec.

The operator L definitely exists in the case of low-frequency vibrations (proposition 3). But since it does not depend explicitly on the frequencies of the vibrations (proposition 1), for the determination of its form it is sufficient to use the low-frequency part of the true displacement field in the plate (proposition 4).

We saw that in the frequency range

$$0 < \nu < \nu_0 = \frac{v_s}{4h} \tag{7}$$

the spectral representation of the complete displacement field in the plate, excited by the application of arbitrary bending forces, is given by formulas (3), in which it is only necessary to cut the upper limit of integration up to the value $k_0 \approx \frac{\pi}{2h}$.

The normal displacements of the middle plane are determined from the last expression of (3), by substituting in it $\operatorname{ch} kz\alpha = \operatorname{ch} kz\beta = 1$. Since the coefficients $C_1(k, \varphi)$ and $C_2(k, \varphi)$ depend on the bending forces acting on the plate, equation (1_1) must be satisfied by the expression

$$w = \exp\left[ik\left(x\cos\varphi + y\sin\varphi + v_s\tau_1 t\right)\right], \tag{8}$$

in which $\tau_1(k)$ is the smallest positive root of the characteristic equation (4). This means that the result of the substitution of (8) in (1_1) must for $\tau_1(k)$ lead to the same (or to a similar) equation as equation (4) of the exact theory.

It is clear from the assumed form of the operator L that the substitution of (8) in (1_1) can lead only to an algebraic equation for $\tau_1(k)$. The characteristic equation (4) being transcendental, it follows that our problem can be solved only approximately, namely, in the frequency range adjoining the zero frequency (proposition 3), in which equation (4) for $\tau_1(k)$ can be replaced by an approximate algebraic equation.

In order to obtain such equations, all we have to do is to expand $\operatorname{th} kh\alpha$ and $\operatorname{th} kh\beta$ from (4) in Taylor series by the powers of $(kh\alpha)$ and $(kh\beta)$. The region of convergence of these series is determined by the inequalities $|kh\beta| < \frac{\pi}{2}$ and $|kh\alpha| < \frac{\pi}{2}$ fulfilled in the entire frequency range (7).

The following equation for the determination of τ is obtained with the aid of these expansions:

$$\tau^2 - \frac{4(1-\gamma^2)}{3}k^2h^2 + \left[\frac{3-4\gamma^2}{3}\tau^2k^2h^2 + \frac{16}{15}(1-\gamma^2)k^4h^4\right] +$$
$$+\left[\frac{2}{15}(4\gamma^4+8\gamma^2-11)\tau^2k^4h^4 + \frac{\gamma^2}{3}\tau^4k^2h^2 - \frac{68}{105}(1-\gamma^2)k^6h^6\right] + \cdots = 0. \tag{9}$$

This equation is equivalent to equation (4) in region (7) only if its solution is sought by successive approximations, assuming $\tau \sim kh$ and $kh \ll 1$. The value $\tau_1(k)$ obtained as a result will be a root of (4) in the whole region (7) of values of ν.

If we group the terms on the left-hand side of (9) according to their order of magnitude relative to $kh \sim \tau$ and keep only the terms up to a certain order, we obtain a series of algebraic "equations" with whose aid we determine the form of the operator L in different approximations. It is

sufficient to require then that the substitution of (8) in the left-hand part of (1_1) lead to the corresponding equation.

In a zero approximation we obtain from (9) and (8):

$$L = \Delta^2 + p_0^2 \frac{\partial^2}{\partial t^2}, \qquad p_0^2 = \frac{3\rho(\lambda + 2\mu)}{4h^2\mu(\lambda+\mu)}, \tag{10}$$

where Δ is the Laplace operator (Kirchhoff).

The next (first) approximation yields

$$L = \Delta^2 + p_0^2 \frac{\partial^2}{\partial t^2} - \varkappa_1 \Delta \frac{\partial^2}{\partial t^2} + \varkappa_2 \Delta^3, \tag{11}$$

where

$$\varkappa_1 = \frac{\rho(3\lambda + 2\mu)}{4\mu(\lambda + 2\mu)}, \qquad \varkappa_2 = \frac{4}{5} h^2, \tag{12}$$

etc. We cannot dwell here on the determination of the left-hand part of (1) and on the derivation of the final equations of motion. Some of the pertinent results are given in /7, 8/.

4. In the determination of the form of the operator L we made use of the explicit expression of the displacement field in a plate corresponding to the frequency range (7). That the region of applicability of any equations of the engineering theory of bending vibrations is not wider than the region of frequencies (7) can be seen from the fact that the field $w_2(x, y, 0, t)$ in (6), entering in the composition of the displacement field, does not satisfy equation (1_1) with the value found for L for frequencies exceeding ν_0.

The accuracy of the different approximations can be assessed on the basis of the data of Table 1, which gives the relative phase velocity $v_\varphi : v_s = \tau_1(k)$ as a function of the ratio $\nu : \nu_0$.

TABLE 1

$\nu:\nu_0$	0.00	0.05	0.10	0.15	0.20
True τ	0.000	0.266	0.365	0.435	0.490
τ_1 by (10)	0.000	0.272	0.385	0.471	0.544
τ_1 by (11)	0.000	0.265	0.359	0.417	0.440

It follows from the data of Table 1 as well as from an analysis of the possibilities of satisfying the requirements of proposition 2 that the region of applicability of the equations of the engineering theory of flexural vibrations must be narrowed somewhat in the side of the lower frequencies.

There remains in conclusion to stress, in the first place, that the additional terms (relative to (10)) in any refined expression for the operator L must be interpreted as small corrective terms. This leads to a certain ambivalence in the form of equivalent (of the same order of approximation) refined equations of the engineering theory, always nonhyperbolic, and imposes a cautious approach to the characteristic equations in problems dealing with finite plates (in order not to introduce "false" roots in the solution).

In the second place, it is necessary to interpret our first-approximation equations from the point of view of the deformations in the plate. This is most easily done on the example of the two-dimensional problem, in which case the displacement field in the plate is a function of the variables x, z, and t only.

An elementary analysis of formulas (3) leads to the conclusion that the displacement field corresponding to the first approximation (11) is of the form

$$\bar{u}(x, z, t) = u_1 + u_2, \quad \bar{w}(x, z, t) = w_1 + w_2, \tag{13}$$

$$w_1 = w + \frac{\lambda z^2}{2(\lambda + 2\mu)} \frac{\partial w}{\partial x^2}, \qquad u_1 = -z \frac{\partial w}{\partial x} - \frac{\lambda z^3}{6(\lambda + 2\mu)} \frac{\partial^3 w}{\partial x^3}, \tag{14}$$

$$w_2 = 0, \quad u_2 = -\frac{2(\lambda + \mu) z}{3(\lambda + 2\mu)} (3h^2 - z^2) \frac{\partial^3 w}{\partial x^3}, \tag{15}$$

where $w = w(x, t)$ are the displacements of the middle plane of the plate.

The terms w and $(-zw'_x)$ in (14) characterize the principal parts of the displacement in the frame of the hypothesis of straight normals. The second term in (14) corresponds to the Poisson compression caused by extensions at bending. This compression leads in turn to an additional extension, characterized by the second term in (14). For the terms u_1, w_1 of field (13) it turns out that $t_{xz} = 0$, i.e., this deformation is not linked with the appearance of shearing forces.

On the other hand, the displacements (15) are linked with the appearance of the shearing force

$$N = \int_{-h}^{h} t_{xz} dz = -\frac{8}{3} h^3 \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} \frac{\partial^3 w}{\partial x^3} \tag{16}$$

and determine the rotation of the (undeformed) normals through the additional angle

$$\beta = -2h^2 \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial^3 w}{\partial x^3}. \tag{17}$$

The ratio between N and β turns out to be exactly the same as in /1/.

Our first-approximation equations allow therefore automatically for all the hypotheses on which Timoshenko's reasoning /1/ is based. The incorrect final result of that author is due to the insufficient (for the derivation of the equations in a first approximation) accuracy in the analysis of the dynamic equilibrium of a plate element.

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REFINED SOLUTION OF THE PROBLEM OF BENDING OF ORTHOTROPIC CYLINDRICAL PANELS AND PLATES

D. V. Peshtmaldzhyan

(Yerevan)

§1.

Finite displacements of very shallow orthotropic shells and plates at small deformations are treated on the basis of the general equations of the nonlinear theory of elasticity /1/, keeping the nonlinear terms containing derivatives of the normal displacement w. The treatment is based on the theory proposed by Ambartsumyan /2/:

a) the deflections w do not vary through the shell thickness;

b) the shearing stresses τ_{xz} and τ_{yz} are represented in the following form:

$$\tau_{xz} = f_1(z)\varphi(x, y) + \frac{z}{h}(X^+ + X^-) + \frac{X^+ - X^-}{2},$$

$$\tau_{yz} = f_2(z)\psi(x, y) + \frac{z}{h}(Y^+ + Y^-) + \frac{Y^+ + Y^-}{2},$$

where $\varphi(x, y)$ and $\psi(x, y)$ are unknown functions and $f_i(z)$ is the given law of distribution of these stresses through the shell thickness;

c) the normal stress σ_{zz} is determined from the linearized equilibrium equation.

The meaning of the symbols used in the text has been defined in /3/.

§2. FUNDAMENTAL RELATIONSHIPS AND EQUATIONS

Let the shearing stresses τ_{xz} and τ_{yz} vary through the shell thickness according to the square parabola law:

$$f_1(z) = f_2(z) = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right),$$

and let there act on the shell a uniform pressure of intensity $q\,(X = Y = 0)$.

The strain components are represented in the form

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{\partial \Phi_1}{\partial x}, \qquad (2.1)$$

$$e_{yy}=\frac{\partial v}{\partial y}+kw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2-z\frac{\partial^2 w}{\partial y^2}+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\frac{\partial\Phi_2}{\partial y},$$

$$e_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2z\frac{\partial^2 w}{\partial x\partial y}+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left(\frac{\partial\Phi_1}{\partial y}+\frac{\partial\Phi_2}{\partial x}\right)+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y},$$

$$(\Phi_1=\varphi(x,\ y)/B_{55},\ \Phi_2=\psi(x,\ y)/B_{44}). \qquad (2.1)\ \text{(cont'd)}$$

The generalized Hooke law, resolved relative to the stresses, has the form

$$\sigma_{xx}=B_{11}e_{xx}+B_{12}e_{yy}-A_1\sigma_{zz},$$
$$\sigma_{yy}=B_{12}e_{xx}+B_{22}e_{yy}-A_2\sigma_{zz}, \qquad (2.2)$$
$$\sigma_{xy}=B_{66}e_{xy},\quad \sigma_{xz}=B_{55}e_{xz},\quad \sigma_{yz}=B_{44}e_{yz}.$$

The normal stress σ_{zz} is determined on the strength of c) by the linear theory:

$$\sigma_{zz}=\frac{q}{2}+z\left(kB_{12}\frac{\partial u}{\partial x}+kB_{22}\frac{\partial v}{\partial y}+k^2B_{22}w\right)+$$
$$+\frac{1}{2}\left(\frac{h^2}{4}-z^2\right)\left(kB_{12}\frac{\partial^2 w}{\partial x^2}+kB_{22}\frac{\partial^2 w}{\partial y^2}\right)-\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}\right). \qquad (2.3)$$

The forces and moments are

$$T_{xx}=C_{11}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+C_{12}\left[\frac{\partial v}{\partial y}+kw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]-A_1T^\circ,$$
$$T_{yy}=C_{22}\left[\frac{\partial v}{\partial y}+kw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+C_{12}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]-A_2T^\circ, \qquad (2.4)$$
$$S=C_{66}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right],$$

$$M_{xx}=-D_{11}\frac{\partial^2 w}{\partial x_2}-D_{12}\frac{\partial^2 w}{\partial y^2}+\frac{h^2}{10}\left(D_{11}\frac{\partial\Phi_1}{\partial x}+D_{12}\frac{\partial\Phi_2}{\partial y}\right)-A_1M^\circ,$$
$$M_{yy}=-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}+\frac{h^2}{10}\left(D_{12}\frac{\partial\Phi_1}{\partial x}+D_{22}\frac{c\Phi_2}{\partial y}\right)-A_2M^\circ, \qquad (2.5)$$
$$H=-2D_{66}\frac{\partial^2 w}{\partial x\partial y}+\frac{h^2}{10}D_{66}\left(\frac{\partial\Phi_1}{\partial y}+\frac{\partial\Phi_2}{\partial x}\right),$$

$$N_x=\frac{h^3}{12}B_{55}\Phi_1,\qquad N_y=\frac{h^3}{12}B_{44}\Phi_2, \qquad (2.6)$$

where

$$T^\circ=k\left(D_{12}\frac{\partial^2 w}{\partial x^2}+D_{22}\frac{\partial^2 w}{\partial y^2}\right),$$
$$M^\circ=k\left(D_{12}\frac{\partial u}{\partial x}+D_{22}\frac{\partial v}{\partial y}+kD_{22}w\right)-\frac{h^5}{120}\left(B_{55}\frac{\partial\Phi_1}{\partial x}+B_{44}\frac{\partial\Phi_2}{\partial y}\right)$$

satisfy the following equilibrium equations:

$$\frac{\partial T_{xx}}{\partial x}+\frac{\partial S}{\partial y}=0,\qquad \frac{\partial M_{xx}}{\partial x}+\frac{\partial H}{\partial y}=N_x,$$

$$\frac{\partial T_{yy}}{\partial y}+\frac{\partial S}{\partial x}=0,\qquad \frac{\partial M_{yy}}{\partial y}+\frac{\partial H}{\partial x}=N_y, \tag{2.7}$$

$$kT_{yy}-\frac{\partial N_x}{\partial x}-\frac{\partial N_y}{\partial y}-\frac{\partial^2 w}{\partial y^2}T_{yy}-\frac{\partial^2 w}{\partial x^2}T_{xx}-2S\frac{\partial^2 w}{\partial x\partial y}=q$$

and the equation of continuity

$$\frac{C_{22}}{\Omega}\frac{\partial^2 T_{xx}}{\partial y^2}-\frac{C_{12}}{\Omega}\frac{\partial^2 T_{xx}}{\partial x^2}+\frac{C_{11}}{\Omega}\frac{\partial^2 T_{yy}}{\partial x^2}-\frac{C_{12}}{\Omega}\frac{\partial^2 T_{yy}}{\partial y^2}-\frac{1}{C_{66}}\frac{\partial^2 S}{\partial x\partial y}-k\frac{\partial^2 w}{\partial x^2}+$$

$$+\left(A_2\frac{C_{11}}{\Omega}-A_1\frac{C_{12}}{\Omega}\right)\left(kD_{12}\frac{\partial^4 w}{\partial x^4}+kD_{22}\frac{\partial^4 w}{\partial x^2\partial y^2}\right)+$$

$$+\left(A_1\frac{C_{22}}{\Omega}-A_2\frac{C_{12}}{\Omega}\right)\left(kD_{12}\frac{\partial^4 w}{\partial x^2\partial y^2}+kD_{22}\frac{\partial^4 w}{\partial y^4}\right)=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \tag{2.8}$$

By introducing the stress function $\eta(x, y)$ by the standard formulas we satisfy the first two equations of system (2.7). The remaining equilibrium equations and the equation of continuity (2.8) form a system of four equations with four unknown functions w, η, Φ_1, Φ_2:

$$k\frac{\partial^2\eta}{\partial x^2}-\frac{h^3}{12}\left(B_{55}\frac{\partial\Phi_1}{\partial x}+B_{44}\frac{\partial\Phi_2}{\partial y}\right)-\frac{\partial^2 w}{\partial y^2}\frac{\partial^2\eta}{\partial x^2}-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\eta}{\partial y^2}+2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\eta}{\partial x\partial y}=q;$$

$$D_{11}\frac{\partial^3 w}{\partial x^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x\partial y^2}-\frac{h^2}{10}\left[D_{11}\frac{\partial^2\Phi_1}{\partial x^2}+D_{66}\frac{\partial^2\Phi_1}{\partial y^2}+\right.$$

$$\left.+(D_{12}+D_{66})\frac{\partial^2\Phi_2}{\partial x\partial y}\right]+\frac{h^3}{12}B_{55}\Phi_1+A_1\left[k\left(D_{12}\frac{\partial^2 u}{\partial x^2}+D_{22}\frac{\partial^2 v}{\partial x\partial y}+\right.\right.$$

$$\left.\left.+kD_{22}\frac{\partial w}{\partial x}\right)-\frac{h^5}{120}\left(B_{55}\frac{\partial^2\Phi_1}{\partial x^2}+B_{44}\frac{\partial^2\Phi_2}{\partial x\partial y}\right)\right]=0;$$

$$D_{22}\frac{\partial^3 w}{\partial y^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x^2\partial y}-\frac{h^2}{10}\left[D_{22}\frac{\partial^2\Phi_2}{\partial y^2}+D_{66}\frac{\partial^2\Phi_2}{\partial x^2}+\right.$$

$$\left.+(D_{12}+D_{66})\frac{\partial^2\Phi_1}{\partial x\partial y}\right]+\frac{h^3}{12}B_{44}\Phi_2+A_2\left[k\left(D_{12}\frac{\partial^2 u}{\partial x\partial y}+D_{22}\frac{\partial^2 v}{\partial y^2}+\right.\right.$$

$$\left.\left.+kD_{22}\frac{\partial w}{\partial y}\right)-\frac{h^5}{120}\left(B_{55}\frac{\partial^2\Phi_1}{\partial x\partial y}+B_{44}\frac{\partial^2\Phi_2}{\partial y^2}\right)\right]=0; \tag{2.9}$$

$$\frac{C_{11}}{\Omega}\frac{\partial^4\eta}{\partial x^4}+\frac{C_{22}}{\Omega}\frac{\partial^4\eta}{\partial y^4}+\left(\frac{1}{C_{66}}-2\frac{C_{12}}{\Omega}\right)\frac{\partial^4\eta}{\partial x^2\partial y^2}-k\frac{\partial^2 w}{\partial x^2}+$$

$$+k\left[\left(A_2\frac{C_{11}}{\Omega}-A_1\frac{C_{12}}{\Omega}\right)\left(D_{12}\frac{\partial^4 w}{\partial x^4}+D_{22}\frac{\partial^4 w}{\partial x^2\partial y^2}\right)+\left(A_1\frac{C_{22}}{\Omega}-\right.\right.$$

$$\left.\left.-A_2\frac{C_{12}}{\Omega}\right)\left(D_{12}\frac{\partial^4 w}{\partial x\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}\right)\right]=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2},$$

where

$$\frac{\partial u}{\partial x}=\frac{C_{22}}{\Omega}\frac{\partial^2\eta}{\partial y^2}-\frac{C_{12}}{\Omega}\frac{\partial^2\eta}{\partial x^2}-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\left(A_1\frac{C_{22}}{\Omega}-A_2\frac{C_{12}}{\Omega}\right)\left[\frac{hq}{2}+\right.$$

$$+k\left(D_{12}\frac{\partial^2 w}{\partial x^2}+D_{22}\frac{\partial^2 w}{\partial y^2}\right)\Bigg],$$

$$\frac{\partial v}{\partial y}=\frac{C_{11}}{\Omega}\frac{\partial^2\eta}{\partial x^2}-\frac{C_{12}}{\Omega}\frac{\partial^2\eta}{\partial y^2}-kw-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\left(A_2\frac{C_{11}}{\Omega}-A_1\frac{C_{12}}{\Omega}\right)\Bigg[\frac{hq}{2}+$$

$$+k\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\Bigg].$$

In the classical formulation, the resolving system consists of two nonlinear equations relative to the deflection and the stress function. In our case we have two additional unknowns, the functions Φ_1 and Φ_2, which appear in all equations of system (2.9) in the case of arbitrary $f_i(z)$. The additional two equations are linear on the strength of assumption c).

§ 3.

Consider the problem of bending of a simply supported orthotropic panel elongated in the x direction ($a \gg b$). Let this panel be subjected to a uniformly distributed pressure applied from the side of the panel concavity. This problem can be exactly solved under some assumptions.* We shall allow for the influence of lateral shears only ($A_i = 0$).

The resolving system of equations (2.9) reduces in our case (when the unknown magnitudes depend on the coordinate y only) to the two equations

$$-\frac{h^3}{12}\frac{d\psi}{dy}-kT_2+T_2\frac{d^2w}{dy^2}=-q,$$

$$D_{22}\frac{d^3w}{dy^3}+\frac{h^5}{120}a_{44}B_{22}\frac{d^2\psi}{dy^2}-\frac{h^3}{12}\psi=0.$$

Having determined the function $\psi(y)$ from the first equation, using the condition $\psi\left(\frac{b}{2}\right)=0$, and having eliminated this function from all the following relationships, we finally reduce the problem to the solution of the equation

$$\frac{d^4w}{dy^4}-\lambda^2\frac{d^2w}{dy^2}=\frac{\lambda^2}{T_{yy}}(q-kT_{yy}), \tag{3.1}$$

where

$$\lambda^2=\frac{12T^*}{b^2(1+mT^*)},\qquad m=\frac{1{,}2B_{22}h^2}{B_{44}b^2},\qquad T^*=\frac{T_{yy}b^2}{B_{22}h^3}$$

with edge conditions

$$y=0,\ b,\qquad w=0,\qquad \frac{T_{yy}}{\lambda^2}\frac{d^2w}{\partial y^2}+\left(\frac{1}{\lambda^2}-\frac{D_{22}}{T_{yy}}\right)(q-kT_{yy})=0. \tag{3.2}$$

* A similar problem was solved in the classical formulation by Bubnov for the case of an isotropic panel /4, 5/.

The edges do not approach.

The following expressions are obtained for the deflection, the circumferential stress, and the maximum bending stress:

$$\zeta = \frac{(q^* - k^*T^*)}{8T^*}\left[1 + \frac{2}{3}\frac{1}{T^*}\left(\frac{1}{\operatorname{ch}\frac{\lambda b}{2}} - 1\right)\right]; \tag{3.3}$$

$$T^* = \frac{k^*\lambda^2 b^2}{12T^*}\left(k^* - \frac{q^*}{T^*}\right)\left(-\frac{2}{\lambda^3 b^3}\operatorname{th}\frac{\lambda b}{2} + \frac{1}{\lambda^2 b^2} - \frac{T^*}{\lambda^2 b^2}\right) +$$

$$+ \frac{\lambda^2 b^2}{144(T^*)^2}\left(k^* - \frac{q^*}{T^*}\right)^2\left(\frac{\operatorname{th}\frac{\lambda b}{2}}{2\lambda b} - \frac{1}{4\operatorname{ch}^2\frac{\lambda b}{2}} - \frac{12T^*}{\lambda^2 b^2} + \right.$$

$$\left. + \frac{24T^*\operatorname{th}\frac{\lambda b}{2}}{b^3\lambda^3} + \frac{6(T^*)^2}{\lambda^2 b^2}\right); \tag{3.4}$$

$$\sigma^* = -\frac{q^* - k^*T^*}{2T^*}\left\{\left(1 + \frac{5m}{6}T^*\right)\left[\frac{1}{(1 + mT^*)\operatorname{ch}\frac{\lambda b}{2}} - 1\right] + \frac{5m}{6}T^*\right\}$$

$$\left(k^* = \frac{kb^2}{h}, \quad q^* = \frac{qb^4}{B_{22}h^4}, \quad \sigma^* = \frac{\sigma b^2}{B_{22}h^2}\right). \tag{3.5}$$

At $m = 0$ we obtain the results of the classical theory of bending. The results obtained for different values of m for the case of a plate ($k^* = 0$) are given in Tables 1 to 3.

$m=0$ TABLE 1

q^*	T^*	ζ	σ^*
2.27	0.2	0.28	1.36
3.83	0.4	0.40	1.20
7.16	0.8	0.56	2.65
8.99	1	0.63	2.25
32.60	3	1.09	4.89
82.3	6	1.53	6.66
168	10	1.97	8.33

$m=0.5$ TABLE 2

q^*	T^*	ζ	σ^*
1.72	0.2	0.28	1.00
3.06	0.4	0.40	1.46
6.09	0.8	0.56	1.98
7.80	1	0.62	2.21
30.49	3	1.07	3.75
79.1	6	1.51	5.23
164	10	1.94	6.71

$m=1$ TABLE 3

q^*	T^*	ζ	σ^*
1.44	0.2	0.28	0.82
2.64	0.4	0.39	1.15
5.51	0.8	0.55	1.61
7.15	1	0.62	1.80
29.3	3	1.07	3.07
77.5	6	1.50	4.32
162	10	1.94	5.55

§ 4.

We shall now solve the same problem for the case of an elongated plate with the long edges clamped. We obtain then instead of (3.2):

at $y = 0, b$ $w = 0$

$$\left(\frac{\partial u_y}{\partial z}\right)_{z=0} = -\frac{dw}{dy} + \frac{3a_{44}q}{2h\left(\frac{3a_{44}}{2h}T_2 + 1\right)}\left(\frac{b}{2} - y\right) = 0. \tag{4.1}$$

The following expressions are obtained instead of (3.3) to (3.5):

$$\zeta = \frac{0.4q^*}{(mT^* + 0.8)\lambda b T^* \operatorname{th}\frac{\lambda b}{2}}\left(\frac{1}{\operatorname{ch}\frac{\lambda l}{2}} - 1\right) + \frac{q^*}{8T^*}, \tag{4.2}$$

$$T^* = \frac{q^{*2}}{T^{*2}} \frac{0.4}{mT^* + 0.8} \left[\frac{2}{\lambda^2 b^2} + \frac{mT^* + 0.8}{9.6} - \frac{0.1}{(mT^* + 0.8)\operatorname{sh}^2 \frac{\lambda b}{2}} - \right.$$

$$\left. - \frac{mT^* + 0.6}{(mT^* + 0.8)\lambda b \operatorname{th} \frac{\lambda b}{2}} \right], \tag{4.3}$$

$$\sigma^* = - \frac{q^*}{2T^*} \left[\left(1 + \frac{5n}{6} T^*\right) \left(\frac{0.4}{mT^* + 0.8} \frac{b\lambda}{\operatorname{th} \frac{\lambda b}{2}} - 1 \right) + \frac{5m}{6} T^* \right]. \tag{4.4}$$

The results obtained from these expressions for different values of m are given in Tables 4 to 6.

$m=0$ TABLE 4

q^*	T^*	ζ	σ^*
25.82	1	0.62	10.90
67.42	3	1.11	22.64
141	6	1.59	38.12
245	10	1.95	54.75

$m=0.5$ TABLE 5

q^*	T^*	ζ	σ^*
10.77	1	0.63	2.08
35.58	3	1.08	3.27
86.39	6	1.52	5.35
182.4	10	2.05	5.75

$m=1$ TABLE 6

q^*	T^*	ζ	σ^*
8.26	1	0.62	0.76
31 2	3	1.06	1.15
80.32	6	1.51	1.52
165	10	1.92	1.89

It is seen by plotting the data of Tables 1 to 6 that the boundary conditions are qualitatively similar in the two cases: the hypothesis of undeformable normals leads to underestimated values of the deflection; as to the stresses, the circumferential stresses are underestimated and the maximum bending stresses overestimated when the Kirchhoff hypothesis is used. Quantitatively, however, the results of the classical theory diverge considerably from the refined results in the case of clamped edges.

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SOME DYNAMIC PROBLEMS OF THE THERMO-ELASTICITY OF THIN SHELLS

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(L'vov)

1. FUNDAMENTAL EQUATIONS OF THE THERMO-ELASTICITY OF THIN SHELLS

In the general case, the temperature of a rigid body subjected to deformation can vary both with time and from point to point. This variation can be caused both by heat exchange with the external medium and by the process of deformation itself, during which a part of the mechanical energy is transformed into heat. The thermoelastic scattering plays an important part in nonstationary problems; it is in particular one of the causes for the dampening of the vibrations in an elastic body.

Consider a thin shell with middle surface expressed through the curvature lines, and let α and β be coordinates of the middle surface point, z the distance of an arbitrary point from the middle surface, and τ the time. Starting from the well-known hypothesis of the unvariability of a normal element, we then obtain the following physical relationships for a shell in a nonstationary temperature field and subjected to an external load:

$$N_1 = D_1\,\{\varepsilon_1 + \nu\varepsilon_2 - \alpha_t(1+\nu)\,T_1\}, \qquad M_1 = D_2\left[\varkappa_1 + \nu\varkappa_2 - \frac{\alpha_t}{h}\,(1+\nu)\,T_2\right],$$

$$N_2 = D_1\,\{\varepsilon_2 + \nu\varepsilon_1 - \alpha_t(1+\nu)\,T_1\}, \qquad M_2 = D_2\left[\varkappa_2 + \nu\varkappa_1 - \frac{\alpha_t}{h}\,(1+\nu)T_2\right], \qquad (1)$$

$$S_{12} = \frac{1}{2}\,D_1\,(1-\nu)\,\varepsilon_{12}, \qquad H_{12} = D_2\,(1-\nu)\,\varkappa_{12}.$$

Here $\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \varkappa_1, \varkappa_2, \varkappa_{12}$ are strain components of the middle surface, h is the half-thickness of the shell, and

$$T_1 = \frac{1}{2h}\int_{-h}^{h} t\,dz, \qquad T_2 = \frac{3}{2h^2}\int_{-h}^{h} tz\,dz, \qquad D_1 = \frac{2hE}{1-\nu^2}, \quad D_2 = \frac{2}{3}\,\frac{h^3E}{1-\nu^2}. \qquad (2)$$

The equations necessary for determining the integral characteristics T_1 and T_2 can be obtained from the law of conservation of energy, written in a differential form /1, 2/:

$$\lambda\left(\Delta t + \frac{\partial^2 t}{\partial z^2}\right) = c_v\frac{\partial t}{\partial \tau} + \frac{\alpha_t ET}{1-2\nu}\,\frac{\partial e}{\partial \tau}. \qquad (3)$$

Here λ is the thermal conductivity; c_v is the specific heat at constant volume; T is the absolute temperature; E and ν are the isothermal modulus of elasticity and the Poisson ratio,

$$\Delta = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right], \quad e = e_1 + e_2 + e_3.$$

In accordance with the assumption $\sigma_{zz} = 0$ made in the theory of shells, the volume expansion of a shell element will be

$$e = \frac{1-2\nu}{1-\nu}(\varepsilon + \varkappa z) + \frac{1+\nu}{1-\nu}\alpha_t t, \tag{4}$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$, $\varkappa = \varkappa_1 + \varkappa_2$.

Substituting (4) in (3) and introducing the designations

$$a_0 = \frac{\lambda}{c_v}, \quad \gamma_0 = \frac{\alpha_t E T}{c_v(1-2\nu)}, \quad a = \frac{a_0}{1+\gamma_0\alpha_t\frac{1+\nu}{1-\nu}},$$

$$\gamma = \frac{\gamma_0(1-2\nu)}{(1-\nu)\left(1+\gamma_0\alpha_t\frac{1+\nu}{1-\nu}\right)}, \quad \Delta - \frac{1}{a}\frac{\partial}{\partial\tau} = p^2, \quad \frac{1}{a}\frac{\partial}{\partial\tau} = p_1^2, \tag{5}$$

we find

$$\frac{\partial^2 t}{\partial z^2} + p^2 t = \gamma p_1^2(\varepsilon + \varkappa z). \tag{6}$$

Using the operatorial method /3/, we obtain from here

$$t = \sin pz\left(\frac{\partial t}{\partial z}_{z=0} - \gamma\frac{p_1^2}{p^2}\varkappa\right) + \cos pz\left(t_{z=0} - \gamma\frac{p_1^2}{p^2}\varepsilon\right) + \gamma\frac{p_1^2}{p^2}(\varepsilon + \varkappa z). \tag{7}$$

The integration of (7) by z in accordance with (2) yields

$$T_1 = \frac{\sin ph}{ph}\left(t_{|z=0} - \gamma\frac{p_1^2}{p^2}\varepsilon\right) + \gamma\frac{p_1^2}{p^2}\varepsilon,$$

$$T_2 = \frac{3}{p^2h}\frac{\sin ph}{ph}\left(\frac{\partial t}{\partial z}_{|z=0} - \gamma\frac{p_1^2}{p^2}\varkappa\right) - \frac{3}{p^2h}\cos ph\left(\frac{\partial t}{\partial z}_{|z=0} - \gamma\frac{p_1^2}{p^2}\varkappa\right) + \gamma\frac{p_1^2}{p^2}h\varkappa. \tag{8}$$

Eliminating $t_{z=0}$ and $\left.\frac{\partial t}{\partial z}\right|_{z=0}$ from (7) and (8), we obtain

$$t = \frac{ph\cos pz}{\sin ph}T_1 + \frac{p^2h^2}{3}\frac{\sin pz}{\sin ph - ph\cos ph}T_2 -$$

$$-\gamma\frac{p_1^2}{p^2}\left(\frac{ph\cos pz}{\sin ph} - 1\right)\varepsilon + \frac{\gamma}{3}\frac{p_1^2}{p^2}\left(\frac{p^2h^3\sin pz}{\sin ph - ph\cos ph} - 3z\right)\varkappa. \tag{9}$$

It is easily seen that (9) satisfies equation (6) at any values of T_1, T_2, ε, and $\varkappa$. T_1 and T_2 are determined from the boundary conditions for the temperature t. In the case of boundary conditions of the third kind, which in the case of shells are of the form

$$\frac{\partial t}{\partial z} + h_1(t - t_1^c) = 0 \text{ at } z = h, \quad \frac{\partial t}{\partial z} - h_2(t - t_2^c) = 0 \text{ at } z = -h, \tag{10}$$

we have the following equations for determining the unknown magnitudes T_1 and T_2:

$$L_1 T_1 - \frac{\mu_2}{3} L T_2 - \gamma \frac{p_1^2}{p^2} (L_1 + \mu_1) \varepsilon + \frac{\gamma \mu_2}{3} \frac{p_1^2}{p^2} (L-3) h\varkappa = -(\mu_1 t_c^+ + \mu_2 t_c^-),$$

$$L_2 T_2 - 3\mu_2 ph \operatorname{ctg} ph\, T_1 - \gamma \frac{p_1^2}{p^2} [L_2 + 3(1+\mu_1)] h\varkappa + 3\mu_2 \gamma \frac{p_1^2}{p^2} (ph \operatorname{ctg} ph - 1) \varepsilon = \tag{11}$$

$$= -3(\mu_1 t_c^- + \mu_2 t_c^+),$$

where h_1 and h_2 are relative heat release coefficients; t_1^c, t_2^c are values of the temperature of the medium washing the surface $z = \pm h$;

$$L = \frac{p^2 h}{1 - ph \operatorname{ctg} ph}, \quad L_1 = p^2 h^2 - \mu_1 ph \operatorname{ctg} ph, \quad L_2 = p^2 h^2 - (1+\mu_1) \frac{p^2 h^2}{1 - ph \operatorname{ctg} ph},$$

$$\mu_1 = \frac{h}{2}(h_1 + h_2), \qquad \mu_2 = \frac{h}{2}(h_1 - h_2), \tag{12}$$

$$t_c^+ = \frac{1}{2}(t_1^c + t_2^c), \qquad t_c^- = \frac{1}{2}(t_1^c - t_2^c).$$

By restricting ourselves to the first terms in the serial expansions of the operators $\operatorname{ctg} ph$ and $(1 - ph \operatorname{ctg} ph)^{-1}$, we obtain the following approximate system of equations, which coincides with the equations obtained in /4/ by a different method:

$$p^2 h^2 T_1 - \mu_1 T_1 - \mu_2 T_2 - \gamma p_1^2 h^2 \varepsilon = -(\mu_1 t_c^+ + \mu_2 t_c^-),$$
$$p^2 h^2 T_2 - 3(1+\mu_1) T_2 - 3\mu_2 T_1 - \gamma p_1^2 h^2 \varkappa h = -3(\mu_1 t_c^- + \mu_2 t_c^+). \tag{13}$$

At $\mu_2 = 0$ systems (11) and (13) split into two independent equations.

Equations (1) form together with the equation of motion, and the physical and geometrical relationships, a complete system of equations of the theory of thermoelasticity of thin shells.

2. EQUATIONS OF THERMOELASTICITY AND RHEOLOGICAL RELATIONSHIPS

It is known that real bodies do not behave as perfectly elastic bodies under deformation. Various models have accordingly been created for describing their mechanical behavior, which differ from each other by the strain law adopted. It turns out that if the variation of the body temperature is subjected to equations (11), then under some restrictions the thermoelastic relationship transforms the shell material into elastic-plastic.

Eliminating the integral temperature characteristics T_1 and T_2 from (1) and (11), and taking $\mu_2 = 0$, $\gamma_1 = \alpha_t \gamma$, we obtain the following differential relationships between the forces, moments, and strains of the middle surface of the shell:

$$L_1 N_1 - D_1 L_1 (\varepsilon_1 + \nu \varepsilon_2) + \gamma_1 D_1 (1+\nu) \frac{p_1^2}{p^2} (L_1 + \mu_1)(\varepsilon_1 + \varepsilon_2) = \mu_1 D_1 \alpha_t (1+\nu) t_c^+,$$

$$L_1 N_2 - D_1 L_1 (\varepsilon_2 + \nu \varepsilon_1) + \gamma_1 D_1 (1+\nu) \frac{p_1^2}{p^2} (L_1 + \mu_1)(\varepsilon_1 + \varepsilon_2) = \mu_1 D_1 \alpha_t (1+\nu) t_c^+, \tag{14}$$

$$L_2M_1-D_2L_2(\varkappa_1+\nu\varkappa_2)+\gamma_1D_2(1+\nu)\frac{p_1^2}{p^2}[L_2+3(1+\mu_1)](\varkappa_1+\varkappa_2)=3\mu_1D_2\alpha_t(1+\nu)t_c^-,$$
$$L_2M_2-D_2L_2(\varkappa_2+\nu\varkappa_1)+\gamma_1D_2(1+\nu)\frac{p_1^2}{p^2}[L_2+3(1+\mu_1)](\varkappa_1+\varkappa_2)=3\mu_1D_2\alpha_t(1+\nu)t_c^-. \quad (15)$$

The connection between the shearing forces S_{12}, the torsional moments H_{12}, and the strains remains elastic.

Relationships (14) and (15) allow for the irreversibility caused by the heat flux through the shell surface (μ_1 finite) and for the irreversibility due to the finite heat fluxes inside the shell (λ finite).

At $\mu_1 = 0$ and finite λ we obtain from (14) and (15) equations which allow only for the irreversibility due to the heat fluxes inside the shell.

If we write in (14) and (15) $\lambda=0\left(\text{then } p^2=-p_1^2=-\frac{1}{a_1}\frac{\partial}{\partial\tau}\right)$ and take μ_1 as finite, we obtain the following strain law:

$$L_3N_1-D_1L_3(\varepsilon_1+\nu\varepsilon_2)-\gamma_1D_1(1+\nu)(L_3-\mu_1)(\varepsilon_1+\varepsilon_1)=-\mu_1D_1\alpha_t(1+\nu)t_c^+,$$
$$L_3N_2-D_1L_3(\varepsilon_2+\nu\varepsilon_1)-\gamma_1D_1(1+\nu)(L_3+\mu_1)(\varepsilon_1+\varepsilon_2)=-\mu_1D_1\alpha_t(1+\nu)t_c^+,$$
$$L_4M_1-D_2L_4(\varkappa_1+\nu\varkappa_2)-\gamma_1D_2(1+\nu)[L_4-3(1+\mu_1)](\varkappa_1+\varkappa_2)=-3\mu_1D_2\alpha_t(1+\nu)t_c^-, \quad (16)$$
$$L_4M_2-D_2L_4(\varkappa_1+\nu\varkappa_1)-\gamma_1D_2(1+\nu)[L_4-3(1+\mu_1)](\varkappa_1+\varkappa_2)=-3\mu_1D_2\alpha_t(1+\nu)t_c^-,$$

where
$$L_3=p_1^2h^2+\mu_1p_1h\operatorname{cth}p_1h,\quad L_4=p_1^2h^2-(1+\mu_1)\frac{p_1^2h^2}{1-p_1h\operatorname{cth}p_1h}.$$

Expanding the operators $\operatorname{cth}p_1h$ and $(1-p_1h\operatorname{cth}p_1h)^{-1}$ in a series, we obtain from (16) differential relationships of infinite order. Restricting ourselves to the first terms only in the expansions, we obtain the following strain law for the shell:

$$N_1+r\dot{N}_1=D_1[\varepsilon_1+\nu\varepsilon_2+r(\gamma_2\varepsilon_1+\gamma_3\varepsilon_2)-(1+\nu)\alpha_t t_c^+],$$
$$N_2+r\dot{N}_2=D_1[\varepsilon_2+\nu\varepsilon_1+r(\gamma_3\varepsilon_1+\gamma_2\varepsilon_2)-(1+\nu)\alpha_t t_c^+], \quad (17)$$

$$M_1+r_1\dot{M}_1=D_2\left[\varkappa_1+\nu\varkappa_2+r_1(\gamma_2\varkappa_1+\gamma_3\varkappa_2)-(1+\nu)\alpha_t\frac{\mu_1}{1+\mu_1}t_c^-\right],$$
$$M_2+r_1\dot{M}_2=D_2\left[\varkappa_2+\nu\varkappa_1+r_1(\gamma_3\varkappa_1+\gamma_2\varkappa_2)-(1+\nu)\alpha_t\frac{\mu_1}{1+\mu_1}t_c^-\right], \quad (18)$$

where the dot stands for differentiation with respect to time,

$$r=\frac{h^2}{a_1\mu_1},\qquad r_2=\frac{h^2}{3a_1(1+\mu_1)},\qquad \gamma_2=1+\gamma(1+\nu),\qquad \gamma_3=\nu+\gamma(1+\nu).$$

Only relationships (17) are taken into account in the case of the membrane theory of thin shells. At $t_c^+ = 0$ relationships (17) coincide in form with the equations given in /5/ for an Ishlinskii medium.

At $\mu_1=\infty$ and $t_c^+=t_c^-=0$ we obtain from (17) and (18) physical relationships for thin elastic shells, which correspond to Hooke's law in the case of isothermal deformation.

It can similarly be shown that in the case when the integral temperature characteristics are determined from boundary conditions of the first kind we obtain the rheological relationships for a shell whose material behaves as a Voigt body.

3. EXAMPLES

A. Consider the forced vibrations of a spherical shell caused by an internal pressure $Q_0(\tau)$ and a temperature t_0. We have in this case $\varkappa_1 = \varkappa_2 = 0$, $M_1 = M_2 = H_{12} = 0$, $T_2 = 0$, $N_1 = N_2 = N$,

$$\varepsilon_1 = \varepsilon_2 = \frac{w_0}{R}, \qquad N = -h\rho R\ddot{w} + \frac{R}{2}Q_0 . \tag{19}$$

Introducing the dimensionless magnitudes $\frac{w_0}{R} = w$, $p_1^2 h^2 = \frac{h^2}{a_1}\frac{\partial}{\partial\tau} = \frac{\partial}{\partial f} = p^2$ and substituting (19) into the first equation of (16), we obtain

$$\{[p^4 + \omega^2(1+2\gamma_1)](p^2 + \mu_1 p \operatorname{cth} p) - 2\mu_1\gamma_1\omega^2\} w = \mu_1\omega^2 t_c + (p^2 + p\operatorname{cth} p) Q. \tag{20}$$

where

$$\omega^2 = \frac{h^2}{R^2}\frac{2v_0^2 h^2}{a_1^2(1-\nu)}, \quad Q = \frac{h^3 Q_0}{2Rp\, a_1^2}, \quad t_c = \alpha_t t_0, \quad v_0^2 = \frac{E}{\rho}.$$

Applying the Laplace transform to equation (20) at homogeneous initial conditions, we find

$$\bar{w}(s) = \frac{\mu_1\omega^2\bar{t}_c + (s + \mu_1\sqrt{s}\operatorname{cth}\sqrt{s})\bar{Q}}{[s^2 + \omega^2(1+2\gamma_1)](s + \mu_1\sqrt{s}\operatorname{cth}\sqrt{s}) - 2\mu_1\gamma_1\omega^2}, \tag{21}$$

where

$$\bar{t}_c = \int_0^\infty t_c e^{-sf}\, df, \qquad \bar{Q} = \int_0^\infty Q e^{-sf}\, df.$$

The denominator in (21) has an infinite number of negative real roots, and also two complex roots with negative real parts.

Taking into account that for rigid bodies γ_1 is small compared with unity, we can expand the roots in a series

$$s = s_0 + s_1\gamma_1 + s_2\gamma_1^2 + \cdots, \tag{22}$$

whose coefficients are determined from the equations

$$(s_0^2 + \omega^2)(s_0 + \mu_1\sqrt{s_0}\operatorname{cth}\sqrt{s_0}) = 0,$$

$$s_1\left[6s_0^2 + 5\mu_1 s_0\sqrt{s_0}\operatorname{cth}\sqrt{s_0} - \mu_0(s_0^2 + \omega^2)\frac{1}{\operatorname{sh}^2\sqrt{s_0}} + \mu_1\omega^2\frac{\operatorname{cth}\sqrt{s_0}}{\sqrt{s_0}} + 2\omega^2\right] = \tag{23}$$

$$= 4\omega^2(\mu_1 - s_0 - \mu_1\sqrt{s_0}\operatorname{cth}\sqrt{s_0}), \text{ etc.}$$

Since the real parts of the denominator roots are negative, we obtain the damping of the natural vibrations caused by thermoelastic dissipation. If we write $\mu_1 = 0$, we obtain, as expected, undampened natural frequencies of the shell.

B. Consider now the longitudinal axisymmetrical vibrations of a heat-insulated infinite cylindrical shell ($\mu_1 = 0$), allowing for the heat fluxes inside it only ($\lambda \neq 0$). Introducing the dimensionless magnitudes

$$x_0 = Rx, \ \frac{v_0}{R}\tau_1 = \tau, \ \frac{u_0}{R} = u, \ \frac{w_0}{R} = w, \ \frac{R^2 h_1}{h} = \mu, \ \frac{Rv_0}{a_1} = \delta, \ \alpha_t T_1 = T, \ T_2 = 0, \tag{24}$$

we can write the equation of motion and the equation of heat conduction (13) in this case in the form

$$\frac{\partial^2 u}{\partial x^2} - (1-\nu^2)\frac{\partial^2 u}{\partial \tau^2} + \nu\frac{\partial w}{\partial x} - (1+\nu)\frac{\partial T}{\partial x} = 0,$$
$$\nu\frac{\partial u}{\partial x} + w + (1-\nu^2)\frac{\partial^2 w}{\partial \tau^2} - (1+\nu)T = 0, \qquad (25)$$
$$\frac{\partial^2 T}{\partial x^2} - \delta\frac{\partial T}{\partial \tau} - \gamma_1\delta\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial x} + w\right) = 0.$$

Writing

$$u = \frac{\partial}{\partial x}\left[1 + (1+\nu)\frac{\partial^2}{\partial \tau^2}\right]\Phi, \quad w = \frac{\partial^2 \Phi}{\partial x^2} - (1+\nu)\frac{\partial^2 \Phi}{\partial \tau^2},$$
$$T = \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial \tau^2} + \frac{\partial^4 \Phi}{\partial x^2 \partial \tau^2} - (1-\nu^2)\frac{\partial^4 \Phi}{\partial \tau^4}, \qquad (26)$$

we obtain from (25)

$$\frac{\partial^4 \Phi}{\partial x^4} + \frac{\partial^6 \Phi}{\partial x^4 \partial \tau^2} - \delta(1+2\gamma_1)\frac{\partial^3 \Phi}{\partial x^2 \partial \tau} - \frac{\partial^4 \Phi}{\partial x^2 \partial \tau^2} - \delta\gamma_2\frac{\partial^5 \Phi}{\partial x^2 \partial \tau^3} -$$
$$-(1-\nu^2)\frac{\partial^6 \Phi}{\partial x^2 \partial \tau^4} + \delta\gamma_2\frac{\partial^3 \Phi}{\partial \tau^3} + \delta(1-\nu^2)\frac{\partial^5 \Phi}{\partial \tau^5} = 0. \qquad (27)$$

We shall now study vibrations of the type

$$\Phi = \Phi_0 e^{i(sx + \omega\tau)}, \qquad (28)$$

where s and ω are generally speaking complex magnitudes, satisfying the equation

$$(1-\omega^2)s^4 - [\omega^2 - (1-\nu^2)\omega^4 + i\delta\omega^3(\gamma_2\omega^2 - 1 - 2\gamma_1)]s^2 - i\delta\omega^3[(1-\nu)\omega^2 - \gamma_2] = 0. \qquad (29)$$

Taking ω as real, we find

$$s_{1,2}^2 = \frac{1}{2(1-\omega^2)}[\omega^2 - (1-\nu^2)\omega^4 + i\delta\omega(\gamma_2\omega^2 - 1 - 2\gamma) \pm \sqrt{a+ib}], \qquad (30)$$

where $\alpha_{1,2} = \pm s_1, \quad \beta_{1,2} = \pm s_2,$

$$a = (1-\nu^2)^2\omega^8 - [\delta^2\gamma_2^2 + 2(1-\nu^2)]\omega^6 + [1 + 2\delta^2\gamma_2(1+2\gamma_1)]\omega^4 - \delta^2\omega^2(1+2\gamma_1),$$
$$b = 2\delta\{(1-\nu^2)(2-\gamma_2)\omega^7 - [\gamma_2 + (1-\nu^2)(1-2\gamma_1)]\omega^5 + 1 + 2\gamma_1\nu\}.$$

It follows from the analysis of (30) that the roots α_1, α_2 correspond to the longitudinal elastic wave, while β_1, β_2 characterize the heat wave. All roots are complex.

In the dimensionless system of coordinates adopted, the phase velocity v and the dampening coefficient q of the elastic wave will be equal to

$$v = \frac{\omega}{\operatorname{Re}\alpha_1}, \quad q = \operatorname{Im}\alpha_1, \qquad (31)$$

where α_1 is the root characterizing the diverging elastic wave.

In the initial system of coordinates we have

$$v = \frac{\omega}{\operatorname{Re}\alpha_1}v_0, \quad q = \operatorname{Im}\frac{\alpha_1}{R}. \qquad (32)$$

The expressions for the roots α_i being very bulky, we shall find the approximate values of v and q at small and large ω. At $\omega \ll 1$ we have

$$v = \left[1 + \frac{1}{2}\gamma_1(1+\nu)\right]v_0, \quad q_0 = \left[1 - \frac{\gamma_1}{2}(9-\nu)\right]\frac{\gamma_1(1-\nu)}{2R\delta}\omega^2. \tag{33}$$

At $\omega \gg \delta$ we obtain

$$v = \frac{v_0}{\sqrt{1-\nu^1}}, \quad q_\infty = \frac{\delta\gamma_1(1+\nu)}{2R\sqrt{1-\nu^2}}. \tag{34}$$

The dampening coefficient of the elastic wave is an increasing function of the square of the frequency for low frequencies ($\omega \ll 1$), and tends asymptotically to q_∞ when $\omega \to \infty$.

The phase velocity for low frequencies is larger by a factor of $1 + \gamma_1(1+\nu)/2$ than the velocity v_0 in an elastic bar, while at large frequencies $\omega \gg \delta$ it becomes equal to the wave velocity in an elastic plate.

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ON THE APPLICATION OF THE METHOD OF SUM REPRESENTATIONS TO THE NUMERICAL SOLUTION OF BIHARMONIC PROBLEMS

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Let
$$\Delta_h\Delta_h u = f(x, y), \tag{1}$$
where

$$\Delta_h\Delta_h u = \frac{1}{h^4}\begin{vmatrix} & & \alpha^4 & & \\ & 2\alpha & -4(1+\alpha^2)\alpha^2 & 2\alpha^2 & \\ 1 & -4(1+\alpha^2) & 6(1+\alpha^2)+8\alpha^2 & -4(1+\alpha^2) & 1 \\ & 2\alpha^2 & -4(1+\alpha^2)\alpha^2 & 2\alpha^2 & \\ & & \alpha^4 & & \end{vmatrix} u \tag{2}$$

($\alpha = \frac{h}{h_1}$, h, and h_1 are mesh sides of the regular rectangular net in the x and y directions) is an equation in finite differences corresponding to the biharmonic differential equation

$$\Delta\Delta U = f(x, y) \quad \left(\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right). \tag{3}$$

Applied to the finite-differences equation (1), the method of sum representations yields a solution for the internal points of an arbitrary rectangle $D(x_i, y_k)$ $(i = 2, 3, \cdots, m-1;\ k = 1, 2, \cdots, n)$ (see figure), either in an explicit form through the boundary conditions of the problem, or in the form of formulas containing a small number of parameters determined from the corresponding system of linear algebraic equations /1/.

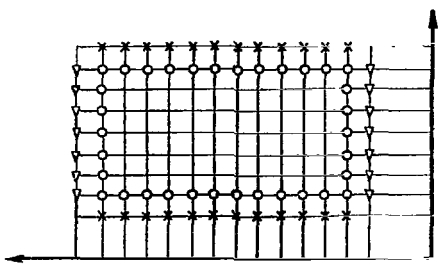

FIGURE.

We give here a certain modification of the formulas of sum representations /1, 2/ which simplifies the computing scheme and reduces the amount of calculations involved in the solution of the system of linear algebraic equations relative to the indeterminate parameters. Several numerical examples are solved for the case of a large number of nodal points (equal to 3600).

We introduce the designations

$$u_k(x) = u(x, y_k), \qquad f_k(x) = f(x, y_k), \tag{4}$$

the square matrix

$$P = \sqrt{\frac{2}{n+1}}\left[\sin\frac{ik\pi}{n+1}\right]_{i,k=1}^{n} \quad P^2 = E, \tag{5}$$

and the n-dimensional vectors

$$\begin{gathered}\vec{u}(x_i) = \{u_1(x_i), \quad u_2(x_i), \cdots, u_n(x_i)\},\\ \vec{f}(x_i) = \{f_1(x_i), \quad f_2(x_i), \cdots, f_n(x_i)\},\\ \vec{\omega}(x_i) = \{Qu_0(x_i), \quad \alpha^4 u_0(x_i), \quad 0, \cdots, 0, \alpha^4 u_{n+1}(x_i)\, Qu_{n+1}(x_i)\},\end{gathered} \tag{6}$$

where

$$\begin{gathered}Qu_k(x) = 2\alpha^2 u_k(x+h) - 4(1+\alpha^2)\alpha^2 u_k(x) + 2\alpha^2 u_k(x-h),\\ (k = 0, \ n+1).\end{gathered} \tag{7}$$

The vector defined by the equality

$$\vec{\hat{\sigma}} = \{\hat{\sigma}_1, \hat{\sigma}_2, \cdots, \hat{\sigma}_n\} = P\vec{\sigma} \tag{8}$$

will be called the P-transformation of the vector $\vec{\sigma} = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$.

The formula of sum representations suggested by us is of the form

$$\begin{gathered}\vec{u}(x_i) = P\alpha(x_i)\vec{A} + P\beta(x_i)\vec{B} + P\gamma(x_i)\vec{C} + P\delta(x_i)\vec{D} +\\ + \sum_{p=2}^{m-1} PD(i, p)\, P[h^4\vec{f}(x_p) - \vec{\omega}(x_p) \to \alpha^4\vec{w}(x_p)]\\ (i = 0, 1, \cdots, m+1),\end{gathered} \tag{9}$$

where

$$\begin{gathered}\vec{A} = \{A_1 A_2, \cdots, A_n\}, \qquad \vec{B} = \{B_1, B_2, \cdots, B_n\},\\ \vec{C} = \{C_1, C_2, \cdots, C_n\}, \qquad \vec{D} = \{D_1, D_2, \cdots, D_n\}\end{gathered} \tag{10}$$

are vectors of indeterminate parameters

$$\vec{w}(x_i) = \{u_1(x_i) + u_{-1}(x_i), \ 0, \cdots, \ 0, \ u_{n+2}(x_i) + u_n(x_i)\}, \tag{11}$$

$$\begin{gathered}\alpha(x_i) = [\alpha_k(x_i)], \quad \beta(x_i) = [\beta_k(x_i)], \quad \gamma(x_i) = [\gamma_k(x_i)],\\ \delta(x_i) = [\delta_k(x_i)],\\ D(i, p) = [D^{(k)}(i, p)],\end{gathered} \tag{12}$$

$(k = 1, 2, \cdots n)$ are diagonal matrices of order n,

$$D^k(i, p) = \begin{cases} \text{at.} \quad i = 0, 1, 2, 3 & p = 2, 3, \cdots, m-1 \quad D_1^{(k)}(i, p), \\ \text{at} \quad i = m+1 & p = 2, 3, \cdots, m-1 \quad D_2^{(k)}(i, p), \\ \text{at} \quad 4 \leqslant i \leqslant m, & \begin{cases} 2 \leqslant p \leqslant i-2 & D_1^{(k)}(i, p), \\ i-1 \leqslant p \leqslant m-1 & D_2^{(k)}(i, p). \end{cases} \end{cases} \tag{13}$$

The other magnitudes introduced here are determined for values of u given at all contour points of the rectangle, and under the condition that at the horizontal sides of the rectangle, the sums of the values of u outside and inside the contour are determined according to the type of the second boundary condition on the vertical sides of the rectangle. If the sums of the values of u inside and outside the contour are specified on the vertical sides of the rectangle, we have

$$\begin{gathered} \alpha_k(x_i) = (i-1)\operatorname{ch}(i-1)\mu_k, \\ \beta_k(x_i) = (i-1)\operatorname{sh}\mu_k \operatorname{ch}(i-1)\mu_k - \operatorname{ch}\mu_k \cdot \operatorname{sh}(i-1)\mu_k, \\ \gamma_k(x_i) = (i-m)\operatorname{ch}(i-m)\mu_k, \\ \delta_k(x_i) = (i-m)\operatorname{sh}\mu_k \cdot \operatorname{ch}(i-m)\mu_k - \operatorname{ch}\mu_k \cdot \operatorname{sh}(i-m)\mu_k, \\ \mu_k = \ln\left[a_k + \sqrt{a_k^2 - 1}\right], \quad a_k = 1 + \alpha^2 + \alpha^2\lambda_k, \quad \lambda_k = \cos\frac{k\pi}{n+1}, \end{gathered} \tag{14}$$

$$\begin{gathered} D_1^{(k)}(i, p) = -\frac{1}{\Delta_k}\{4(i-1)\operatorname{sh}\mu_k \operatorname{ch}(i-1)\mu_k \operatorname{sh}(m-p)\mu_k \operatorname{sh}(m-1)\mu_k + \\ + \operatorname{sh}(i-1)\mu_k[(2m-p)\operatorname{ch}p\mu_k - p\operatorname{ch}(2m-p)\mu_k + \\ + (p-2)\operatorname{ch}(2m-p-2)\mu_k - (2m-p-2)\operatorname{ch}(p-2)\mu_k]\}, \end{gathered} \tag{15}$$

$$\begin{gathered} D_2^{(k)}(i, p) = -\frac{1}{\Delta_k}\{4(i-m)\operatorname{sh}\mu_k \cdot \operatorname{ch}(i-m)\mu_k \operatorname{sh}(1-p)\mu_k \operatorname{sh}(m-1)\mu_k + \\ + \operatorname{sh}(i-m)\mu_k[(m-p+1)\operatorname{ch}(m+p-1)\mu_k - \\ - (m+p-1)\operatorname{ch}(m-p+1)\mu_k + (m+p-3)\operatorname{ch}(m-p-1)\mu_k - \\ - (m-p-1)\operatorname{ch}(m+p-3)\mu_k]\}, \\ \Delta_k = 8\operatorname{sh}^3\mu_k \cdot \operatorname{sh}^2(m-1)\mu_k. \end{gathered}$$

Actually, proceeding in the same way as in /1, 2/, we reduce the solution of the equation in finite differences (1) to the solution of the system of linear algebraic equations

$$\begin{gathered} \hat{u}_k(x+4h) - 4a_k\hat{u}_k(x+3h) + 4(a_k^2+2)\hat{u}_k(x+2h) - \\ - 4a_k\hat{u}_k(x+h) + \hat{u}_k(x) = h^4\hat{f}_k(x+h) - \\ - \hat{w}_k(x+h) - \alpha^4\hat{w}_k(x+h) \qquad (k = 1, 2, \cdots, n). \end{gathered} \tag{16}$$

Each of these equations will now be examined separately.

We take the functions (14) as linearly independent particular solutions of the homogeneous equation corresponding to (16). A particular solution of the nonhomogeneous equation will be sought, by the method of variation

of arbitrary constants, in the form

$$\hat{u}_k^r(x_i) = A_k(x_i)\alpha_k(x_i) + B_k(x_i)\beta_k(x_i) + C_k(x_i)\gamma_k(x_i) + D_k(x_i)\delta_k(x_i).$$

($A_k(x_i)$, $B_\nu(x_i)$, $C_k(x_i)$, $D_k(x_i)$ are some functions of x_i), requiring the satisfaction of the boundary conditions

$$\hat{u}_k^r(x_1) = 0, \qquad \hat{u}_k^r(x_0) + \hat{u}_k^r(x_2) = 0,$$

$$\hat{u}_k^r(x_n) = 0, \qquad \hat{u}_k^r(x_{m+1}) + \hat{u}_k^r(x_{m-1}) = 0.$$

We find

$$\hat{u}_k^r(x_i) = \sum_{p=2}^{m-1} D^{(k)}(i, p)\,[h^4 \hat{f}_k(x_p) - \hat{\omega}_k(x_p) - \alpha^4 \hat{w}_k(x_p)].$$

The general solution of (16) will therefore be

$$\hat{u}_k(x_i) = A_k\alpha_k(x_i) + B_k\beta_k(x_i) + C_k\gamma_k(x_i) + D_k\delta_k(x_i) +$$
$$+ \sum_{p=2}^{m-1} D^{(k)}(i, p)[h^4 \hat{f}_k(x_p) - \hat{\omega}_k(x_p) - \alpha^4 \hat{w}_k(x_p)].$$

The general solution of the system of n equations (16) will be

$$\vec{\hat{u}}(x_i) = \alpha(x_i)\vec{A} + \beta(x_i)\vec{B} + \gamma(x_i)\vec{C} + \delta(x_i)\vec{D} +$$
$$+ \sum_{p=2}^{m-1} D(i, p)\,[h^4 \vec{\hat{f}}(x_p) - \vec{\hat{\omega}}(x_p) - \alpha^4 \vec{\hat{w}}(x_p)]. \tag{17}$$

By multiplying both sides of (17) by the matrix P we obtain the wanted formula of sum representations (9).

The vectors of the indeterminate parameters are easily determined through the boundary conditions on the vertical sides of the rectangle, and namely:

$$\vec{A} = \Delta^{-1}\{[\beta(x_{m+1}) + \beta(x_{m-1})]\vec{\hat{u}}(x_m) - \beta(x_m)\vec{\hat{\beta}}(x_m)\},$$

$$\vec{B} = \Delta^{-1}\{\alpha(x_m)\vec{\hat{\mu}}(x_m) - [\alpha(x_{m+1}) + \alpha(x_{m-1})]\vec{\hat{u}}(x_m)\}, \tag{18}$$

$$\vec{C} = \Delta^{-1}\{-\delta(x_1)\vec{\hat{\lambda}}(x_1) + [\delta(x_0) + \delta(x_2)]\vec{\hat{u}}(x_1)\},$$

$$\vec{D} = \Delta^{-1}\{-[\gamma(x_0) + \gamma(x_2)]\vec{\hat{u}}(x_1) + \gamma(x_1)\vec{\hat{\lambda}}(x_1)\},$$

$$\Delta_k = \operatorname{sh} 2\mu_k \operatorname{sh}^2(m-1)\mu_k, \quad \vec{\lambda}(x_1) = \vec{u}(x_0) + \vec{u}(x_2), \tag{19}$$

$$\vec{\mu}(x_m) = \vec{u}(x_{m+1}) + \vec{u}(x_{m-1}).$$

We shall now consider several numerical examples.

Numerical example 1. Let it be required to solve the finite-differences biharmonic problem $\Delta_h\Delta_h u = 0$ in the rectangle $D(x_i, y_k)$ ($i = 1, 2, \cdots, 59$, $k = 1, 2, \cdots, 59$) at boundary conditions at the horizontal sides of the

rectangle

$$u_0(x_i) = 0, \quad u_{-1}(x_i) + u_1(x_i) = 0 \quad (i = 1, 2, \cdots, 59), \tag{20}$$

$$u_{n+1}(x_i) = 60(x^2 - 60x), \quad u_{n+2}(x_i) + u_n(x_i) = 120(x^2 - 60x)$$
$$(i = 1, 2, \cdots, 59), \tag{21}$$

and at the vertical sides of the rectangle

$$u_k(x_0) = 0, \quad u_k(x_{-1}) + u_k(x_1) = 2y \quad (y = 1, 2, \cdots, 59), \tag{22}$$

$$u_k(x_{m+1}) = 0, \quad u_k(x_{m+2}) + u_k(x_m) = 2y \quad (y = 1, 2, \cdots, 59). \tag{23}$$

We obtain from the formula of sum representations (9), in accordance with (18), (19):

$$\vec{u}(x_i) = P\alpha(x_i)P\vec{u}(x_0) + P\beta(x_i)P\vec{u}(x_{m+1}) + P\gamma(x_i)P[\vec{u}(x_{-1}) +$$
$$+ \vec{u}(x_1)] + P\delta(x_i)P[\vec{u}(x_m) + \vec{u}(x_{m+2})] + \vec{u}_r(x_i), \tag{24}$$

where $\alpha(x_i)$, $\beta(x_i)$, $\gamma(x_i)$, $\delta(x_i)$ are diagonal matrices with elements respectively

$$\alpha_k(x_i) = \frac{1}{\operatorname{sh}\mu_k \operatorname{sh}^2(m+1)\mu_k}\{(i-m-1)\operatorname{ch}\mu_k \operatorname{sh}(m+1)\mu_k \operatorname{ch}(i-m-1)\mu_k -$$
$$- \operatorname{sh}(i-m-1)\mu_k[m\operatorname{ch}\mu_k \operatorname{ch}(m+1)\mu_k + \operatorname{ch}(m+2)\mu_k]\},$$
$$\beta_k(x_i) = \frac{1}{\operatorname{sh}\mu_k \operatorname{sh}^2(m+1)\mu_k}\{-i\operatorname{ch}\mu_k \operatorname{ch} i\mu_k \operatorname{sh}(m+1)\mu_k + \tag{25}$$
$$+ \operatorname{sh} i\mu_k[m\operatorname{ch}\mu_k \operatorname{ch}(m+1)\mu_k + \operatorname{ch}(m+2)\mu_k]\},$$
$$\gamma_k(x_i) = \frac{1}{2\operatorname{sh}\mu_k \operatorname{sh}^2(m+1)\mu_k}\{-(i-m-1)\operatorname{sh}(m+1)\mu_k \operatorname{ch}(i-m-1)\mu_k +$$
$$+ (m+1)\operatorname{ch}(m+1)\mu_k \operatorname{sh}(i-m-1)\mu_k\},$$
$$\delta_k(x_i) = \frac{1}{2\operatorname{sh}\mu_k \operatorname{sh}^2(m+1)\mu_k} \times \tag{26}$$
$$\times\{i\operatorname{ch}\mu_k \operatorname{sh}(m+1)\mu_k - (m+1)\operatorname{sh} i\mu_k \operatorname{ch}(m+1)\mu_k\},$$
$$\vec{u}_r(x_i) = \sum_{p=1}^{m-1} PD(i,p)P[-\vec{\omega}(x_p) - \alpha^4\vec{w}(x_p)].$$

Allowing for (6) and (11), we see that $\vec{u}_r(x_i) = 0$ if the boundary conditions are zero on the horizontal sides of the rectangle:

$$u_0(x_i) = 0, \quad u_{n+1}(x_i) = 0, \quad u_{-1}(x_i) + u_1(x_i) = 0,$$
$$u_{n+2}(x_i) + u_n(x_i) = 0. \tag{27}$$

When these last conditions are fulfilled formula (24) acquires an exceedingly simple form:

$$\vec{u}_x(x_i) = P\alpha(x_i)P\vec{u}(x_0) + P\beta(x_i)P\vec{u}(x_{m+1}) + P\gamma(x_i)P[\vec{u}(x_{-1}) + \vec{u}(x_1)] +$$
$$+ P\delta(x_i)P[\vec{u}(x_m) + \vec{u}(x_{m+2})], \tag{24}$$

where the index x means that the zero boundary conditions (27) are satisfied on the horizontal sides of the rectangle.

It is easily seen that the wanted solution $\vec{u}(x_i)$ can be represented in the form

$$u_k(x_i) = u_k^{(1)}(x_i) + u_{60-i}^{(2)}(y_k) \tag{28}$$

$$(i = -1, 0, 1, \cdots, 61, \quad k = -1, 0, 1, \cdots, 61),$$

where $u_k^{(1)}(x_i)$ $(k = 1, 2, \cdots, 59)$ are components of the vector $\vec{u}_x(x_i)$ defined by equality (24); $u_{60-i}^{(2)}(y_k)$ $(60-i = 1, 2, \cdots, 59)$ are components of the vector $\vec{u}_y(y_k)$, defined by an equality similar to (24).

Thus, in order to calculate $\vec{u}(x_i, y_k)$, it suffices to calculate the k-th component of the vector $\vec{u}_x$ at point x_i and the $(60-i)$-th component of the vector $\vec{u}_y$ at point y_k, and to add them. The fundamental calculating operation is then, according to (24), the multiplication of the vectors of the boundary conditions by the matrix P, then by a diagonal matrix, and once more by the matrix P. This computing process is easily programed and, due to the character of the matrix P, does not require a large memory. We give here the values obtained by the method indicated on the "Ural" computer, and also obtained with the aid of desk calculators:

$$u(x_5, y_1) = -276, \quad u(x_5, y_2) = -555, \quad u(x_5, y_3) = -832,$$

$$u(x_5, y_4) = -1109, \quad u(x_5, y_5) = -1387,$$

$$u(x_5, y_6) = -1663, \quad u(x_5, y_7) = -1938, \quad u(x_5, y_8) = -2218,$$

$$u(x_5, y_9) = -2492, \quad u(x_5, y_{10}) = -2767.$$

The comparison of these numbers with the values of the exact solution of the corresponding differential biharmonic problem shows that the total error of the approximate values found lies within the limits of 0.1 to 0.9%. It follows in particular from here that the computing error is sufficiently small, in spite of the large number of nodal points (3600). This is due to the fact that, in accordance with the general idea underlying the method of sum representations /2/, the solution at a given nodal point is independent of the values obtained at most of the other nodal points of the grid.

Numerical example 2. Let the problem conditions be the same as in the preceding case, with the sole difference that at the points of the left-hand vertical side of the rectangle (x_0, y_{27}), (x_0, y_{28}), (x_0, y_{29}), (x_0, y_{30}), (x_0, y_{31}) are given the following differences between the values of the solution outside the contour and inside it, respectively: 3240, 3360, 3480, 3600, 3720.

By reasoning as in the preceding case we obtain a system of linear algebraic equations for determining the additional indeterminate parameters, representing the values of the solution at five points inside the contour. This system is:

$$64.85247\alpha_1 + 30.79238\alpha_2 + 26.72852\alpha_3 + 23.53494\alpha_4 + 21.04222\alpha_5 = -123798.96,$$

$$30.79238\alpha_1 + 64.92061\alpha_2 + 30.84745\alpha_3 + 26.77080\alpha_4 + 23.56329\alpha_5 = -11959.97,$$

$$26.72852\alpha_1 + 30.84745\alpha_2 + 64.96294\alpha_3 + 30.87578\alpha_4 + 26.78472\alpha_5 = -135084.70,$$

$$23.53494\alpha_1 + 26.77080\alpha_2 + 30.87578\alpha_3 + 64.97682\alpha_4 + 30.87578\alpha_5 = -133474.75,$$

$$21.04222\alpha_1 + 23.56329\alpha_2 + 26.78472\alpha_3 + 30.87578\alpha_4 + 64.96294\alpha_5 = -126691.22.$$

By solving this system we find the numerical values of the sums of the values of the solution inside the contour and outside it at the points

$$(x_1, y_{27}), \ (x_1, y_{28}), \ (x_1, y_{29}), \ (x_1, y_{30}), \ (x_1, y_{31}). \tag{29}$$

The calculation proceeds then in the same way as in the preceding example. The calculations conducted for points (29) of the rectangle $D(x_i, y_k)$ show that the total error remains sufficiently small and lies within the limits of 0.1 to 0.9%.

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STABILITY OF RECTANGULAR FLEXIBLE PLATES MADE FROM NONLINEARLY ELASTIC MATERIALS

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This paper treats the problem of determining the critical stresses and the pattern of deformation after buckling of a flexible rectangular plate made from a nonlinearly elastic material.

The theory of stability of flexible plates, which is being widely applied lately, is based on Hooke's law, i.e., on a linear stress-strain relationship.

There exist, however, many materials (cast iron, aluminum alloys, high-strength alloys, certain plastics) which do not behave according to this law. The stress-strain relationship for such materials is nonlinear and is established experimentally. The empirically obtained stress-strain diagram is then approximated by an analytical law, as shown in /4/, and it is this law which is used in the theoretical calculations.

In the solution of this problem use is made of all the initial assumptions and hypotheses of the theory of lateral bending of flexible plates made from a nonlinearly elastic material.

The stressed state of the plate is characterized by the theory of small elastic-plastic deformations, with the addition of the following hypotheses:

a) the plate material is incompressible;

b) the loading is assumed to be simple and the strains can be only active;

c) the stress-strain relationship is assumed to be exponential

$$\sigma_i = Ae_i^k, \quad \text{where } 0 \leqslant k \leqslant 1, \tag{1}$$

and

$$e_i = \frac{2}{\sqrt{3}} \sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_x \varepsilon_y + \frac{1}{4}\gamma_{xy}^2}\,. \tag{2}$$

The parameters A and k are determined from the equality of the ultimate strengths of the experimental and analytical curves and the equality of their specific energies, in the manner shown in /4/.

This law reduces in the particular case $k = 1$, $A = E$ to Hooke's law, and in the particular case $k = 0$, $A = \sigma_Y$ to the law of a rigid-plastic body.

Under these hypotheses

$$\begin{aligned} \sigma_x &= \frac{2}{3} A e_i^{k-1} (2\varepsilon_x + \varepsilon_y), \\ \sigma_y &= \frac{2}{3} A e_i^{k-1} (2\varepsilon_y + \varepsilon_x), \\ \tau_{xy} &= \frac{1}{3} A e_i^{k-1} \gamma_{xy}\,. \end{aligned} \tag{3}$$

The deformed state of the plate is characterized by the Kirchhoff hypothesis

$$\begin{aligned} \varepsilon_x &= e_x - z\varkappa_x, \\ \varepsilon_y &= e_y - z\varkappa_y, \\ \gamma_{xy} &= 2(\gamma_{xy}^0 - z\varkappa_{xy}), \end{aligned} \tag{4}$$

where z is the distance from the neutral layer to the fiber considered; e_n and γ_n are strains of the middle plane of the plate, equal respectively to

$$\begin{aligned} e_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ e_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial x}, \end{aligned} \tag{5}$$

where u, v, and w are displacements of the points of the middle plane of the plate in the direction of the corresponding coordinate axes.

We shall use the energy method in solving this problem.

The total energy of the plate is made of the strain energy U and the potential energy V of the load acting on the plate,

$$W = U + V. \tag{6}$$

Under the above-adopted hypotheses the strain energy is written in the following form:

$$U = \frac{A}{k+1}\left(\frac{2}{\sqrt{3}}\right)^{k+1}\iiint\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_x\varepsilon_y + \frac{1}{4}\gamma_{xy}^2\right)^{\frac{k+1}{2}} dxdydz. \tag{7}$$

Using relationships (5), the total energy of the plate is expressed as a function of the three displacements:

$$W = W(u, v, w). \tag{8}$$

Consider a square plate simply supported in the direction of the x axis. The coordinate axes are oriented as shown in Figure 1.

Lateral expansion in the y direction is prevented by a rigid frame.

The plate being square, it can be assumed that buckling occurs in the direction of each side of the plate in one half-wave; the longitudinal and lateral edges of the plate remain straight throughout the deformation, the longitudinal edges sliding freely along the supports.

As a first approximation of the displacements, satisfying the boundary conditions, we take the following expressions:

$$\begin{aligned} u &= c\sin\frac{\pi x}{a}\cos\frac{\pi y}{2a} - ex, \\ v &= c\sin\frac{\pi y}{a}\cos\frac{\pi x}{2a}, \\ w &= w_0\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2a}, \end{aligned} \tag{9}$$

where c, w_0, and e are constants. It is obvious that the displacements v become equal to zero at the edges $y = \pm a$, and that at the boundaries $x = \pm a$

the displacements u are equal to the displacements caused by the uniform compression strain e in the x direction. The plate edges $x = \pm a$ are subjected to compressive forces, whose mean value per unit length will be designated by N_1. The edges $y = \pm a$ are subjected to internal forces of mean intensity N_2.

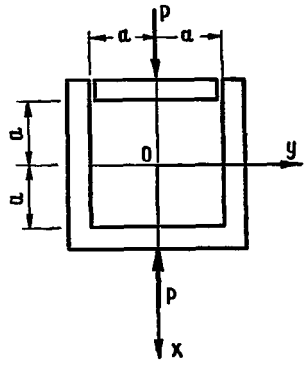

FIGURE 1.

At given displacement functions (9) the work of the external forces is written in the following form:

$$V = 4a^2 N_1 e. \tag{10}$$

Using expressions (9) for the displacements u, v, and w, and integrating expression (8) with the aid of the cubature formula

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x, y, z)\, dxdydz = \frac{4}{3}\Big[f(1,0,0) + f(-1,0,0) + f(0,-1,0) + f(0,1,0) + f(0,0,1) + f(0,0,-1)\Big] \tag{11}$$

we obtain the total energy of the plate, written in dimensionless parameters:

$$W = \frac{Ah}{k+1}\left(\frac{2}{\sqrt{3}}\right)^{k+1}\frac{2}{3}a^2\, 0.5^{\frac{k+1}{2}}\left(\frac{\pi}{2a}\right)^{2(k+1)} h^{2(k+1)}\Big\{\Big[0.125 f_0^4 - 1.4142 c_1 f_0^2 + 4.5 c_1^2 - f_0^2 e_1 + 5.6568 c_1 e_1 + 2e_1^2\Big]^{\frac{k+1}{2}} + \Big[0.125 f_0^4 + 0.7071 c_1 f_0^2 + 4 c_1^2 - 0.5 f_0^2 e_1 - 5.6568 c_1 e_1 + 2e_1^2\Big]^{\frac{k+1}{2}} + \Big[0.125 f_0^4 - 1.4142 c_1 f_0^2 + 4.5 c_1^2 - 0.5 f_0^2 e_1 + 2.8284 c_1 e_1 + 2e_1^2\Big]^{\frac{k+1}{2}} + \Big[0.125 f_0^4 + 0.7071 c_1 f_0^2 + 4 c_1^2 - f_0^2 e_1 - 2.8284 c_1 e_1 + 2e_1^2\Big]^{\frac{k+1}{2}} + \Big[(0.125 f_0^4 - 0.35355 c_1 f_0^2 + c_1^2 + 0.75 f_0^2 + 2e_1^2) + (0.25 - 1.5 f_0 e_1 + 0.7071 c_1 f_0) + 0.5 f_0^2\Big]^{\frac{k+1}{2}} + \Big[(0.125 f_0^4 - 0.35355 c_1 f_0^2 + c_1^2 + 0.75 f_0^2 + 2e_1^2) - (0.25 - 1.5 f_0 e_1 + 0.7071 c_1 f_0) + 0.5 f_0^2\Big]^{\frac{k+1}{2}}\Big\} - \pi^2 h^2 N_1 e_1, \tag{12}$$

where

$$c_1=\frac{2ac}{\pi h^2};\quad e_1=\frac{4a^2e}{\pi^2h^2};\quad f_0=\frac{w_0}{h}. \tag{13}$$

At given unit compression e the constants c and w_0 in (9) can be found from the condition of minimum of the total energy of the plate:

$$\frac{\partial W}{\partial c_1}=0, \tag{14}$$

$$\frac{\partial W}{\partial f_0}=0. \tag{15}$$

These two equations lead to a system of two nonlinear algebraic equations.

At $k=1$, $A=E$ it follows from (14) that the parameter

$$c=0.0055823\,f_0^2, \tag{16}$$

and equation (16) yields the relationship between the parameters e and f_0^2:

$$e_1=0.222+0.3719\,f_0^2. \tag{17}$$

Timoshenko's solution for such a plate at $\mu = 0.3$ yields the following expression for e_1:

$$e_1=0.2565+0.36\,f_0^2.$$

A system of equations of this type cannot be solved rigorously in the general case for an arbitrary value of k. We shall proceed as follows in order to solve it approximately.

We assume for c_1 and f_0, with an accuracy of up to a constant factor, the relationship of the elastic problem:

$$c_1=m\,f_0^2, \tag{18}$$

where the parameter "m" is independent of the parameter f_0.

It also follows from the solution of the elastic problem for $k = 1$ that relationship (18) will remain correct at any value of e_1, including $e_1 = 0$.

Under the assumptions adopted equations (14) and (15) become

$$\frac{\partial W}{\partial c_1}=f_1\,(m,\,k)=0, \tag{19}$$

$$\frac{\partial W}{\partial f_0}=f_2\,(m,\,f_0,\,e,\,k)=0. \tag{20}$$

In other words, the system of equations (14) and (15) splits into two independent equations, the first relative to the parameter "m", while the second one gives the connection between e and f_0.

The parameter "m" can be found by solving approximately equation (19) for every value of the index k.

Equation (20) can likewise be solved for each value of the index k by plotting the e_1-f_0 curve. It is then necessary to find the value of e_1 corresponding to every value of the deflection parameter. A large amount of calculations is involved in this procedure. Every value of e_1 must be determined by trial and error, leading to great inaccuracy.

There exists, however, a second procedure for solving the problem, which consists in the following:

In the theory proposed the deformed state of the plate is described by the same relationships as in the theory of elastic plates. It can therefore be assumed that the final relationship between the parameters e_1 and f_0 in the general case of a nonlinearly elastic plate will be the same, up to constant factors, as in the elastic problem.

The first term on the right hand of (17) represents the value of the critical strain, while the second term characterizes the strain due to the bending of the plate.

The solution of the system of equations (19) and (20) for an arbitrary value of the index k can be written in the form

$$e_1 = e_1^{cr} + Lf_0^2, \tag{21}$$

where e_1^{cr} is the value of the critical strain of a nonlinearly elastic plate, depending on the index k (its determination will be considered later), and L is a magnitude likewise dependent on the index k and independent of the parameter f_0.

Substituting $f_0 = 1$ in (20), we obtain

$$\frac{\partial W}{\partial f_0} = f_3\ (m,\ e_1^{cr},\ L,\ k) = 0. \tag{22}$$

In order to determine the strain behavior of the plate after buckling it remains to find e_1^{cr} as a function of k. The magnitude e_1^{cr} represents the plate strain corresponding to the beginning of the bending, i.e., the point $f_0 = 0$ will be one of the extreme points of the $e_1 - f_0$ curve, and therefore the condition $\left.\frac{\partial e_1}{\partial f_0}\right|_{(f_0 \to 0)} = 0$ must be fulfilled at this point.

Taking the derivative of (20) as of an implicit function of f_0, and equating $\frac{\partial e_1}{\partial f_0}$ and f_0 to zero, we obtain the value of the critical strain:

$$e_1^{cr} = 0.222 - 0.125\ (1-k). \tag{23}$$

The critical strain of a plate made from a nonlinearly elastic material thus represents a certain part of the critical strain calculated by Hooke's law.

The solution of (22) at $k = 0$ yields the value $L = 0.3511$. When the index k varies from 1 to 0 this coefficient will vary within the limits

$$0.3719 > L > 0.3511.$$

The range of variation of L being small, we can assume with a certain approximation that L is independent of k and determine its value from the solution of the elastic problem. The error allowed as a result in the determination of the strain will be largest at $k = 0$ and $f_0 = 5$, when it will be equal to 5.92%. At other values $k \neq 1$ and $k \neq 0$ it will be considerably smaller, decreasing still further in the determination of the stresses.

Thus, at the price of a negligible error in the determination of the strain, we avoid the numerical solution of a complex system of equations.

It follows that we can write relationship (21), which gives the strain law after buckling, in the form

$$e_1 = 0.222 - 0.125\ (1-k) + 0.3719 f_0^2. \tag{24}$$

In other words, the strain of a nonlinearly elastic plate can be considered as the strain of some equivalent elastic plate, and it can be assumed in addition that the plate strain caused by the bending is independent of the physical properties of the material, and depends on the boundary conditions only.

In the initial parameters relationship (24) will be written as follows:

$$e=\frac{\pi^2 h^2}{(2a)^2}\ [0.222-0.125\ (1-k)+0.3719 f_0^2]. \tag{25}$$

Knowing the value of the strain, it is now easy to determine the forces acting at the middle points of the plate edges.

According to the general theory of nonlinearly elastic bending of a plate /6/, the expressions for N_1 and N_2 are

$$N_x=\left(\frac{2}{\sqrt{3}}\right)^{k+1} A\ [J_1\ (e_x+0.5 e_y)-J_2\ (\varkappa_x+0.5\varkappa_y)],$$

$$N_y=\left(\frac{2}{\sqrt{3}}\right)^{k+1} A\ [J_1\ (0.5 e_x+e_y)-J_2 (0.5\ \varkappa_x+\varkappa_y)],$$

where

$$J_1=\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \psi^{\frac{k-1}{2}}\, dz, \qquad J_2=\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \psi^{\frac{k-1}{2}}\, z dz,$$

and $\qquad \psi=[(e_x-z\varkappa_x)^2+(e_y-z\varkappa_y)^2+(e_x-z\varkappa_x)(e_y-z\varkappa_y)+(\overset{0}{\gamma}_{xy}-z\varkappa_{xy})^2].$

In our case $e_x=e,\ e_y=\overset{0}{\gamma}_{xy}=0.$

At the points $x=\pm a$ and $y=\pm a$ $\quad \varkappa_x=\varkappa_y=\varkappa_{xy}=0$

$y=0 \qquad x=0$

Then $\psi=e_x^{k-1}, \quad J_1=e^{k-1} h, \quad J_2=0.$

We obtain for N_1 and N_2:

$$N_1=\left(\frac{2}{\sqrt{3}}\right)^{k+1} Ahe^k,$$

$$N_2=0.5\left(\frac{2}{\sqrt{3}}\right)^{k+1} Ahe^k, \tag{26}$$

or, with the aid of (25)

$$N_1=\left(\frac{2}{\sqrt{3}}\right)^{k+1} \frac{\pi^{2k} A h^{2k+1}}{(2a)^{2k}}\ [0.222-0.125\ (1-k)+0.3719 f_0^2]^k, \tag{27}$$

$$N_2=0.5\ N_1.$$

At $k=1$, $A=E$ we obtain the value of the longitudinal force for the elastic problem:

$$N_1=0.296\frac{\pi^2 E h^3}{(2a)^2}+0.49586\frac{\pi^2 E h^3}{(2a)^2} f_0^2 . \tag{28}$$

The first term of the equation obtained corresponds to the value of the critical force and coincides with the value given by Timoshenko /7/ and

by Vol'mir /2/. The second term, which corresponds to the value of the longitudinal force appearing as a result of the plate bending, coincides with the Timoshenko solution and differs from the Vol'mir solution. This discrepancy is due to our use of boundary conditions at the lateral edges different from those used by Vol'mir.

At $k=0$, $A=\sigma_Y$ we obtain the case of a rigid-plastic body:

$$N_1 = \frac{2}{\sqrt{3}} \sigma_Y h. \tag{29}$$

This result coincides with the result for a plate undergoing small deflections as ought to be expected.

The same result is also obtained by using the theory of limiting equilibrium.

In order to pass to the stressed state of the plate we introduce the additional dimensionless parameters

$$\sigma_1^* = \frac{N}{Eh}\left(\frac{2a}{h}\right)^2 \beta = \frac{E}{A}; \qquad a_* = \frac{2a}{h}. \tag{30}$$

Relationship (28) then becomes

$$\sigma_1^* = \left(\frac{2}{\sqrt{3}}\right)^{k+1} \frac{\pi^{2k}}{a_*^{2k-2}\,\beta} [0.222-0.125\,(1-k)+0.3719\, f_0^2]^k. \tag{31}$$

Curves of the critical stress parameter as a function of the deflection parameter have been plotted in Figure 2 for several values of a_* and for k equal to 1, 0.137, and 0. It is seen from them that the critical stress parameter increases with the index k and the parameter a_*. Both these conclusions agree with the physical interpretation of the problem.

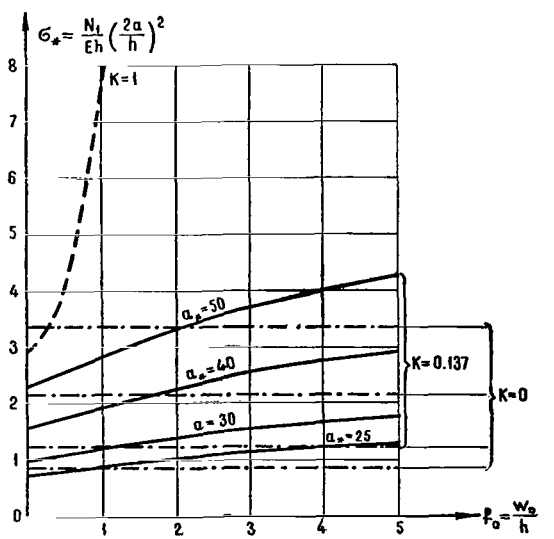

FIGURE 2.

The shape of the curves given is similar to that of the curves obtained by Bozhinskii /8/.

The method described here can be applied to other similar problems as well. This lies, however, outside the scope of this paper.

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ON THE CALCULATION OF HANGING ROOFINGS

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The fundamental equations of the problem of a hanging roofing circular in the plan and made of two systems of prestressed shallow fibers (upper and lower) were given in /1/ and solved in the axisymmetrical case.*

We consider here the nonaxisymmetrical case of such a roofing subjected to the simultaneous action of a uniformly distributed and a snow load:

$$q = q_0 + q_1 \cos\varphi, \tag{1}$$

where q_0 and q_1 are constant magnitudes, and $P_2 = M_2 = 0$ (for the meaning of the symbols, cf. /1/).

It follows in this case from the condition of symmetry relative to the axis $\varphi = 0$ that $u_2 = \gamma_2 = 0$ and that the equilibrium equation of the upper fiber has the following particular solution:

$$w^* = -\frac{qx^3}{6N_1}.$$

The problem considered reduces to solving a system of seven equations: the geometrical boundary conditions at the external and internal contours

$$\begin{gathered} u(R, \varphi) = w(R, \varphi) = 0, \\ w(r, \varphi) = w_0 + (r - \alpha_1 h)\gamma_1 \cos\varphi, \\ u(r, \varphi) = w_0\alpha_1 + u_1 \cos\varphi + (h + \alpha_1 r)\gamma_1 \cos\varphi; \end{gathered} \tag{2}$$

the equations expressing the conditions of equilibrium of the internal supporting ring

$$\begin{gathered} P_1 = -\int_0^{2\pi} (N_1 + N_2)\cos\varphi\, d\varphi, \\ P_3 = r\int_0^{2\pi} (Q_1 + Q_2)\, d\varphi, \\ M_1 = -\int_0^{2\pi} (N_1\delta_1 - N_2\delta_2)\cos\varphi\, d\varphi. \end{gathered} \tag{3}$$

* The expression for ρ_1 in /1/ contains a misprint. The correct expression is

$$\rho_1^2 = \frac{1}{360}(4 + 7k + 4k^2)(1 - k)^2.$$

Equations (2) and (3) contain the following unknowns:

$$C_1(\varphi), \quad C_2(\varphi), \quad C_3(\varphi), \quad N_1(\varphi), \quad u_1, \quad w_0, \quad \gamma_1.$$

By eliminating $C_1(\varphi)$, $C_2(\varphi)$, and $C_3(\varphi)$ from the four equations of (2) we obtain a cubic equation for the determination of the relative elongation of the upper ropes:

$$\varepsilon^3 + (b_0 + b_1 \cos\varphi)\,\varepsilon^2 - aq^2 = 0, \tag{4}$$

where

$$b_0 = -\varepsilon_1 - \frac{1}{2}\zeta^2 + \alpha_1\zeta,$$

$$b_1 = \xi + \frac{h + \alpha_1 r}{R - r}\gamma_1 - \frac{r - \alpha_1 h}{R - r}\zeta\gamma_1,$$

$$a = \frac{\beta_1^2 R^4}{360 D_1^2}(4 + 7k + 4k^2)(1 - k)^2,$$

$$\zeta = \frac{w_0}{R - r}, \qquad \xi = \frac{u_1}{R - r}, \qquad \varepsilon = \frac{\beta_1 N_1}{D_1}.$$

It was assumed in the derivation of (4) that the magnitude $(r - \alpha_1 h)^2 \gamma_1 \cos\varphi$ can be neglected in comparison with $2(R - r)(h + \alpha_1 r)$.

Equations (3) and (4) represent thus a system of four equations with unknowns ζ, ξ, γ_1, ε.

This complex system of nonlinear equations can be solved in the following way. We assume some values for the magnitudes ζ, ξ, γ_1, and calculate from (4) the value of ε corresponding to given values of φ. We obtain as a result the values of the integrands of (3) at different points, using which it is possible to calculate approximately the integrals. The left- and right-hand sides of these equations are then compared, new values of the magnitudes ζ, ξ, γ_1 are selected on the basis of this comparison, and the process is repeated.

In a first approximation the values of the unknowns ζ, ξ, γ_1 can be found under the condition that ε can be represented in the form of the sum

$$\varepsilon = \varepsilon_0 + f(\varphi), \tag{5}$$

in which ε_0 is a certain constant magnitude, and $\left(\frac{f}{\varepsilon_0}\right)^2 \ll 1$. As will be shown by example, such a representation will be correct for the elongation of the upper ropes even in the case when the shell is loaded by a surface load very different from an axisymmetrical load, and the solution obtained need not be refined.

In this case ε_0 must satisfy in accordance with (4) the equation

$$\varepsilon_0^3 + b_0\varepsilon_0^2 - aq_0 = 0, \tag{6}$$

while for the function characterizing the deviation of ε from ε_0 we obtain

$$f = \frac{a(q^2 - q_0^2) - b_1\varepsilon_0^2\cos\varphi}{3\varepsilon_0^2 + 2b_0\varepsilon_0 + 2b_1\varepsilon_0\cos\varphi}. \tag{7}$$

All this remains correct even in the more general case of shell loading by a surface load, namely in the case when $q(\varphi) = q(-\varphi)$; in this case q_0 represents some "mean" value of $q(\varphi)$.

By substituting (5) in (3) and integrating we obtain

$$\frac{P_1}{\pi} = -\frac{D_2\mu}{\beta_2} - \frac{2D_1J_2}{\beta_1},$$

$$\frac{P_3}{\pi} = \frac{2D_1\gamma_1J_2(r-\alpha_1h)}{(R-r)\beta_1} - \frac{2D_1(\alpha_1-\zeta)(\varepsilon_0+J_1)}{\beta_1} +$$

$$+\frac{D_2}{\beta_2}\left[2\psi(\alpha_2+\zeta)+\frac{r-\alpha_2h}{R-r}\mu\gamma_1\right]-\frac{q_0R^2}{3}(1+k-2k^2), \qquad (8)$$

$$\frac{M_1}{\pi} = -\frac{2D_1J_2}{\beta_1}[h+r(\alpha_1-\zeta)]+\frac{rRD_1\gamma_1}{(R-r)\beta_1}(\varepsilon_0+J_3)+$$

$$+\frac{D_2}{\beta_2}\left[(h+\alpha_2r+\zeta r)\mu+\frac{rR\psi\gamma_1}{R-r}\right]-\frac{rR^2q_1}{6}(1+k-2k^2),$$

where

$$J_1 = \frac{Bl-atq_1^2}{l^2}\left(1-\frac{t}{\sqrt{t^2-l^2}}\right),$$

$$J_2 = \frac{aq_1^2}{2l}+\frac{(atq_1^2-Bl)t}{l^3}\left(1-\frac{t}{\sqrt{t^2-l^2}}\right),$$

$$J_3 = \frac{Bl-atq_1^2}{l^4}\left[2t^2\left(1-\frac{t}{\sqrt{t^2-l^2}}\right)+l^2\right], \qquad (9)$$

$$\mu = \frac{h+\alpha_2r}{R-r}\gamma_1+\frac{r-\alpha_2h}{R-r}\zeta\gamma_1-\xi, \qquad \psi = \varepsilon_2+\alpha_2\zeta+\frac{1}{2}\zeta^2,$$

$$B = 2aq_0q_1-b_1\varepsilon_0^2, \qquad t = 3\varepsilon_0^2+2b_0\varepsilon_0, \qquad l = 2b_1\varepsilon_0,$$

and it is assumed that $t > l$.

In order to simplify the system of equations still further we shall assume that, from physical considerations, the terms in the second equation of (8) which contain γ_1 and ξ are small and can be neglected. This equation reduces then to

$$\zeta^3+3\alpha_2\zeta^2+2(\varepsilon_2+\alpha_2^2)\zeta+2\nu\varepsilon_0(\zeta-\alpha_1)+2\alpha_2\varepsilon_2 =$$

$$= \frac{\beta_2}{\pi D_2}\left[P_3+\frac{\pi q_0R^2}{3}(1+k-2k^2)\right]. \qquad (10)$$

By solving simultaneously (6) and (10) we find the unknown magnitudes ζ and ε_0. This system can be solved in the following way. We assume a certain value of ζ and find from (6) the single positive value of ε_0; the left-hand part of (10) is then calculated and compared with its right-hand part. A new value of ζ is selected if necessary, and the process is repeated.

After the magnitudes ζ and ε_0 have been found, the remaining two unknowns ξ and γ_1 are determined from the first and third equations of (8). Applying to (9) the rule of approximate calculation of the square root, we obtain

$$J_2 \approx \frac{B}{2t}.$$

Assuming further that $J_3 \ll \varepsilon_0$, we obtain the following system of linear

algebraic equations for the determination of ξ and γ_1:

$$\left(\frac{D_1\varepsilon_0^2}{t\beta_1}+\frac{D_2}{\beta_2}\right)\xi+\frac{\gamma_1}{R-r}\left\{\frac{D_1\varepsilon_0^2}{t\beta_1}[h+r(\alpha_1-\zeta)]-\right.$$

$$\left.-\frac{D_2}{\beta_2}[h+r(\alpha_2+\zeta)]\right\}=\frac{P_1}{\pi}+\frac{2aq_0q_1D_1}{t\beta_1},$$

$$\left\{\frac{D_1\varepsilon_0^2}{t\beta_1}[h+r(\alpha_1-\zeta)]-\frac{D_2}{\beta_2}[h+r(\alpha_2+\zeta)]\right\}\xi+$$

$$+\frac{\gamma_1}{R-r}\left\{\frac{D_1\varepsilon_0^2}{t\beta_1}[h+r(\alpha_1-\zeta)]^2+\frac{D_2}{\beta_2}[h+r(\alpha_2+\zeta)]^2+\right.$$

$$\left.+rR\left(\psi+\frac{D_1\varepsilon_0}{\beta_1}\right)\right\}=\frac{M_1}{\pi}+\frac{q_1rR^2}{6}(1+k-2k^2)+$$

$$+\frac{2aq_0q_1D_1}{t\beta_1}[h+r(\alpha_1-\zeta)]. \qquad (11)$$

After the unknowns ζ, ξ, γ_1, ε_0 have been thus determined, it remains to check the accuracy of the assumptions used for simplifying the problem. To that end, the values of P_1, P_3, M_1 are calculated by (8) and compared with the specified values. It is necessary in addition to test the accuracy of the assumption $\left(\frac{f}{\varepsilon_0}\right)^2 \ll 1$.

Example of calculation. It is required to check the section of the ropes of a hanging roofing circular in the plan and its rigidity for the following values: R = 32.0 m; r = 2.5 m, α_1 = 0.07; α_2 = 0.10; $\beta_1 = \beta_2$ = 0.224; upper ropes Φ 55 mm (F_1 = 13.78 cm²), lower ones Φ 59 mm (F_2 = 15.96 cm); collapsing forces for the upper and lower ropes, respectively, equal to P_u = 169 t and P_l = 195 t; modulus of elasticity of the steel ropes $E = 1.7\times10^6$ kg/cm², constant vertical load 400 kg/m², snow load 140 kg/m²; dead weight of the inner ring P = 5 t; prestressing of the lower rope H_2 = 40 t (the load is given with the corresponding overload coefficients).

We first determine the value of the tension of the upper ropes which ensures the specified structure geometry before its loading by a surface load. Using the last formula of /1/ we find

$$H_1=\frac{1}{\alpha_1}\left(\alpha_2H_2-\frac{P\beta_1}{2\pi}\right)=\frac{1}{0.07}\left(0.1\cdot40-\frac{0.224\cdot5}{2\cdot3.14}\right)=54.6\ \text{t}.$$

1. When the shell is calculated for the case of the simultaneous action of a constant load and a uniformly distributed snow load, we have: $q_0 = 400+140 = 540$ kg/cm²; $q_1 = 0$, $P_1 = P_2 = M_1 = M_2 = 0$, $P_3 = 15.6$ t (axisymmetrical loading).

In this case we obtain from the simultaneous solution of (6) and (10): $\zeta = 4.43\cdot10^{-2}$; $\varepsilon_0 = 0.67\cdot10^{-2}$. The force acting in the upper rope will be

$$T_1+H_1=\varepsilon_0EF_1=0.67\cdot10^{-2}\cdot1.7\cdot10^7\cdot13.78\cdot10^{-4}=157\ \text{t}<169\ \text{t}$$

and that acting in the lower rope (cf. /1/)

$$T_2+H_2=D_2\left(\alpha_2\zeta+\frac{1}{2}\zeta^2\right)+H_2=1.7\cdot10^7\cdot15.96\cdot10^{-4}\,(0.1\cdot4.43\cdot10^{-2}+$$

$$+\frac{1}{2}4.43^2\cdot10^{-4})+40=187\ \text{t}<195\ \text{t}.$$

The deflection of the inner ring of the roofing will be

$$w_0 = \zeta (R - r) = 4.43 \cdot 10^{-2} (32 - 2,5) = 1.31 \text{ m}.$$

2. We shall now calculate the case when the snow load is given by the formula $70(1 + \cos\varphi)$ kg/m². We must write then in (1): $q_0 = 470$ kg/m²; $q_1 = 70$ kg/m². Then $P_1 = 0$, $P_3 = 14.1$ t,

$$M_1 = \frac{\pi q_1 r^3}{3} = \frac{3.14 \cdot 0.07 \cdot 2.5^3}{3} = 1.15 \text{ t}\cdot\text{m}.$$

The simultaneous solution of (6) and (10) yields

$$\zeta = 4.02 \cdot 10^{-2}; \quad \varepsilon_0 = 0.612 \cdot 10^{-2}.$$

By solving system (11) allowing for the values obtained for ζ and ε_0 we obtain

$$\xi = 0.986 \cdot 10^{-3}; \quad \gamma_1 = 1.04 \cdot 10^{-2}.$$

From (5) and (7) we obtain

$$\varepsilon = \left(0.612 + \frac{-0.028 \cos\varphi + 0.051 \cos^2\varphi}{10.8 + 2.32 \cos\varphi}\right) 10^{-2}.$$

The value of ε will be maximum at $\varphi = -\pi$: $\varepsilon_{max} = 0.621 \times 0^{-2}$; the maximum tension of the upper rope will be

$$T_1 + H_1 = 145 \text{ t} < 169 \text{ t}.$$

The forces acting in the lower ropes, determined by the formula /1/

$$T_2 + H_2 = D_2 \left[a_2 \zeta + \frac{1}{2} \zeta^2 + \left(\frac{h + r a_2}{R - r} \gamma_1 - \xi + \frac{r - h a_2}{R - r} \zeta \gamma_1 \right) \cos\varphi + \right.$$
$$\left. + \frac{(r - h a_2)^2}{2 (R - r)^2} \gamma_1^2 \cos^2\varphi \right] + H_2,$$

will be equal (in tons) to

$$T_2 + H_2 = 171 - 0.27 \cos\varphi + 0.08 \cos^2\varphi.$$

It is thus seen that the forces acting in the upper and lower ropes vary very little with the variation of the angle φ.

By substituting the values found for the unknowns in the system of equations (8) we observe that no further refinement of the solution is necessary. In addition, $\left(\frac{f}{\varepsilon_0}\right)^2_{max} = 2.2 \times 10^{-4}$, and therefore the assumption $\left(\frac{f}{\varepsilon_0}\right)^2 \ll 1$ is fulfilled.

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ON THE CALCULATION OF PLATES ON AN ELASTIC FOUNDATION

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The calculation of plates on an elastic foundation is usually based on the assumption that the foundation can be considered as a linearly deformable medium /1/. Use is made of the combined-foundation model proposed by Shtaerman /2/, of the Fuss-Winkler hypothesis (interest in which has been renewed as a result of new experimental data), of the Pasternak hypothesis of a foundation with two foundation moduli, etc. The values obtained for the stresses and strains depend strongly on the foundation model selected.

Lately there has arisen a trend toward a more accurate calculation of structures on an elastic foundation and toward the study of soils possessing very varied mechanical characteristics (the papers of Korenev /3/, Cherkasov /4/, etc.).

The relationship between the contact pressure $q(x, y)$ applied to the foundation surface and the depression of the surface points $w(x, y)$ can be represented in a general form by the following linear integral law:

$$w(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x-\xi, y-\eta)\, q(\xi, \eta)\, d\xi d\eta, \qquad (1)$$

where $K(x-\xi, y-\eta)$, the kernel characterizing the foundation properties, represents the deflection produced at point x, y by a unit force applied at point ξ, η.

Two problems are of interest.

1. The foundation properties are known, i.e., the kernel $K(x-\xi, y-\eta)$ is specified. The depression surface $w(x, y)$ is specified. It is required to find the contact pressure $q(x, y)$.

2. The pressure distribution $q(x, y)$ is given for a certain foundation, and the depressions $w(x, y)$ of the surface points have been determined experimentally. It is required to find the kernel $K(x-\xi, y-\eta)$ which represents the foundation properties.

Both of these are inverse problems relative to the usual statement of the problem of determining the depression surface under the action of a given load, and consist in solving the linear integral equation (1).

This equation will be solved by the methods of operatorial calculation, using a two-dimensional two-sided Laplace transform /5/; a solution is obtained for a large class of problems, including as a particular case many solutions described in the literature /6, 7, 8/.

Consider now the following rule of operational calculus /5/. Let

$$f(x, y) \doteqdot \Phi(p, q), \quad g(x, y) \doteqdot G(p, q).$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)\, g(\xi, \eta)\, d\xi\, d\eta \doteqdot \frac{\Phi(p, q)\, G(p, q)}{pq}.$$

The double integral on the left appears in the integral equation (1), which shows the expediency of applying the two-dimensional two-sided Laplace transform to the solution of this equation. It should also be noted that the functions $f(x, y)$ and $g(x, y)$ must be of bounded variation in any finite region. This is the condition for a two-dimensional two-sided Laplace transformation. It is satisfied by kernels reflecting the properties of various foundations met in practice (the Fuss-Winkler foundation, the foundation with two foundation moduli, etc.). We shall not deal here with the elastic half-space, the combined foundation model, and other foundations to which this condition is inapplicable.

By applying the convolution theorem to equation (1), in those cases when this is possible, we obtain

$$W(p, q) = \frac{1}{pq} K(p, q)\, Q(p, q), \tag{2}$$

where

$$q(x, y) \doteqdot Q(p, q), \quad w(x, y) \doteqdot W(p, q), \quad K(x, y) \doteqdot K(p, q).$$

Using relationship (4) it is possible to solve the two problems formulated.

1. We obtain from (2)

$$Q(p, q) = \frac{W(p, q)\, pq}{K(p, q)}. \tag{3}$$

Returning to the original space, we obtain

$$q(x, y) = -\frac{1}{4\pi^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} e^{px+qx}\, \frac{W(p, q)}{K(p, q)}\, dp\, dq. \tag{3a}$$

2. We obtain from (2)

$$K(p, q) = \frac{W(p, q)\, pq}{Q(p, q)} \tag{4}$$

and, using the inversion formula:

$$K(x, y) = -\frac{1}{4\pi^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} e^{px+qx}\, \frac{W(p, q)}{Q(p, q)}\, dp\, dq. \tag{4a}$$

The following problem will be considered as an example of the application of the method proposed to the second type of problem.

A uniform pressure q_0 applied on the region $-a \leqslant x \leqslant a$, $-a \leqslant y \leqslant a$ of a certain foundation causes a depression of the surface points expressed by the formula

at $-a \leqslant x \leqslant a$, $-a \leqslant y \leqslant a$

$$w(x, y) = \frac{4mq_0}{k^2}(1 - e^{-ka}\operatorname{ch} kx - e^{-ka}\operatorname{ch} ky + e^{-2ka}\operatorname{ch} kx \operatorname{ch} ky),$$

at

$$x > a, \quad y > a, \qquad x < -a, \quad y > a,$$
$$x > a, \quad y < -a, \quad x < -a, \quad y < -a,$$

$$w(x, y) = \frac{4mq_0}{k^2} e^{-k|x|-k|y|} \operatorname{sh}^2 ka,$$

at

$$x > a, \qquad -a \leqslant y \leqslant a,$$
$$x < -a, \quad -a \leqslant y \leqslant a,$$

$$w(x, y) = \frac{4mq_0}{k^2} \operatorname{sh} ka\, e^{-k|x|}(1 - e^{-ka}\operatorname{ch} ky),$$

at

$$-a \leqslant x \leqslant a, \quad y > a,$$
$$-a \leqslant x \leqslant a, \quad y < -a,$$

$$w(x, y) = \frac{4mq_0}{k^2} \operatorname{sh} ka\, e^{-k|y|}(1 - e^{-ka}\operatorname{ch} kx).$$

It is required to find the kernel of the integral equation (1) corresponding to the foundation considered.

Passing from the functions $w(x, y)$ and $q(x, y)$ to their transforms, we find

$$W(p, q) = \frac{4mq_0k^2}{(p^2 - k^2)(q^2 - k^2)}(e^{ap+aq} - e^{ap-aq} - e^{-ap+aq} + e^{-ap-aq}),$$
$$Q(p, q) = q_0(e^{ap+aq} - e^{ap-aq} - e^{-ap+aq} + e^{-ap-aq}).$$

The transform of the kernel of integral equation (1) is obtained by applying formula (4).

$$K(p, q) = \frac{4mk^2pq}{(p^2 - k^2)(q^2 - k^2)},$$

which corresponds to the original

$$K(x, y) = me^{-k|x|-k|y|}.$$

The kernel is therefore of the form

$$K(x - \xi, y - \eta) = me^{-k|x-\xi|-k|y-\eta|}, \tag{5}$$

i.e., is similar to the kernel $K(x - \xi) = me^{-k|x-\xi|}$ considered by Wieghardt for the one-dimensional problem /6/.

As an example of the solution of the first of the problems formulated by the method described, consider the problem of an ideally rigid square punch of side a penetrating to the elastic foundation described by kernel (5)

of the preceding example. It is required to determine the foundation reaction.

Before proceeding to the solution of the problem, note that if the surface of the foundation considered is free of load except for some portion $-a \leqslant x \leqslant a$, $-a \leqslant y \leqslant a$ then the depression surface outside of this portion is expressed by an exponential law. Thus, at $x > a$, $-a \leqslant y \leqslant a$ we have the cylindrical surface

$$w(x, y) = w_0 e^{-k(x-a)},$$

where w_0 is the depression under the punch; at

$$x > a, \quad y > a \quad w(x, y) = w_0 e^{-k(x-a)-k(y-a)} \quad \text{etc.}$$

The depression surface can be represented in this case by the following formula, introducing into consideration unit functions:

$$\begin{gathered}
w(x, y) = w_0 [V(x+a) V(y+a) - V(x-a) V(y+a) - \\
- V(x+a) V(y-a) + V(x-a) V(y-a)] + \\
+ w_0 e^{-k(x-a)-k(y-a)} V(x-a) V(y-a) + \\
+ w_0 e^{k(x+a)+k(y+a)} V(-x-a) V(-y-a) + \\
+ w_0 e^{-k(x-a)+k(y+a)} V(x-a) V(-y-a) + \\
+ w_0 e^{k(x+a)-k(y-a)} V(-x-a) V(y-a) + \\
+ w_0 e^{-k(x-a)} [V(x-a) V(y+a) - V(x-a) V(y-a)] + \\
+ w_0 e^{k(x+a)} [V(-x-a) V(y+a) - V(-x-a) V(y-a)] + \\
+ w_0 e^{-k(y-a)} [V(x+a) V(y-a) - V(x-a) V(y-a)] + \\
+ w_0 e^{k(y+a)} [V(x+a) V(-y-a) - V(x-a) V(-y-a)],
\end{gathered}$$

where $V(x)$ and $V(y)$ are unit functions, whose transform is 1.

The transform of the depression surface is

$$\begin{gathered}
W(p, q) = \frac{w_0 k^2}{(p^2-k^2)(q^2-k^2)} [e^{ap+aq}(k^2+pq+pk+qk) + \\
+ e^{-ap-aq}(k^2+pq-pk-qk) + e^{ap-aq}(-k^2+pq-pk+qk) + \\
+ e^{-ap+aq}(-k^2+pq+pk-qk)],
\end{gathered}$$

and the transform of the kernel is

$$K(p, q) = \frac{4mk^2pq}{(p^2-k^2)(q^2-k^2)}.$$

From here, allowing for formula (3), we find the transform of the reaction pressure

$$\begin{gathered}
Q(p, q) = \frac{w_0}{4m} [e^{ap+aq}(k^2+pq+pk+qk) + \\
+ e^{-ap-aq}(k^2+pq-pk-qk) + e^{ap-aq}(-k^2+pq-pk+qk) + \\
+ e^{-ap+aq}(-k^2+pq+pk-qk)],
\end{gathered}$$

which corresponds to the original

$$q(x, y) = \frac{w_0 k^2}{4m}[V(x+a)V(y+a) - V(x+a)V(y-a) - V(x-a)V(y+a) + V(x-a)V(y-a)] +$$
$$+ \frac{w_0 k}{4m}[\delta(x+a)V(y+a) - \delta(x+a)V(y-a) + \delta(x-a)V(y+a) - \delta(x-a)V(y-a) + V(x+a)\delta(y+a) - V(x-a)\delta(y+a) + V(x+a)\delta(y-a) - V(x-a)\delta(y-a)] +$$
$$+ \frac{w_0}{4m}[\delta(x+a)\delta(y+a) + \delta(x-a)\delta(y+a) + \delta(x+a)\delta(y-a) + \delta(x-a)\delta(y-a)].$$

Here $\delta(x)$ and $\delta(y)$ are Dirac impulse functions, whose physical application to the case considered is unit concentrated forces applied at the coordinate origin.

The reaction under the punch thus represents a uniformly distributed pressure of intensity $\frac{w_0 k^2}{4m}$ acting in the portion $-a \leqslant x \leqslant a$, $-a \leqslant y \leqslant a$, a uniformly distributed pressure of intensity $\frac{w_0 k}{4m}$ acting along the punch perimeter, and concentrated forces $\frac{w_0}{4m}$ acting at the corners.

It is obvious that the class of foundations considered includes the Winkler foundation and the foundation with two foundation moduli, whose model can be represented as an inextensible diaphragm lying on a system of springs (the Filonenko-Borodich model) /8/, and whose load-deflection relationship can be written in the form

$$B\nabla^2 w(x, y) + cw(x, y) = q(x, y),$$

where $\nabla^2 = \frac{\partial^2}{dx^2} + \frac{\partial^2}{\partial y^2}$ is the differential Laplacian.

The class of foundations (1) considered also includes all the foundations for which the load is determined as the right-hand part of a linear differential equation in partial derivatives with constant coefficients relative to the depressions of the surface points, i.e., the foundations whose surface deformation is described by the equation

$$L[w(x, y)] = q(x, y), \tag{6}$$

where L is a linear differential operator with constant coefficients.

It is obvious that the class of foundations (1) is larger than class (6). Thus, the foundation with kernel $K(x-\xi, y-\eta) = Ae^{-[m(x-\xi)+n(y-\eta)]^2}$ is not described by equation (6), but belongs to the class of foundations (1).

Consider the case when the load is distributed along a straight line. In this case the function w is function of one variable only, and equation (6) can be written as follows:

$$w^{(n)}(x) + c_1 w^{(n-1)}(x) + \cdots + c_n w(x) = mq(x). \tag{6a}$$

The kernel corresponding to the foundation described by equation (6a) is of the form

$$K(x) = \sum_i (A_{0i} + A_{1i}|x| + \cdots + A_{ni}|x|^n)e^{-m_i|x|}. \tag{7}$$

The transform of a kernel of the form of (7) is a rational-fractional function. The kernel itself can therefore be found from its transform, without using the inversion formulas, by expanding the transform in elementary fractions and finding the originals corresponding to them from standard handbooks.

This suggests the advisability of approximating the kernels of arbitrary foundations by kernels of the form of (7). In other words, an arbitrary function is approximated by means of an exponential function by suitably selecting the factors A_{ni} and m_i.

The following approximating formulas can be proposed in the three-dimensional case:

$$K(x, y) = \sum (A_{0i} + A_{1i}|x| + A_{2i}|y| + \cdots + A_{ni}|x^k y^l|)\, e^{-m_i|x| - r_i|y|}$$

or

$$K(x, y) = \sum_i \left[A_{0i} + A_{1i}\sqrt{x^2 + y^2} + \cdots + A_{ni}(x^2 + y^2)^{n/2} \right] e^{-m_i\sqrt{x^2+y^2}} \text{etc.} \quad (8)$$

Using such an approximation it is possible to solve problem 2, approximately in the following way.

We specify the form of the kernel $K(x, y)$ in accordance with formulas (7) or (8), leaving the values of the parameters indeterminate at this stage. Knowing by hypothesis the function $q(x, y)$, we determine the function $w(x, y)$ by formula (1), and select then the indeterminate parameters in such a way as to make the function $w(x, y)$ obtained reflect best the given depression surface.

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STABILITY OF REINFORCED CYLINDRICAL SHELLS UNDER AXISYMMETRICAL LOADS

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The problem of the stability of a cylindrical shell reinforced by stringers and enclosed between stiffeners under axial compression or axial compression with an internal or external pressure is solved by the energy method in a nonlinear formulation. It is assumed that the skin and stringers buckle simultaneously. The study of the stability of a reinforced cylindrical shell reduces, as a result, to the study of the stability of the skin, along whose lines of contact act radial forces depending on the stringers rigidity.

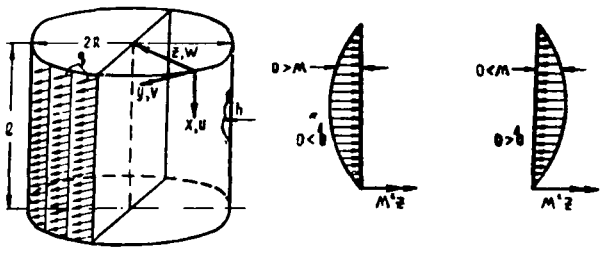

FIGURE.

The necessary number of stringers is determined on the basis of the results of the studies given in /1/, i.e., the number of half-waves in the circumferential direction is determined from the given R, l, h. In order to increase the critical stress in the skin the stringers must be arranged at a distance smaller than the half-wave in the circumferential direction; more precisely, the number of stringers must be no less than four times the number of half-waves.

1. As shown by the results of experiments, a cylindrical shell, reinforced by a set of stringers, buckles in one half-wave along the generator.

The radial displacement can therefore be selected in the same form as assumed in /1/

$$w = f_0 + f_1 \sin\frac{\pi x}{l}\sin\frac{ny}{R} + f_2 \sin^2\frac{\pi x}{l}. \qquad (1.1)$$

The results of the studies given in /1/ satisfactorily agreed with the experimental data for the case of a smooth cylindrical shell.

Using (1.1) we determine the strain energy of the middle surface at bending, the work of the external forces applied to the shell ends, the work of the internal or external pressure, and also the work of the radial forces from the stringers to the skin.

The last is given in an expanded form by the following expression:

$$U_q=\sum_{i=1}^{s}(U_q)_i=\frac{\pi^2 S}{8l}\left\{\left[EJ\left(\frac{\pi}{l}\right)^2-\sigma F\right]f_1^2+2\left[4EJ\left(\frac{\pi}{l}\right)^2-\sigma F\right]f_2^2\right\}. \qquad (1.2)$$

Using the fact that the critical stress of the skin and stringer are equal, we can write the expression for the total energy in the following form:

$$\bar{U}=\frac{4U}{\pi E R l h}=\frac{m^4+n^4}{32}g_1^4+n^4\left[\frac{m^4}{(m^2+n^2)^2}+\frac{m^4}{(9m^2+n^2)^2}\right]g_1^2g_2^2-$$
$$-\frac{1}{4}n^2\left[1+\frac{8m^4}{(m^2+n^2)^2}\right]g_1^2g_2+\frac{m^4}{(m^2+n^2)^2}g^2+\frac{1}{2}g_2^2+$$
$$+\frac{h^2}{12(1-\nu)R^2}[(m^2+n^2)^2g_1^2+8m^4g_2^2]-m^2\frac{\sigma}{E}(ag_1^2+2bg_2^2)\pm$$
$$\pm h^2\frac{PR}{Eh}g_1^2-4\left[\pm\frac{PR}{Eh}+\frac{\sigma}{E}\right]^2\pm\frac{\sigma}{E}\frac{PR}{Eh}. \qquad (1.3)$$

By varying the expression for the total energy by g_1, g_2, and n we obtain the following expressions for determining the critical stress:

$$\bar{\sigma}=\frac{1}{12(1-\nu^2)\psi}+\psi(B_1\gamma^2-F_1\gamma+1)+A_1\psi\gamma\frac{\gamma}{\omega^2}\pm\frac{\bar{P}}{\mu^2 a}, \qquad (1.4a)$$

$$\bar{\sigma}=\frac{1}{\beta}\left[\frac{G_1}{12(1-\nu^2)\psi}+\frac{\psi}{G_1}+\psi\left(\frac{1}{2}B_1\gamma-\frac{1}{4}F_1\right)\frac{\gamma}{\omega^2}\right], \qquad (1.4b)$$

$$0=\frac{P_1}{12(1-\nu^2)\psi}+\psi(2M_1\gamma^2-N_1\gamma-P_1)+\psi\frac{\gamma}{4G_1}\cdot\frac{\gamma}{\omega^2}\pm\frac{\bar{P}}{\mu^2 a}, \qquad (1.4c)$$

where F, J, and S are the area, moment of inertia, and number of stringers; E and ν are the modulus of elasticity and the Poisson ratio.

The dimensionless parameters of the shape of the reinforced shell after buckling are

$$\mu=\frac{\pi R}{ln};\quad \omega=\frac{f_2}{f_1};\quad \gamma=n^2\frac{f_2}{R};\quad \psi=\frac{\mu^2}{n^2(1+\mu^2)^2}\cdot\frac{R}{ah},$$

where n is the number of half-waves in the circumferential direction.

The expressions linking the geometrical parameters of the skin and stringers are

$$a=1+\xi\left(1-\frac{1}{\alpha}\right);\quad b=1+\xi\left(1-\frac{4}{\alpha}\right);\quad \xi=\frac{F\cdot S}{2\pi Rh};\quad \beta=\frac{b}{a},$$

where α is a coefficient characterizing the support conditions of the stringer ends.

The dimensionless value of the critical stress at compression and the value of the internal or external pressure are determined by the formulas

$$\bar{\sigma}=\frac{\sigma R}{Eh};\quad \bar{P}=\frac{PR^2}{Eh^2}.$$

The coefficients, which are functions of the parameter μ, have the following values:

$$A_1=\frac{(1+\mu^4)(1+\mu^2)^2}{16\mu^4};\quad B_1=1+\left(\frac{1+\mu^2}{1+9\mu^2}\right)^2;\quad F_1=\frac{1}{4}\left[8+\frac{(1+\mu^2)^2}{\mu^4}\right],$$

$$G_1=\frac{4\mu^4}{(1+\mu^2)^2};\quad P_1=\frac{2}{(1+\mu^2)};\quad M_1=\frac{\mu^2}{1+\mu^2}+\frac{9\mu^2(1+\mu^2)^2}{(1+9\mu^2)^3};$$

$$N_1=\frac{1}{4}\left[\frac{(1+\mu^2)^2}{\mu^4}-8\frac{1-\mu^2}{1+\mu^2}\right].$$

The following obvious relationship exists between the parameters μ and ψ and the geometrical parameters of the skin and stringers R, l, h, and ξ:

$$\Theta=\left(\frac{l}{R}\right)^2\frac{R}{h}=\frac{\pi^2(1+\mu^2)^2\psi}{\mu^4}a. \tag{1.5}$$

The following cubic equation gives us the value of γ:

$$\left[8M_1G_1\left(\frac{1}{2}B_1+\frac{\beta}{4G_1}-A_1\beta\right)-\beta(2M_1-B_1)\right]\gamma^3-$$
$$-\left[4G_1N_1\left(\frac{1}{2}B_1+\frac{\beta}{4G_1}-A_1\beta\right)+2M_1G_1F_1-\beta(N_1-F_1)\right]\gamma^2+$$
$$+\left[4G_1\left(\frac{P_1}{12(1-\nu^2)\psi^2}-P_1\pm\frac{P}{a\psi\mu^2}\right)\left(\frac{1}{2}B_1+\frac{\beta}{4G_1}-A_1\beta\right)+G_1N_1F_1-\right.$$
$$\left.-\frac{G_1+\beta(P_1-1)}{12(1-\nu^2)\psi^2}+\beta P_1+\beta-\frac{1}{G_1}\right]\gamma-$$
$$-\left[G_1F_1\left(\frac{P}{12(1-\nu^2)\psi^2}-P_1\pm\frac{\bar{P}}{a\psi\mu^2}\right)\right]=0. \tag{1.6}$$

The magnitude $\frac{\gamma}{\omega_2}$ is determined from the expression

$$\frac{\gamma}{\omega^2}=-\frac{\frac{P}{12(1-\nu^2)\psi^2}-P_1\pm\frac{P}{a\psi\mu^2}+2M_1\gamma^2-N_1\gamma}{\gamma}. \tag{1.7}$$

At $\xi=0$ equations (1.4) to (1.7) coincide with the equation obtained in /1/.

At given R, l, h, and ξ we can determine the value of σ, and therefore also the value of the critical axial force of the reinforced shell:

$$P_{cr}=2\pi E\bar{\sigma}h^2(1+\xi), \tag{1.8}$$

and the geometrical parameters of the stringer

$$F=\frac{2\pi Rh}{S}\cdot\xi,\qquad J=\frac{\bar{\sigma}Fl^2}{\pi^2\alpha}\cdot\frac{h}{R}, \tag{1.9}$$

where $\alpha=1$ for hinged support; $\alpha=2$ for support at the end; $\alpha=4$ for clamped ends; $\alpha=2.5$ is the approximate value corresponding to (1.1).

2. It is now possible to determine the minimum weight of the reinforced cylindrical shell.

Let the value of the axial force P and the shell dimensions L and R be given. It is required to select the skin thickness h, the distance between

the stiffeners l, and the magnitude ξ in such a way that the shell be of minimum weight; the following conditions must therefore be fulfilled:

$$P_{cr}=2\pi E\bar{\sigma}h^2(1+\xi)=P, \tag{2.1}$$

$$G=d\cdot 2\pi RLh\ (1+\xi+\xi_1)=G_{min}, \tag{2.2}$$

where d is the specific weight of the material; $\xi_1=\frac{F_{st}}{lh}$, F_{st} is the area of the stiffener. The minimum weight of the shell is determined in the following way: a number of values of h and ξ are selected for each value of l, and the corresponding values of P_{cr} and G are determined. From the set of values of l, h, and ξ we select the values l^*, h^*, and ξ^* for which (2.1) is satisfied, and the value of G in (2.2) is smallest. The following condition must be imposed on the selection of the stiffener area:

$$F_{st} \geqslant 1.5\, F_{str}.$$

If an internal pressure acts in addition to the axial force, we must impose an additional condition imposed on the selection of the shell thickness:

$$\sigma_l=\sqrt{\sigma_{cr}^2+\sigma_{cr}\sigma_2+\sigma_2^2}\leqslant\sigma_s. \tag{2.3}$$

It is seen from expressions (2.1) to (2.3) that the selection of the minimum weight of the shell depends on the mechanical characteristics of the material and its unit weight.

3. If the stringer is of rectangular section, which is characteristic for shells made by chemical milling, its cross-sectional dimensions are easily determined by the formulas

$$h_1=c_1 l\sqrt{\frac{\sigma_{cr}}{E}}, \tag{3.1}$$

$$b_1=c_2\frac{F_{str}}{l}\sqrt{\frac{E}{\sigma_{cr}}}, \tag{3.2}$$

where the coefficients c_1, c_2 depend on the conditions of fastening of the stringer ends.

Conditions of fastening of the ends:

	α	c_1	c_2
Hinged support	1.0	1.09	0.92
Support at the end	2.0	0.77	1.30
Clamped edges	4.0	0.55	1.82
By formula (1.1)	2.5	0.68	1.45

It is seen from (3.1) and (3.2) that at ξ = const the area of the stringer cross section decreases with the increase of the number of stringers; the height remaining constant, this means that the stringer width decreases, which can lead to a local buckling of the stringer. It follows that the number of stringers is restricted from above by the local buckling of the stringer, and from below by the buckling of the skin between the stringers, i.e.,

$$4n\leqslant S<S_1, \tag{3.3}$$

where S_1 is the number of stringers for each of which occurs a local buckling.

4. The modeling of the stability of smooth cylindrical shells was considered in /1/.

A similar approach can be used in solving the problem of modeling reinforced cylindrical shells. Use is made in the modeling of equations (1.4) to (1.9). The modeling is based on the condition that the critical compression stresses and the stresses due to the internal or external pressure in the original shell and in its model are equal:

$$\sigma_{cr} = (\sigma_{cr})_1; \qquad \sigma_2 = (\sigma_2)_1. \tag{4.1}$$

The following condition must also be satisfied:

$$\theta = \theta_1. \tag{4.2}$$

It follows from (4.2) that

$$\frac{l}{R} = \frac{l_1}{R_1}; \qquad \frac{R}{h} = \frac{R_1}{h_1}; \qquad a_1 = a.$$

When condition $a = a_1$ is satisfied it is necessary that the following equality be satisfied:

$$\xi = (\xi)_1 \qquad \alpha = \alpha_1, \tag{4.3}$$

i.e., at the same number of stringers, the loads transmitted to the full-scale stringer and skin must be equal to the loads transmitted to the stringer and skin in the model. The conditions of fastening of the stringers must be identical, both in the full-scale shell and in the model. Thus, if conditions (4.1) to (4.3) are fulfilled, a model of a reinforced shell subjected to axisymmetrical loads is created.

5. The results of the comparison of the theory exposed in Section 1 with the experimental results are presented below. The shell tested at axial compression had the following geometrical characteristics:

$$\frac{h}{R} = 2.664; \qquad \frac{R}{h} = 1340; \qquad F_{str} = 0.651 \text{ m}^2; \qquad S = 100.$$

Mechanical characteristics of the material:

$$E = 7.2 \cdot 10^5 \frac{\text{kg}}{\text{cm}^2} \qquad \nu = 0.3.$$

Results of the calculation. Take $\alpha = 2.5$; $\xi = 0.773$; $a = 1.464$; $b = 0.536$; $\beta = 0.366$; $\theta = 9510$; at $\mu = 0.16$; $\psi = 0.319$; $\gamma = 0.303$; $\omega = 1.481$, $\bar{\sigma}_{min} = 0.349$.

Minimum calculated value of the axial critical force is $P_{cr} = 28{,}000$ kg, number of dents $n = 8.36 \approx 9$, $f_0 = 12h$; $f_1 = 9h$; $f_2 = 13h$.

Experimental results. Buckling starts under a load of 26,300 kg without disturbing the carrying capacity. At a load of 30,500 kg the skin and stringers buckle with a disturbance of the carrying capacity; this buckling is accompanied by the formation of one half-wave along the generator and nine dents in the circumferential direction.

The comparison of the experimental results with the calculation shows

that the discrepancy between the calculated and experimental critical loads does not exceed 6%. A satisfactory agreement between the calculated and experimental number of dents was obtained, which proves the correctness of the premises of the theoretical studies.

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ON THE RIGIDITY OPERATORS OF A STIFFENED PLATE AT CREEP

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1. RIGIDITY OPERATIONS

Consider the problem of the creep at bending of a plate reinforced in the two directions by mutually perpendicular ribs arranged close to each other and symmetrically on the two sides of the middle plane.

Let the plate be made of $2n+1$ isotropic homogeneous layers rigidly fastened to each other and symmetrical relative to the middle layer, with transient and steady-state Young moduli E_{0k} and $E_{\infty k}$, transient Poisson ratios ν_{0k}, and heredity (relaxation) characteristics β_k and χ_k $(k=1, 2, 3, \cdots, 2n+1)$.

Introduce the operator

$$\overline{E}_k = E_{0k}[1-\chi_k Q^*(-\beta_k)], \tag{1.1}$$

where $Q^*(-\beta_k)$ is the integral operator with relaxation kernel $Q(-\beta_k; t, s)$, acting in the following way on some function ζ of the coordinates and the time t:

$$Q^*(-\beta_k)\zeta = \int_0^t Q(-\beta_k; t, s)\zeta(s)\,ds. \tag{1.2}$$

Let the operator (1.2) satisfy the condition

$$Q^*(x_1) - Q^*(x_2) = (x_1 - x_2)\,Q^*(x_1)\,Q^*(x_2), \tag{1.3}$$

where x_1 and x_2 are arbitrary numbers, with $x_1 \neq x_2$.

Condition (1.3) is exactly satisfied by the integral operator with kernel of Rabotnov's type /1/:

$$E_\alpha(x_1; t-s) = (t-s)^{-\alpha}\sum_{n=0}^{n}\frac{x_1^n (t-s)^{n(1-\alpha)}}{\Gamma[(n+1)(1-\alpha)]}, \quad (\alpha > -1), \tag{1.4}$$

and by the integral operator with simple exponential kernel

$$E_0(x_1; t-s) = \exp[x_1(t-s)].$$

A kernel of Arutyunyan's type /2/ satisfies condition (1.3) only approximately.

According to Rabotnov /1/, it can be assumed that the dilatation operator does not vary with time, i.e.,

$$\frac{1-2\bar{\nu}_k}{\bar{E}_k}=\frac{1-2\nu_{0k}}{E_{0k}}. \tag{1.5}$$

The substitution of (1.1) in (1.5) yields

$$\bar{\nu}_k=\nu_{0k}\left[1+\frac{1-2\nu_{0k}}{2\nu_{0k}}\chi_k Q^*(-\beta_k)\right], \tag{1.6}$$

where ν_k is the operator Poisson ratio, $\chi_k=\lambda_k\beta_k$, $\lambda_k=(E_{0k}-E_{\infty k})E_{0k}^{-1}$, where $E_{\infty k}$ is the steady-state Young modulus, and $E_{\infty k}<E_{0k}$.

Designate the moments of inertia of the sections of the ribs by J_1 and J_2, the distance between the ribs by d_1 and d_2, the transient Young moduli by E'_{0i}, and the heredity (relaxation) characteristics of the material of the ribs by β'_i and χ'_i, $\chi'_i=\lambda'_i\beta'_i$, $\lambda'_i=(E'_{0i}-E'_{\infty i})/E'_{0i}$, where $E'_{\infty i}$ is the steady-state Young modulus of the material of the ribs, $E'_{\infty i}<E'_{0i}$ $(i=1,2)$.

The operator of normal deformation of the ribs is

$$\bar{E}'_i=E'_{0i}[1-\chi'_i Q^*(-\beta'_i)] \quad (i=1,2). \tag{1.7}$$

Using formulas (65.6) and (66.6) from Lekhnitskii's book /3/, and operators (1.1), (1.6), and (1.7), we obtain by applying Volterra's principle /1/ the rigidity operators of a sandwich plate stiffened by ribs placed close to each other:

$$\bar{D}_i=\frac{2}{3}\left[\sum_{k=1}^{n}\bar{T}_k(h_k^3-h_{k+1}^3)+h_{n+1}^3\bar{T}_{n+1}\right]+E'_{0i}J_i d_i^{-1}[1-\chi Q^*(-\beta'_i)], \tag{1.8}$$

$$(i=1,2),$$

where

$$\bar{T}_k=\frac{\bar{E}_k}{(1-\bar{\nu}_k)(1+\bar{\nu}_k)}=\frac{T_{0k}}{1-\chi_k\gamma_{1k}Q^*(-\beta_k)}\cdot\frac{1-\chi_k Q^*(-\beta_k)}{1+\chi_k\gamma_{2k}Q^*(-\beta_k)}, \tag{1.9}$$

$$T_{0k}=\frac{E_{0k}}{1-\nu_{0k}^2},\qquad \gamma_{ik}=\frac{1-2\nu_{0k}}{2[1+(-1)^i\nu_{0k}]},$$

$$(i=1,2;\ k=1,2,3,\cdots,n,n+1). \tag{1.10}$$

The operator $\bar{T}_{n+1}$ and the magnitude $T_{0,n+1}$ are determined by formulas (1.9) and (1.10), respectively, for $k=n+1$.

Formulas (1.8) to (1.10) correspond to the case when the sandwich plate as a whole represents a body possessing not only geometrical symmetry but also elastic-heredity symmetry relative to the middle plane. Here $h_1, h_2, \cdots, h_{2n+1}$ are distances from the middle plane to the boundaries of the layers; $(h_1-h_2), (h_3-h_4), \cdots, (h_n-h_{n+1})$, and $2h_{n+1}$ are thicknesses of layers one to n respectively (cf. Figure 131 in /3/); E_{0k}, $E_{\infty k}$, ν_{0k}, β_k, χ_k are the elastic-heredity characteristics of these layers. We have $E_{01}=E_{0;2n+1}$, $E_{\infty 1}=E_{\infty;2n+1}$, $\nu_{01}=\nu_{0;n+1}$, $\chi_1=\chi_{2n+1}$, $\beta_1=\beta_{2n+1}$, etc.

In order to be able to use formulas (1.8) in practice, the operator (1.9) must be reduced to a nonfractional form which does not contain products of the operator Q^* with different values of the parameters or powers of

this operator. This can be realized by using formula (1.3) and its corollaries

$$[1-\chi Q^*(-\beta)]^{-1}=1+\chi Q^*(\chi-\beta). \tag{1.11}$$

Having found the inverse operators of those in the denominators of (1.9), we obtain from (1.11)

$$\overline{T}_k=T_{0k}[1-\chi_k Q^*(-\beta_k)]\,[1+\chi_k\gamma_{1k}Q^*(\chi_k\gamma_{1k}-\beta_k)]\times$$
$$\times[1-\chi_k\gamma_{2k}Q^*(-\chi_k\gamma_{2k}-\beta_k)]. \tag{1.12}$$

By twice applying formula (1.3) in (1.12) we obtain the required expression for the operator $\overline{T}_k$:

$$\overline{T}_k=T_{0k}[1-b_{1k}Q^*(-m_{1k})-b_{2k}Q^*(-m_{2k})], \tag{1.13}$$

where

$$m_{ik}=\beta_k+(-1)^i\chi_k\gamma_{ik};\qquad b_{ik}=\frac{(2i-1)\chi_k[1+(-1)^{i-1}\nu_{0k}]}{4[1+(-1)^i\nu_{0k}]}\quad (i=1,2); \tag{1.14}$$

since $0\leqslant\nu_{0k}\leqslant 0.5$, then $b_{ik}>0$ and $m_{ik}>0$.

The third rigidity operator $\overline{D}_3$ is of the form

$$\overline{D}_3=\frac{2}{3}\left[\sum_{k=1}^{n}\overline{T}_k(h_k^3-h_{k+1}^3)+h_{n+1}^3\overline{T}_{n+1}\right], \tag{1.15}$$

which naturally enters into the composition of operator (1.8).

2. SOME APPLICATIONS

The solution of any problem relative to the determination of the deflection $w(x, y, t)$ of a sandwich plate reinforced by ribs placed close to each other at creep reduces to replacing the rigidities D_1, D_2, and D_3 in the formula determining the deflection of an ideally elastic orthotropic plate by the operators (1.8) and (1.15) with subsequent determination of the function of the operator Q^* thus formed.

Take for instance formula (72.5) in /3/, which determines the deflection of an orthotropic plate, supported on its four sides, under the action of a normal load distributed according to some arbitrary law (varying in the general case with time). We have

$$W(x,y,t)=\frac{b^4}{\pi^4}\sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\frac{q_{mi}}{\overline{H}}\sin\frac{m\pi x}{a}\sin\frac{i\pi y}{b}. \tag{2.1}$$

The operator $\overline{H}$ can be presented in the form

$$\overline{H}=H_0\left[1-\sum_{k=1}^{n+1}\sum_{i=1}^{2}\delta_{ik}Q^*(-m_{ik})-\sum_{i=1}^{2}p_iQ^*(-\beta_i')\right], \tag{2.2}$$

where

$$H_0 = \xi\left[\sum_{k=1}^{n} T_{0k}(h_k^3 - h_{k+1}^3) + h_{n+1}^3 T_{0;n+1}\right] + \sum_{i=1}^{2} \rho_i,$$

$$\zeta = \frac{2}{3}\left[\left(\frac{m}{c}\right)^4 + 2n^2\left(\frac{n}{c}\right)^2 + n^4\right], \quad \rho_1 = \left(\frac{m}{c}\right)^4 E'_{01}\frac{J_1}{d_1},$$

$$\rho_2 = n^4 E'_{02}\frac{J_2}{d_2}, \qquad \nu_i = \rho_i\frac{\chi'_i}{H_0},$$

$$\delta_{ik} = b_{ik}\xi(h_k^3 - h_{k+1}^3)T_{0k}H_0^{-1}, \quad (k = 1, 2, \cdots, n), \quad \delta_{i;n+1} = b_{i;n+1}\xi h_{n+1}^3 T_{0;n+1},$$

a and b are lengths of the plate sides; $c = a/b$.

To find the deflection W it suffices to find the operator inverse to $\overline{H}$. The latter can be realized by using the theorem of /4/. We have

$$\frac{q_{mi}}{\overline{H}} = \frac{q_{mi}}{H_0}\left[1 + \sum_{k=1}^{n+1}\sum_{i=1}^{2} a_{ik}Q^*(r_{ik}) + \sum_{i=1}^{2}\alpha_i Q^*(r'_i)\right] =$$

$$= \frac{1}{H_0}\left\{q_{mi}(t) + \int_0^t\left[\sum_{k=1}^{n+1}\sum_{i=1}^{2} a_{ik}Q(r_{ik};t,s) + \sum_{i=1}^{2}\alpha_i Q(r'_i;t,s)\right]q_{mi}(s)\,ds, \right. \tag{2.3}$$

where r_{ik} and r'_i are the roots of the equation

$$1 - \sum_{k=1}^{n+1}\sum_{i=1}^{2}\frac{\delta_{ik}}{r + m_{ik}} - \sum_{i=1}^{2}\frac{\rho_i}{r + \beta'_i} = 0, \tag{2.4}$$

and the coefficients a_{ik} and α_i are determined from the system of linear equations

$$1 - \sum_{k=1}^{n+1}\sum_{i=1}^{2}\frac{a_{ik}}{r_{ik} + m_{jl}} - \sum_{i=1}^{2}\frac{\alpha_i}{\beta'_i + r'_i} = 0, \quad (j = 1, 2, \cdots, n+1;\ l = 1, 2). \tag{2.5}$$

For a concentrated load $P(t) = M\cos\omega t$, applied at point (x_0, y_0), and the case when $Q(-\beta; t, s) = \exp(-\beta; t - s)$ it follows from (2.3) that

$$\frac{q_{mi}}{\overline{H}} = \frac{4M}{abH_0}\sin\frac{m\pi x_0}{a}\sin\frac{i\pi y_0}{b}\left\{\cos\omega t - \sum_{k=1}^{n+1}\sum_{i=1}^{2}\frac{a_{ik}}{r_{ik}^2 + \omega^2}[\omega\sin\omega t - \right.$$

$$- r_{ik}\cos\omega t + r_{ik}\exp(r_{ik}t)] +$$

$$\left. + \sum_{i=1}^{2}\frac{\alpha_i}{r_i'^2 + \omega^2}[\omega\sin\omega t - r'_i\cos\omega t + r'_i\exp(r'_i t)]\right\}. \tag{2.6}$$

The deflection W will be rheologically stable at $t \to \infty$, if $\mathrm{Re}(r_{ik}) < 0$ and $\mathrm{Re}(r'_i) < 0$ $(k = 1, 2, \cdots, n+1)$, $(i = 1, 2)$.

As known (/3/, p. 289), at large sides ratio $c = a/b$ the deflection at the center of an ideally elastic orthotropic plate, supported by the sides $x = \pm 0.5a$ and $y = 0$, $y = b$, under a uniformly distributed load q, is approximately determined by the formula

$$W_c = 5qb^4/384\,D_2. \tag{2.7}$$

By substituting the rigidity operator (1.8) instead of D_2 in (2.7) at $i = 2$, we obtain an operational expression for the deflection W_c, which remains

to be deciphered. In a three-layer sandwich plate of thickness h (the thickness of each layer is $h/3$) with elastic-heredity characteristics E_{01}, χ_1, β_1, $\nu_{01}=0.5$ (the outer layer), E_{02}, $\chi_2=0$, ν_{02} (the middle layer), E'_{02}, χ'_2, β'_2 (the strip) we obtain for $q(t)=q_0=\text{const}$

$$\frac{W_c(t)}{W_c(0)}=\left[1-\frac{\chi_1 Q^*(-\beta_1)}{1+\mu_1+\mu_2}-\frac{\mu_2\chi'_2 Q^*(-\beta'_2)}{1+\mu_1+\mu_2}\right]^{-1}, \tag{2.8}$$

where

$$\mu_1=\frac{3E_{02}}{104E_{01}(1-\nu_{02}^2)}, \qquad \mu_2=\frac{243E'_{02}J_2}{26E_{01}d_2h^3},$$

$$W_c(0)=\frac{65q_0b^4}{46656E_{01}(1+\mu_1+\mu_2)}. \tag{2.9}$$

Let $Q^*=E^*_\alpha$, where the kernel of operator E^*_α is of the form (1.4). It follows then from (2.8) that

$$W_c(t)=W_c(0)\,[1+a_1E^*_\alpha(r_1)+a_2E^*_\alpha(r_2)], \tag{2.10}$$

where r_1 and r_2 are the roots of the quadratic equation

$$r^2+p_1r+p_2=0, \tag{2.11}$$

$$p_1=\beta_1+\beta'_2-\frac{\chi_1+\chi'_2\mu_2}{1+\mu_1+\mu_2}, \qquad p_2=\beta_1\beta'_2-\frac{\beta_1\mu_2\chi'_2+\beta'_2\nu_1}{1+\mu_1+\mu_2}. \tag{2.12}$$

The coefficients a_1 and a_2 are determined by the formula

$$a_k=(-1)^{k-1}\frac{(r_k+\beta_1)(r_k+\beta'_2)}{r_1-r_2} \qquad (r_1\neq r_2), \quad (k=1,2). \tag{2.13}$$

At $r_1=r_2$ expression (2.10) becomes indeterminate, and is solved by l'Hospital's rule (cf. /4/).

It follows from (2.9) that

$$\frac{W_c(t)}{W_c(0)}=1+\sum_{k=1}^{2}\frac{a_k}{r_k}\,[E_{1-\alpha}(r_kt^{1-\alpha})-1]\approx$$

$$\approx 1+\sum_{k=1}^{2}\frac{a_k}{r_k}\,[\exp(\gamma r_kt^{1-\alpha})-1], \text{ where } \gamma=(1-\alpha)^{1-\alpha}. \tag{2.14}$$

The expression was obtained from the relationship between the operator $E^*_\alpha(r_k)$ and the function of Mittag-Löffler $E_{1-\alpha}(r_kt^{1-\alpha})$, and also its approximation according to /4/.

It follows from (2.14) that at $t\to\infty$

$$W_c(\infty)=\frac{405q_0b^4}{4992E_{01}(1+\mu_1+\mu_2)}\left(1-\sum_{k=1}^{2}a_kr_k^{-1},\right), \text{where } \operatorname{Re}(r_k)<0. \tag{2.15}$$

Formulas (2.8) to (2.16) correspond to the case when the outer layers and the ribs display the property of creep, while the filler is ideally elastic. The same formulas can be used for determining the deflection W_c

of a plate with the following elastic-heredity characteristics E_{0k}, χ_k, β_k $(k=1, 2)$, $\nu_{01}=\nu_{02}=0.5$, E'_{02}, i.e., for the case when all the layers possess the property of creep, while the ribs are ideally elastic. It suffices to that end to write $\nu_{02}=0.5$ in the calculation of μ_1 and to replace the parameter β'_2 and the product $\mu_2\chi'_2$ by β_2 and $\mu_1\chi_2$, respectively, in the formulas given above.

Consider the problem of creep of a rectangular simple plate, stiffened by ribs and with four built-in sides, bent by a uniformly distributed normal load q. The elastic-heredity characteristics of the plate are E_0, ν_0, χ, β, and of the ribs E'_{0i}, χ'_i $(i=1, 2)$.

The formula approximately determining the deflection W of the corresponding ideally elastic orthotropic plate is, as known /3/,

$$W(x, y)=\frac{49q}{8}\frac{\left(x^2-\frac{a^2}{4}\right)\left(y^2-\frac{b^2}{4}\right)}{7D_1b^4+4D_3a^2b^2+7D_2a^4} \tag{2.16}$$

$$(-0.5a\leqslant x\leqslant 0.5a;\quad -0.5b\leqslant y\leqslant 0.5b).$$

By substituting the operators $\overline{D}_1$, $\overline{D}_2$, and $\overline{D}_3$ from (1.8) and (1.15) instead of D_1, D_2, and D_3 in (2.16) for $n=0$ and $2h_1=h$ (where h is the thickness of the plate considered) we reduce the problem to finding the operator inverse to $7\overline{D}_1b^4+4\overline{D}_3a^2b^2+7\overline{D}_2a^4$.

In particular, at $a=b$, $\chi=\chi'_1=\chi'_2$, $\beta=\beta'_1=\beta'_2$, $E_0=E'_{01}=E'_{02}$, $d_1=d_2$, $J_1=J_2$, i.e., in the case of a square plate made from the same material as the ribs, we have at $Q^*=E^*_\alpha$ and $q(t)=q_0=\text{const}$

$$\frac{W(x, y, t)}{W(x, y, 0)}=1+\sum_{k=1}^{3}\frac{a_k}{r_k}\left[E_{1-\alpha}\left(r_kt^{1-\alpha}\right)-1\right], \tag{2.17}$$

where r_k $(k=1, 2, 3)$ are determined from the equation

$$r^3+q_1r^2+q_2r+q_3=0,$$

where

$$q_1=m_1+m_2+m_3-\alpha_1-\alpha_2-\alpha_3,$$

$$q_2=m_1m_2+m_1m_3+m_2m_3-\alpha_1(m_2+m_3)-\alpha_2(m_1+m_3)-\alpha_3(m_1+m_2), \tag{2.18}$$

$$q_3=m_1m_2m_3-\alpha_1m_2m_3-\alpha_2m_1m_3-\alpha_3m_1m_2;\qquad D_{03}=E_0h^3/(1-\nu_0^2),$$

$$\eta_0=\frac{9D_{03}d_1+7E_0J_1}{d_1},\qquad \alpha_i=\frac{9D_{03}b_i}{\eta_0}\quad (i=1, 2),$$

$$\alpha_3=\frac{7E_0J_1\chi}{d_1\eta_0},\qquad m_3=\beta,$$

m_i and b_i $(i=1, 2)$ are determined by formulas (1.14), dropping the index k; $W(x, y; 0)=\frac{49q_0}{16a^4\eta_0}\left(x^2-\frac{a^2}{4}\right)\left(y^2-\frac{a^2}{4}\right)$.

The coefficients a_k $(k=1, 2, 3)$ are determined from the system of equations

$$1-\sum_{k=1}^{3}\frac{a_k}{m_i+r_k}=0\qquad (i=1, 2, 3). \tag{2.19}$$

At $q_1 > 0$, $q_1 q_2 - q_3 > 0$, $q_3 > 0$ we have $\mathrm{Re}\,(r_k) < 0$. In this case

$$\frac{W(x, y, \infty)}{W(x, y, 0)} = 1 - \sum_{k=1}^{3} \frac{a_k}{r_k}. \tag{2.20}$$

Let $E_0 \neq E'_{01} \neq E'_{02} \neq 0$ and $\chi_1 \neq \chi'_1 = \chi'_2 = 0$, i.e., the plate material possesses the property of creep, while the ribs are ideally elastic. Let also $E'_{01} J_1 d_1^{-1} = E'_{02} J_2 d_2^{-1}$, $E'_{01} J_1 d_1^{-1} = \lambda E_0 h^3/12$, where λ is specified. Then

$$\frac{W(x, y, t)}{W(x, y, 0)} = \left[1 - \frac{9 b_1 E_\alpha^*(-m_1)}{9 + 7\lambda(1 - \nu_0^2)} - \frac{9 b_2 E_\alpha^*(-m_2)}{9 + 7\lambda(1 - \nu_0^2)} \right]^{-1} =$$

$$= 1 + \sum_{k=1}^{2} a_k E_\alpha^*(r_k) = 1 + \sum_{k=1}^{2} \frac{a_k}{r_k} [E_{1-\alpha}(r_k t^{1-\alpha}) - 1], \tag{2.21}$$

where

$$a_k = (-1)^{k-1} \frac{(r_k + m_1)(r_k + m_2)}{r_1 - r_2} \quad (r_1 \neq r_2), \quad (k = 1, 2), \tag{2.22}$$

and r_1 and r_2 are the roots of the equation

$$r^2 + \left[m_1 + m_2 - \frac{9(b_1 + b_2)}{9 + 7\lambda(1 - \nu_0^2)} \right] r + m_1 m_2 - \frac{9(m_1 b_2 + b_1 m_2)}{9 + 7\lambda(1 - \nu_0^2)} = 0. \tag{2.23}$$

At $\mathrm{Re}\,(r_k) < 0$ it follows from (2.22) that at $t \to \infty$

$$\frac{W(x, y; \infty)}{W(x, y; 0)} = 1 - \sum_{k=1}^{2} a_k r_k^{-1}. \tag{2.24}$$

The magnitudes m_i and b_i $(i = 1, 2)$ are determined by the same formulas as in the preceding case. The elastic-transient deflection $w(x, y; 0)$ depends on the value of λ and is determined by the formula

$$W(x, y; 0) = \frac{49}{4} q_0 D_{03} [9 + 7\lambda(1 - \nu_0^2)] \left(x^2 - \frac{a^2}{4}\right)\left(y^2 - \frac{a^2}{4}\right). \tag{2.25}$$

We give as an illustration the results of the calculations corresponding to lead and synthetic rubber at room temperature, and copper at a temperature of 165°C.

The values of the rheological parameters $\chi_1 = 0.1979$, $\beta_1 = 0.3299$ for lead and $\chi_2 = 0.2099$, $\beta_2 = 0.4665$ for rubber, were obtained on the basis of Bronskii's experimental data /5/, and the values $\chi = 0.500$ and $\beta = 0.862$ for copper on the basis of the experimental curve of simple relaxation represented in Figure 598 in /6/ at $\alpha = 0.3$.

TABLE 1a

$\mu_1 = 0.2$

μ_2	0	0.2	0.4	0.6	0.8	1.0
$\frac{W_C(\infty)}{W'_0}$	1.3931	1.2033	1.0983	1.0112	0.9327	0.8685

TABLE 1b

$\mu_1 = 0.4$

μ_2	0	0.2	0.4	0.6	0.8	1.0
$\frac{W_c(\infty)}{W_0'}$	1.0526	0.9699	0.8999	0.8403	0.7870	0.7439

TABLE 1c

$\mu_1 = 0.6$

μ_2	0	0.2	0.4	0.6	0.8	1.0
$\frac{W_c(\infty)}{W_0'}$	0.8695	0.8123	0.7635	0.7207	0.6801	0.6449

Tables 1a, 1b, and 1c give the results of calculations by formula (2.15), allowing for (2.11) to (2.13), for the case of lead supporting ribs and rubber outer layers of a sandwich plate at various values of μ_2 for $\mu_1 = 0.2$ (Table 1a), $\mu_1 = 0.4$ (Table 1b), and $\mu_1 = 0.6$ (Table 1c). Here $W_0' = 405\ q_0 b^4/4992\, E_{01}$.

Table 2 gives the results of the calculations on the basis of (2.25), (2.23), and (2.24) for a simple copper plate at $\nu_0 = 0.3$, reinforced by elastic ribs. Here W_0^* is the elastic-transient deflection of a nonreinforced plate.

TABLE 2

λ	0	1	2	3	4	5	6	7	8	9	10
$W(\infty)/W_0^*$	0.264	0.097	0.060	0.044	0.034	0.028	0.024	0.020	0.017	0.016	0.015

Table 3 gives the results of calculations by (2.20), (2.18), and (2.19) at $\nu_0 = 0.3$ for a simple copper plate reinforced by ribs made from the same copper.

TABLE 3

λ	0	1	2	3	5	7	10
$W(\infty)/W(0)$	2.379	2.567	2.345	2.382	2.384	2.333	2.360

It follows from a comparison of Tables 2 and 3 that stiffening by ribs made from the same material as the plate gives the poorest results.

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ON THE BENDING OF LENGTHY ANISOTROPIC PLATES MOVING IN A GAS AT CONSTANT SUPERSONIC VELOCITY

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The state of stress and strain of a thin elastic rectangular plate of uniform thickness hinged on two opposite sides and moving at constant supersonic velocity in a gaseous medium was treated in /2/ on the basis of Il'yushin's piston theory /1/.

Numerous papers have been devoted to the subject of the stability of deformation processes of solid bodies moving in a gas (cf. /3/).

The nonlinear problems of the stability of plane panels at high supersonic velocities were considered in /4/ and other papers.

The problem of the bending of lengthy elastic plates moving in a gas at constant supersonic velocity was solved in /5/.

We use here the small-parameter method for an approximate solution of the problem of the bending of a long and narrow anisotropic (nonorthotropic) plate of arbitrary symmetrical shape clamped by the edges, moving at constant supersonic velocity in a gaseous medium, and inclined under an angle α to the direction of motion.

The solution is represented in the form of a power series of the small parameter λ, introduced in the equation of the region contour. The question of the asymptotic convergence of this series is examined.

1. STATEMENT OF THE PROBLEM

Consider an elastic homogeneous anisotropic plate of uniform thickness deformed under the action of a bending load. It is assumed that the plate is not orthotropic in the general case, but possesses at each point a plane of elastic symmetry parallel to the middle plane.

The problem of finding the deflections of a nonorthotropic plate bent under the action of forces perpendicular to its plane reduces, as known /6/, to solving the equation*

$$D_{11}\cdot\frac{\partial^4 w}{\partial x^4}+4D_{16}\cdot\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\cdot\frac{\partial^4 w}{\partial x\partial y^3}+$$
$$+D_{22}\cdot\frac{\partial^4 w}{\partial y^4}=q \qquad (1.1)$$

* We use here and below the generally accepted symbols.

at given boundary conditions.

Let a nonorthotropic plate clamped by the edges, and moving at a constant supersonic velocity in a gaseous medium, be inclined under an angle α to the direction of motion. In this case $q(x_1 y)$ is the aerostatic load on the plate, calculated by Il'yushin's formula /1/

$$q(x,y)=\frac{p_0 \varkappa c}{c_0}\cdot\left(\alpha+\frac{\partial w}{\partial x}\right). \tag{1.2}$$

In the case of a plate clamped by the edges we have at the boundary

$$w=0, \quad \frac{\partial w}{\partial n}=0.$$

2. THE SMALL-PARAMETER METHOD FOR THE PROBLEM CONSIDERED

Let the region Ω be bounded by the curves (Figure 1)

$$y=\pm\varphi(x). \tag{2.1}$$

Designate by Ω_1 a region contained in Ω and bounded by the curves

$$y=\pm\lambda\varphi(x) \quad (0<\lambda<1). \tag{2.2}$$

The boundary-value problem for region Ω_1 will then be of the form

$$D_{11}\cdot\frac{\partial^4 w}{\partial x^4}+4D_{16}\cdot\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\cdot\frac{\partial^4 w}{\partial x\partial y^3}+$$
$$+D_{22}\cdot\frac{\partial^4 w}{\partial y^4}+A\cdot\frac{\partial w}{\partial x}=B, \tag{2.3}$$
$$w=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=\pm\lambda\varphi(x),$$

where $A=-\frac{p_0\varkappa c}{c_0}$, $B=-A\alpha$.

Note that to each value of the parameter λ there corresponds a single solution $w=w(x_1 y;\lambda)$ regular in region Ω_1 of the boundary-value problem (2.3).

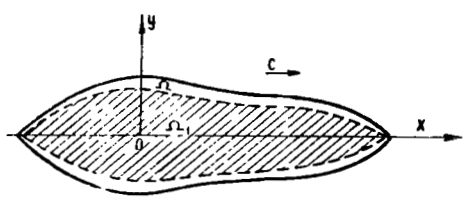

FIGURE 1.

Passing to new variables by the formulas

$$x=x, \quad y=\lambda\eta, \tag{2.4}$$

we can represent equation (2.3) in the form

$$D_{11}\cdot\frac{\partial^4 w}{\partial x^4}+\frac{4D_{16}}{\lambda}\cdot\frac{\partial^4 w}{\partial x^3\partial\eta}+\frac{2\,(D_{12}+2D_{66})}{\lambda^2}\cdot\frac{\partial^4 w}{\partial x^2\partial\eta^2}+\frac{4D_{26}}{\lambda^3}\cdot\frac{\partial^4 w}{\partial x\partial\eta^3}+$$
$$+\frac{D_{22}}{\lambda^4}\cdot\frac{\partial^4 w}{\partial\eta^4}+A\cdot\frac{\partial w}{\partial x}=B, \qquad (2.5)$$

and the boundary conditions in the form

$$w=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial\eta}=0 \quad \text{at} \quad \eta=\pm\varphi(x). \qquad (2.6)$$

We shall represent the solution of the boundary-value problem in the form of a series by the powers of a small parameter:

$$w(x,\eta;\lambda)=\sum_{k=0}^{\infty} w_k(x,\eta)\lambda^k. \qquad (2.7)$$

By substituting the value of w from (2.7) in (2.5), multiplying by λ^4 and requiring the identical satisfaction by λ, we obtain the system of differential equations

$$\frac{\partial^4 w_0}{\partial\eta^4}=0;$$
$$4D_{26}\cdot\frac{\partial^4 w_0}{\partial x\partial\eta^3}+D_{22}\cdot\frac{\partial^4 w_1}{\partial\eta^4}=0;$$
$$2\,(D_{12}+2\,D_{66})\cdot\frac{\partial^4 w_0}{\partial x^2\partial\eta^2}+4D_{26}\cdot\frac{\partial^4 w_1}{\partial x\partial\eta^3}+D_{22}\cdot\frac{\partial^4 w_2}{\partial\eta^4}=0;$$
$$4D_{16}\cdot\frac{\partial^4 w_0}{\partial x^3\partial\eta}+2(D_{12}+2D_{66})\cdot\frac{\partial^4 w_1}{\partial x^2\partial\eta^2}+4D_{26}\cdot\frac{\partial^4 w_2}{\partial x\partial\eta^3}+D_{22}\cdot\frac{\partial^4 w_3}{\partial\eta^4}=0; \qquad (2.8)$$
$$D_{11}\cdot\frac{\partial^4 w_0}{\partial x^4}+4D_{16}\cdot\frac{\partial^4 w_1}{\partial x^3\partial\eta}+2(D_{12}+2D_{66})\cdot\frac{\partial^4 w_2}{\partial x^2\partial\eta^2}+4D_{26}\cdot\frac{\partial^4 w_3}{\partial x\partial\eta^3}+$$
$$+D_{22}\cdot\frac{\partial^4 w_4}{\partial\eta^4}+A\cdot\frac{\partial w_0}{\partial x}=B;$$
$$D_{11}\cdot\frac{\partial^4 w_k}{\partial x^4}+4D_{16}\cdot\frac{\partial^3 w_{k+1}}{\partial x^3\partial\eta}+2(D_{12}+2D_{66})\cdot\frac{\partial^4 w_{k+2}}{\partial x^2\partial\eta^2}+4D_{26}\cdot\frac{\partial^4 w_{k+3}}{\partial x\partial\eta^3}+$$
$$+D_{22}\cdot\frac{\partial^4 w_{k+4}}{\partial\eta^4}+A\frac{\partial w_k}{\partial x}=0 \qquad (k=1,2,3,\ldots).$$

In order that the series (2.7) satisfy the boundary conditions (2.6) it is sufficient that these latter be satisfied by each of the functions $w_k(x,\eta)$.

3. SOLUTION OF THE SYSTEM OF EQUATIONS (2.8)

We obtain from the first four equations of system (2.8) and the boundary conditions (2.6)

$$w_0=w_1=w_2=w_3=0. \qquad (3.1)$$

The fifth equation reduces to

$$D_{22}\cdot\frac{\partial^4 w_4}{\partial\eta^4}=B. \tag{3.2}$$

By solving this equation at boundary conditions (2.6) we obtain

$$w_4(x,\eta)=\frac{B}{24D_{22}}\cdot(\eta^2-\varphi^2)^2.$$

From the sixth equation of system (2.8) and the boundary conditions (2.6) it is clear that

$$w_5(x,\eta)=0.$$

The following differential equation is obtained for determining $w_6(x,\eta)$:

$$2(D_{12}+2D_{66})\cdot\frac{\partial^4 w_6}{\partial x^2\partial\eta^2}+D_{22}\cdot\frac{\partial^4 w_6}{\partial\eta^4}=0.$$

Its solution yields

$$w_6(x,\eta)=\frac{(D_{12}+2D_{66})B}{72\cdot D_{22}^2}\cdot(\varphi^2)''\cdot(\eta^2-\varphi^2)^2. \tag{3.3}$$

The following differential equation is obtained from (2.8) for the determination of $w_7(x,\eta)$:

$$D_{22}\cdot\frac{\partial^4 w_7}{\partial\eta^4}+4D_{26}\cdot\frac{\partial^4 w_6}{\partial x\partial\eta^3}+4D_{16}\cdot\frac{\partial^4 w_4}{\partial x^3\partial\eta}=0.$$

We find from here, with the aid of (2.6):

$$w_7(x,\eta)=\frac{c_7\cdot(\varphi^2)''\cdot\eta}{5!}\cdot(\eta^2-\varphi^2)^2, \tag{3.4}$$

where

$$c_7=\frac{2B}{3D_{22}^3}\cdot[D_{16}\cdot D_{22}-2D_{26}(D_{12}+2D_{66})].$$

We next obtain for $w_8(x,\eta)$ the expression

$$w_8(x,\eta)=\frac{c_8(x)}{6!}(\eta^6-3\varphi^4\cdot\eta^2+2\varphi^6)+\frac{B_8(x)}{4!}(\eta^2-\varphi^2)^3, \tag{3.5}$$

where

$$c_8(x)=\frac{AB(\varphi^2)'}{3!D_{22}}-\frac{4D_{26}c_7}{D_{22}}\cdot(\varphi^2)^{\mathrm{IV}}-\frac{2(D_{12}+2D_{66})^2\cdot B}{D_{22}^3}\cdot(\varphi^2)^{\mathrm{IV}}+\frac{D_{11}B}{3!D_{22}^2}\cdot(\varphi^2)^{\mathrm{IV}}$$

$$B_8(x)=-\frac{AB}{12D_{22}}\cdot\varphi^2\cdot(\varphi^2)'+\frac{2D_{26}\cdot c_7}{5D_{22}}[(\varphi^2)'\cdot(\varphi^2)'''+\varphi^2\cdot(\varphi)^{\mathrm{IV}}]+$$

$$+\frac{(D_{12}+2D_{66})^2\cdot B}{9D_{22}^3}\,[(\varphi^2)^{\mathrm{IV}}\cdot\varphi^2+2(\varphi^2)'''\cdot(\varphi^2)'+((\varphi^2)'')^2]+\frac{D_{11}B}{41D_{22}^2}\cdot(\varphi^4)^{\mathrm{IV}}. \qquad (3.6)$$

The following expression is obtained thus for determining the plate deflection $w(x,\eta;\lambda)$ with an accuracy of the order of λ^9:

$$w(x,\eta;\lambda)=\frac{\lambda^4 B}{24D_{22}}\cdot(\eta^2-\varphi^2)^2+\lambda^6\cdot\frac{(D_{12}+2D_{66})B}{72D_{22}^2}\cdot(\varphi^2)''\cdot(\eta^2-\varphi^2)^2+$$
$$+\lambda^7\cdot\frac{c_7\cdot\eta\,(\varphi^2)'''\cdot(\eta^2-\varphi^2)^2}{5!}+\lambda^8\cdot\left\{\frac{c_8(x)}{6!}\cdot(\eta^6-3\varphi^4\eta^2+2\varphi^6)+\right.$$
$$\left.+\frac{B_8(x)}{4!}\cdot(\eta^2-\varphi^2)^2\right\}+0(\lambda^9). \qquad (3.7)$$

Knowing w, we calculate by the standard methods the bending moments, the torque, and the shearing forces (cf. /6/).

The following expression is obtained from (1.2) and (3.7) for the aerostatic load:

$$q=B-\lambda^4\cdot\frac{B^2}{12D_{22}a}\cdot(\eta^2-\varphi^2)\cdot(\varphi^2)'+\lambda^6\cdot\frac{(D_{12}+2D_{66})B^2}{72D_{22}^2a}\cdot[(\varphi^2)'''\cdot(\eta^2-\varphi^2)^2-$$
$$-2(\varphi^2)''\cdot(\varphi^2)']+\lambda^7\cdot\frac{Bc_7\eta\,[(\varphi^2)^{\mathrm{IV}}\cdot(\eta^2-\varphi^2)^2-2(\varphi^2)'\cdot(\varphi^2)''']}{5!\,a}+$$
$$+\lambda^8\cdot\left\{\frac{c_8(x)}{6!}\cdot[2(\varphi^6)'-3(\varphi^4)'\cdot\eta^2]-\frac{B_8(x)}{12}(\varphi^2)'\cdot(\eta^2-\varphi^2)+\right.$$
$$\left.+\frac{c_8'(x)}{6!}(\eta^6-3\varphi^4\eta^2+2\varphi^6)+\frac{B_8'(x)}{4!}\cdot(\eta^2-\varphi^2)^2\right\}\cdot\frac{B}{a}+0(\lambda^9). \qquad (3.8)$$

The results obtained become simpler in the case of an orthotropic plate.

4. PARTICULAR CASES

1. Infinite strip. Consider the action of an aerostatic load on an infinite strip of width 2a (Figure 2).

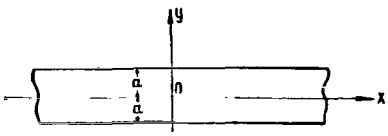

FIGURE 2.

The equation of the infinite strip is $y=\pm\lambda a$ or $\eta=\pm a$. It is easily seen that all w_k with the exception of w_4 vanish. We obtain then from (3.7) the exact solution of the boundary-value problem (2.3):

$$w(x,y)=\frac{B}{24D_{22}}\cdot(y^2-a^2)^2. \qquad (4.1)$$

The bending moments, the torsional moment, the shearing forces, and the aerostatic load are determined as follows:

$$M_x = -\frac{BD_{12}}{6D_{22}}\cdot(3y^2-a^2), \qquad M_y = -\frac{B}{6}\cdot(3y^2-a^2),$$

$$H_{xy} = -\frac{BD_{26}}{6D_{22}}\cdot(3y^2-a^2), \qquad N_x = -\frac{BD_{26}\cdot y}{D_{22}},$$

$$N_y = -By, \qquad q = \frac{p_0 x c}{c_0}\cdot\alpha.$$

2. Elliptical plate (Figure 3). The contour equation is

$$y = \pm\lambda\varphi(x), \text{ where } \varphi(x) = \sqrt{a^2-x^2}, \ \lambda = \frac{b}{a}.$$

From (3.7) we obtain

$$w(x,\eta;\lambda) = \lambda^4\cdot\left[\frac{B}{24D_{22}} - \frac{\lambda^2\cdot(D_{12}+2D_{66})B}{36D_{22}^2}\right]\times$$

$$\times(\eta^2+x^2-a^2)^2 - \lambda^8\cdot\frac{ABx}{3D_{22}}\cdot[\eta^6-3\eta^2\cdot(a^2-x^2)^2+2(a^2-x^2)^3]+0(\lambda^9). \qquad (4.2)$$

The following expression is obtained for the aerostatic load:

$$q = B + \lambda^4\cdot x\cdot\left[\frac{B^2}{6D_{22}\alpha} - \lambda^2\cdot\frac{B(D_{12}+2D_{66})}{9D_{22}^2\alpha}\right]\cdot(\eta^2+x^2-a^2) - \lambda^8\cdot\frac{AB^2}{3D_{22}\alpha}\times$$

$$\times\{[\eta^6-3\eta^2(a^2-x^2)^2+2(a^2-x^2)^3]+12x^2(a^2-x^2)\cdot(2\eta^2+a^2-x^2)\}+0(\lambda^9). \qquad (4.3)$$

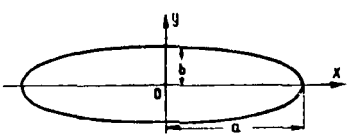

FIGURE 3.

3. Streamline profile (Figure 4). The contour equation is

$$y = \pm\lambda\varphi(x), \text{where } \varphi^2(x) = c_1x(a-x)^2, \ c_1 = \frac{27}{16a}.$$

rom (3.7) we obtain the expression for the deflection

$$w(x,\eta;\lambda) = \lambda^4\cdot\left[\frac{B}{24D_{22}} - \lambda^2\frac{(D_{12}+2D_{66})Bc_1x}{36D_{22}^2}\right]\cdot[\eta^2-c_1x(a-$$

$$-x)^2]^2 - \lambda^7\cdot\frac{2c_7\eta c_1}{5!}\cdot[\eta^2-c_1x(a-x)^2]^2+0(\lambda^8).$$

The expression for the aerostatic load is

$$q = B - \lambda^4\cdot\left[\frac{B^2c_1(a^2-x^2)}{12D_{22}\alpha} + \lambda^2\cdot\frac{(D_{12}+2D_{66})B^2c_1}{36D_{22}^2\alpha}\right]\cdot[\eta^2-c_1x(a-x)^2]^2+$$

$$+\lambda^6\cdot\frac{(D_{12}+2D_{66})B^2c_1^2x(a^2-x^2)}{18D_{22}^2\alpha} + 0(\lambda^8).$$

As seen from the examples considered, the question of the convergence of series (3.7) is essential.

FIGURE 4.

We shall study the asymptotic convergence of series (2.7), which is more general.

5. PROOF OF THE ASYMPTOTIC CONVERGENCE OF SERIES (2.7)

We formulate the following lemma in order to prove the asymptotic convergence of series (2.7).*

Lemma. If $w(x, y)$ satisfies the equation

$$L_4(w) = B$$

$$\left[L_4 = D_{11}\cdot \frac{\partial^4}{\partial x^4} + 4D_{16}\cdot \frac{\partial^4}{\partial x^3 \partial y} + 2(D_{12}+2D_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + \right.$$

$$\left. + 4D_{26}\cdot \frac{\partial^4}{\partial x \partial y^3} + D_{22}\cdot \frac{\partial^4}{\partial y^4} + A\cdot \frac{\partial}{\partial x} \right], \qquad (5.1)$$

inside region D and the conditions

$$w|_\Gamma = \frac{\partial w}{\partial x}\Big|_\Gamma = \frac{\partial w}{\partial y}\Big|_\Gamma = 0,$$

on the boundary (Γ), then for any function $F(x,y)$ which satisfies the equation

$$L_4(F) = f(x,y)$$

in region $D_1 \subset D$ and the conditions

$$F|_{\Gamma_1} = \frac{\partial F}{\partial x}\Big|_{\Gamma_1} = \frac{\partial F}{\partial y}\Big|_{\Gamma_1} = 0,$$

at the boundary (Γ_1) of region D_1 we have $|F(x,y)| < |w(x,y)|$ inside and on the boundary of D_1, only if $|f(x,y)| < B$ in this region and on its boundary.

We shall now prove the asymptotic convergence of series (2.7). To that end we shall study the solution of our problem as a function of λ.

Form the difference

$$\rho_n(x,y;\lambda) = w(x,y;\lambda) - \sum_{k=0}^{n} \lambda^k \cdot w_k(x,\eta). \qquad (5.2)$$

* Equation (5.1) is reduced by an affine transformation to the equation considered in /5/, for which the lemma was proved.

Apply on it the operator L_4:

$$L_4(\rho_n) = L_4(w) - \sum_{k=0}^{n} \lambda^k L_4(w_k). \tag{5.3}$$

Allowing for (5.1) and (2.9), we find from (5.3)

$$L_4(\rho_n) = -\lambda^{n-3}\cdot\left[D_{11}\cdot\frac{\partial^4 w_{n-3}}{\partial x^4} + 4D_{16}\cdot\frac{\partial^4 w_{n-2}}{\partial x^3\partial\eta} + 2(D_{12}+2D_{66})\frac{\partial^4 w_{n-1}}{\partial x^2\partial\eta^2} + \right.$$
$$\left. + A\cdot\frac{\partial w_{n-3}}{\partial x}\right] - \lambda^{n-2}\cdot\left[D_{11}\cdot\frac{\partial^4 w_{n-2}}{\partial x^4} + 4D_{16}\cdot\frac{\partial^4 w_{n-1}}{\partial x^3\partial\eta} + \right.$$
$$\left. + 2(D_{12}+2D_{66})\cdot\frac{\partial^4 w_n}{\partial x^2\partial\eta^2} + A\cdot\frac{\partial w_{n-2}}{\partial x}\right] - \lambda^{n-1}\cdot\left[D_{11}\cdot\frac{\partial^4 w_{n-1}}{\partial x^4} + \right.$$
$$\left. + 4D_{16}\frac{\partial^4 w_n}{\partial x^3\partial\eta} + A\cdot\frac{\partial w_{n-1}}{\partial x}\right] - \lambda^n\cdot\left[D_{11}\frac{\partial^4 w_4}{\partial x^4} + A\cdot\frac{\partial w_n}{\partial x}\right] \equiv g_n(x,\eta;\lambda). \tag{5.4}$$

We shall now assess expression (5.4):

$$|g_n(x,\eta;\lambda)| \leqslant \lambda^{n-3}\cdot\left\{\left|\frac{\partial^4 w_{n-3}}{\partial x^4}\right|\cdot D_{11} + 4D_{16}\cdot\left|\frac{\partial^4 w_{n-2}}{\partial x^3\partial\eta}\right| + 2(D_{12}+2D_{66})\times\right.$$
$$\left.\times\left|\frac{\partial^4 w_{n-1}}{\partial x^2\partial y^2}\right| + \left|A\cdot\frac{\partial w_{n-3}}{\partial x}\right|\right\} + \lambda^{n-2}\cdot\left\{D_{11}\cdot\left|\frac{\partial^4 w_{n-2}}{\partial x^4}\right| + 4D_{16}\cdot\left|\frac{D^4 w_{n-1}}{\partial x^3\partial\eta}\right| + \right.$$
$$\left. + 2(D_{12}+2D_{66})\cdot\left|\frac{\partial^4 w_n}{\partial x^2\partial\eta^2}\right| + \left|A\cdot\frac{\partial w_{n-2}}{\partial x}\right|\right\} + \lambda^{n-1}\cdot\left\{D_{11}\cdot\left|\frac{\partial^4 w_{n-1}}{\partial x^4}\right| + \right.$$
$$\left. + 4D_{16}\cdot\left|\frac{\partial^4 w_n}{\partial x^2\partial\eta}\right| + A\cdot\frac{\partial w_{n-1}}{\partial x}\right|\right\} + \lambda^n\cdot\left\{D_{11}\cdot\left|\frac{\partial^4 w_n}{\partial x^4}\right| + \left|A\frac{\partial w_n}{\partial x}\right|\right\}. \tag{5.5}$$

The expression $\dfrac{\partial^4 w_k\left(x, \frac{y}{\lambda}\right)}{\partial i^2\partial j^2}$ and $A\dfrac{\partial w_k}{\partial x}$ are regular functions of the argument x and simply polynomials of the argument η inside and on the boundary of D_1. Introducing in region D the symbols

$$r_n = \max_D\left|A\cdot\frac{\partial w_n(x,\frac{y}{\lambda})}{\partial x}\right|, \quad p_n = \max_D\left|\frac{\partial^4 w_n\left(x,\frac{y}{\lambda}\right)}{\partial x^4}\right|,$$

$$l_n = \max_D\left|\frac{\partial^4 w_n(x,\frac{y}{\lambda})}{\partial x^3\partial\eta}\right|, \quad q_n = \max_D\left|\frac{\partial^4 w_n}{\partial x^2\partial\eta^2}\right|,$$

we obtain the following estimates for the function $g_n(x,\eta;\lambda)$ in region D_1:

$$|g_n(x,\eta;\lambda)| \leqslant \lambda^{n-3}\cdot[D_{11}\cdot p_{n-3} + 4D_{16}\cdot l_{n-2} + 2(D_{12}+2D_{66})\cdot q_{n-1} + r_{n-3}] +$$
$$+ \lambda^{n-2}\cdot[D_{11}\cdot p_{n-2} + 4D_{16}\cdot l_{n-1} + 2(D_{12}+2D_{66})\cdot q_n + r_{n-2}] + \lambda^{n-1}\cdot[D_{11}\cdot p_{n-1} +$$
$$+ 4D_{16}\cdot l_n + r_{n-1}] + \lambda^n\cdot[D_{11}\cdot p_n + r_n] \equiv g_n^*(\lambda). \tag{5.6}$$

From the boundary conditions for $w(x,y)$ and expression (5.2) we obtain

$$\left.\begin{aligned} &\rho_n(x,y;\lambda) = 0 \\ &\frac{\partial\rho_n}{\partial m} = 0 \end{aligned}\right\} \text{ at the boundary of region } D_1,$$

where m is an inner normal to the contour Γ_1.

Introduce a new function, defined as follows:

$$\overset{*}{\rho}_n(x,y;\lambda) = \frac{B}{\overset{*}{g}_n(\lambda)} \cdot \rho_n(x,y;\lambda). \tag{5.7}$$

Form

$$\overset{*}{R}_n = L_4(\overset{*}{\rho}_n).$$

We shall obviously have, in a manner similar to (5.4),

$$|\overset{*}{R}_n| = \frac{B}{\overset{*}{g}_n} |g_n(x,\eta;\lambda)| \leqslant B. \tag{5.8}$$

The determination of the function $\overset{*}{\rho}_n(x,y;\lambda)$ reduces thus to solving the following boundary-value problem:

$$\left.\begin{aligned} &L_4(\overset{*}{\rho}_n) = \overset{*}{R}_n(x,y;\lambda) \text{ in } D_1 \subset D, \\ &\overset{*}{\rho}_n\Big|_{\Gamma_1} = 0, \quad \frac{\partial \overset{*}{\rho}_n}{\partial x}\Big|_{\Gamma_1} = \frac{\partial \overset{*}{\rho}_n}{\partial y}\Big|_{\Gamma_1} = 0. \end{aligned}\right\} \tag{5.9}$$

The conditions of the lemma are obviously satisfied. Therefore,

$$|\overset{*}{\rho}_n| < |w(x,y)|$$

or, allowing for (5.7):

$$|\rho_n(x,y;\lambda)| < \frac{\overset{*}{g}_n(\lambda)}{B} \cdot |w(x,y)|. \tag{5.10}$$

Let an infinite strip of width $2a$ contain the region D within it. It is easily seen that we can select an infinite strip of width $2\lambda a$, which will contain the region D_1. Then, according to (4.1), the deflection for D_1 will satisfy the inequality

$$|w| = \frac{B}{24D_{22}} \cdot (y^2 - \lambda^2 a^2)^2 \leqslant \frac{B\lambda^4 a^4}{24D_{22}}. \tag{5.11}$$

Using (5.11), we finally find from (5.10)

$$|\rho_n(x,y;\lambda)| < \frac{\overset{*}{g}_n(\lambda)}{B} \cdot \frac{B\lambda^4 a^4}{24D_{22}}$$

or

$$\begin{aligned} |\rho_n(x,y;\lambda)| < \frac{a^4\lambda^{n+1}}{24D_{22}} \cdot \{&[D_{11}p_{n-3} + 4D_{16}\cdot l_{n-2} + 2(D_{12}+2D_{66})\cdot q_{n-1} + \\ &+ r_{n-3}] + \lambda[D_{11}\cdot p_{n-2} + 4D_{16}\cdot l_{n-1} + 2(D_{12}+2D_{66})\cdot q_n + r_{n-2}] + \\ &+ \lambda^2\cdot[D_{11}\cdot p_{n-1} + 4D_{16}\cdot l_n + r_{n-1}] + \lambda^3\cdot(D_{11}\cdot p_n + r_n)\}, \quad n \geqslant 4. \end{aligned} \tag{5.12}$$

The estimate (5.12) obtained solves the problem of the asymptotic convergence (in the Poincaré sense) of series (2.7), since it follows from it that for any value of n

$$\lim_{\lambda\to 0} \rho_n(x,y;\lambda) = 0, \qquad \lim_{\lambda\to 0} \frac{|\rho_n(x,y;\lambda)|}{\lambda^n} = 0.$$

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USE OF THE METHOD OF DYNAMIC RIGIDITIES FOR CALCULATING THE FREQUENCIES OF NATURAL VIBRATIONS OF BUILT-UP SHELLS

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(Moscow)

The determination of the frequencies of vibrations of built-up shells is of considerable practical interest. It makes sense to calculate them by the method of dynamic rigidities used in the theory of vibrations of bar systems /1/. This method has already been used by the author for solving the problems of axisymmetrical vibrations of annular plates and cylindrical shells /2/. The region of application of the method of dynamic rigidities can be broadened still further and made to include the study of arbitrary vibrations of built-up shells.

The dynamic rigidity is by definition the ratio of the amplitude of the generalized force, acting according to some harmonic law, to the amplitude of the generalized displacement caused by it and varying likewise harmonically.

The scheme for solving problems on the determination of the frequencies of natural vibrations by the method of dynamic rigidities is as follows: by means of several cuts we split the built-up shell into several shells of simple types (cylindrical, spherical, conical, etc.), and introduce at the planes of separation forces and moments varying harmonically with an as yet unknown frequency ω. We then solve the problem of the forced vibrations of each of the simple shells under the action of the end forces. Using the values obtained for the displacements at the ends we form the dynamic rigidities as the ratio of the amplitudes of the forces to the amplitudes of the displacements caused by them. Next, using the equality of the interactions and the conditions of strain compatibility, we obtain the equation for determining the frequencies of natural vibrations of built-up shells.

1. DYNAMIC RIGIDITY OF A CYLINDRICAL SHELL SUBJECTED TO DISTRIBUTED BENDING MOMENTS ACTING AT ITS EDGES

Consider an isotropic cylindrical shell (the same treatment is applicable to an orthotropic cylindrical shell as well) of radius b, length $2l$, thickness h, Young's modulus and Poisson ratio of the shell material E and σ, and mass of unit surface of the shell M.

Introduce the dimensionless coordinates $\xi = \frac{x}{l}$ and φ. The x-axis is directed along the shell generator; φ is the central angle in the shell cross section.

We shall restrict ourselves to the case of vibrations which are symmetrical relative to the plane perpendicular to the shell axis and passing through its middle (the problem of antisymmetrical vibrations is similarly solved).

The equations of vibrations will be derived from Gol'denveizer's equations /3/, by introducing the inertia forces $M\frac{\partial^2 u}{\partial t^2}$, $-M\frac{\partial^2 v}{\partial t^2}$, $-M\frac{\partial^2 w}{\partial t^2}$ instead of X, Y, Z, respectively.

We have the boundary conditions

$$w = 0, \qquad v = 0, \qquad T_1 = \frac{Eh}{(1-\sigma^2)\,b}\frac{\partial u}{\partial \xi}\Big|_{\xi_0} = 0,$$

$$M_1 = -\frac{Eh^3}{12(1-\sigma^2)}\frac{1}{b^2}\frac{\partial^2 w}{\partial \xi^2}\Big|_{\xi_0} = M_0 \cos m\varphi \cos \omega t \quad \text{at} \quad \xi = \pm\xi_0 = \pm\frac{l}{b}.$$

We shall look for a solution in the form

$$u = e^{k\xi}\cos m\varphi \cos \omega t; \quad v = e^{k\xi}\sin m\varphi \cos \omega t; \quad w = e^{k\xi}\cos m\varphi \cos \omega t.$$

The value of k is determined from the characteristic equations

$$(1+4\nu^2)k^8 - \left[4(1+\nu^2)m^2 - \left(\frac{3-\sigma}{1-\sigma} + 4\nu^2\right)\Omega^2\right]k^6 + \Big\{[6+\nu^2(1-\sigma^2)]\,m^2 -$$
$$-2(4-\sigma^2)m^2 + (1-\sigma^2)(4-\nu^{-2}) + \frac{2}{1-\sigma}\Omega^4 -$$
$$-\left[(4+\nu^{-2}) + 2(1+\sigma)\nu^2 m^2 + 3\frac{3-\sigma}{1-\sigma}m^2\right]\Omega^2\Big\}k^4 -$$
$$-\Big\{4m^2(m^2-1)^2 + \left[4 + 4\frac{m^2}{1-\sigma} + \frac{3-\sigma}{1-\sigma}\nu^{-2}\right]\Omega^4 -$$
$$-\left[(3+2\sigma)\nu^{-2} + 4 + 2m^2\nu^{-2} + 3\frac{3-\sigma}{1-\sigma}m^4 - 2\frac{2-\sigma^2}{1-\sigma}m^2\right]\Omega^2\Big\}k^2 +$$
$$+ m^4(m^2-1)^2 - \frac{2}{1-\sigma}\nu^{-2}\Omega^6 + \frac{1}{1-\sigma}\{[2+m^2(3-\sigma)]\,\nu^{-2} + 2m^2(m^2+$$
$$+1)\}\,\Omega^4 - \left\{m^2(m^2+1)\nu^{-2} + \frac{m^2}{1-\sigma}[2+(3-\sigma)m^2 - (3+\sigma)m^4]\right\}\Omega^2 = 0.$$

Here

$$\Omega^2 = \frac{(1-\sigma^2)\,b^2 M}{Eh}\,\omega^2, \qquad \nu^2 = \frac{h^2}{12\,b^2}.$$

The general solution will be written as sum of the eight particular solutions corresponding to the different k. The eight arbitrary constants are determined from the boundary conditions. Form the expression for the deflection. Its differentiation by ξ and the substitution $\xi = \xi_0$ yield the value of the slope at the shell edge. By dividing M_0 by the amplitude of the slope

at the edge we obtain the expression for the dynamic rigidity

$$K_c(\omega) = \frac{M_0}{\frac{1}{b}\frac{dW}{d\xi}\Big|_{\xi_0}} = \frac{D}{b}\frac{\Delta}{\Delta^*}.$$

Here

$$\Delta^* = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix},$$

where

$$a_{1i} = (1+4\nu^2)\varphi_i^{(4)} + \left\{-2m^2\left[1+\nu^2\frac{2-2\sigma+\sigma^2}{1-\sigma}\right] + \left(\frac{3-\sigma}{1-\sigma}+4\nu^2\right)\Omega^2\right\}\varphi_i^{(4)} +$$
$$+\left[m^4(1+\nu^2) + \frac{2}{1-\sigma}\Omega^4 - (3-\sigma+2\nu^2)\frac{m^2}{1-\sigma}\Omega^2\right]\varphi_i,$$
$$a_{2i} = 2\nu^2 m^2\frac{2-\sigma}{1-\sigma}\varphi_i^{(4)} - \left[(2+\sigma)m + \nu^2 m^3\frac{4-3\sigma+\sigma^2}{1-\sigma} -\right.$$
$$\left. - 2\nu^2 m\frac{2-\sigma}{1-\sigma}\Omega^2\right]\varphi_i^{(2)} + \left[m^3(1+\nu^2m^2) - 2m\frac{1+\nu^2m^2}{1-\sigma}\Omega^2\right]\varphi_i;$$
$$a_{3i} = \left[\sigma + 4\sigma\nu^2 - \nu^2m^2\frac{(1+\sigma)(2-\sigma)}{1-\sigma}\right]\varphi_i^{(4)} +$$
$$+\left[m^2 - \frac{1+\sigma}{1-\sigma}\nu^2m^2 - 2\frac{\sigma^2\nu^2m^2}{1-\sigma} + 2\frac{\sigma}{1-\sigma}\Omega^2\right]\varphi_i^{(2)};$$
$$a_{4i} = (1+4\nu^2)\varphi_i^{(5)} + \left\{-2m^2\left[1+\nu^2\frac{2-2\sigma+\sigma^2}{1-\sigma}\right] + \left(\frac{3-\sigma}{1-\sigma}+4\nu^2\right)\Omega^2\right\}\varphi_i^{(3)} +$$
$$+\left[m^4(1+\nu^2) + \frac{2}{1-\sigma}\Omega^4 - (3-\sigma+2\nu^2)\frac{m^2}{1-\sigma}\Omega^2\right]\varphi_i^{(1)};$$
$$a_{5i} = (1+4\nu^2)\varphi_i^{(6)} + \left\{-2m^2\left[1+\nu^2\frac{2-2\sigma+\sigma^2}{1-\sigma}\right] +\right.$$
$$\left. + \left(\frac{3-\sigma}{1-\sigma}+4\nu^2\right)\Omega^2\right\}\varphi_i^{(4)} + \left[m^4(1+\nu^2) +\right.$$
$$\left. + \frac{2}{1-\sigma}\Omega^4 - (3-\sigma+2\nu^2)\frac{m^2}{1-\sigma}\Omega^2\right]\varphi_i^{(2)} \qquad (i=1,\ 2,\ 3,\ 4).$$

Here φ_i $(i = 1,\ 2,\ 3,\ 4)$ (and their derivatives) are functions of the form $\operatorname{ch} p\xi_0$, $\cos q\xi_0$, $\operatorname{sh} p\xi_0$, $\sin q\xi_0$, $\operatorname{ch}\lambda\xi_0$, $\cos\mu\xi$ depending on the form of the roots of the characteristic equation $(p \pm iq,\ \lambda,\ i\mu)$.

2. DYNAMIC RIGIDITY OF A SPHERICAL SHELL SUBJECTED TO DISTRIBUTED BENDING MOMENTS ACTING AT THE EDGE

Consider a spherical shell of radius R, thickness δ, Young's modulus E, Poisson ratio μ, and mass per unit surface M. The middle surface

will be characterized by the geographical coordinates θ, φ, where θ is the angle complementary to the latitude angle and φ is the longitude.

The vibrations equations will be /4/

$$\Delta\Delta w + 2\Delta w + c^2 w - \frac{c^2}{1+\mu}\Theta + \frac{MR^2}{E\delta}\frac{\partial^2}{\partial t^2}\{[c^2 + 2(1-\mu^2)(1+\mu)]w +$$

$$+ 2(1+\mu)\Theta\} = 0,$$

$$\Delta\Theta + 2\Theta - (1-\mu^2)\frac{MR^2}{E\delta}\frac{\partial^2}{\partial t^2}[(2+\mu)w + \Theta] = 0,$$

$$\Delta\chi + 2\chi - 2(1-\mu)\frac{MR^2}{E\delta}\frac{\partial^2\chi}{\partial t^2} = 0,$$

$$\Theta = \frac{\partial u}{\partial\theta} + \operatorname{ctg}\theta u + \frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi} - (1+\mu)w = 0,$$

$$\chi = \frac{\partial v}{\partial\theta} + \operatorname{ctg}\theta v - \frac{1}{\sin\theta}\frac{\partial u}{\partial t}.$$

Here Δ is the Laplacian in spherical coordinates

$$\Delta = \frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}, \quad \text{and} \quad c^2 = \frac{12R^2}{\delta^2}(1-\mu^2).$$

The boundary conditions are (we restrict ourselves to the case of a spherical shell closed at the top at $\theta = 0$):

$$v = 0, \quad -u\cos\alpha + w\sin\alpha = 0, \quad T_1\sin\alpha + Q_1^*\cos\alpha = 0,$$
$$M_1 = M_0\cos m\varphi\cos\omega t \quad \text{at} \quad \theta = \alpha.$$

Eliminating Θ from the first two equations, we obtain an equation for w. The solution will be represented in the form

$$w = W(\theta)\cos m\varphi\cos\omega t.$$

The function W is expressed in the form of a sum of the attached Legendre functions $P_{nl}^m(\cos\theta)$

$$W(\theta) = A_1P_{n_1}^m(\cos\theta) + A_2P_{n_2}^m(\cos\theta) + A_3P_{n_3}^m(\cos\theta).$$

They represent solutions of the equation for W if k

$$\left(k^2 = \frac{(1-\mu^2)R^2M}{E\delta}\omega^2\right)$$

satisfies the characteristic equation

$$p^3 - p^2(4+k^2) + pc^2\left(1 - \frac{k^2}{1-\mu^2}\right) + c^2\left(-2 - \frac{1-3\mu}{1-\mu^2}k^2 + \frac{k^4}{1-\mu^2}\right) = 0.$$

Here $p = n(n+1)$.

The solution of the equation for χ is likewise an attached Legendre function. We then find the expression for Θ, and obtain from the formulas determining Θ, χ expressions for u and v in the form of sums of spherical

functions, with n_4 for the fourth function found from the equation

$$p-2\left(1+\frac{k^2}{1-\mu}\right)=0.$$

The arbitrary constants are determined from the boundary conditions. The expression for the dynamic rigidity is formed by the formula

$$K_s(\omega)=\frac{M_0}{\frac{1}{R}\left(-\frac{dW}{d\theta}+U\right)_\alpha}=\frac{D}{R}\frac{\Delta}{\Delta^*},$$

$$a_{11}=P_{n_1}^m+\eta_1\operatorname{ctg}\alpha P_{n_1}^{m'},\quad a_{12}=w_2^m+\operatorname{ctg}\alpha\,[d_2w_2^m-d_3w_3^m],$$

$$a_{13}=w_3^m+\operatorname{ctg}\alpha\,(d_3w_2^m+d_2w_2^m),\quad a_{14}=-m\frac{\operatorname{ctg}\alpha}{\sin\alpha}P_{n_4}^m,$$

$$a_{21}=\eta_1P_{n_1}^m,\quad a_{22}=d_2w_2^m-d_3w_3^m,\quad a_{23}=d_3w_2^m+d_2w_3^m,$$

$$a_{24}=-\frac{\sin\alpha}{m}P_{n_4}^{m'},\quad a_{31}=\Big\{(1-\mu)\,\eta_1\operatorname{ctg}\alpha+\lambda^2\operatorname{ctg}\alpha\Big[(1-2\eta_1)P_1+2\mu+$$

$$+(1-\mu)(1-2\eta_1)\frac{m^2}{\sin^2\alpha}\Big]\Big\}P_{n_1}^{m'}+$$

$$+\Big\{-(1+\mu)+p_1\eta_1-(1-\mu)\frac{m^2}{\sin^2\alpha}[\eta_1+\lambda^2(1-2\eta_1)\operatorname{ctg}^2\alpha]\Big\}P_{n_1}^m,$$

$$a_{32}=\Big\{-(1+\mu)+(1-\mu)\frac{m^2}{\sin^2\alpha}[\operatorname{ctg}\alpha-d_2(1-2\lambda^2\operatorname{ctg}^2\alpha)]+$$

$$+(ad_2-bd_3)\Big\}w_2^m+\Big\{(1-\mu)\frac{m^2}{\sin^2\alpha}d_3(1-2\lambda^2\operatorname{ctg}^2\alpha-(ad_3+bd_2)\Big\}w_2^m+$$

$$+\Big\{(1-\mu)\,d_2\operatorname{ctg}\alpha+\lambda^2\operatorname{ctg}\alpha\Big[2\mu+a-2(ad_2-bd_3)+$$

$$+(1-\mu)(1-2d_2)\frac{m^2}{\sin^2\alpha}\Big\}w_2^{m'}+\Big\{-(1-\mu)\,d_3\operatorname{ctg}\alpha+$$

$$+\lambda^2\operatorname{ctg}\alpha\Big[-b+2(ad_3+bd_2)+2(1-\mu)d_3\frac{m^2}{\sin^2\alpha}\Big]\Big\}w_3^{m'},$$

$$a_{33}=\Big\{(ad_3+bd_2)-(1-\mu)\frac{m^2}{\sin^2\alpha}d_3(1-2\lambda^2\operatorname{ctg}^2\alpha)\Big\}w_2^m+$$

$$+\Big\{-(1+\mu)+(1-\mu)\frac{m^2}{\sin^2\alpha}[\operatorname{ctg}\alpha-d_2(1-2\lambda^2\operatorname{ctg}^2\alpha)]+$$

$$+(ad_2-bd_3)\Big\}w_3^m+\Big\{(1-\mu)\,d_3\operatorname{ctg}\alpha-$$

$$-\lambda^2\operatorname{ctg}\alpha\Big[-b+2(ad_3+bd_2)+2(1-\mu)d_3\frac{m^2}{\sin^2\alpha}\Big]\Big\}w_2^{m'}+$$

$$+\Big\{(1-\mu)\,d_2\operatorname{ctg}\alpha+\lambda^2\operatorname{ctg}\alpha\Big[2\mu+a-2(ad_2-bd_3)+(1-\mu)\frac{m^2}{\sin^2\alpha}\times$$

$$\times(1-2d_2)\Big]\Big\}w_3^{m'},$$

$$a_{34}=\frac{(1-\mu)\,m}{\sin\alpha}\Big\{(1-2\lambda^2\operatorname{ctg}^2\alpha)P_{n_4}^{m'}-\operatorname{ctg}\alpha\left(1-2\lambda^2\frac{m^2}{\sin^2\alpha}\right)P_{n_4}^m\Big\},$$

$$a_{41}=(1-\mu)(1-2\eta_1)\operatorname{ctg}\alpha\, P_{n_1}^{m'}+\left[(1+\mu)+(1-2\eta_1)\left(p_1-m^2\frac{1-\mu}{\sin^2\alpha}\right)\right]P_{n_1}^{m},$$

$$a_{42}=\left[a-2(ad_2-bd_3)+(1+\mu)-(1-\mu)(1-2d_2)\frac{m^2}{\sin^2\alpha}\right]w_2^m+$$

$$+\left[-b+2(ad_3+bd_2)-2d_3(1-\mu)\frac{m^2}{\sin^2\alpha}\right]w_3^m+$$

$$+[(1-\mu)(1-2d_2)\operatorname{ctg}\alpha]\,w_2^{m'}+2(1-\mu)\,d_3\operatorname{ctg}\alpha\, w_3^{m'},$$

$$a_{43}=\left[b-2(ad_3+bd_2)+2(1-\mu)\,d_3\frac{m^2}{\sin^2\alpha}\right]w_2^m+$$

$$+\left[a-2(ad_2-bd_3)+(1+\mu)-(1-\mu)(1-2d_2)\frac{m^2}{\sin^2\alpha}\right]w_3^m-$$

$$-2(1-\mu)\,d_3\operatorname{ctg}\alpha\, w_2^{m'}+(1-\mu)(1-2d_2)\operatorname{ctg}\alpha\, w_3^{m'},$$

$$a_{51}=-(1+\eta_1)\,P_{n_1}^{m'},\qquad a_{52}=-(1-d_2)\,w_2^{m'}-d_3w_3^{m'},$$

$$a_{53}=d_3w_2^{m'}-(1-d_2)\,w_3^{m'},\qquad a_{54}=\frac{m}{\sin\alpha}P_{n_1}^{m}.$$

Here $\lambda^2=\dfrac{\delta^2}{12R^2}$, $P_{n_1}^m(\cos\theta)$ is the Legendre function corresponding to the real value of the root of the characteristic equation; $w_2^m(\cos\theta)$, $w_3^m(\cos\theta)$ are the real and imaginary parts of the Legendre function in the case of complex conjugate roots $p_{2,3}=a\pm ib$ of the characteristic equation;

$$P_{n_{2,3}}^m=w_2^m(\cos\theta)\pm iw_3^m(\cos\theta),$$

$$\eta_1=\frac{(1-\mu^2)\,p_1(p_1-2)+c^2[2(1+\mu)-k^2]}{(1+\mu)[c^2+2(1-\mu)k^2]\,p_1},$$

if p_1 is real and $\eta_{2,3}=d_1\pm id_2$, if $p_{2,3}$ are complex conjugate roots.

3. DYNAMIC RIGIDITY OF AN ANNULAR PLATE SUBJECTED TO DISTRIBUTED BENDING MOMENTS ACTING ON THE OUTWARD EDGE

Consider an annular plate of thickness δ and external radius b. E, μ are Young's modulus and the Poisson ratio of the plate material, and M is the mass of unit area of the plate.

The equation of vibrations will be /7/

$$D\nabla^4u+M\frac{\partial^2u}{\partial t^2}=0\qquad\left(D=\frac{E\delta}{12(1-\mu^2)}\right).$$

The boundary conditions are

$$u=0,\quad M_1=M_0\cos m\varphi\cos\omega t\qquad\text{at}\qquad r=b,$$

$$M_1=-D\left[\frac{\partial^2u}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\varphi^2}\right)\right]=0,$$

$$\lambda_1^* = -D\left[\frac{\partial^3 u}{\partial r^3} + \frac{1}{r}\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r} + (2-\mu)\frac{1}{r^2}\frac{\partial^3 u}{\partial r \partial \varphi^2} - \right.$$

$$\left. -(3-\mu)\frac{1}{r^3}\frac{\partial^2 u}{\partial \varphi^2}\right] = 0 \qquad \text{at} \quad r = a.$$

The solution will be represented in the form

$$u(r, \varphi, t) = \{C_1 J_m(\lambda\xi) + C_2 Y_m(\lambda\xi) + C_3 I_m(\lambda\xi) + C_4 K_m(\lambda\xi)\} \cos m\varphi \cos \omega t.$$

Here $\lambda^4 = \frac{Mb^4}{D}\omega^2$, $\xi = \frac{r}{b}$, $J_m(\lambda\xi)$, $Y_m(\lambda\xi)$, $I_m(\lambda\xi)$, $K_m(\lambda\xi)$ are Bessel functions of the first and second kind of order m of real and imaginary arguments. We shall consider in what follows the case $m = 2$.

The arbitrary constants are determined from the boundary conditions. The expression for the dynamic rigidity is formed by the formula

$$K_a(\omega) = \frac{M_0}{\frac{1}{b}\frac{\partial W}{\partial \xi}\Big|_\alpha} = \frac{D(1-\mu)}{b}\frac{\Delta}{\Delta^*} \qquad \left(\alpha = \frac{a}{b}\right).$$

The expressions determined Δ^* and Δ are given in /5/.

4. EXAMPLE OF DETERMINATION OF THE FREQUENCY OF NATURAL VIBRATIONS OF A BUILT-UP SHELL

The dynamic rigidities reduce, at a frequency of the exciting force $\omega = 0$, to the static rigidity of the corresponding type of loading. In the numerator of the dynamic rigidity there appears an expression which gives the equation of natural vibration frequencies of a hinged shell. In the denominator there appears an expression which gives the antiresonance frequencies, i.e., the frequencies of vibrations corresponding to the case when the section on which the external force is applied is built-in.

Knowing the dynamic rigidities, we can find the frequency of natural vibrations of a cylindrical shell closed at the ends by parts of spherical shells or by annular plates. In the case of spherical bottoms it is assumed that a supporting ring is placed at the place of contact. By cutting the shell by the ends we obtain one cylindrical shell and two spherical shells, loaded at the ends by distributed bending moments.

The moment acting on the end of a cylindrical shell is expressed through the dynamic rigidity and the slope $M^c = K_c \vartheta^c$. The moment acting on a spherical shell will be $M^s = K_s \vartheta^s$. But the interactions are equal, and the slopes at the place of contact are the same. The conditions $K_c + K_s = 0$ will therefore yield an equation for calculating the frequencies of natural vibrations of a built-up shell. The problem is solved graphically.

Consider as an example the problem of the frequency of natural vibrations of a cylindrical shell with flat bottoms. The dimensions of the shell are

$$b = 35 \text{ cm}, \; 2l = 21 \text{ cm}, \; h = 1.7 \text{ cm}.$$

The material characteristics are $E = 2.1 \times 10^6$ kg/cm^2, $\sigma = 0.3$, $\gamma = 0.00785$ kg/cm. The dimensions of the annular plate are: $b = 35$ cm, $a = 28$ cm, $\delta = 1.2$ cm. The material is the same as that of the shell. The calculations were conducted for $m = 2$. The results are presented below in tabular and graphical forms.

TABLE. Frequency of vibrations in cycles/sec

$m = 2$ frequency number	Type of structure				
	Plate		Cylindrical shell		Built-up shell
	hinged	built-in	hinged	built-in	
I	685	2,370	2,150	2,820	865 I' 2,800 II''
II	9,400	12,500	8,730	11,700	9,280 II' — II''

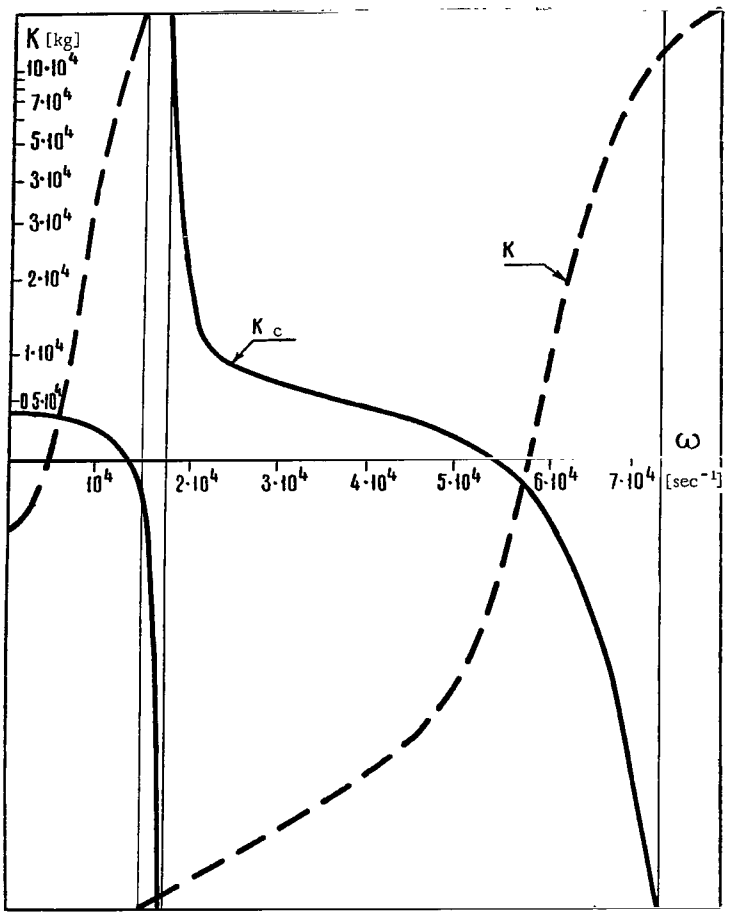

FIGURE.

The dotted line on the graph represents the dynamic rigidity of the plate, and the unbroken line the dynamic rigidity of the cylindrical shell. The abscissas of the intersection points give the frequencies of the built-up shell. It is seen from the table that the frequencies of the built-up shells can differ considerably from the frequencies of the separate elements. Thus, the frequency of vibrations of the cylindrical shell f_{21} = 2150 cycles/sec splits in two, f'_{21} = 865 and f''_{21} = 2800 cycles/sec. They differ in that at the first vibration mode the cylindrical shell and the plate vibrate in phase, and in the second one in antiphase.

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GENERALIZED PROBLEMS OF THE WATER HAMMER IN AN ELASTIC CONDUIT

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The problem of the water hammer has been studied by numerous authors, who have generalized in different manners Joukowski's solution /1/. Lately, there have appeared papers in which more and more general equations of the theory of shells are being drawn to describe the motion of the conduit /2, 4, 5, etc./. This paper treats the propagation of small disturbances in a system consisting of a conduit and of the liquid filling it. The conduit motion is described by a system of two differential equations of the hyperbolic type, correct for thick shells /3/. The hydrodynamic effects are studied in an acoustic approximation.

We introduce the following designations: p is the liquid pressure; v is the mean velocity of the liquid in the conduit; γ is the liquid density; φ is the velocity potential; x and r are the axial and radial coordinates; t is the time; r_0 is the shell radius; l is the wave length; $c_1 = \sqrt{\frac{k}{\gamma_0}}$ is the velocity of sound in the liquid; k is the modulus of elasticity of the liquid; u_0 and w_0 are the axial and radial displacements of the shell; $2h$ is the shell thickness; ρ is the density of the shell material; σ is the Poisson ratio; λ and μ are the elastic constants; $c_s = \sqrt{\frac{\mu}{\rho}}$ is the velocity of the distortion wave; c is the phase velocity; q is the pressure on the surface of separation between the liquid and the shell.

Dimensionless magnitudes $(p^*, q^*) = \frac{1}{\rho c_s^2}(p, q)$,

$$(v^*, c^*) = \frac{1}{c_s}(v, c), \qquad l^* = \frac{l}{r_0}, \qquad r^* = \frac{r}{r_0}, \qquad \varphi^* = \frac{1}{r_0 c_s}\varphi,$$

$$(u_0^*, w_0^*) = \frac{1}{r_0}(u_0, w_0), \qquad x^* = \frac{x}{r_0}, \qquad t^* = \frac{c_s}{r_0}t, \qquad \xi = \frac{2h}{r_0}.$$

1. Consider a liquid-filled circular cylindrical shell. The action of the shearing forces on the surface of separation between the liquid and the shell will be neglected. The differential equations of motion of the shell are then /3/

$$\left\{(\xi a_4 + \xi^3 a_5) - \xi^3 a_6 \frac{\partial^2}{\partial x^{*2}} + (\xi + \xi^3 a_7)\frac{\partial^2}{\partial t^{*2}} + \xi^3 a_1 \frac{\partial^4}{\partial x^{*4}} - \right.$$
$$\left. - \xi^3 a_2 \frac{\partial^4}{\partial t^{*2}\partial x^{*2}} + \xi^3 a_3 \frac{\partial^4}{\partial t^{*4}}\right\} w_0^* + \left\{\xi^3 a_8 \frac{\partial^3}{\partial x^{*3}} - \xi^3 a_9 \frac{\partial^3}{\partial t^{*2}\partial x^*}\right\} u_0^* =$$
$$= \left\{\left(2 + \xi^2 \frac{1}{2}\right) - \xi^2 d_1 \frac{\partial^2}{\partial x^{*2}} + \xi^3 d_2 \frac{\partial^2}{\partial t^{*2}}\right\} \frac{1}{2} q^*; \qquad (1)$$

$$\left\{(-b_4-\xi^2 b_5)\frac{\partial^2}{\partial x^{*2}}+(1+\xi^2 b_6)\frac{\partial^2}{\partial t^{*2}}+\xi^2 b_1\frac{\partial^4}{\partial x^{*4}}-\xi^2 b_2\frac{\partial^4}{\partial t^{*2}\partial x^{*2}}+\right.$$
$$\left.+\xi^2 b_2\frac{\partial^4}{\partial t^{*4}}\right\}\dot{u}_0+\left\{(-b_7-\xi^2 b_8)\frac{\partial}{\partial x^*}+\xi^2 b_9\frac{\partial^3}{\partial x^{*3}}-\xi^2 b_{10}\frac{\partial^3}{\partial t^{*2}\partial x^*}\right\}\dot{w}_0=0. \qquad (2)$$

The coefficients a_i, b_j, and d_k depend on the Poisson ratio and are given in /3/.

The motion of the liquid is described by the wave equation for the velocity potential

$$\nabla^2\varphi^*=\frac{1}{c_1^{*2}}\frac{\partial^2\varphi^*}{\partial t^{*2}}, \qquad (3)$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial}{\partial r^*}+\frac{\partial^2}{\partial x^{*2}}.$$

If the system liquid–shell is at rest at $t^*=0$, the initial conditions corresponding to this case can be written in the form

$$\dot{u}_0|_{t^*=0}=\dot{w}_0|_{t^*=0}=0 \quad \text{at} \quad -\infty<x^*<\infty;$$
$$p^*|_{t^*=0}=v^*|_{t^*=0}=0 \quad \text{at} \quad 0<x^*<\infty;$$
$$p^*|_{t^*=0}=\dot{p}_0, \qquad v^*|_{t^*=0}=\dot{v}_0 \quad \text{at} \quad -\infty<x^*<0. \qquad (4)$$

We also assume that on the external surface of the conduit, at $-\infty<x^*<0$, a uniformly distributed load $\dot{p}_0$ acts constantly. The initial conditions assumed correspond approximately to a reservoir ($x^*<0$) with a conduit ($x^*>0$), in which a flap is suddenly opened at moment $t^*=0$ in section $x^*=0$ /5/.

Finally, the condition

$$\frac{\partial\dot{w}_0}{\partial t^*}=-\frac{\partial\varphi^*}{\partial r^*}\bigg|_{r^*=1} \qquad (5)$$

must be satisfied at the boundary of separation between the liquid and the shell.

2. An approximate solution of the problem can be achieved in two stages. We first solve the hydrodynamics problem for an absolutely rigid conduit. The velocity potential is determined from equation (3), and the functions v^* and p^* are equal to

$$\dot{v}_1=\dot{v}_0, \qquad \dot{p}_1=\dot{p}_0 \quad \text{at} \quad x^*<\dot{c}_1 t^*;$$
$$\dot{v}_1=0, \quad \dot{p}_1=0 \quad \text{at} \quad x^*>\dot{c}_1 t^*. \qquad (6)$$

Using the solution obtained we determine the pressure at the boundary of separation between the liquid and the shell, which represents the load acting on the shell:

$$q_2^*=\frac{\gamma_0}{\rho}\frac{\partial\dot{\varphi}_1}{\partial t^*}\bigg|_{r^*=1}+\dot{p}_k, \qquad (7)$$

where $\dot{p}_k=\dot{p}_0$ at $0<x^*<\dot{c}_1 t^*$ and $\dot{p}_k=0$ for the other values of x^*. By solving the system of equations (1) to (3) with the aid of conditions (4)

and (5), it is now possible to determine the function φ_2, and therefore v_2^*, p_2^*, and also the functions u_0^* and w_0^*.

The problem will be solved by passing to a Fourier transform for x^* and to a Laplace transform for t^*, with a transformation parameter multiplied by $\sqrt{-1}$ for convenience. The relationships between the originals and the transforms are determined by the formulas

$$\bar{f}(s, t^*) = \int_{-\infty}^{\infty} f(x^*, t^*)\, e^{-isx^*} dx^*, \quad f(x^*, t^*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s, t^*)\, e^{isx^*} ds;$$

$$F(s, \omega) = \int_0^{\infty} \bar{f}(s, t^*) e^{-i\omega t^*} dt^*, \quad \bar{f}(s, t^*) = \frac{1}{2\pi} \int_{-\infty - i\alpha}^{\infty - i\alpha} F(s, \omega) e^{i\omega t^*} d\omega. \tag{8}$$

The system of equations (1) to (3), (5), allowing for (4), (7), has the following form in the transform space:

$$\frac{d^2\Phi}{dr^{*2}} + \frac{1}{r^*}\frac{d\Phi}{dr^*} + \left(\frac{\omega^2}{c_1^{*2}} - s^2\right)\Phi = 0; \tag{9}$$

$$\{(\xi a_4 + \xi^3 a_5) + \xi^3 a_6 s^2 - (\xi + \xi^3 a_7)\,\omega^2 + \xi^3 a_1 s^4 - \xi^3 a_2 \omega^2 s^2 + \xi^3 a_6 \omega^4\}\, W_0 +$$
$$+ \{-\xi^3 a_8 i s^3 + \xi^3 a_9 i \omega^2 s\}\, U_0 = \left\{\left(2 + \xi^2 \frac{1}{2}\right) + \xi^2 d_1 s^2 -\right.$$
$$\left. - \xi^2 d_2 \omega^2\right\} \frac{1}{2}\left[\frac{\gamma_0}{\rho}\, i\omega\Phi\Big|_{r^*=1} + p_k\right]; \tag{10}$$

$$\{(-b_4 - \xi^2 b_5)\, s^2 - (1 + \xi^2 b_6)\,\omega^2 + \xi^2 b_1 s^4 - \xi^2 b_2 \omega^2 s^2 + \xi^2 b_3 \omega^4\}\, U_0 +$$
$$+ \{(-b_7 - \xi^2 b_8)\, is - \xi^2 b_9 i s^3 + \xi^2 b_{10} i \omega^2 s\}\, W_0 = 0; \tag{11}$$

$$i\omega\, W_0 = -\frac{d\Phi}{dr^*}\Big|_{r^*=1} \tag{12}$$

By solving the system of equations (9) to (12) and passing back to the originals we obtain

$$w_0^*(x^*, t^*) = \frac{p_0^*}{8\pi^2} \int_{-\infty}^{\infty} e^{isx^*} ds \int_{-\infty - i\alpha}^{\infty - i\alpha} \frac{2M_0\, e^{i\omega t^*}}{L_0 + \frac{\gamma_0}{2\rho}\,\omega^2 N 2M_0}\, d\omega; \tag{13}$$

$$\varphi_2(r^*, x^*, t^*) = \frac{p_0^*}{8\pi^2} \int_{-\infty}^{\infty} e^{isx^*} ds \times$$
$$\times \int_{-\infty - i\alpha}^{\infty - i\alpha} \frac{i\omega 2M_0 J_0\left(r^* \sqrt{\left(\frac{\omega}{c_1^*}\right)^2 - s^2}\right) e^{i\omega t^*}}{\sqrt{\left(\frac{\omega}{c_1^*}\right)^2 - s^2}\; J_1\left(\sqrt{\left(\frac{\omega}{c_1^*}\right)^2 - s^2}\right)\left(L_0 + \frac{\gamma_0}{2\rho}\,\omega^2 N 2M_0\right)}\, d\omega, \tag{14}$$

$$p_2^*(r^*, x^*, t^*) = -\frac{p_0}{4\pi^2}\,\frac{\gamma_0}{2\rho} \int_{-\infty}^{\infty} e^{isx^*} ds \times$$
$$\times \int_{-\infty - i\alpha}^{\infty - i\alpha} \frac{\omega^2 2M_0 J_0\left(r^* \sqrt{\left(\frac{\omega}{c_1^*}\right)^2 - s^2}\right) e^{i\omega t^*}}{\sqrt{\left(\frac{\omega}{c_1^*}\right)^2 - s^2}\; J_1\left(\sqrt{\left(\frac{\omega}{c_1^*}\right)^2 - s^2}\right)\left(L_0 + \frac{\gamma_0}{2\rho}\,\omega^2 N 2M_0\right)}\, d\omega, \tag{15}$$

where

$$L_0=\omega^8-\left[\left(\frac{1}{\xi^2}\alpha_1+\alpha_2\right)+\alpha_3 s^2\right]\omega^6+\left[\left(\frac{1}{\xi^4}\alpha_4+\frac{1}{\xi^2}\alpha_5+\alpha_6\right)+\right.$$
$$\left.+\left(\frac{1}{\xi^2}\alpha_7+\alpha_8\right)s^2+\alpha_9 s^4\right]\omega^4-\left[\left(\frac{1}{\xi^4}\alpha_{10}+\frac{1}{\xi^2}\alpha_{11}+\alpha_{12}\right)+\left(\frac{1}{\xi^4}\alpha_{13}+\right.\right.$$
$$\left.+\frac{1}{\xi^2}\alpha_{14}+\alpha_{15}\right)s^2+\left(\frac{1}{\xi^2}\alpha_{16}+\alpha_{17}\right)s^4+\alpha_{18}s^6\Big]\omega^2+\left[\left(\frac{1}{\xi^4}\alpha_{19}+\frac{1}{\xi^2}\alpha_{20}+\right.\right.$$
$$\left.\left.+\alpha_{21}\right)s^2+\left(\frac{1}{\xi^2}\alpha_{22}+\alpha_{23}\right)s^4+\left(\frac{1}{\xi^2}\alpha_{24}+\alpha_{25}\right)s^6+\alpha_{26}s^8\right]; \tag{16}$$

$$\xi 2M_0=-\gamma_{16}\omega^6+\left[\left(\frac{1}{\xi^2}\gamma_{11}+\gamma_{12}\right)+\gamma_{15}s^2\right]\omega^4-\left[\left(\frac{1}{\xi^4}\gamma_4+\frac{1}{\xi^2}\gamma_5+\gamma_6\right)+\right.$$
$$\left.+\left(\frac{1}{\xi^2}\gamma_9+\gamma_{10}\right)s^2+\gamma_{14}s^4\right]\omega^2+\left[\left(\frac{1}{\xi^4}\gamma_1+\frac{1}{\xi^2}\gamma_2+\gamma_3\right)s^2+\right.$$
$$\left.+\left(\frac{1}{\xi^2}\gamma_7+\gamma_8\right)s^4+\gamma_{13}s^6\right]; \tag{17}$$

$$N=\frac{J_0\left(\sqrt{\left(\frac{\omega}{c_1}\right)^2-s^2}\right)}{\sqrt{\left(\frac{\omega}{c_1}\right)^2-s^2}\,J_1\left(\sqrt{\left(\frac{\omega}{c_1}\right)^2-s^2}\right)}; \tag{18}$$

$$\alpha_1=\frac{a_3+b_3}{a_3b_3},\quad \alpha_2=\frac{a_3b_6+a_7b_3}{a_3b_3},\quad \alpha_3=\frac{a_3b_2+a_2b_3}{a_3b_3},$$
$$\alpha_4=\frac{1}{a_3b_3},\quad \alpha_5=\frac{a_7+b_6+a_4b_3}{a_3b_3},\quad \alpha_6=\frac{a_7b_6+a_5b_3}{a_3b_3},$$
$$\alpha_7=\frac{a_3b_4+a_2+b_2}{a_3b_3},\quad \alpha_8=\frac{a_3b_5+a_2b_6+a_7b_2+a_6b_3+a_9b_{10}}{a_3b_3},$$
$$\alpha_9=\frac{a_3b_1+a_2b_2+a_1b_3}{a_3b_3},\quad \alpha_{10}=\frac{a_4}{a_3b_3},\quad \alpha_{11}=\frac{a_5+a_4b_6}{a_3b_3},$$
$$\alpha_{12}=\frac{a_5b_6}{a_3b_3},\quad \alpha_{13}=\frac{b_4}{a_3b_3},\quad \alpha_{14}=\frac{a_7b_4+b_5+a_6+a_4b_2+a_9b_7}{a_3b_3},$$
$$\alpha_{15}=\frac{a_7b_5+a_6b_6+a_3b_2+a_9b_8}{a_3b_3},\quad \alpha_{16}=\frac{a_2b_4+a_1+b_1}{a_3b_3}, \tag{19}$$
$$\alpha_{17}=\frac{a_2b_5+a_1b_6+a_7b_1+a_6b_2+a_8b_{10}+a_9b_9}{a_3b_3},$$
$$\alpha_{18}=\frac{a_2b_1+a_1b_2}{a_3b_3},\quad \alpha_{19}=\frac{a_4b_4}{a_3b_3},\quad \alpha_{20}=\frac{a_5b_4+a_4b_5}{a_3b_3},$$
$$\alpha_{21}=\frac{a_5b_5}{a_3b_3},\quad \alpha_{22}=\frac{a_6b_4+a_3b_1+a_8b_7}{a_3b_3},$$
$$\alpha_{23}=\frac{a_6b_5+a_5b_1+a_8b_8}{a_3b_3},\quad \alpha_{24}=\frac{a_1b_4}{a_3b_3},$$
$$\alpha_{25}=\frac{a_1b_5+a_6b_1+a_8b_9}{a_3b_3},\quad \alpha_{26}=\frac{a_1b_1}{a_3b_3},$$
$$\gamma_1=\frac{2b_4}{a_3b_3},\quad \gamma_2=\frac{2b_5+0{,}5b_4}{a_3b_3},\quad \gamma_3=\frac{0.5b_5}{a_3b_3},$$

$$\gamma_4=\frac{2}{a_3b_3},\quad \gamma_5=\frac{2b_6+0.5}{a_3b_3},\quad \gamma_6=\frac{0.5b_6}{a_3b_3},$$

$$\gamma_7=\frac{b_4d_1+2b_1}{a_3b_3},\quad \gamma_8=\frac{b_5d_1+0.5b_1}{a_3b_3},\quad \gamma_9=\frac{b_4d_2+d_1+2b_2}{a_3b_3},$$

$$\gamma_{10}=\frac{b_5d_2+b_6d_1+0.5b_2}{a_3b_3},\quad \gamma_{11}=\frac{d_2+2b_3}{a_3b_3}, \tag{19 (cont'd)}$$

$$\gamma_{12}=\frac{b_6d_2+0.5b_3}{a_3b_3},\quad \gamma_{13}=\frac{b_1d_1}{a_3b_3},\quad \gamma_{14}=\frac{b_1d_2+b_2d_1}{a_3b_3},$$

$$\gamma_{15}=\frac{a_2d_2+b_3d_1}{a_3b_3},\quad \gamma_{16}=\frac{b_3d_2}{a_3b_3}.$$

3. The examination of the solution for the displacements w_0^* (13) shows that the denominator of the integrand has an infinite number of roots which satisfy the equation

$$L_0+\frac{\gamma_0}{2\rho}\omega^2 N2M_0=0. \tag{20}$$

It can be concluded from physical considerations that at real s all the roots ω of the transcendental equation (20) are real (at values of ξ which do not exceed the limits of applicability of equations (1) and (2)). This is corroborated by the calculations. The results of the calculation of the phase velocities corresponding to the lower mode of vibrations for a water-filled duralumin shell, with parameters

$$\xi=\frac{1}{30},\quad \sigma=0.3,\quad \frac{\gamma_0}{2\rho}=0.178,\quad c_1^*=0.453,$$

are given in the figure. Curve 1 corresponds to equation (20), curve 2 to the theory which allows for the radial inertia and the flexural rigidity of the shell walls /2/, curve 3 to the Joukowski formula, which allows only for the circumferential stresses in the shell /1/. It is seen that the velocities corresponding to a given value of ξ differ considerably from each other. There exist apparently values of ξ for which the limiting values of the velocities ($l^*\to\infty$) coincide.

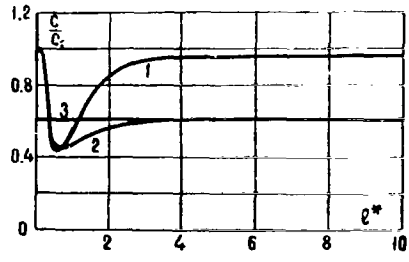

FIGURE.

All the roots of equation (20) lie on the real axis of the complex plane, and the internal integral of solution (13) can be represented in the

form of an infinite sum of residues. The expression for $\dot{w}_0$ then becomes of the form

$$\dot{w}_0 (x^*, t^*) = -\frac{\dot{p}_0}{\pi} \int_0^\infty \sum_{\omega_n} \frac{2M_0 \cos sx^* \sin \omega t^* \, ds}{\frac{d}{d\omega}\left\{L_0 + \frac{\gamma_0}{2\rho}\omega^2 N 2 M_0\right\}} . \tag{21}$$

The integrand function in (21) depends on the real variable s, and an approximate solution can be obtained by numerical methods. Solutions (14) and (15) can be similarly represented. The solutions obtained for the functions $\dot{w}_0$, φ^*, and p^* are more general than those found in the literature /2, 4, 5/.

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ON THE INFLUENCE OF CREEP ON THE STATE OF THIN SANDWICH SHELLS

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The allowance for the creep of the material plays an important part in the assessment of the reliability of a given structure. This paper considers the influence of creep on the state of stress of reinforced shells, in particular those made of reinforced concrete. A computing scheme is proposed which can be applied in many cases of reinforcement.

Consider a shell with a double framework in the directions of the lines of principal curvature. We shall distinguish between two fundamental variants of reinforcement: a) reinforcement by a net with an equal number of fittings in the two directions; b) reinforcement in one direction (the influence of the distributing framework will be neglected). The concrete (or any other material of which the shell is made) will be considered as an isotropic homogeneous body, possessing the property of linear creep. Neglecting the protective layer of concrete, we shall initially assume that cracks in the elongated zone during the service of the structure are inadmissible.

As computing model of such a shell we take a shell made of layers of isotropic or orthotropic material. The framework will be replaced in the two basic reinforcement variants by isotropic or orthotropic reinforcing layers, respectively. In the latter case it will be considered that the orthotropy axes coincide with the lines of principal curvature. In the first case we obtain a shell made of isotropic layers of different mechanical and rheological characteristics. In the second case the reinforcing orthotropic layers in one of the directions must have a rigidity equal to zero.

This condition can be satisfied if we assume that the reinforcing layer has the following elastic constants:

$$E_1 = 0, \quad \nu_{12} = 0, \quad E_2 = E_a, \quad \nu_{21} = \nu_a, \quad G_{12} = G_a, \tag{1}$$

where G_a is the modulus of shear; E_a is the modulus of elasticity; ν_a is the Poisson ratio of the framework.

The thickness of the reinforcing layer is selected in accordance with the framework area per running meter, its center of gravity coinciding with the center of gravity of the framework.

The computing model proposed for a reinforced shell makes it possible to elucidate the influence of creep on the distribution of the internal forces and stresses, allowing for the nonhomogeneity and orthotropy caused by the reinforcement in one direction. Using this model it is possible to conduct a study on the basis of the experimental data on the creep of

concrete, without introducing average rheological characteristics of the reinforced structures.

It was shown in /1/ that the equations describing the state of stress and strain of bodies possessing the property of linear creep do not differ from the corresponding equations of the theory of linear creep, if we replace the functions entering in them by their Laplace-Carson transforms, with the part of the elastic constants played by the functions

$$\bar{E} = \frac{E}{1 + E\bar{K}/p}, \qquad \bar{\mu}_\delta = \frac{\mu + \nu E\bar{K}/p}{1 + E\bar{K}/p}, \tag{2}$$

where E and μ are the modulus of elasticity and the Poisson ratio, respectively, of the material; ν is the coefficient of lateral creep strains.

$$\bar{K} = p\int_0^\infty K(t)\, e^{-pt}\, dt;$$

$K(t)$ is the influence function (creep kernel).

Thus, by transforming the equations of the theory of shells and the boundary conditions according to Laplace-Carson it is possible to solve the problem in transforms, and then, using the well-known inversion formula /2/, to find the components of the stresses, strains, and displacements.

When considering a sandwich shell it is necessary to satisfy the conditions between the layers in addition to the surface conditions. By writing these conditions in curvilinear coordinates α, β, γ (the normal to the surface of separation between the layers coincides with γ direction) we obtain

$$u_\alpha^i = u_\alpha^{i+1}, \quad u_\beta^i = u_\beta^{i+1}, \quad u_\gamma^i = u_\gamma^{i+1}; \tag{3}$$

$$\tau_{\alpha\gamma}^i = \tau_{\alpha\gamma}^{i+1}, \quad \tau_{\beta\gamma}^i = \tau_{\beta\gamma}^{i+1}, \quad \sigma_\gamma^i = \sigma_\gamma^{i+1}. \tag{4}$$

In the case of a thin sandwich shell it follows from the hypothesis of undeformable normals that conditions (3) are satisfied identically /3/, and conditions (4) reduce to the equilibrium equations. The stress-strain relationship is expressed in the case of identical layers symmetrically arranged relative to the middle surface by the following relationships, transformed according to Laplace-Carson:

$$\left.\begin{aligned} \bar{T}_1 &= \bar{C}_{11}\bar{\varepsilon}_1 + \bar{C}_{12}\bar{\varepsilon}_2, & M_1 &= \bar{D}_{11}\bar{\varkappa}_1 + \bar{D}_{12}\bar{\varkappa}_2, \\ \bar{T}_2 &= \bar{C}_{22}\bar{\varepsilon}_2 + \bar{C}_{12}\bar{\varepsilon}_1, & \bar{M}_2 &= \bar{D}_{22}\bar{\varkappa}_2 + \bar{D}_{12}\bar{\varkappa}_1, \\ \bar{S}_{12} &= \bar{S}_{21} = \bar{C}_{66}\bar{\omega}, & \bar{H}_{12} &= \bar{H}_{21} = \bar{D}_{66}\bar{\tau}, \end{aligned}\right\} \tag{5}$$

where

$$\bar{C}_{jk} = 2\,[\bar{B}_{jk}^{i+1} h_{i+1} + \sum \bar{B}_{jk}^i (h_i - h_{i+1})], \tag{6}$$

$$\bar{D}_{jk} = 2/3[\bar{B}_{jk}^{i+1} h_{i+1}^3 + \sum \bar{B}_{jk}^i (h_i^3 - h_{i+1}^3)]. \tag{7}$$

The coefficients $\bar{B}_{jk}$ are given by the following formulas in the different cases:

a) in an isotropic reinforcing layer

$$\overline{B}_{11}=\overline{B}_{22}=\frac{E_a}{1-\nu_a^2}; \quad \overline{B}_{12}=\frac{\nu_a E_a}{1-\nu_a^2}; \quad \overline{B}_{66}=\frac{E_a}{2(1+\nu_a)}, \tag{8}$$

b) in an orthotropic reinforcing layer

$$\overline{B}_{11}=0; \quad \overline{B}_{22}=\frac{E_a}{1-\nu_a^2}; \quad \overline{B}_{12}=0; \quad \overline{B}_{66}=\frac{E_a}{2(1+\nu_a)}, \tag{9}$$

c) in a layer of concrete

$$\overline{B}_{11}=\overline{B}_{22}=\frac{\overline{E}_\delta}{1-\overline{\mu}_\delta^2}; \quad \overline{B}_{12}=\frac{\overline{\nu}_\delta\,\overline{E}_\delta}{1-\overline{\mu}_\delta^2}; \quad \overline{B}_{66}=\frac{\overline{E}_\delta}{2(1+\mu_\delta)}. \tag{10}$$

Here E_a, ν_a are the elastic constants of the framework; $\overline{E}_\delta$, $\overline{\mu}_\delta$ are the above-mentioned functions, which replace the elastic constants of the concrete.*

The problem of the calculation of a shell in the framework of the engineering theory reduces, as known /3/, to solving an equation of the eighth order relative to some potential function ψ. By transforming this equation according to Laplace-Carson we obtain, in the case of a shell with symmetrically arranged layers of equal modulus of elasticity and thickness, subjected to a normal load Z ($X=0$, $Y=0$):

$$\overline{P}_1\frac{\partial^8\overline{\psi}}{\partial\alpha^8}+\overline{P}_3\frac{\partial^8\overline{\psi}}{\partial\alpha^6\partial\beta^2}+\overline{P}_5\frac{\partial^8\overline{\psi}}{\partial\alpha^4\partial\beta^4}+\overline{P}_4\frac{\partial^8\overline{\psi}}{\partial\alpha^2\partial\beta^6}+\overline{P}_2\frac{\partial^8\overline{\psi}}{\partial\beta^8}+\frac{1}{R^2}\frac{\partial^4\overline{\psi}}{\partial\alpha^4}=\overline{Z}, \tag{11}$$

where the coefficients $\overline{P}_j$ are given by the following formulas:

$$\left.\begin{aligned}
&\overline{P}_1=\frac{\overline{D}_{11}\overline{C}_{11}}{\overline{\Omega}}, \quad \overline{P}_2=\frac{\overline{D}_{22}\overline{C}_{22}}{\overline{\Omega}}, \quad \overline{P}_3=2\frac{\overline{G}_{11}}{\overline{\Omega}}(\overline{D}_{12}+2\overline{D}_{66})+\\
&\qquad+\frac{\overline{D}_{11}}{\overline{C}_{66}}-2\frac{\overline{C}_{12}\overline{D}_{11}}{\overline{\Omega}},\\
&\overline{P}_4=2\frac{\overline{C}_{22}}{\overline{\Omega}}(\overline{D}_{12}+2\overline{D}_{66})+\frac{\overline{D}_{22}}{\overline{C}_{66}}-2\frac{\overline{D}_{22}\overline{C}_{12}}{\overline{\Omega}},\\
&\overline{P}_5=\frac{\overline{D}_{11}\overline{C}_{22}}{\overline{\Omega}}+2(\overline{D}_{12}+2\overline{D}_{66})\left(\frac{1}{\overline{C}_{66}}-2\frac{\overline{C}_{12}}{\overline{\Omega}}\right)+\\
&\qquad+\frac{\overline{D}_{22}\overline{C}_{11}}{\overline{\Omega}}; \quad \overline{\Omega}=\overline{C}_{11}\overline{C}_{22}-\overline{C}_{12}^2.
\end{aligned}\right\} \tag{12}$$

The rigidities C_{jk} and D_{jk} here are determined by formulas (6) and (7). The formulas for determining the rigidities are considerably simplified if it is assumed that the concrete and reinforcing layers have the same Poisson ratio. Such an assumption drastically reduces the amount of calculations involved, while the error introduced by it does not exceed 5% for practical values of the reinforcement coefficient. It can also be assumed, without reducing the accuracy of the calculations, that $\nu_\delta^2 = 0$.

* It will be assumed in what follows that $\mu=\nu=\overline{\mu}_\delta$.

On the strength of these assumptions, the resolving equation in the case of isotropic reinforcing layers can be presented in the form

$$\left.\begin{aligned} &\frac{\overline{D}_{11}}{\overline{C}_{11}} L(\overline{\psi}) + \frac{1}{R^2}\frac{\partial^4\overline{\psi}}{\partial\alpha^4} = \overline{Z}, \\ &\text{where} \\ &L(\ ) = \frac{\partial^8}{\partial\alpha^8} + 4\frac{\partial^8}{\partial\alpha^6\partial\beta^2} + 6\frac{\partial^8}{\partial\alpha^4\partial\beta^4} + 4\frac{\partial^8}{\partial\alpha^2\partial\beta^6} + \frac{\partial^8}{\partial\beta^8}. \end{aligned}\right\} \tag{13}$$

If the shell is reinforced in one direction (or if the influence of the distributive framework can be neglected), it is necessary to take into account the orthotropy of the reinforcing layer. Introducing the additional assumption that the Poisson ratio of concrete is zero, and taking (12) into account, we obtain the following relationships:

$$\left.\begin{aligned} &\overline{P}_1 = \frac{\overline{D}_{11}}{\overline{C}_{11}}, \quad \overline{P}_2 = \frac{\overline{D}_{22}}{\overline{C}_{11}}, \quad \overline{P}_3 = 4\frac{\overline{D}_{66}}{\overline{C}_{22}} + \frac{\overline{D}_{11}}{\overline{C}_{66}}, \\ &\overline{P}_4 = 4\frac{\overline{D}_{66}}{\overline{C}_{11}} + \frac{\overline{D}_{22}}{\overline{C}_{66}}, \quad \overline{P}_5 = \frac{\overline{D}_{11}}{\overline{C}_{11}} + 4\frac{\overline{D}_{66}}{\overline{C}_{66}} + \frac{\overline{D}_{22}}{\overline{C}_{22}}. \end{aligned}\right\} \tag{14}$$

Note that expressions similar to (14) are obtained for the coefficients entering in the resolving equation in the calculation of very shallow shells.

Consider a cylindrical shell of open profile reinforced symmetrically relative to the middle surface. The integration of equation (13) in the case of hinged shell, i.e., at boundary conditions of the form

$$\left.\begin{aligned} &\alpha = 0, \quad \alpha = \alpha_0; \quad v = w = T_1 = M_1 = 0; \\ &\beta = 0, \quad \beta = \beta_0; \quad u = w = T_2 = M_2 = 0, \end{aligned}\right\} \tag{15}$$

can be realized by representing the functions by double trigonometric series. If the shell is loaded by a concentrated force Q applied statically at a point of coordinates $\alpha = \alpha_1$, $\beta = \beta_1$, the function $\overline{\psi}$ can be represented in the form

$$\overline{\psi} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \overline{A}_{mn} \sin\frac{m\pi\alpha}{\alpha_0} \sin\frac{n\pi\beta}{\beta_0}. \tag{16}$$

Then, by expanding the load in a double Fourier series, we obtain from equation (13)

$$\overline{\psi} = -\frac{4Q\alpha_0^8}{\pi^8\omega}\sum_m\sum_n \frac{\sin\frac{m\pi\alpha}{\alpha_0}\sin\frac{m\pi\beta}{\beta_0}\sin\frac{m\pi\alpha_1}{\alpha_0}\sin\frac{n\pi\beta_1}{\beta_0}}{\overline{\Delta}_{mn}}, \tag{17}$$

where

$$\overline{\Delta}_{mn} = \frac{\overline{D}_{11}}{\overline{C}_{11}}(m^8 + 4m^6n^2\lambda^2 + 6m^4n^4\lambda^4 + 4m^2n^2\lambda^6 + n^8\lambda^8) + \frac{\alpha_0^4 m^4}{R^2\pi^4}; \tag{18}$$

$$\lambda = \alpha_0/\beta_0; \quad \omega = \alpha_0\beta_0.$$

In accordance with this, we shall have the following expressions for the transforms of the internal forces and moments:

$$\left.\begin{aligned}
\overline{T}_1 &= -\frac{4Qa^4\lambda^2}{\pi^4 abR}\sum_m\sum_n \frac{m^2n^2}{\overline{\Delta}_{mn}}\sin\frac{m\pi\alpha}{\alpha_0}\sin\frac{n\pi\beta}{\beta_0}\sin\frac{m\pi\alpha_1}{\alpha_0}\sin\frac{n\pi\beta_1}{\beta_0},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\overline{M}_1 &= -\frac{4Qa^2}{\pi^2 ab}\sum_m\sum_n (\overline{D}_{11}m^2+\overline{D}_{12}n^2\lambda^2)\times\\
&\times\frac{\overline{\Delta}'_{mn}}{\overline{\Delta}_{mn}}\sin\frac{m\pi\alpha}{\alpha_0}\sin\frac{n\pi\beta}{\beta_0}\sin\frac{m\pi\alpha_1}{\alpha_0}\sin\frac{n\pi\beta_1}{\beta_0},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\qquad(19)$$

The following relationships, similar to those introduced in solving the problem of an elastic shell, are obtained for the transforms of the stress components in the shell layers:

$$\left.\begin{aligned}
\overline{\sigma}^l_\alpha &= -\frac{4Q\alpha_0^4}{\pi^4\omega R}\sum_m\sum_n \frac{\overline{B}^l_{11}}{\overline{\Delta}_{mn}}\left[\frac{n^2m^2\lambda^2}{\overline{C}_{11}}-\frac{\gamma R^2\pi^2}{\alpha_0^2}\overline{\Delta}'_{mn}(m^2+\mu^2n^2\lambda^2)\right]\times\\
&\times\sin\frac{m\pi\alpha}{\alpha_0}\sin\frac{n\pi\beta}{\beta_0}\sin\frac{m\pi\alpha_1}{\alpha_0}\sin\frac{n\pi\beta_1}{\beta_0},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\qquad(20)$$

where

$$\overline{\Delta}'_{mn}=\frac{1}{\overline{C}_{11}}(m^4+n^4\lambda^4+2m^2n^2\lambda^2).\qquad(21)$$

Formulas (19) remain correct in the case of an orthotropic reinforcing layer, when the shell as a whole likewise becomes orthotropic; in accordance with (14), we obtain in this case

$$\overline{\Delta}_{mn}=\overline{P}_1l_1+\overline{P}_2l_2+\overline{\Lambda}l_3+\frac{1}{R^2}\frac{a^4}{\pi^4}m^4,\qquad(22)$$

$$\overline{\Delta}'_{mn}=\frac{1}{\overline{C}_{22}}[m^4+2m^2n^2\lambda^2]+\frac{1}{\overline{C}_{11}}n^4\lambda^4,\qquad(23)$$

where

$$l_1=m^8+2m^6n^2\lambda^2(1+3h_2^2)+5m^4n^4\lambda^4+2m^2n^2\lambda^6,$$

$$l_2=n^8\lambda^8+2m^2n^2\lambda^6,\quad l_3=2m^6n^2\lambda^2+5m^4n^4\lambda^4+2m^2n^6\lambda^6,$$

$\overline{\Lambda}=\dfrac{D^a_{22}}{3[\overline{E}_\delta h_2+C^a_{22}]}$, D^a, C^a are rigidities of the reinforcing layers.

The determination of the internal forces and stresses in the shell layers, allowing for the creep of concrete, reduces to applying the inversion formula to (19) and (20) and calculating the following time-dependent integrals:

$$J_1=\frac{1}{2\pi i}\int_{(L)}\frac{e^{pt}}{p}\frac{1}{\overline{\Delta}_{mn}}dp,\qquad J_2=\frac{1}{2\pi i}\int_{(L)}\frac{e^{pt}}{p}\frac{\overline{D}_{11}}{\overline{\Delta}_{mn}}\overline{\Delta}'_{mn}dp.\qquad(24)$$

The calculation of the stresses and forces, allowing for the creep of concrete, can be conducted only after the analytical form of the creep kernel $K(t)$ has been established. The following form of this function was recommended by Rzhanitsyn for concrete:

$$K(t) = ae^{-bt}/t^{1-c}, \tag{25}$$

where a, b, c are some constant positive magnitudes, with $0 < c < 1$. Such a relationship, as was noted in /1/, approximates satisfactorily the empirical curves of creep, and makes it possible to solve the problem in a closed form; the coefficients a, b, c are obtained by an elementary processing of the experimental data.

In accordance with (25), we obtain for the function K

$$\bar{K} = a\Gamma(c) \frac{p}{(p+b)^c}, \tag{26}$$

where $\Gamma(c)$ is the gamma function.

By substituting (26) in the expression for the rigidities (6), (7), and then in (24), we reduce the integrals, whose calculation is necessary for determining the forces, stresses, and displacements, to the form

$$J = \frac{1}{2\pi i} \int_{(L)} \frac{e^{pt}}{p} \frac{\sum_0^k a_k (p+b)^{kc}}{\sum_0^{k_1} a_{k_1} (p+b)^{k_1 c}} dp. \tag{27}$$

The coefficients a_k here depend on the mechanical and rheological constants of the concrete and the reinforcing layers, and also on their thickness.

Contour integrals of the type of (27) are calculated by the standard methods used in the case of existence of points of bifurcation and poles of the integrand.

The following conclusions can be drawn from the results of our study.

1. In isotropic and orthotropic reinforced shells at stationary boundary conditions the creep causes a redistribution of the internal forces produced by the action of an external load.

2. The creep of concrete causes a redistribution of the stresses and eventually leads to an increase of the stresses in the reinforcing layers and a decrease of the stresses in the concrete.

3. Considering a homogeneous isotropic shell as a particular case of a shell reinforced by layers with mechanical characteristics coinciding with the corresponding characteristics of concrete, we obtain that in a homogeneous isotropic shell the creep does not lead to a redistribution of the stresses and forces. Only an increase of the displacements takes place in this case.

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FUNDAMENTAL EQUATIONS OF THE SEMIMEMBRANE NONLINEAR THEORY OF PRISMATIC SHELLS AND FOLDS

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Consider a prismatically folded shell of average length, consisting of a finite number (m) of thin rectangular plates (figure). Let the load applied to the fold be large, so that its deformation is considerable. The stressed-strained state of each plate must then be described by Karman's system of nonlinear equations

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,$$

$$2\omega = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y};$$

$$\varkappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \varkappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \tau = -\frac{\partial^2 w}{\partial x \partial y}; \tag{1}$$

$$T_x = K(\varepsilon_x + \nu\varepsilon_y), \quad T_y = K(\varepsilon_y + \nu\varepsilon_x), \quad S = K(1-\nu)\omega,$$

$$M_x = D(\varkappa_x + \nu\varkappa_y), \quad M_y = D(\varkappa_y + \nu\varkappa_x), \quad H = D(1-\nu)\tau \tag{2}$$

or

$$\varepsilon_x = K'(T_x - \nu T_y), \quad \varepsilon_y = K'(T_y - \nu T_x), \quad \omega = K'(1+\nu)S,$$

$$\varkappa_x = D'(M_x - \nu M_y), \quad \varkappa_y = D'(M_y - \nu M_x), \quad \tau = D'(1+\nu)H \tag{2'}$$

and

$$\frac{\partial T_x}{\partial x} + \frac{\partial S}{\partial y} + X = 0, \quad \frac{\partial S}{\partial x} + \frac{\partial T_y}{\partial y} + Y = 0,$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 H}{\partial x \partial y} - (\varkappa_x T_x + \varkappa_y T_y + 2\tau S) + F_z = 0, \tag{3}$$

$$N_x = \frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} + L_x, \quad N_y = \frac{\partial H}{\partial x} + \frac{\partial M_y}{\partial y} + L_y,$$

where

$$K = \frac{Et}{1-\nu^2}, \quad D = \frac{Et^3}{12(1-\nu^2)}; \quad K' = \frac{1}{Et}, \quad D' = \frac{12}{Et^3};$$

$$F_z = Z + \frac{\partial L_x}{\partial x} + \frac{\partial L_y}{\partial y}.$$

Note that, on the basis of (2) and (2'), the third equation of (3) can also be written in the form

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 H}{\partial x \partial y} - \frac{D'}{K'}(\varepsilon_x M_x + \varepsilon_y M_y + 2\omega H) + F_z = 0. \qquad (3')$$

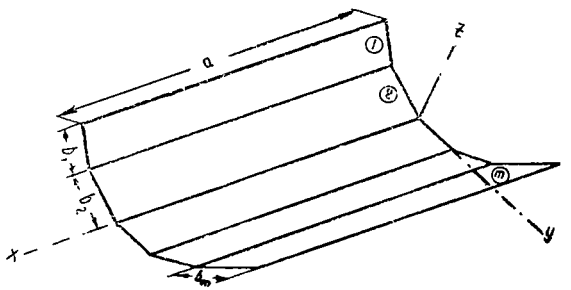

FIGURE.

We adopt a priori the hypotheses of Vlasov's semimembrane theory $M_x = H = 0$, $\varepsilon_y = \omega = 0$, and determine the 13 remaining unknown functions from the six relationships (1), the first relationship of (2), the fifth relationship of (2'), the first, second, fourth, and fifth relationships of (3), and (3'). It is easily seen that, by arranging them in the following order

$$\begin{gathered}
1)\ N_x = L_x, \quad 2)\ \frac{\partial^2 M_y}{\partial y^2} = -F_z, \quad 3)\ N_y = L_y + \frac{\partial M_y}{\partial y}, \quad 4)\ \varkappa_y = D' M_y, \\
5)\ \frac{\partial^2 w}{\partial y^2} = -\varkappa_y, \quad 6)\ \varkappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad 7)\ \tau = -\frac{\partial^2 w}{\partial x \partial y}, \\
8)\ \frac{\partial v}{\partial y} = -\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad 9)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} - \frac{\partial w}{\partial y} \cdot \frac{\partial w}{\partial x}, \\
10)\ \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad T_x = K\varepsilon_x, \\
12)\ \frac{\partial S}{\partial y} = -X - \frac{\partial T_x}{\partial y}, \quad 13)\ \frac{\partial T_y}{\partial y} = -Y - \frac{\partial S}{\partial x},
\end{gathered} \qquad (4)$$

we can find all the magnitudes interesting us by solving the equations (4) one after the other. The general solution of these equations can be represented in the following form $(i = 1, 2, \cdots, m)$:

$$\begin{gathered}
1)\ N_{x,i} = L_{x,i}, \quad 2)\ M_{y,i} = \varphi_i(x, y) + [\alpha_{1,i}(x)(1 - \eta_i) + \alpha_{2,i}(x)\ \eta_i], \\
3)\ N_{y,i} = \chi_i(x, y) - \frac{1}{b_i}[\alpha_{1,i}(x) - \alpha_{2,i}(x)], \\
4)\ \varkappa_{y,i} = \varphi_{1,i}(x, y) + D_i'[\alpha_{1,i}(x)(1 - \eta_i) + \alpha_{2,i}(x)\eta_i], \\
5)\ w_i = \psi_i(x, y) - D_i' b_i^2 [\alpha_{1,i}(x) a_1(\eta_i) + \alpha_{2,i}(x) a_2(\eta_i)] + \\
+ [\beta_{1,i}(x)(1 - \eta_i) + \beta_{2,i}(x)\eta_i],
\end{gathered} \qquad (5)$$

$$6)\ \varkappa_{x,i} = -\psi_{xx,i}(x, y) + D_i' b_i^2 [\alpha_{1,i}'(x) a_1(\eta_i) + \alpha_{2,i}'(x) a_2'(\eta_i)] - $$
$$- [\beta_{1,i}''(x)(1-\eta_i) + \beta_{2,i}''(x)\eta_i],$$
$$7)\ \tau_i = -\psi_{xy,i}(x, y) + D_i' b_i [\alpha_{1,i}'(x) a_1'(\eta_i) + \alpha_{2,i}'(x) a_2'(\eta_i)] + $$
$$+ \frac{1}{b_i}[\beta_{1,i}'(x) - \beta_{2,i}'(x)],$$

$$8)\ v_i = V_i(x, y) + \frac{1}{b_i}\int_0^x \left\{ \int_0^{x_2} [\gamma_{1,i}(x_1) - \gamma_{2,i}(x_1)]\, dx_1 \right\} dx_2 + $$
$$+ \frac{1}{b_i}(\xi_{1,i} - \xi_{2,i})x + l_i,$$
$$9)\ u_i = U_i(x, y) + \int_0^x [\gamma_{1,i}(x_1)(1-\eta_i) + \gamma_{2,i}(x_1)\eta_i]\, dx_1 + \qquad (5)\ \text{(cont'd)}$$
$$+ [\xi_{1,i}(1-\eta_i) + \xi_{2,i}\eta_i],$$
$$10)\ \varepsilon_{x,i} = U_{1,i}(x, y) + [\gamma_{1,i}(x)(1-\eta_i) + \gamma_{2,i}(x)\eta_i],$$
$$11)\ T_{x,i} = U_{2,i}(x, y) + K_i[\gamma_{1,i}(x)(1-\eta_i) + \gamma_{2,i}(x)\eta_i],$$
$$12)\ S_i = W_i(x, y) - K_i b_i [\gamma_{1,i}'(x) a_1'(\eta_i) + \gamma_{2,i}'(x) a_2'(\eta_i)] + $$
$$+ \frac{1}{b_i}\int_0^x [\delta_{1,i}(x_1) - \delta_{2,i}(x_1)]\, dx_1 + \zeta_i,$$
$$13)\ T_{y,i} = W_{1,i}(x, y) + K_i b_i^2 [\gamma_{1,i}''(x) a_1(\eta_i) + \gamma_{2,i}''(x) a_2(\eta_i)] + $$
$$+ [\delta_{1,i}(x)(1-\eta_i) + \delta_{2,i}(x)\eta_i],$$

where

$$a_1(\eta_i) = -\frac{1}{6}(2 - 3\eta_i + \eta_i^2)\eta_i, \quad a_2(\eta_i) = -\frac{1}{6}(1-\eta_i^2)\eta_i, \quad \eta_i = \frac{y}{b_i};$$
$$\varphi_i(x, y) = G_i[F_{z,i}], \quad \chi_i(x, y) = L_{y,i} + \varphi_{y,i}, \quad \varphi_{1,i}(x, y) = D_i'\varphi_i,$$
$$\psi_i(x, y) = G_i[\varphi_{1,i}],$$
$$V_i(x, y) = B_i[q_i, p_i], \quad U_i(x, y) = G_i[r_i], \quad U_{1,i}(x, y) = s_i + U_{x,i}, \qquad (6)$$
$$U_{2,i}(x, y) = K_i U_{1,i}, \quad W_i(x, y) = -B_i[Y_i, X_{1,i}], \quad W_{1,i}(x, y) = G_i[R_i],$$

with

$$G_i[f(x, y)] = \eta_i \int_0^{b_i} (b_i - y_1) f(x, y_1)\, dy_1 - \int_0^y (y - y_1) f(x, y_1)\, dy_1,$$
$$B_i[g(x, y), h(x, y)] =$$
$$= \frac{1}{b_i}\int_0^x \left\{ \int_0^{b_i} \left[g(x_1, y_1) - (b_i - y_1)\frac{\partial h(x_1, y_1)}{\partial x_1} \right] dy_1 \right\} dx_1 + \int_0^y h(x, y_1)\, dy_1$$

and further on

$$p_i(x, y) = -\frac{1}{2} w_{y,i}^2, \quad q_i(x, y) = -w_{y,i} w_{x,i}, \quad r_i(x, y) = w_{yy,i} w_{x,i},$$

$$s_i(x, y) = \frac{1}{2} w_{x,i}^2,$$

$$X_{1,i} = X_i + U_{2,x,i}, \quad R_i(x, y) = Y_{y,i} - X_{1,x,i}; \tag{7}$$

$\alpha_{1,i}(x)$, $\alpha_{2,i}(x)$; $\beta_{1,i}(x)$, $\beta_{2,i}(x)$; $\gamma_{1,i}(x)$, $\gamma_{2,i}(x)$; $\delta_{1,i}(x)$, $\delta_{2,i}(x)$ and $\xi_{1,i}$, $\xi_{2,i}$; l_i, ζ_i are arbitrary functions and integration constants, determined from the conditions of connection of adjacent plates, the boundary conditions at the extreme longitudinal edges of the fold, and the boundary conditions at its lateral edges.

The connection conditions are written in the form $(i = 1, 2, \cdots, m-1)$

$$\begin{gathered} u_{i+1}(x, 0) = u_i(x, b), \quad v_{i+1}(x, 0) = +v_i(x, b)\cos\vartheta_i + w_i(x, b)\sin\vartheta_i; \\ w_{i+1}(x, 0) = -v_i(x, b)\sin\vartheta_i + w_i(x, b)\cos\vartheta_i, \quad w_{y,i+1}(x, 0) = w_{y,i}(x, b); \\ S_{i+1}(x, 0) = S_i(x, b), \quad T_{y,i+1}(x, 0) = +T_{y,i}(x, b)\cos\vartheta_i + N_{y,i}(x, b)\sin\vartheta_i; \\ N_{y,i+1}(x, 0) = -T_{y,i}(x, b)\sin\vartheta_i + N_{y,i}(x, b)\cos\vartheta_i; \\ M_{y,i+1}(x, 0) = M_{y,i}(x, b); \end{gathered} \tag{8}$$

where ϑ_i is the angle between the planes of the i-th and the $(i+1)$-th plates. The substitution in the first, second, third, sixth, seventh, and eighth equations of system (8) of the corresponding expressions of (5) makes it possible to express the $8m$ functions $\alpha_{1,i}(x)$, $\alpha_{2,i}(x)$; $\beta_{1,i}(x)$, $\beta_{2,i}(x)$; $\gamma_{1,i}(x)$, $\gamma_{2,i}(x)$; $\delta_{1,i}(x)$, $\delta_{2,i}(x)$ and the $2m$ constants $\xi_{1,i}$, $\xi_{2,i}$ through $2m + 6$ new unknown functions $m_i(x)$, $\sigma_i(x)$ $(i = 0, 1, \cdots, m)$; $w_1(x, 0)$, $w_m(x, b)$; $T_{y,1}(x, 0)$, $T_{y,m}(x, b)$ and $m + 1$ new unknown constants k_i $(i = 0, 1, \cdots, m)$ by the formulas

$$\begin{gathered} \alpha_{1,i}(x) = m_{i-1}(x), \quad \alpha_{2,i}(x) = m_i(x); \quad \gamma_{1,i}(x) = \frac{1}{E}\sigma_{i-1}(x), \\ \gamma_{2,i}(x) = \frac{1}{E}\sigma_i(x); \\ \beta_{1,i}(x) = -P_{1,i}(x) + \sum_{j=i-2}^{i} f_{ij}\left\{\frac{1}{E}\int_0^x\left[\int_0^{x_2}\sigma_j(x_1)\,dx_1\right]dx_2 + k_j x\right\} - \\ -(\operatorname{cosec}\vartheta_{i-1} l_{i-1} - \operatorname{ctg}\vartheta_{i-1} l_i); \\ \beta_{2,i}(x) = -P_{2,i}(x) + \sum_{j=i-1}^{i+1} g_{ij}\left\{\frac{1}{E}\int_0^x\left[\int_0^{x_2}\sigma_j(x_1)\,dx_1\right]dx_2 + k_j x\right\} - \\ -(\operatorname{ctg}\vartheta_i l_i - \operatorname{cosec}\vartheta_i l_{i+1}); \\ \delta_{1,i}(x) = Q_{1,i}(x) + \sum_{j=i-2}^{i} f_{ij} m_j(x), \\ \delta_{2,i}(x) = Q_{2,i}(x) + \sum_{j=i-1}^{i+1} g_{ij} m_j(x); \quad \xi_{1,i} = k_{i-1}, \quad \xi_{2,i} = k_i; \end{gathered} \tag{9}$$

where

$$P_{1,1}(x) = -w_1(x, 0), \quad P_{1,i}(x) = \operatorname{cosec}\vartheta_{i-1} v_{i-1}(x, b) - \operatorname{ctg}\vartheta_{i-1} v_i(x, 0)$$
$$(i = 2, 3, \cdots, m);$$
$$P_{2,i}(x) = \operatorname{ctg}\vartheta_i v_i(x, b) - \operatorname{cosec}\vartheta_i v_{i+1}(x, 0) \quad (i = 1, 2, \cdots, m-1),$$
$$P_{2,m}(x) = -w_m(x, b);$$
$$Q_{1,1}(x) = T_{y,1}(x, 0), \quad Q_{1,i}(x) = \operatorname{cosec}\vartheta_{i-1}\chi_{i-1}(x, b) - \operatorname{ctg}\vartheta_{i-1}\chi_i(x, 0) \tag{10}$$
$$(i = 2, 3, \cdots, m);$$
$$Q_{2,i}(x) = \operatorname{ctg}\vartheta_i \chi_i(x, b) - \operatorname{cosec}\vartheta_i \chi_{i+1}(x, 0) \quad (i = 1, 2, \cdots, m-1),$$
$$Q_{2,m}(x) = T_{y,m}(x, b)$$

and

$$f_{i,i-2} = -\frac{\operatorname{cosec}\vartheta_{i-1}}{b_{i-1}}, \quad f_{i,i-1} = \frac{\operatorname{cosec}\vartheta_{i-1}}{b_{i-1}} + \frac{\operatorname{ctg}\vartheta_{i-1}}{b_i},$$
$$f_{i,i} = -\frac{\operatorname{ctg}\vartheta_{i-1}}{b_i}; \tag{11}$$

$$g_{i,i-1} = -\frac{\operatorname{ctg}\vartheta_i}{b_i}, \quad g_{i,i} = \frac{\operatorname{ctg}\vartheta_i}{b_i} + \frac{\operatorname{cosec}\vartheta_i}{b_{i+1}}, \quad g_{i,i+1} = -\frac{\operatorname{cosec}\vartheta_i}{b_{i+1}},$$

with

$$\operatorname{cosec}\vartheta_0 = \operatorname{ctg}\vartheta_0 = \operatorname{cosec}\vartheta_m = \operatorname{ctg}\vartheta_m = 0.$$

The substitution of expressions (5) for $w_{y,i}$ and S_i in the fourth and fifth equations of (8) leads to a system of $2(m-1)$ integral-differential equations $(i = 1, 2, \cdots, m-1)$

$$\sum_{j=i-1}^{i+1} C_{ij} m_j(x) + \sum_{j=i-2}^{i+2} d_{ij} \int_0^x \left[\int_0^{x_2} \sigma_j(x_1)\, dx_1 \right] dx_2 =$$
$$= E\left\{ \psi_{y,i}(x, b) - \psi_{y,i+1}(x, 0) + \frac{1}{b_i}[P_{1,i}(x) - P_{2,i}(x)] - \right.$$
$$\left. - \frac{1}{b_{i+1}}[P_{1,i+1}(x) - P_{2,i+1}(x)] - \left(\sum_{j=i-2}^{i+2} d_{ij} k_j\right) x - \sum_{j=i-1}^{i+2} h_{ij} l_j \right\}, \tag{12}$$
$$- \sum_{j=i-2}^{i+2} d_{ij} \int_0^x m_j(x_1)\, dx_1 + \sum_{j=i-1}^{i+1} e_{ij} \sigma_j'(x) = \int_0^x \left\{ \frac{1}{b_i}[Q_{1,i}(x_1) - Q_{2,i}(x_1)] - \right.$$
$$\left. - \frac{1}{b_{i+1}}[Q_{1,i+1}(x_1) - Q_{2,i+1}(x_1)] \right\} dx_1 + W_i(x, b) - W_{i+1}(x, 0) + \zeta_i - \zeta_{i+1},$$

connecting the unknown magnitudes with the load acting on the fold. The following symbols are used in (12):

$$C_{i,i-1} = \frac{ED_i' b_i}{6}, \quad C_{i,i} = \frac{E(D_i' b_i + D_{i+1}' b_{i+1})}{3}, \quad C_{i,i+1} = \frac{ED_{i+1}' b_{i+1}}{6}; \tag{13}$$

$$d_{i,i-2} = -\frac{\operatorname{cosec}\vartheta_{i-1}}{b_{i-1}b_i},$$

$$d_{i,i-1} = \frac{\operatorname{cosec}\vartheta_{i-1}}{b_{i-1}b_i} + \frac{\operatorname{ctg}\vartheta_{i-1} + \operatorname{ctg}\vartheta_i}{b_i^2} + \frac{\operatorname{cosec}\vartheta_i}{b_i b_{i+1}};$$

$$d_{i,i} = -\left(\frac{\operatorname{ctg}\vartheta_{i-1} + \operatorname{ctg}\vartheta_i}{b_i^2} + \frac{2\operatorname{cosec}\vartheta_i}{b_i b_{i+1}} + \frac{\operatorname{ctg}\vartheta_i + \operatorname{ctg}\vartheta_{i+1}}{b_{i+1}^2}\right); \qquad (13)\ \text{(cont'd)}$$

$$d_{i,i+1} = \frac{\operatorname{cosec}\vartheta_i}{b_i b_{i+1}} + \frac{\operatorname{ctg}\vartheta_i + \operatorname{ctg}\vartheta_{i+1}}{b_{i+1}^2} + \frac{\operatorname{cosec}\vartheta_{i+1}}{b_{i+1}b_{i+2}},$$

$$d_{i,i+2} = -\frac{\operatorname{cosec}\vartheta_{i+1}}{b_{i+1}b_{i+2}};$$

$$e_{i,i-1} = \frac{K_i b_i}{6E}, \quad e_{i,i} = \frac{K_i b_i + K_{i+1}b_{i+1}}{3E}, \quad e_{i,i+1} = \frac{K_{i+1}b_{i+1}}{6E}$$

and

$$h_{i,i-1} = -\frac{\operatorname{cosec}\vartheta_{i-1}}{b_i}, \quad h_{i,i} = \frac{\operatorname{ctg}\vartheta_{i-1} + \operatorname{ctg}\vartheta_i}{b_i} + \frac{\operatorname{cosec}\vartheta_i}{b_{i+1}}, \qquad (14)$$

$$h_{i,i+1} = -\left(\frac{\operatorname{cosec}\vartheta_i}{b_i} + \frac{\operatorname{ctg}\vartheta_i + \operatorname{ctg}\vartheta_{i+1}}{b_{i+1}}\right), \quad h_{i,i+2} = \frac{\operatorname{cosec}\vartheta_{i+1}}{b_{i+1}};$$

E is an arbitrary magnitude having the dimensions of stress (the "reduced modulus of elasticity" of the system).

Obviously, the $2(m-1)$ equations of (12) are insufficient for determining all $2m+6$ unknown functions. The remaining 8 equations can always be obtained from the conditions of fastening and loading of the extreme longitudinal edges of the fold. By substituting (9) in (5) we obtain the following expressions for the fundamental geometrical and dynamic characteristics at the left edge of the first plate and the right edge of the m-th one:

$$u_1(x,0) = \frac{1}{E}\int_0^x \sigma_0(x_1)\,dx_1 + k_0, \quad u_m(x,b) = \frac{1}{E}\int_0^x \sigma_m(x_1)\,dx_1 + k_m;$$

$$v_1(x,0) = V_1(x,0) + \frac{1}{Eb_1}\int_0^x \left\{\int_0^{x_2} [\sigma_0(x_1) - \sigma_1(x_1)]\,dx_1\right\} dx_2 +$$

$$+ \frac{1}{b_1}(k_0 - k_1)\,x + l_1;$$

$$v_m(x,b) = V_m(x,b) + \frac{1}{Eb_m}\int_0^x \left\{\int_0^{x_2} [\sigma_{m-1}(x_1) - \sigma_m(x_1)]\,dx_1\right\} dx_2 + \qquad (15)$$

$$+ \frac{1}{b_m}(k_{m-1} - k_m)\,x + l_m;$$

$$w_{y,1}'(x,0) = \psi_{y,1}(x,0) - \frac{1}{b_1}\,[w_1(x,0) + P_{2,1}(x)] + \frac{1}{E}\sum_{j=0}^{1} C_{0j}\,m_j(x) +$$

$$+ \sum_{j=0}^{2} d_{0j}\left\{\frac{1}{E}\int_0^x \left[\int_0^{x_2} \sigma_j(x_1)\,dx_1\right] dx_2 + k_j x\right\} + \sum_{j=1}^{2} h_{0j}\,l_j;$$

$$w_{y,m}(x,b)=\psi_{y,m}(x,b)+\frac{1}{b_m}[P_{1,m}(x)+w_m(x,b)]-\frac{1}{E}\sum_{l=m-1}^{m}C_{ml}m_l(x)-$$

$$-\sum_{l=m-2}^{m}d_{ml}\left\{\frac{1}{E}\int_0^x\left[\int_0^{x_2}\sigma_l(x_1)\,dx_1\right]dx_2+k_l x\right\}-\sum_{l=m-1}^{m}h_{ml}l_l;$$

$$S_1(x,0)=W_1(x,0)+\frac{1}{b_1}\int_0^x[T_{y,1}(x_1,0)-Q_{2,1}(x_1)]\,dx_1-$$

$$-\sum_{l=0}^{2}d_{0l}\int_0^x m_l(x_1)\,dx_1+\sum_{l=0}^{1}e_{0l}\sigma_l'(x)+\zeta_1,$$

$$S_m(x,b=W_m(x,b)+\frac{1}{b_m}\int_0^x[Q_{1,m}(x_1)-T_{y,m}(x_1,b)]\,dx_1+ \qquad (15)\ \text{(cont'd)}$$

$$+\sum_{l=m-2}^{m}d_{ml}\int_0^x m_l(x_1)\,dx_1-\sum_{l=m-1}^{m}e_{ml}\sigma_l'(x)+\zeta_m;$$

$$N_{y,1}(x,0)=\chi_1(x,0)-\frac{1}{b_1}[m_0(x)-m_1(x)],$$

$$N_{y,m}(x,b)=\chi_m(x,b)-\frac{1}{b_m}[m_{m-1}(x)-m_m(x)];$$

$$M_{y,1}(x,0)=m_0(x),\qquad M_{y,m}(x,b)=m_m(x).$$

Using these expressions it is possible to form additional equations for any combination of kinematic and static boundary conditions at the extreme longitudinal edges of the fold. Together with the fundamental equations (12) they form a complete system, whose solution yields all the unknown fuctions of the problem.

The solution of such a system obviously necessitates the knowledge of the constants $k_0, k_l, l_l, \zeta_i\ (i=1,2,\cdots,m)$ and the initial values $\sigma_i(0)\ (i=0,1,\cdots,m)$. In order to find the latter it is necessary to consider the conditions of fastening and loading of the lateral edges of the fold. Since this scheme is incapable of fulfilling these conditions exactly, we shall require in what follows only their approximate fulfillment in the part connected with the deformation of the middle surface of the fold. Substituting once more (9) in (5), we obtain the following expressions for the middle values of the displacements u, v and forces T_x, S at the lateral edges:

$$u_i(0)=k_i,\quad u_i(a)=\frac{1}{E}\int_0^a\sigma_i(x_1)\,dx_1+k_i\quad (i=0,1,\cdots,m);$$

$$v_i(0)=\frac{1}{2}V_i(0,b)+l_i,\quad v_i(a)=\frac{1}{2}[V_i(a,0)+V_i(a,b)]+$$

$$+\frac{1}{Eb_i}\int_0^a\left\{\int_0^{x_2}[\sigma_{i-1}(x_1)-\sigma_i(x_1)]\,dx_1\right\}dx_2+\frac{a}{b_i}(k_{i-1}-k_i)+l_i\quad (i=1,2,\cdots,m);\qquad (16)$$

$$T_{x,0}(0)=U_{2,1}(0,0)+\frac{K_1}{E}\sigma_0(0),\quad T_{x,i}(0)=\frac{1}{2}[U_{2,i}(0,b)+U_{2,i+1}(0,0)]+$$

$$+\frac{K_i+K_{i+1}}{2E}\sigma_i(0)\quad (i=1,2,\cdots,m-1),\quad T_{x,m}(0)=U_{2,m}(0,b)+\frac{K_m}{E}\sigma_m(0);$$

$$T_{x,0}(a) = U_{2,1}(a, 0) + \frac{K_1}{E}\sigma_0(a), \quad T_{x,i}(a) = \frac{1}{2}[U_{2,i}(a, b) + U_{2,i+1}(a, 0)] +$$

$$+ \frac{K_i + K_{i+1}}{2E}\sigma_i(a) \quad (i = 1, 2, \cdots, m-1),$$

$$T_{x,m}(a) = U_{2,m}(a, b) + \frac{K_m}{E}\sigma_m(a);$$

$$S_i(0) = \frac{1}{2}W_i(0, b) + \frac{K_i b_i}{12E}[\sigma'_{i-1}(0) - \sigma'_i(0)] + \zeta_i,$$

$$S_i(a) = \frac{1}{2}[W_i(a, 0) + W_i(a, b)] + \tag{16 (cont'd)}$$

$$+ \frac{1}{b_i}\int_0^a [Q_{1,i}(x_1) - Q_{2,i}(x_1)]\,dx_1 + \frac{1}{b_i}\int_0^a [f_{i,i-2}m_{i-2}(x_1) +$$

$$+ (f_{i,i-1} - g_{i,i-1})m_{i-1}(x_1) + f_{i,i} - g_{i,i})m_i(x_1) - g_{i,i+1}m_{i+1}(x_1)]\,dx_1 +$$

$$+ \frac{K_i b_i}{12E}[\sigma'_{i-1}(a) - \sigma'_i(a)] + \zeta_i \quad (i = 1, 2, \cdots, m).$$

Using these expressions it is always possible to find the values of k_0, k_i, l_i, ζ_i, $\sigma_i(0)$, and thus make the solution of the problem wholly determinate.

In conclusion, we note on the most efficient method of integrating the system of nonlinear equations derived. If the fold deformations under a load are not too large, the method of successive approximations will obviously be the simplest. The solution of the problem reduces then to a repetition of the procedure worked out by Vlasov for calculating linear systems. In the case of weak nonlinearity the convergence of the process is fairly rapid, so that the solution can be obtained relatively simply.

It is easily seen that the assumption $M_x = H = \varepsilon_y = \omega \equiv 0$ adopted by us at the beginning is not essential. Results similar to those given above are also obtained at different values (specified beforehand) of M_x, H, ε_y, ω. The only difference lies in the fact that in the case of values of these magnitudes different from zero it is necessary to use expressions for φ_i, χ_i, $\varphi_{1,i}, \cdots$ which are somewhat more complex than those given in (6). The fundamental equations (12), and also relationships (15) and (16), which are necessary for forming the additional equations and boundary conditions, remain unaltered. This means that the results obtained above can be used also for studying prismatic shells and folds in which all the static and kinematic factors of deformation without exception are of essential importance.

The actual values of the moments M_x, H and the components ε_y, ω can be determined by the iteration method. Assuming first arbitrary acceptable expressions for these 4 functions, we conduct the calculation described above, and determine the corresponding values of the 13 remaining characteristics of the state of stress and strain of each edge of the system. New (refined) values of the moments M_x, H and the strains ε_y, ω can then be found from the fourth and sixth relationships of (2), and the second and third relationships of (2'). If a considerable discrepancy exists between those values and the starting ones, the process described is repeated.

VIBRATIONS AND STABILITY OF ANISOTROPIC AND SANDWICH CYLINDRICAL SHELLS OF ARBITRARY CROSS SECTION

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All the studies conducted so far on the subject of the vibrations and stability of simple anisotropic and sandwich cylindrical shells have dealt only with the case of shells of circular section. The question of the vibrations and stability of shells having the shape of a cylinder of arbitrary cross section has been ignored.

Such shells are, however, being used in some ship structures, and the development of methods for calculating their stability and vibrations seems therefore desirable.

An approximate solution of the problem considered can be obtained relatively easily if it is assumed that the initial stressed state of these shells can be determined by the membrane theory.

This paper deals with the problem of the free flexural vibrations and the stability of a simple anisotropic shell having the form of an elliptic cylinder simply supported by the end sections and loaded by an all-round uniformly distributed pressure.

A similar problem is considered for the case of a sandwich elliptical shell with rigid incompressible filler.

Both problems are considered in a linear formulation, and the approximate solutions obtained are presented in the form of relatively simple formulas for the frequency of flexural vibrations and the critical pressure.

No particular necessity arises for determining the so-called lower critical pressure, since the latter does not differ in the given case by more than 30% from the upper critical pressure obtained in the linear formulation.

1. Consider first the problem of the free vibrations and stability of a simple anisotropic elliptic shell. We shall start from the equations of motion of a cylindrical shell of arbitrary cross section, in which the tangential inertia forces are neglected and only the inertia forces due to the normal deflection are allowed for.

It is assumed that the shell is initially in a membrane state of stress. The corresponding forces and moments are assumed to be given by the expressions obtained in /1/.

The system of equations for the displacements u, v and w is

$$\frac{E_1}{1-\mu_1\mu_2}\frac{\partial^2 u}{\partial x^2}+\frac{E_2}{2(1+\mu_2)}\frac{\partial^2 u}{\partial s^2}+\left[\frac{E_1\mu_2}{1-\mu_1\mu_2}+\frac{E_2}{2(1+\mu_2)}\right]\frac{\partial^2 v}{\partial x\partial s}-\frac{E_1\mu_2}{1-\mu_1\mu_2}\frac{1}{r}\frac{\partial w}{\partial x}=0, \tag{1}$$

$$\frac{E_1}{2(1+\mu_1)}\frac{\partial^2 v}{\partial x^2}+\frac{E_2}{1-\mu_1\mu_2}\frac{\partial^2 v}{\partial s^2}+\left[\frac{E_2\mu_1}{1-\mu_1\mu_2}+\frac{E_1}{2(1+\mu_1)}\right]\frac{\partial^2 u}{\partial x\partial s}-$$

$$-\frac{E_2}{1-\mu_1\mu_2}\frac{\partial}{\partial s}\left(\frac{w}{r}\right)=0,$$

$$\nabla_2^4 w+\frac{E_1\mu_2}{r^2}\frac{\partial^2 w}{\partial x^2}+E_2\frac{\partial^2}{\partial s^2}\left(\frac{w}{r^2}\right)-\frac{12E_2}{h^2 r}\left(\frac{\partial v}{\partial s}-\frac{w}{r}+\mu_1\frac{\partial u}{\partial x}\right)+ \qquad (1)\ (\text{cont'd})$$

$$+\frac{12p(1-\mu_1\mu_2)}{h^3}\left[r\frac{\partial^2 w}{\partial s^2}+\frac{w}{r}-(2x-1)\left(\frac{\partial^2 w}{\partial x\partial s}-\frac{\partial v}{r\partial x}\right)\frac{dr}{ds}-\right.$$

$$\left.-\frac{x(L-x)}{2}\frac{\partial^2 w}{\partial x^2}\frac{d^2 r}{ds^2}+\frac{\pi\partial b}{s_0}\frac{\partial^2 w}{\partial x^2}\right]+\frac{12\rho_c(1-\mu_1\mu_2)}{h^2}\frac{\partial^2 w}{\partial t^2}=0,$$

where ∇_2^4 is the differential operator /2/,

$$\nabla_2^4=E_1\frac{\partial^4}{\partial x^4}+E_3\frac{\partial^4}{\partial x^2\partial s^2}+E_2\frac{\partial^4}{\partial s^4};\quad E_3=4G(1-\mu_1\mu_2)+E_1\mu_2+E_2\mu_1,$$

u, v, and w are displacements along the generator, the directrix, and the normal, respectively; x, s are coordinates measured along the generator from one of the end sections of the shell and along the cross-sectional arc from the fixed generator; L, h are the length and thickness of the shell; a, b are semiaxes of the shell cross section; s_0 is its perimeter, expressed as known through a complete elliptic integral of the second kind; $r=r(s)$ is the radius of curvature of the cross section; E_1 and μ_1 are Young's modulus and the Poisson ratio in the direction of the x-axis; E_2 and μ_2 are the corresponding magnitudes in the direction of the s-axis; $G=\frac{E_1}{2(1+\mu_1)}=\frac{E_2}{2(1+\mu_2)}$ is the modulus of shear; ρ_c is the mass density of the shell material; p is the uniformly distributed pressure; t is the time.

The system of equations (1) can be reduced by the standard method to one differential equation relative to w, which is written in dimensionless coordinates ξ and η in the form

$$\nabla_1^4\left\{\rho\left[\nabla_2^4 w-q\left(\frac{\xi(\xi_0-\xi)}{2}\frac{\partial^2 w}{\partial\xi^2}\frac{d^2\rho}{d\eta^2}+(2\xi-\xi_0)\frac{\partial^2 w}{\partial\xi\partial\eta}\frac{d\rho}{d\eta}-\right.\right.\right.$$

$$\left.\left.\left.-\frac{\pi ab}{s_0 r_0}\frac{\partial^2 w}{\partial\xi^2}-\rho\frac{\partial^2 w}{\partial\eta^2}\right)+\frac{\rho_c h^*}{D_1}\frac{\partial^2 w}{\partial t^2}\right]\right\}+\frac{4b_1^4}{\rho}\frac{\partial^4 w}{\partial\xi^4}=0, \qquad (2)$$

where ∇_1^4 and ∇_2^4 are differential operators /2/ in the dimensionless coordinates ξ and η:

$$\nabla_1^4=\frac{1}{r_0^4}\left[E_1\frac{\partial^4}{\partial\xi^4}+E_4\frac{\partial^4}{\partial\xi^2\partial\eta^2}+E_2\frac{\partial^4}{\partial\eta^4}\right];\quad E_4=\frac{E_1E_2}{G}-E_1\mu_2-E_2\mu_1;$$

$$\xi=\frac{x}{r_0},\quad \eta=\frac{s}{r_0},\quad \xi_0=\frac{L}{r_0},\quad \rho=\frac{r}{r_0};$$

r_0 is the maximum radius of curvature of the shell cross section;

$$q=12p(1-\mu_1\mu_2)\left(\frac{r_0}{h}\right)^3;\quad 4b_1^4=12(1-\mu_1\mu_2)E_1E_2\left(\frac{r_0}{h}\right)^2;$$

$$D_1=\frac{E_1h^3}{12(1-\mu_1\mu_2)};\quad h^*=hr_0^4E_1.$$

Equation (2) will be solved by the Bubnov-Galerkin method. In the case of a shell simply supported by the end sections, we adopt for the deflection w the expression

$$w = \sin \lambda\xi \sum_k T_k(t) \sin k\nu\eta \sin \Omega_k t, \tag{3}$$

where $\lambda = \frac{\pi r_0}{L} = \frac{\pi}{\xi_0}$; $\nu = \frac{4\pi}{\eta_0}$; $\eta_0 = \frac{s_0}{r_0}$; Ω_k is the unknown frequency. In accordance with the Bubnov-Galerkin method we can write

$$\int_0^{\eta_0}\int_0^{\xi_0} \left[\nabla_1^4 + \frac{4b_1^4}{\rho}\frac{\partial^4 w}{\partial \xi^4}\right] \sin \lambda\xi \sin k\nu\eta \, d\xi d\eta = 0, \tag{4}$$

where

$$f(\xi, \eta, t) = \rho\left\{\nabla_2^4 w - q\frac{\xi(\xi_0 - \xi)}{2}\frac{\partial^2 w}{\partial \xi^2}\frac{d^2\rho}{d\eta^2} + (2\xi - \xi_0)\frac{\partial^2 w}{\partial\xi\partial\eta}\frac{d\rho}{d\eta} - \right.$$
$$\left. - \frac{\pi ab}{s_0 r_0}\frac{\partial^2 w}{\partial \xi^2} - \rho\frac{\partial^2 w}{\partial \eta^2}\right] + \frac{\rho_c h^*}{D_1}\frac{\partial^2 w}{\partial t^2}\Big\}, \tag{5}$$

or after integrating by parts and allowing for the boundary conditions:

$$\int_0^{\eta_0}\int_0^{\xi_0}\left[(E_1\lambda^4 + E_4\lambda^2 k^2\nu^2 + E_2 k^4\nu^4) f + \frac{4b_1^4\lambda^4}{\rho} w\right] \sin \lambda\xi \sin k\nu\eta \, d\xi d\eta = 0. \tag{6}$$

Assuming that the functions ρ and $\frac{1}{\rho}$ are continuous and differentiable, we can represent them in the form of series

$$\rho = \frac{1}{2}\rho_0 + \sum_{i=1} \rho_i \cos i\nu\eta, \qquad \frac{1}{\rho} = \frac{1}{2}\left(\frac{1}{\rho}\right)_0 + \sum_{i=1}\left(\frac{1}{\rho}\right)_i \cos i\nu\eta, \tag{7}$$

whose coefficients can be determined by one of the numerical methods.

It is now necessary to substitute series (7) and (3) in equation (6), and to calculate the expressions obtained.

We proceed to the selection of the approximating function for the deflection. Instead of retaining in series (3) the terms of lowest index, as is usual when using the Bubnov-Galerkin method, we retain only the k-th term, which corresponds to the principal harmonic, and the subsequent terms located near it.

The calculations showed that in a first approximation, when only the k-th term in (3) is retained, the k-th frequency is determined with an error not exceeding 4 to 5%.

When, in addition to the k-th term, we retain the $(k-1)$-th and $(k+1)$-th terms as well, the error in the determination of the frequency is reduced to 1 to 2%.

The following approximate expression for the square of the frequency of the flexural vibrations of an elliptic shell was obtained by the method described above:

$$\Omega_k^2 = \frac{D_1}{\rho_c h^* \rho_0}\left\{(E_1\lambda^4 + E_3\lambda^2k^2\nu^2 + E_2k^4\nu^4)\rho_0 + \frac{4b_1^4\lambda^4}{E_1\lambda^4 + E_4\lambda^2k^2\nu^2 + E_2k^4\nu^4} \times \right.$$
$$\times\left[\left(\frac{1}{\rho}\right)_0 - \left(\frac{1}{\rho}\right)_{2k}\right] - q\nu^2\left[k^2\left(\frac{1}{2}\rho_0^2 + \sum_{i=1,2}\rho_i^2\right) + \right.$$
$$\left.\left. + \frac{1}{4}\left(\frac{\pi^2}{3}+1\right)\sum_{i=1,2} i^2\rho_i^2 + \frac{\pi ab}{s_0 r_0}\frac{\lambda^2}{\nu^2}\rho_0\right]\right\}. \qquad (8)$$

In the case $q = 0$ (unloaded shell) the square of the frequency is given by the expression

$$\Omega_k^2 = \frac{D_1}{\rho_c h^* \rho_0}\left\{(E_1\lambda^4 + E_3\lambda^2k^2\nu^2 + E_2k^4\nu^4)\rho_0 + \right.$$
$$\left. + \frac{4b_1^4\lambda^4}{E_1\lambda^4 + E_4\lambda^2k^2\nu^2 + E_2k^4\nu^4}\left[\left(\frac{1}{\rho}\right)_0 - \left(\frac{1}{\rho}\right)_{2k}\right]\right\}. \qquad (9)$$

By writing $\Omega_k = 0$ in (8), we can obtain an approximate expression for determining the critical load of an elliptic shell. This expression is

$$q_{cr} = \frac{(E_1\lambda^4 + E_3\lambda^2k^2\nu^2 + E_2k^4\nu^4)\rho_0 + \dfrac{4b_1^4\lambda^4}{E_1\lambda^4 + E_4\lambda^2k^2\nu^2 + E_2k^4\nu^4}\left[\left(\dfrac{1}{\rho}\right)_0 - \left(\dfrac{1}{\rho}\right)_{2k}\right]}{k^2\nu^2\left(\dfrac{1}{2}\rho_0^2 + \sum\limits_{i=1,2}\rho_i^2\right) + \dfrac{\nu^2}{4}\left(\dfrac{\pi^2}{3}+1\right)\sum\limits_{i=1,2} i^2\rho_i^2 + \dfrac{\pi ab}{s_0 r_0}\lambda^2\rho_0}. \qquad (10)$$

The minimum critical load can be found by assuming some values for "k", calculating the corresponding values of the critical load, and selecting the smallest one.

If a more rigorous determination of the critical load is required, the terms of index $k+1$ and $k-1$ are allowed for in the system of equations relative to T_k, in addition to the term of index k. Assuming that in this system of equations the frequency terms vanish, and equating the determinant of the system to zero, we obtain an expression from which the critical load is determined by assuming several values for "k". The error of such a refined value does not exceed 2 to 3% relative to the higher approximations, obtained by retaining in the system several more additional terms.

In the case of an isotropic shell, when $E_1 = E_2 = E$ and $\mu_1 = \mu_2 = \mu$, formulas (8) to (10) reduce to

$$\Omega_k^2 = \frac{D}{\rho_c h r_0^4 \rho_0}\left\{(\lambda^2 + k^2\nu^2)^2\rho_0 + \frac{4b^4\lambda^4}{(\lambda^2 + k^2\nu^2)^2}\left[\left(\frac{1}{\rho}\right)_0 - \left(\frac{1}{\rho}\right)_{2k}\right] - \right.$$
$$\left. - q\nu^2\left[k^2\left(\frac{1}{2}\rho_0^2 + \sum_{i=1,2}\rho_i^2\right) + \frac{1}{4}\left(\frac{\pi^2}{3}+1\right)\sum_{i=1,2} i^2\rho_i^2 + \frac{\pi ab}{s_0 r_0}\frac{\lambda^2}{\nu^2}\rho_0\right]\right\}, \qquad (11)$$

$$\Omega_k^2 = \frac{D}{\rho_c h r_0^4 \rho_0}\left\{(\lambda^2 + k^2\nu^2)^2\rho_0 + \frac{4b^4\lambda^4}{(\lambda^2 + k^2\nu^2)^2}\left[\left(\frac{1}{\rho}\right)_0 - \left(\frac{1}{\rho}\right)_{2k}\right]\right\}, \qquad (12)$$

$$q_{cr} = \frac{(\lambda^2 + k^2\nu^2)^2\rho_0 + \dfrac{4b^4\lambda^4}{(\lambda^2 + k^2\nu^2)^2}\left[\left(\dfrac{1}{\rho}\right)_0 - \left(\dfrac{1}{\rho}\right)_{2k}\right]}{k^2\nu^2\left(\dfrac{1}{2}\rho_0^2 + \sum\limits_{i=1,2}\rho_i^2\right) + \dfrac{\nu^2}{4}\left(\dfrac{\pi^2}{3}+1\right)\sum\limits_{i=1,2} i^2\rho_i^2 + \dfrac{\pi ab}{s_0 r_0}\lambda^2\rho_0}, \qquad (13)$$

where

$$4b^4 = 12(1-\mu^2)\left(\frac{r_0}{h}\right)^2, \qquad q = \frac{12p(1-\mu^2)}{E}\left(\frac{r_0}{h}\right)^3.$$

In order to test the reliability of the computing formulas obtained for determining the critical pressure, we conducted several experiments with celluloid models of elliptic shells, in which the ratios of the cross-section semiaxes were equal to 1.2 and 1.7. The thickness, length, and large semiaxes of the cross section were the same in all the models tested, and only the magnitude of the small semiaxis was varied.

The calculated and experimental data obtained are given in the following table:

TABLE

a/b	a cm	b cm	L cm	δ cm	p_{cr}^{th} atm	k^{th}	p_{cr}^{exp} atm	k^{exp}
1.2	10	8.33	11.3	0.15	0.62	3	0.62	3
1.7	10	5.89	11.3	0.15	0.48	3	0.47	3

It is seen from the table that the experimental and theoretical values for both the number of waves into which the shell buckles (k) and the critical load coincide practically. It can therefore be concluded that the computing formulas obtained for determining the critical load of elliptical shells are sufficiently reliable.

2. The problem of a sandwich elliptical shell with a rigid incompressible filler is treated in the same way as that of a simple anisotropic elliptical shell.

The results obtained for a rigid filler can be transferred to the case of light filler by neglecting the tensile and flexural rigidities of the filler.

Consider the case of a rigid filler whose lateral deformation can be neglected. The problem will be solved in a linear formulation, using the theory of shallow shells. We adopt the standard assumptions:

a) the material of the load-carrying layers is homogeneous and isotropic;

b) the load-carrying layers are symmetric relative to the middle surface of the shell;

c) the Poisson ratios of the load-carrying layers and the filler are equal;

d) the Kirchhoff-Love hypotheses are applicable to the load-carrying layers;

e) the straight-lines hypothesis is used for the filler.

Consider a sandwich cylindrical shell of arbitrary cross section with rigid incompressible filler, subjected to the action of an all-round uniform pressure. We take the system obtained in /3/ and /4/ as the initial system of stability differential equations, keeping in mind that (unlike /3/ and /4/) the radius of curvature of the shell is an arbitrary function of the coordinate s, and the initial membrane state of stress is determined by the expressions already used in the treatment of a simple anisotropic shell.

$$
\left.
\begin{aligned}
&\frac{\partial^2 u_\alpha}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 u_\alpha}{\partial s^2}+\frac{1+\mu}{2}\frac{\partial^2 v_\alpha}{\partial x\partial s}+\frac{\mu}{R}\frac{\partial w}{\partial x}=0,\\
&\frac{1+\mu}{2}\frac{\partial^2 u_\alpha}{\partial x\partial s}+\frac{1-\mu}{2}\frac{\partial^2 v_\alpha}{\partial x^2}+\frac{\partial^2 v_\alpha}{\partial s^2}+\frac{\partial}{\partial s}\left(\frac{w}{R}\right)=0,\\
&A\left[\frac{\partial^2 u_\beta}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 v_\beta}{\partial x\partial s}+\frac{1-\mu}{2}\frac{\partial^2 u_\beta}{\partial u^2}\right]+B^*\nabla^2\frac{\partial w}{\partial x}-\\
&\qquad -h_1\frac{\partial w}{\partial x}-u_\beta=0,\\
&A\left[\frac{\partial^2 v_\beta}{\partial s^2}+\frac{1+\mu}{2}\frac{\partial^2 u_\beta}{\partial x\partial s}+\frac{1-\mu}{2}\frac{\partial^2 v_\beta}{\partial x^2}\right]+B^*\nabla^2\frac{\partial w}{\partial s}-\\
&\qquad -h_1\frac{\partial w}{\partial s}-v_\beta=0,\\
&\frac{B_1}{R}\left(\frac{\partial v_\alpha}{\partial s}+\mu\frac{\partial u_\alpha}{\partial x}+\frac{w}{R}\right)+\Gamma\nabla^4 w+F\nabla^2\left(\frac{\partial u_\beta}{\partial x}+\frac{\partial v_\beta}{\partial s}\right)-\\
&\quad -G_1h_1\left[\frac{\partial}{\partial x}\left(u_\beta+h_1\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial s}\left(v_\beta+h_1\frac{\partial w}{\partial s}\right)\right]+\\
&\quad +\frac{1}{2}\rho_c\frac{\partial^2 w}{\partial t^2}+\frac{K}{2}\left[R\frac{\partial^2 w}{\partial s^2}+\frac{w}{R}-(2x-L)\frac{\partial^2 w}{\partial x\partial s}\frac{dR}{ds}-\right.\\
&\qquad \left.-\frac{x(L-x)}{2}\frac{\partial^2 w}{\partial x^2}\frac{d^2R}{ds^2}+\frac{\pi ab}{s_0}\frac{\partial^2 w}{\partial x^2}\right]=0,
\end{aligned}
\right\}
\tag{14}
$$

where

$$
A=\frac{h}{G}\left(B+\frac{1}{3}B_c\right),\qquad B^*=\frac{D_c\delta}{2Gh},\qquad B_1=B+B_c,
$$

$$
\Gamma=D+\frac{D_c\delta^2}{4h^2},\qquad F=\frac{D_c\delta}{2h^2},\qquad K=\frac{12p(1-\mu^2)}{E\delta^3}\Gamma,\qquad G_1=\frac{G}{h},
$$

$$
h_1=h+\frac{\delta}{2},\qquad D=\frac{E\delta^3}{12(1-\mu^2)},\qquad D_c=\frac{E_c h^3}{3(1-\mu^2)},
$$

$$
B=\frac{E\delta}{1-\mu^2},\qquad B_c=\frac{E_c h}{1-\mu^2},
$$

ρ_c is the density of the material of the load-carrying layer per unit middle surface (it is assumed that all the layers are made of the same material); t is time;

$$
u_\alpha=\frac{u_1+u_2}{2},\qquad u_\beta=\frac{u_1-u_2}{2},\qquad v_\alpha=\frac{v_1+v_2}{2},\qquad v_\beta=\frac{v_1-v_2}{2};
$$

u_1, v_1; u_2, v_2 are displacements of the points of the middle surfaces along the coordinates x, s of the upper and lower load-carrying layers, respectively; δ is the thickness of the load-carrying layer; $2h$ is the thickness of the filler; $R=R(s)$ is the radius of the middle surface of the filler; L is the shell length; E is the modulus of elasticity of the material of the load-carrying layers; E_c, G are moduli of elasticity and shear of the filler; and μ is the Poisson ratio.

The system of equations (14) (which can likewise be obtained directly by the variational method) will now be reduced to one differential equation relative to w, by eliminating all the unknown functions except w.

Such an equation can be written in the form

$$\nabla_1^4 f_1 + \frac{B_1(1-\mu^2)}{R_0^5}\left(\frac{A}{R_0^2}\nabla^2 - 1\right)\left(\frac{1}{\rho}\frac{\partial^4 w}{\partial \xi^4}\right) = 0, \tag{15}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}, \qquad \nabla^4 = \frac{\partial^4}{\partial \xi^4} + 2\frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4}{\partial \eta^4};$$

$$f_1(\xi, \eta, t) = \frac{1}{R_0^7}\left(D - \frac{D_c\delta}{2h}\right)\left(\frac{A}{R_0^2}\nabla^2 - 1\right)(\rho\nabla^4 w) -$$

$$- \frac{1}{R_0^7}\left(Bh_1 + \frac{B_c h}{3}\right)\rho\nabla^4\left(h_1 - \frac{B^*}{R_0^2}\nabla^2\right)w + \frac{1}{2}\frac{\rho_c}{R_0^3}\left(\frac{A}{R_0^2}\nabla^2 - 1\right)\left(\rho\frac{\partial^2 w}{\partial t^2}\right) +$$

$$+ \frac{K}{2R_0^4}\left(\frac{A}{R_0^2}\nabla^2 - 1\right)\left\{\rho\left[\rho\frac{\partial^2 w}{\partial \eta^2} + \frac{w}{\rho} - (2\xi - \xi_0)\frac{\partial^2 w}{\partial \xi d\eta}\frac{d\rho}{d\eta} -\right.\right.$$

$$\left.\left. - \frac{\xi(\xi_0 - \xi)}{2}\frac{\partial^2 w}{\partial \xi^2}\frac{d^2\rho}{d\eta^2} + \frac{\pi ab}{s_0 R_0}\frac{\partial^2 w}{\partial \xi^2}\right]\right\}. \tag{16}$$

The problem has thus been reduced to integrating equation (15) at the corresponding boundary conditions.

In the case of a shell simply supported by the end sections we adopt for w expression (3).

Integrating (15) by the Bubnov-Galerkin method, we obtain

$$\int_0^{\eta_0}\int_0^{\xi_0}\left[(\lambda^2 + k^2\nu^2)^2 f_1 + \frac{B_1(1-\mu^2)\lambda^4}{R_0^5}\left(\frac{A}{R_0^2}\nabla^2 - 1\right)\left(\frac{w}{\rho}\right)\right]\sin\lambda\xi\sin k\nu\eta\, d\xi\, d\eta = 0. \tag{17}$$

The following approximate expression is obtained for the frequency by using expressions (7) for ρ and $\frac{1}{\rho}$ and taking into account the considerations developed in Section 1 in the selection of the approximating function in the form (3):

$$\frac{1}{2}\rho_c\Omega_k^2 = \frac{D - \frac{D_c\delta}{2h}}{R_0^4}(\lambda^2 + k^2\nu^2)^2 + \frac{Bh_1 + \frac{B_c h}{3}}{R_0^4}\,\frac{h_1 + \frac{B^*}{R_0^2}(\lambda^2 + k^2\nu^2)}{1 + \frac{A}{R_0^2}(\lambda^2 + k^2\nu^2)}(\lambda^2 + k^2\nu^2)^2 -$$

$$- \frac{K}{2\rho_0 R_0}\left[k^2\nu^2\left(\frac{1}{2}\rho_0^2 + \sum_{i=1,2}\rho_i^2\right) + \frac{1}{1 + \frac{A}{R_0^2}(\lambda^2 + k^2\nu^2)}\left(\frac{A}{R_0^2}\sum_{i=1,2}C_i\rho_i^2 +\right.\right.$$

$$\left.\left. + \sum_{i=1,2}F_i\rho_i^2\right)\right] - \frac{K\pi ab\lambda^2}{2s_0 R_0} + \frac{B_1(1-\mu^2)\lambda^4}{R_0^2\rho_0(\lambda^2 + k^2\nu^2)^2}\left[\left(\frac{1}{\rho}\right)_0 - \left(\frac{1}{\rho}\right)_{2k}\right], \tag{18}$$

where

$$C_i = \frac{5}{2}ik^3\nu^4 - \frac{3}{2}ik\nu^2\lambda^2 + 2i^3k\nu^4 +$$

$$+\frac{1}{4}\left(\frac{\pi^2}{3}+1\right)(-i^4\nu^4+2i^3k\nu^4+i^2\nu^2\lambda^2+i^2k^2\nu^4);$$

$$F_i=\frac{1}{2}ik\nu^2+\frac{1}{4}\left(\frac{\pi^2}{3}+1\right)i^2\nu^2.$$

In the case of a circular shell $\rho_0=\left(\frac{1}{\rho}\right)_0=2;\ \rho_i=\left(\frac{1}{\rho}\right)_i=0\ (i\geqslant 1)$ and formula (18), reducing to

$$\frac{1}{2}\rho_c\,\Omega_k^2=\frac{D-\frac{D_c\delta}{2h}}{R^4}(n^2+\lambda^2)^2+$$

$$+\frac{Bh_1+\frac{B_ch}{3}}{R^4}\cdot\frac{h_1+\frac{B^*}{R^2}(n^2+\lambda^2)}{1+\frac{A}{R^2}(n^2+\lambda^2)}(n^2+\lambda^2)^2-\frac{K}{2R}\left(n^2+\frac{\lambda^2}{2}\right)+\frac{B_1(1-\mu^2)\lambda^4}{R^2(n^2+\lambda^2)^2},\quad(19)$$

coincides with the formula obtained in /4/. The latter formula reduces to the formula obtained in /5/ at $p=0$.

From (18) one passes to the case of a shell with light filler by writing $B_c=D_c=0$.

The following approximate formula can be written for the critical load of a sandwich shell having the shape of an elliptical cylinder, in the case of rigid incompressible filler:

$$p_{cr}=\frac{E\delta^3}{12(1-\mu)\Gamma}\frac{D_1^*}{D_2^*},$$

where

$$D_1^*=\frac{D-\frac{D_c\delta}{2h}}{R_0^4}(\lambda^2+k^2\nu^2)^2+\frac{Bh_1+\frac{B_ch}{3}}{R_0^4}\cdot\frac{h_1+\frac{B^*}{R_0^2}(\lambda^2+k^2\nu^2)}{1+\frac{A}{R_0^2}(\lambda^2+k^2\nu^2)}(\lambda^2+k^2\nu^2)^2+$$

$$+\frac{B_1(1-\mu^2)\lambda^4}{R_0^2\rho_0(\lambda^2+k^2\nu^2)^2}\left[\left(\frac{1}{\rho}\right)_0-\left(\frac{1}{\rho}\right)_{2k}\right],$$

$$D_2^*=\frac{1}{2R_0\rho_0}\left[k^2\nu^2\left(\frac{1}{2}\rho_0^2+\sum_{i=1,2}\rho_i^2\right)+\frac{1}{1+\frac{A}{R_0^2}(\lambda^2+k^2\nu^2)}\left(\frac{A}{R_0^2}\sum_{i=1,2}C_i\rho_i^2+\right.\right.$$

$$\left.\left.+\sum_{i=1,2}F_i\rho_i^2\right)\right]+\frac{\pi ab}{2s_0R_0}\lambda^2.\quad(20)$$

These formulas yield as particular cases the formulas for an elliptical shell with light filler, for a circular sandwich shell with rigid or light filler, and for a simple circular shell (the Mises formula).

It follows that the approximate computing formulas obtained for determining the critical pressure and frequency of free flexural vibrations of the shell represent a generalization of the similar formulas for circular

cylindrical shells, reducing to the latter by passing from the elliptical cross section to the particular case of a circular cross section.

The procedure used in the case of an elliptical shell can likewise be applied to the case of a circular shell of arbitrary cross section, provided the radius of curvature of the shell section and its derivatives are continuous.

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COMPRESSION AND FLEXURE OF HYPERBOLIC SHELLS

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1. Shells having the shape of a hollow single-cavity hyperboloid of revolution are used in certain structures, such as the ventilation pipes of thermoelectric power plants, whose height attains 100 m. The pipes are designed and manufactured to have a variable wall thickness. Their external and internal surfaces (see figure) are formed by rotating cofocal hyperbolas about an imaginary axis. It follows that the general equations of the three-dimensional theory of elasticity can be used in calculating their strength.

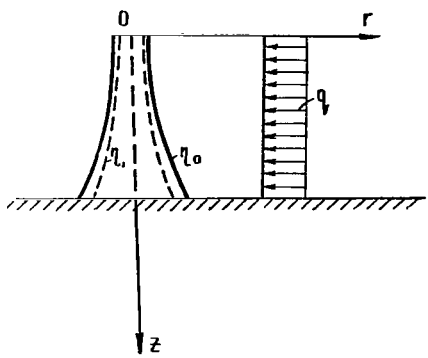

FIGURE.

The loads acting on a shell of the type considered are mainly the wind forces and the dead weight of the structure. The design standards assume a linear distribution of the wind forces along the shell height, with an intensity increasing vertically. The law of variation of the load along the cross-section circumference is piecewise continuous and can be represented by a Fourier series by the meridional coordinate φ.

In the present paper we restrict ourselves to two terms of the series — the free term and the term containing $\cos\varphi$. The load corresponding to the first term is axisymmetrical and produces therefore an axisymmetrical deformation of the shell. The load corresponding to the second term causes a general bending of the shell. The shell deformation in this case will be called bending in what follows.

2. In the case of axisymmetrical deformation of bodies of revolution, the solution is represented by one biharmonic function (Michell /1/, Love /2/), or by two harmonic functions (Papkovich /3/, Naiber /4/). We shall make use here of the solution given in /5/, according to which the stresses and displacements can be expressed through two stress functions ψ and φ, defined by the differential equations in partial derivatives

$$\nabla^2\psi = 0, \qquad \nabla^2\varphi = 0, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \tag{1}$$

where r and z are cylindrical coordinates (the z-axis coincides with the shell axis). Designating by Φ and Ω the following combinations of the functions ψ and φ:

$$\Phi = \psi + z\frac{\partial\varphi}{\partial z}, \qquad \Omega = \frac{1}{r^2}[\Phi + 2(1-\nu)\varphi] - \frac{2}{r}\frac{\partial\varphi}{\partial r}, \tag{2}$$

and introducing the curvilinear orthogonal isothermal coordinate system ξ, η in such a way as to make the contour in the axial section of the body of revolution coincide with the coordinate line $\eta = \eta_0 =$ const, we obtain the following relationships between the functions Φ and Ω, and the external axial loads p_z and radial loads p_r

$$\frac{\partial\Phi}{\partial\xi} = r\sqrt{H}\cdot p_z, \qquad \frac{\partial\Phi}{\partial\eta} = r\left(\Omega\frac{\partial r}{\partial\eta} + \sqrt{H}\cdot p_r\right). \tag{3}$$

Here H is Lamé's differential operator of the curvilinear coordinates.

In the case of a hollow body of revolution conditions (3) must be satisfied on both the external ($\eta = \eta_0$) and internal ($\eta = \eta_1$) surfaces.

Note further that the function Φ is connected by the following simple formula with the normal forces in the cross section of the body of revolution:

$$\Phi(\eta_0) - \Phi(\eta_1) = -\frac{N}{2\pi}. \tag{4}$$

The surfaces of the hyperbolic shell are defined by constant values of η ($\eta = \eta_0$ and $\eta = \eta_1$) of the elliptic coordinates

$$r = c\,\mathrm{ch}\,\xi\cos\eta, \qquad z = c\,\mathrm{sh}\,\xi\sin\eta, \tag{5}$$

whose differential parameter is

$$H = c^2(\mathrm{ch}^2\xi - \cos^2\eta). \tag{6}$$

The solution of equations (1) in these coordinates in the form $\psi = f_1(\xi)f_2(\eta)$ leads to the following values for the stress functions ψ and φ:

$$\begin{aligned} &\psi_0, \varphi_0 = (A_0\,\mathrm{sh}\,\xi + B_0)(C_0\sin\eta + D_0), \\ &\psi_k, \varphi_k = [A_kP_k'(x) + B_kQ_k'(x)][C_kP_k'(y) + D_kQ_k'(y)](\mathrm{ch}\,\xi\cos\eta)^2, \\ &\qquad\qquad k = 1, 2, 3, \ldots \end{aligned} \tag{7}$$

Here $y = \sin\eta$; $x = i\,\mathrm{sh}\,\xi$; $P_k'(x)$, $Q_k'(x)$, $P_k'(y)$, $Q_k'(y)$ are derivatives of the Legendre functions of the first and second kind of order k by x and y, respectively.

By selecting in different manners the stress functions from (7) and using an assumption (to be explained below) obvious in the case of thin shells, we obtain values of the stresses and displacements in hyperbolic shells of the form considered which correspond to their loading by axisymmetrical forces of different types. We shall consider the case of radial compression of a hyperbolic shell by a load uniform along the height (see figure) and distributed along the external surface ($p_r = p$ at $\eta = \eta_0$ and $p_r = 0$ at $\eta = \eta_1$). Writing in this case

$$\begin{aligned} \psi &= A_0 \sin\eta + [A_1 + B_1 Q_1'(y)] (\operatorname{ch}\xi \operatorname{ccs}\eta)^2, \\ \varphi &= C_0 \sin\eta + D_0 + [C_1 + D_1 Q_1'(y)] (\operatorname{ch}\xi \cos\eta)^2 \end{aligned} \tag{8}$$

and requiring that the function Φ on the contour be independent of H^{-1} we obtain $C_0 = -2D_1$ and

$$\Phi = (A_0 - 2D_0) \sin\eta + S(y) \operatorname{ch}^2\xi, \tag{9}$$

where

$$S(y) = [A_1 + B_1 Q_1'(y)] \cos^2\eta + 2D_1 \sin\eta. \tag{10}$$

Since in the loading case considered the normal force $N = 0$, the following equalities must be satisfied:

$$A_0 = 2D_1, \quad S(b_0) = S(b_1) = 0, \tag{11}$$

where $b_0 = \sin\eta_0$ and $b_1 = \sin\eta_1$.

The fulfillment of the second contour condition of (3) leads to the following two equalities:

$$\begin{aligned} S'(b_0) - 2(1+\nu)[C_1 + D_1 Q_1'(b_0)]\, b_0 + \frac{4D_1 b_0^2}{1-b_0^2} &= pc^2 b_0, \\ S'(b_1) - 2(1+\nu)[C_1 + D_1 Q_1'(b_1)]\, b_1 + \frac{4D_1 b_1^2}{1-b_1^2} &= 0 \end{aligned} \tag{12}$$

and to the requirement

$$C_0 \sin\eta + D_0 = 0 \tag{13}$$

at $\eta = \eta_0$ and $\eta = \eta_1$.

The values of the constants A_1, B_1, C_1, and D_1 are obtained by solving the system of four equations (11) and (12). Equality (13) cannot be fulfilled simultaneously on the two contours (the external and internal). In the case of thin shells we can replace it, with an accuracy depending on the shell thickness, by the condition

$$C_0 b + D_0 = 0, \tag{14}$$

where b is some average value of $\sin\eta$ between b_0 and b_1, for instance

$$b = \frac{1}{2}(b_0 + b_1)$$

(the surface corresponding to b can be considered by convention as the middle surface of the shell).

The use of this assumption leads to an approximate solution. The contour conditions are satisfied "on the average." The stresses σ_ξ and $\tau_{\xi\eta}$

are not equal to zero on the contour $\eta = \eta_1$ and are not equal to the corresponding projections of the external load on the contour $\eta = \eta_0$. Thus, the stresses σ_η have on the contour $\eta = \eta_1$ the value

$$\sigma_\eta = \frac{2(1-\nu)}{H(b_1)} \frac{C_0 b_1^2}{1-b_1^2} \cdot \varepsilon,$$

where $\varepsilon = b_1 - b = b - b_0$. The stresses $\tau_{\xi\eta}$ on the contour $\eta = \eta_1$ and the complements of the exact values σ_η and $\tau_{\xi\eta}$ on the contour $\eta = \eta_0$ appear multiplied by the factor ε. The stresses σ_ξ and σ_η (which are fundamental in the loading case considered) do not have such a factor, and therefore are $\frac{1}{\varepsilon}$ times larger than σ_η and $\tau_{\xi\eta}$ or their corrections. It should be stressed that 2ε is not the thickness of the shell, but the difference $b_1 - b_0$ of the sine defining the shell surface; this is a small magnitude even for shells of relatively thick walls.

A similar solution can likewise be obtained for the case of loading of the shell by symmetrical surface forces varying along the height according to an arbitrary polynomial law, and for the case of central compression.*

3. The solution of the problem of shell bending is obtained by simplifying one of the boundary conditions, as in the axisymmetrical problem given above. In this case, the stresses and displacements of deformation of the body of revolution can be represented through the derivatives of the three functions χ, ψ, and f, each of which is defined by a differential equation in partial derivatives of the second order

$$\nabla_3^2 \chi = 0, \quad \nabla_3^2 \psi = 0, \quad \nabla_3^2 f = 0, \quad \nabla_3^2 = \frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{15}$$

By introducing the symbols

$$\Psi = \chi + z\psi, \qquad \Omega = \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{\partial f}{\partial z},$$

$$F = \frac{\partial \Psi}{\partial z} - 2\psi - f + r\frac{\partial f}{\partial r}, \tag{16}$$

$$T_\xi = \frac{\partial \Psi}{\partial \xi} + [2\nu\psi + (1-2\nu)f]\frac{\partial r}{\partial \eta},$$

we can write the contour conditions in a concise form. Next, introducing a curvilinear orthogonal isothermal system of coordinates ξ and η, whose coordinate lines (for instance $\eta = \eta_0 = \text{const}$) define the surface of the body of revolution, one obtains easily the relationships

$$\frac{1}{\sqrt{H}}\frac{\partial \Omega}{\partial \xi} = -r(p_r - p_\varphi), \qquad \frac{1}{\sqrt{H}} T_\xi = -r^2 p_\varphi, \tag{17}$$

$$\frac{1}{\sqrt{H}}\left(\frac{\partial F}{\partial \xi} + \Omega\frac{\partial r}{\partial \eta}\right) = r^2 p_z$$

* In addition to the solutions given here, the author has also obtained solutions for the cases of central compression of the shell, pure bending, bending by a lateral force, axisymmetrical loading of the external surface by radial forces varying linearly with height, and bending by radial linear forces.

at $\eta = \eta_0$ where p_r, p_φ, and p_z are the radial, tangential, and axial components, respectively, of the forces distributed over the shell surface.

The differences of the contour values of the functions Ω and F are connected with the shearing force and the bending moment in the cross sections of the body of revolution by the equalities

$$\Omega(\eta_0) - \Omega(\eta_1) = \frac{1}{\pi} Q_x, \qquad F(\eta_0) - F(\eta_1) = -\frac{1}{\pi} M_y. \tag{18}$$

This fact facilitates establishing solutions of particular cases of shell bending.

In elliptical coordinates (5) the solution of equations (16) in the form of a product of a function of ξ by a function of η yields the following values of the stress functions χ, ψ, and f:

$$\begin{gathered}\chi_0,\ \psi_0,\ f_0 = [A_0 \operatorname{sh}\xi + B_0(1 - \operatorname{sh}^2\xi)]\,[C_0 \sin\eta + D_0(1 + \sin^2\eta)],\\ \chi_1,\ \psi_1,\ f_1 = [A_1 + B_1(3 + \operatorname{sh}^2\xi)\operatorname{sh}\xi]\,[C_1 + D_1(3 - \sin^2\eta)\sin\eta],\\ \chi_k,\ \psi_k,\ f_k = [A_k P_k''(x) + B_k Q_k''(x)]\,[C_k P_k''(y) + D_k Q_k''(y)]\,(\operatorname{ch}\xi\cos\eta)^4,\\ k = 2, 3, 4, \cdots\end{gathered} \tag{19}$$

$P_k''(x)$ and $Q_k''(x)$ are second derivatives by $x = i \operatorname{sh}\xi$, and $P_k''(y)$ and $Q_k''(y)$ are second derivatives by $y = \sin\eta$ of the Legendre functions of x or y, respectively of the first kind and of order k.

We shall solve the problem of the loading of the shell by radial forces q uniform along the height and distributed along the external surface, i.e., the problem corresponding to the following external forces:

$$p_r = p_\varphi = p_z = 0 \text{ at } \eta = \eta_1 \text{ (on the internal surface)},$$
$$p_r = p\cos\varphi, \quad p_\varphi = p_z = 0 \text{ at } \eta = \eta_0 \text{ (on the external surface)},$$

with $p = -\dfrac{q}{\pi r}$, $q = \text{const}$ (see figure).

Taking as fundamental stress functions the values

$$\begin{gathered}\chi = c\{D_1(1 + \sin^2\eta) + [A_1 + B_1(3 - \sin^2\eta)\sin\eta]\,(3 + \operatorname{sh}^2\xi)\}\operatorname{sh}\xi,\\ \psi = C_3 + D_2(3 - \sin^2\eta)\sin\eta + [A_2\sin\eta + B_2(1 + \sin^2\eta)]\,(1 - \operatorname{sh}^2\xi),\\ f = C_3 + D_3(3 - \sin^2\eta)\sin\eta + [A_3\sin\eta + B_3(1 + \sin^2\eta)]\,(1 - \operatorname{sh}^2\xi),\end{gathered} \tag{20}$$

and successively forming the expressions for the functions Ψ (not given here) and Ω, we obtain

$$\begin{aligned}\Omega = \frac{1}{c}\Big\{&3A_1 - A_2\sin^2\eta + A_3(1 + \sin^2\eta) + 2(3B_1 - B_2 + 2B_3)\sin\eta +\\ &+ \frac{c^2}{H}(3A_1 - A_2\sin^2\eta - A_3\cos^2\eta + D_1 - 3D_2\sin^2\eta - 3D_3\cos^2\eta)\cos^2\eta\Big\}\operatorname{sh}\xi.\end{aligned} \tag{21}$$

Since the function Ω does not depend on H^{-1} (the shearing force is a linear function of z), the constants must be linked by the equalities

$$3A_1 + D_1 = A_2 + 3D_2 = A_3 + 3D_3. \tag{22}$$

Using the first relationship of (18), we obtain

$$\Omega = \frac{1}{c}\left[\frac{1}{\pi}\frac{qc^2 b_0}{b_0 - b_1} + (A_2 - A_3)(b_0 - \sin\eta)\right](\sin\eta - b_1)\operatorname{sh}\xi \tag{23}$$

and two relationships between the constants

$$2(3B_1 - B_2 + 2B_3) = \frac{1}{\pi}\frac{qc^2 b_0}{b_0 - b_1} + (A_2 - A_3)(b_0 + b_1),$$
$$3A_1 = -\frac{1}{\pi}\frac{qc^2 b_0 b_1}{b_0 - b_1} - A_2 b_0 b_1 - A_3(1 - b_0 b_1). \tag{24}$$

We write the functions F and T_ξ in the form

$$F = S_1(y) + S_2(y)\,\mathrm{sh}^2\xi, \quad T_\xi = c\,[V_1(y) + V_2(y)\,\mathrm{sh}^2\xi]\,\mathrm{ch}\,\xi, \tag{25}$$

where

$$\begin{aligned}
S_1(y) &= -(C_2 + C_3) - 3A_1 \sin\eta + D_2 \sin^3\eta + D_3(3 - 2\sin^2\eta)\sin\eta + \\
&\quad + (3B_1 - B_3)(3 - \sin^2\eta) - B_2(1 + \sin^2\eta), \\
S_2(y) &= 3A_1 \sin\eta + A_2(4 - 5\sin^2\eta)\sin\eta + A_3 \sin^2\eta - \\
&\quad - 3B_1(1 - 5\sin^2\eta + 2\sin^4\eta) + (B_2 - B_3)(1 - 3\sin^2\eta), \\
V_1(y) &= (1 - 2\nu)(C_2 - C_3)\sin\eta - 3A_1 \sin^2\eta + (A_2 + 3D_2)(1 + \sin^2\eta) - \\
&\quad - (1 - 2\nu)(D_2 - D_3)\sin^4\eta + 3B_1(3 - \sin^2\eta)\sin\eta + \\
&\quad + (1 - 2\nu)(B_2 - B_3)(1 + \sin^2\eta)\sin\eta, \\
V_2(y) &= 3A_1 - (3 - 2\nu)A_2 \sin^2\eta + (1 - 2\nu)A_3 \sin^2\eta + \\
&\quad + 3B_1(3 - \sin^2\eta)\sin\eta - [(3 - 2\nu)B_2 - (1 - 2\nu)B_3](1 + \sin^2\eta)\sin\eta.
\end{aligned} \tag{26}$$

The following equalities must be fulfilled on the strength of contour conditions (17):

$$\begin{aligned}
S_1(b_0) &= 0, & S_2(b_0) &= \frac{1}{2\pi} qc^2 b_0^2, \\
S_1(b_1) &= 0, & S_2(b_1) &= 0, \\
V_1(b_0) &= V_1(b_1) = 0, & V_2(b_0) &= V_2(b_1) = 0.
\end{aligned} \tag{27}$$

We thus have twelve equalities (22), (24), and (27) connecting eleven constants. This means that all the boundary conditions cannot be exactly satisfied. Therefore, as in the axisymmetrical problem, we shall satisfy one of the conditions approximately — "on the average" — on some middle surface, defined by

$$\sin\eta = b = \frac{1}{2}(b_0 + b_1),$$

instead of on each of the surfaces $\eta = \eta_0$ and $\eta = \eta_1$.

Instead of $V_1(b_0) = V_1(b_1) = 0$ we write

$$V_1(b) = 0, \tag{28}$$

then $V_1(b_0)$ and $V_1(b_1)$ can be written in the form

$$V_1(b_0) = -V_1(b_1) = \frac{1}{b}\{3A_1 + 6B_1 b^3 + D_1(1 - b^2) - (1 - 2\nu)[(A_2 - A_3)b + \\ + 2(B_2 - B_3)]b^3\}\varepsilon, \tag{29}$$

where $\varepsilon = b - b_0 = b_1 - b$ is a small magnitude in the case of thin shells.

As a result of the assumption adopted, the stresses σ_η, $\tau_{\xi\eta}$, and $\tau_{\eta\varphi}$ do not vanish on the surface $\eta = \eta_1$ as expected from the statement of the

problem, and differ from the corresponding projections of the load on the surface $\eta = \eta_0$. On surface $\eta = \eta_1$, for instance, the stresses σ_η can be written in the form

$$\sigma_\eta = \frac{1}{rH(b_1)} \frac{b_1 V_1(b_1)}{1 - b_1^2} \cos\varphi,$$

i.e., have as a factor the magnitude ε small in the case of thin shells (cf. (29)). A similar formula, and a small factor ε, are also obtained in the expressions for the stresses $\tau_{\xi\eta}$ and $\tau_{\eta\varphi}$ at $\eta = \eta_1$ and for the complements which distinguish the approximate values of the stresses σ_η, $\tau_{\xi\eta}$, and $\tau_{\eta\varphi}$ from the exact ones at $\eta = \eta_0$. The expressions for the fundamental stresses σ_ξ, σ_φ, and $\tau_{\xi\varphi}$ do not contain the factor ε, i.e., these stresses are $\frac{1}{\varepsilon}$ times larger than σ_η, $\tau_{\xi\eta}$, and $\tau_{\eta\varphi}$ or their corrections. It follows that the number ε represents the measure of the error of the approximate solution.

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EXPERIMENTAL INVESTIGATION OF CYLINDRICAL AND SPHERICAL REINFORCED-CONCRETE SHELLS

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A. EXPERIMENTAL INVESTIGATION OF A REINFORCED-CONCRETE SHALLOW CYLINDRICAL PANEL-SHELL FOR ROOFS OF INDUSTRIAL BUILDINGS

Description of the design

The dimensions of the panel-shell are 6 × 6 meters in plan, the camber [highest point] is 0.32 m (1/20), and the slab thickness 0.04 m. The contour is reinforced by flanges along the generatrix and by variable-height diaphragms in the direction perpendicular to it. The boundary conditions are: the flanges are freely supported along their entire length and the diaphragms are suspended (Figure 1).

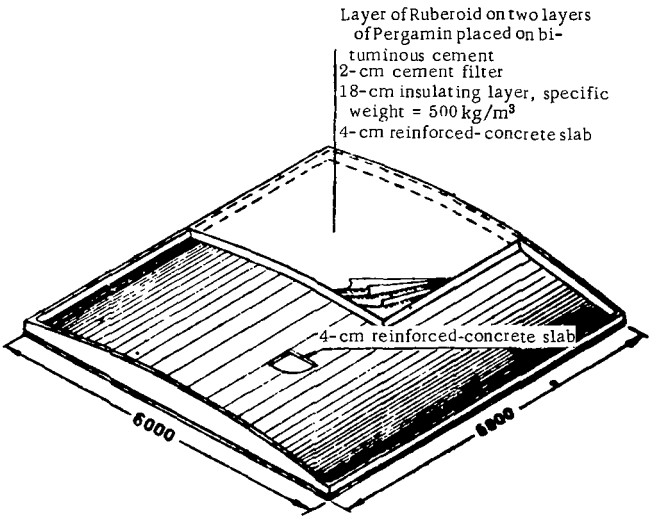


FIGURE 1.

Aim of investigation

This investigation was intended to check experimentally the strength and rigidity of the design, to determine the effect of precompressing the outermost elements of the shell on the stresses and ultimate carrying capacity of the cylindrical slab, and to establish the zone in which the edge stresses act.

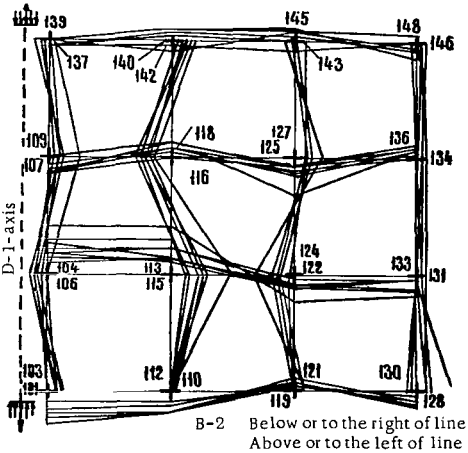
It was planned to subject the shell to a uniformly distributed static load, gradually increasing until ultimate failure.

Experimental results

The following experiments were performed:

1. Precompression of outermost elements (edge beams and diaphragms) in 2-ton intervals from 0 to 12 tons by a high-strength wire. Since the behavior of the diaphragms in compression was of primary interest here, these were equipped with strain gages, deflectometers, and sensors. One quarter of the upper surface of the cylindrical slab was also equipped with sensors.

During the three loading cycles the deflections of the diaphragms did not exceed 0.83 mm under a force of 12 tons. Up to 10 tons the load was applied eccentrically; however, the strain gages on the upper reinforcement rod showed that the edge elements were fully compressed. The torsional deflections of the diaphragms were measured by dial gages. The compression forced both diaphragms inward. The horizontal displacements of the lower flanges were between 2 and 4 mm.



B-2 Below or to the right of line — compression
Above or to the left of line — tension

FIGURE 2.

Figure 2 shows the stresses and strains at the upper surface of the slab during the compression of No. 1 diaphragm. The stresses in the slab

[are partially given in the table below]. The part of the slab adjoining the diaphragm is under tensile stresses. In the direction of the shell apex, the zone of tensile stresses spreads into the interior of the slab. At most points of the slab the transverse stresses do not endanger the shell, since they do not exceed 30 kg/cm^2 either in compression or in tension (see table).

Stresses, kg/cm^2

Sensor No.	139	142	145	148	109	118	127	136	106	115	124	133	103	112	121	130
Load 4t	-4	4	0	0	-8	4	-11	4	19	11	-8	-4	-8	-8	4	-4
" 6t	-4	-4	-8	0	-15	0	-8	4	15	15	-8	-4	-19	-19	4	0
" 8t	-4	8	4	-4	-8	8	-19	8	11	15	-15	-4	-15	-11	8	-15
" 9t	0	11	8	-4	-15	8	-8	8	23	23	-8	-4	-23	-19	4	0
" 10t	0	-8	8	-4	-11	8	-15	8	19	23	-8	-11	-23	-19	8	-11
" 11t	0	8	8	8	-15	15	-8	11	30	30	-4	-4	-23	-19	4	0
" 12t	-4	11	8	8	-15	8	-11	8	26	15	-11	-15	-30	-19	4	-8

Sensor No.	101	104	107	137	110	113	116	140	119	122	125	143	128	131	134	146
Load 4t	0	4	-8	0	0	-11	26	-4	-4	15	-8	4	-8	0	0	4
" 6t	-8	0	-15	-4	-4	-26	19	-4	0	11	-8	-8	4	0	0	-4
" 8t	0	4	-4	0	0	-15	30	-11	-4	19	-15	0	-11	-4	0	-4
" 9t	-15	8	-11	8	0	-23	23	-8	0	11	-11	4	0	0	-4	-4
" 10t	-8	11	-8	4	0	-23	23	-11	0	19	-15	0	-11	-4	-4	-4
" 11t	-8	15	-8	4	0	-30	23	-15	4	4	-8	0	4	0	-4	-8
" 12t	-11	11	-8	0	-4	-23	23	-8	-8	11	-15	0	-8	-8	-8	-4

Longitudinal stresses in the shell, due to the compression of the diaphragms, are nonuniformly distributed and differ in sign. These stresses (23 kg/cm^2) are not dangerous.

Precompression of the edge elements by forces of up to 10 tons did not appreciably affect the stresses in the slab. Owing to the edges being supported along their entire length, these forces were to a considerable degree counterbalanced by friction between the lower surface of the edge elements and the supporting beam.

The transverse stresses in the cylindrical part of the shell [due to precompression of the edges] are opposite in sign to the stresses in the shell due to its own weight and to a uniformly distributed vertical load. Precompression of the diaphragms thus gives rise to stresses which affect favorably the subsequent behavior of the slab when an external load is applied.

2. Short-time uniformly distributed load applied to the shell. The loads in these experiments were 150, 330, 450, 330, and 150 kg/cm^2. Three such loading cycles were carried out, each cycle lasting a day.

The compressive stresses in the upper [reinforcement] rod of the diaphragm did not exceed 700 kg/cm^2; the additional tensile stresses in the high-strength wire mesh, subjected to a pretension of 12,000 kg did not exceed 1750 kg/cm^2. In the same experiment the stresses in No. 2. diaphragm were higher; the additional stresses in its upper and lower reinforcement rods were 2000 and 2940 kg/cm^2, respectively. The diaphragm thus behaved satisfactorily at loads of up to 450 kg/cm^2.

The transverse stresses in the edge elements were low. The stress in the reinforcement rod of the slab near the edge did not exceed 170 kg/cm^2. The stresses in the shell at points adjacent to the diaphragms could not be determined. Most strain gages placed on the reinforcement rods at the bottom of the slab indicated tension, while most strain gages at the top

indicated compression. On the basis of the majority of the instrument readings it could be concluded that at the points adjacent to the diaphragms the slab was mainly in tension, in correspondence to the central parts of the diaphragms at points adjacent to the slab.

The stresses in the reinforcement rods in the center of the shell did not exceed 170 kg/cm^2 but were extremely nonuniform. Tension was observed close to the diaphragms; toward the center the stresses decreased gradually, becoming zero at the quarter-span. Compressive stresses appeared close to the center. The longitudinal stresses in the shell were insignificant, since in the center of the slab the compressive stress in the concrete did not exceed 16 kg/cm^2.

As already stated, in one quarter of the slab the longitudinal and transverse stresses were investigated by sensors. On the basis of their readings the following conclusions can be drawn:

a. The upper part of the slab near the diaphragms and edge elements was in nonuniform tension. Toward the center of the shell the stresses in the slab became compressive. The highest compressive stress did not exceed 60 kg/cm^2.

b. The lower surface of the slab near No. 1 diaphragm was in tension in all three test cycles. Parts of the slab adjoining the edge elements were insignificantly compressed during the first cycle and in tension during the second cycle, while both tensile and compressive stresses appeared in them in the third cycle.

In all three cycles the bottom surface of the slab, away from the edge, was mainly in tension in both longitudinal and transverse directions.

3. Investigation of the shell rigidity. Short-time loadings (test-cycle duration = 1 day) gave the following results. The maximum deflections of Nos 1 and 2 diaphragms were 7.7 and 8.59 mm, respectively, this being 1/780 and 1/700, respectively, of the total span. The largest relative deflection of the slab, not counting the deflections of the diaphragms, was 5.17 mm, or 1/1160 of the span. Together with the deflection of diaphragms, the deflection of the slab was 12.82 mm, or 1/468 of the span. The overall rigidity of the shell was thus found to be satisfactory.

At a load of 450 kg/m^2 torsion in the diaphragms was negligible.

In the long-time experiments the shell was instantaneously subjected to a load of 450 kg/m^2 and kept thus for 11 days. The deflections increased everywhere, though only by a few millimeters. For example, the deflection in the center of the shell was 12.65 mm on the first day and 14.05 mm on the ninth day. Consequently, the total deflection was 1.4/600 = 1/430 of the span.

4. Determination of the ultimate load of the cylindrical slab. Further experiments on the slab were performed only with the edges undeformed in the vertical direction. For this purpose three supports were put under each diaphragm. The test consisted of three consecutive loading cycles, the shell being subjected to the highest load for 20 hours.

In the first and second cycles the loads were increased from 630 kg/m^2 in steps of 100 kg/m^2 with subsequent load removal. In the first and second cycles the load exceeded by 60 % the design load; in the third cycle, by 90 %. At a load of 630 kg/m^2 the largest deflection after 20 hours was

33.2 mm or 1/180 of the span. The residual deflection after load removal was 5.8 mm, or 1/1030 of the span. Thus, a 90% overload did not surpass the load-carrying capacity of the shell. The deformations of the slab were in the elastic-plastic range.

A further aim of the experiments was to find the pattern of ultimate failure of the slab. For this purpose the load was increased to 730 kg/m^2 (120% overload). As a result the cylindrical slab gradually assumed a new equilibrium shape in the form of a hyperbolic paraboloid and acquired a double-negative curvature. In this new shape the shell withstood a load of 450 kg/m^2 during several loading cycles; when the load was removed the curved surface rose by about 1.5 to 2.0 cm.

CONCLUSIONS

1. The experimental investigation of a cylindrical shell showed a positive influence of the edge effect, which in this case consisted in precompressing the diaphragms and edge elements.
2. The cylindrical slab of the shell behaved like a three-dimensional system of moments, characterized by bending in two planes and the appearance of shearing forces at the edges of the shell.
3. When the edges are precompressed the ultimate equilibrium state of a reinforced-concrete shallow cylindrical shell is characterized by the fact that before ultimate failure the cylindrical surface of the shell becomes a surface with a double-negative curvature. When the load is further increased, this equilibrium shape becomes a suspended system having a considerable load-carrying capacity, whose deformations are in the elastic-plastic range.

B. EXPERIMENTAL INVESTIGATION OF A PRECAST REINFORCED-CONCRETE SPHERICAL SHELL FOR FLOORS OF INDUSTRIAL BUILDINGS

Description of the design

The shell is a floor element consisting of: a) a precast spherical slab assembled from four 4-cm-thick square spherical elements measuring 3×3 m, bordered by curved 10×10 cm ribs; b) four precast 6-m-long edge diaphragms with full cross sections and curved upper flanges, measuring 15×30 cm at the supports and 15×60 cm in the center; and c) four precast reinforced-concrete columns 45×45 cm in cross section (Figure 3).

Aim of investigation

The main purpose of the investigation was to check experimentally the strength and rigidity of the design, intended for mass production for floor

assemblies of industrial buildings. These elements can be preassembled in various ways. It was already known that a monolithic shell of these dimensions has sufficient strength. However, in the assembled version it was of interest to check the behavior, under load, of the central joints and, in particular, of the joints between the spherical elements and the diaphragms where, as is known from theory, considerable shearing forces appear.

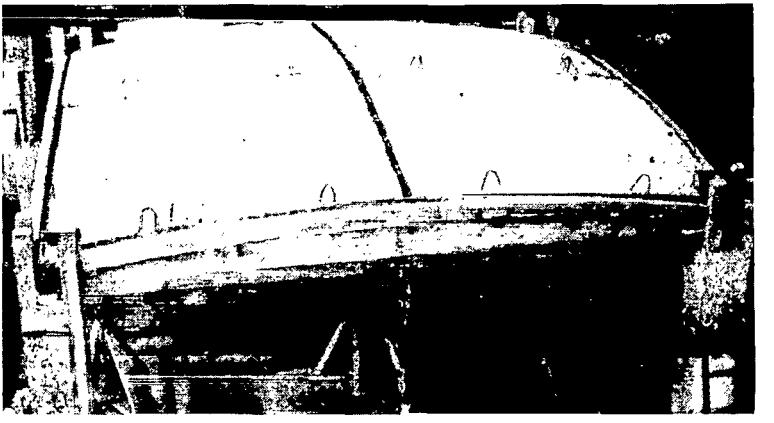


FIGURE 3.

Although the experiments were undertaken for this practical purpose, they were also valuable from the scientific aspect, since the edge effects of the shearing forces on the contour and their influence on the behavior of the shell flanges (diaphragms) could be investigated experimentally.

The existence of shear at the contour of shells, although proved theoretically, has not yet been verified experimentally. In our experiments on a spherical shell whose surface is nondevelopable, the edge shear effect should appear most clearly; it was also of interest to investigate experimentally other edge effects, namely, the flexural effect in the slab and the torsional effect in the diaphragms.

Experimental results

The components of the experimental element were cast at the construction site and then brought to the site of the experiments. Some of the slabs developed cracks during transit and were assembled in this state. All elements were made from grade-200 concrete.

To investigate the effect of tangential shear and its influence on the edge elements of the shell (diaphragms) the upper reinforcement rods of the diaphragms were equipped with resistance gages which were inserted together with the reinforcement rods into the upper flanges of the

diaphragms before the concrete was poured, and connected to measuring instruments.

The experiments consisted of 0-750-0 kg/m^2 long cycles in steps of 250 kg/m^2. After two runs for instrument adjustment, the first test cycle was carried out. The shell was then carefully inspected: no cracks except those present before the experiment were found in the precast elements of the slab. However, cracks of insignificant size were found in the joints between the edges of the slab and the diaphragms.

In the second test cycle it was intended to increase the load by the same steps to 1000 kg/m^2, subsequently reducing it to zero. However, this could not be done, since failure of the shell commenced at a load of 850 kg/m^2. This took place as follows.

After no-load readings had been taken the load was increased by steps to 750 kg/m^2. At this load the cracks in the edge joints became larger. At the beginning of the following state, at a load slightly in excess of 800 kg/m^2, the cracks in the edge joints began to open out and to spread rapidly over the entire circumference of the shell. At the same time the old cracks in the slab also became larger, while new cracks appeared as prolongations of the old ones. Parts of the slab became considerably deformed and failed gradually.

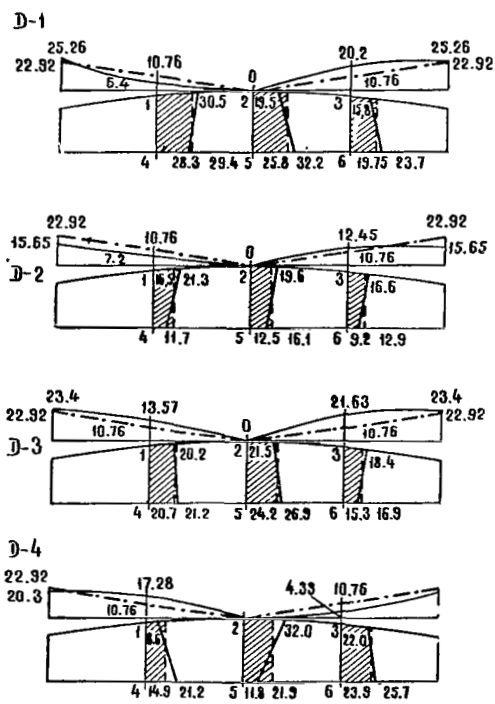


FIGURE 4.

During the experiments we studied in detail the behavior of the center joints, of a quarter-slab, and, in particular, of the diaphragms. The center joints functioned satisfactorily. The center ribs of the spherical

elements were nonuniformly compressed. Nowhere did joints between ribs show cracks. The spherical slab was more highly stressed than the framing ribs. The slab was bent by compression. The edge joints between the slab and the diaphragms were most highly stressed. Considerable shearing forces causing shearing of edge joints, radial cracks in the spherical slab, and an appreciable elongation of the diaphragms, developed on the contours of the shell.

The shell failed through shearing of the edge joints, after which cracks developed rapidly in all parts of the slab in the direction from the periphery toward the center of the shell.

Of greatest interest was the behavior of the diaphragms. In the first test cycle, the results of which are given in Figure 4, the diaphragms functioned satisfactorily. They were mainly in tension and only slightly bent in the vertical plane. Theoretical and experimental values of the stresses in the upper flanges of the diaphragm showed good agreement.

In the second test cycle one diaphragm was more highly stressed (Figure 5); the experimental stress values were much higher than the theoretical values. This diaphragm failed but the three remaining diaphragms remained in a state of elastic strain; theoretical and experimental stress values were in good agreement. Over their height the diaphragms were mostly in tension, showing moderate deflections. The edge flexure effect and the torsion in the diaphragms were negligible.

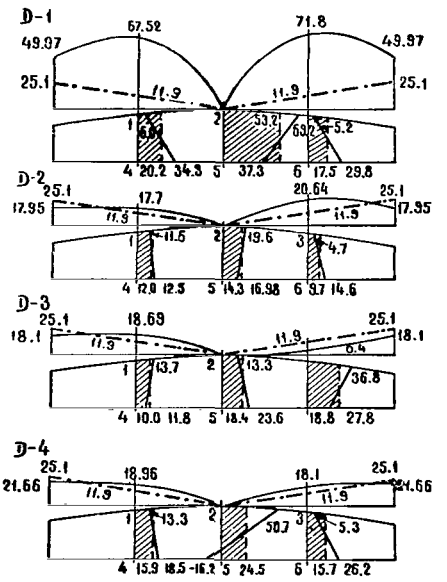


FIGURE 5.

CONCLUSIONS

1. The main result of these experiments is a clear verification of the edge effect of tangential shear between the spherical slab and the diaphragms, causing shearing of joints at the edges and gradual separation of the spherical parts of the slab from the diaphragms.

2. The edge elements of the shell were mainly in tension accompanied by insignificant deflections in the vertical plane.

THE LIMITING STATE OF THIN-WALLED SHAPES IN BIAXIAL BENDING WITH TORSION

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In conjunction with the extensive use of thin-walled beams and shells in various branches of technology, investigation of their behavior beyond the elastic limit, and establishment of their load-carrying capacity acquire great importance.

As an extension and development of previous studies /4-5/ we investigated, at the Strength of Materials Division of the Institute of Mechanics of the Ukrainian Academy of Sciences, the combined stresses in thin-walled beams under plastic biaxial flexure with torsion. We established relationships between the forces giving rise to normal stresses, and known formulas for calculating the ultimate loads. The theoretical assumptions made for channel beams subjected at the flange to external loads applied at various angles to the vertical, were checked experimentally.

a. THEORETICAL DATA

The following assumptions are made: a) the simplified stress-strain diagram corresponds to that of a material exhibiting a yield-point plateau; b) the limiting state is defined as that in which the entire cross section begins to yield; c) tangential stresses are neglected. Assumptions usually made when analyzing thin-walled beams /2/ are retained.

We consider the normal stresses in a channel beam at the limiting state (Figure 1). A force P, inclined at an angle α to the y-axis- acts either away from or toward the channel web.

The distances of the points of zero plastic stress in the flanges and web from the principal axes of the cross sections are denoted by u, d, and v; x_0 is the distance from the y-axis to the center line of the web; h, b, δ_w, and δ_{fl} are the channel dimensions.

The coordinates of A, the pole of the sectorial areas a_x and a_y, are assumed to be known. The sectorial area ω is regarded as positive when the radius vector is rotated counterclockwise (in the general case the zero points of ω will not coincide with the points of zero stress. The yield points $\sigma_{y,w}$ and $\sigma_{y,f}$ of web and flanges, respectively, differ in rolled sections.

From Figure 1a we obtain the following expressions for the bending moments M_x and M_y, in the principal planes and for the bimoment B:

$$M_x = \sigma_{y,f}\delta_f h\,(2x_0 - b + u + d) + \sigma_{y,w}\delta_w\left(\frac{h^2}{4} - v^2\right), \qquad (1)$$

$$M_y = \sigma_{y,f}\delta_f (u^2 - d^2) - 2\sigma_{y,w}\delta_w x_0 v, \tag{2}$$

$$B = M_x a_x - M_y a_y + \sigma_{y,f}\delta_f \frac{h}{2} [b^2 - (x_0 + u)^2 - (x_0 + d)^2], \tag{3}$$

$$N = 2\sigma_{y,f}\delta_f (u - d) + 2\sigma_{y,w}\delta_w v = 0. \tag{4}$$

The last equation defines the condition of equilibrium, i.e., the resultant normal force N acting on the cross section is zero.

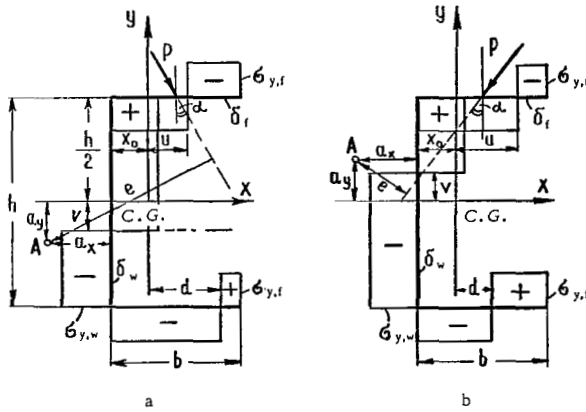

FIGURE 1.

Equations (1) through (4) contain four unknowns (the distances u, d, and v, and the limiting load in terms of which we can express M_x, M_y, and B).

From (2), (3), and (4) we find

$$u = -x_0 \pm \sqrt{\frac{b^2}{2} - R_1 + S_1}; \qquad d = -x_0 \pm \sqrt{\frac{b^2}{2} - R_1 + S_1},$$
$$v = -\frac{\sigma_{y,f}\delta_f}{\sigma_{y,w}\delta_w}\left(\pm\sqrt{\frac{b^2}{2} - R_1 + S_1} \mp \sqrt{\frac{b^2}{2} - R_1 - S_1}\right), \tag{5}$$

where

$$R_1 = \frac{B - M_x a_x + M_y a_y}{\sigma_{y,f}\delta_f h}; \qquad S_1 = \frac{M_y}{2\sigma_{y,f}\delta_f}. \tag{6}$$

Substitution of these values in (1) yields the following relationship between the three moments in the limiting state of the channel:

$$\frac{M_x}{\sigma_{y,f}} - 2\frac{\sigma_{y,f}\delta_f^2}{\sigma_{y,w}\delta_w} R_1 \mp \delta_f h\left(\sqrt{\frac{b^2}{2} - R_1 + S_1} + \sqrt{\frac{b^2}{2} - R_1 - S_1}\right) \mp$$
$$\mp 2\frac{\sigma_{y,f}\delta_f^2}{\sigma_{y,w}\delta_w}\sqrt{\left(\frac{b^2}{2} - R_1 + S_1\right)\left(\frac{b^2}{2} - R_1 - S_1\right)} = \frac{\sigma_{y,w}}{\sigma_{y,f}}\delta_w \frac{h^2}{2} - \delta_f bh -$$
$$- \frac{\sigma_{y,f}\delta_f}{\sigma_{y,w}\delta_w} b^2. \tag{7}$$

We now express the moments in terms of M and the angle α between

the applied force and the y-axis

$$M_x = M\cos\alpha; \quad M_y = -M\sin\alpha; \quad B = Me\gamma, \tag{8}$$

where e is the eccentricity of the load with respect to A

$$e = \left(a_x + \frac{b}{2}\right)\cos\alpha + \left(\frac{h}{2} + a_y\right)\sin\alpha; \tag{9}$$

and γ is a coefficient which depends on the kind of load, beam length, and support conditions at the ends /2/.

Inserting (8) into (7), we obtain

$$\frac{M}{\sigma_{y,f}} A_1 \mp \delta_f h \left(\sqrt{\frac{b^2}{2} + \frac{M}{\sigma_{y,f}} C_1} + \sqrt{\frac{b^2}{2} + \frac{M}{\sigma_{y,f}} D_1}\right) \mp$$
$$\mp 2\frac{\sigma_{y,f}\delta_f^2}{\sigma_{y,w}\delta_w}\sqrt{\left(\frac{b^2}{2} + \frac{M}{\sigma_{y,f}} C_1\right)\left(\frac{b^2}{2} + \frac{M}{\sigma_{y,f}} D_1\right)} = \frac{\sigma_{y,w}}{\sigma_{y,f}}\delta_w \frac{h^2}{4} - \delta_f bh -$$
$$- \frac{\sigma_{y,f}\delta_f^2}{\sigma_{y,w}\delta_w} b^2; \tag{10}$$

here

$$A_1 = \cos\alpha + \frac{2}{h}\frac{\sigma_{y,f}\delta_f}{\sigma_{y,w}\delta_w}(a_x\cos\alpha + a_y\sin\alpha - e\gamma),$$
$$C_1 = \frac{1}{\delta_f h}\left(a_x\cos\alpha + a_y\sin\alpha - e\gamma - \frac{h}{2}\sin\alpha\right), \tag{11}$$
$$D_1 = \frac{1}{\delta_f h}\left(a_x\cos\alpha + a_y\sin\alpha - e\gamma + \frac{h}{2}\sin\alpha\right).$$

After rationalization of the radicals we arrive at a fourth-degree equation for $\frac{M}{\sigma_{y,f}}$, which contains the limiting load. Equation (10) can be solved easily in the form given above.

If the force P is directed toward the channel web (Figure 1b), the signs of terms containing M_y and v must in (1) through (4) be changed, as must be certain signs in subsequent equations. Equation (10) remains valid, but the expressions for A_1, C_1, and D_1 become

$$A_2 = \cos\alpha + \frac{2}{h}\frac{\sigma_{y,f}\delta_f}{\sigma_{y,w}\delta_w}(a_x\cos\alpha + a_y\sin\alpha - e\gamma),$$
$$C_2 = \frac{1}{\delta_f h}\left(a_x\cos\alpha + a_y\sin\alpha - e\gamma + \frac{h}{2}\sin\alpha\right), \tag{12}$$
$$D_2 = \frac{1}{\delta_f h}\left(a_x\cos\alpha + a_y\sin\alpha - e\gamma - \frac{h}{2}\sin\alpha\right).$$

The eccentricity of the load is

$$e = \left(a_x + \frac{b}{2}\right)\cos\alpha - \left(\frac{h}{2} - a_y\right)\sin\alpha. \tag{13}$$

If in one flange the plastic stress is nowhere zero (Figure 2), i.e., if there are only two parameters (u and v, or v and d), (1) through (4) become simplified.

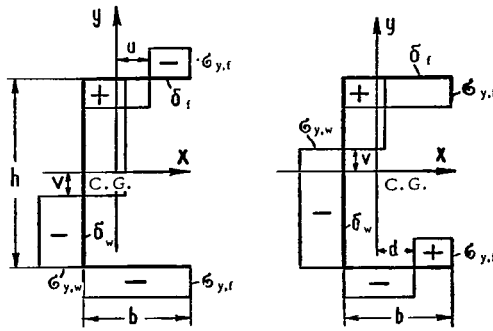

FIGURE 2.

For the stress diagram in Figure 2a ($d = b - x_0$) we obtain the following relationship between M_x, M_y, and B:

$$\frac{M_x}{\sigma_{y,f}} + \frac{\sigma_{y,f}\delta_f^2}{\sigma_{y,w}\delta_w}(-R_1 + S_1) \mp F_1 \frac{\sigma_{y,f}}{\sigma_{y,w}} \frac{\delta_f}{\delta_w} \sqrt{\frac{b^2}{2} - R_1 + S_1} =$$

$$= \frac{\sigma_{y,w}}{\sigma_{y,f}} \delta_w \frac{h^2}{4} - \frac{3}{2} \frac{\sigma_{y,f}}{\sigma_{y,w}} \frac{\delta_f^2 b^2}{\delta_w}, \tag{14}$$

where

$$F_1 = 2\delta_f b + \frac{\sigma_{y,w}}{\sigma_{y,f}} \delta_w h. \tag{15}$$

In this case the limiting load is found from the quadratic equation.

Using the above method, similar equations can be obtained for the limiting state of other sections subjected to biaxial flexure with torsion.

b. EXPERIMENTAL DATA

To check the theoretical assumptions we tested three cantilever beams (l = 130 cm) consisting of No. 12 channel iron (OST* 10017-39). The external force acted at angles α = 15, 30, and 90 degrees to the y-axis.

The mechanical properties of the steel used are given in Table 1, separately for flange and web, as the average of tensile tests of eight flat specimens.

The tests of channels in biaxial flexure with torsion were performed on a five-ton hydraulic action beam-bending testing machine made by the Detroit Testing Machine Co. (Figure 3). The end of the beam being tested was secured at the specified angle in a 48-cm-long double support by inserts, wedges, and screws. To transmit the load to the free end of the beam, a device was fastened to the crosshead of the machine; the load was vertically transmitted to the channel flange through a 14-mm-diameter

* [Obshchezoyuznyi Standart — All-Union Standard.]

ball. To maintain the shape of the section, a plate was welded to its inside at the point of load application (Figure 3).

TABLE 1

Average for 8 specimens	σ_T, kg/cm^2	σ_{ult}, kg/cm^2	δ, %	ψ, %	Length of yield-point plateau, %	E, kg/cm^2
Flange	2500	3660	22.2	52.1	0.80	$2.13 \cdot 10^6$
Web	2530	3990	27.4	55.8	0.70	$2.07 \cdot 10^6$

The channels were marked out and measured before the tests; it was found that the cross sections of channels being tested were almost identical with those specified by OST 10017-39.

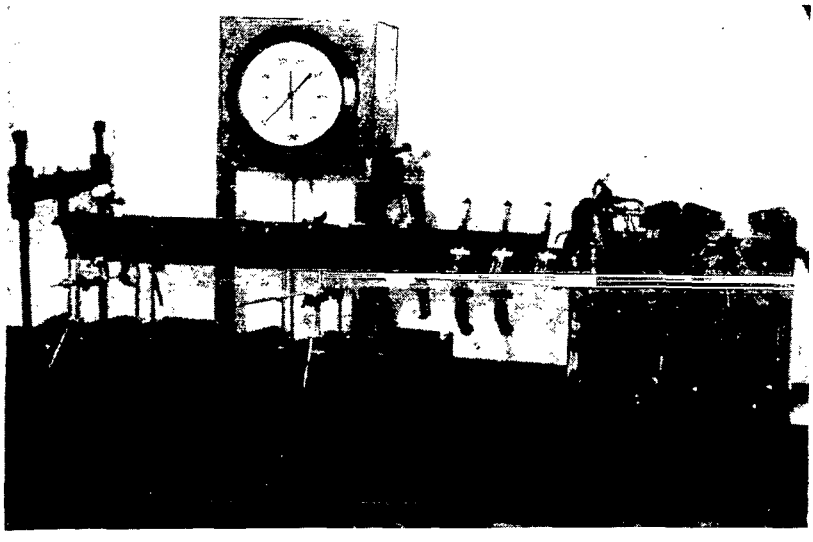

FIGURE 3.

The strains in the longitudinal fibers of the beams were measured by wire strain gages and tensometers; the angles of twist and deflection were measured by dial gages.

The gages measured the elongations in the cross section adjoining the point of embedment, where the stresses are highest. Twenty resistance gages were glued to the contour of the beam at a distance of 1.5 cm from the support. All the gages had a gage length of 10 mm and were 4 mm wide; they had six coils and their sensitivity was 2. The strains measured were recorded by two SD-1 instruments.

The tensometers were used to measure the strains in the outermost fibers of the flanges in several cross sections of the beam to determine the length of the yield zone along the span. Ten to twelve tensometers were fastened by single or paired screw clamps on a base of 10 mm.

The dial gages measured the angles of twist and deflection near the free end and in the center of the cantilever. Three dial gages were placed at each cross section; two were used to measure the transverse displacements of the webs to determine the angles of twist; the third was used to measure the deflection parallel to the web. Displacements normal to the plane of the web were measured by two additional dial gages which, due to lack of space, had to be placed 6 cm away from the previously mentioned cross sections. The ninth, control, dial gage was placed behind the rigid support at the beam end.

The initial load was 25 kg. The load was increased by steps of 100 or 50 kg in the elastic range, and of 50 or 25 kg in the elastic-plastic range. The load was indicated by the pressure gage. Loading was carried out in 12 to 17 steps, unloading in 6 to 8 steps. In all tests yielding occurred first in the upper part of the section. The limiting state of the beam was characterized by a noticeable increase in the deformation, extending over the entire cross section at the point of fixation.

The test results are given in tables and diagrams. In analyzing the experimental data we took into account the differences in the yield points of the flanges and the web.

The strains in channel sections in biaxial bending with torsion are not symmetrical; the upper flange and the web of beams 2-3 ($\alpha = 15°$) and 2-2 ($\alpha = 30°$) were both under tension and compression, whereas the lower flange was only under compression. The strains in both flanges of beam 2-1 ($\alpha = 90°$) were of different signs, while in the web they had the same sign. The strain diagrams are not linear.

The relative elongations of the fibers were used to determine the normal stresses in the section at the point of fixation and then at the flange edges in other cross sections. These data were obtained for individual fibers in the form of σ, P curves at all stages of loading and unloading, and also in the form of stress-distribution curves for the entire cross section at discrete loading stages.

Figure 4 shows the stresses at the point of fixation in the limiting state. With increasing angle between the y-axis and the line of action of the applied force, the range of elastic strains and the load-carrying capacity of the beams decrease.

Table 2 gives the theoretical and experimental limiting loads. The theoretical values were obtained from (10) on the assumption that the coordinates of the pole A, from which the sectorial areas are measured, are the same as in the elastic state. The experimental data are in satisfactory agreement with the theoretical values.

After unloading the residual stresses changed sign in the most highly stressed fibers (in channels 2-3 and 2-2 these are located at the junction between flange and web; in beam 2-1 they are at the flange edges).

The spreading of the yield zone over the flange at its free edge depends on the angle at which the external force is applied. The yield zones are most noticeable in beam 2-1 along the edges of both flanges.

We obtained the angles of twist and deflection as functions of the force P for all loading and unloading stages from dial-gage readings. We also obtained the distribution of deflections and angles of twist along the beam for individual loading stages (not reproduced here due to lack of space).

The vertical component of the deflection varies inversely with the angle at which the force acts, while its horizontal component varies

directly. The change in the angles of twist is a function of the eccentricity of the force plane relative to the pole A, and of the magnitude of the load. The maximum angle of twist, obtained for channel 2-3 (α = 15°), was 13°10'.

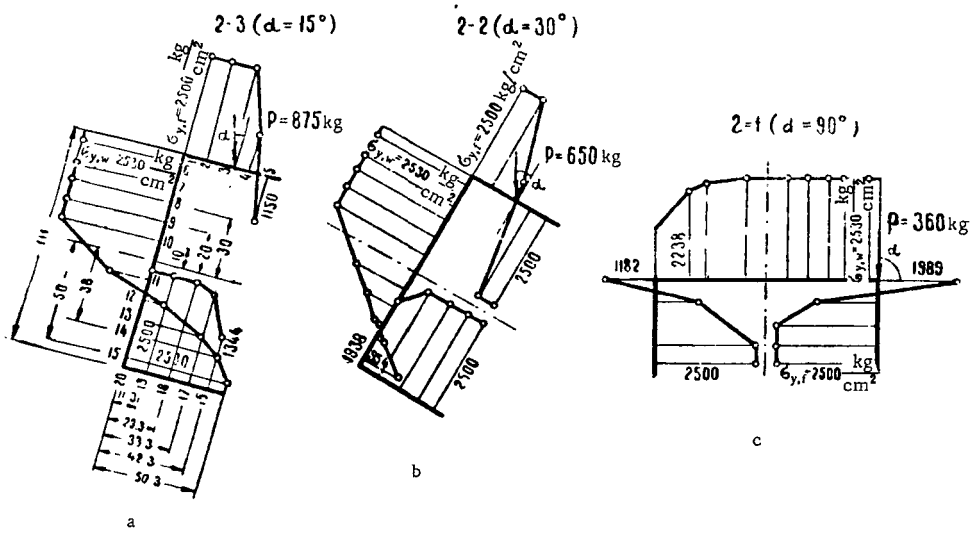

FIGURE 4.

After unloading residual stresses and twisting were observed.

TABLE 2

Channel	α°	l_p, cm	$P_{T\,th}$, kg	$P_{T\,ex}$, kg	Difference, %	$\frac{P_{T\,ex}}{P_{T\,th}}$
2—3	15	129.0	841	875	3.9	1.04
2—2	30	128.5	624	650	4.0	1.04
2—1	90	128.5	349	360	3.1	1.03

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NATURAL VIBRATIONS AND STABILITY OF RECTANGULAR PLATES OF VARIABLE THICKNESS

V.G. Tret'yak

(Khar'kov)

This paper attempts to extend Prof. Kan's method of shell calculation /2/ to the calculation of rectangular plates with thickness constant ($\delta = \delta_0$) or variable ($\delta = \delta_0 \cdot \delta(v)$) in one direction, and various boundary conditions (Figure 1). The problem is considered in a linear formulation. The plate material is assumed to be ideally elastic and homogeneous. The general hypotheses of the theory of thin shells are used in the study. The plate position in space is defined by means of the rectangular system of coordinates xyz. The displacements of the middle-surface points in the direction of the z-axis will be designated by w.

The results of the studies are reduced to simple computing formulas and curves, which are easy to use in actual designing.

1. NATURAL VIBRATIONS

In the process of natural vibrations the plate is loaded by inertia forces. The corresponding state of stress and strain is determined by calculating the plate as a multiply statically indeterminate system. As fundamental statically possible system we take a structure having the stressed state obtained in an ordinary thin-walled beam of constant cross section. In this case all the beamlike strips separated by planes perpendicular to the x-axis are equally deformed. The mode of vibrations of the strips $f(y)$ is easily determined from the calculation of the fundamental system.

Due to the existence of different boundary conditions at $x=0$, $x=L$, the strips obtained thus will be differently deformed. The conditions of strain compatibility will be satisfied, if we allow for the additional displacements of the middle-surface points w_{ad}. If it is assumed that the law of variation of the additional displacements is approximately the same as the law of variation of the fundamental system, we can write

$$w_{ad} = \Psi(x)_{ad} \cdot f(y) \cdot \sin \omega t,$$

where $\Psi(x)_{ad}$ is a statically indeterminate function; ω is the frequency of natural vibrations; t is time.

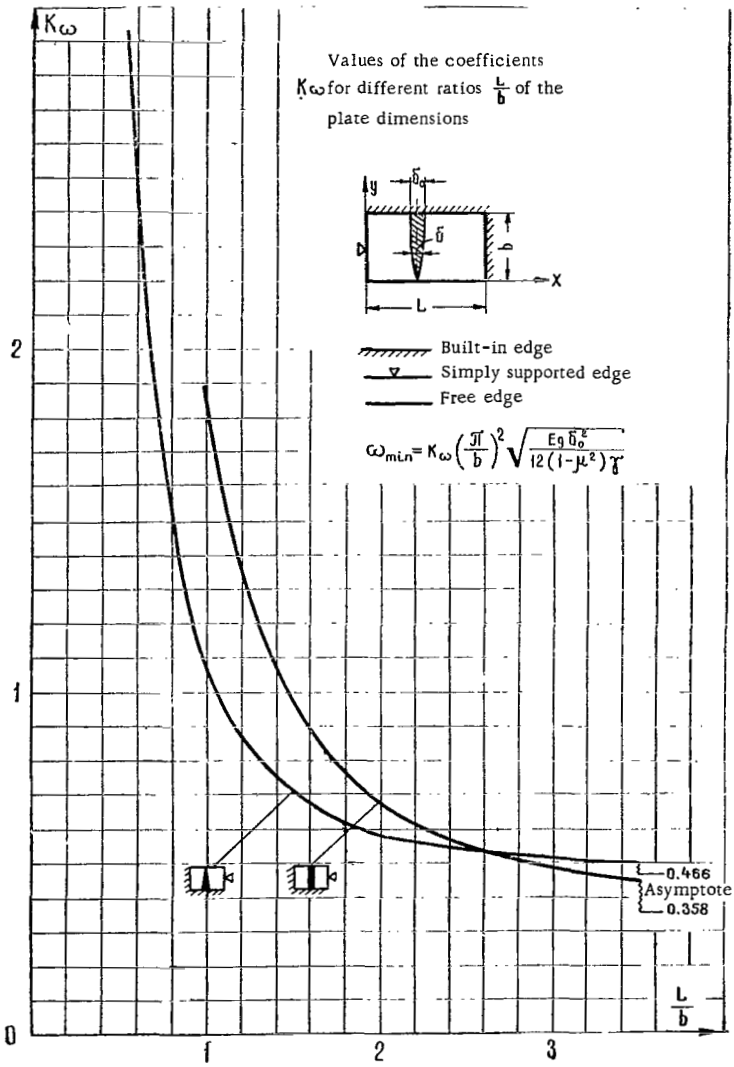

FIGURE 1.

By summing the displacements of the fundamental and additional systems we obtain

$$w = \Psi(x) \cdot f(y) \sin \omega t, \tag{1}$$

where the statically indeterminate function $\Psi(x)$ and the frequency ω are unknown. We shall determine them from the minimum condition of the potential energy of the system. The external forces are in our case the inertia forces of the masses; their value per unit plate length is

$$q = \frac{\gamma \cdot \delta}{g} \cdot \frac{\partial^2 w}{\partial t^2} = -\omega^2 \frac{\gamma \delta}{g} f(y) \cdot \Psi(x) \cdot \sin \omega t, \tag{2}$$

where γ is the unit weight of the plate material; g is the gravity acceleration; δ is the plate thickness.

On the strength of (2) and of d'Alembert's principle we can write the following expression for the potential energy of the system per unit length:

$$\Gamma=\int_0^b\left[\frac{m_x\varkappa_x}{2}+\frac{m_y\varkappa_y}{2}+m_{xy}\varkappa_{xy}-\frac{1}{2}\frac{\gamma\delta}{g}\left(-\frac{\partial^2 w}{\partial t^2}\right)w\right]dy, \tag{3}$$

where m_x, m_y, m_{xy} are bending and torsional moments per unit length; $\varkappa_x$, $\varkappa_y$, $\varkappa_{xy}$ are curvatures and relative angle of twist (m_n and $\varkappa_n$ are determined by the known formulas of the theory of plates and expressed through w).

Using Euler's variational-problem equation

$$\frac{\partial\Gamma}{\partial\Psi(x)}-\frac{d}{dx}\left(\frac{\partial\Gamma}{\partial\Psi'(x)}\right)+\frac{d^2}{dx^2}\left(\frac{\partial\Gamma}{\partial\Psi''(x)}\right)=0, \tag{4}$$

we obtain the resolving differential equation

$$\frac{d^4\Psi(x)}{dx^4}-2\nu^2\frac{d^2\Psi(x)}{dx^2}-\chi^4\Psi(x)=0, \tag{5}$$

where

$$2\nu^2=\frac{2(1-\mu)\int_0^b\bar{D}(y)\left[\frac{df(y)}{dy}\right]^2dy-2\mu\int_0^b\bar{D}(y)\frac{d^2f(y)}{dy^2}\cdot f(y)dy}{\int_0^b\bar{D}(y)\cdot f^2(y)\,dy}; \tag{6}$$

$$\chi^4=\frac{\frac{\gamma\delta_0\omega^2}{gD_0}\int_0^b\bar{\delta}(y)\cdot f^2(y)\,dy-\int_0^b\bar{D}(y)\left[\frac{d^2f(y)}{dy^2}\right]^2dy}{\int_0^b\bar{D}(y)\cdot f^2(y)\,dy}; \tag{7}$$

$$D(y)=\frac{E\delta^3}{12(1-\mu^2)}=\frac{E\delta_0^3}{12(1-\mu^2)}\cdot\bar{\delta}^3(y)=D_0\cdot\bar{D}(y); \tag{8}$$

μ is the Poisson ratio.

To determine $\Psi(x)$, consider the characteristic equation

$$\lambda^4-2\nu^2\lambda^4-\chi^4=0. \tag{9}$$

We obtain

$$\lambda_{1,2}=\pm\gamma_1;\quad \lambda_{3,4}=\pm i\gamma_2,$$

where

$$\gamma_1=\sqrt{\sqrt{\nu^4+\chi^4}+\nu^2}; \tag{10}$$

$$\gamma_2=\sqrt{\sqrt{\nu^4+\chi^4}-\nu^2}. \tag{11}$$

Since equation (9) has two real and two imaginary roots, by placing the

coordinate origin at the plate edge we obtain the solution of (5) in the form

$$\Psi(x) = C_1 \operatorname{ch}\gamma_1 x + C_2 \cos\gamma_2 x + C_3 \operatorname{sh}\gamma_1 x + C_4 \sin\gamma_2 x. \tag{12}$$

The determination of the constants C_1, C_2, C_3, C_4 from the boundary conditions $x=0$, $x=L$ yields a system of algebraic linear homogeneous equations. The condition of existence of natural vibrations is satisfied as known only if the determinant of this system is equal to zero. By equating the determinant of the system to zero and expanding it we obtain a transcendental equation which yields the relationship between $\gamma_1 L$ and $\gamma_2 L$.

From (10) and (11) we find

$$(\gamma_1 L)^2 - (\gamma_2 L)^2 = 2\nu^2 L^2; \tag{13}$$

$$(\gamma_1 L \cdot \gamma_2 L)^2 = \chi^4 L^4. \tag{14}$$

By multiplying and dividing the right-hand side of (13) by b^2 we obtain

$$\frac{L}{b} = \sqrt{\frac{(\gamma_1 L)^2 - (\gamma_2 L)^2}{2\nu^2 \cdot b^2}}. \tag{15}$$

By multiplying and dividing the right-hand side of (14) by b^4 and substituting the values of χ^4 from (7) and $\left(\frac{L}{b}\right)^4$ from (15) we obtain

$$\omega = K_\omega \cdot \left(\frac{\pi}{b}\right)^2 \sqrt{\frac{Eg\delta_0^2}{12\gamma(1-\mu^2)}}, \tag{16}$$

where

$$K_\omega = \frac{1}{\pi^2}\left\{\left[\frac{(2\nu^2 \cdot b^2)^2}{\left(\frac{\gamma_1 L}{\gamma_2 L} - \frac{\gamma_2 L}{\gamma_1 L}\right)^2} + \frac{b^4 \int_0^b \bar{D}(y)\left[\frac{d^2 f(y)}{dy^2}\right]^2 dy}{\int_0^b \bar{D}(y) \cdot f^2(y)\, dy}\right] \frac{\int_0^k \bar{D}(y) \cdot f^2(y)\, dy}{\int_0^b \bar{\delta}(y) f^2(y)\, dy}\right\}^{1/2}. \tag{17}$$

To simplify the calculations during design, we have plotted the curves $K_\omega = f\left(\frac{L}{b}\right)$ on the basis of (15), (17), and the solutions of the transcendental equations (see Figures 1 and 2). Figure 1 gives the relationships $K_\omega = f\left(\frac{L}{b}\right)$ for the frequency of the fundamental tone of plates with thickness constant $\delta = \delta_0$ or variable in one direction $\delta = \delta_0 \sqrt{\frac{y}{b}} = \delta_0 \bar{\delta}(y)$ with nonsymmetrical boundary conditions. The analysis of the curves shows that the frequency of the fundamental tone ω of short plates $\left(\frac{L}{b} < 2.6\right)$ of uniform thickness is higher than the frequency ω of plates of variable thickness. This is due to the fact that the rigidity of the longitudinal strips, which is higher in plates of uniform thickness, has a stronger influence in the case of short plates.

Figure 2 gives the computing formula and the curves $K_\omega = f\left(\frac{L}{b}\right)$ for the frequency of the fundamental tone of plates of uniform thickness. The examples which the author did not find in the literature are represented by rhombes. Warburton's results /4/ are represented by tiny circles. It is seen from the curve that the results of the proposed solution coincide practically with the known ones.

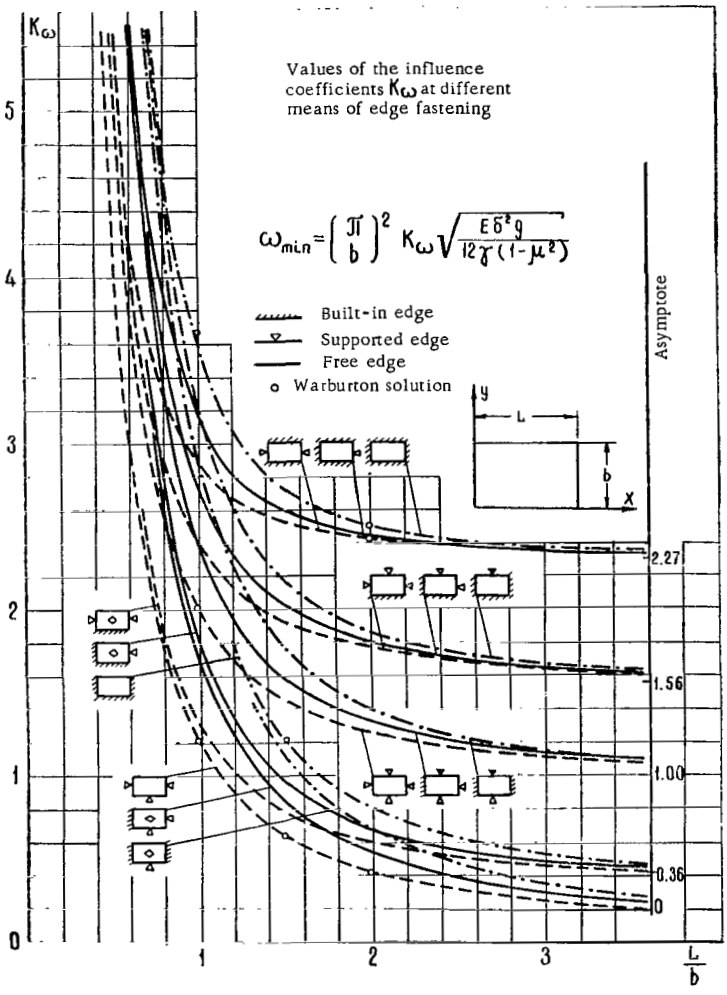

FIGURE 2.

The approach proposed for determining ω was also used in determining the frequencies of natural vibrations of plates loaded in the middle plane, and of plates supported on elastic ribs.

The problem solution is considerably simplified if the edges $x = 0$, $x = L$ are simply supported. We obtain in this case the simple transcendental equation

$$(\gamma_1^2 + \gamma_2^2)\cdot \mathrm{sh}\gamma_1 L \cdot \sin_2 L = 0. \tag{18}$$

Since $\gamma_1^2 + \gamma_2^2 \neq 0$, $\mathrm{sh}\gamma_1 L \neq 0$, then $\sin \gamma_2 L = 0$. This is possible at $\gamma_2 L = m\pi$ where $m = 1, 2, 3, \cdots$ is the number of half-waves along the plate. With the aid of (11) we obtain

$$\gamma_2^2 = \left(\frac{m\pi}{L}\right)^2 = \sqrt{\nu^4 + \chi^4} - \nu^2;$$

whence

$$\left(\frac{m\pi}{L}\right)^4 + 2\nu^2\left(\frac{m\pi}{L}\right)^2 - \chi^4 = 0. \tag{19}$$

By substituting χ^4 in (19) we find (16), where

$$K_\omega = \left\{\left[\left(\frac{m\pi}{L}\right)^4 + \frac{2\nu^2\cdot b^2}{\pi^2}\left(\frac{m\pi}{L}\right)^2 + \frac{\left(\frac{b}{\pi}\right)^4 \int\limits_0^b \bar{D}(y)\left[\frac{d^2 f(y)}{dy^2}\right]^2 dy}{\int\limits_0^b \bar{D}(y)\cdot f^2(y)\, dy}\right]\Bigg/\left[\frac{\int\limits_0^b \bar{D}(y) f^2(y) dy}{\int\limits_0^b \bar{\delta}(y) f^2(y)\, dy}\right]\right\}^{1/2}. \tag{20}$$

Thus, for the plate represented in Figure 1 $\left(\delta = \delta_0\sqrt{\frac{y}{b}}\right)$, but simply supported by the edges $x = 0$, $x = L$, the frequency of the fundamental tone is given by

$$K_\omega = \sqrt{\frac{1}{4.272}\left\{\left(\frac{b}{L}\right)^4 + \left(\frac{b}{L}\right)^2\left[\frac{22.966 - 32.536\,\mu}{\pi^2}\right] + \frac{90.226}{\pi^4}\right\}}. \tag{21}$$

If $\delta = \delta_0$ and all the edges of the plate are simply supported, (20) yields easily the Timoshenko formula /3/

$$K_{\omega_2} = \left(\frac{mb}{L}\right)^2 + n^2, \tag{22}$$

by writing $f(y) = \sin\frac{n\pi y}{b}$. Here $n = 1, 2, 3, \ldots$ is the number of half-waves across the plate (when comparing with the formula of /3/ it is necessary to take into account (16) and (22)).

2. STABILITY

The method exposed is easily applied to the calculation of the stability of plates with different boundary conditions. Then

$$w = \psi(x)\cdot f(y)$$

and the potential of the external forces in (3) must be altered by:

$-\frac{\sigma_{cr}\delta}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2$, if in the direction of the x-axis the plate is compressed in the middle plane by a uniformly distributed load $\sigma_{cr}\delta$;

$\frac{\sigma_{cr}\delta}{2}\left(\frac{\partial^2 w}{\partial y^2}\right)^2$, if in the direction of the y-axis the plate is compressed in the middle plane by a load of $\sigma_{cr}\cdot\delta$;

$\tau_{cr}\delta\ \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}$, if uniformly distributed shearing forces $\tau_{cr}\cdot\delta$ act along the edges.

Formulas (5), (10) to (14) will then be written, respectively, as follows:

$$\frac{d^4\Psi(x)}{dx^4}+2\nu^2\frac{d^2\Psi(x)}{dx^2}+\chi^4\Psi(x)=0; \tag{23}$$

$$\gamma_1=\sqrt{\nu^2+\sqrt{\nu^4-\chi^4}}; \tag{24}$$

$$\gamma_2=\sqrt{\nu^2-\sqrt{\nu^4-\chi^4}}; \tag{25}$$

$$\Psi(x)=C_1\cos\gamma_1 x+C_2\sin\gamma_1 x+C_3\cos\gamma_2 x+C_4\sin\gamma_2 x; \tag{26}$$

$$(\gamma_1 L)^2+(\gamma_2 L)^2=2\nu^2\cdot L^2, \tag{27}$$

$$(\gamma_1 L\cdot\gamma_2 L)^2=\chi^4\cdot L^4. \tag{28}$$

If the plate is compressed in the direction of the x-axis by a uniformly distributed load $\sigma_{cr}\cdot\delta$, then

$$2\nu^2=\frac{1}{\int_0^b\bar{D}(y)\cdot f(y)dy}\left[\frac{\sigma_{cr}\cdot\delta}{D_0}\int_0^b\bar{\delta}(y)\cdot f^2(y)dy-2(1-\mu)\int_0^b\bar{D}(y)\left(\frac{df(y)}{dy}\right)^2dy+\right.$$

$$\left.+2\mu\int_0^b\bar{D}(y)\frac{d^2f(y)}{dy^2}f(y)dy\right], \tag{29}$$

$$\chi^4=\frac{\int_0^b\bar{D}(y)\left[\frac{d^2f(y)}{dy^2}\right]^2\cdot dy}{\int_0^b\bar{D}(y)\cdot f^2(y)dy}. \tag{30}$$

Using (27), (28), (29), and (30), we obtain

$$\sigma_{cr}=\frac{\pi^2 E}{12(1-\mu^2)\left(\frac{b}{\delta_0}\right)^2}\cdot K, \tag{31}$$

$$\frac{L}{B}=\frac{\sqrt{\gamma_1 L\cdot\gamma_2 L}}{\sqrt[4]{\chi^4 b^4}}, \tag{32}$$

where K is the stability coefficient, determined from the expression

$$K=\frac{1}{\pi^2}\left[\left(\frac{\gamma_1 L}{\gamma_2 L}+\frac{\gamma_2 L}{\gamma_1 L}\right)\sqrt{\chi^4 b^4}+\bar{A}\right]\cdot\bar{B}; \tag{33}$$

where

$$\bar{A}=\frac{2(1-\mu)b^2\int_0^b \bar{D}(y)\left(\frac{df(y)}{dy}\right)^2 dy-2\mu b^2\int_0^b \bar{D}(y)\frac{d^2f(y)}{dy^2}f(y)dy}{\int_0^b \bar{D}(y)\cdot f^2(y)\,dy}; \qquad (34)$$

$$\bar{B}=\frac{\int_0^b \bar{D}(y)\cdot f^2(y)dy}{\int_0^b \bar{\delta}(y)\,f^2(y)dy}. \qquad (35)$$

Using the solutions of the transcendental equations (32) and (33), it is easy to plot the curves $K=f\left(\frac{L}{b}\right)$. Such curves for plates of uniform thickness are given in Figure 3. The solutions represented by rhombes have not been met by the author in the literature. The problem designated "H", in /1/, was solved incorrectly there. Since at $\frac{L}{b}\to\infty$ the value of K is independent of the boundary conditions at $x=0$, $x=L$, then for "H" $K_{\frac{L}{b}=\infty}=0.426$.

Formulas (32) and (33) simplify if the edges $x=0$, $x=L$ are simply supported. In this case the transcendental equation reduces to

$$(\gamma_1^2-\gamma_2^2)\cdot\sin\gamma_1 L\cdot\sin\gamma_2 L=0, \qquad (36)$$

whence $\gamma_1 L=m\pi$, $\gamma_2 L=m_1\pi$, where m and m_1 are positive integers. Using (24) or (25), we find

$$\left(\frac{m\pi}{L}\right)^5-2\nu^2\left(\frac{m\pi}{L}\right)^2+\chi^4=0. \qquad (37)$$

From (37) we obtain (31), where

$$K=\bar{B}\left[\left(\frac{mb}{L}\right)^2+\frac{\bar{A}}{\pi^2}+\frac{\chi^4 b^4}{\pi^4\left(\frac{mb}{L}\right)^2}\right]. \qquad (38)$$

In the case of a plate compressed in the direction of the x-axis, of thickness varying according to the law $\delta=\delta_0\sqrt{\frac{y}{b}}$, and having the side $y=b$ clamped, the side $y=0$, free, and the other sides simply supported,

$$K=\frac{1}{4.272}\left[\left(\frac{mb}{L}\right)^2+\frac{22.966-32536\mu}{\pi^2}+\frac{90.226}{\pi^4\left(\frac{mb}{L}\right)^2}\right]. \qquad (39)$$

At $\mu=0.3$ $K_{min}=0.764$.

The first mode of vibration of beams of uniform thickness was taken as $f(y)$ in plotting the curves of Figures 2 and 3. In the derivation of formulas (21) and (39) and the plotting of the curve of Figure 1, $f(y)$ was determined in a first approximation from the calculation of a strip of variable thickness loaded by a uniformly distributed load.

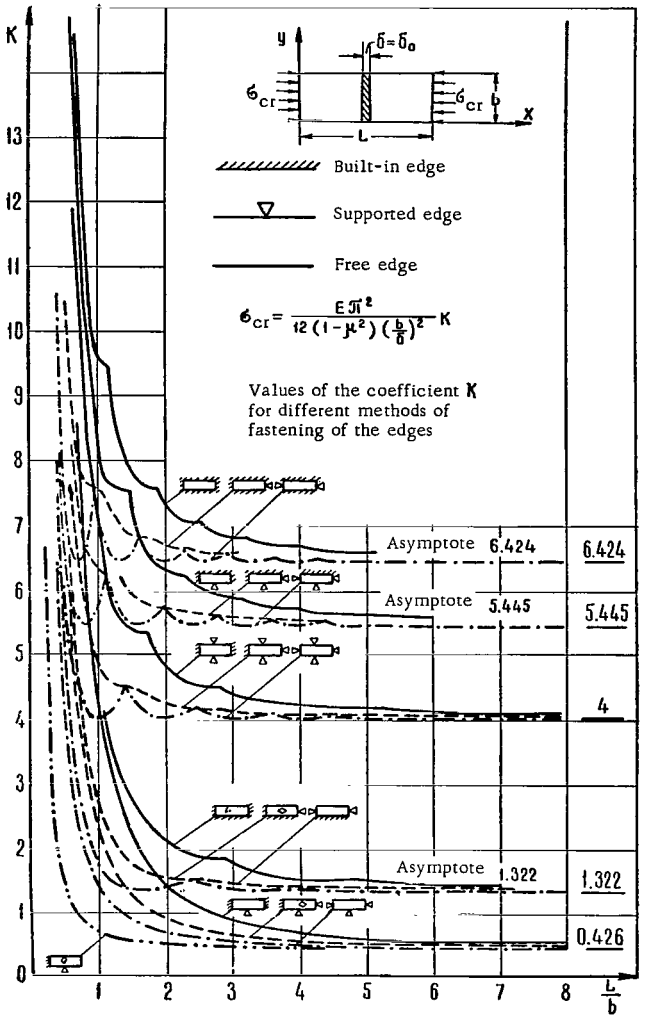
K
y
δ=δ₀
σcr
σcr
b
x
L
Built-in edge
Supported edge
Free edge
σcr = Eπ² / 12(1-μ²)(b/δ)² K
Values of the coefficient K for different methods of fastening of the edges
Asymptote 6.424
Asymptote 5.445
Asymptote 1.322
6.424
5.445
4
1.322
0.426
L/b

FIGURE 3.

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THERMAL STRESSES IN AN ANISOTROPIC CIRCULAR CYLINDRICAL VAULT

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(Saratov)

The two-dimensional problem of thermoelasticity of an infinite cylindrical vault of orthotropic elastic and thermal properties is considered. The stresses appearing in the vault under the influence of the temperature field are determined. The problem of the bending of a curved bar with cylindrical anisotropy under the action of a load was solved in Lekhnitskii's paper /1/.

§ 1.

The vault represents a part of an infinite hollow circular cylinder limited by two axial sections forming an angle of 2α. The vault material is subjected to the generalized Hooke law and possesses cylindrical anisotropy of a particular type, i.e., is orthotropic; the anisotropy axis coincides with the cylinder axis. The elastic and thermal characteristics of the material are independent of the temperature (the range of variation of the temperature is narrow). The inertia forces and the volume forces are not taken into account.

The deformations and rotations are assumed to be small.

The temperature function $T(r,\theta,t)$ and the stress function $F(r,\theta,t)$ satisfy, in a cylindrical system of coordinates, the following differential equations /3/:

$$k_{11}\cdot\left(\frac{1}{r}\cdot\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}\right)+k_{22}\cdot\frac{1}{r^2}\cdot\frac{\partial^2 T}{\partial\theta^2}=\frac{\partial T}{\partial t}, \tag{1.1}$$

$$\beta_{22}\frac{\partial^4 F}{\partial r^4}+(2\beta_{12}+\beta_{66})\frac{1}{r^2}\frac{\partial^4 F}{\partial r^2\partial\theta^2}+\beta_{11}\frac{1}{r^4}\frac{\partial^4 F}{\partial\theta^4}+2\beta_{22}\frac{1}{r}\frac{\partial^3 F}{\partial r^3}-$$
$$-(2\beta_{12}+\beta_{66})\frac{1}{r^3}\cdot\frac{\partial^3 F}{\partial r\partial\theta^2}+(2\beta_{11}+2\beta_{12}+\beta_{66})\frac{1}{r^4}\cdot\frac{\partial^2 F}{\partial\theta^2}-\beta_{11}\frac{1}{r^2}\frac{\partial^2 F}{\partial r^2}+$$
$$+\beta_{11}\frac{1}{r^3}\cdot\frac{\partial F}{\partial r}=(\beta_1-2\beta_2)\cdot\frac{1}{r}\frac{\partial T}{\partial r}-\beta_1\frac{1}{r^2}\cdot\frac{\partial^2 T}{\partial\theta^2}-\beta_2\cdot\frac{\partial^2 T}{\partial r^2}, \tag{1.2}$$

where

$$\beta_i=\alpha_i-\frac{\alpha_3\alpha_{13}}{\alpha_{33}}\qquad(i=1,2). \tag{1.3}$$

α_1, α_2, α_3 are temperature coefficients of linear expansion of the material in directions r, θ, and z; k_{11}, k_{22} are diffusivities in the r and θ directions; t is time. The remaining symbols are generally accepted /1/.

The stress components are expressed through the stress function F by the well-known formulas:

$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 F}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial^2}{\partial r \partial \theta}\left(\frac{F}{r}\right), \tag{1.4}$$

$$\sigma_z = -\frac{1}{a_{33}}(a_{13}\sigma_r + a_{23}\sigma_\theta + \alpha_3 T). \tag{1.5}$$

Let the boundary surface of the vault be free of the action of an external load:

$$\sigma_r = \tau_{r\theta} = 0 \quad \text{at } r = a \text{ and } r = b, \tag{1.6}$$

$$\int_a^b \sigma_\theta \, dr = 0, \quad \int_a^b \sigma_\theta \, r dr = 0, \quad \int_a^b \tau_{r\theta} \, dr = 0 \tag{1.7}$$

at $\theta = \alpha$ and $\theta = 2\alpha$.

Here a and b are the internal and external radii of the cylinder cross section.

The conditions on the vault ends (1.7) are presented in integral form.

§ 2.

We shall now determine the law of distribution of the temperature stresses for the following two cases of boundary conditions.

I. A constant temperature T_1 exists on the internal surface of the vault and at the ends, while the temperature on the external surface varies in the tangential direction according to a law $\varphi(\theta)$:

$$T = T_1 \text{ at } r = a, \; T = \varphi(\theta) \text{ at } r = b, \; T = T_1 \text{ at } \theta = 0 \text{ and } \theta = 2\alpha. \tag{2.1}$$

II. The internal surface of the vault is thermally insulated, on the external surface the temperature varies in the tangential direction according to the law $\varphi(\theta)$, and a constant temperature T_1 is maintained at the ends:

$$\frac{\partial T}{\partial r} = 0 \text{ at } r = a, \; T = \varphi(\theta) \text{ at } r = b, \; T = T_1 \text{ at } \theta = 0 \text{ and } \theta = 2\alpha. \tag{2.2}$$

In the given cases the functions T and F are independent of the time t. The solution of the equation of heat conduction (1.1) at conditions (2.1) or (2.2) is

$$T(r,\theta) = T_1 + \sum_{n=1}^{\infty} A_n \cdot \left(\frac{b}{r}\right)^{\nu_n \cdot s} \cdot \left(\frac{r^{2\nu_n \cdot s} \mp a^{2\nu_n \cdot s}}{b^{2\cdot\nu_n \cdot s} \mp a^{2\nu_n \cdot s}}\right) \cdot \sin \nu_n \theta, \tag{2.3}$$

where

$$s = \sqrt{\frac{k_{22}}{k_{11}}}, \quad \nu_n = \frac{n\pi}{2\alpha}, \quad A_n = \frac{1}{\alpha}\int_0^{2\alpha} [\varphi(\theta) - T_1] \cdot \sin \nu_n \theta \cdot d\theta \quad (n = 1, 2, 3, \cdots). \tag{2.4}$$

In formula (2.3) and in what follows the upper sign corresponds to the case of boundary conditions (2.1), and the lower sign to the case of boundary conditions (2.2).

By integrating equation (1.2) at $\alpha \neq \frac{\pi}{2}$ we obtain

$$F(r,\theta)=\sum_{n=1}^{\infty}\Big\{N_{1n}\cdot r^{1+\gamma_n}+N_{2n}\cdot r^{1-\gamma_n}+N_{3n}\cdot r^{1+\sigma_n}+N_{4n}\cdot r^{1-\sigma_n}+$$

$$+l_n\cdot r^{2+\nu_n\cdot s}\mp l_{-n}\cdot r^{2-\nu_n\cdot s}\Big\}\sin\nu_n\theta. \tag{2.5}$$

Here N_{in} are arbitrary integration constants,

$$l_n=\frac{D_n\cdot[\beta_1\nu_n(\nu_n+s)-\beta_2\cdot\nu_n s(1+\nu_n s)]}{(1+\nu_n\cdot s)^2\cdot\{\beta_{22}(1+\nu_n\cdot s)^2-[\beta_{22}+\beta_{11}+(2\beta_{12}+\beta_{66})\cdot\nu_n^2]\}+\beta_{11}(\nu_n^2-1)^2},$$

$$l_{-n}=\frac{D_n\cdot a^{2\nu_n s}[\beta_1\nu_n(\nu_n-s)-\beta_2\cdot\nu_n\cdot s\cdot(\nu_n\cdot s-1)]}{(1-\nu_n\cdot s)^2\cdot\{\beta_{22}(1-\nu_n s)-[\beta_{22}+\beta_{11}+(2\beta_{12}+\beta_{66})\cdot\nu_n^2]\}+\beta_{11}(\nu_n^2-1)^2},$$

$$D_n=\frac{A_n\cdot b^{\nu_n\cdot s}}{b^{2\nu_n s}\mp a^{2\nu_n s}}, \tag{2.6}$$

$$\gamma_n=\sqrt{\frac{\beta_{22}+\beta_{11}+2\beta_{12}\cdot\nu_n^2+\beta_{66}\nu_n^2}{2\beta_{22}}}+\sqrt{\left(\frac{\beta_{22}+\beta_{11}+2\beta_{12}\cdot\nu_n^2+\beta_{66}\nu_n^2}{2\beta_{22}}\right)^2-\frac{\beta_{11}}{\beta_{22}}(\nu_n^2-1)^2}.$$

The expression for the constant σ_n differs from γ_n only by the minus sign before the second root.

The stress components are, on the strength of (1.4) and (2.5):

$$\sigma_r=\sum_{n=1}^{\infty}[N_{1n}\cdot(1+\gamma_n-\nu_n^2)\cdot r^{\gamma_n-1}+N_{2n}\cdot(1-\gamma_n-\nu_n^2)\cdot r^{-\gamma_n-1}+$$

$$+N_{3n}\cdot(1+\sigma_n-\nu_n^2)\,r^{\sigma_n-1}+N_{4n}\cdot(1-\sigma_n-\nu_n^2)\cdot r^{-\sigma_n-1}+$$

$$+l_n\cdot(2+\nu_n\cdot s-\nu_n^2)\cdot r^{\nu_n s}\mp l_{-n}\cdot(2-\nu_n\cdot s-\nu_n^2)\cdot r^{-\nu_n s}]\sin\nu_n\theta,$$

$$\sigma_\theta=\sum_{n=1}^{\infty}[N_{1n}(1+\gamma_n)\cdot\gamma_n r^{\gamma_n-1}-N_{2n}(1-\gamma_n)\cdot\gamma_n\cdot r^{-\gamma_n-1}+N_{3n}\cdot(1+ \tag{2.7}$$

$$+\sigma_n)\cdot\sigma_n\cdot r^{\sigma_n-1}-N_{4n}\cdot(1-\sigma_n)\cdot\sigma_n\cdot r^{-\sigma_n-1}+l_n\cdot(2+\nu_n\cdot s)(1+\nu_n\cdot s)\cdot r^{\nu_n\cdot s}\mp$$

$$\mp l_{-n}\cdot(2-\nu_n\cdot s)(1-\nu_n\cdot s)\cdot r^{-\nu_n s}]\sin\nu_n\theta,$$

$$\tau_{r\theta}=-\sum_{n=1}^{\infty}[N_{1n}\cdot\gamma_n\cdot r^{\gamma_n-1}-N_{2n}\cdot\gamma_n r^{-\gamma_n-1}+N_{3n}\cdot\sigma_n\cdot r^{\sigma_n-1}-$$

$$-N_{4n}\,\sigma_n\cdot r^{-\sigma_n-1}+l_n(1+\nu_n\cdot s)\cdot r^{\nu_n s}\mp l_{-n}\cdot(1-\nu_n\cdot s)\cdot r^{-\nu_n s}]\nu\cdot\cos\nu_n\theta.$$

The constants N_{in} are determined from conditions (1.6):

$$N_{1n}=\frac{1}{\gamma_n}[N_{2n}\cdot\gamma_n\cdot b^{-2\gamma_n}-N_{3n}\cdot\sigma_n b^{\sigma_n-\gamma_n}+N_{4n}\cdot\sigma_n\cdot b^{-\sigma_n-\gamma_n}-$$

$$-l_n\cdot\omega_n\cdot b^{\omega_n-\gamma_n}\pm l_{-n}\cdot\varkappa_n\cdot b^{\varkappa_n-\gamma_n}],$$

$$N_{2n}=\frac{\Delta_{2n}}{\Delta_n},\qquad N_{3n}=\frac{\Delta_{3n}}{\Delta_n},\qquad N_{4n}=\frac{\Delta_{4n}}{\Delta_n}, \tag{2.8}$$

$$\Delta_{2n}=l_n\cdot[(a^{\omega_n-\sigma_n}\cdot b^{\gamma_n+\sigma_n}+a^{\gamma_n+\sigma_n}b^{\omega_n-\sigma_n})(\gamma_n-\sigma_n)(\omega_n+\sigma_n)-$$

$$-(a^{\gamma_n-\sigma_n}b^{\omega_n+\sigma_n}+a^{\omega_n+\sigma_n}\cdot b^{\gamma_n-\sigma_n})(\gamma_n+\sigma_n)(\omega_n-\sigma_n)+$$

$$+2(a^{\omega_n+\gamma_n}+b^{\omega_n+\gamma_n})\cdot(\omega_n-\gamma_n)\cdot\sigma_n]\mp l_{-n}\cdot[\cdots]_{2n},$$

$$\Delta_{3n} = l_n[(a^{\omega_n+\gamma_n}\cdot b^{-(\sigma_n+\gamma_n)} + a^{-(\sigma_n+\gamma_n)} b^{\omega_n+\gamma_n})\cdot(\gamma_n-\sigma_n)(\omega_n-\gamma_n)+$$
$$+(a^{\omega_n-\gamma_n} b^{\gamma_n-\sigma_n} + a^{\gamma_n-\sigma_n}\cdot b^{\omega_n-\gamma_n})\cdot(\gamma_n+\sigma_n)(\omega_n+\gamma_n)-$$
$$-2(a^{\omega_n-\sigma_n}+b^{\omega_n-\sigma_n})\cdot(\omega_n+\sigma_n)\cdot\gamma_n] \mp l_{-n}\cdot[\cdots]_{3n}\,.$$
$$\Delta_{4n} = l_n\cdot[(a^{\sigma_n+\gamma_n}\cdot b^{\omega_n-\gamma_n} + a^{\omega_n-\gamma_n}\cdot b^{\sigma_n+\gamma_n})(\sigma_n-\gamma_n)(\omega_n+\gamma_n)-$$
$$-(a^{\sigma_n-\gamma_n}\cdot b^{\omega_n+\gamma_n} + a^{\omega_n+\gamma_n}\cdot b^{\sigma_n-\gamma_n})\cdot(\sigma_n+\gamma_n)(\omega_n-\gamma_n)+ \qquad (2.8)$$
$$+2(a^{\omega_n+\sigma_n}+b^{\omega_n+\sigma_n})\cdot(\omega_n-\sigma_n)\cdot\gamma_n] \mp l_{-n}[\cdots]_{4n}\,, \qquad \text{(cont'd)}$$
$$\Delta_n = 8\cdot\gamma_n\cdot\sigma_n + (\gamma_n-\sigma_n)^2\cdot(a^{2(\gamma_n+\sigma_n)}+b^{2(\gamma_n+\sigma_n)})\cdot a^{-(\gamma_n+\sigma_n)}\cdot b^{-(\gamma_n+\sigma_n)}-$$
$$-(\gamma_n+\sigma_n)^2\cdot(a^{2(\sigma_n-\gamma_n)}+b^{2(\sigma_n-\gamma_n)}\cdot a^{\gamma_n-\sigma_n}\cdot b^{\gamma_n-\sigma_n}\,,$$
$$\omega_n = 1+\nu_n\cdot s, \qquad \varkappa_k = 1-\nu_n\cdot s.$$

In the formula for Δ_{2n} the factor following l_{-n}, designated by $[\cdots]_{2n}$, differs from the factor following l_n only in that ω_n is replaced by $\varkappa_n$. The same is true relative to the formulas for Δ_{3n} and Δ_{4n}. It is easily proved that the conditions at the vault ends (1.7) are identically satisfied.

If the vault is defined by the angle $\alpha=\frac{\pi}{2}$, then, as follows from formula (2.4), $\nu_n = n$ $(n = 1, 2, \cdots)$. At $n = 1$ the constants γ_n and σ_n are equal to

$$\gamma_1 = \sqrt{\frac{\beta_{22}+\beta_{11}+2\beta_{12}+\beta_{66}}{\beta_{22}}}\,, \quad \sigma_1 = 0 \qquad (2.9)$$

The first term of sum (2.5) for $F(r, \theta)$ will then be equal to

$$\left\{\frac{N_{11}}{\gamma_1}\cdot r^{1+\gamma_1} + \frac{N_{21}}{\gamma_1}\cdot r^{1-\gamma_1} + N_{31}\cdot r\cdot\ln r + N_{41} r + l_1\cdot r^{2+s} \mp l_{-1}\cdot r^{2-s}\right\}\cdot\sin\theta. \qquad (2.10)$$

The following terms of sum (2.5) remain unaltered. The constant N_{41} has no influence on the stressed state; N_{11}, N_{21}, N_{31} are determined from the boundary conditions (1.6) and (1.7).

§ 3.

Consider the case when a constant temperature T_1 is maintained on the internal surface of the vault, and a constant temperature T_2 on the external surface, and the vault ends are heat-insulated:

$$T = T_1 \text{ at } r = a, \quad T = T_2 \text{ at } r = b. \qquad (3.1)$$

The temperature distribution in the body is defined by the function

$$T(r) = A_0\cdot\ln r + B_0, \qquad (3.2)$$

where

$$A_0 = \frac{T_2 - T_1}{\ln b - \ln a}\,, \quad B_0 = \frac{T_1\ln b - T_2\ln a}{\ln b - \ln a}\,. \qquad (3.3)$$

The formulas for the stress function and the stress components are

$$F(r) = \frac{M_1}{1+k}\cdot r^{1+k} + \frac{M_2}{1-k}\cdot r^{1-k} + \frac{M_3}{2}\cdot r^2 + M_4 - Ar^2\ln r; \qquad (3.4)$$

$$\sigma_r = M_1\cdot(r^{k-1}-a^{k-1})+M_2\cdot(r^{-(k+1)}-a^{-(k+1)})-2A\cdot(\ln r-\ln a),$$
$$\sigma_\theta = M_1\cdot(k\cdot r^{k-1}-a^{k-1})-M_2\cdot(kr^{-(k+1)}-a^{-(k+1)})-2\cdot A\cdot[(\ln r-\ln a)+1], \quad (3.5)$$
$$\tau_{r\theta}=0,$$

where

$$k=\sqrt{\frac{\beta_{11}}{\beta_{22}}}, \quad A=\frac{A_0}{2}\left(\frac{\beta_1-\beta_2}{\beta_{11}-\beta_{22}}\right). \quad (3.6)$$

By using the conditions at the lateral surfaces of the vault (1.6) and at the ends (1.7), we determine the arbitrary integration constants

$$M_1=\frac{A}{\Delta}\cdot[2\cdot(1+k)^2\cdot(b^{1-k}-a^{1-k})\cdot(\ln b-\ln a)+$$
$$+(1-k^2)\cdot(b^{-(k+1)}-a^{-(k+1)})(b^2-a^2)],$$
$$M_2=\frac{-A}{\Delta}[2\cdot(1-k)^2\cdot(b^{1+k}-a^{1+k})\cdot(\ln b-\ln a)+$$
$$+(1-k^2)(b^{k-1}-a^{k-1})(b^2-a^2)],$$
$$\Delta=(1+k)^2\cdot(b^{1-k}-a^{1-k})(b^{k-1}-a^{k-1})-$$
$$-(1-k)^2(b^{k+1}-a^{k+1})(b^{-(k+1)}-a^{-(k+1)}). \quad (3.7)$$

Example. The vault is made of concrete. Its elastic constants are equal to /4/

$$a_{11}=\frac{1}{E}, \quad a_{22}=\frac{1}{2E}, \quad a_{33}=\frac{3}{4E}, \quad a_{12}=a_{23}=-\frac{1}{10\cdot E}, \quad a_{13}=-\frac{3}{20E},$$

where E is Young's modulus in the radial direction /1/.

The constant a_{66} is taken by us as equal to $\frac{56}{25\cdot E}$; $\alpha_1=\alpha_2=\alpha_3=\alpha$, $a=240$ cm, $b=300$ cm.

The numerical values of the dimensionless magnitude $\frac{100\cdot\sigma_\theta}{(T_2-T_1)E\alpha}$ at different points of the vault are given in the table. The stress component σ_r is very small compared with σ_θ.

TABLE

r cm	240	250	260	270	280	290	300
$\frac{100\cdot\sigma_\theta}{(T_2-T_1)\cdot E\cdot\alpha}$	0.30	0.05	−0.11	−0.12	−0.08	0.04	0.23

§ 4.

Consider a vault in a nonstationary stressed state. This state appears under the following conditions. At the initial moment the temperature field is defined by the function $f(r,\theta)$, and at the subsequent moments the specified constant temperature is maintained at the vault boundary:

$$T=f(r,\theta) \text{ at } t=0. \quad (4.1)$$

$$T = T_1 \text{ at } r = a,\ T = T_2 \text{ at } r = b,\ T = T_1 \text{ at } \theta = 0 \text{ and } \theta = 2\alpha. \tag{4.2}$$

We shall look for a temperature function in the form

$$T = T_s(r, \theta) + T_t(r, \theta, t), \tag{4.3}$$

where T_s is the stationary temperature and T_t is the deviation from the stationary temperature. The function T_s satisfies the homogeneous equation corresponding to (1.1) at boundary conditions

$$T_s = T_1 \text{ at } r = a,\ T_s = T_2 \text{ at } r = b,\ T_s = T_1 \text{ at } \theta = 0 \text{ and } \theta = 2\alpha. \tag{4.4}$$

This function is determined in the same way as in the first considered case of Section 2.

The function T_t satisfies equation (1.1), the initial condition

$$T_t = f(r, \theta) - T_s(r, \theta) \quad \text{at } t = 0 \tag{4.5}$$

and the homogeneous boundary conditions

$$T_t = 0 \text{ at } r = a \text{ and } r = b,\ T_t = 0 \text{ at } \theta = 0 \text{ and } \theta = 2\alpha. \tag{4.6}$$

The integration of the heat conduction equation (1.1) at conditions (4.5) and (4.6) yields

$$T_t(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} A_{ni} \cdot Z_m(\lambda_{ni} \cdot r) \cdot \sin \nu_n \theta \cdot e^{-\lambda_{ni}^2 k_{11} \cdot t}. \tag{4.7}$$

The function $Z_m(\lambda_{ni} r)$ here is determined by the formula

$$Z_m(\lambda_{ni} r) = J_m(\lambda_{ni} \cdot r) \cdot V_m(\lambda_{ni} a) - V_m(\lambda_{ni} r) \cdot J_m(\lambda_{ni} a); \tag{4.8}$$

$J_m(\lambda_{ni} r)$ is a cylindrical function of the first kind of order m;
$V_m(\lambda_{ni} r)$ is a cylindrical function of the second kind of order m;

$$m = \nu_n \sqrt{\frac{k_{22}}{k_{11}}} = \frac{n \cdot \pi}{2\alpha} \sqrt{\frac{k_{22}}{k_{11}}} \quad (n = 1, 2, 3, \cdots). \tag{4.9}$$

At fixed n the magnitudes $\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \cdots$ are positive roots of the equation

$$Z_m(\lambda_{ni} b) = 0. \tag{4.10}$$

The constants A_{ni} are the Fourier coefficients of the expansion of the function $f(r, \theta) - T_s(r, \theta)$ in a double series by the eigenfunctions $Z_m(\lambda_{ni} r) \cdot \sin \nu_n \theta$, and are determined by the formulas

$$A_{ni} = \frac{2}{\alpha} \cdot \frac{\int_a^b \int_0^{2\alpha} [f(r, \theta) - T_s(r, \theta)] \cdot Z_m(\lambda_{ni} r) \cdot \sin \nu_n \theta \cdot r \cdot dr \cdot d\theta}{b^2 \cdot [Z_m'(\lambda_{ni} b)]^2 - a^2 \cdot [Z_m'(\lambda_{ni} a)]^2} \tag{4.11}$$

It will be assumed that this expansion is possible.

The nonstationary stressed state at $\alpha \neq \frac{\pi}{2}$ is determined by the function F, which is equal to

$$F = F_s(r, \theta) + F_t(r, \theta, t). \tag{4.12}$$

The function F_s is found by formula (2.5). The function F_t has the

following form (cf. /2/, pp. 135-136):

$$F_t = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \{[M_{nl} + C_n \cdot \int \psi_{nl} \cdot r^{-(\gamma_n+2)} \cdot dr] \cdot r^{1+\gamma_n} + [N_{nl} -$$

$$- C_n \int \psi_{nl}\, r^{\gamma_n - 2} \cdot dr] \cdot r^{1-\gamma_n} + [K_{nl} + D_n \cdot \int \psi_{nl} \cdot r^{-(\sigma_n+2)} \cdot dr] \cdot r^{1+\sigma_n} +$$

$$+ [L_{nl} - D_n \int \psi_{nl} \cdot r^{\sigma_n - 2} \cdot dr] \cdot r^{1-\sigma_n} \} \sin \nu_n \theta \cdot e^{-\lambda_{nl}^2 \cdot k_{11} \cdot t}, \tag{4.13}$$

where

$$\psi_{nl} = A_{nl} \left\{ (\beta_1 - \beta_2) \cdot r^3 \cdot \frac{g Z_m(\lambda_{nl} r)}{dr} + \left[\beta_2 \lambda_{nl}^2 + \right.\right.$$

$$\left.\left. + \left(\beta_1 - \beta_2 \cdot \frac{k_{22}}{k_{11}}\right) \frac{\nu_n^2}{r^2} \right] \cdot r^4 \cdot Z_m(\lambda_{nl} r) \right\}, \tag{4.14}$$

$$C_n = \frac{1}{2 \cdot \gamma_n \cdot (\gamma_n^2 - \sigma_n^2) \cdot \beta_{22}}, \qquad D_n = \frac{1}{2\sigma_n (\sigma_n^2 - \gamma_n^2) \cdot \beta_{22}}.$$

The arbitrary integration constants M_{nl}, N_{nl}, K_{nl}, L_{nl} can be determined from conditions (1.6) with the aid of formulas (1.4) and (4.13).

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SOME PROBLEMS OF THE EQUILIBRIUM OF ORTHOTROPIC CYLINDRICAL SHELLS AT LARGE DEFORMATIONS

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Similar problems for an isotropic material at nonlinear stress-strain relationship were solved in /1, 2/. This paper, just as /8/, is devoted to the study of the membrane state of stress of orthotropic shells with rigid bottoms at a linear stress-true strain relationship.

It is assumed that the shell material allows very large deformations. The standard assumptions of the membrane theory of shells are used.

1. INITIAL RELATIONSHIPS AND EQUATIONS

Consider a cylindrical (before the deformation) shell subjected to the action of an internal pressure q and of forces F applied to the centers of the rigid bottoms (figure). We shall represent it in the cylindrical frame of coordinates $r\theta Y$ fixed to one of the bottoms.

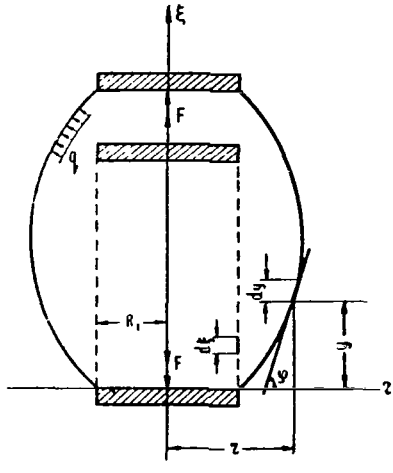

FIGURE.

Let R_1, L_1, and H_1 be the initial radius, length, and thickness of the shell, L and H its length and thickness after the deformation, φ the angle between the normal to the shell axis and the tangent to the meridian, ζ the value of Y for the undeformed shell, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ the true principal elongations, and e_1, e_2, e_3 the usual principal elongations.

Introduce the following dimensionless magnitudes:

$$x=\frac{r}{R_1},\ y=\frac{Y}{R_1},\quad \eta=\frac{\zeta}{R_1},\quad h=\frac{H}{H_1},$$
$$l=\frac{L}{R_1},\quad l_1=\frac{L_1}{R_1}. \tag{1.1}$$

As seen from the figure

$$e_1=\frac{dx}{d\eta}\frac{1}{\cos\varphi}-1,\quad e_2=x-1.\quad e_3=h-1. \tag{1.2}$$

But since $\varepsilon=\ln(e+1)$ we have

$$\varepsilon_1=\ln\left(\frac{dx}{d\eta}\frac{1}{\cos\varphi}\right),\quad \varepsilon_2=\ln x,\quad \varepsilon_3=\ln h. \tag{1.3}$$

The equilibrium equations in the given case are /1, 3/

$$H\sigma_2=\frac{d}{dr}(rH\sigma_1),\ \sigma_1=\frac{F+\pi qr^2}{2\pi rH\sin\varphi}. \tag{1.4}$$

Here σ_1 is the meridional stress and σ_2 is the circumferential stress.

We shall write the relationships between the stresses and the true strains in a form similar to Hooke's law:

$$\varepsilon_1=\frac{1}{E_1}\sigma_1-\frac{\nu_{21}}{E_2}\sigma_2-\frac{\nu_{31}}{E_3}\sigma_3,$$
$$\varepsilon_2=-\frac{\nu_{12}}{E_1}\sigma_1+\frac{1}{E_2}\sigma_2-\frac{\nu_{32}}{E_3}\sigma_3, \tag{1.5}$$
$$\varepsilon_3=-\frac{\nu_{13}}{E_1}\sigma_1-\frac{\nu_{23}}{E_2}\sigma_2+\frac{1}{E_3}\sigma_3.$$

Here E_1, E_2, E_3 are the moduli of elasticity in the meridional and circumferential directions, and the direction perpendicular to them; ν_{ij} are the Poisson ratios /4/ representing the ratio of the strain in the j direction to the strain in the i direction when the force acts in the j direction. In formulas (1.5) we shall neglect in what follows the stress σ_3 since it is small compared with σ_1 and σ_2.

Introducing the dimensionless stresses

$$p_1=\frac{\sigma_1}{E_1},\quad p_2=\frac{\sigma_2}{E_2} \tag{1.6}$$

and using the relationship for the constants of an orthotropic material /4/

$$E_j\nu_{ij}=E_i\nu_{ji}, \tag{1.7}$$

we obtain in dimensionless coordinates the following system of equations from (1.3) to (1.7):

$$
\begin{gathered}
\frac{d}{dx}(xhp_1) = \frac{\nu_{21}}{\nu_{12}} hp_2, \\
\sin\varphi = \frac{F + \pi q R_1^2 x^2}{2\pi R_1 H_1 E_1 x \cdot p_1}, \\
\ln\left(\frac{dx}{d\eta}\frac{1}{\cos\varphi}\right) = p_1 - \nu_{21} p_2, \\
\ln x = -\nu_{12} p_1 + p_2, \\
\ln h = -\nu_{13} p_1 - \nu_{23} p_2, \\
\frac{dy}{dx} = \operatorname{tg}\varphi.
\end{gathered}
\tag{1.8}
$$

It follows from the fourth and fifth equations of (1.8) that

$$
p_1 = -\frac{\nu_{23}\ln x + \ln h}{\nu_{13} + \nu_{12}\nu_{23}}, \quad p_2 = \frac{\nu_{13}\ln x - \nu_{12}\ln h}{\nu_{13} + \nu_{12}\nu_{23}} \tag{1.9}
$$

The substitution of (1.9) in the first equation of (1.8) yields

$$
\frac{dh}{dx} = -\frac{h}{x}\frac{a\ln x + b\ln h + c}{c\ln x + \ln h + 1}. \tag{1.10}
$$

where

$$
a = \frac{\nu_{21}}{\nu_{12}}\nu_{13} + \nu_{23}, \quad b = 1 - \nu_{21}, \quad c = \nu_{23} \tag{1.11}
$$

Introduce the designations $\alpha = \ln x$, $\beta = \ln h$; (1.10) will then be written in the following form:

$$
\frac{d\beta}{d\alpha} = -\frac{a\alpha + b\beta + c}{c\alpha + \beta + 1}. \tag{1.12}
$$

We shall write the solution of this equation under the assumption that $a - bc \neq 0$. Introduce the designation

$$
\Delta = 4a - (b + c)^2. \tag{1.13}
$$

If we use the condition $\beta = \beta_0 = \ln h_0$ at $\alpha = 0$, the solution of (1.12) can be written in the following form.

At $\Delta > 0$ we have

$$
\begin{gathered}
\frac{b - c}{\sqrt{\Delta}} \times \\
\times \operatorname{arc\,tg}\left\{\sqrt{\Delta}\,\frac{n[2(\beta - k) + (b + c)(\alpha - n)] + [2(\beta_0 - k) - n(b + c)](\alpha - n)}{n(\alpha - n)\,\Delta - [2(\beta - k) + (b + c)(\alpha - n)][2(\beta_0 - k) - n(b + c)]}\right\} - \\
- \frac{1}{2}\ln\frac{(\beta - k)^2 + (b + c)(\beta - k)(\alpha - n) + a(\alpha - n)^2}{(\beta_0 - k)^2 - n(b + c)(\beta_0 - k) + an^2} = 0.
\end{gathered}
\tag{1.14}
$$

If $\Delta = 0$, then

$$(b-c)\left[\frac{\alpha-n}{2(\beta-k)+(b+c)(\alpha-n)}-\frac{n}{2(\beta_0-k)-(b+c)}\right]+$$
$$+\ln\left[\frac{2(\beta-k)+(b+c)(\alpha-n)}{2(\beta_0-k)-n(b+c)}\right]=0. \qquad (1.15)$$

When $\Delta < 0$, then

$$\frac{b-c}{\sqrt{-\Delta}}\times$$
$$\times\ln\frac{[2(\beta-k)+(b+c-\sqrt{-\Delta})(\alpha-n)][2(\beta_0-k)-n(b+c+\sqrt{-\Delta})]}{[2(\beta-k)+(b+c+\sqrt{-\Delta})(\alpha-n)][2(\beta_0-k)-n(b+c-\sqrt{-\Delta})]}-$$
$$-\ln\frac{(\beta-k)^2+(b+c)(\alpha-n)(\beta-k)+a(\alpha-n)^2}{(\beta_0-k)^2-(b+c)(\beta_0-k)n+an^2}=0. \qquad (1.16)$$

Here

$$n=\frac{b-c}{a-bc},\quad k=\frac{c_2-a}{a-bc}. \qquad (1.17)$$

In the particular case when $b-c=0$, but $\Delta \neq 0$, we obtain from (1.17) that $n = 0$, $k = -1$, and equation (1.14) or (1.16) yields

$$(\beta+1)^2+2b(\beta+1)\alpha+a\alpha^2=(\beta_0+1)^2 \qquad (1.18)$$

or in x, h variables

$$(\ln h+1)^2+2b(\ln h+1)\ln x+a\ln^2 x=(\ln h_0+1)^2. \qquad (1.19)$$

Expression (1.18) coincides with the formula obtained in /1/ if we substitute there $\mu = 1$, i.e., if we replace the exponential stress-strain relationship by a linear one and assume that all Poisson ratios are equal to 0.5.

Equation (1.18) represents a family of ellipses with parameter $\beta_0 = \ln h_0$ and center $(0, -1)$.

We have not considered so far the case $a - bc = 0$, i.e., when the determinant of equation (1.12) vanishes. Note that if $b = c$ and $\Delta = 0$, this is equivalent to saying that $a - bc = 0$. The condition $a - bc = 0$ corresponds to the expression

$$\frac{\nu_{21}}{\nu_{12}}\nu_{13}+\nu_{23}\nu_{21}=0, \qquad (1.20)$$

which is possible at either $\nu_{21} = 0$ or $\nu_{13} = \nu_{23} = 0$.

The latter condition corresponds to $E_3 = \infty$, and therefore $h = 1$. Equation (1.12) becomes meaningless in this case. Condition $\nu_{21} = 0$ means that either $E_1 = \infty$ or $E_2 = \infty$. It is then possible to eliminate from equations (1.8) the coefficients ν_{21} and ν_{12}. The procedure to be used in these two cases is illustrated below by means of practical examples.

2. PROBLEMS

In the general case, when not all the Young moduli tend to infinity, we determine Δ by formula (1.13) and select the corresponding relationship

for x and h. Specifying the parameter h_0, we plot the curves in x, h coordinates. We next use the second formula of (1.8), which gives on the strength of (1.9):

$$\sin\varphi = -\frac{Q_1 + Q_2 x^2}{xh\,(c\ln x + \ln h)}. \tag{2.1}$$

where

$$Q_1 = \frac{F(\nu_{13} + \nu_{12}\nu_{23})}{2\pi R_1 H_1 E_1}, \quad Q_2 = \frac{qR_1(\nu_{13} + \nu_{12}\nu_{23})}{2H_1 E_1}. \tag{2.2}$$

From the symmetry, the extreme values for x and h will be at the middle of the shell height /1/, at $\eta = 0.5\,l$, $\sin\varphi = 1$.

The extreme values x_m and h_m are determined from the system

$$\begin{gathered} xh\,(c\ln x + \ln h) = -(Q_1 + Q_2 x^2), \\ f(x,\ h,\ h_0) = 0, \end{gathered} \tag{2.3}$$

where the last equation is understood to mean one of the equations (1.14) to (1.16) or (1.18), namely the equation corresponding to the specific value of Δ.

The following integrals are obtained for η and y as a function of x:

$$\eta = \int_1^x \frac{x^\gamma h^\delta}{\cos\varphi}\,dx, \qquad y = \int_1^x \operatorname{tg}\varphi\,dx, \tag{2.4}$$

where

$$\gamma = \frac{\nu_{23} + \nu_{21}\nu_{13}}{\nu_{13} + \nu_{12}\nu_{23}}, \qquad \delta = \frac{1 - \nu_{21}\nu_{12}}{\nu_{13} + \nu_{12}\nu_{23}}, \tag{2.5}$$

and h and φ are understood to be functions of x.

The dimensionless shell length before and after the deformation l_1 and l at given h_0 is found by taking in (2.4) and (2.5) x_m as upper limit of integration and multiplying the right-hand parts by two.

At given Q_1 and Q_2 it is, however, impossible to specify arbitrarily the value of h_0 /1/. By substituting in (2.1) $\sin\varphi = 1$, $x = 1$, $h = h_0$ we obtain the following condition for determining the upper boundary h_0:

$$h_0 \ln h_0 = -(Q_1 + Q_2). \tag{2.6}$$

As easily seen, the upper boundary h_0 corresponds to a shell of zero length.

The lower boundary will be determined as in /1/: at given Q_1 and Q_2 we substitute in the right-hand part of (2.1) the values of x and h at which this expression has the largest value, and then require that this value be smaller than or equal to unity.

Having solutions for a number of values of h_0 at given Q_1 and Q_2, it is easy to solve the direct problem, i.e., to calculate a shell of given initial length.

Consider the particular cases which follow from the condition $a - bc = 0$. Let E_1 be infinite or very large as compared with E_2 and E_3. In this

case $\varepsilon_1 = 0$. System (1.8) yields

$$\frac{d}{dx}(xhp_1') = hp_2, \qquad \sin\varphi = \frac{F + \pi q R_1 x^2}{2\pi H_1 R_1 E_2 x h p_1'},$$
$$p_2 = \ln x = -\frac{1}{\nu_{23}} \ln h, \quad \frac{dx}{d\eta} = \cos\varphi, \quad \frac{dy}{dx} = \operatorname{tg}\varphi. \tag{2.7}$$

Here $p_1' = \sigma_1/E_2$. This magnitude enters in the equilibrium equations only. Eliminating it from the first two equations, using the third and fourth ones, we obtain

$$\sin\varphi = \frac{(Q_1' + Q_2' x^2)\sin\varphi_0}{x^\lambda(\lambda \ln x - 1)\sin\varphi_0 + Q_1' + Q_2' + \sin\varphi_0}, \tag{2.8}$$

where

$$Q_1' = \frac{\lambda^2 F}{2\pi H_1 E_2 R_1}, \qquad Q_2' = \frac{\lambda^2 q R_1}{2 H_1 E_2}, \qquad \lambda = 1 - \nu_{23},$$

and φ_0 is the value of φ at $x = 1$.

Formula (2.8) can be obtained from (2.1) also, by integrating (1.12) under the assumption that $a - bc = 0$. We obtain:

$$x = h^{-1/\nu_{23}}, \tag{2.9}$$

i.e., the same relationship as was obtained in (2.7). Note that solutions for the particular cases $E_2 = \infty$ and $E_3 = \infty$ cannot be obtained directly from (1.12), since then either x = const or h = const, and (1.12) becomes meaningless.

As seen from formulas (2.7) and (2.8), the determination of the functions η and y leads as above to integrals. It is necessary to specify here the parameter φ_0.

In the case $E_2 = \infty$ ($\varepsilon_2 = 0$) the shell preserves the cylindrical shape and only increases its length. The following relationship exists between the lengths of the deformed and undeformed shells:

$$l = l_1 h^{-1/\nu_{13}}, \tag{2.10}$$

where h = const and is determined from the condition

$$h \ln h = -\nu_{13}(Q_1^* + Q_2^*), \tag{2.11}$$

where

$$Q_1^* = \frac{F}{2\pi R_1 H_1 E_1}, \quad Q_2^* = \frac{q R_1}{2 H_1 E_1}.$$

If $E_3 = \infty$ then $h \equiv 1$. The first and second equations of (1.8) yield

$$\sin\varphi = \frac{(Q_1''' + Q_2''' x^2)\sin\varphi_0}{x^{1-b}(Q_1''' + Q_2''' + \sin\varphi_0) + x^b(b \ln x - 1)}. \tag{2.12}$$

Here

$$Q_1''' = \frac{b^2 \nu_{12} F}{2\pi R_1 E_1 H_1 \nu_{21}}, \quad Q_2''' = \frac{b^2 \nu_{12} R_1 q}{2 H_1 E_1 \nu_{21}}.$$

The functions η and y are once more given by integrals similar to (2.4):

$$\eta = \int_1^x \frac{x^{\nu_{21}} e^{-(1-\nu_{21}\nu_{12})p_1}}{\cos\varphi}\, dx, \qquad y = \int_1^x \operatorname{tg}\varphi\, dx,$$

where

$$p_1 = \frac{\nu_{21}}{b^2\nu_{12}}\left[\frac{Q_1''' + Q_2''' + \sin\varphi_0}{\sin\varphi_0} x^{-b} + b\ln x - 1\right],$$

and $\cos\varphi$ is determined from (2.12).

3. LARGEST POSSIBLE LOADS

We define the largest possible load as the limiting load under which the shell shape obtained remains stable. It is assumed that no restrictions are imposed on the magnitude of the deformations (cf. /1/, /2/, /6/).

Taking into account that the dangerous section will be at $\varphi = \pi/2$, we find the largest load by writing $\sin\varphi = 1$. The values of x and h corresponding to $\varphi = \pi/2$ will be designated by x_k and h_k.

Consider the case when $Q_2 = 0$ and $a - bc \neq 0$. We shall determine the largest value $Q_1 = Q_{1\max}$. Equating the derivative of dQ_1 by dx zero, we obtain from (2.1) and (1.10)

$$h_k = x_k^{\nu_{12}/\nu_{12}}. \tag{3.1}$$

By substituting (3.1) into the equation connecting x and h which is correct for the given case, we obtain the values of x_k and h_k for the given h_0. If for instance $E_2 = \infty$, we have for the values x_k and h_k

$$x_k = 1, \qquad h_x = \frac{1}{e}. \tag{3.2}$$

and the largest load will be

$$(Q_1^* + Q_2^*)_{\max} = e^{-\nu_{12}}. \tag{3.3}$$

Similar analyses are possible in all the particular cases mentioned above.

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USE OF ELECTRONIC DIGITAL COMPUTERS IN STRENGTH CALCULATION OF THIN-WALLED AIRCRAFT STRUCTURES

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Structurally, an aircraft usually represents a reinforced shell. Its surface has a complex shape, and the thickness of the shell and the dimensions of its supporting ribs are variable. The great difficulty involved in the strength calculation of aircraft structures is due to these peculiar features, and also to the great variety of types of loading to which they are subjected. The use of computers in conducting the separate computing operations does not always lead to a substantial saving in time, due to the large amount of work necessary for preparing the input data and for analyzing the intermediate results of the calculation. That is the reason why the programing of the entire process of structure calculation for computer solution is of great importance.

Prior to the appearance of electronic digital computers, the usual procedure in calculating complex structures was to split the calculation into separate parts, and to calculate each of these parts with the aid of narrowly specialized means (calculating methods). It now becomes necessary to reassemble in a unique process these different operations of design and calculation of structures, and to create a system for the automatization of the calculations.

This paper attempts to outline a first approximation for the complex solution of several interconnected problems of the design of the force-transmitting members of structures and of the strength calculation. This method must be further refined and developed. We treat here the particular case of strength calculation of the aircraft structure at static loading. The automatization of the following calculations is envisaged: 1) calculating the coordinates of the surface points of the aircraft structure; 2) determining the dimensions of the force-transmitting members of the structure in such a way as to ensure its minimum relative weight; 3) conducting a check calculation of the structure strength on the basis of the applied theory of thin-walled bars (by the beam theory); 4) refining the calculation of stresses and strains of the structure by means of a method of calculation in which the equilibrium equations of a reinforced shell are written in finite differences relative to the displacements.

Universal computer programs were established for conducting certain calculations relative to the strength of aircraft structures. These programs envisage the possibility of automatic fulfillment of the calculations of certain types of structures.

Consider a shell reinforced by stiffening ribs. The shell is represented in the rectangular system of coordinates xyz and in a Gaussian system of coordinates α, β. The coordinate lines α and β are taken on the middle surface of the shell in the direction of its lines of connection with the stiffening ribs (with the stringers or wing ribs). The coordinate lines are designated by the indices kj $(\alpha_{kj}, \beta_{kj})$. The first index designates the number of the coordinate line (the number of the stringer, stiffener, or wing rib), and the second index the number of the point on this line. The shell region limited by two adjacent coordinate lines α_k, α_{k-1} or β_j, β_{j-1} is called the span (k or j), and the skin region enclosed between these two couples of lines is called the panel kj. The position of a point on the middle surface is defined by the vector $\overline{M}_{kj} = M_{kj}(\alpha_{kj}\beta_{jk}) = M_{kj}(x_{kj}\, y_{kj}\, z_{kj})$, and its coordinates in the rectangular system of coordinates are

$$x_{kj} = x(\alpha_{kj}\beta_{jk}), \quad y_{kj} = y(\alpha_{kj}\beta_{jk}), \quad z_{kj} = z(\alpha_{kj}\beta_{jk}).$$

The coordinate lines α and β are replaced by broken lines with corners at their intersection points, and instead of the surface we consider a thin-walled system having the shape of a polyhedron. Each side (panel) at point M_{kj} is characterized by the coefficients of the first quadratic form of the surface, which is expressed in finite differences as follows:

$$A_{kj}^2 = 1, \quad B_{kj}^2 = 1, \qquad \cos\chi_{kj} = \left(\frac{\Delta_j \overline{M}_{kj}}{\Delta_j \alpha_{kj}} \; \frac{\Delta_k \overline{M}_{kj}}{\Delta_k \beta_{jk}}\right). \tag{1}$$

The expression with the parentheses is the scalar product of vectors, which in the given cases represent the finite differences of the vector $\overline{M}$ corresponding to unit finite differences of the coordinates α and β. The symbols Δ_j and Δ_k designate the first differences by the corresponding parameter, for instance $\Delta_j \alpha_{kj} = \alpha_{kj} - \alpha_{k,\, j-1}$.

The components of the displacements of the middle-surface points in the direction of the x, y, z axes are designated by u_x, u_y, u_z, respectively, and the displacement vector by $\overline{u} = u(u_x u_y u_z)$. The unit elongations of the shell elements in the directions of the coordinate lines α and β are designated by ε_α and ε_β, and the variation of the angle between the tangents to these lines by γ.

$$\left.\begin{aligned} &\varepsilon_{\alpha kj} = \left(\frac{\Delta_j \overline{M}_{kj}}{\Delta_j \alpha_{kj}} \; \frac{\Delta_j \overline{u}_{kj}}{\Delta_j \alpha_{kj}}\right), \qquad \varepsilon_{\beta kj} = \left(\frac{\Delta_k \overline{M}_{kj}}{\Delta_k \beta_{jk}} \; \frac{\Delta_k \overline{u}_{kj}}{\Delta_k \beta_{jk}}\right), \\ &\gamma_{kj} = \frac{1}{\sin\chi_{kj}}\left[\left(\frac{\Delta_j \overline{M}_{kj}}{\Delta_j \alpha_{kj}} \; \frac{\Delta_k \overline{u}_{kj}}{\Delta_k \beta_{jk}}\right) + \left(\frac{\Delta_k \overline{M}_{kj}}{\Delta_k \beta_{jk}} \; \frac{\Delta_j \overline{u}_{kj}}{\Delta_j \alpha_{kj}}\right)\right] - \\ &\qquad - (\varepsilon_{\alpha kj} - \varepsilon_{\beta jk}) \frac{\cos\chi_{kj}}{\sin\chi_{kj}}. \end{aligned}\right\} \tag{2}$$

A wide class of problems relative to the strength of thin-walled aircraft structures are solved on the basis of the theory of calculation of thin-walled bars, which makes use of the hypothesis of invariability of the shell section contour /1/. The fundamental equations of such a theory for the case of a thin-walled bar of variable cross section were established. The x axis is directed along the bar length. Instead of the coordinate lines β = const we consider the lines of intersection of the shell surface with the planes x_k = const. The coordinates of point M_{kj} are written as follows:

$$x_k, \quad y_{kj} = y(x_k \alpha_{kj}), \quad z_{kj} = z(x_k \alpha_{kj}). \tag{3}$$

Each stringer in span k_j can be adjoined by an arbitrary number of panels of the shell, designated by the indices $o\varkappa$ ($\varkappa = 1, 2, 3, \cdots$). It is assumed that the shell thickness remains invariable within the limits of one panel. It is also assumed that on the line connecting the shell and the stringer there act only shearing forces $\delta_{kj}^{o\varkappa}$. The tangential stresses do not vary through the shell thickness and along the span length. It is assumed that the shell temperature remains constant within the limits of one panel. The stringer temperature can differ from the temperature of the skin, but it is assumed to be constant along the length of one span.

Relationships (2) for a thin-walled bar with undeformable contour are

$$\left.\begin{aligned}
&\varepsilon_{akj}^{o\varkappa} = \alpha_{Tkj}^{o\varkappa}\, T_{kj}^{o\varkappa},\\
&\varepsilon_{\beta jk} = \left(\frac{\Delta_k x_k}{\Delta_k \beta_{jk}}\right)^2 \left[\frac{\Delta_k u_{xkj}}{\Delta_k x_k} + \left(\frac{\Delta_k \bar{w}_{kj}}{\Delta_k x_k}\,\frac{\Delta_k \bar{\eta}_k}{\Delta_k x_k}\right)\right],\\
&\gamma_{kj}^{o\varkappa} = \frac{\bar{\gamma}_{kj}^{o\varkappa}}{\sin \chi_{kj}^{o\varkappa}} - \varepsilon_{\beta jk}\frac{\cos \chi_{kj}^{o\varkappa}}{\sin \chi_{kj}^{o\varkappa}},\\
&\bar{\gamma}_{kj} = \left(\frac{\Delta_k x_k}{\Delta_k \beta_{jk}}\right)\left[\frac{\Delta_j u_{xkj}}{\Delta_j \alpha_{kj}} + \left(\frac{\Delta_j \bar{w}_{kj}}{\Delta_j \alpha_{kj}}\,\frac{\Delta_k \bar{\eta}_k}{\Delta_k x_k}\right)\right],
\end{aligned}\right\} \qquad (4)$$

were α_i is the coefficient of linear expansion and T the temperature of the structure element; $\bar{\eta}_k = \eta(\eta_{2k}\eta_{3k}\eta_{4k})$ is the displacement vector of the point of intersection of section x_k = const with the x axis; η_{2k}, η_{3k} are the y and z components of the displacements of this point; η_{4k} is the angle of rotation of the section about the x axis; $\bar{w}_{kj} = w_{kj}(y_{kj}\, z_{kj}\, \omega_k)$ is the vector of the point in the $yz\omega$ coordinate axes, where ω is the sectorial coordinate of the point lying on the contour of the bar cross section. In writing equations (4) it was assumed that the contour of the bar cross section is inextensible, but that it expands freely when heating the skin.

The projection of all the forces acting on the stringer in one span on the x axis is

$$\frac{\Delta_k P_{kj}}{\Delta_k \beta_{jk}} + \sum_{\varkappa} \frac{\Delta_j \alpha_{kj}^{o\varkappa}}{\Delta_j \alpha_{k-1,j}^{o\varkappa}}\, q_{kj}^{o\varkappa} - p_{kj} = 0, \qquad (5)$$
$$k = 1, 2, \cdots m; \qquad j = 1, 2, \cdots n.$$

where $P_{kj} = P_{kj}^{c} + \sum_{\varkappa} P_{kj}^{o\varkappa}$ is the axial force acting in the stringer section (P_{kj}^{c}) and in the shell panels adjoining it ($P_{kj}^{o\varkappa}$); p is the external load per unit length acting along the stringer axis; m is the number of calculated cross sections of the thin-walled bar, and n the number of stringers in the cross section.

The equilibrium equations of the external and internal forces acting on the thin-walled bar on one side of the section k are

$$\left.\begin{aligned}
&\sum_{j=1}^{n} \Delta_j y_{kj}^{o\varkappa}\, q_{kj}^{o\varkappa} + \sum_{j=1}^{n} \frac{\Delta_k y_{kj}}{\Delta_k \beta_{jk}} P_{kj} = -Q_{yk},\\
&\sum_{j=1}^{n} \Delta_j z_{kj}^{o\varkappa}\, q_{kj}^{o\varkappa} + \sum_{j=1}^{n} \frac{\Delta_k z_{kj}}{\Delta_k \beta_{jk}} P_{kj} = -Q_{zk},\\
&\sum_{j=1}^{n} \Delta_j \omega_{kj}^{o\varkappa}\, q_{kj}^{o\varkappa} + \sum_{j=1}^{n} \frac{\Delta_k \omega_{kj}}{\Delta_k \beta_{jk}} P_{kj} = -M_{xk},
\end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned}
&\sum_{j=1}^{n}\frac{\Delta_k x_k}{\Delta_k \beta_{jk}} P_{kj} = -Q_{xk},\\
&\sum_{j=1}^{n}\frac{\Delta_k x_k}{\Delta_k \beta_{jk}} P_{kj} = -M_{zk},\\
&\sum_{j=1}^{n}\frac{\Delta_k x_k}{\Delta_k \beta_{jk}} P_{kj} = -M_{yk},
\end{aligned}\right\} \tag{7}$$

where Q_x, Q_y, Q_z are the components of the axial and lateral forces along the coordinate axes and M_x, M_y, M_z are the torsional and bending moments in the cross section of the thin-walled bar.

The Hooke's law relationships for the skin of a thin-walled bar with rigid cross-sectional contour are written as follows in an oblique system of coordinates:

$$\left.\begin{aligned}
p_{kj}^{ox} &= \frac{E_{kj}^{ox}\delta_{kj}^{ox}\Delta_j a_{kj}^{ox}}{2(1-\mu)\sin^3\chi_{kj}^{ox}}\left(\varepsilon_{\beta jk} - \bar{\gamma}^{ox}\cos\chi_{kj}^{ox} + \alpha_{Tkj}^{ox}T_{kj}^{ox}\cos 2\chi_{kj}^{ox}\right),\\
q_{kj}^{ox} &= \frac{E_{kj}^{ox}\delta_{kj}^{ox}}{2(1-\mu^2)\sin^3\chi_{kj}^{ox}}\left[\bar{\gamma}_{kj}^{ox}(1+\cos^2\chi_{kj}^{ox} - \mu\sin^2\chi_{kj}^{ox}) - 2\varepsilon_{\beta jk}\cos\chi_{kj}^{ox}\right].
\end{aligned}\right\} \tag{8}$$

The corresponding relationship for the stringer is

$$P_{kj}^{c} = E_{kj}^{c} f_{kj}^{c}\left(\varepsilon_{\beta jk} - \alpha_{Tkj}^{c} T_{kj}^{c}\right).$$

Here f is the area of the stringer section. The superscript "c" designates that the parameter corresponds to the stringer.

Introducing (4) in (8) and the result in (5) and (6), we obtain a system of algebraic equations which is written as follows in a matrix form:

$$\bar{A}\,\bar{u}_x + \bar{B}\,\bar{\eta} = \bar{R}. \tag{9}$$

The solution of this system of equations, using the boundary conditions, makes it possible to determine the state of stress and strain of the thin-walled bar.

The curved lines on the middle surface of the shell can be described by means of polynomials, curves of second order, or by means of the radius-graphic method. All these methods are used in designing the surfaces of aircraft. A program for computerized calculation of the surface points was worked out on the basis of the radius-graphic method.

The design calculation on the determination of the dimensions of the force set in span "k" for given external loads is conducted on the basis of the beam theory. In accordance with the plane-sections hypothesis we have

$$u_{xkj} = \eta_{1k} + \eta_{5k} y_{kj} + \eta_{6k} z_{kj}, \tag{10}$$

where η_{1k}, η_{5k}, η_{6k} are components of the displacement of the section as a rigid body. The axial force per unit length in the longitudinal set is determined by the formula

$$T_{kj}^{*} = c_{kj}\frac{2P_{kj}}{\Delta_j a_{kj} + \Delta_j a_{k,j+1}}, \tag{11}$$

where c_{kj} are coefficients allowing for the nonhomogeneity of the force-transmitting members. Their value is selected from the condition of minimum weight of the force-transmitting members of the given span for given systems of external loads.

The internal forces per unit length T_{kj}^{*} and q_{kj} for each system of external loads and combination of coefficients are determined from equations (6) and (7), allowing for relationships (4), (8), (10), and (11). Next, the parameters of the optimum panels corresponding to these values of the internal forces are selected by the method exposed in /2/, and the weight of the force-transmitting members of the span is calculated. A set of values of the coefficients c_{kj} is selected which satisfies the condition of relative extremum of the weight function of the force-transmitting members (the stringers and the skin).

The check calculations of the structure is conducted in several stages. In the first stage we make use of the theory of thin-walled beams based on the hypothesis of plane sections for closed shells, or on the hypothesis of deplanation by the sectorial-area law for open shells. In the case of a thin-walled beam the hypothesis of deplanation by the sectorial-area law, allowing for the initial and temperature deformations, is written as follows:

$$\varepsilon_{kj} = (\bar{g}_k \bar{\omega}_{kj}) - \dot{\varepsilon}_{kj}, \tag{12}$$

where $\dot{\varepsilon}_{kj} = \varepsilon_{kj}^{H} + \alpha_{Tkj} T_{kj}$; ε_{kj}^{H} is the initial strain; $\bar{\omega}_{kj} = \omega(1 y_{kj} z_{kj} \omega_{kj})$ is the radius-vector of point j on the contour of section k; $\bar{g}_k = g(g_{1k} g_{2k} g_{3k} g_{4k})$ is the matrix-column of constant coefficients, which remain to be determined.

The normal stress σ_{kj} is calculated by the formula

$$\sigma_{kj} = E_{kj}(\bar{g}_k \bar{\omega}_{kj}) - E_{kj}\varepsilon_{kj}^{c}, \tag{13}$$

where E_{kj} is the secant modulus of the diagram $\sigma_{kj}(\varepsilon)$.

Using equations (5) to (7), the normal stresses σ_{kj} and the shearing stresses $\tau_{kj} = \frac{q_{kj}}{\delta_{kj}}$ (kj is the thickness of the δ_{kj} shell-panel) are calculated for each case of loading. Using the method exposed in /3/, the critical and collapsing stresses of the panel corresponding to the values obtained for σ_{kj} and τ_{kj} are calculated.

The skin and the thin-walled elements of the aircraft structure usually buckle before the carrying capacity of the structure as a whole is exhausted. The magnitude of the secant modulus E_{kj} for these magnitudes is unknown beforehand, and the determination of the stresses and strains of the thin-walled beam is realized by successive approximations, using the so-called method of reduction coefficients or the method of secant moduli.

On the basis of the method described, Sokolov wrote a computer program capable of calculating without stoppage all the design sections of the structure for all given cases of loading by the method of reduction coefficients. The machine time of the calculation of one section of the aircraft fuselage for one case of loading is less than a minute. The accuracy in the determination of the stresses is higher than the accuracy of the method used in manual calculation by about 10 to 20%, due to the more accurate allowance for the skin work after buckling. Shin wrote a program for determining the stresses by the method of secant moduli, allowing for the temperature strains and the influence of the temperature on the mechanical properties of the material. The program was written for the Ural-2 computer.

Owing to the above-mentioned beam-theory hypotheses, the local stresses cannot be determined in the first stage of the check calculations. The next state in the refinement of the calculation of the stresses and strains must

consist in the calculation of a thin-walled structure on the basis of the theory of thin-walled bars, which leads to solving system (9). A program for calculating a cylindrical reinforced shell by this method was written by Batalov; by this program the solution of system (9) is conducted by the method of the fastest descent.

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STRESS CONCENTRATION IN A THIN PLATE WEAKENED BY AN INFINITE NUMBER OF CIRCULAR HOLES UNDER ELASTIC-PLASTIC STRAINS

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The elastic-plastic problem in a thin plate weakened by an infinite number of identical circular holes is considered by the approximate method proposed by Kosmodamianskii /2, 3/, and by the method of approximating the stress function in the plastic region by a biharmonic function /4/.

Consider a thin plate weakened by an infinite line of identical circular holes of centers lying on the x axis at a distance l from each other.

Let the plate be extended at infinity by constant forces C and D in the x and y directions, respectively, and let the plastic zones at the given forces completely enclose the holes, but without merging. On the strength of the dynamic and geometric symmetry, the shape of the lines separating the elastic and plastic zones will be the same about each hole. As known /2/, the stresses in plastic zones are independent of the forces at infinity and the influence of the adjacent holes, and therefore they will be the same as in the case of a single hole. This makes it possible to reduce the problem considered to the corresponding problem for a plate with single hole.

Take the origin of coordinates at the hole center, and designate by L the contour separating the elastic and plastic zones.

The stress components in the plastic region must satisfy the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{2}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \qquad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} = 0 \tag{1}$$

and the plasticity condition

$$(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2 = [2\sigma_s - |\sigma_r + \sigma_\theta|]^2, \tag{2}$$

where σ_s is the yield point.

Equations (1) and (2) will be satisfied at zero boundary conditions $\sigma_r = 0$, $\tau_{r\theta} = 0$ at $r = R$, if the stress components are represented in the form

$$\sigma_r = \sigma_s\left(1 - \frac{R}{r}\right), \qquad \sigma_\theta = \sigma_s, \quad \tau_{r\theta} = 0. \tag{3}$$

The stress function corresponding to these stress components is

$$U_1 = \sigma_s\left(\frac{r^2}{2} - rR\right). \tag{4}$$

Following /4/, we approximate this function by the biharmonic function

$$U_1^+ = \sigma_s\left[Fr^2 - NR^2 \ln\frac{r}{R} + Gr^2 \ln\frac{r}{R} + M\right], \tag{5}$$

where F, N, G, M are constants which must be selected in such a way as to minimize the error in the determination of the contour L and the stress components in the elastic region.

Calculating now the stress components by function (5), we write them in the following combinations:

$$\begin{aligned} \sigma_x^{(1)} + \sigma_y^{(1)} &= 4\sigma_s\left[F + G + G\ln\frac{r}{R}\right], \\ \sigma_y^{(1)} - \sigma_x^{(1)} + 2i\tau_{xy}^{(1)} &= 2\sigma_s\left[N\frac{R^2}{r^2} + G\right]. \end{aligned} \tag{6}$$

The stress components in the elastic region are expressed through two holomorphous functions $\Phi(z)$ and $\Psi(z)$. The function $\Phi(z)$, which is directly linked with the stress components, will be periodic in the direction of the x axis. As was done in /2/, we introduce the periodic function

$$\Psi^*(z) = \Psi(z) + z\Phi'(z). \tag{7}$$

The stress components in the elastic region will then be expressed by the formulas

$$\begin{aligned} \sigma_x^{(2)} + \sigma_y^{(2)} &= 4\,\mathrm{Re}\,[\Phi(z)], \\ \sigma_y^{(2)} - \sigma_x^{(2)} + 2i\tau_{xy}^{(2)} &= 2\,[(\bar{z} - z)\Phi'(z) + \Psi^*(z)]. \end{aligned} \tag{8}$$

Since the functions $\Phi(z)$ and $\Psi^*(z)$ are analytical in the elastic region, they can be represented in the form /2/

$$\begin{aligned} \Phi(z) &= \sum_{n=1}^{\infty} \frac{\alpha_n}{[\zeta(z)]^n} + \sum_{k=-\infty}^{\infty}{}^{*} \sum_{n=1}^{\infty} \frac{\alpha_n}{[\zeta(z-kl)]^n}, \\ \Psi^*(z) &= \sum_{n=1}^{\infty} \frac{\beta_n}{[\zeta(z)]^n} + \sum_{k=-\infty}^{\infty}{}^{*} \sum_{n=1}^{\infty} \frac{\beta_n}{[\zeta(z-kl)]^n}, \end{aligned} \tag{9}$$

where

$$z - kl = \omega(\zeta) = c\zeta + \sum_{m=1}^{\infty} c_m \zeta^{-m}. \tag{10}$$

The first sums of (9) are holomorphous in the region outside contour L. They will be the components of some biharmonic function, which we shall designate by $U_2(x, y)$. The second sums are holomorphous inside the contour L and will be the components of some biharmonic function, which we shall designate by $\bar{U}_2(x, y)$. They depend in addition on the

small parameter $\varepsilon = \frac{R}{l}$. By expanding them in a series by the powers of this small parameter and retaining the terms containing ε^2 we obtain

$$\begin{aligned} \tilde{\Phi}_2(\zeta) &= 2\alpha_2 c^2 \varepsilon^2 \delta_2 - 2\alpha_1 c^2 \varepsilon^2 \delta_2 \omega(\zeta), \\ \tilde{\Psi}_2(\zeta) &= 2\beta_2 c^2 \varepsilon^2 \delta_2 - 2\beta_1 c^2 \varepsilon^2 \delta_2 \omega(\zeta), \end{aligned} \tag{11}$$

where

$$\delta_2 = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Introduce a new biharmonic function

$$U_3(x, y) = U_2(x, y) - [U_1^*(x, y) + \tilde{U}_2(x, y)]. \tag{12}$$

From the continuity conditions of the stress components at the boundary γ between the elastic and plastic zones in the plane ζ transformed by means of the function $\omega(\zeta)$ and the conditions at infinity the following boundary conditions are obtained:

$$4\,\mathrm{Re}\,[\Phi_3(\zeta)] = \begin{cases} 0 \quad \text{on} \quad \gamma, \\ C + D - 4\sigma_s(F + G) - 4\sigma_s \ln|\zeta| - 4\sigma_s G \ln c + \\ + 4\sigma_s G \ln R + 8\alpha_2 c^2 \varepsilon^2 \delta_2 - 8\alpha_1 c^3 \varepsilon^2 \delta_2 \zeta \quad \text{at} \quad \zeta \to \infty, \end{cases} \tag{13}$$

$$2\left[\frac{\overline{\omega(\zeta)} - \omega(\zeta)}{\omega'(\zeta)} \Phi_3'(\zeta) + \Psi_3(\zeta)\right] = \begin{cases} 0 \quad \text{on} \quad \gamma, \\ D - C - 2\sigma_s G e^{-2i\theta_1} + 4\alpha_1 c^3 \varepsilon^2 \delta_2 (\bar{\zeta} - \zeta) + \\ + 4\beta_2 c^2 \varepsilon^2 \delta_2 - 4\beta_1 c^3 \varepsilon^2 \delta_2 \zeta \quad \text{at} \quad \zeta \to \infty. \end{cases} \tag{14}$$

Conditions (13) will be satisfied if the function $\Phi_3(\zeta)$ is represented in the form

$$\Phi_3(\zeta) = -\sigma_s G \ln \zeta - 2\alpha_1 c^3 \varepsilon^2 \delta_2 \left(\zeta - \frac{1}{\zeta}\right) \tag{15}$$

and the constant c is determined so as to satisfy the equation

$$\frac{D + C}{4} - \sigma_s(F + G) + \sigma_s G \ln R + 2\alpha_2 c^2 \varepsilon^2 \delta_2 - \sigma_s G \ln c = 0,$$

whence

$$c = R \exp\left\{\frac{1}{\sigma_s G}\left[\frac{D + C}{4} - \frac{F + G}{G} + 2\sigma_s G \alpha_2 c^2 \varepsilon^2 \delta_2\right]\right\}. \tag{16}$$

On the strength of (11) and (12) the function $\Phi_2(\zeta)$ will be written as follows:

$$\begin{aligned} \Phi_2(\zeta) = &-\sigma_s G \ln \zeta - 2\alpha_1 c^3 \varepsilon^2 \delta_2 \left(\zeta - \frac{1}{\zeta}\right) + \sigma_s(F + G) - \sigma_s G \ln R - \\ &- 2\alpha_2 c^2 \varepsilon^2 \delta_2 + \sigma_s G \ln \omega(\zeta) + 2\alpha_1 c^2 \varepsilon^2 \delta_2 \omega(\zeta). \end{aligned}$$

By equating the coefficients of equal negative powers of ζ we obtain

$$\alpha_1 = 2\alpha_1 c^2 \varepsilon^2 \delta_2 (c + c_1), \tag{17}$$

$$\alpha_2 = \sigma_s G \frac{c_1}{c} + 2\alpha_1 c^2 \varepsilon^2 \delta_2 c_2. \tag{18}$$

Since ε can take arbitrary values, equality (17) will be satisfied if $\alpha_1 = 0$.

The substitution of (15) in boundary conditions (14) leads to the following equality:

$$\Psi_3(\zeta) = \frac{D-C}{2} - \sigma_s G + 2\beta_2 c^2 \varepsilon^2 \delta_2 - 2\beta_1 c^3 \varepsilon^2 \delta_2 \zeta + Q(\zeta), \tag{19}$$

where $Q(\zeta)$ is holomorphous outside of γ, and $Q(\infty) = 0$.

The first condition of (14) yields

$$[\overline{\omega(\zeta)} - \omega(\zeta)]\, \Phi_3'(\zeta) = -\omega'(\zeta)\, \Psi_3(\zeta). \tag{20}$$

A functional equation has thus been obtained from which the transformation function $\omega(\zeta)$ is determined. By comparing the coefficients of equal positive powers of ζ we obtain

$$\bar{c}_1 = \frac{c}{\sigma_s G}\left[\frac{D-C}{2} + 2\beta_2 c^2 \varepsilon^2 \delta_2\right], \tag{21}$$

$$\bar{c}_2 = -\frac{2}{\sigma_s G}\beta_1 c^4 \varepsilon^2 \delta_2, \tag{22}$$

$$\sigma_s G \bar{c}_n = 0 \quad (n \geqslant 3).$$

It follows that the transformation function $\omega(\zeta)$ will be of the form

$$\omega(\zeta) = c\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2}. \tag{23}$$

The expression of the function $\Psi_2(\sigma)$ on the contour of the unit circle is:

$$\Psi_2(\sigma) = -\frac{\bar{\omega}\left(\frac{1}{\sigma}\right) - \omega(\sigma)}{\omega'(\sigma)}\left[\sigma_s G \frac{\omega'(\sigma)}{\omega(\sigma)} - \sigma_s \frac{G}{\sigma}\right] + \sigma_s N \frac{R^2}{\omega^2(\sigma)} + \\ + \sigma_s G \frac{\bar{\omega}\left(\frac{1}{\sigma}\right)}{\omega(\sigma)} - 2\beta_2 c^2 \varepsilon^2 \delta_2 + 2\beta_1 c^2 \varepsilon^2 \delta_2 \omega(\sigma). \tag{24}$$

From here we find $\Psi_2(\zeta)$ by the Muskhelishvili method. By comparing the coefficients of equal negative powers of ζ and taking (21) and (22) into account we obtain

$$\beta_1 = 0 \tag{25}$$

$$\beta_2 = \frac{2\sigma_s G c^2 + (D-C)\, c c_1 - 4\sigma_s G c c_1 + 2\sigma_s N R^2}{2c^2(1 + 2c\varepsilon^2 \delta_2 c_1)}. \tag{26}$$

It is now seen from (22) that $c_2 = 0$. It follows that the contour L will be an ellipse with semiaxes

$$a = c\left(1 + \frac{c_1}{c}\right), \qquad b = c\left(1 - \frac{c_1}{c}\right). \tag{27}$$

The stress components are determined by the formulas

$$\sigma_x^{(2)} + \sigma_y^{(2)} = 4\sigma_s \operatorname{Re}\left[(F+G) + G\ln\frac{\omega(\zeta)}{R\zeta}\right],$$

$$\sigma_y^{(2)} - \sigma_x^{(2)} + 2i\tau_{xy}^{(2)} = 2\sigma_s\left[G\frac{\overline{\omega(\zeta)}}{\omega(\zeta)} + \frac{G}{\zeta}\frac{\bar{\omega}\left(\frac{1}{\zeta}\right) - \omega(\zeta)}{\omega'(\zeta)} + N\frac{R^2}{\omega^2(\zeta)}\right]. \tag{28}$$

and the constants F, N, G, M are determined from the equality of the first four terms in the expansion of U_1 and U_1^* in a Taylor series near the point whose radius-vector is equal to the half-sum of the minimum and maximum radius-vectors of the points of contour L /4/.

$$F = \frac{1}{2}\left[1 - \frac{R}{c} - \frac{R}{2c}\ln\frac{c}{R}\right], \qquad N = \frac{c}{4R},$$

$$G = \frac{R}{4c}, \qquad M = \frac{Rc}{4}\ln\frac{c}{R} - \frac{cR}{2}. \tag{29}$$

The substitution of the values of the constants (29) in expressions (16), (18), (21), and (26), respectively, yields

$$c = \frac{\sigma_s R}{2\sigma_s - (D+C) - 2\sigma_s R\varepsilon^2\delta_2 c_1},$$

$$c_1 = \frac{4c^2}{\sigma_s R}\left[\frac{D-C}{2} + 2\beta_2 c^2\varepsilon^2\delta_2\right],$$

$$\alpha_2 = \frac{\sigma_s R c_1}{4c^2}, \tag{30}$$

$$\beta_2 = \frac{\sigma_s R c + (D-C)cc_1 - \sigma_s R c_1}{2c^2(1 - 2c\varepsilon^2\delta_2 c_1)}.$$

We shall consider now a numerical example. Let

$$R = 1, \quad l = 10, \quad C = 0.7375\,\sigma_s, \quad D = 0.7125\,\sigma_s.$$

The calculations yield

$$\omega(\zeta) = 1.84\zeta + \frac{0.195}{\zeta}.$$

The table contains the values of the stress components on contour L, calculated from the plastic side by the exact expression (3), and from the elastic side by the approximate expression (28).

TABLE

θ°	$\frac{1}{\sigma_s}\sigma_r^{(1)}$	$\frac{1}{\sigma_s}\sigma_r^{(2)}$	$\frac{1}{\sigma_s}\sigma_\theta^{(1)}$	$\frac{1}{\sigma_s}\sigma_\theta^{(2)}$
0	0.5098	0.5097	1	1.0031
15	0.5037	0.5039	1	1.0023
30	0.4864	0.4865	1	1.0009
45	0.4595	0.4595	1	1.0002
60	0.4279	0.4280	1	1.0009
75	0.4012	0.4012	1	1.0029
90	0.3903	0.4027	1	0.9919

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INTEGRAL-DIFFERENTIAL EQUATIONS OF PLATES WHOSE MATERIAL BEHAVES ACCORDING TO LINEAR RHEOLOGICAL RELATIONSHIPS

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1. The author obtained in /1/ the following linear relationships in a first approximation by the methods of the thermodynamics of irreversible processes, allowing for the mechanical and thermal phenomena only:

$$\sigma_{ij} - \sigma_{ij}^c = 2a\left(\dot{\varepsilon}_{ij} - \frac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}\right) + b\delta_{ij}\dot{\varepsilon}_{kk};$$

$$\sigma_{ij}^c = 2c\left(\dot{\varepsilon}_{ij}^p - \frac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}^p\right) + d\delta_{ij}\dot{\varepsilon}_{kk}^p \qquad (i, j = 1, 2, 3); \tag{1}$$

$$\sigma_{ij}^c = 2\mu\left(\varepsilon_{ij}^e - \frac{1}{3}\delta_{ij}\varepsilon_{kk}^e\right) + K\delta_{ij}\varepsilon_{kk}^e - K\alpha(T - T_0)\delta_{ij}.$$

Here ε_{ij}, σ_{ij} are strains and stresses; ε_{ij}^e are elastic strains; σ_{ij}^c are stresses taking place at quasistatic strains; $\dot{\varepsilon}_{ij}^p = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^e$; T is absolute temperature; δ_{ij} is the Kronecker delta; a, b, c, d are viscous constants; μ, K are elastic constants; α is the coefficient of volume temperature expansion.

Let the x_1, x_2 coordinate axes lie in the middle plane of the plate, and the x_3 axis be perpendicular to them. In accordance with Kirchhoff-Love's hypothesis we assume that in relationships (1) $\sigma_{33} = \sigma_{33}^c = 0$. Then, by eliminating from each system the magnitudes $\dot{\varepsilon}_{33}$, $\dot{\varepsilon}_{33}^p$, ε_{33}^e, respectively, we obtain the relationships

$$\sigma_{ij} - \sigma_{ij}^c = \frac{G}{1 - m^2}[(1 - m)\dot{\varepsilon}_{ij} + m\dot{\varepsilon}_{kk}\delta_{ij}];$$

$$\sigma_{ij}^c = \frac{H}{1 - n^2}[(1 - n)\dot{\varepsilon}_{ij}^p + n\dot{\varepsilon}_{kk}^p\delta_{ij}]; \qquad \begin{pmatrix}\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}; \\ i, j = 1, 2;\end{pmatrix} \tag{2}$$

$$\sigma_{ij}^c = \frac{E}{1 - \nu^2}[(1 - \nu)\varepsilon_{ij}^e + \nu\varepsilon_{kk}^e\delta_{ij}] - \frac{E\alpha}{3(1 - \nu)}(T - T_0)\delta_{ij},$$

where

$$K = \frac{E}{3(1 - 2\nu)}; \qquad \mu = \frac{E}{2(1 + \nu)}; \qquad b = \frac{G}{3(1 - 2m)}; \qquad a = \frac{G}{2(1 + m)};$$

$$d = \frac{H}{3(1 - 2n)}; \qquad c = \frac{H}{2(1 + n)}, \tag{3}$$

and

$$E > 0; \quad G > 0; \quad H > 0; \quad -1 \leqslant \nu \leqslant \frac{1}{2}; \quad -1 < m < \frac{1}{2}; \quad -1 < n \leqslant \frac{1}{2}.$$

Express the strains through the deflections:

$$\varepsilon_{ij} = -x_3 \frac{\partial^2 w}{\partial x_i \partial x_j}; \qquad \varepsilon^p_{ij} = -x_3 \frac{\partial^2 w^p}{\partial x_i \partial x_j}; \qquad \varepsilon^e_{ij} = -x_3 \frac{\partial^2 w^e}{\partial x_i \partial x_j}, \tag{4}$$

substitute these values in (2), multiply by x_3, and integrate in the interval $\left[-\frac{h}{2}, \frac{h}{2}\right]$; the following expressions for the moments are then obtained:

$$\begin{aligned} M_{ij} - M^c_{ij} &= -A\left[(1-m)\frac{\partial^2 \dot{w}}{\partial x_i \partial x_j} + m\frac{\partial^2 \dot{w}}{\partial x_k \partial x_k}\delta_{ij}\right]; \\ M^c_{ij} &= -B\left[(1-n)\frac{\partial^2 \dot{w}^p}{\partial x_i \partial x_j} + n\frac{\partial^2 \dot{w}^p}{\partial x_k \partial x_k}\delta_{ij}\right]; \\ M^c_{ij} &= -D\left[(1-\nu)\frac{\partial^2 w^e}{\partial x_i \partial x_j} + \nu\frac{\partial^2 w^e}{\partial x_k \partial x_k}\delta_{ij}\right] - \frac{E\alpha}{3(1-\nu)}\tau\delta_{ij}, \end{aligned} \tag{5}$$

where

$$A = \frac{Gh^3}{12(1-m^2)}; \quad B = \frac{Hh^3}{12(1-n^2)}; \quad D = \frac{Eh^3}{12(1-\nu^2)}; \quad \tau = \int_{-\frac{h}{2}}^{\frac{h}{2}} (T - T_0) x_3 dx_3.$$

By applying the Laplace transform to relationships (5), eliminating the transforms of the magnitudes M^c_{ij}, w^e, and returning to the original functions, we obtain

$$\begin{aligned} M_{ij} = &-D\left[(1-\nu)\frac{\partial^2}{\partial x_i \partial x_j}(w - w^p(0)) + \nu\frac{\partial^2}{\partial x_k \partial x_k}(w - w^p(0))\delta_{ij}\right] - \\ &-A\left[(1-m)\frac{\partial^2 \dot{w}}{\partial x_i \partial x_j} + m\frac{\partial^2 \dot{w}}{\partial x_k \partial x_k}\delta_{ij}\right] - \frac{E\alpha}{3(1-\nu)}\tau\delta_{ij} + \\ &+\int_0^t \Big\{D(1-\nu)k_1 l^{-k_1(t-t_1)}\frac{\partial^2}{\partial x_i \partial x_j}(w - w^p(0)) + \\ &+\frac{D}{2}\left[(1+\nu)k_2 l^{-k_2(t-t_1)} - (1-\nu)k_1 l^{-k_1(t-t_1)}\right] \times \\ &\times \frac{\partial^2}{\partial x_k \partial x_k}(w - w^p(0))\delta_{ij} + \frac{E\alpha k_2}{3(1-\nu)} l^{-k_2(t-t_1)}\tau\delta_{ij}\Big\} dt_1, \end{aligned} \tag{6}$$

where

$$w^p(0) = w\Big|_{t=0} - w^e\Big|_{t=0}; \qquad k_1 = \frac{D(1-\nu)}{B(1-n)}; \qquad k_2 = \frac{D(1+\nu)}{B(1+n)}.$$

By substituting (6) in the equation of motion of the plate we obtain an integral-differential equation for the deflection:

$$D\Delta\Delta(w - w^p(0)) + A\Delta\Delta w - \frac{D(1-\nu)k_1}{2}\int_0^t l^{-k_1(t-t_1)}\Delta\Delta(w - w^p(0))\,dt_1 -$$

$$-\frac{D(1+\nu)k_2}{2}\int_0^t e^{-k_2(t-t_1)}\Delta\Delta\,(w-w^p(0))+\frac{E\alpha}{3(1-\nu)}\Delta\tau-$$

$$-\frac{E\alpha k_2}{3(1-\nu)}\int_0^t e^{-k_2(t-t_1)}\Delta\tau dt_1=N_1\frac{\partial^2 w}{\partial x_1^2}+2N_{12}\frac{\partial^2 w}{\partial x_1\partial x_2}+N_2\frac{\partial^2 w}{\partial x_2^2}+q-\rho h\frac{\partial^2 w}{\partial t^2},\quad(7)$$

where N_1, N_2, N_{12} are forces acting in the middle plane; q is the load; and ρ is the density of the material.

In what follows the terms containing the stresses and the temperature refer to the load q, i.e., we shall consider the equation

$$D\Delta\Delta w+A\Delta\Delta\dot{w}-\frac{D(1-\nu)k_1}{2}\int_0^t e^{-k_1(t-t_1)}\Delta\Delta w dt_1-$$

$$-\frac{D(1+\nu)k_2}{2}\int_0^t e^{-k_2(t-t_1)}\Delta\Delta w dt_1=N_1\frac{\partial^2 w}{\partial x_1^2}+2N_{12}\frac{\partial^2 w}{\partial x_1\partial x_2}+N_2\frac{\partial^2 w}{\partial x_2^2}+q-\rho h\frac{\partial^2 w}{\partial t^2}.\quad(8)$$

2. Consider the vibrations of a hinged rectangular plate of sides l_1 and l_2, and let $N_1=N_2=N_{12}=0$. Representing the deflection by the expansion

$$w=\sum_{\alpha,\beta=1,2\ldots}w_{\alpha\beta}(t)\sin\frac{\alpha\pi x_1}{l_1}\sin\frac{\beta\pi x_2}{l_2},\quad(9)$$

we obtain the following ordinary integral-differential equation for $w_{\alpha\beta}$:

$$\rho h\ddot{w}_{\alpha\beta}+Ar_{\alpha\beta}^2\dot{w}_{\alpha\beta}+Dr_{\alpha\beta}^2 w_{\alpha\beta}-\frac{D(1-\nu)k_1}{2}r_{\alpha\beta}^2\int_0^t e^{-k_1(t-t_1)}w_{\alpha\beta}(t_1)\,dt_1-$$

$$-\frac{D(1+\nu)k_2}{2}r_{\alpha\beta}^2\int_0^t e^{-k_2(t-t_1)}w_{\alpha\beta}(t_1)\,dt_1=q_{\alpha\beta},\quad(10)$$

where $q_{\alpha\beta}$ are the expansion coefficients of the load, $r_{\alpha\beta}^2=\pi^4\left(\frac{\alpha^2}{l_1^2}+\frac{\beta^2}{l_2^2}\right)^2$.

Equation (10) will be solved by means of the Laplace transform, with $\tilde{w}_{\alpha\beta}$ and $\tilde{q}_{\alpha\beta}$ standing for the transforms of the functions $w_{\alpha\beta}$ and $q_{\alpha\beta}$. The solution in the transform space will then be

$$\tilde{w}_{\alpha\beta}(s)=[\tilde{q}_{\alpha\beta}+(\rho hs+Ar_{\alpha\beta}^2)w_{\alpha\beta}(0)+\rho h\dot{w}_{\alpha\beta}(0)](s+k_1)(s+k_2)s^{-1}\Big\{\rho hs^2+$$

$$+[\rho h(k_1+k_2)+Ar_{\alpha\beta}^2]s^2+[\rho hk_1k_2+(Ak_1+Ak_2+D)r_{\alpha\beta}^2]s+$$

$$+\left[Ak_1k_2+\frac{D(1+\nu)k_1}{2}+\frac{D(1-\nu)k_2}{2}\right]r_{\alpha\beta}^2\Big\}^{-1}.\quad(11)$$

We shall now examine the roots of the denominator of (11). One root is equal to zero; of the three remaining roots, they can either all be real, or one real and two complex. In both cases, however, the real parts of the roots are negative, as can be seen by using the Gurvitz criterion. Since a more detailed analysis of the roots would be cumbersome, we shall not elucidate the conditions of existence of periodic solutions, noting only that in the case of complex roots the solution of the homogeneous equation corresponding to (10) will be

$$w_{\alpha\beta}(t)=C_0+C_1e^{m_rt}+e^{n_rt}(C_2\sin p_rt+C_3\cos p_rt),\quad(12)$$

where m_r $n_r + ip_r$, $n_r - ip_r$ are the roots of the denominator of (11); C_0, C_1, C_2, C_3 are constants dependent on the initial conditions.

Assume that $k_1 = k_2 = k$, i.e., $n = \nu$. This is a reasonable assumption, the values of n and ν being very close. Expression (11) simplifies then to

$$\bar{w}_{\alpha\beta}(s) = \frac{[\bar{q}_{\alpha\beta} + (\rho hs + Ar^2_{\alpha\beta})\, w_{\alpha\beta}(0) + \rho h\dot{w}_{\alpha\beta}(0)]\,(s+k)}{s\,[\rho hs^2 + (\rho hk + Ar^2_{\alpha\beta})\, s + Akr^2_{\alpha\beta} + Dr^2_{\alpha\beta}]} \tag{13}$$

and the denominator roots will be 0, $n_r + ip_r$, $n_r - ip_r$, where

$$n_r = -\frac{\rho hk + Ar^2_{\alpha\beta}}{2\rho h}; \qquad p_r = \frac{\sqrt{4\rho hDr^2_{\alpha\beta} - (\rho hk - Ar^2_{\alpha\beta})^2}}{2\rho h}. \tag{14}$$

The necessary condition of existence of periodic solutions of (10) will thus be

$$4\rho hDr^2_{\alpha\beta} - (\rho hk - Ar^2_{\alpha\beta})^2 > 0. \tag{15}$$

This inequality will always be satisfied at $A = 0$, beginning with some $r_{\alpha\beta}$.

If $\bar{q}_{\alpha\beta} = 0$, we obtain the following solution of (13) by passing to the original space:

$$w_{\alpha\beta}(t) = C_0 + l^{n_r t}(C_1 \sin p_r t + C_2 \cos p_r t), \tag{16}$$

where

$$C_0 = \frac{k}{\rho h\,(n_r^2 + p_r^2)}\,(Ar^2_{\alpha\beta} w_{\alpha\beta}(0) + \rho h\dot{w}_{\alpha\beta}(0));$$

$$C_1 = \left(\rho hn_r + Ar^2_{\alpha\beta} + \rho hk + \frac{An_r r^2_{\alpha\beta} k}{n_r^2 + p_r^2}\right) w_{\alpha\beta}(0) + \rho h\left(1 + \frac{n_r k}{n_r^2 + p_r^2}\right)\dot{w}_{\alpha\beta};$$

$$C_2 = \left(\rho hp_r - \frac{Ap_r r^2_{\alpha\beta} k}{n_r^2 + p_r^2}\right) w_{\alpha\beta}(0) - \frac{\rho hp_r k}{n_r^2 + p_r^2}\,\dot{w}_{\alpha\beta}(0).$$

Consider now the forced vibrations of the plate under the action of the force

$$q_{\alpha\beta}(t) = q^0_{\alpha\beta}\, l^{i\omega t}, \tag{17}$$

considering that $w_{\alpha\beta}(0) = \dot{w}_{\alpha\beta}(0) = 0$. The transform of the solution will then be the function

$$\bar{w}_{\alpha\beta}(s) = \frac{q_{\alpha\beta}(s+k)}{\rho hs\,(s - i\omega)\,(s - n_r - ip_r)\,(s - n_r + ip_r)}. \tag{18}$$

By returning to the original space we obtain

$$w_{\alpha\beta}(t) = \frac{q^0_{\alpha\beta}}{\rho h}\left\{\frac{a_0}{n_r^2 + p_r^2} + a_1 l^{i\omega t} + \frac{l^{n_r t}}{p_r}\left[\left(a_1 n_r + a_2 + \frac{a_0 n_r}{n_r^2 + p_r^2}\right)\sin p_r t + \right.\right.$$
$$\left.\left. + \left(a_1 p_r - \frac{a_0 n_r}{n_r^2 + p_r^2}\right)\cos p_r t\right]\right\}. \tag{19}$$

where

$$a_0 = \frac{ik}{\omega};$$

$$a_1 = -\frac{\omega(-\omega^2 + 2n_r k + n_r^2 + p_r^2) + i[\omega^2(2n_r + k) - k(n_r^2 + p_r^2)]}{\omega[\omega^4 + 2\omega^2(n_r^2 - p_r^2) + (n_r^2 + p_r^2)^2]};$$

$$a_2 = \frac{\omega[\omega^2 k + (n_r^2 + p_r^2)(2n_r + k) + 2k(n_r^2 - p_r^2)]}{\omega[\omega^4 + 2\omega^2(n_r^2 - p_r^2) + (n_r^2 + p_r^2)]} +$$

$$+ i\frac{\omega^4 + \omega^2(3n_r^2 - p_r^2) - 2n_r k(n_r^2 + p_r^2)}{\omega[\omega^4 + 2\omega^2(n_r^2 - p_r^2) + (n_r^2 + p_r^2)^2]}.$$

At $t \to \infty$ we obtain the following steady forced vibrations:

$$w_{\alpha\beta}(t) = i\frac{q_{\alpha\beta}^0 k}{\rho h\omega(n_r^2 + p_r^2)} + Rl^{i(\omega t - \gamma)}. \tag{20}$$

The amplitude here is determined by the formula

$$R = \frac{q_{\alpha\beta}^0}{\rho h\omega}\sqrt{\frac{\omega^2 + k}{(\omega^2 - n_r^2 - p_r^2)^2 + 4n_r^2\omega^2}}, \tag{21}$$

and the phase shift by the formula

$$\operatorname{tg}\gamma = \frac{-(k + 2n_r)\omega^2 + k(n_r^2 + p_r^2)}{\omega(-\omega^2 + 2n_r k + n_r^2 + p_r^2)}. \tag{22}$$

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STUDY OF THE STRESSED AND STRAINED STATE OF A PHYSICALLY NONLINEARLY ELASTIC PLATE WITH REINFORCED CIRCULAR HOLE

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The problem of the elastic equilibrium of a monoaxially extended plate with reinforced circular hole in the case of small strains and a linear stress-strain relationship was solved by Savin /1/ by the method of the theory of functions of a complex variable. The stress functions characterizing the state of stress and strain in the plate and the ring are:

for the plate

$$\varphi(z)=\frac{PR}{4}\left(z+\alpha_{-1}\frac{R}{z}\right);$$
$$\psi(z)=-\frac{PR}{4}\left(2\frac{z}{R}+\beta_{-1}\frac{R}{z}+\beta_{-3}\frac{R^3}{z^3}\right); \tag{1}$$

for the ring

$$\varphi(z)=\frac{PR}{4}\left(a_3\frac{z^3}{R^3}+a_1\frac{z}{R}+a_{-1}\frac{R}{z}\right);$$
$$\psi(z)=-\frac{PR}{4}\left(b_1\frac{z}{R}+b_{-1}\frac{R}{z}+b_{-3}\frac{R^3}{z^3}\right). \tag{2}$$

The coefficients of these functions were determined from the boundary conditions and the conditions of contact between the ring and the plate /2/.

Consider this problem for small deformations in the case of a nonlinear stress-strain relationship. The elastic law for physically nonlinear elastic materials (nonferrous metals, alloys, polymers) will be represented in the form /3, 4/

$$D=\frac{1}{3K}k(s_0)\,T_0+\frac{1}{2G}g(t_0^2)\,T', \tag{3}$$

where D is the strain tensor; T_0, T' are the tensor of the hydrostatic state of stress and the deviator of the state of stress, respectively; $k(s_0)$ and $g(t_0^2)$ are functions of the reduced average stress s_0 and the reduced intensity of the shearing stresses t_0, characterizing the physical nonlinearity of the material; the constants K and G are, respectively, the modulus of volume compression and the modulus of shear.

The differential equations and the computing formulas for the state of stress and strain will be determined for the particular case of the nonlinear elastic law (3), when the functions $k(s_0)$ and $g(t_0^2)$ are given in the form

$$k(s_0) \equiv 1; \quad g(t_0^2) \equiv 1 + g_2 t_0^2, \tag{4}$$

where g_2 is the constant of the material; $s_0 = \frac{\sigma_0}{9K}$; $t_0^2 = \frac{\tau_0^2}{G^2}$

In this case the resolving quasilinear compatibility equation of order four /5/ will contain one small parameter λ, characterizing the deviation of the nonlinear elastic law from Hooke's law. We shall represent the solution of this equation in the form of a power series in λ:

$$F(r, \varphi; \lambda) = \sum_{k=0}^{\infty} \lambda^k F^k(r, \varphi), \tag{5}$$

where $F^{(0)}(r, \varphi)$, $F^{(1)}(r, \varphi), \cdots$ are Airy stress functions of zero, first, etc. approximations.

The determination of the zero approximation $F^{(0)}(r, \varphi)$ is equivalent to solving the linear problem obtained by (1), (2), and the Goursat formula

$$2F(z, \bar{z}) = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \varkappa(z) + \overline{\varkappa(z)}, \text{where } \psi(z) = \frac{d\varkappa}{dz}.$$

The following nonhomogeneous biharmonic equations are obtained for determining the higher approximations:

$$\Delta\Delta F^{(n)} + L_{n-1}\left(F^{(0)}, F^{(1)}, \cdots, F^{(n-1)}; \frac{\partial F^{(0)}}{\partial r}, \cdots, \frac{\partial^2 F^{(n-1)}}{\partial r \partial \varphi}, \cdots\right) = 0. \tag{6}$$

For the first approximation we obtain the expression

$$\Delta\Delta F^{(1)} + L_0\left(F^{(0)}, \frac{\partial F^{(0)}}{\partial r}, \frac{\partial F^{(0)}}{\partial \varphi}, \frac{\partial^2 F^{(0)}}{\partial r \partial \varphi}, \frac{\partial^2 F^{(0)}}{\partial r^2}, \frac{\partial^2 F^{(0)}}{\partial \varphi^2}\right) = 0, \tag{7}$$

where the operator $L_0(F^{(0)})$ has the form

$$\begin{aligned} L_0(F^{(0)}) = {} & \frac{2}{r^4}\frac{\partial H^{(0)}}{\partial \varphi}\frac{\partial F^{(0)}}{\partial \varphi} - \frac{2}{r^3}\left(\frac{\partial H^{(0)}}{\partial \varphi}\frac{\partial^2 F^{(0)}}{\partial r \partial \varphi} + \frac{\partial^2 H^{(0)}}{\partial r \partial \varphi}\frac{\partial F^{(0)}}{\partial \varphi}\right) - \\ & - \frac{1}{r^2}\left(\frac{\partial^2 H^{(0)}}{\partial r^2}\frac{\partial^2 F^{(0)}}{\partial \varphi^2} + \frac{\partial^2 F^{(0)}}{\partial r^2}\frac{\partial^2 H^{(0)}}{\partial \varphi^2} - 2\frac{\partial^2 H^{(0)}}{\partial r \partial \varphi}\frac{\partial^2 F^{(0)}}{\partial r \partial \varphi}\right) - \\ & - \frac{1}{r}\left(\frac{\partial^2 H^{(0)}}{\partial r^2}\frac{\partial F^{(0)}}{\partial r} + \frac{\partial H^{(0)}}{\partial r}\frac{\partial^2 F^{(0)}}{\partial r^2}\right) + \\ & + \left(\frac{2}{3r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{2}{3r}\frac{\partial}{\partial r} + \frac{2}{3}\frac{\partial^2}{\partial r^2}\right)(H^{(0)}\Delta F^{(0)}). \end{aligned} \tag{8}$$

The material constant λ and the function $H^{(0)}(r, \varphi)$ in expressions (5) and (8) are determined by the relationships

$$H^{(0)}(r, \varphi) = \frac{9}{2}G^2 t_0^2; \qquad \lambda = \frac{g_2 K}{(3K+G)G^2}. \tag{9}$$

It is seen from (6), (7), and (8) that each step of the successive approximations basically reduces to solving the classic biharmonic problem: it is required to find a particular solution of the nonhomogeneous fourth-order equation (6), and then to determine with the aid of the boundary conditions and the conditions of connection the biharmonic function of the first, second, etc. approximations.

The first-approximation function $F^{(1)}$ must satisfy, in accordance with the statement of the problem and the method of solution, the following conditions:

a) The contour conditions.* If the inner contour L_1 of the ring (the circle of radius R_1) is free of external forces, while at infinity the plate is extended by forces p along the x axis, then

$$\frac{1}{r^2}\frac{\partial^2 F_k^{(1)}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial F_k^{(1)}}{\partial r}=0;\qquad \frac{1}{r^2}\frac{\partial F_k^{(1)}}{\partial\varphi}-\frac{1}{r}\frac{\partial^2 F_k^{(1)}}{\partial r\partial\varphi}=0 \text{ at } r=R_1;$$

$$\frac{1}{r^2}\frac{\partial F^{(1)}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial E^{(1)}}{\partial r}=0;\qquad \frac{\partial^2 F^{(1)}}{\partial r^2}=0;\qquad \frac{1}{r^2}\frac{\partial F^{(1)}}{\partial\varphi}-\frac{1}{r}\frac{\partial^2 F^{(1)}}{\partial r\partial\varphi}=0 \qquad (10)$$

$$\text{at } r=\infty.$$

b) The connection conditions. If the ring is soldered in a circular hole of the plate of radius $R=R_2$ (R_2 is the external radius of the ring), then on the contour L of contact between the plate and the ring we shall have

$$\frac{1}{r^2}\frac{\partial^2 F_k^{(0)}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial F_k^{(0)}}{\partial r}+\lambda_k\left(\frac{1}{r^2}\frac{\partial^2 F_k^{(1)}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial F_k^{(1)}}{\partial r}\right)=$$

$$=\frac{1}{r^2}\frac{\partial^2 F^{(0)}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial F^{(0)}}{\partial r}+\lambda\left(\frac{1}{r^2}\frac{\partial^2 F^{(1)}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial F^{(1)}}{\partial r}\right);$$

$$\frac{1}{r^2}\frac{\partial F_k^{(0)}}{\partial\varphi}-\frac{1}{r}\frac{\partial^2 F_k^{(0)}}{\partial r\partial\varphi}+\lambda\left(\frac{1}{r^2}\frac{\partial F_k^{(1)}}{\partial\varphi}-\frac{1}{r}\frac{\partial^2 F_k^{(1)}}{\partial r\partial\varphi}\right)= \qquad (11)$$

$$=\frac{1}{r^2}\frac{\partial F^{(0)}}{\partial\varphi}-\frac{1}{r}\frac{\partial^2 F^{(0)}}{\partial r\partial\varphi}+\lambda\left(\frac{1}{r^2}\frac{\partial F^{(1)}}{\partial\varphi}-\frac{1}{r}\frac{\partial^2 F^{(1)}}{\partial r\partial\varphi}\right);$$

$$u_k^{(0)}+\lambda_k u_k^{(1)}=u^{(0)}+\lambda u^{(1)};$$

$$v_k^{(0)}+\lambda_k v_k^{(1)}=v^{(0)}+\lambda v^{(1)}.$$

Using (3), (4), and (5), the displacement functions $u(r,\varphi)$ and $v(r,\varphi)$ at $\sigma_0=\frac{1}{3}(\sigma_r+\sigma_\varphi)$ are found from the equations based on the nonlinear elastic law (3) for the particular case (4):

$$\frac{\partial u}{\partial r}=\frac{1}{9K}(\sigma_r+\sigma_\varphi)+\frac{1}{6G}(2\sigma_r-\sigma_\varphi)+\frac{g_2}{27G^3}(\Phi^{(0)}+\lambda\Phi^{(1)}+\cdots)(2\sigma_2-\sigma_\varphi),$$

$$\frac{1}{r}\frac{\partial v}{\partial\varphi}+\frac{u}{r}=\frac{1}{9K}(\sigma_r+\sigma_\varphi)+\frac{1}{6G}(2\sigma_\varphi-\sigma_r)+$$

$$+\frac{g_2}{27G^3}(\Phi^{(0)}+\lambda\Phi^{(1)}+\cdots)(2\sigma_\varphi-\sigma_r). \qquad (12)$$

where the stresses and displacements are expanded in a series by

* The superscripts "p" (plate) of the components of the stresses, displacements, etc. have been omitted for the sake of simplicity of the notation.

the powers of λ, and

$$\Phi^{(0)} = \sigma_r^{(0)2} + \sigma_\varphi^{(0)2} + 3\tau_{r\varphi}^{(0)2} - \sigma_r^{(0)}\sigma_\varphi^{(0)};$$
$$\Phi^{(1)} = 2\sigma_r^{(0)}\sigma_r^{(1)} + 2\sigma_\varphi^{(0)}\sigma_\varphi^{(1)} + 6\tau_{r\varphi}^{(0)}\tau_{r\varphi}^{(1)} - \sigma_\varphi^{(0)}\sigma_r^{(1)} - \sigma_r^{(0)}\sigma_\varphi^{(1)}. \tag{13}$$

Consider the particular case when the plate material is subjected to the nonlinear elastic law (3), (4), and the ring material follows Hooke's law.

The following differential equation for the function $F^{(1)}$ is obtained by substituting (1), (8), (9) in (7):

$$\begin{aligned}
&\Delta\Delta F^{(1)} + p^3 \Bigg\{ \Bigg[\left(\frac{1}{2}\beta_{-1}^2 - 2\alpha_{-1}\beta_{-1} + 2\alpha_{-1}^2\right)\frac{R^4}{r^6} + \\
&+ \left(9\alpha_{-1}\beta_{-3} + \frac{9}{8}\beta_{-1}^3 + \frac{15}{2}\alpha_{-1}^2\beta_{-1} - \frac{27}{2}\beta_{-1}\beta_{-3}\right)\frac{R^8}{r^8} + \\
&+ (54\beta_{-3}^2 + 66\alpha_{-1}\beta_{-1}\beta_{-3})\frac{R^8}{r^{10}} + \frac{135}{2}\beta_{-1}\beta_{-3}^2\frac{R^{10}}{r^{12}}\Bigg] + \\
&+ \Bigg[\left(-18\alpha_{-1}^2 - \frac{9}{2}\beta_{-1}^2\right)\frac{R^4}{r^6} + \left(\frac{103}{4}\beta_{-1}\beta_{-3} - 114\alpha_{-1}\beta_{-3} + \frac{71}{8}\alpha_{-1}\beta_{-1}^2\right)\frac{R^6}{r^8} + \\
&+ (18\beta_{-1}^2\beta_{-3} - 135\beta_{-3}^2 + 24\alpha_{-1}^2\beta_{-3})\frac{R^8}{r^{10}} + \frac{189}{2}\alpha_{-1}\beta_{-3}^2\frac{R^{10}}{r^{12}} + \\
&+ \frac{405}{4}\beta_{-3}^3\frac{R^{12}}{r^{14}}\Bigg]\cos 2\varphi + \Bigg[\left(6\alpha_{-1} + \frac{9}{2}\beta_{-1}\right)\frac{R^2}{r^4} - 18\alpha_{-1}\beta_{-1}\frac{R^4}{r^6} + \\
&+ \left(\frac{9}{8}\alpha_{-1}^2\beta_{-1} - 45\beta_{-1}\beta_{-3} + \frac{9}{4}\alpha_{-1}\beta_{-3}\right)\frac{R^6}{r^8} + \frac{81}{8}\beta_{-1}\beta_{-3}^2\frac{R^{10}}{r^{12}}\Bigg]\cos 4\varphi + \\
&+ \left[18\alpha_{-1}\frac{R^2}{r^4} + \left(60\beta_{-3} - \frac{1}{2}\alpha_{-1}^2\right)\frac{R^4}{r^6}\right]\cos 6\varphi \Bigg\} = 0.
\end{aligned} \tag{14}$$

The solution of (14) can be represented as sum of a particular solution of (14) and of the general solution of the homogeneous equation $\Delta\Delta F^{(1)} = 0$. The latter is equal to the following, at boundary conditions (10) at $r = \infty$:

$$F_{\text{hom}}^{(1)} = C_{01} + C_{02}\ln r + \sum_{m=2}^{n}(C_{m3}r^{-m+2} + C_{m4}r^{-m})\cos m\varphi, \tag{15}$$

while the following expression is a particular solution of (14):

$$\begin{aligned}
F_r^{(1)} = &-\frac{p^3R^2}{8}\Bigg\{\Bigg[-\frac{8C_{01}}{p^3R^2}\ln\frac{r}{R} + \frac{1}{4}\left(\frac{1}{4}\beta_{-1}^2 - \alpha_{-1}\beta_{-1} + \alpha_{-1}^2\right)\frac{R^2}{r^2} + \\
&+ \frac{1}{8}\left(\alpha_{-1}\beta_{-3} + \frac{1}{8}\beta_{-1}^3 + \frac{5}{6}\alpha_{-1}^2\beta_{-1} - \frac{3}{2}\beta_{-1}\beta_{-3}\right)\frac{R^4}{r^4} + \\
&+ \frac{1}{288}(54\beta_{-3}^2 + 66\alpha_{-1}\beta_{-1}\beta_{-3})\frac{R^6}{r^6} + \frac{27}{320}\beta_{-1}\beta_{-3}^2\frac{R^8}{r^8}\Bigg] + \\
&+ \Bigg[-\frac{8C_{23}}{p^3R^2} - \frac{8C_{24}}{p^3R^4}\frac{R^2}{r^2} + \frac{1}{2}\left(6\alpha_{-1}^2 + \frac{3}{2}\beta_{-1}^2\right)\frac{R^2}{r^2}\ln\frac{r}{R} + \frac{1}{48}\Bigg(\frac{103}{4}\beta_{-1}\beta_{-3} - \\
&- 114\alpha_{-1}\beta_{-3} + \frac{71}{8}\alpha_{-1}\beta_{-1}^2 + \frac{1}{240}(18\beta_{-1}^2\beta_{-3} - 135\beta_{-3}^2 + 24\alpha_{-1}^2\beta_{-3})\frac{R^6}{r^6} +
\end{aligned}$$

$$+\frac{21}{160}\alpha_{-1}\beta_{-3}^2\frac{R^8}{r^8}+\frac{27}{448}\beta_{-3}^3\frac{R^{10}}{r^{10}}\Big]\cos 2\varphi+\Big[\frac{1}{8}\Big(2\alpha_{-1}+\frac{3}{2}\beta_{-1}\Big)-$$

$$-\frac{8C_{43}}{p^3R^4}\frac{R^2}{r^2}-\frac{3}{2}\alpha_{-1}\beta_{-1}\frac{R^2}{r^2}\ln\frac{r}{R}-\frac{8C_{44}}{p^3k^6}\frac{R^4}{r^4}-$$

$$-\frac{1}{20}\Big(\frac{9}{8}\alpha_{-1}^2\beta_{-1}-45\beta_{-1}\beta_{-3}+\frac{9}{4}\alpha_{-1}\beta_{-3}\Big)\frac{R^4}{r^4}\ln\frac{r}{R}+$$

$$+\frac{9}{448}\beta_{-1}\beta_{-3}^2\frac{R^8}{r^8}\Big]\cos 4\varphi+\Big[\frac{1}{8}\alpha_{-1}+\frac{1}{80}\Big(60\beta_{-3}-\frac{1}{2}\alpha_{-1}^2\Big)\frac{R^2}{r^2}-$$

$$-\frac{8C_{63}}{p^3R^6}\frac{R^4}{r^4}-\frac{8C_{64}}{p^3R^8}\frac{R^6}{r^6}\Big]\cos 6\varphi\Big\}. \qquad (16)$$

The integration constants C_{01}, C_{i3}, C_{i4} ($i = 2, 4, 6$) are determined from the contour conditions (10) and the connection conditions (11). The expressions obtained will not be given here for lack of space. The expression obtained for the coefficient of stress concentration on the basis of the first approximation is

$$\left(\frac{\sigma_\varphi}{p}\right)_{r=R}=\frac{1}{2}\Big[\Big(1+\frac{1}{2}\beta_{-1}\Big)-\Big(1-\frac{3}{2}\beta_{-3}\Big)\cos 2\varphi\Big]-$$

$$-\lambda\frac{p^2}{8}\Big[\Big(\alpha_{-1}^2+\frac{1}{4}\beta_{-1}^2+\frac{27}{4}\beta_{-3}^2-\alpha_{-1}\beta_{-1}+2\alpha_{-1}\beta_{-3}-3\beta_{-1}\beta_{-3}+$$

$$+\frac{5}{3}\alpha_{-1}^2\beta_{-1}+\frac{27}{5}\beta_{-1}\beta_{-3}^2+\frac{33}{4}\alpha_{-1}\beta_{-1}\beta_{-3}+\frac{1}{4}\beta_{-1}^3\Big)+\Big(-6\alpha_{-1}^2-$$

$$-\frac{3}{2}\beta_{-1}^2-\frac{27}{2}\beta_{-3}^2-19\alpha_{-1}\beta_{-3}+\frac{103}{24}\beta_{-1}\beta_{-3}+\frac{71}{48}\alpha_{-1}\beta_{-1}^2+$$

$$+\frac{63}{10}\alpha_{-1}\beta_{-3}^2+\frac{12}{5}\alpha_{-1}^2\beta_{-3}+\frac{9}{5}\beta_{-1}^2\beta_{-3}+\frac{135}{28}\beta_{-3}^3\Big)\cos 2\varphi+$$

$$+\Big(2\alpha_{-1}+\frac{3}{2}\beta_{-1}-3\alpha_{-1}\beta_{-1}+\frac{9}{40}\alpha_{-1}\beta_{-3}-\frac{9}{2}\beta_{-1}\beta_{-3}+$$

$$+\frac{9}{80}\alpha_{-1}^2\beta_{-1}+\frac{27}{56}\beta_{-1}\beta_{-3}^2\Big)\cos 4\varphi+\Big(6\beta_{-3}+3\alpha_{-1}-\frac{1}{20}\alpha_{-1}^2\Big)\cos 6\varphi\Big]. \qquad (17)$$

We shall consider several particular examples for a copper plate with soldered steel ring (washer), and determine the stress-concentration coefficient σ_φ in a first approximation.

I. Absolutely flexible ring, i.e., no ring has been soldered in the plate hole of radius R. The coefficients are then equal, respectively, to $\alpha_{-1} = 2$, $\beta_{-1} = 2$, $\beta_{-3} = -2$ and formula (17) yields

$$\left(\frac{\sigma_\varphi}{p}\right)_{r=R}=1-2\cos 2\varphi-\lambda p^2(3.066-4.338\cos 2\varphi+2.107\cos 4\varphi-$$

$$-0.775\cos 6\varphi). \qquad (18)$$

II. Absolutely rigid ring. The coefficients are then equal to $\alpha_{-1}=-\frac{2}{\varkappa}=-1.247$; $\beta_{-1}=1-\varkappa=-0.604$; $\beta_{-3}=\frac{2}{\varkappa}=1.247$ and (17) yields

$$\left(\frac{\sigma_\varphi}{p}\right)_{r=R}=0.349+0.423\cos 2\varphi-\lambda\frac{p^2}{8}(11.595-1.503\cos 2\varphi-$$

$$-3.179\cos 4\varphi+3.663\cos 6\varphi). \qquad (19)$$

III. Solid elastic circular washer. The coefficients are equal to $\alpha_{-1} = 2 - \frac{2(1+\varkappa)}{\varkappa+\mu} = -0.442$; $\beta_{-1} = 2\left[1 - \frac{1+\varkappa}{(1-\mu)+(1+\varkappa_1\mu)}\right] = -0.138$; $\beta_{-3} = 2\left[\frac{1+\varkappa}{\varkappa+\mu} - 1\right] = 0.442$ and (17) yields

$$\left(\frac{\sigma_\varphi}{p}\right)_{r=R} = 0.466 - 0.169\cos 2\varphi - \lambda\frac{p^2}{8}(7.748 - 0.219\cos 2\varphi - 1.059\cos 4\varphi + 1.316\cos 6\varphi). \tag{20}$$

Here $\varkappa = 3 - 4\nu$ for the plate, $\varkappa_1 = 3 - 4\nu_1$ for the ring, $\mu = \frac{G}{G_1}$; ν, ν_1 are the Poisson ratios for the plate and ring, respectively, determined from the formula $\nu = \frac{3K - 2G}{6K + 2G}$. The elastic constants for these materials are: copper $K = 1.37\times10^6$ kg/cm^2; $G = 0.46\times10^6$ kg/cm^2; $g_1 = 0.18\times10^6$; $\lambda = 0.255\times10^{-6}$ cm^4/kg; open-hearth steel $K = 1.821\times10^6$ kg/cm^2; $G_1 = 0.87\times10^6$ kg/cm^2; $g_2 = 0.085\times10^6$; $\lambda = 0.032\times10^{-6}$ cm^4/kg^2.

In the table are given the values of the stress-concentration coefficient (18), (19), (20) as a function of the magnitude of the external load for the copper plate. The relationship between the concentration coefficient $\frac{\sigma_\varphi}{p}$ and the external load for an unsupported circular hole has been plotted in Figure 1.

TABLE

φ		p kg/cm^2							
		100	200	300	400	450	500	550	600
0°	I	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000
	II	0.768	0.758	0.741	0.717	0.703	0.687	0.669	0.650
	III	0.295	0.291	0.285	0.275	0.271	0.265	0.258	0.251
45°	I	0.998	0.990	0.978	0.961	0.951	0.939	0.926	0.912
	II	0.344	0.330	0.306	0.273	0.253	0.231	0.206	0.179
	III	0.463	0.459	0.451	0.440	0.433	0.425	0.417	0.408
90°	I	2.973	2.894	2.762	2.578	2.469	2.341	2.203	2.051
	II	−0.075	−0.081	−0.091	−0.105	−0.114	−0.123	−0.134	−0.145
	III	0.633	0.631	0.628	0.624	0.622	0.619	0.616	0.613

The report dealt with the case of elastic equilibrium of an axisymmetric physically nonlinear plate with reinforced circular hole. We shall give here the values of the stress components for the plate:

$$\begin{aligned} \sigma_r &= \sigma_r^{(0)} + \lambda\sigma_r^{(1)} = \frac{p}{2}\left(2 - \beta_{-1}\frac{R^2}{r^2}\right) + \lambda\left[\frac{c}{r^2} + \frac{p^3\beta_{-1}^2}{8}\left(\frac{R^4}{r^4} + \frac{1}{2}\beta_{-1}\frac{R^6}{r^6}\right)\right], \\ \sigma_\varphi &= \sigma_\varphi^{(0)} + \lambda\sigma_\varphi^{(1)} = \frac{p}{2}\left(2 + \beta_{-1}\frac{R^2}{r^2}\right) + \lambda\left[-\frac{c}{r^2} - \frac{p^3\beta_{-1}^2}{8}\left(3\frac{R^4}{r^4} + \frac{5}{2}\beta_{-1}\frac{R^6}{r^6}\right)\right]. \end{aligned} \tag{21}$$

Using these formulas, curves of the variation of the stress σ_φ as a function of the external load p have been plotted in Figure 2 for different cases of reinforcement; here σ_φ is the value corresponding to the linear theory, and $\sigma_\varphi^{(0)}$ the value corresponding to the nonlinear theory, allowing for the first approximation.

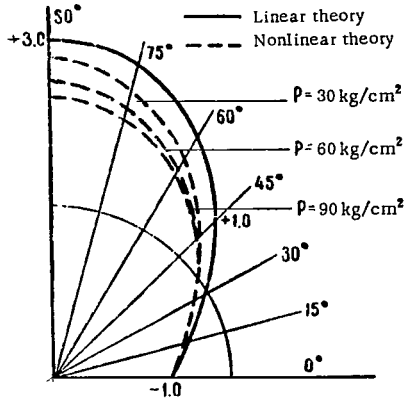

FIGURE 1.

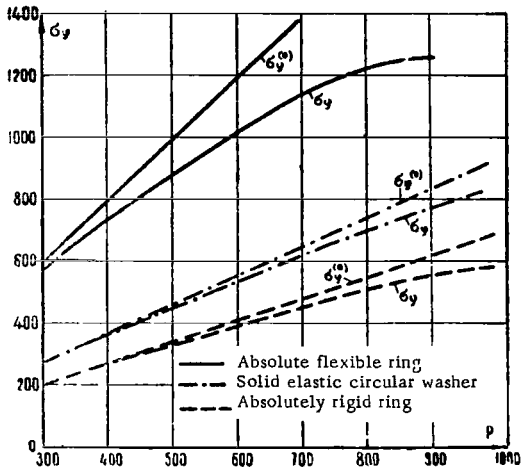

FIGURE 2.

A study of the influence of the second approximation on the state of stress of a biaxially extended plate with circular hole has also been conducted. This problem was treated in a first approximation in /6/. The process of determining the second approximation does not differ basically from the already described /5, 6/ process of obtaining the first approximation. We shall give here only the expression for the stress function allowing for the zero, first, and second approximations:

$$F = F^{(0)} + \lambda F^{(1)} + \lambda^2 F^{(2)} = \frac{p}{2}(r^2 - 2R^2 \ln r) - \lambda p^2 R^2 \left[\ln r + \right.$$

$$\left. + \frac{1}{4}\left(\frac{R^2}{r^2} + \frac{1}{2}\frac{R^4}{r^4}\right)\right] + \lambda^2 p^5 R^2 \left[\frac{13}{5}\ln r + \right.$$

$$\left. + \frac{1}{4}\left(-\frac{R^2}{r^2} - \frac{1}{3}\frac{R^4}{r^4} + \frac{47}{36}\frac{R^6}{r^6} + \frac{59}{80}\frac{R^8}{r^8}\right)\right]. \qquad (22)$$

Figure 3 represents the influence of the first and second approximations on the stress σ_φ, calculated by (22). The unbroken line corresponds to the linear theory, the dotted line to the first approximation, and the third line to the second approximation.

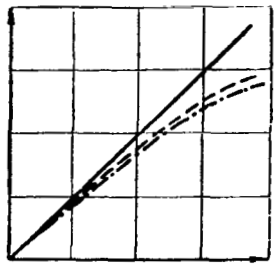

FIGURE 3.

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CALCULATION OF A RIBBED TOROIDAL SHELL UNDER A SYMMETRICAL LOAD

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A symmetrically loaded circular toroidal shell, with identical meridional ribs of uniform cross section disposed close to each other and spaced through the same angle φ, is considered (Figure 1). The calculation is based on the model of structural anisotropy, which consists in replacing the original shell by an equivalent smooth orthotropic shell. The main symbols used are the same as those in /5/, /6/.

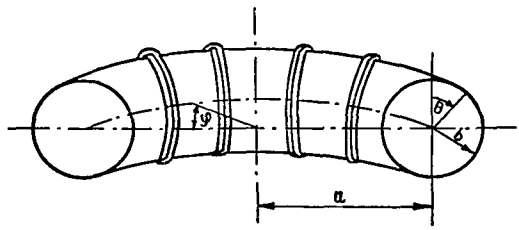

FIGURE 1.

1. Shells are usually stiffened by ribs in order to increase their flexural rigidity only. Hence the cross-sectional area of the rib is small compared with the area of the shell belt, a fact which makes it possible to use the usual expression for the stresses in the skin (cf. /3/)

$$\sigma_{11} = \frac{E}{1-\nu^2}\,[\varepsilon_1 + \nu\varepsilon_2 + z(\varkappa_1 + \nu\varkappa_2)],$$
$$\sigma_{22} = \frac{E}{1-\nu^2}\,[\varepsilon_2 + \nu\varepsilon_1 + z(\varkappa_2 + \nu\varkappa_1)], \qquad -\frac{h}{2} \leqslant z \leqslant \frac{h}{2}. \tag{1.1}$$

The stresses in the rib are, assuming that the ribs are strained in their planes only:

$$\sigma_{11}^{p} = E(\varepsilon_1 + z\varkappa_1). \tag{1.2}$$

Here z represents the distance, measured from the middle surface.

The radius of curvature of the meridian is usually considerably larger than the dimensions of the rib cross section. In other words, in our treatment of the rib we meet the problem of the bending of a bar of small

curvature. It is then possible to use for the equivalent orthotropic shell the simplified elastic relationships

$$T_1 = \frac{Eh}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2); \quad T_2 = \frac{Eh}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1);$$

$$M_1 = \frac{Eh^3}{12(1-\nu^2)}\left\{\left[1 + \frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}\right]\varkappa_1 + \nu\varkappa_2\right\}; \tag{1.3}$$

$$M_2 = \frac{Eh^3}{12(1-\nu^2)}(\varkappa_2 + \nu\varkappa_1),$$

where r is the radius of the parallel circle; J is the moment of inertia of the rib cross section relative to the axis $z = 0$ (Figure 2); $\varphi_0 = \frac{2\pi}{n}$; n is the number of ribs; ε_1, ε_2, $\varkappa_1$, $\varkappa_2$ are the strain parameters, calculated by the standard formulas (cf. /5/).

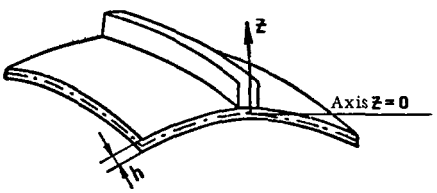

FIGURE 2.

With the aid of (1.3), the equations of symmetrical deformation of an orthotropic shell given in /1/ and /2/ can be transformed, with an accuracy up to the order of h/R, to the form

$$\frac{d^2\bar{\vartheta}}{d\theta^2} - \frac{d\bar{\vartheta}}{d\theta}\left[\frac{R_1\cos\theta}{r} + \frac{1}{R}\frac{dR_1}{d\theta}\right] +$$

$$+\left\{\bar{\vartheta}\left(\frac{R_1\cos\theta}{r}\right)^2 \frac{3\left[\frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}\right]^2 + 2\left[\frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}\right]}{4\left[1 + \frac{J\cdot 12(1-\nu)}{r\varphi_0 h^3}\right]^2}\right\} -$$

$$-\frac{\sqrt{12(1-\nu^2)}}{h}\frac{R_1^2\sin\theta}{r\sqrt{1 + \frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}}}\bar{Y} = \tag{1.4}$$

$$= \frac{12(1-\nu^2)}{Eh^3}\frac{R_1}{\sqrt{1 + \frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}}}\left[\frac{dr M_1^*}{d\theta} - M_2^* R_1\cos\theta\right],$$

$$\frac{d^2\bar{Y}}{d\theta^2} - \frac{d\bar{Y}}{d\theta}\left[\frac{R_1\cos\theta}{r} + \frac{1}{R_1}\frac{dR_1}{d\theta}\right] +$$

$$+\frac{\sqrt{12(1-\nu^2)}}{h}\frac{R_1^2\sin\theta}{r\sqrt{1 + \frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}}}\bar{\vartheta} = 0.$$

Here $r = R_2\sin\theta$; R_1, R_2 are the principal radii of curvature of the middle

surface; $\vartheta = \frac{\bar{\vartheta}}{r}\left[1 + \frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}\right]^{-1/2}$ is the slope of the tangent to the meridian; $Y = \frac{\bar{Y}}{r}$ is the stress function, with whose aid the forces T_1 and T_2 are calculated by the formulas

$$T_1 = T_1^* + \frac{Eh^2}{\sqrt{12(1-\nu^2)}}\frac{\cos\theta}{r}Y, \quad T_2 = T_2^* + \frac{Eh^2}{\sqrt{12(1-\nu^2)}}\frac{1}{R_1}\frac{dY}{d\theta}. \tag{1.5}$$

The asterisks in (1.4) and (1.5) designate the functions of the static system /5/, representing some solution of the system of equations of equilibrium.

By neglecting in the first relationship of (1.4) the term between braces we obtain the following single equation relative to the complex function:

$$\tilde{\vartheta} = \bar{\vartheta} + i\bar{Y}; \tag{1.6}$$

$$\frac{d\tilde{\vartheta}}{d\theta^2} - \frac{d\tilde{\vartheta}}{d\theta}\left[\frac{R\cos\theta}{r} + \frac{1}{R_1}\frac{dR_1}{d\theta}\right] + i\frac{\sqrt{12(1-\nu^2)}}{h}\frac{R_1^2\sin\theta}{r\sqrt{1+\frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}}}\tilde{\vartheta} =$$

$$= \frac{12(1-\nu^2)}{Eh^3}\frac{R_1}{\sqrt{1+\frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}}}\left[\frac{drM_1^*}{d\theta} - M_2^* R_1\cos\theta\right]. \tag{1.7}$$

The reduction of (1.4) to (1.7) is possible in view of the fact that the coefficient

$$\frac{\sqrt{12(1-\nu^2)}}{h}\frac{R_1^2\sin\theta}{r\sqrt{1+\frac{J\cdot 12(1-\nu^2)}{r\varphi_0 h^3}}}$$

represents, in a majority of cases, the product of some function of the order of unity and of a large factor $2k^2$, i.e., that the asymptotic methods are applicable to system (1.4). It is easily checked that the error introduced when replacing (1.4) by (1.7) does not exceed the error of the asymptotic solution of (1.7).

2. In the case of a circular toroidal shell (Figure 1)

$$R_1 = b = \text{const}; \quad r = a(1+\alpha\sin\theta); \quad \alpha = \frac{b}{a}. \tag{2.1}$$

By substituting (2.1) in (1.7) and calculating the function of the static system as done in /6/ we obtain the equation

$$\frac{d^2\tilde{\vartheta}}{d\theta^2} - \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{d\tilde{\vartheta}}{d\theta} + i2k^2\frac{\sin\theta}{\sqrt{1+\alpha\sin\theta+l^2(1+\alpha\sin\theta)^2}}\tilde{\vartheta} =$$

$$= \frac{a\sqrt{12(1-\nu^2)}}{Eh^2}\frac{P}{2\pi}2k^2\frac{\sqrt{1+\alpha\sin\theta}}{\sqrt{1+l^2(1+\alpha\sin\theta)}}\cos\theta, \tag{2.2}$$

where P is the axial force at section $\theta = 0$,

$$\frac{1}{l^2} = \frac{J \cdot 12(1-\nu^2)}{a\varphi_0 h^3},$$

$$2k^2 = \frac{\sqrt{12(1-\nu^2)}}{h}\,\frac{b^2 l}{a} \gg 1. \tag{2.3}$$

The asymptotic solution of equation (2.2) will be determined by the method of "standard equations" exposed in /6/. By repeating the calculations conducted in /6/ we obtain

$$\frac{Eh^2}{a\sqrt{12(1-\nu^2)}}\,\tilde{\vartheta} = \varphi(\theta)\{\tilde{C}_1 h_1[-i\lambda\omega(\theta)] + \tilde{C}_2 h_2[-i\lambda\omega(\theta)]\} + \Phi^{\mathrm{I}}(\theta), \tag{2.4}$$

$$\Phi^{\mathrm{P}}(\theta) = \frac{P}{2\pi}\left\{i\left[\frac{1+\alpha\sin\theta}{\sin\theta}\cos\theta - \frac{\varphi(\theta)}{\omega(\theta)}\right] - \lambda\varphi(\theta)E[-\lambda\omega(\theta)]\right\}, \tag{2.5}$$

$$\omega(\theta) = \operatorname{sign}\theta\left[\frac{3}{2}\int_0^{|\theta|}\sqrt[4]{\frac{\sin^2\theta}{1+\alpha\sin\theta+l^2(1+\alpha\sin\theta)^2}}\,d\theta\right]^{2/3}; \tag{2.6}$$

$$\varphi(\theta) = \sqrt{\frac{1+\alpha\sin\theta}{\omega'(\theta)}}. \tag{2.7}$$

$$\lambda = \sqrt[3]{2k^2}.$$

The Airy functions h_1, h_2 and the Clark function E entering in (2.4) and (2.5) are tabulated (the tables and the asymptotic representations are given in /6/).

Thus, in the case of a ribbed torus, the solution is represented through the same functions as in the case of a smooth torus, the only difference being the relationship between the argument $\omega(\theta)$ and the variable θ. It follows that the ribs do not introduce any basic difficulty in the calculation of the torus.

As in the case of a smooth torus, the variability of the smooth function $\varphi(\theta)$ can be neglected, and the following simplified relationships can be used instead of (2.4) and (2.5):

$$\frac{Eh^2}{a\sqrt{12(1-\nu^2)}}\,\tilde{\vartheta} \approx \tilde{C}_1 h_1[-i\lambda\omega(\theta)] + \tilde{C}_2 h_2[-i\lambda\omega(\theta)] + \Phi^{\mathrm{P}}(\theta) \tag{2.8}$$

$$\Phi^{\mathrm{P}}(\theta) \approx \frac{P}{2\pi}\left\{i\left[\frac{1+\alpha\sin\theta}{\sin\theta}\cos\theta - \frac{1}{\omega(\theta)}\right] - \lambda E[-\lambda\omega(\theta)]\right\}. \tag{2.9}$$

This does not introduce a considerable error in the calculation of the stresses, since the expression

$$\varphi(\theta)\{\tilde{C}_1 h_1[-i\lambda\omega(\theta)] + \tilde{C}_2 h_2[-i\lambda\omega(\theta)]\}$$

characterizes a rapidly dampened edge effect, and the magnitude

$$\varphi(\theta)\left\{\frac{i}{\omega(\theta)} + \lambda E[-\lambda\omega(\theta)]\right\}$$

is substantial only near zero.

Consider the particular solution (2.9). It corresponds to a complete toroidal shell, cut by the parallel circle $\theta = -\frac{\pi}{2}$ and loaded by the axial force P. At the edges the slope of the tangent to the meridian and the

thrust are equal to zero. Starting from (2.9), we find the stresses and the divergence of the shell ends:

$$\delta_V \approx P \frac{b\sqrt{12(1-\nu^2)}}{Eh^2\sqrt{1+\frac{J\cdot 12(1-\nu^2)}{\alpha\tau_0 h^3}}}. \tag{2.10}$$

The basic stresses in the shell σ_{11}^{τ} (flexural meridional in the external fibers) and σ_{22}^{τ} (circumferential) are calculated by the formulas

$$\begin{gathered}
\frac{\sigma_{11}^{\tau}}{\sigma_V} \approx \frac{3}{\sqrt{3(1-\nu^2)}} \frac{\lambda^2\omega'(\theta)\,\mathrm{Re}\,E'[-\lambda\omega(\theta)]}{\alpha(1+\alpha\sin\theta)\sqrt{1+\frac{1}{l^2(1+\alpha\sin\theta)}}}, \\
\frac{\sigma_{22}^{\tau}}{\sigma_V} \approx -\alpha\sin\theta - \frac{1}{\sin^2\theta} + \frac{\omega'(\theta)}{\omega^2(\theta)} + \lambda^2\omega'(\theta)\,\mathrm{Jm}\,E'[-\lambda\omega(\theta)], \\
\sigma_V = \frac{P}{2\pi a h}.
\end{gathered} \tag{2.11}$$

Equalities (1.1) and (1.5) were used in the derivation of (2.11).

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CALCULATING ARCH DAMS

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Contemporary arch dams represent a thick plate of doubly varying curvature and variable thickness, clamped elastically by the contour, or with an unbroken perimetric joint. Arch dams are more flexible and stronger than buttress and gravity dams; they resist overturning and sliding, and are economical.

A study of the materials of the Fifth International Congress on High Dams shows that in Europe, and in particular in France and Italy, there is a noticeable trend to use arch dams instead of gravity dams. In the USA, on the other hand, gravity dams are predominant, a fact explained by the specific economic conditions in that country. In the USSR arch dams are still a rarity, due to the fact that hydraulic structures have mainly been built on the rivers of the flat north country.

The object of this paper is to illustrate several approximate methods of calculation used by us in designing the shells of the Ladzhanuri and Inguri hydroelectric power plants.

In a first approximation the circumferential stresses in the dams were determined on the basis of the membrane scheme of a thin-walled cylinder, by the Mariotte formula

$$\sigma = \frac{PR}{\delta} \leqslant [\sigma_0], \tag{1}$$

where P is the hydrostatic pressure per unit radius of the axial line; R is the radius of the axial line; δ is the constant thickness of the ring; $[\sigma_0]$ is the allowable compressive stress of the dam material per unit length. For relatively thick rings we used the solution of the Lamé problem, considering each arch section of the dam as closed hollow cylinders.

In order to allow for the actual boundary conditions, the temperature stresses, and the other computing factors, the dams were then calculated by the "pure arch" principle proposed by Ritter /1/, allowing for the elastic strains of the supports according to Voigt.

The transition from the one-dimensional problem to the two-dimensional was realized on the basis of the Kalinovich-Rezal assumptions /2/ relative to the distribution of the external load between the arches and the wall. By Kalinovich's method, the dam calculation was reduced to an approximate integration of the equation of the bent axis of a cantilever beam of variable rigidity lying on a Winkler elastic foundation:

$$E\frac{d^2}{dx^2}\left[J(x)\frac{d^2w}{dx^2}\right] = q(x) - \frac{w}{k(x)}, \tag{2}$$

where E is the modulus of elasticity; $J(x)$ is the moment of inertia; w is the normal deflection of the cantilever beam; $q(x)$ is the external load; $k(x)$ is the coefficient of elastic yielding of the arc.

Equation (2) was solved both numerically with the aid of a computer (by the method of finite differences) and by the direct variational method of Bubnov-Galerkin.

At the limit (infinite number of arches and walls), expression (2) becomes

$$D\nabla^4 w + 2\frac{\partial D}{\partial \alpha}\frac{\partial}{\partial \alpha}\nabla^2 w + 2\frac{\partial D}{\partial \beta}\frac{\partial}{\partial \beta}\nabla^2 w -$$

$$-(1-\nu)\left(\frac{\partial^2 D}{\partial \alpha^2}\frac{\partial^2 w}{\partial \beta^2} - 2\frac{\partial^2 D}{\partial \alpha \partial \beta}\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{\partial^2 D}{\partial \beta^2}\frac{\partial^2 w}{\partial \alpha^2}\right) = q(\alpha, \beta) - \frac{w}{k(\alpha, \beta)}. \qquad (3)$$

where D is the cylindrical rigidity; ν is the Poisson ratio; α, β are the curvilinear orthogonal coordinate lines of the middle surface of the dam shell.

Equations (2) and (3) can be refined by passing to the Filonenko-Borodich elastic foundation with two foundation moduli or to the generalized Vlasov elastic foundation /3, 4/.

The method of "arches and walls" described by the equilibrium equation (2) does not allow for the influence of the shearing forces $S_{12} \approx S_{21} = S$, the normal forces along the height of the dam T, and the torsional moments $H_{12} \approx H_{21} = H$. More correct results are obtained when the dams are calculated by Vlasov's engineering moment theory of shells /3/, based on the Kirchhoff-Love hypotheses.

The following system of equations relative to the components u, v, w of the principal displacement vector was obtained for calculating shallow arch dam shells:

$$\frac{E\delta}{1-\nu^2}\left[\left(\frac{\partial u}{\partial \alpha} + k_1 w\right)(k_1 + \nu k_2) + \left(\frac{\partial v}{\partial \beta} + k_2 w\right)(k_2 + \nu k_1)\right] + L_1 - Z = 0,$$

$$\frac{E}{1-\nu^2}L_5 + \frac{E}{2(1+\nu)}L_6 + X = 0, \qquad (4)$$

$$\frac{E}{1-\nu^2}L_7 + \frac{E}{2(1+\nu)}L_8 + Y = 0,$$

where

$$L_1 = 2L_2 + L_3 + L_4,$$

$$L_2 = \frac{\partial^2 D}{\partial \alpha \partial \beta}\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{\partial D}{\partial \alpha}\frac{\partial^3 w}{\partial \alpha \partial \beta^2} + \frac{\partial D}{\partial \beta}\frac{\partial^3 w}{\partial \alpha^2 \partial \beta} +$$

$$+ D\frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} - \nu\frac{\partial^2 D}{\partial \alpha \partial \beta}\frac{\partial^2 w}{\partial \alpha \partial \beta}, \qquad (5)$$

$$L_3 = \frac{\partial^2 D}{\partial \alpha^2}\left(\frac{\partial^2 w}{\partial \alpha^2} + \nu\frac{\partial^2 w}{\partial \beta^2}\right) + 2\frac{\partial D}{\partial \alpha}\frac{\partial^3 w}{\partial \alpha^3} + D\frac{\partial^4 w}{\partial \alpha^4},$$

$$L_4 = \frac{\partial^2 D}{\partial \beta^2}\left(\frac{\partial^2 w}{\partial \beta^2} + \nu\frac{\partial^2 w}{\partial \alpha^2}\right) + 2\frac{\partial D}{\partial \beta}\frac{\partial^3 w}{\partial \beta^3} + D\frac{\partial^4 w}{\partial \beta^4}.$$

$$L_5 = \frac{\partial\delta}{\partial\alpha}\left[\frac{\partial u}{\partial\alpha} + \nu\frac{\partial v}{\partial\beta} + w(k_1 + \nu k_2)\right] +$$

$$+ \delta\left[\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial w}{\partial\alpha}(k_1 + \nu k_2) + \nu\frac{\partial^2 v}{\partial\alpha\partial\beta} + w\left(\frac{\partial k_1}{\partial\alpha} + \nu\frac{\partial k_2}{\partial\alpha}\right)\right],$$

$$L_6 = \frac{\partial\delta}{\partial\beta}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha}\right) + \delta\left(\frac{\partial^2 u}{\partial\beta^2} + \frac{\partial^2 v}{\partial\alpha\partial\beta}\right), \qquad (5)$$

$$L_7 = \frac{\partial\delta}{\partial\beta}\left[\frac{\partial v}{\partial\beta} + \nu\frac{\partial u}{\partial\alpha} + w(k_2 + \nu k_1)\right] + \qquad \text{(cont'd)}$$

$$+ \delta\left[\frac{\partial^2 v}{\partial\beta^2} + \frac{\partial w}{\partial\beta}(k_2 + \nu k_1) + \nu\frac{\partial^2 u}{\partial\alpha\partial\beta} + w\left(\frac{\partial k_2}{\partial\beta} + \nu\frac{\partial k_1}{\partial\beta}\right)\right],$$

$$L_8 = \frac{\partial\delta}{\partial\alpha}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha}\right) + \delta\left(\frac{\partial^2 u}{\partial\alpha\partial\beta} + \frac{\partial^2 v}{\partial\alpha^2}\right).$$

In equations (4) $k_1 = k_1(\alpha, \beta)$; $k_2 = k_2(\alpha, \beta)$ are the principal curvatures of the coordinate surface on the lines β = const, α = const, respectively; $X = X(\alpha, \beta)$, $Y = Y(\alpha, \beta)$, $Z = Z(\alpha, \beta)$ are the components of the vector of intensity of the surface load per unit length in the axes of a mobile trihedron; $\delta = \delta(\alpha, \beta)$ is the dam thickness.

In order to obtain equations (4), the values of the lateral forces Q_1, Q_2 are eliminated from the equations of equilibrium of the shell element, and the forces M_1, M_2, H, T_1, T_2, S are then expressed through u, v, w by the standard relationships of the theory of elasticity.

The same method was used to obtain the basic equations for sloping dam shells of conic and spherical contours. These equations were formed relative to u, v, w and also of a mixed type — relative to the normal displacements w and the generalized stress function φ, similar to the Airy function in the two-dimensional problem of the theory of elasticity.

To equations (4) are added the boundary conditions. In the case of absolutely rigid rock these conditions are:

a) in the plane of connection between the dam and the foundation, and at the inclined shores of the dam

$$u = v = w = \left.\frac{\partial w}{\partial n}\right|_L = 0, \qquad (6)$$

b) on the crest level

$$\frac{\partial u}{\partial\alpha} + \nu\frac{\partial v}{\partial\beta} + w(k_1 + \nu k_2) = 0,$$

$$\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} = 0, \qquad (7)$$

$$\frac{\partial^2 w}{\partial\alpha^2} + \nu\frac{\partial^2 w}{\partial\beta^2} = 0,$$

$$\frac{\partial^3 w}{\partial\alpha^3} + (2 - \nu)\frac{\partial^3 w}{\partial\alpha\partial\beta^2} = 0.$$

In equations (7) the coordinate lines α and β are directed along the dam height and the ravine width, respectively.

The above-considered approximate methods of calculating arch dams, based on the Navier and Kirchhoff-Love hypotheses can be insufficiently accurate at certain values of the dam parameters. The calculation of the dam by means of the general equations of the theory of elasticity is therefore of practical importance.

As known /5/, the six conditions of equilibrium of an elementary curvilinear parallelepiped in space relative to an arbitrary system of triorthogonal curvilinear coordinates α, β, γ are represented by the following three equations, written in stress components:

$$
\begin{gathered}
\frac{\partial}{\partial\alpha}(h_2^{-1}h_3^{-1}\sigma_{\alpha\alpha})+\frac{\partial}{\partial\beta}(h_1^{-1}h_3^{-1}\sigma_{\alpha\beta})+\frac{\partial}{\partial\gamma}(h_1^{-1}h_2^{-1}\sigma_{\alpha\gamma})-\sigma_{\beta\beta}h_3^{-1}\frac{\partial h_2^{-1}}{\partial\alpha}- \\
-\sigma_{\gamma\gamma}h_2^{-1}\frac{\partial h_3^{-1}}{\partial\alpha}+\sigma_{\alpha\beta}h_3^{-1}\frac{\partial h_1^{-1}}{\partial\beta}+\sigma_{\alpha\gamma}h_2^{-1}\frac{\partial h_1^{-1}}{\partial\gamma}+h_1^{-1}h_2^{-1}h_3^{-1}P_\alpha=0, \\
\frac{\partial}{\partial\beta}(h_3^{-1}h_1^{-1}\sigma_{\beta\beta})+\frac{\partial}{\partial\gamma}(h_2^{-1}h_1^{-1}\sigma_{\beta\gamma})+\frac{\partial}{\partial\alpha}(h_2^{-1}h_3^{-1}\sigma_{\beta\alpha})-\sigma_{\gamma\gamma}h_1^{-1}\frac{\partial h_3^{-1}}{\partial\beta}- \\
-\sigma_{\alpha\alpha}h_3^{-1}\frac{\partial h_1^{-1}}{\partial\beta}+\sigma_{\gamma\beta}h_1^{-1}\frac{\partial h_2^{-1}}{\partial\gamma}+\sigma_{\alpha\beta}h_3^{-1}\frac{\partial h_2^{-1}}{\partial\alpha}+h_1^{-1}h_2^{-1}h_3^{-1}P_\beta=0, \\
\frac{\partial}{\partial\gamma}(h_1^{-1}h_2^{-1}\sigma_{\gamma\gamma})+\frac{\partial}{\partial\alpha}(h_3^{-1}h_2^{-1}\sigma_{\gamma\alpha})+\frac{\partial}{\partial\beta}(h_3^{-1}h_1^{-1}\sigma_{\gamma\beta})-\sigma_{\alpha\alpha}h_2^{-1}\frac{\partial h_1^{-1}}{\partial\gamma}- \\
-\sigma_{\beta\beta}h_1^{-1}\frac{\partial h_2^{-1}}{\partial\gamma}+\sigma_{\alpha\gamma}h_2^{-1}\frac{\partial h_3^{-1}}{\partial\alpha}+\sigma_{\beta\gamma}h_1^{-1}\frac{\partial h_3{}^{1}}{\partial\beta}+h_1^{-1}h_2^{-1}h_3^{-1}P_\gamma=0,
\end{gathered}
\tag{8}
$$

where $h_1=h_1(\alpha,\beta,\gamma)$, $h_2=h_2(\alpha,\beta,\gamma)$, $h_3=h_3(\alpha,\beta,\gamma)$ are differential parameters of the first kind, representing the inverse values of the Lamé coefficients.

If the concrete dams are built by the block system, in order to stabilize the slump strains, then in equations (8) the components of the volume forces P_α, P_β, P_γ vanish, since deformations up to the closing of the last block occur due to the dead weight. The influence of the dead weight must be allowed for, however, in a separate way (initial stresses).

By approximating the dam surface by a system of stepped cylindrical shells we shall attempt to obtain initial equations for an approximate calculation of a dam of arbitrary contour.

Neglecting the squares and products of the displacement components u, v, w, we obtain for the six independent components of the strain tensor /5/

$$
\begin{gathered}
e_{\alpha\alpha}=h_1\frac{\partial u}{\partial\alpha}+h_1h_2v\frac{\partial h_1^{-1}}{\partial\beta}+h_1h_3w\frac{\partial h_1^{-1}}{\partial\gamma}, \\
e_{\beta\beta}=h_2\frac{\partial v}{\partial\beta}+h_2h_3w\frac{\partial h_2{}^{1}}{\partial\gamma}+h_1h_2u\frac{\partial h_2^{-1}}{\partial\alpha}, \\
e_{\gamma\gamma}=h_3\frac{\partial w}{\partial\gamma}+h_1h_3u\frac{\partial h_3^{-1}}{\partial\alpha}+h_2h_3v\frac{\partial h_3^{-1}}{\partial\beta}, \\
e_{\alpha\beta}=\frac{h_1}{h_2}\frac{\partial}{\partial\alpha}(h_2v)+\frac{h_2}{h_1}\frac{\partial}{\partial\beta}(h_1u), \\
e_{\beta\gamma}=\frac{h_2}{h_3}\frac{\partial}{\partial\beta}(h_3w)+\frac{h_3}{h_2}\frac{\partial}{\partial\gamma}(h_2v), \\
e_{\gamma\alpha}=\frac{h_3}{h_1}\frac{\partial}{\partial\gamma}(h_1u)+\frac{h_1}{h_3}\frac{\partial}{\partial\alpha}(h_3w).
\end{gathered}
\tag{9}
$$

The generalized Hooke law for an isotropic and homogeneous elastic body in an arbitrary system of orthogonal coordinates is

$$\begin{aligned} \sigma_{\alpha\alpha} &= \lambda\Delta + 2\mu e_{\alpha\alpha}, & \sigma_{\alpha\beta} &= \mu e_{\alpha\beta}, \\ \sigma_{\beta\beta} &= \lambda\Delta + 2\mu e_{\beta\beta}, & \sigma_{\beta\gamma} &= \mu e_{\beta\gamma}, \\ \sigma_{\gamma\gamma} &= \lambda\Delta + 2\mu e_{\gamma\gamma}, & \sigma_{\gamma\alpha} &= \mu e_{\gamma\alpha}, \end{aligned} \tag{10}$$

where λ and μ are the Lamé elastic constants.

In accordance with (9) we obtain for the unit volume expansion of the dam material at small deformation

$$\Delta = e_{\alpha\alpha} + e_{\beta\beta} + e_{\gamma\gamma} = h_1 h_2 h_3 \left[\frac{\partial}{\partial\alpha}\left(\frac{u}{h_2 h_3}\right) + \frac{\partial}{\partial\beta}\left(\frac{v}{h_1 h_3}\right) + \frac{\partial}{\partial\gamma}\left(\frac{w}{h_1 h_2}\right)\right]. \tag{11}$$

By replacing in equations (9) the independent variable β by the dimensionless coordinate θ and taking the Dupin theorem into account (/5/, § 19), we obtain

$$\begin{aligned} e_{\alpha\alpha} &= \frac{\partial u}{\partial\alpha}; \\ e_{\theta\theta} &= \frac{1}{R}\left(\frac{\partial v}{\partial\theta} + w\right). \\ e_{\gamma\gamma} &= \frac{\partial w}{\partial\gamma}, \\ e_{\alpha\theta} &= \frac{\partial v}{\partial\alpha} + \frac{1}{R}\frac{\partial u}{\partial\theta}. \\ e_{\theta\gamma} &= \frac{\partial v}{\partial\gamma} + \frac{1}{R}\frac{\partial w}{\partial\theta} - \frac{v}{R}. \\ e_{\gamma\alpha} &= \frac{\partial u}{\partial\gamma} + \frac{\partial w}{\partial\alpha}. \end{aligned} \tag{12}$$

In accordance with (11) we obtain for the volume expansion

$$\Delta = \frac{\partial u}{\partial\alpha} + \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial\theta} + \frac{w}{R}. \tag{13}$$

By substituting (12) and (13) in (10) we obtain for the stress components

$$\begin{aligned} \sigma_{\alpha\alpha} &= 2\mu\frac{\partial u}{\partial\alpha} + \lambda\left(\frac{\partial u}{\partial\alpha} + \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial\gamma} + \frac{w}{R}\right), \\ \sigma_{\theta\theta} &= 2\mu\left(\frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{w}{R}\right) + \lambda\left(\frac{\partial u}{\partial\alpha} + \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial\gamma} + \frac{w}{R}\right), \\ \sigma_{\gamma\gamma} &= 2\mu\frac{\partial w}{\partial\gamma} + \lambda\left(\frac{\partial u}{\partial\alpha} + \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial\gamma} + \frac{w}{R}\right), \\ \sigma_{\alpha\theta} &= \mu\left(\frac{\partial v}{\partial\alpha} + \frac{1}{R}\frac{\partial u}{\partial\theta}\right), \\ \sigma_{\theta\gamma} &= \mu\left(\frac{\partial v}{\partial\gamma} + \frac{1}{R}\frac{w}{\partial\theta} - \frac{v}{R}\right), \end{aligned} \tag{14}$$

$$\sigma_{\gamma\alpha} = \mu\left(\frac{\partial u}{\partial \gamma} + \frac{\partial w}{\partial \alpha}\right). \tag{14 (cont'd)}$$

By eliminating from equations (8) the values of the stress components with the aid of (14), we obtain the following initial conditions for calculating a thick-walled dam shell:

$$\begin{gathered}
(1-\nu)\frac{\partial^2 u}{\partial \alpha^2} + \frac{1-2\nu}{2}\left(\frac{\partial^2 u}{\partial \gamma^2} + \frac{1}{R}\frac{\partial u}{\partial \gamma} + \frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2}\right) + \frac{1}{2R}\frac{\partial w}{\partial \alpha} + \\
+ \frac{1}{2R}\frac{\partial^2 v}{\partial \alpha \partial \theta} + \frac{1}{2}\frac{\partial^2 w}{\partial \alpha \partial \gamma} + \frac{\nu}{\lambda} P_\alpha = 0, \\
\frac{1-\nu}{R^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1-2\nu}{2}\left(\frac{\partial^2 v}{\partial \gamma^2} + \frac{1}{R}\frac{\partial v}{\partial \gamma} - \frac{v}{R^2} + \frac{\partial^2 v}{\partial \alpha^2}\right) + \\
+ \frac{3-4\nu}{2R^2}\frac{\partial w}{\partial \theta} + \frac{1}{2R}\frac{\partial^2 w}{\partial \theta \partial \gamma} + \frac{1}{2R}\frac{\partial^2 u}{\partial \alpha \partial \theta} + \frac{\nu}{\lambda} P_\beta = 0; \\
(1-\nu)\left(\frac{\partial^2 w}{\partial \gamma^2} + \frac{1}{R}\frac{\partial w}{\partial \gamma} - \frac{w}{R^2}\right) + \frac{1-2\nu}{2}\left(\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial \alpha^2}\right) - \\
- \frac{3-4\nu}{2R^2}\frac{\partial v}{\partial \theta} + \frac{1}{2}\frac{\partial^2 u}{\partial \alpha \partial \gamma} + \frac{1}{2R}\frac{\partial^2 v}{\partial \theta \partial \gamma} + \frac{\nu}{\lambda} P_\gamma = 0.
\end{gathered} \tag{15}$$

The connection between the external load and the stress components will be realized through the boundary conditions. These conditions at the head- and tailraces will be

$$\begin{aligned}
X_\xi &= \sigma_{\alpha\alpha}\cos(\alpha, \xi) + \sigma_{\alpha\theta}\cos(\theta, \xi) + \sigma_{\alpha\gamma}\cos(\gamma, \xi), \\
Y_\xi &= \sigma_{\theta\alpha}\cos(\alpha, \xi) + \sigma_{\theta\theta}\cos(\theta, \xi) + \sigma_{\theta\gamma}\cos(\gamma, \xi), \\
Z_\xi &= \sigma_{\gamma\alpha}\cos(\alpha, \xi) + \sigma_{\gamma\theta}\cos(\theta, \xi) + \sigma_{\gamma\gamma}\cos(\gamma, \xi).
\end{aligned} \tag{16}$$

In formulas (16) X_ξ, Y_ξ, Z_ξ represent the projections of the surface load in the direction of the tangents to the coordinate lines α, β, γ. The letter ξ designates the direction of the normal to the inclined plane of the elementary tetrahedron. At the crest we have

$$\sigma_{\alpha\alpha} = \sigma_{\theta\gamma} = \sigma_{\alpha\gamma} = 0. \tag{17}$$

The boundary conditions (6) remain in force in the plane of connection between the dam and the foundation and on the inclined shores of the dam.

If the dam has a uniform thickness in the horizontal sections, without thickening toward the abutments (this condition is usually satisfied in practice), the elementary tetrahedron degenerates into an elementary prism, and the conditions on surfaces (16) simplify, taking the form

$$\begin{gathered}
\sin\psi\left[2\mu\frac{\partial u}{\partial \alpha} + \lambda\left(\frac{\partial u}{\partial \alpha} + \frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \gamma} + \frac{w}{R}\right)\right] + \\
+ \cos\psi\left[\mu\left(\frac{\partial v}{\partial \gamma} + \frac{\partial w}{\partial \alpha}\right)\right] = X_\xi, \\
\cos\psi\left[\mu\left(\frac{\partial v}{\partial \gamma} + \frac{1}{R}\frac{\partial w}{\partial \theta} - \frac{v}{R}\right)\right] + \sin\psi\left[\mu\left(\frac{\partial v}{\partial \alpha} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)\right] = Y_\xi, \\
\cos\psi\left[2\mu\frac{\partial w}{\partial \gamma} + \lambda\left(\frac{\partial u}{\partial \alpha} + \frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \gamma} + \frac{w}{R}\right)\right] + \\
+ \sin\psi\left[\mu\left(\frac{\partial u}{\partial \gamma} + \frac{\partial w}{\partial \alpha}\right)\right] = Z_\xi,
\end{gathered} \tag{18}$$

where ψ is the angle of inclination of the boundary curve toward the direction of the coordinate line α.

By replacing equations (15) by the corresponding equations in finite differences, using a net of nonuniform mesh side, we obtained for the parameters of the Inguri hydroelectric power plant dam a system of linear algebraic equations with 93 unknowns.

In a first approximation it was assumed that there are no vertical displacements u. The simplified system with 62 unknowns was solved by the relaxation method. The ordinary iteration and the Seidel iterative method turned out to be unsuitable for the system considered.

The table gives the values of the normal deflections w at the crest level of the central cantilever of the Inguri hydroelectric power plant dam, calculated by different means.

TABLE

Normal deflection w, in meters				
By the Mariotte formula	By the "pure arch" principle	By the bar method of "arches and walls"	By the shell theory, on the basis of the Kirchhoff-Love hypotheses	On the basis of the general equations of the theory of elasticity
0.03934	0.13912	0.11985	0.07030	0.08140

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REFINING THE LINEAR MOMENT THEORY OF THIN SHELLS

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(L'vov)

The exposition of the moment theory of shells in treatises and articles is usually based on the Kirchhoff-Love hypothesis.

The theory obtained as a result has many shortcomings. Thus, it is impossible to transfer the variational principles of the theory of elasticity to the theory of shells. When establishing the elastic relationships it is always necessary to take into account a sixth equilibrium equation. There are many other generally known incompatibilities between the theory of elasticity and the theory of shells.

We adopt here the hypothesis /3, 6/ according to which the normal to the middle surface remains straight during the shell deformation, does not suffer extensions or compressions, but does not remain perpendicular to the middle surface after the deformation. When this hypothesis is used the position (after deformation) of the normal to the middle surface of the shell is determined by the angles of rotation γ_3 and γ_4 relative to the unit vectors $\vec{\tau}_\beta$ and $\vec{\tau}_\alpha$, and by the displacement vector

$$\vec{\delta} = u\vec{\tau}_\alpha + v\vec{\tau}_\beta + w\vec{n}$$

of its middle point.

(The surface is represented here in orthogonal Gaussian coordinates. The vectors $\vec{\tau}_\alpha$ and $\vec{\tau}_\beta$ are tangential to the α and β lines, respectively; $\vec{n}$ represents the unit vector of the normal.)

On the basis of the hypothesis adopted we shall have the following strain components on the middle surface of the shell:

$$e_{\alpha\alpha} = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{\partial A}{\partial \beta}\frac{v}{AB} + \frac{w}{R_1}, \quad e_{\beta\beta} = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{\partial B}{\partial \alpha}\frac{u}{AB} + \frac{w}{R_2},$$

$$e_{\alpha\beta} = \omega_1 + \omega_2, \quad e_{\alpha n} = \gamma_3 + \frac{1}{A}\frac{\partial w}{\partial \alpha} - \frac{u}{R_1}, \quad e_{\beta n} = \gamma_4 + \frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_2}. \tag{1}$$

Here

$$\omega_1 = \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{\partial A}{\partial \beta}\frac{u}{AB}, \quad \omega_2 = \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{\partial B}{\partial \alpha}\frac{v}{AB},$$

$$\gamma_3 = \frac{\partial u^*}{\partial z}, \quad \gamma_4 = \frac{\partial v^*}{\partial z}, \quad \omega_1 - \omega_2 = 2\omega_n.$$

On the equidistant surface the corresponding strain components will be

$$e^*_{\alpha\alpha} = \frac{e_{\alpha\alpha} + z\varkappa_1}{1 + \frac{z}{R_1}}, \qquad e^*_{\beta\beta} = \frac{e_{\beta\beta} + z\varkappa_2}{1 + \frac{z}{R_2}}.$$

$$e^*_{\alpha\beta} = \frac{e_{\alpha\beta} + z\left(\frac{\omega_1}{R_2} + \frac{\omega_2}{R_1}\right) + z\tau_1\left(1 + \frac{z}{R_2}\right) + z\tau_2\left(1 + \frac{z}{R_1}\right)}{\left(1 + \frac{z}{R_1}\right)\left(1 + \frac{z}{R_2}\right)}, \tag{2}$$

$$e^*_{\alpha n} = \frac{e_{\alpha n}}{1 + \frac{z}{R_1}}, \qquad e^*_{\beta n} = \frac{e_{\beta n}}{1 + \frac{z}{R_2}}.$$

It is seen from here that the assumption of a linear or parabolic distribution of the shearing stresses through the shell thickness contradicts in the given case the hypothesis of straight normal adopted.

The bending strain components $\varkappa_1$, $\varkappa_2$, τ_1, τ_2 are defined by the following expressions through the derivatives of the angles of rotation of the normal:

$$\varkappa_1 = \frac{1}{A}\frac{\partial\gamma_3}{\partial\alpha} + \frac{\partial A}{\partial\beta}\frac{\gamma_4}{AB}, \qquad \varkappa_2 = \frac{1}{B}\frac{\partial\gamma_4}{\partial\beta} + \frac{\partial B}{\partial\alpha}\frac{\gamma_3}{AB},$$

$$\tau_1 = \frac{1}{A}\frac{\partial\gamma_4}{\partial\alpha} - \frac{\partial A}{\partial\beta}\frac{\gamma_3}{AB}, \qquad \tau_2 = \frac{1}{B}\frac{\partial\gamma_3}{\partial\beta} - \frac{\partial B}{\partial\alpha}\frac{\gamma_4}{AB}. \tag{3}$$

The equations of strain compatibility are obtained from the condition

$$\frac{\partial^2\vec{\delta}^*}{\partial\beta\partial\alpha} = \frac{\partial^2\vec{\delta}^*}{\partial\alpha\partial\beta},$$

where $\vec{\delta}^*$ is the displacement vector on the equidistant surface

$$\vec{\delta}^* = \vec{\delta} + z\,\vec{\theta}\times\vec{n}, \tag{4}$$

$$\vec{\theta} = -\gamma_4\vec{\tau}_\alpha + \gamma_3\vec{\tau}_\beta + \omega_n\vec{n}, \tag{5}$$

and has the form

$$\begin{aligned}
1.\ & AB\left(\frac{\varkappa_1}{R_2} + \frac{\varkappa_2}{R_1}\right) = AB\left[\frac{1}{R_2}\left(\frac{1}{A}\frac{\partial e_{\alpha n}}{\partial\alpha} + \frac{\partial A}{\partial\beta}\frac{e_{\beta n}}{AB}\right) + \right.\\
& \left. + \frac{1}{R_1}\left(\frac{1}{B}\frac{\partial e_{\beta n}}{\partial\beta} + \frac{\partial B}{\partial\alpha}\frac{e_{\alpha n}}{AB}\right)\right] + \frac{\partial^2 e_{\alpha\beta}}{\partial\alpha\partial\beta} + \\
& + \frac{\partial}{\partial\alpha}\left[\frac{\partial A}{\partial\beta}\frac{e_{\alpha\beta}}{A} + \frac{1}{A}\frac{\partial B}{\partial\alpha}e_{\alpha\alpha} - \frac{1}{A}\frac{\partial(Be_{\beta\beta})}{\partial\alpha}\right] + \\
& + \frac{\partial}{\partial\beta}\left[\frac{\partial B}{\partial\alpha}\frac{e_{\alpha\beta}}{B} + \frac{\partial A}{\partial\beta}\frac{e_{\beta\beta}}{B} - \frac{1}{B}\frac{\partial(Ae_{\alpha\alpha})}{\partial\beta}\right],
\end{aligned} \tag{6}$$

$$\begin{aligned}
2.\ & -\frac{\partial(B\varkappa_2)}{\partial\alpha} + \frac{\partial B}{\partial\alpha}\varkappa_1 + \frac{\partial(A\tau_1)}{\partial\beta} + \frac{\partial A}{\partial\beta}\tau_2 - \\
& - \frac{1}{R_1}\left[\frac{AB}{R_2}\cdot e_{\alpha n} + \frac{\partial B}{\partial\alpha}e_{\alpha\alpha} + \frac{\partial(A\omega_1)}{\partial\beta} + \frac{\partial A}{\partial\beta}\omega_2 - \frac{\partial(Be_{\beta\beta})}{\partial\alpha}\right] = 0,
\end{aligned}$$

$$3.\quad \frac{\partial(A\varkappa_1)}{\partial\beta} - \frac{\partial A}{\partial\beta}\varkappa_2 - \frac{\partial(B\tau_2)}{\partial\alpha} - \frac{\partial B}{\partial\alpha}\tau_1 +$$

$$+\frac{1}{R_2}\left[\frac{AB}{R_1}e_{\beta n} + \frac{\partial A}{\partial\beta}e_{\beta\beta} + \frac{\partial(B\omega_2)}{\partial\alpha} + \frac{\partial B}{\partial\alpha}\omega_1 - \frac{\partial(Ae_{\alpha\alpha})}{\partial\beta}\right] = 0, \qquad (6)\ \text{(cont'd)}$$

$$4.\quad \frac{\omega_1}{R_2} + \tau_2 = \frac{\omega_2}{R_1} + \tau_1 + \frac{1}{AB}\left[\frac{\partial(Ae_{\alpha n})}{\partial\beta} - \frac{\partial(Be_{\beta n})}{\partial\alpha}\right].$$

If $e_{\alpha n} = e_{\beta n} = 0$ we obtain from (6) the compatibility relationships which correspond to the Kirchhoff-Love hypothesis.

Introduce new bending strain components

$$\tau_1^* = \tau_1 + \frac{\omega_2}{R_1}, \qquad \tau_2^* = \tau_2 + \frac{\omega_1}{R_2}. \qquad (7)$$

Then

$$e_{\alpha\beta}^* = \frac{\left(1 - \frac{z^2}{R_1R_2}\right)e_{\alpha\beta} + z\left(1 + \frac{z}{R_2}\right)\tau_1^* + z\left(1 + \frac{z}{R_1}\right)\tau_2^*}{\left(1 + \frac{z}{R_1}\right)\left(1 + \frac{z}{R_2}\right)}. \qquad (8)$$

The compatibility conditions 2, 3, 4 can be written through $e_{\alpha\beta}$, τ_1^*, τ_2^*, instead of through ω_1, ω_2, τ_1, τ_2.

The work increment in the shell in the given case will be determined by the formula

$$\delta L = \iint\limits_{S_{(\alpha\beta)}} (T_1\delta e_{\alpha\alpha} + T_2\delta e_{\beta\beta} + N_1\delta e_{\alpha n} + N_2\delta e_{\beta n} + G_1\delta\varkappa_1 +$$

$$+G_2\delta\varkappa_2)\,AB d\alpha d\beta + \iint\limits_{S_{(\alpha\beta)}} (S_1\delta\omega_1 + S_2\delta\omega_2 + H_1\delta\tau_1 + H_2\delta\tau_2)\,AB d\alpha d\beta. \qquad (9)$$

From (7), the second term of (9) reduces to

$$\iint\limits_{S_{(\alpha\beta)}} (S_{\alpha\beta}\delta e_{\alpha\beta} + H_1\delta\tau_1^* + H_2\delta\tau_2^*)\,AB d\alpha d\beta,$$

where

$$S_{\alpha\beta} = S_1 - \frac{H_2}{R_2} = S_2 - \frac{H_1}{R_1},$$

and therefore

$$S_1 = S_{\alpha\beta} + \frac{H_2}{R_2}, \qquad S_2 = S_{\alpha\beta} + \frac{H_1}{R_1}. \qquad (10)$$

Using (10), we eliminate the forces S_1 and S_2 from the five differential equations of equilibrium and the boundary conditions; the sixth (algebraic) equation is transformed into an identity as a result.

We proceed to the derivation of the elastic relationships.

Let w_0 be the potential energy per unit area of the middle surface of the shell. By reasoning in the same way as in the theory of elasticity we obtain the following formulas, similar to Green's formulas:

$$T_1 = \frac{\partial w_0}{\partial e_{\alpha\alpha}}, \quad T_2 = \frac{\partial w_0}{\partial e_{\beta\beta}}, \quad S_{\alpha\beta} = \frac{\partial w_0}{\partial e_{\alpha\beta}},$$

$$N_1 = \frac{\partial w_0}{\partial e_{\alpha n}}, \quad N_2 = \frac{\partial w_0}{\partial e_{\beta n}}, \quad G_1 = \frac{\partial w_0}{\partial \varkappa_1}, \tag{11}$$

$$G_2 = \frac{\partial w_0}{\partial \varkappa_2}, \quad H_1 = \frac{\partial w_0}{\partial \tau_1^*}, \quad H_2 = \frac{\partial w_0}{\partial \tau_2^*}.$$

The unit potential energy of the shell w_0 is obtained by integrating by z the expression

$$2w_0 = \int_{-h}^{h} \left[\frac{E}{1-\nu^2} (e_{\alpha\alpha}^{*2} + e_{\beta\beta}^{*2}) + \frac{2E\nu}{1-\nu^2} e_{\alpha\alpha}^* e_{\beta\beta}^* + \right.$$

$$\left. + \frac{E}{2(1+\nu)} (e_{\alpha\beta}^{*2} + e_{\alpha n}^{*2} + e_{\beta n}^{*2}) \right] \left(1 + \frac{z}{R_1}\right)\left(1 + \frac{z}{R_2}\right) dz. \tag{12}$$

The integrand of (12) was obtained by writing $e_{zz} = \sigma_{zz} = 0$ in the expression for the unit potential energy used in the linear theory of elasticity. By substituting in (12) the values of the strain components according to (2) and (8), and integrating, we can represent the unit potential energy of the shell in the form $w_0 = w_1 + w_2$, where w_1 depends on $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha n}$, $e_{\beta n}$, $\varkappa_1$, $\varkappa_2$, and w_2 on $e_{\alpha\beta}$, τ_1^*, τ_2^*.

The unit potential energy of a spherical shell, obtained by this method, is

$$w_0^{(\mathrm{sp})} = \frac{Eh}{1-\nu^2} \left[e_{\alpha\alpha}^2 + e_{\beta\beta}^2 + \frac{h^2}{3} (\varkappa_1^2 + \varkappa_2^2) \right] + \frac{2E\nu h}{1-\nu^2} \left(e_{\alpha\alpha} e_{\beta\beta} + \frac{h^2}{3} \varkappa_1 \varkappa_2 \right) +$$

$$+ \frac{Eh}{2(1+\nu)} \left(e_{\alpha\beta}^2 + e_{\alpha n}^2 + e_{\beta n}^2 + \frac{h^2}{3} \tau_0^2 \right), \tag{13}$$

where $\tau^0 = \tau_1 + \tau_2$.

The corresponding elastic law is defined by the equalities

$$T_1 = \frac{2Eh}{1-\nu^2} (e_{\alpha\alpha} + \nu e_{\beta\beta}), \quad T_2 = \frac{2Eh}{1-\nu^2} (e_{\beta\beta} + \nu e_{\alpha\alpha}),$$

$$S_1 = S_2 = \frac{Eh}{1+\nu} e_{\alpha\beta}, \quad N_1 = \frac{Eh}{1+\nu} e_{\alpha n}, \quad N_2 = \frac{Eh}{1+\nu} e_{\beta n},$$

$$G_1 = \frac{2Eh^3}{3(1-\nu^2)} (\varkappa_1 + \nu\varkappa_2), \quad G_2 = \frac{2Eh^3}{3(1-\nu^2)} (\varkappa_2 + \nu\varkappa_1), \tag{14}$$

$$H_1 = H_2 = \frac{Eh^3}{3(1+\nu)} \tau_0.$$

The comparison of this elastic law with the elastic law proposed for shells by Novozhilov and Bolabukh shows that their law is correct for spherical shells, within the limits of the hypothesis adopted.

In the case of nonspherical shells the coefficients in the expression for w_0 become more complicated.

The elastic relationships for any shell can be obtained from (12) with the aid of formulas (11). We give as an example the elastic relationships obtained from (12) by expanding the coefficients of the quadratic form in series by the powers of $\frac{h}{R}$ and retaining the terms up to the order of $\frac{h^2}{R^2}$:

$$
\begin{gathered}
T_1 = \frac{2Eh}{1-\nu^2}\left[e_{\alpha\alpha} + \nu e_{\beta\beta} + \frac{h^2}{3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\varkappa_1\right], \\
T_2 = \frac{2Eh}{1-\nu^2}\left[e_{\beta\beta} + \nu e_{\alpha\alpha} - \frac{h^2}{3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\varkappa_2\right], \\
S_{\alpha\beta} = \frac{Eh}{1+\nu}\left\{\left[1 + \frac{h^2}{3}\left(\frac{1}{R_1^2} - \frac{1}{R_1R_2} + \frac{1}{R_2^2}\right)\right]e_{\alpha\beta} - \frac{h^2}{3}\left(\frac{\tau_1^*}{R_1} + \frac{\tau_2^*}{R_2}\right)\right\}, \\
N_1 = \frac{Eh}{1+\nu}e_{\alpha n}, \qquad N_2 = \frac{Eh}{1+\nu}e_{\beta n}, \\
G_1 = \frac{2Eh^2}{3(1-\nu^2)}\left[\varkappa_1 + \nu\varkappa_2 + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)e_{\alpha\alpha}\right], \qquad (15) \\
G_2 = \frac{2Eh^3}{3(1-\nu^2)}\left[\varkappa_2 + \nu\varkappa_1 - \left(\frac{1}{R_2} - \frac{1}{R_1}\right)e_{\beta\beta}\right], \\
H_1 = \frac{Eh^3}{3(1+\nu)}\left(\tau_1^* + \tau_2^* - \frac{e_{\alpha\beta}}{R_1}\right), \\
H_2 = \frac{Eh^3}{3(1+\nu)}\left(\tau_1^* + \tau_2^* - \frac{e_{\alpha\beta}}{R_2}\right).
\end{gathered}
$$

The dependence of S_1 and S_2 on the strain components is obtained from formulas (10).

Relationships (15) are resolvable relative to the strain components.

If the quadratic form (12) is not simplified, the stress-strain relationship will be of the type of (15), the only difference being that the coefficients will be more complex in the general case.

When this hypothesis relative to the normal element is used, the theory becomes free of inner contradictions. The variational principles of the theory of elasticity are transferred to the theory of shells. From Castigliano's variational principle the equations of strain compatibility follow, the problems of stress concentration in the shell are formulated more rigorously, etc.

Naturally, the question arises how much this formulation complicates the resolving equation of specific shells — spherical, cylindrical, etc.

A resolving equation was obtained for the case of a spherical shell. It coincides in form with Gol'denveizer's equation (cf. DAN SSSR, Vol. 49, No. 5. 1945), but differs from it by the constants

$$R^2\Delta u + (-\nu + ik)\,u = 0,$$

where

$$u = (1 + ik)\frac{G}{R} - (1+\nu)\,T, \quad k^2 = (1-\nu^2)\frac{3R^2}{h^2} - 1.$$

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DYNAMICS OF ELASTIC AND ELASTIC-VISCOUS SHELLS FILLED WITH AN IDEAL LIQUID

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The vibrations of elastic and elastic-viscous cylindrical shells partially filled with an ideal liquid are considered. The hypotheses of the nonlinear theory of shells are used in solving the nonlinear problem of the vibrations of a liquid-filled shell. The frequencies of the small coupled vibrations of the shell and liquid, the frequencies of the nonlinear vibrations, and the critical frequencies of the exciting loads are determined.

1. EQUATIONS OF MOTION OF THE ELEMENTS OF THE SHELL AND LIQUID

Let a circular cylindrical shell of middle-surface radius R and length l be filled up to height h by an ideal liquid, loaded by a uniformly distributed load q and a longitudinal force P. Let w be the radial displacement of the middle surface points, w_{in} the initial deviation from the regular shape, μ_{sh} the surface density of the shell, $D = Eh_*^3/12(1-\nu^2)$ the cylindrical rigidity, E the modulus of elasticity, h_* the cross-sectional height, ν the Poisson ratio, and Z the inertia force and reduced load, normal to the deformed surface. The shell bottom is flat and undeformable (figure).

In accordance with /1/ we have

$$L(F) - R^4 Z = 0, \tag{1}$$

$$L(F) = D(\nabla^2+1)^2\nabla^4 F - 2D(1-\nu)\left(\frac{\partial^4}{\partial\alpha^4} - \frac{\partial^4}{\partial\alpha^2\partial\beta^2}\right)\nabla^2 F + h_* ER\frac{\partial^4 F}{\partial\alpha^4}, \tag{2}$$

where F is a resolving function; $w = \nabla^4 F$; $\alpha = z/R$; $\beta = s/R$; z is the longitudinal coordinate; s is the length of the arc;

$$Z = \frac{1}{R^2}\left[N_\alpha\frac{\partial^2}{\partial\alpha^2} + N_\beta\left(1+\frac{\partial^2}{\partial\beta^2}\right)\right]w - \mu_{\text{sh}}\frac{\partial^2 w}{\partial t^2} -$$

$$-p(t)\left[1 - \frac{1}{R}\left(1+\frac{\partial^2}{\partial\beta^2}\right)w\right]; \quad N_\alpha = \frac{P_0 + P(t)}{2\pi R}; \; N_\beta = [q_0 + q(t)]R,$$

$$p(t) = -\rho\frac{\partial\Phi}{\partial t}. \tag{3}$$

Here P_0 and q_0 are constant components of the loads; $p(t)$ is the disturbed hydrodynamic pressure; ρ is the density of the liquid; Φ is the velocity of

the liquid particles. The potential will be sought in the form

$$\Phi = \Phi_1 + \Phi_2, \tag{4}$$

where Φ_1 and Φ_2 satisfy the Laplace equation and the boundary conditions /2/

$$\left.\begin{aligned} &\frac{\partial \Phi_1}{\partial \alpha}_{\alpha=0} = \frac{\partial \Phi_1}{\partial \alpha}_{\alpha=h/R} = 0; \qquad \frac{\partial \Phi_1}{\partial r}_{r=R} = \dot{w}; \\ &\frac{\partial \Phi_2}{\partial \alpha}_{\alpha=0} = \frac{\partial \Phi_2}{\partial r}_{r=R} = 0; \quad \rho g \frac{\partial \Phi}{\partial z} + \rho \frac{\partial^2 \Phi}{\partial t^2} + \gamma \frac{\partial^3 \Phi}{\partial z^3}_{z=h} = 0. \end{aligned}\right\} \tag{5}$$

Here g is acceleration due to gravity; γ is the coefficient of surface tension. By representing Φ_1 and Φ_2 as products of functions of α, β, and r, we obtain

$$\Phi_1 = \sum_{m,n,k} \frac{C_n}{\lambda_n} \frac{I_m(\lambda_k r)}{\lambda_k I'_m(\lambda_k R)} \cos \lambda_n \alpha, \qquad \Phi_2 = \sum_{m,n} \dot{f}_n(t) J_m(\lambda_n^* r) \operatorname{ch} \lambda_n^* R\alpha \sin m\beta, \tag{6}$$

where $I_m(\lambda_k r)$ are Bessel functions of an imaginary argument; $\lambda_n = \frac{n\pi R}{h}$; $\lambda_k = \frac{k\pi R}{l}$; λ_n^* is a root of the equation $J'_m(\lambda_n^* R) = 0$; C_n is the expansion coefficient of $\dot{w}$ in a series by $\cos \lambda_{n\alpha}$; $C_n = \frac{2R}{h} \int_0^{\alpha_h} w \cos \lambda_n \alpha d\alpha$.

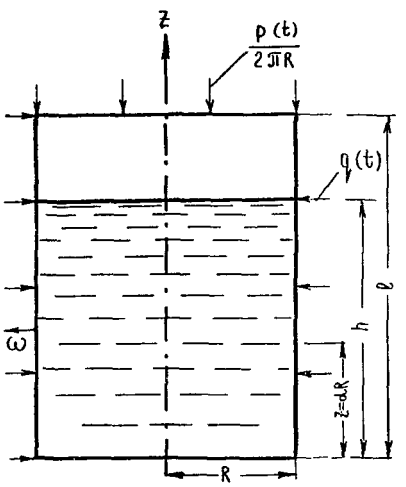

FIGURE.

From the last equation of (5) we obtain for the function $f(t)$ a second-order equation, which is the equation of forced vibrations of the liquid in a cylindrical vessel:

$$\ddot{f}_{mn} + \omega_{mn}^2 f_n = -\sum_{n,k} e_{nk} \ddot{y}_{km}. \tag{7}$$

Here ω_{mn} is the frequency of free vibrations of the liquid; y_{mk} are generalized coordinates.

$$\omega_{mn}^2 = g\lambda_n^* R \operatorname{th} \lambda_n^* h \left(1 + \frac{\gamma}{g\rho} \lambda_n^{*2} R^2\right),$$

$$e_{nk} = \frac{2(\lambda_k^2 + m^2)^2 \lambda_k R J_m(\lambda_k R) J_m'(\lambda_n^* R)}{\lambda_n^* h \operatorname{ch} \lambda_n^* h J_m'(\lambda_k R)(\lambda_k^2 + \lambda_n^{*2}) N_m} \int_0^{\alpha_h} \cos \lambda_n \alpha \sin \lambda_k \alpha d\alpha, \tag{8}$$

N_m is the norm of the functions $J_m(\lambda_n^* r)$.

2. SMALL VIBRATIONS AND PARAMETRIC RESONANCE

Write for a hinged shell

$$F = F(\alpha, \beta\ t) = \sum_{k,m} y_{km}(t) \sin \lambda_k \alpha \sin m\beta. \tag{9}$$

In accordance with the Bubnov-Galerkin method

$$\int_0^{2\pi} \int_0^{l/R} [L(F) - R^4 Z] \sin \lambda_j \alpha \sin m\beta d\alpha d\beta = 0.$$

We obtain as a result, together with equation (7), the system of equations

$$\left.\begin{aligned} &\ddot{y}_{jm} + \omega_{jm}^2 \left(1 - \frac{P_t}{P_* \frac{\omega_{jm}^2}{\bar{\omega}_{jm}^2}} - \frac{q_t}{q_* \frac{\omega_{jm}^2}{\bar{\omega}_{jm}^2}}\right) y_{jm} + \sum_{n,k} a_{nk} \ddot{y}_{km} + \sum_{n,k} b_{n,k} \ddot{f}_{nm} = 0, \\ &\ddot{f}_{nm} + \omega_{nm}^2 f + \sum_{n,k} e_{nk} \ddot{y}_{km} = 0, \end{aligned}\right\} \tag{10}$$

where $\bar{\omega}_{jm}$ is the frequency of free vibrations of the empty shell; ω_{jm} is the frequency of free vibrations of the loaded shell

$$\bar{\omega}_{jm}^2 = \frac{D(1 - \lambda_j^2 - m^2)^2 (\lambda_j^2 + m^2)^2 + 2D(1 - \nu) \lambda_j^2 (\lambda_j^4 - m^4) + h_* E R^2 \lambda_j^4}{M_{sh} R^4 (\lambda_j^2 + m^2)^2};$$

$$\omega_{jm}^2 = \bar{\omega}_{jm}^2 \left(1 - \frac{P_0}{P_*} - \frac{q_0}{q_*}\right);$$

$$P_* = \frac{2\pi \bar{\omega}_{jm}^2 \mu_{sh} R^3}{\lambda_k^2}; \qquad q_* = \frac{\bar{\omega}_{jm}^2 \mu_{sh} R^3}{(m^2 - 1)};$$

$$a_{nk} = \frac{4\rho R^2}{\mu_{sh} lh} \frac{I_m(\lambda_k R)}{\lambda_k \lambda_n I_m'(\lambda_k R)} \int_0^{\alpha_h} \cos \lambda_n \alpha \sin \lambda_k \alpha d\alpha \cdot \int_0^{\alpha_h} \cos \lambda_n \alpha \sin \lambda_j \alpha d\alpha,$$

$$b_{nk} = \frac{2\rho R}{l \mu_{sh}} J_m(\lambda_n^* R) \int_0^{\alpha_h} \operatorname{ch} \lambda_n^* \alpha R \sin \lambda_j \alpha d\alpha.$$

In the case of free vibrations of a shell ($P_t = q_t = 0$) loaded by constant components of longitudinal and radial forces, we obtain the system of equations

$$\left.\begin{aligned} \ddot{y}_{jm} + \omega_{jm}^2 y_{jm} &= \sum_{n,k} a_{nk} \ddot{y}_{km} + \sum_{n,k} b_{nk} \ddot{f}_{nm} = 0, \\ \ddot{f}_{nm} + \omega_{nm}^2 f_{nm} &+ \sum_{n,k} e_{nk} \ddot{y}_{km} = 0. \end{aligned}\right\} \tag{11}$$

The equations for the frequencies Ω of coupled vibrations of the shell and liquid can be presented in the matrix form

$$|C - \Omega^2 D| Y = 0, \tag{12}$$

where C is a matrix made of the elements $\omega_{jm}^2 \cdots \omega_{nm}^2$; Y is a matrix of the generalized coordinates y_{jm}, f_{nm}. The matrix D is of the form

$$\begin{vmatrix} 1 + a_{11} & a_{12} & \cdot & b_{11} & b_{21} & \cdot \\ a_{21} & 1 + a_{22} & \cdot & b_{12} & b_{22} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e_{11} & e_{12} & \cdot & 1 & & \cdot \\ e_{21} & e_{22} & \cdot & & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

By successively considering determinants of increasing order we obtain successive approximations for determining the frequencies of coupled vibrations of the elastic body and the liquid. The vibration modes do not separate due to the presence of the liquid. When determining the approximations it is possible to use the conditions of interconnection of the modes, i.e., to allow for the interaction of modes with near partial frequencies.

We obtained above an infinite system of differential equations of vibrations of the liquid under the assumption that the displacement vector of the ideal liquid particles can be expanded in a series by a complete system of functions. If the real liquid represents a system with a limited number of lower modes of vibrations, of frequencies considerably lower than the lowest frequencies of elastic vibrations, then the latter have almost no influence on the frequency of vibrations of the liquid. As to the dynamics of the elastic cavity, the lower modes of the liquid only slightly influence the frequency of vibrations of the cavity. In addition to the magnitudes of the attached liquid masses corresponding to each mode of elastic vibrations, it is also necessary to estimate the interaction between them, caused by the liquid motion.

Taking into account the matrix equation (12), we equate the fourth-order determinant to zero

$$\begin{vmatrix} \omega_{mj(1)}^2 - (1 + a_{11})\Omega^2 & -a_{12}\Omega^2 & -b_{11}\Omega^2 & -b_{12}\Omega^2 \\ -a_{21}\Omega^2 & \omega_{mj(2)}^2 - (1 + a_{22})\Omega^2 & -b_{21}\Omega^2 & -b_{22}\Omega^2 \\ -e_{11}\Omega^2 & -e_{12}\Omega^2 & \omega_{mn(1)}^2 - \Omega^2 & 0 \\ e_{21}\Omega^2 & -e_{22}\Omega^2 & 0 & \omega_{mn(2)}^2 - \Omega^2 \end{vmatrix} = 0.$$

With an accuracy of up to a second-order determinant (in the middle part), allowing for the interaction between one harmonic of the shell

vibrations and one harmonic of the liquid vibrations, we obtain the following expression for the square of the frequency of the coupled vibrations:

$$\Omega_{1,2}^2 = \frac{1}{2(1+a_{22}-b_{21}e_{12})}\left[(1+a_{22})\,\omega_{nm(1)}^2 + \omega_{jm(2)}^2 \pm \right.$$

$$\left. \pm\,\omega_{jm(2)}^2 \sqrt{(1+a_{22})^2 \frac{\omega_{n(1)}^2}{\omega_{jm(2)}^4} + 1 - 2(1+a_{22})\frac{\omega_{n(1)}^2}{\omega_{jm(2)}^4} + 4b_{21}e_{12}\frac{\omega_{n(1)}^2}{\omega_{jm(2)}^4}}\right]. \qquad (13)$$

Since $\omega_{mn(1)}^2 \ll \omega_{jm(2)}^2$, we obtain from (13)

$$\Omega_{1,2}^2 = \frac{(1+a_{22})\,\omega_{mn(1)}^2 + \omega_{jm(2)}^2 \pm \omega_{jm(2)}^2}{2(1+a_{22}-b_{21}e_{12})}. \qquad (13')$$

The plus sign corresponds to the shell, and the minus sign to the liquid. The frequency of vibrations of the shell increases in the case of a wavy motion on the liquid surface.

By equating the second-order determinant in the upper left-hand corner to zero we obtain for the square of the frequency of the coupled vibrations of the shell (allowing for the interaction between two harmonics of the vibrations of the shell with the attached masses):

$$\Omega_{1,2}^2 = \frac{1}{2[(1+a_{11})(1+a_{22})-a_{12}^2]}\left[(1+a_{11})\,\omega_{jm(2)}^2 + (1+a_{22})\,\omega_{jm(1)}^2 \pm \right.$$

$$\left. \pm\sqrt{[(1+a_{11})\,\omega_{jm(2)}^2 - (1+a_{22})\,\omega_{jm(1)}^2]^2 - 4a_{12}^2\omega_{jm(1)}^2\,\omega_{jm(2)}^2}\right]. \qquad (14)$$

Since $\omega_{jm(1)}^2 < \omega_{jm(2)}^2$,

$$\Omega_{1,2}^2 \approx \frac{(1+a_{11})\,\omega_{jm(2)}^2 + (1+a_{22})\,\omega_{jm(1)}^2 \pm (1+a_{11})\,\omega_{jm(2)}^2}{2[(1+a_{11})(1+a_{22})-a_{12}^2]}, \qquad (14')$$

$$\Omega_2 \approx \frac{\omega_{jm(2)}}{\sqrt{(1+a_{22}) - \dfrac{a_{12}^2}{1+a_{11}}}}\left[1 + \frac{1}{4}\frac{\omega_{jm(1)}^2\left(1+a_{22}-\dfrac{a_{12}^2}{1+a_{11}}\right)}{\omega_{jm(2)}^2\left(1+a_{11}-\dfrac{a_{12}^2}{1+a_{22}}\right)}\right]. \qquad (14'')$$

The correction of this approximation is of about 2%. If the internal pressure in the shell is so high that the frequency of flexural vibrations of the shell is considerably lower than the frequency of vibrations of a string, then the interconnection between the vibration modes increases and the correction of the approximation for the lowest tone is of about 10%.

Assuming that the undisturbed state is practically identical with the undeformed state, consider the parametric resonance of the filled shell. The system of equations (10) will be, in a matrix form,

$$D\ddot{Y} + C[E - A\varphi(t)]\,Y = 0, \qquad (15)$$

where A is the diagonal matrix of the elements

$$a_{mj} = \frac{P_t}{P_* \dfrac{\omega_{jm}^2}{\omega_{jm}^2}} + \frac{q_t}{q_* \dfrac{\omega_{jm}^2}{\omega_{jm}^2}}.$$

Let $\varphi(t) = \cos\Theta t$, where Θ is the frequency of the external load. By representing the solution of system (15) in the form

$$y(t) = \sum_i \left(a_i \sin\frac{i\Theta t}{2} + b_i \cos\frac{i\Theta t}{2} \right), \tag{16}$$

we obtain for the principal region $(i = 1)$ the following matrix equation relative to Θ:

$$C\left[E - \frac{1}{2} A \right] - \frac{\Theta^2}{4} D = 0. \tag{17}$$

With an accuracy of up to a second-order matrix, we obtain in accordance with expression (13')

$$\Theta^2 = 4\Omega^2_{*1,2}; \quad \Omega^2_{*1,2} \approx \frac{(1 + a_{22})\,\omega^2_{mn(1)} + \omega^{*2}_{jm(2)} \pm \omega^{*2}_{jm(2)}}{2(1 + a_{22} - b_{21}e_{12})};$$
$$\omega^{*2}_{j(2)} = \omega^2_{j(2)}\left(1 \pm \frac{a_{j(2)}}{2} \right). \tag{18}$$

Here $\Omega_{*1,2}$ is the frequency of the coupled vibrations of the system shell — liquid loaded by constant and transient components of the loads and the hydrostatic pressure.

The lowest critical frequency is determined by the formula

$$\Theta_* = \omega_{mn(1)} \sqrt{\frac{2(1 + a_{22})}{(1 + a_{22} - b_{21}e_{12})}}. \tag{18'}$$

Allowing for the interaction between the harmonics of the shell vibrations in accordance with (14'), we obtain

$$\Theta^2 = 4\Omega^2_{*1,2}; \quad \Omega^2_{*1,2} \approx \frac{(1 + a_{11})\,\omega^{*2}_{jm(2)} + (1 + a_{22})\,\omega^{*2}_{jm(1)} \pm (1 + a_{11})\,\omega^{*2}_{jm(2)}}{2\left[(1 + a_{11})(1 + a_{22}) - a_{12}^2\right]}. \tag{19}$$

Successive approximations for Θ are obtained by considering determinants of increasing order.

3. THE NONLINEAR PROBLEM

In accordance with /3, 4/, the equations for the function F and for the function Φ of the two-dimensional state of stress, averaged over the shell thickness, can be presented in the form:

$$\left.\begin{aligned} &L(F) + \varkappa_T(1 + \nu) D\nabla^2\psi - R^4 Z = 0, \\ &\frac{1}{Eh_*}\nabla^4\Phi = L(w) - R^2\varkappa_T \nabla^2 T_0. \end{aligned}\right\} \tag{20}$$

Here $L(F)$ is the operator (2); $\varkappa_T$ is the coefficient of temperature expansion; ψ is the temperature gradient; $T_0(\alpha, \beta)$ is the temperature of the middle

surface;

$$Z=\frac{1}{R^2}\left[\frac{\partial^2\Phi}{\partial\beta^2}\frac{\partial^2 w}{\partial\alpha^2}+\left(1+\frac{\partial^2}{\partial\beta^2}\right)w\frac{\partial^2\Phi}{\partial\alpha^2}+R\frac{\partial^2\Phi}{\partial\alpha^2}-2\frac{\partial^2 w}{\partial\alpha\partial\beta}\frac{\partial^2\Phi}{\partial\alpha\partial\beta}\right]+$$

$$+F_{Tp}+\mu_{sh}\ddot{w}+p(t)\left[1-\frac{1}{R}\left(1+\frac{\partial^2}{\partial\beta^2}\right)w\right],$$

$$L(w)=\left(\frac{\partial^2 w}{\partial\alpha\partial\beta}\right)^2-\frac{\partial^2 w}{\partial\alpha^2}\frac{\partial^2 w}{\partial\beta^2}-R\frac{\partial^2 w}{\partial\alpha^2};$$

F_{Tp} is the dissipative force.

Let F_{in}, Φ_{in}, and w_{in} be the functions determined by the intial factors (temperature, static lateral load, initial deviations), so that

$$F=F_*+F_{in};\quad \Phi=\Phi_*+\Phi_{in};\quad w=w_*+w_{in}. \tag{21}$$

The functions F_{in}, Φ_{in}, and w_{in} satisfy the equations

$$\left.\begin{aligned}&L(F_{in})+\varkappa_T(1+\nu)D\nabla^2\psi-R^2\left[\frac{\partial^2\Phi_{in}}{\partial\beta^2}\frac{\partial^2 w_{in}}{\partial\alpha^2}+\right.\\&\left.+\left(1+\frac{\partial^2}{\partial\beta^2}\right)w_{in}\frac{\partial^2\Phi_{in}}{\partial\alpha^2}-2\frac{\partial^2\Phi_{in}}{\partial\alpha\partial\beta}\frac{\partial^2 w_{in}}{\partial\alpha\partial\beta}\right]=0,\\&\frac{1}{Eh_*}\nabla^4\Phi_{in}+R^2\varkappa_T\nabla^2T_0=\left(\frac{\partial^2 w_{in}}{\partial\alpha\partial\beta}\right)^2-\frac{\partial^2 w_{in}}{\partial\alpha^2}\frac{\partial^2 w_{in}}{\partial\beta^2}-R\frac{\partial^2 w_{in}}{\partial\alpha^2}.\end{aligned}\right\} \tag{22}$$

The functions F_*, Φ_*, w_* are determined from the equations

$$\left.\begin{aligned}&L(F_*)-R^4Z=0,\\&\frac{1}{Eh_*}\nabla^4\Phi_*=L(w_*,w_{in}),\end{aligned}\right\} \tag{20'}$$

where $L(F_*)$ is an operator of the form of (2),

$$Z=\frac{1}{R^2}\left\{\frac{\partial^2\Phi_{in}}{\partial\beta^2}\frac{\partial^2 w_*}{\partial\alpha^2}+\left(1+\frac{\partial^2}{\partial\beta^2}\right)w_*\frac{\partial^2\Phi_{in}}{\partial\alpha^2}+\frac{\partial^2 w}{\partial\alpha^2}\frac{\partial^2\Phi_*}{\partial\beta^2}+\right.$$

$$\left.+\left(1+\frac{\partial^2}{\partial\beta^2}\right)w\frac{\partial^2\Phi_*}{\partial\alpha^2}+R\frac{\partial^2\Phi_*}{\partial\alpha^2}-2\frac{\partial^2\Phi_*}{\partial\alpha\partial\beta}\frac{\partial^2 w}{\partial\alpha\partial\beta}-2\frac{\partial^2 w_*}{\partial\alpha\partial\beta}\frac{\partial^2\Phi_{in}}{\partial\alpha\partial\beta}\right\}+$$

$$+p(t)\left[1-\frac{1}{R}\left(1+\frac{\partial^2}{\partial\beta^2}\right)w_*\right]+F_{Tp}+\mu_{sh}\ddot{w},$$

$$L(w_*,w_{in})=\left(\frac{\partial^2 w_*}{\partial\alpha\partial\beta}\right)^2-\frac{\partial^2 w}{\partial\alpha^2}\frac{\partial^2 w_*}{\partial\beta^2}-\left(\frac{\partial^2 w_{in}}{\partial\beta^2}+R\right)\frac{\partial^2 w_*}{\partial\alpha^2}+2\frac{\partial^2 w_*}{\partial\alpha\partial\beta}\frac{\partial^2 w_{in}}{\partial\alpha\partial\beta}.$$

The function Φ_{in} is an integral of the second equation of (22) and is made of the integral Φ_{0in} of the homogeneous biharmonic equation $\nabla^4\Phi_{0in}=0$ and the integral Φ_{1in} of the nonhomogeneous second equation of (22). The initial deflections of the shell being small, we can write $\Phi_{1in}=0$, so that $\Phi_{in}=\Phi_{0in}$ $w_{in}=w_{0in}$. The function Φ_* is an integral of the second equation of (20') and, just as Φ_{in}, is made of Φ_{0*} and Φ_{1*}. We then have

$$\Phi_0=\Phi_{0in}+\Phi_{0*}=\frac{P}{2\pi R}\frac{\beta^2}{2}+q\frac{R\alpha^2}{2}.$$

For a cylindrical shell hinged by the ends take:

$$F_* = \sum_{k,m} y_{*km} \sin \lambda_k \alpha \sin m\beta, \qquad F_{\text{in}} = \sum y_{ckm} \sin \lambda_k \alpha \sin m\beta. \tag{23}$$

Only one of the parameters y_{*km} (for instance y_{*1}) is not small. We obtain as a result for Φ

$$\Phi = \Phi_0 + \frac{Eh_*}{16} \eta^2 \left(\frac{m^2}{\lambda_k^2} \cos 2\lambda_k \alpha + \frac{\lambda_k^2}{m^2} \cos 2m\beta \right) \left(\frac{y_{*1}^2}{2} + y_{*1} y_0 \right) +$$

$$+ \frac{R}{2} \sum \frac{n}{m^2} y_{mk} \sin \lambda_k \sin m\beta, \tag{24}$$

$$\eta = (\lambda_k^2 + m^2)^2.$$

By the Bubnov-Galerkin method we obtain from (23) and (24) the following equation for determining y_{*1}:

$$L(y_{*1}, t) = \ddot{y}_{*1} + \omega_{mj(1)}^2 \left(1 - \frac{P_i}{P_* \frac{\omega_{jm(1)}^2}{\omega_{jm(1)}^2}} - \frac{q_t}{q_* \frac{\omega_{jm(1)}^2}{\omega_{mj(1)}^2}} \right) y_{*1} +$$

$$+ \sum_{k,m} a_{kn} \ddot{y}_{km} + \sum_{k,n} b_{kn} \ddot{f}_{nm} + 2(\delta + \delta_1 y_{*1}^2) \dot{y}_{*1} + \xi_1 y_{*1}^3 + \xi_2 y_{*1}^2 + \xi_3 y +$$

$$+ \xi_0 + y_0^* \cos \Theta t = 0. \tag{25}$$

Here $2(\delta + \delta_1 y_{*1}^2) \dot{y}_{*1}$ is the generalized dissipative force in the scale of the density; ε and ε_1 are constants allowing for the viscous friction. Equations (25) and (7) form the system of equations of vibrations of the shell — liquid at not-small deflections of the shell.

$$\xi_1 = \frac{R^2 E h_*}{16} \eta^3 [\lambda_j^4 + (m^2 - 1) m^2], \quad \xi_2 = 3\xi_1 y_{01}, \quad \xi_3 = \xi_1 y_{01}^2.$$

$$\xi_0 = \eta \left[\frac{\lambda_j^2}{2\pi R} P_0 + (m^2 - 1) R q_0 \right] y_{01},$$

$$y_0^* = \eta \left[\frac{\lambda_j^2}{2\pi R} P_i + (m^2 - 1) R q_t \right] y_{01}.$$

Consider the natural not-small vibrations ($y_0 = 0$). Represent the solution of (25) in the form

$$y_{*1} = A_* \sin \tau + B_* \cos \tau, \quad \tau = \Omega t. \tag{26}$$

Substituting (26) in (25) and calculating the integrals

$$\int_0^{2\pi} L(y_{*1}, \tau) \sin \tau d\tau = 0 \text{ and } \int_0^{2\pi} L(y_{*1}, \tau) \cos \tau d\tau = 0,$$

we obtain the system

$$\left[\Omega^4 G_1 - \Omega^2 \left(M_1 + \frac{3}{4} \xi_1 A^2 \right) + M_2 + \frac{3}{4} \xi_1 \omega_{mn(1)}^2 A^2 \right] A_* -$$

$$- \left(2\delta\Omega + \frac{\delta_1}{2} A^2 \Omega \right) B_* = 0, \tag{27}$$

$$\left[\Omega^4 G_1 - \Omega^2\left(M_1 + \frac{3}{4}\xi_1 A^2\right) + M_2 + \frac{3}{4}\xi_1\omega^2_{mn(1)} A^2\right] B_* +$$

$$+\left(2\delta\Omega + \frac{\delta_1}{2} A^2\Omega\right) A_* = 0, \qquad (27)$$

(cont'd)

$$G_1 = 1 + a_{11} - b_{11}e_{11}; \quad A^2 = A_*^2 + B_*^2; \quad M_1 = (1+a)\omega^2_{mn(1)} + \omega^2_{mj(2)};$$

$$M_2 = \omega^2_{mn(1)}\omega^2_{jm(1)}.$$

By equating the determinant of the system to zero we obtain an algebraic equation of the eighth degree relative to Ω, which expresses the amplitude-frequency relationship in the presence of linear and nonlinear damping. In the case of undampened vibrations system (27) splits into separate equations of the form

$$G_1\Omega^4 - \left(\frac{3}{4}\xi_1 A^2 + M_1\right)\Omega^2 + M_2 + \frac{3}{4}\xi_1 A^2\omega^2_{mn(1)} = 0. \qquad (28)$$

By writing $\Omega^2 = \Omega_0^2 + \sigma_1^* A^2$, where Ω_0 is the frequency of the small coupled vibrations of the system, we obtain by substituting in (28) and equating the coefficients

$$\Omega_0^2 = \frac{M_1 + \sqrt{M_1 - 4G_1M_2}}{2G_1}, \qquad \sigma_1^* = \frac{\frac{3}{4}\xi_1(\Omega_0^2 - \omega^2_{mn(1)})}{2\Omega_0^2 G_1 - M_1}. \qquad (29)$$

For the shell $\sigma_1^* > 0$, since $\Omega^2_{0sh} > \omega^2_{mn(1)}$; $2\Omega^2_{0sh}G_1 > M_1$.
For the liquid $\sigma_1^* > 0$, since $\Omega^2_{01} < \omega^2_{mn(1)}$; $2\Omega^2_{01}G_1 < M_1$.
The frequency of vibrations thus continues to increase.

The solution of equation (25) at parametrically excited vibrations is represented in the form (16). We obtain as a result a system of the form of (27), which expresses the amplitude-frequency relationship. At $\delta = \delta_1 = 0$ we obtain for the solution of period $T(i = 2)$ the following simplified relationship between Θ and A, on the one hand, and y_0 on the other:

$$\left(\frac{i\Theta}{2}\right)^4 G_1 - \left(\frac{i\Theta}{2}\right)^2\left(M_1 + \frac{3}{4}\xi_1 A^2 + \frac{y_0^*}{A}\right) +$$

$$+ M_2 + \frac{3}{4}\xi_1 A^2\omega^2_{mn(1)} + \frac{y_0^*\omega^2_{mn(1)}}{A} = 0. \qquad (30)$$

By representing Θ^2 in the form .

$$\Theta^2 = \Theta_0^2 + \sigma_1^{**}A^2 + \sigma_2^{**}\frac{1}{A}$$

and determining the coefficient, we obtain

$$\Theta_0 = \Omega_0; \quad \sigma_1^{**} = \frac{\frac{3}{4}\xi_1(\Theta_0^2 - \omega^2_{mn(1)})}{2\Theta_0^2 G_1 - M_1}; \quad \sigma_2^{**} = \frac{y_0^*(\Theta_0^2 - \omega^2_{mn(1)})}{2\Theta_0^2 G_1 - M_1}. \qquad (31)$$

Here $\sigma_1^{**} > 0$ and $\sigma_2^{**} > 0$. From the condition $\frac{d(\Theta^2)}{dA} = 0$ we find the value of A corresponding to the extremum value of Θ. The lowest critical frequency Θ is equal to the frequency of vibrations of the liquid obtained by solving

the linear problem of parametric resonance of the shell—liquid system.

4.

Consider the vibrations of an elastic-viscous shell under the assumption that linear relationships exist between the tensors of stress, rate of stress, and rate of strain. For a homogeneous and isotropic Maxwell medium, in the case of a two-dimensional state of stress /5, 6/, we have

$$\left.\begin{aligned} \dot{\varepsilon}_1 &= \frac{D_0}{2\mu}\sigma_1 + \frac{1}{3}\left(\frac{D_1}{3K_v} - \frac{D_0}{2\mu}\right)(\sigma_1+\sigma_2), \quad \gamma_{12} = \frac{D_0}{\mu}\tau_{12}, \\ \dot{\varepsilon}_2' &= \frac{D_0}{2\mu}\sigma_2 + \frac{1}{3}\left(\frac{D_1}{3K_v} - \frac{D_0}{2\mu}\right)(\sigma_1+\sigma_2), \end{aligned}\right\} \tag{32}$$

where

$$K_v = \frac{2}{3}\mu + \lambda, \qquad K_e = \frac{2G(1+\nu)}{3(1-2\nu)},$$

G is the modulus of shear; μ, λ are rigidity coefficients; and D_0 and D_1 are time operators:

$$D_0 = 1 + \frac{\mu}{G}\frac{\partial}{\partial t}, \qquad D_1 = 1 + \frac{K_v}{K_e}\frac{\partial}{\partial t}.$$

By solving relationships (32) relative to the stresses, determining the forces $\overline{M}$, $\overline{H}$, and $\overline{N}$, and using the equilibrium equation

$$\frac{\partial^2\overline{M}_1}{\partial z^2} + 2\frac{\partial^2\overline{H}_{12}}{\partial z\partial s} + \frac{\partial^2\overline{M}_2}{\partial s^2} + \overline{N}_s\left(\frac{1}{R} + \frac{\partial^2 w}{\partial s^2}\right) + \overline{N}_z\frac{\partial^2 w}{\partial z^2} + Z = 0,$$

we obtain finally one equation for the rate of deflections:

$$\begin{aligned} &\left(\frac{D_0}{\mu} + \frac{D_1}{3K_v}\right)\nabla^2\dot{w} - \frac{1}{R}\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_v}\right)\frac{\partial^2}{\partial s^2}\nabla^4\dot{w} + \left(\frac{D_0}{\mu} - \right.\right. \\ &\left. - \frac{D_1}{3K_v}\right)\frac{\partial^2}{\partial z^2}\nabla^4\dot{w} + \frac{6}{h_v^3}\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_v}\right)\frac{D_0}{\mu}\left[\frac{3h}{R^2}\frac{\partial^4\dot{w}}{\partial z^4} - \right. \\ &\left. - \left(\frac{D_0}{\mu} + \frac{D_1}{3K_v}\right)\nabla^4\left(N_z^0\frac{\partial^2 w}{\partial z^2} + N_s^0\frac{\partial^2 w}{\partial s^2} + \mu_{sh}\ddot{w} + p(t)\right)\right] = 0. \end{aligned} \tag{33}$$

By substituting the solution $w = w_0 e^{i(\Omega t - kz)}\cos m\beta$ and allowing for equation (7), we obtain for each tone of vibrations of a liquid-filled elastic-viscous shell an equation of the seventh order relative to Ω.

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ON THE CRITERION OF DYNAMIC STABILITY OF SHALLOW SHELLS OF REVOLUTION

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The question of the stability of shallow shells of revolution at axisymmetrical vibrations was treated in /1/. It was established that for a shell of given geometry there exists a definite initial velocity distribution, such that its potential energy attains an extremal value when the shell velocity becomes equal to zero.

A simple criterion was proposed in /2/ for determining the minimum value of a uniformly distributed transient impulse causing the popping of a shallow shell, considered as a system with one degree of freedom. This criterion was of the form:

$$K_* = \max \{\Pi\}, \tag{1}$$

Using it, the critical values of the impulse for a shallow spherical dome and for a lengthy cylindrical panel were determined in /2/ for some particular cases of boundary conditions.

In the present paper the solution of Humphreys and Bodner is generalized to the case of a solid shallow shell of revolution with meridian described by an exponential function at arbitrary elastic clamping of the edge.

1. Symbols. K_* represents kinetic energy of the minimum initial impulse causing the popping of the shell, Π potential strain energy of the shell, ν the Poisson ratio, E Young's modulus, $D = \dfrac{Eh^3}{12(1-\nu^2)}$ the cylindrical rigidity, T_1, T_2 the meridional and circumferential forces per unit length, M_1 the meridional bending moment per unit length, r the radius of the point of the middle surface of the shell, b its maximum value, z the equation of the middle surface of the shell, h the shell thickness, H the deflection of the middle surface, u, w the displacements of the middle-surface points in the radial and axial directions.

$$\rho = \frac{r}{b}, \quad f = \frac{z}{H},$$

$$U = \frac{u}{H}, \quad W = \frac{w}{H},$$

$$\lambda^2 = \frac{1}{12(1-\nu^2)}\left(\frac{h}{H}\right)^2, \quad \mu = \frac{H}{b},$$

$$T_* = \frac{K_*}{2\pi\mu^2 H^2 E h}, \quad \Omega = \frac{\Pi}{2\pi\mu^2 H^2 E h}.$$

The dot will designate differentiation by ρ.

2. Having in view an axisymmetrical deformation of a shallow shell of revolution, we neglect the inertial forces in the radial direction, which makes it possible to introduce the force function ψ by means of the relationships

$$\frac{T_1}{Eh} = -\mu^2 \frac{\psi}{\rho}, \quad \frac{T_2}{Eh} = -\mu^2 \psi.$$

As a result, the expression for the dimensionless potential energy can be represented in the form

$$\Omega = \frac{\lambda^2}{2} \int_0^1 \left[(W^{\cdot\cdot})^2 + \left(\frac{W^{\cdot}}{\rho}\right)^2 + 2\nu W^{\cdot\cdot} \frac{W^{\cdot}}{\rho} \right] \rho d\rho +$$

$$+ \frac{1}{2} \int_0^1 \left[(\psi^{\cdot})^2 + \frac{\psi^2}{\rho^2} - 2\nu\psi^{\cdot} \frac{\psi}{\rho} \right] \rho d\rho. \qquad (2)$$

The condition of strain continuity gives the following relationship between the functions ψ and W:

$$\psi^{\cdot\cdot} + \frac{\psi^{\cdot}}{\rho} - \frac{\psi}{\rho^2} = f^{\cdot} \frac{W^{\cdot}}{\rho} + \frac{\rho}{2} \left(\frac{W^{\cdot}}{\rho}\right)^2. \qquad (3)$$

Knowing f and W as functions of ρ, the function ψ is easily found from (3). The two constants appearing as a result of the integration of (3) are determined from the condition of finite meridional force at the shell top and the condition of elastic displacement of the edge in the radial direction:

$$U(1) = -\beta \frac{T_1(1)}{\mu Eh}$$

(β is the proportionality factor).

3. The equation of the middle surface of the shell will be taken in the form

$$f = \rho^n \quad (n \geqslant 1). \qquad (4)$$

We shall also specify the post-buckling shape of the deflection. The shell edge is considered as unmovable in the axial direction and elastically clamped, so that

$$W^{\cdot}(1) = -\gamma \frac{bM_1(1)}{\mu D}$$

(γ is the proportionality factor). In addition, for a solid shell $W^{\cdot}(0) = 0$. Such conditions are satisfied by the expression /1/

$$W = W_0 (1 - \rho^2) \left(1 - \frac{\rho^2}{2k-1}\right). \qquad (5)$$

where

$$k = \frac{(3+\nu)\gamma - 1}{(1+\nu)\gamma - 1}$$

By substituting (5) and (4) in (3) we obtain

$$\psi = \frac{W_0}{2k-1}\psi_1 + \frac{1}{2}\frac{W_0^2}{(2k-1)^2}\psi_2 \tag{6}$$

Here

$$\psi_1 = \frac{4}{m}\left(\rho^{n+3} - \frac{n+4}{n}k\rho^{n+1} + A_n\rho\right),$$

$$\psi_2 = \frac{1}{3}(\rho^7 - 4k\rho^5 + 6k^2\rho^3 - A\rho),$$

$$A_n = B_n + \frac{nB_n - 2}{1+\beta-\nu},$$

$$A = B + \frac{2(B - 4k + 2)}{1+\beta-\nu},$$

$$B_n = \frac{n+4}{n}k - 1,$$

$$B = 6k^2 - 4k + 1,$$

$$m = \frac{(n+2)(n+4)}{n}.$$

At $n = 1$ (conical shell) and $n = 2$ (spherical shell) expression (6) coincides, up to a constant factor, with the corresponding expression of /1/.

4. Substituting now (5) and (6) in (2), we obtain

$$\Omega = (\lambda^2\chi_0 + \omega_0)\frac{W_0^2}{2} - 2\omega_1\frac{W_0^3}{3} + \omega_2\frac{W_0^4}{4}, \tag{7}$$

where

$$\chi_0 = \frac{16}{(2k-1)^2}\left[(1+\nu)(k-1)^2 + \frac{2}{3}\right],$$

$$\omega_0 = \frac{16}{m^2(2k-1)^2}\left\{(1-\nu)A_n^2 - 2(1-\nu)A_nB_n + \right.$$

$$+ k^2\left(\frac{n+4}{n}\right)^2\left[\frac{(n+1)^2+1}{2(n+1)} - \nu\right] - 2k\frac{n+4}{n}\left(\frac{n+2}{2} - \nu\right) +$$

$$\left. + \frac{(n+3)^2+1}{2(n+3)} - \nu\right\},$$

$$\omega_1 = \frac{1}{m(2k-1)^3}\left\{(1-\nu)(A-B)A_n - (1-\nu)AB_n + \right.$$

$$+ 6k^3\left[\frac{3(n+1)+1}{n} - \nu\frac{n+4}{n}\right] - 2k^2\left[\frac{19n^2+82n+48}{n(n+6)} - \nu\frac{5n+8}{n}\right] +$$

$$\left. + k\left[\frac{27n^2+100n+32}{n(n+8)} - \nu\frac{5n+4}{n}\right] - \frac{7(n+3)+1}{n+10} + \nu\right\},$$

$$\omega_2 = \frac{1}{18(2k-1)^4}\Big[(1-\nu)A^2 - 2(1-\nu)AB + 12(5-3\nu)k^4 -$$
$$-48(2-\nu)k^3 + 4(17-7\nu)k^2 - 8(3-\nu)k + \frac{25}{7} - \nu\Big]$$

(note that χ_0 and ω_2 are independent of n).

The procedure for finding the maximum of Ω as a function of W_0 leads to the equation

$$W_0(\omega_2 W_0^2 - 2\omega_1 W_0 + \omega_0 + \lambda^2\chi_0) = 0,$$

whose solutions are

$$W_{01} = 0,$$
$$W_{0(2,3)} = \frac{\omega_1}{\omega_2} \mp \sqrt{\left(\frac{\omega_1}{\omega_2}\right)^2 - \frac{\omega_0}{\omega_2} - \lambda^2\frac{\chi_0}{\omega_2}}. \tag{8}$$

The first solution corresponds to the initial state of equilibrium. The second and third ones are meaningful if the following condition is satisfied:

$$\lambda^2 \leqslant \lambda_*^2 = \frac{\omega_2}{\chi_0}\left[\left(\frac{\omega_1}{\omega_2}\right)^2 - \frac{\omega_0}{\omega_2}\right] \tag{9}$$

(it becomes clear from a careful analysis that $\chi_0 > 0$ and $\omega_2 > 0$).

With the aid of (9) the second formula of (8) can be represented in the form

$$W_{0(2,3)} = \frac{\omega_1}{\omega_2} \mp \sqrt{\frac{\chi_0}{\omega_2}(\lambda_*^2 - \lambda^2)}. \tag{10}$$

The analysis of this formula leads to the conclusion that at $\lambda^2 < \lambda_*^2$ formula (10) gives two extremal points of expression (7), the first of them (W_{02}) representing a maximum, and the second one (W_{03}) a minimum. At $\lambda^2 = \lambda_*^2$ the two points coalesce and form an inflection point of curve (7).

Thus, according to criterion (1):

1) elastic popping of the shells considered is possible only for $\lambda^2 < \lambda_*^2$;

2) the minimum value of the kinetic energy which causes this popping is calculated by the formula

$$T_* = (\lambda^2\chi_0 + \omega_0)\frac{W_*^2}{2} - 2\omega_1\frac{W_*^3}{3} + \omega_2\frac{W_*^4}{4}, \tag{11}$$

where

$$W_* = \frac{\omega_1}{\omega_2} - \sqrt{\frac{\chi_0}{\omega_2}(\lambda_*^2 - \lambda^2)}$$

represents the maximum displacement of the shell top.

5. Numerical calculations were conducted for conic and spherical shells at the following boundary conditions:

1) clampled edge ($\beta = \infty$, $k = 1$),

2) simply supported edge $\left(\beta = \infty,\ k = \frac{3+\nu}{1+\nu}\right)$,

TABLE 1

No. of boundary conditions	χ_0	ω_0	ω_1	ω_2	λ_*^2
			Conic shell ($n=1$)		
1	10.6667	0.16889	0.16303	0.14286	0.0018012
2	3.6037	0.12282	0.10809	0.08669	0 0033163
3	10.6667	0.57524	0.54488	0.46032	0.0065378
4	3.6037	0.69697	0.68989	0.61072	0.0228465
			Spherical shell ($n=2$)		
1	10.6667	0.17778	0.16667	0.1 286	0.0015625
2	3.6037	0.16734	0.12751	0.08669	0.0056072
3	10 6667	0.81270	0.64286	0.46032	0.0079767
4	3.6037	1.37185	0.97019	0.61072	0.0469996

TABLE 2

Values of W_*

No. of boundary conditions	λ^2					
	0	$0.2\lambda_*^2$	$0.4\lambda_*^2$	$0.6\lambda_*^2$	$0.8\lambda_*^2$	λ_*^2
			Conic shell ($n=1$)			
1	0.781	0.819	0.863	0.916	0.983	1.147
2	0.876	0.915	0.859	1.012	1.081	1.247
3	0.794	0.836	0.882	0.938	1.010	1.184
4	0.762	0.801	0.845	0.897	0 965	1.130
			Spherical shell ($n=2$)			
1	0.825	0.861	0.902	0.951	1.014	1.167
2	0.988	1.039	1.097	1.165	1.255	1.471
3	0.967	1.012	1.064	1.125	1.204	1.397
4	1.062	1.118	1.181	1.256	1.353	1.589

TABLE 3

Value of T_*

No. of boundary conditions	λ^2					
	0	$0.2\lambda_*^2$	$0.4\lambda_*^2$	$0.6\lambda_*^2$	$0.8\lambda_*^2$	λ_*^2
			Conic shell ($n=1$)			
1	0.0127	0.0140	0.0153	0.0168	0.0186	0.0206
2	0.0115	0.0124	0.0135	0.0146	0.0159	0.0175
3	0.0452	0.0499	0.0550	0.0608	0.0673	0.0753
4	0.0503	0.0554	0.0609	0.0672	0.0743	0.0829
			Spherical shell ($n=2$)			
1	0.0147	0.0158	0.0171	0.0186	0.0202	0.0221
2	0.0203	0.0224	0.0247	0.0273	0.0302	0.0338
3	0.0 31	0.1014	0.1105	0.1207	0.1322	0.1459
4	0.1931	0.2132	0.2356	0.2607	0.2894	0.3241

3) restrained edge ($\beta = 0$, $k = 1$),

4) rigidly supported edge $\left(\beta = 0,\ k = \frac{3+\nu}{1+\nu}\right)$

(the Poisson ratio was taken as equal to 0.3).

Table 1 gives the values of the coefficients entering in (11), and the magnitudes λ_*^2, and Tables 2 and 3 the values of W_* and T_*, respectively. It is seen from the last table that the restriction of the radial displacement of the edge causes a considerable increase of the critical value of the kinetic energy.

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EXPLANATORY LIST OF ABBREVIATED NAMES OF USSR INSTITUTIONS AND PERIODICALS APPEARING IN THE BOOK

Abbreviation	Full name (transliterated)	Translation
AN SSSR	Akademiya Nauk SSSR	Academy of Sciences of the USSR
BSE	Bol'shaya Sovetskaya Entsiklopediya	Large Soviet Encyclopedia
DAN SSSR	Doklady Akademii Nauk SSSR	Reports of the Academy of Sciences of the USSR
FMET	Fiziko-Matematicheskie, Estestvennye i Tekhnicheskie Nauki	Physical-Mathematical, Natural and Technical Sciences
FMI	Fiziko-Matematicheskii Institut	Physical-Mathematical Institute
IMASh	Institut Mashinovedeniya	Institute of Machine Science
IVUZ	Izvestiya Vysshikh Uchebnykh Zavedenii	Bulletin of the Institutions of Higher Education
KF AN SSSR	Kazanskii Filial AN SSSR	Kazan' Branch of the USSR Academy of Sciences
KGU	Kievskii Gosudarstvennyi Universitet im. T.G. Shevchenko	Kiev State University im. T.G. Shevchenko
KPI	Kievskii Politekhnicheskii Institut	Kiev Polytechnic Institute
LGU	Leningradskii Gosudarstvennyi Universitet	Leningrad State University
LKI	Leningradskii Korablestroitel'nyi Institut	Leningrad Shipbuilding Institute
LPI	Leningradskii Politekhnicheskii Institut	Leningrad Polytechnic Institute
MGU	Moskovskii Gosudarstvennyi Universitet	Moscow State University

Abbreviation	Full name (transliterated)	Translation
MIIT	Moskovskii Institut Inzhenerov Zheleznodorozhnogo Transporta	Moscow Institute of Railroad Engineering
NDVSh	Nauchnye Doklady Vysshei Shkoly	Scientific Reports of Institutions of Higher Education
NIIG	Nauchno-Issledovatel'skii Institut Gidrotekhniki	Hydroengineering Scientific Research Institute
NIIM	Nauchno-Issledovatel'skii Institut Metallurgii	Metallurgical Scientific Research Institute
N. T. O.	Nauchno-Tekhnicheskoe Obshchestvo	Scientific and Technical Society
OMEN	Otdelenie Matematicheskikh i Estestvennykh Nauk	Department of Mathematical and Natural Sciences
OTN	Otdelenie Tekhnicheskikh Nauk (Akademii Nauk SSSR)	Department of Technical Sciences (of the USSR Academy of Sciences)
PMM	Prikladnaya Matematika i Mekhanika	[Journal of] Applied Mathematics and Mechanics
TsAGI	Tsentral'nyi Aero-Gidrodinamicheskii Institut im. N. E. Zhukovskogo	Central Aero-Hydrodynamical Institute im. N. E. Joukowski
TsNIPS	Tsentral'nyi Nauchno-Issledovatel'skii Institut Promyshlennykh Sooruzhenii	Central Scientific Research Institute of Industrial Structures
VIAM	Vsesoyuznyi Nauchno-Issledovatel'skii Institut Aviatsionnykh Materialov	All-Union Scientific Research Institute of Aviation Materials
VVIA	Voenno-Vozdushnaya Inzhenernaya Akademiya	Air Force Engineering Academy
ZhETF	Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki	Journal of Experimental and Theoretical Physics

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