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ABSTRACT

The interaction between galactic cosmic rays and the steady solar wind is studied. In the steady state the cosmic rays form a stationary cosmic ray gas through which massive magnetic irregularities are carried by the solar wind. Collisions between the streaming irregularities and the stationary cosmic ray gas transfer energy and momentum to the cosmic rays. The cosmic ray gas in the solar environment is thereby heated by friction with the solar wind that flows through it. In the steady state the average cosmic rays near the earth are more energetic than they were in interstellar space by a factor that is less than three.

INTRODUCTION

If we look at local cosmic rays on a scale somewhat larger than the solar system we see a star imbedded in the cosmic ray gas and emitting a stream of magnetized plasma in all directions. The stellar plasma carries irregularities which appear as small kinks or knots in the magnetic field. There are also inhomogeneities in density and velocity. The magnetic irregularities scatter individual cosmic rays with the result that the stellar plasma tends to push the cosmic ray gas away from the star. However the cosmic ray gas diffuses upstream toward the star until a balance is established in which outward convection is matched by inward diffusion. A density gradient is thereby set up in which the cosmic ray gas has a lower density near the star than in deep interstellar space. In the steady state the cosmic ray gas at any point has an isotropic velocity distribution on the microscopic scale. That is, the bulk velocity of the cosmic ray gas vanishes, i.e.

$$\overline{\mathbf{w}} = \int f(\mathbf{w})\, \mathbf{w}\, d\mathbf{w} = 0$$

where $f(\mathbf{w})$ is the distribution function.

Friction (by way of Coulomb interactions) between the plasma particles and the cosmic rays is weak and will be neglected. The cosmic rays interact with the wind chiefly by way of the electromagnetic fields carried by the wind. These fields can be separated into smooth parts $\mathbf{E}_0$ and $\mathbf{B}_0$ which are time invariant in the steady wind, plus fluctuating, spatially

irregular parts. $\mathbf{E}_0$ arises from the polarization of the flowing plasma. If $\mathbf{v}$ is the velocity of the plasma then

$$\mathbf{E}_0 = -\mathbf{v} \times \mathbf{B}_0 / c \tag{1}$$

The smooth, steady fields do not destroy the isotropy of the cosmic rays[1] although the kinetic energies of the individual particles do vary as they move about in the potential field (1) and a pressure gradient normal to the ecliptic tends to appear[2]. The effect of the smooth field on the energy of the average particle is

$$q \int f(\mathbf{w})\, \mathbf{w} \cdot \mathbf{E}_0 \, d\mathbf{w} = q\, \overline{\mathbf{w}} \cdot \mathbf{E}_0$$

which vanishes since $\overline{\mathbf{w}} = 0$. The smooth, large-scale time-invariant, average electromagnetic fields produce no continual deceleration in the steady state.

Now consider the interaction between the cosmic rays and the irregular components of the field. The irregularities will scatter the cosmic rays. Now an irregularity is very massive when compared to an individual cosmic ray. The irregularity has the mass of the plasma to which the magnetic field is tied (by the high electrical conductivity of the plasma). Observations of cosmic ray diffusion[3-9] suggest a size of the order of $\ell \simeq 10^{11}$ cm for the typical irregularity responsible for the diffusion. The mass is therefore approximately $\ell^3 \times$ (hydrogen mass) x (number density) $\simeq 10^9$ grams. The momentum of such

a scattering center is about 10^{27} times greater than the momentum of a 10-Gev cosmic ray proton. It is therefore a very good approximation to regard the solar wind-cosmic ray interaction as being the interaction between a gas of very massive particles (the magnetic irregularities) streaming through a stationary gas of light particles (the cosmic rays). The light gas maintains its stationary distribution in space by diffusing upstream toward the sun.

We now see a gas (consisting of clumps of magnetic field) flowing rapidly outward through a stationary gas consisting of cosmic rays. Collisions between the massive irregularities and the cosmic rays transfer momentum and energy to the cosmic ray gas. The momentum transferred to the cosmic ray gas is balanced by the gradient in the pressure of the cosmic ray gas[10]. The energy given to the cosmic ray gas leaks away to interstellar space carried by individual cosmic rays which escape from the solar wind with more energy than they had when entering it. The energy transfer can be regarded as the conversion of the streaming energy of the solar wind into heat by the action of friction between two interpenetrating fluids.

ACCELERATION BY FRICTION WITH SOLAR WIND

An estimate of the rate at which cosmic rays gain energy due to collisions with the magnetic scattering centers that stream through the stationary cosmic ray gas is easily made: reduce the problem to one dimension. At a given point in space the cosmic rays are maintained in an isotropic distribution by scattering. Equal numbers of cosmic rays are moving upstream and downstream so that the average velocity is zero. In a collision with a scattering center moving with velocity v, a particle whose <u>total</u> energy is $\mathcal{E}$ and whose speed is w will suffer a change in energy given by

$$\Delta\mathcal{E} = 2\mathcal{E}(v \pm w)v \Big/ (1 - v^2/c^2)c^2 \qquad (2)$$

where the + (-) sign refers to a head-on (tail-on) collision. If the number density of cosmic rays is n, the number density of scattering centers N and the cross section for total reflection σ, then the number of head-on (tail-on) collisions per unit volume per unit time is

$$\frac{1}{2} n N \sigma (w \pm v)$$

Multiplying the number of each kind of collision by the energy change for that kind of collision and adding gives the energy change suffered per unit time by all the cosmic rays in a unit volume. Dividing by n then gives the average energy change per unit time experienced by a

single particle. Neglecting terms in v^2/c^2 compared to unity the result is

$$\frac{d\mathcal{E}}{dt} = 4\mathcal{E}N\sigma v^2 w/c^2$$

The calculation is a bit more involved in three dimensions and therefore we shall outline it here, leaving the details to the Appendix. We let $f(\theta)$ be the angular part of the particle velocity distribution function, $\rho(\underset{\sim}{u})$ the velocity distribution function of the scattering centers with density N, V_R the magnitude of the relative velocity of the particles and scattering centers, $\sigma(\Theta)$ the differential scattering cross section for the particle-scattering center interaction with a center-of-mass scattering angle Θ, and let $\Delta\mathcal{E}$ be the change in energy suffered by the particle which had velocity $\underset{\sim}{w}$ before the collision. The mean rate of change of energy of a particle is given by multiplying $\Delta\mathcal{E}$ times the frequency of collision ν. ν is equal to the product of the two distribution functions times the collision cross section and the relative velocity. Since we want the mean rate of change of energy averaged over all collisions per unit time per unit volume we must then integrate over all scattering center velocities $\underset{\sim}{u}$, scattering angles Θ, and incident angles θ, between $\underset{\sim}{u}$ and $\underset{\sim}{w}$, the particle velocity. One should note that the distribution functions are normalized so that

$$\int f(\theta)\, 2\pi\, d(\cos\theta) = 1$$

while

$$\int \rho(\underset{\sim}{u})\, d\underset{\sim}{u} = N$$

where N is the density of scattering centers. Thus we wish to calculate

$$\frac{d\mathcal{E}}{dt} = \int_{-1}^{1} 2\pi\, d(\cos\theta) \int d\underset{\sim}{u} \int_{-1}^{1} 2\pi\, d(\cos\Theta)\, f(\theta)\, g(\underset{\sim}{u})\, V_R\, \sigma(\Theta)\, \Delta\mathcal{E} \tag{3}$$

The case we will consider is that in which the scattering centers are streaming radially outward from the sun with a constant velocity $\underset{\sim}{w}$, equal to the solar wind velocity. This approximation neglects the random magnetohydrodynamic waves which one expects to propagate with the Alfvén speed in a frame of reference moving with the velocity $\underset{\sim}{w}$. The Alfvén speed is about one tenth the solar wind speed which is 300-500 km/sec. Thus any second-order Fermi effects from the random motion of these MHD waves will be 100 times smaller than the friction caused by the streaming irregularities.

The scattering model chosen is 180° scattering in the center-of-mass system. One should note that the relativistic expression for the relative velocity should be used in equation (3). $\Delta\mathcal{E}$ is found by making a Lorentz transformation to the center-of-mass frame, letting the direction (but not the magnitude) of the particle's momentum change in the scattering process, and then transforming back to the laboratory system.

The situation of particular interest is that in which $v^2 \ll w^2 \simeq c^2$. For this case substitution in equation (3) yields, after integration,

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \frac{8}{3}\sigma N \mathcal{E} \frac{v^2 w}{c^2}\left[1 + O(v^2/c^2)\right] \\ &= \frac{8}{9}\mathcal{E}\frac{v^2 w^2}{\kappa c^2} \end{aligned} \tag{4}$$

where $\kappa = \lambda w/3$ is the diffusion coefficient for isotropic diffusion in three dimensions and $\lambda = 1/\sigma N$ is the mean free path for scattering.

ADIABATIC DECELERATION

The tendency for the magnetic irregularities to convect cosmic rays along with the solar wind has led to the idea that cosmic rays in the interplanetary medium experience a systematic deceleration known as "adiabatic deceleration".[11] According to this idea, the kinetic energy T of a particle located at $\mathbf{r}$ is continually decreasing at the rate

$$\frac{1}{T}\frac{dT}{dt} = -\frac{\alpha(T)}{3}\nabla\cdot\mathbf{v}(\mathbf{r}) \tag{5}$$

where $\alpha = 2$ for nonrelativistic particles, $\alpha = 1$ for extreme relativistic particles and $\mathbf{v}$ is the solar wind velocity.

Now Eq. (5) derives from considering a fluid in thermodynamic equilibrium at temperature T, undergoing an adiabatic expansion so that

$$T \propto n^{\gamma-1} \tag{6}$$

where n is the density and γ is the ration of specific heats, <u>and having a fluid velocity</u> $\mathbf{v}$. The temperature of a fluid element then varies as

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T \tag{7}$$

while the density and velocity field are coupled by the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0 \tag{8}$$

Eqns. (6,7,8) may be combined to yield

$$\frac{1}{T}\frac{dT}{dt} = -\frac{2}{3}\nabla \cdot \mathbf{v} \tag{9}$$

in the steady state, which is the nonrelativistic form of (5). We have set $\gamma = 5/3$.

We emphasize that the velocity appearing in (5) is the bulk velocity of the fluid itself. Therefore, if we want to apply eqn. (9), or its generalized form (5), to cosmic rays then $\mathbf{v}$ should be the bulk velocity of the cosmic ray gas -- not the velocity of the solar wind.

Now, the average motion, as shown by Stern [1] , is zero if the magnetic field is smooth. If the field is very irregular the average motion is $\mathbf{\Omega} \times \mathbf{r}$, where $\mathbf{\Omega}$ is the angular velocity of the sun [2,12] . This represents rigid rotation of the cosmic ray gas with the sun.* Therefore, apart from the tendency to corotate, the bulk velocity of the solar wind is zero in the steady state: The cosmic ray distribution in the solar system is the result of a competition between outward convection and inward diffusion. In the steady state, the convective flux is just balanced by the diffusive flux so that the bulk velocity appearing in eg. (5) vanishes. This implies that there is a non-zero bulk velocity in the coordinate frame moving with the wind. The fact that this velocity

*If we drop the assumption of spherical symmetry then streaming motions in the radial direction become possible [2] . However extremely distorted geometries must be assumed in order to achieve streaming velocities approaching the solar wind velocity.

$$\overline{\mathbf{w}}' = -\kappa \nabla \ln n$$

is small compared to the mean particle velocity justifies the use of diffusion theory and implies that the distribution function is <u>nearly</u> isotropic in the frame of the wind. But we would not be justified in entirely neglecting $\overline{\mathbf{w}}'$, thereby assuming isotropy in the frame of the wind, since to do so would leave us with a bulk velocity in the frame of the <u>sun</u>

$$\overline{\mathbf{w}} = \mathbf{v}_{wind} + \overline{\mathbf{w}}'$$

equal to $\mathbf{v}_{wind}$. In this case the solar system, lacking any sources, would be emptied of galactic cosmic rays in a few days. Therefore eq (5) cannot be directly applied to galactic cosmic rays in the steady state.

It should also be clear that acceleration and deceleration of cosmic rays, galactic particles included, is certainly to be expected when the large scale structure of the interplanetary magnetic field is changing in time, as it does after a large solar flare[13-15]. In this paper we are considering stationary conditions while in the latter references transient conditions were considered. A large flare generates a disturbance which propagates into interplanetary space. This disturbance is variously described as a cloud of turbulent plasma bearing a disordered magnetic field[16] or a blast wave expanding away from the sun[14]. In the former case the irregular magnetic field tends to exclude galactic cosmic rays leading to a Forbush decrease when the cloud sweeps over the earth. As the cloud grows from solar flare dimensions to astronomical-unit dimensions the mean magnetic field at a given point within the cloud decreases with time. The number of magnetic irregularities per unit volume also decreases with time. Under these transient conditions there is a deceleration of cosmic rays.

The blast wave[14] is a region or shell of compressed interplanetary magnetic field and gas at the head of the cloud or disturbance. The advancing blast wave is capable of accelerating cosmic rays and other particles in its path. Cosmic rays that find themselves behind the blast wave are reflected by a "magnetic piston" which expands away from the sun. The magnetic structure, as characterized by the size of the region between the blast wave and the sun, is growing in time and, by Fermi collisions of the overtaking type, this transient expansion produces a deceleration of cosmic rays.

These phenomena should be contrasted with the conditions existing in interplanetary space when the solar wind is steady. Except for the reversal of field direction when a sector boundary[17] sweeps by, the average magnetic field at a point is constant in strength and direction. Even if we take note of the existence of fluctuations in $\mathbf{B}_m$ representing scattering centers for cosmic rays, we must recognize that the number of scattering centers per unit volume is constant in time in the quiet wind. Therefore collisions between cosmic rays and scattering centers in the steady solar wind will not produce the deceleration described by Laster et al[15].

Instead, the streaming of the scattering centers throught the stationary cosmic ray gas will heat the cosmic ray gas, accelerating individual cosmic rays. The acceleration will be enhanced by any random motion which the scattering centers may have. The problem is analagous to a slowly turning perforated paddle wheel moving through a volume of air. The paddles tend to convect the air along with them. The air leaks through the holes in the paddles and maintains a constant spatial distribution. But friction between the paddles and air molecules heats the air.

DISCUSSION

The typical cosmic ray seen at the earth has spent a time of the order of L^2/K diffusing around in the solar wind, where L is the radius of the cavity which the wind carves out of the interstellar medium[18]. Integrating (4) over the time interval L^2/K, neglecting the variation of K, yields

$$\mathcal{E} = \mathcal{E}_0 \exp\left\{ \frac{8}{9}\left(\frac{Lvw}{Kc}\right)^2 \right\} \tag{10}$$

for the mean total energy of a cosmic ray inside the cavity in terms of its total energy $\mathcal{E}_0$ outside the cavity.

The density n of cosmic rays near the sun is reduced below the density in interstellar space n_0 by the competition between convection and diffusion[19]. Parker's theory yields

$$n = n_0 \exp\left\{ -\frac{Lv}{K} \right\} \tag{11}$$

If we neglect the fact that cosmic rays display a distribution in energy and consider a monoenergetic cosmic ray gas then a limit on the coefficient Lv/K can be deduced. The particles are assumed to be relativistic, therefore $w \simeq c$ and we may write

$$\frac{Lvw}{Kc} \simeq \frac{Lv}{K} \equiv x$$

Then

$$\mathcal{E} \simeq \mathcal{E}_0 e^{x^2} \tag{12}$$

and

$$n = n_0 e^{-x} \tag{13}$$

Now momentum conservation applied to the interaction between the solar wind and the cosmic ray gas requires that the pressure of the cosmic ray gas shall increase with distance from the sun. Since the pressure of the cosmic ray gas is proportional to $n\mathcal{E}$ we must have

$$n\mathcal{E}/n_0\mathcal{E}_0 < 1 \tag{14}$$

or $x \lesssim 1$. Thus for the average cosmic ray energy we must have $Lv/K \lesssim 1$.

Analysis of the behavior of "cosmic rays" generated at the sun suggests that the mean free path for cosmic rays with kinetic energies of the order of a few hundred Mev to ~ 1 Gev is about 10^{12} cm yielding $K \simeq 10^{22}$ cm^2 sec^{-1}(3-8). For kinetic energies of the order of a few Gev the mean free path may be somewhat longer. However, if we take $K = 10^{22}$ cm^2 s^{-1} and $v = 3 \times 10^7$ cm s^{-1} (the solar wind velocity) we get an upper limit on L which is ~ 20 astronomical units. The actual boundary between the solar wind plasma and interstellar gas may be located considerably further away from the sun since the dimension L

defines only the region within which the diffusion coefficient is small enough to cause significant scattering. The diffusion coefficient is likely to vary with heliocentric distance partly because of the radial divergence of the plasma flow and partly because of the generation and dissipation of the turbulence which manifests itself as magnetic irregularities.

It is instructive to compute the total amount of energy dissipated from the wind by this friction. This will be of the order of

$$Q = \frac{4\pi}{3} L^3 n_{av} \frac{d\mathcal{E}}{dt} \tag{15}$$

$$= \frac{32\pi}{9} L^3 n_{av} \mathcal{E}_{av} v^2 w / \lambda c^2$$

With the parameters mentioned above and taking $\mathcal{E}_{average}$ = 2 Gev and $n_{av} = 10^{-10}$ cm^{-3} we find $Q \simeq 10^{24}$ erg s^{-1}. This should be compared to the energy transported away from the corona by the solar wind which is of the order of $\pi r_e^2 N_e M v^3/2$ where r_e is the astronomical unit, N_e is the solar wind density at one a.u. and M is the hydrogen mass. For N_e = 4 cm^{-3} this is of the order of 10^{26} erg s^{-1}. Thus about one percent of the solar wind's streaming energy is transferred to cosmic rays by collisions with magnetic irregularities. However, some care should be exercised here since several of the parameters in (15) are highly uncertain, particularly the size of the cavity and the mean free path for scattering.

We conclude that (apart from the effect of the polarization electric field) cosmic rays in the vicinity of the earth have a mean density that is smaller than that in interstellar space by a factor somewhat less than about three, while they are on the average more energetic than they are in interstellar space by a factor also somewhat less than about three.

CONCLUSION

The idea that cosmic rays are adiabatically decelerated in the steady solar wind is based on the assumption that all cosmic rays are convected along with the wind. Those cosmic rays that are convected along with the solar wind will in fact be decelerated by adiabatic deceleration. However all cosmic rays are not convected along with the wind. The average motion of individual cosmic ray particles is given by the bulk velocity of the cosmic ray gas. In the equilibrium state this bulk velocity is zero. Some particles are being convected outward at any instant and these are losing energy. But other particles are working their way upstream against the wind and these latter particles are gaining energy. The net effect is a stationary cosmic ray gas through which massive scattering centers are flowing. The energy gained in "head-on" Fermi collisions is then slightly greater than the energy lost in "tail-on" collisions so that on the average energy is being fed <u>into</u> the cosmic rays by the irregularities in the solar wind.

The cosmic ray gas in the solar environment is therefore being heated by friction with the solar wind which flows through it. This heat leaks away to interstellar space (and to the sun) by diffusion. That is, individual cosmic rays, having been "heated" by their contact with the solar wind, ultimately escape back to interstellar space or are absorbed by the sun or a planet. In this manner an equilibrium is established in which the cosmic ray gas within the solar wind region is slightly "hotter" than the surrounding interstellar cosmic ray gas.

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APPENDIX

The expression to be calculated is

$$\frac{d\mathcal{E}}{dt} = 4\pi^2 \int_{-1}^{1} d(\cos\theta) \int d\mathbf{u} \int_{-1}^{1} d(\cos\Theta)\, f(\theta)\, \rho(\mathbf{u})\, V_R\, \sigma(\Theta)\, \Delta\mathcal{E} \tag{A1}$$

where

$$f(\theta) = \frac{1}{4\pi}$$

$$\rho(\mathbf{u}) = N\, \delta(\mathbf{u} - \mathbf{v})$$

$$V_R = \left\{ (\mathbf{w} - \mathbf{u})^2 - |\mathbf{w} \times \mathbf{u}|^2 \right\}^{1/2} (1 - \mathbf{w} \cdot \mathbf{u})^{-1}$$

$$\sigma(\Theta) = \frac{\sigma}{2\pi}\, \delta(\cos\Theta + 1)$$

$$\Delta\mathcal{E} = \Gamma^2 \mathcal{E} \left\{ u^2 - \mathbf{w} \cdot \mathbf{u} + u \cos\theta_c \left[(\mathbf{w} - \mathbf{u})^2 - |\mathbf{w} \times \mathbf{u}|^2 \right] \right\}^{1/2}$$

We are now using units in which $c = 1$. σ is the total scattering cross section, $\mathcal{E}$ is the initial energy of the particle, $\Gamma^2 = (1 - u^2)^{-1}$, θ_c is the angle between $\mathbf{u}, \mathbf{w}$ in the center-of-mass system after the collision, and θ is the angle between $\mathbf{u}, \mathbf{w}$ in the laboratory system before the collision.

Substitution into equation (A1) yields

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2} \sigma N \mathcal{E} \Gamma^2 \left(I_1 + I_2 + I_3 \right) \tag{A2}$$

with

$$I_1 = \int_{-1}^{1} d\mu \, v^2 \, \frac{\left[w^2+v^2-2wv\mu - w^2v^2(1-\mu^2)\right]^{1/2}}{1-wv\mu} \tag{A3}$$

$$I_2 = -\int_{-1}^{1} d\mu \, wv\mu \, \frac{\left[w^2+v^2-2wv\mu - w^2v^2(1-\mu^2)\right]^{1/2}}{1-wv\mu} \tag{A4}$$

$$I_3 = \int_{-1}^{1} d\mu \, v \cos\theta_c \, \frac{\left[w^2+v^2-2wv\mu - w^2v^2(1-\mu^2)\right]^{1/2}}{1-wv\mu} \tag{A5}$$

where $\mu = \cos\theta$.

For the scattering process which we are considering, $\cos\theta_c = -\cos\varphi_c$ where φ_c is the angle between the particle velocity and the scattering center velocity in the center-of-mass frame before the interaction. When one then relates $\cos\varphi_c$ to $\cos\theta$ via the usual Lorentz transformation one finds that

$$I_3 = I_1 + I_2 \tag{A6}$$

These integrals can be computed exactly; however it is instructive to first examine the expressions in three limiting cases, viz. Case I: $v^2 \ll w^2 \ll 1$; Case II: $v^2 \ll w^2 \simeq 1$ (this is the physical situation under discussion in the body of the paper); and Case III: $v^2 \lesssim w^2 \ll 1$.

The ultrarelativistic case, $v \lesssim w \simeq 1$, will be discussed separately below.

In order to solve (A2) for cases I-III it is sufficient to make a few simplifying approximations. Ignore the cross product terms in V_R and $\Delta\mathcal{E}$ as these are of order $1/c^2$ times the other terms. One can further allow

$$V_R \simeq |\mathbf{w}-\mathbf{v}|\left(1-\mathbf{w}\cdot\mathbf{v}\right)^{-1} \simeq |\mathbf{w}-\mathbf{v}|\left(1+\mathbf{w}\cdot\mathbf{v}\right) \tag{A7}$$

Substitution of (A3)-(A7) into (A2) and integrating yields, after some elementary algebra,

$$\frac{d\mathcal{E}}{dt} \simeq \frac{8}{3}\sigma N\mathcal{E}\Gamma^2 v^2 w\left[1+\frac{1}{5}\frac{v^2}{w^2}-\frac{v^2}{4}\left(1-\frac{1}{5}\frac{v^2}{w^2}\right)\right]$$

Thus, one has for Case I and Case II $(v^2 \ll w^2)$

$$\frac{d\mathcal{E}}{dt} = \frac{8}{3}\sigma N\mathcal{E}v^2 w\left\{1+O(v^2 \ll 1)\right\} \tag{A8}$$

while for Case III $(v^2 \lesssim w^2 \ll 1)$

$$\frac{d\mathcal{E}}{dt} \simeq \frac{16}{5}\sigma N\mathcal{E}v^2 w \tag{A9}$$

The salient feature of these limiting cases is that at least one velocity is nonrelativistic. It is this fact which allows the approximations to be made. It should be pointed out that in Case III the

series in v^2/w^2 converges rapidly even when $v^2 \simeq w^2$.

To evaluate (A2) in the ultrarelativistic case when $v^2 \lesssim w^2 \simeq 1$ we are allowed no approximations and (A3)-(A5) must be computed exactly. This is easily done, but at the expense of not being readily able to recover the nonrelativistic results in (A8)-(A9).

The exact expression for (A2) is

$$\frac{d\mathcal{E}}{dt} = \sigma N \mathcal{E} \Gamma^2 \left[\frac{2v^2}{w} + \frac{1}{w}(1-w^2) - \frac{1}{2}(1-w^2-v^2+w^2v^2) \ln \left| \frac{1-wv+w-v}{1+wv+w+v} \right| \right.$$

$$\left. + \frac{v}{w}\sqrt{1-w^2-v^2+w^2v^2} \left\{ \cos^{-1}\left| \frac{\sqrt{1-w^2-v^2+w^2v^2}}{1-wv} \right| - \cos^{-1}\left| \frac{\sqrt{1-w^2-v^2+w^2v^2}}{1+wv} \right| \right\} \right]$$

which reduces to

$$\frac{d\mathcal{E}}{dt} \simeq 2\sigma N \mathcal{E} \Gamma^2 v^2/w$$

when $v^2 \lesssim w^2 \simeq 1$.

In order to verify this limit, notice that the logarithmic term becomes

$$\approx \frac{-1}{2}(1-w^2)^2 \ln \left| \frac{1-w^2}{(1+w)^2} \right| \tag{A10}$$

By L'Hopital's rule $\lim_{y \to \infty} \frac{\ln 1/y}{y^2} = 0$, which yields equation (A10).