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A STUDY OF LONGITUDINAL OSCILLATIONS OF PROPELLANT TANKS AND WAVE PROPAGATIONS IN FEED LINES

PART II - Longitudinal Oscillation of a Liquid-Filled Elastic Cylindrical Tank with a Flexible Inverted Conical Bulkhead

3 April 1967

NAS8-11490

Prepared by

Shoichi Uchiyama, Staff Investigator
Clement L. Tai, Principal Investigator

Approved by

F. C. Hung, Program Manager
Structures and Materials

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Science and Technology

NORTH AMERICAN AVIATION, INC.
SPACE and INFORMATION SYSTEMS DIVISION

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FOREWORD

This report was prepared by the Space and Information Systems Division of North American Aviation, Inc., Downey, California, for the George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, under the Supplemental Agreement Modification No. 2 of Contract NAS8-11490, "Study of Longitudinal Oscillations of Propellant Tanks and Wave Propagations in Feed Lines," dated April 6, 1966. Dr. George F. McDonough (Principal) and Mr. Robert S. Ryan (Alternate) of Aero-Astrodynamics Laboratory, MSFC, are Contracting Officer Representatives. The work is published in two separate parts:

PART I - Propagating Pressure Waves in a Fluid Filled Cylindrical Shell

PART II - Longitudinal Oscillation of A Liquid-Filled Elastic Cylindrical Tank with A Flexible Inverted Conical Bulkhead

The project was carried out by the Vehicle Dynamics Branch, Structures and Materials Department of Research, Engineering, and Testing Division, S&ID. Dr. F. C. Hung was the Program Manager for North American Aviation, Inc. The study was conducted by Dr. Clement L. Tai (Principal Investigator), Dr. Shoichi Uchiyama, and Mr. John S. Kanno. The computer program was developed by Mr. Shigeo Miyashiro.

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ABSTRACT

A procedure has been formulated to determine the natural frequencies and the corresponding mode shapes of a liquid-filled elastic cylindrical tank with a flexible inverted conical bulkhead. It is assumed that the liquid is incompressible and inviscid and its motion is irrotational. Under such assumptions, the velocity potential is obtained from the solution of Laplace's equation in both circular cylindrical and spherical polar coordinates. This velocity potential, together with Bernoulli's equation, permits the evaluation of the fluctuating liquid pressure at the liquid-shell interfaces.

The interface pressure is treated as a forcing function in the shell equations, and the shell displacement components are then determined analytically. An eigenvalue problem was constructed by the least square technique through the boundary conditions and interface conditions. The solution of such an eigenvalue problem are the desired natural frequencies and the corresponding mode shapes of the system.

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CONTENTS

Section		Page
	NOMENCLATURE	xi
I	INTRODUCTION	1
II	EQUATIONS OF MOTION AND BOUNDARY CONDITIONS	3
	1. Liquid L_1	4
	2. Liquid L_2	5
	3. Circular Cylindrical Shell Filled with Liquid L_1	6
	4. Circular Conical Shell	8
	5. Circular Cylindrical Shell Filled with Liquid L_2	10
III	SOLUTIONS OF VELOCITY POTENTIALS AND SHELL DISPLACEMENTS	11
	1. Liquid L_1	11
	2. Liquid L_2	14
	3. Circular Cylindrical Shell Partially Filled with Liquid L_1	15
	4. Circular Conical Shell	26
	5. Circular Cylindrical Shell Filled with Liquid L_2	34
IV	MATRIX EIGENVALUE PROBLEM	41
V	PROCEDURE FOR NUMERICAL CALCULATIONS	47
	CONCLUDING REMARKS AND RECOMMENDATIONS	51
	REFERENCES	53

Section Page

APPENDICES

A. Liquid Velocity Potentials, $\bar{\phi}_1(\bar{r},\bar{z})$ and $\bar{\phi}_2(\bar{s},\mu)$ and Shell Displacement Components $\bar{w}_{11}(\bar{z})$, $\bar{w}_{12}(\bar{z})$ and $\bar{w}_2(\bar{s})$ 59

B. Functional Errors $\bar{\varepsilon}_i$ (i = 1, 2, ---5) . . . 77

C. Computer Program Description 85

ILLUSTRATIONS

Figure		Page
1	Liquid-Filled Circular Cylindrical Tank with An Inverted Conical Bulkhead at the Bottom . . .	3
2	Axisymmetric Mode Shapes for A Liquid-Filled Circular Cylindrical Tank with An Inverted Conical Bulkhead, f = 2.23 cps	55
	(a) Normal Displacement of A Circular Cylindrical Tank	55
	(b) Normal Displacement of An Inverted Conical Bulkhead	55
	(c) Pressure Distribution of Liquid at the Interface, $\bar{z}=\bar{b}$	55
3	Axisymmetric Mode Shapes for A Liquid-Filled Circular Cylindrical Tank with An Inverted Conical Bulkhead, f = 3.98 cps	56
	(a) Normal Displacement of A Circular Cylindrical Tank	56
	(b) Normal Displacement of An Inverted Conical Bulkhead	56
	(c) Pressure Distribution of Liquid at the Interface, $\bar{z}=\bar{b}$	56
4	Axisymmetric Mode Shapes for A Liquid-Filled Circular Cylindrical Tank with An Inverted Conical Bulkhead f = 6.21 cps	57
	(a) Normal Displacement of A Circular Cylindrical Tank	57
	(b) Normal Displacement of An Inverted Conical Bulkhead	57
	(c) Pressure Distribution of Liquid at the Interface, $\bar{z}=\bar{b}$	57
5	Frequencies and Modes Program Logic	
	(a) Flow Diagram for Evaluation of Frequencies . .	86
	(b) Flow Diagram for Evaluation of Mode Shapes . .	87
Table		
1	Physical Characteristics of A Liquid-Filled Circular Cylindrical Tank with An Inverted Conical Bulkhead	54

NOMENCLATURE

A_n, B_n	Arbitrary constants
a	Radius of cylindrical shell
b	Height of cone
C_n, $C_{\hat{n}}$, $C_{\bar{n}}$	Undetermined constants $n = 0, 1, ---, N$; $\hat{n} = 2n$; $\bar{n} = 2n + 1$
e_s, e_ϕ	Extensional and angular (hoop) strain.
D	$Eh/(1 - \nu^2)$, shell extension rigidity.
E	Young's modulus
F, f	Defined functions
h_1, h_2	Thickness of cylindrical shell and cone respectively.
$J_n(\mu, \bar{s})$	Bessel's function of nth order
N_s, N_ϕ, N_z	Stress resultant in the direction of s, ϕ, z respectively
$P_n(\mu)$	Legendre polynomials
p	Pressure
r	Cylindrical radial coordinate
s	Spherical radial coordinate
t	Time
u_1, v_1, w_1	Axial, meridional and radial displacement component, respectively of cylindrical shell.
u_2, w_2	Displacement component of cone in the direction of s and θ
z	Axial coordinate

$\bar{r}$, $\bar{s}$, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{z}$, etc.	Nondimensional quantity of r, s, u, v, w, z, etc.
g_{mn}, h_{mn}, ℓ_{mn}	Defined constant
α	Semivortex angle of cone
α_n,	Positive root of Bessel's function of first order
β_n	Undetermined constant
$\Gamma(x)$	Gamma function
ε_1, ε_2, ...	Functional errors
θ	Meridional spherical coordinate
κ_1, κ_2	$\rho_1 h_1 a^2 / D_1$ and $\rho_2 h_2 a^2 / D_2$ respectively
λ	Defined constant
μ	$\cos\theta$
ν	Poisson's ratio
ρ_f, ρ_1, ρ_2	Density of fluid, cylindrical shell, and cone respectively
σ	Defined constant
$\bar{t}$	Nondimensional quantity of t
Φ_1, Φ_2	Velocity potential
ϕ_1, ϕ_2	Velocity potential for steady flow
ω	Natural frequency

I. INTRODUCTION

In the development of launch vehicle and spacecraft design, a knowledge of the dynamic behavior of thin-walled fuel tanks is of great importance. Of prime importance are the natural frequencies and the corresponding mode shapes of vibrations. In a recent investigation, the author (Reference 1) studied the coupled oscillations of a liquid with a free surface in a flexible oblate spheroidal tank. For a better understanding of such coupling of oscillation, the study of the interaction of dynamic behavior of a flexible liquid-filled cylindrical tank with a flexible inverted conical bulkhead is developed.

A number of studies have been made during the past years of the dynamic behavior of cylindrical tanks with various shapes of bulkhead. Bhuta and Koval (Reference 2) studied the problem of sloshing of a liquid in a cylindrical tank with rigid walls and a flexible bottom. Bleich (Reference 3) investigated longitudinal forced vibrations of cylindrical fuel tanks with rigid walls and elastic bottoms of arbitrary shape. Coale and Nagano (Reference 4) dealt with the axisymmetric dynamic behavior of a cylindrical shell with a hemispherical shell bottom. The present study is concerned with the axisymmetric dynamic behavior of an elastic, liquid-filled cylindrical tank with a flexible inverted conical bulkhead.

The present method of analysis is based on two previous papers, one for a hemispherical tank (Reference 5) and one for an oblate spheroidal tank (Reference 1), both of which are fully filled by the liquid. The mathematical model for the analysis of such a system is considered and its geometry is described by a circular cylindrical coordinate system for the cylindrical shell and a spherical polar coordinate system for the inverted conical shell as shown in Figure 1. The motion of the liquid is represented by a velocity potential which satisfies the continuity equation stated by Laplace's equation. The motion of the shell is represented by two displacement components, one in the normal direction and one in the tangential direction. The liquid velocity potential in the cylindrical shell is represented by a series of the product of Bessel functions and trigonometric functions. The liquid velocity potential in the inverted conical shell is represented by a series of Legendre functions. The shell displacement components for both the cylindrical shell and the conical shell are expressed in terms of the liquid velocity potential. Coefficients in these series are selected to satisfy the liquid-shell interface conditions by the least squared error technique. For the numerical calculation of the natural frequencies and the corresponding mode shapes, the procedure is described in detail at the end of this report.

II. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The system to be analyzed is a thin, flexible, liquid-filled circular cylindrical shell with a thin, flexible, inverted conical shell. The configuration is shown in Figure 1. For analysis, the system is divided into four elements: (1) a cylinder of liquid with a free surface, (2) an inverted cone of liquid, (3) a circular cylindrical shell, and (4) a circular conical shell. The liquid in elements (1) and (2) is assumed to be incompressible, inviscid, and irrotational. The shell in elements (3) and (4) is treated as a membrane and hence the bending effects are ignored. For a thin shell whose thickness-to-radius ratio is very small, such as those in the present case, this assumption is justified. The inertial forces of elements (3) and (4) are included in the equations of motion. Only a small-amplitude, longitudinal, axisymmetric motion is considered. As shown in Figure 1, the system is restrained against axial motion at the lower end of the cylindrical shell but is unrestrained otherwise. The upper end of the cylindrical shell is unrestrained longitudinally.

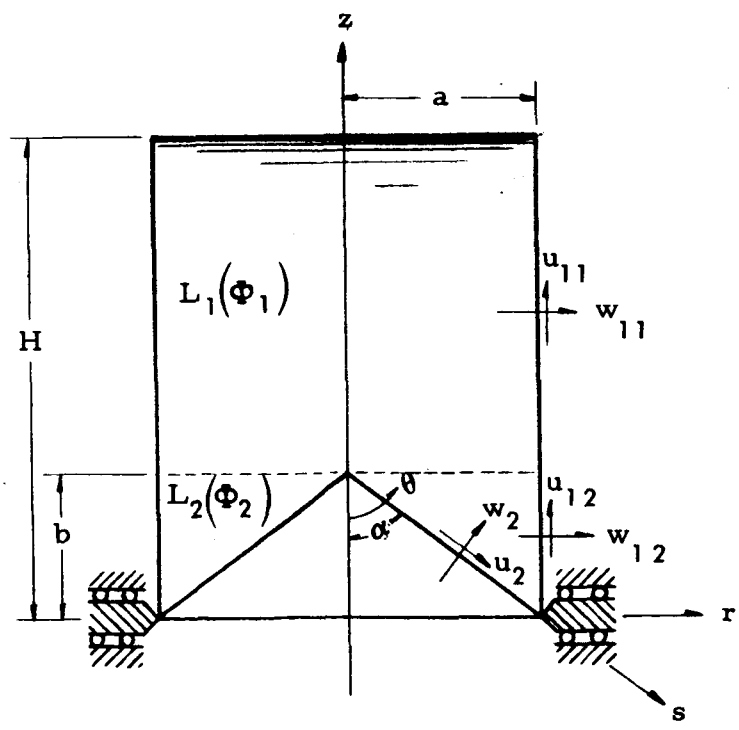

Figure 1. Liquid-Filled Circular Cylindrical Tank with An Inverted Conical Bulkhead at the Bottom

1. Liquid L_1

Motion of the liquid L_1 can be expressed in terms of a velocity potential $\Phi_1(r,z,t)$ which satisfies the axisymmetric Laplace equation in circular cylindrical coordinates

$$\nabla^2 \Phi_1 = \frac{\partial^2 \Phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_1}{\partial r} + \frac{\partial^2 \Phi_1}{\partial z^2} = 0 \tag{1}$$

where r and z are circular cylindrical coordinates and shown in Figure 1.

Two of the boundary conditions associated with this element are

$$\left. \frac{\partial \Phi_1}{\partial r} \right]_{r=0} = 0 \tag{2}$$

which is required to avoid a singular solution at the axis, and

$$\left[\frac{\partial^2 \Phi_1}{\partial t^2} + g \frac{\partial \Phi_1}{\partial z} \right]_{z=H} = 0 \tag{3}$$

which is the linearized boundary condition that is to be satisfied at the free surface.

The remaining boundary conditions are interface conditions coupling this element to other elements of the system. At the liquid-shell interface, the boundary condition is

$$\left. \frac{\partial \Phi_1}{\partial r} \right]_{r=a} = \frac{\partial w_{11}}{\partial t} \tag{4}$$

where $\frac{\partial \Phi_1}{\partial r}$ is taken positively in the direction of increasing r and w_{11} is the radial displacement of the cylindrical shell to be determined in terms of Φ_1. Between the two elements of liquid, pressure, and normal velocity of the two elements must match, namely,

$$\left.\frac{\partial \Phi_1}{\partial t}\right]_{z=b} = \left.\frac{\partial \Phi_2}{\partial t}\right]_{\theta=\frac{\pi}{2}} \tag{5}$$

and

$$\left.\frac{\partial \Phi_1}{\partial z}\right]_{z=b} = \frac{1}{s}\left.\frac{\partial \Phi_2}{\partial \theta}\right]_{\theta=\frac{\pi}{2}} \tag{6}$$

where Φ_2 is a velocity potential of the liquid L_2, and s and θ are spherical polar coordinates.

2. Liquid L_2

Motion of the liquid L_2 can be expressed in terms of a velocity potential $\Phi_2(s,\theta,t)$ which satisfies the Laplace equation in spherical polar coordinates

$$\nabla^2 \Phi_2 = \frac{\partial^2 \Phi_2}{\partial s^2} + \frac{2}{s}\frac{\partial \Phi_2}{\partial s} + \frac{1}{s^2}\frac{\partial^2 \Phi_2}{\partial \theta^2} + \frac{\cot\theta}{s^2}\frac{\partial \Phi_2}{\partial \theta} = 0 \tag{7}$$

The boundary conditions for Φ_2 are interface conditions and two of them are given by Eqs. (5) and (6). The other interface conditions are

$$\left[\frac{r}{\sqrt{r^2 + (b-z)^2}} \frac{\partial \Phi_2}{\partial s} + \frac{b - z}{r^2 + (b-z)^2} \frac{\partial \Phi_2}{\partial \theta} \right]_{\substack{r = a \\ s = \sqrt{a^2 + (b-z)^2} \\ \theta = \tan^{-1} \frac{a}{b-z}}}$$

$$= \frac{\partial w_{12}}{\partial t}_{\substack{s = \sqrt{a^2 + (b-z)^2} \\ \theta = \tan^{-1} \frac{a}{b-z}}} \tag{8}$$

and

$$\frac{1}{s} \left. \frac{\partial \Phi_2}{\partial \theta} \right]_{\theta = \alpha} = \frac{\partial w_2}{\partial t} \tag{9}$$

where $\frac{\partial \Phi_2}{\partial s}$ and $\frac{1}{s} \frac{\partial \Phi_2}{\partial \theta}$ are taken positively in the direction of increasing s and θ respectively and the left-hand side of Eq. (8) represents the component of Φ_2 in the direction normal to the cylindrical shell surface. It is obtained from

$$\frac{\partial \Phi_2}{\partial n} = \vec{n} \cdot \nabla \Phi_2 = \frac{\partial \Phi_2}{\partial s} \sin\theta + \frac{1}{s} \frac{\partial \Phi_2}{\partial \theta} \cos\theta \tag{10}$$

where $\vec{n}$ is the unit outward normal vector.

3. Circular Cylidrical Shell Filled with Liquid L_1

Let the coordinate axes be chosen so that z is in the direction of the generatrix and r is in the direction of the normal to middle surface of the shell as shown in Figure 1. The axisymmetric differential equations (Reference 6) for this shell can then be expressed in terms of displacements $u_{11}(z,t)$ and $w_{11}(z,t)$ as

$$a \frac{\partial^2 u_{11}}{\partial z^2} + \nu_1 \frac{\partial w_{11}}{\partial z} = \frac{\rho_1 h_1 a}{D_1} \frac{\partial^2 u_{11}}{\partial t^2} \tag{11}$$

$$\nu_1 a \frac{\partial u_{11}}{\partial z} + w_{11} = - \frac{\rho_1 h_1 a^2}{D_1} \frac{\partial^2 w_{11}}{\partial t^2} + \frac{a^2 p_1}{D_1} \tag{12}$$

where ν_1 is the Poisson's ratio, ρ_1 is the mass density, h_1 is the thickness, and D_1 is the extensional modulus $\left(= E_1 h_1 \Big/ 1-\nu_1^2 \right)$ of the circular cylindrical shell, and p_1 is the liquid pressure acting at the shell wall.

From the linearized Bernoulli equation, the liquid pressure p_1 may be expressed as

$$p_1 = - \rho_f \frac{\partial \Phi_1}{\partial t} \tag{13}$$

where ρ_f is the mass density of the liquid.

The boundary conditions are

$$\left. \begin{aligned} u_{11} &= u_{12} \\ \frac{\partial u_{11}}{\partial z} &= \frac{\partial u_{12}}{\partial z} \end{aligned} \right\} \text{ at } z = b \tag{14}$$

which indicates that the shell is continuous at $z = b$ and

$$N_z \Big]_{z=H} = D_1 \left(\frac{\partial u_{11}}{\partial z} + \nu \frac{w_{11}}{a} \right) \Bigg]_{z=H} = 0 \tag{15}$$

which specifies that there is no axial force acting on the upper end of the shell. There is no boundary condition which can be applied to w_{11} because with bending stiffness neglected the differential equation is of zero order in w_{11}.

4. Circular Conical Shell

Let s be chosen in the direction of the generatrix and θ be measured in the direction of the normal to the middle surface of the shell as shown in Figure 1. In this case, the curvilinear coordinates are s and θ. Let $R_1 = R_s$ and $R_2 = R_\theta$ be the principal radii of curvature. Then $R_s = \infty$ and $R_\theta = s \tan\alpha$ where α is a semi-vertex angle of the cone.

From the general expression of the equations of motion of a thin shell in terms of curvilinear coordinates (Reference 5), the equations of motion of the circular conical membrane under axially symmetric conditions are obtained as

$$\frac{\partial s N_s}{\partial s} - N_\phi + s f_s = 0 \tag{16}$$

$$\frac{N_\phi}{\tan\alpha} + s\left(p_2 - f_\theta\right) = 0 \tag{17}$$

where N_s and N_ϕ are the stress resultants, f_s and f_θ are the inertia forces, and p_2 is the liquid dynamic pressure acting at both the circular cylindrical shell wall and the circular conical shell wall.

The stress-strain relations give

$$N_s = D_2\left(e_s + \nu_2 e_\phi\right) \tag{18}$$

$$N_\phi = D_2\left(e_\phi + \nu_2 e_s\right) \tag{19}$$

where e_s and e_ϕ are the extensional and angular strain, respectively, and D_2 is the extensional modulus $\left(= E_2 h_2 / 1 - \nu_2^2 \right)$ of the circular conical shell.

The strain-displacement relations are obtained from the general expression (Reference 7):

$$e_s = \frac{\partial u_2}{\partial s} \tag{20}$$

$$e_\phi = \frac{1}{s}\left(u_2 + w_2 \cot\alpha\right) \tag{21}$$

Substituting Eqs. (20) and (21) into Eqs. (18) and (19) gives

$$N_s = D_2 \left[\frac{\partial u_2}{\partial s} + \frac{\nu}{s} \left(u_2 + w_2 \cot\alpha \right) \right] \tag{22}$$

$$N_\phi = D_2 \left[\frac{1}{s} \left(u_2 + w_2 \cot\alpha \right) + \nu \frac{\partial u_2}{\partial s} \right] \tag{23}$$

The inertia forces f_s and f_θ are given by

$$f_s = -\rho_2 h_2 \frac{\partial^2 u_2}{\partial t^2} \tag{24}$$

$$f_\theta = -\rho_2 h_2 \frac{\partial^2 w_2}{\partial t^2} \tag{25}$$

where ρ_2 is the mass density and h_2 is the thickness of the circular conical shell.

From the linearized Bernoulli equation, the liquid pressure p_2 may be expressed as

$$p_2 = -\rho_f \frac{\partial \Phi_2}{\partial t} \tag{26}$$

Substituting Eqs. (22) through (26) into Eqs. (16) and (17) gives

$$s \frac{\partial^2 u_2}{\partial s^2} + \frac{\partial u_2}{\partial s} - \frac{u_2}{s} + \nu \frac{\partial w_2}{\partial s} \cot\alpha - \frac{w_2}{s} \cot\alpha = \frac{\rho_2 h_2 s}{D_2} \frac{\partial^2 u_2}{\partial t^2} \tag{27}$$

$$\nu \frac{\partial u_2}{\partial s} + \frac{u_2}{s} + \frac{w_2}{s} \cot\alpha = -\frac{\rho_2 h_2 s}{D_2} \frac{\partial^2 w_2}{\partial t^2} \tan\alpha - \frac{s p_{2\alpha}}{D_2} \tan\alpha \tag{28}$$

where $p_{2\alpha}$ is the pressure p_2 at $\theta = \alpha$.

The boundary conditions are

$$u_2 = \text{finite} \qquad \text{at } s = 0 \tag{29}$$

which is required to avoid a singular solution at the apex and

$$u_2 - w_2 \tan\alpha = 0 \qquad \text{at } s = \sqrt{a^2 + b^2} \tag{30}$$

which assumes that there is no axial displacement at the bottom of the conical shell.

5. Circular Cylindrical Shell Filled with Liquid L_2

The axisymmetric differential equations for this shell can be expressed as

$$a \frac{\partial^2 u_{12}}{\partial z^2} + \nu_1 \frac{\partial w_{12}}{\partial z} = \frac{\rho_1 h_1 a}{D_1} \frac{\partial^2 u_{12}}{\partial t^2} \tag{31}$$

$$\nu_1 a \frac{\partial u_{12}}{\partial z} + w_{12} = - \frac{\rho_1 h_1 a^2}{D_1} \frac{\partial^2 w_{12}}{\partial t^2} + \frac{a^2 p_2}{D_1} \tag{32}$$

The boundary conditions are Eqs. (14) and

$$u_{12} \Big]_{z=0} = 0 \tag{33}$$

which indicates that there is no axial motion of the base of the shell. There is again no boundary condition which can be applied to w_{12}.

III. SOLUTIONS OF VELOCITY POTENTIALS AND SHELL DISPLACEMENTS

1. Liquid L_1

The equations for determining the shell displacements are found by assuming a simple harmonic motion of the system. For such a simple harmonic motion, the velocity potential, Φ_1, may be taken in the form

$$\Phi_1 = a^2\omega\,\bar{\phi}_1(r,z)\cos\omega t \tag{34}$$

where $\bar{\phi}_1(r,z)$ is the dimensionless velocity potential for steady flow, ω is the natural frequency of the system.

By separation of variables, $\Phi(r,z)$ is obtained in the form

$$\bar{\phi}_1(r,z) = e_{10} + e_{20}\,\ell n\left(\frac{r}{a}\right) + \left[e_{30} + e_{40}\,\ell n\left(\frac{r}{a}\right)\right]\frac{z}{a}$$

$$+\sum_{n=1}^{\infty}\frac{a}{g}A_n\left[a_n\sinh\left(\frac{\alpha_n z}{a}\right) + \cosh\left(\frac{\alpha_n z}{a}\right)\right]J_0\left(\frac{\alpha_n r}{a}\right)$$

$$+\sum_{n=1}^{\infty}B_n\left[b_n\sin\left(\frac{\beta_n z}{a}\right) + \cos\left(\frac{\beta_n z}{a}\right)\right]I_0\left(\frac{\beta_n r}{a}\right),$$

$$\begin{pmatrix} m = 1, 2, 3, - - - \\ n = 1, 2, 3, - - - \end{pmatrix} \tag{35}$$

where e_{10}, e_{20}, e_{30}, e_{40}, A_n, a_n, α_n, B_n, and β_n are constant parameters, and $J_0\left(\alpha_n r/a\right)$ and $I_0\left(\beta_n r/a\right)$ are the Bessel function of the first kind, of order zero and the modified Bessel function of the first kind, of order zero, respectively.

By the boundary condition, Eq. (2),

$$e_{20} = e_{40} = 0 \tag{36}$$

The solution $\bar{\phi}_1(r,z)$, Eq. (35), may be split into two parts as

$$\bar{\phi}_1(r,z) = \bar{\phi}_{1A}(r,z) + \bar{\phi}_{1B}(r,z) \tag{37}$$

where $\bar{\phi}_{1A}(r,z)$ corresponds to the terms $e_{10} + e_{30}\frac{z}{a}$ and the series with coefficients A_n and represents the flow through the intersection between two liquids L_1 and L_2, while $\bar{\phi}_{1B}(r,z)$ corresponds to the series with coefficients B_n and represents the flow through the side of the cylinder. However, both velocity potentials produce pressure on both the intersection and the side.

The boundary conditions for $\bar{\phi}_{1A}(r,z)$ and $\bar{\phi}_{1B}(r,z)$ are, respectively,

$$\left.\frac{\partial\bar{\phi}_{1A}}{\partial r}\right]_{r=a} = 0 \tag{38}$$

and

$$\left.\frac{\partial\bar{\phi}_{1B}}{\partial z}\right]_{z=b} = 0 \tag{39}$$

By the boundary condition, Eq. (38), we have

$$J_1\left(\alpha_n\right) = 0 \tag{40}$$

from which the constant parameter α_n can be determined. By the boundary condition, Eq. (39), we find

$$b_n\cos\left(\frac{\beta_n b}{a}\right) - \sin\left(\frac{\beta_n b}{a}\right) = 0 \tag{41}$$

from which

$$b_n = \tan\left(\frac{\beta_n b}{a}\right) \tag{42}$$

Define

$$e_{30} = A_0 \tag{43}$$

Substituting Eqs. (36), (42), and (43) into Eq. (35) gives

$$\begin{aligned}\bar{\Phi}_1(r,z) = {} & e_{10} + A_0 \frac{z}{a} \\ & + \sum_{n=1}^{\infty} A_n \left[a_n \sinh\left(\frac{\alpha_n t}{a}\right) + \cosh\left(\frac{\alpha_n z}{a}\right)\right] J_0\left(\frac{\alpha_n r}{a}\right) \\ & + \sum_{n=1}^{\infty} B_n \frac{\cos\left(\beta_n \frac{z-b}{a}\right)}{\cos\left(\frac{\beta_n b}{a}\right)} I_0\left(\frac{\beta_n r}{a}\right)\end{aligned} \tag{44}$$

The velocity potential $\Phi_1(r,z)$ is then obtained by substituting Eq. (44) into Eq. (34), as

$$\begin{aligned}\Phi_1(r,z) = a^2\omega \Bigg\{ & e_{10} + A_0 \frac{z}{a} + \sum_{n=1}^{\infty} B_n \frac{\cos\left(\beta_n \frac{z-b}{a}\right)}{\cos\left(\frac{\beta_n b}{a}\right)} I_0\left(\frac{\beta_n r}{a}\right) \\ & + \sum_{n=1}^{\infty} A_n \left[a_n \sinh\left(\frac{\alpha_n z}{a}\right) + \cosh\left(\frac{\alpha_n z}{a}\right)\right] J_0\left(\frac{\alpha_n r}{a}\right) \Bigg\} \cos\omega t\end{aligned} \tag{45}$$

The constant parameters e_{10}, β_n, and a_n are obtained in terms of ω from the boundary condition, Eq. (3), as follows:

$$\left.\begin{aligned} e_{10} &= \frac{A_0}{a}\left(\frac{g}{\omega^2} - H\right) \\ g\,\frac{\beta_n}{a}\tan\left(\beta_n\,\frac{H-b}{a}\right) + \omega^2 &= 0 \\ A_n &= \frac{\omega^2 - g\,\dfrac{\alpha_n}{a}\tanh\left(\dfrac{\alpha_n h}{a}\right)}{g\,\dfrac{\alpha_n}{a} - \omega^2\tanh\left(\dfrac{\alpha_n h}{a}\right)} \end{aligned}\right\} \tag{46}$$

The remaining unknown values, A_0, A_n and B_n of Eq. (45), will be determined later in the eigenvalue problem.

2. Liquid L_2

For a simple harmonic motion, the velocity potential, Φ_2, may be taken in the form

$$\Phi_2 = a^2\omega\,\bar{\phi}_2(s,\theta)\cos\omega t \tag{47}$$

where $\bar{\phi}_2(s,\theta)$ is the velocity potential for steady flow.

By separation of variables, $\bar{\phi}_2(s,\theta)$ is obtained in the form

$$\bar{\phi}_2(\bar{s},\theta) = C_0 + \frac{F_0}{\bar{s}} + \sum_{n=1}^{\infty}\left[C_n\bar{s} + F_n\bar{s}^{-(n+1)}\right]\left[P_n(\mu) + G_nQ_n(\mu)\right] \tag{48}$$

where $\mu = \cos\theta$.

In order to avoid the singularity at $s = 0$ so that the velocity potential $\bar{\phi}_2$ will remain finite, the coefficients F_0, F_n and G_n are assumed zero. In this case, Eq. (48) becomes

$$\bar{\phi}_2(s,\theta) = C_0 + \sum_{n=0}^{\infty} C_n \bar{s}^n P_n(\mu) \tag{49}$$

The remaining unknown values, C_0 and C_n will be determined later in the eigenvalue problem.

3. Circular Cylindrical Shell Partially Filled with Liquid L_1

Let the nondimensionalized displacement components $\bar{u}_{1i}$ and $\bar{w}_{1i}$ ($i = 1$ and 2), pressure $\bar{p}_1$ and axial coordinate $\bar{z}$ be defined by the relations:

$$u_{1i} = \frac{\rho_1 a^4 \omega^2}{D_1} \bar{u}_{1i} \sin\omega t, \qquad i = 1, 2, - - - \tag{50}$$

$$w_{1i} = \frac{\rho_1 a^{4\omega 2}}{D_1} \bar{w}_{1i} \sin\omega t, \qquad i = 1, 2, - - - \tag{51}$$

$$p_1 = \rho_1 a^2 \omega^2 \bar{p}_1 \sin\omega t, \tag{52}$$

$$z = a \bar{z} \tag{53}$$

$$t = \frac{1}{\omega} \bar{t} \tag{54}$$

Substituting these expressions into Eqs. (11) and (12) gives, respectively,

$$\frac{d^2\bar{u}_{11}}{d\bar{z}^2} + \nu \frac{d\bar{w}_{11}}{d\bar{z}} = - \kappa_1 \omega^2 \bar{u}_{11} \tag{55}$$

$$\nu \frac{d\bar{u}_{11}}{d\bar{z}} + \bar{w}_{11} = \kappa_1 \omega^2 \bar{w}_{11} + \bar{p}_{11} \tag{56}$$

where

$$\kappa_1 = \frac{\rho_1 h_1 a^2}{D_1} \tag{57}$$

Let the nondimensionalized liquid density $\bar{\rho}_{f1}$, velocity potentials $\bar{\phi}_1$ and $\bar{\Phi}_1$ be defined by the relations:

$$\rho_f = \rho_1 \bar{\rho}_{f1} \tag{58}$$

$$\phi_1 = a^2 \omega \bar{\phi}_1 \tag{59}$$

$$\bar{\Phi}_1 = \bar{\phi}_1 \cos\omega t \tag{60}$$

Substituting these expressions into Eq. (13) gives

$$p_1 = \rho_1 a^2 \omega^2 \, \bar{\rho}_{f1} \bar{\phi}_1 \, \sin\omega t \tag{61}$$

From Eqs. (52) and (61),

$$\bar{p}_1 = \bar{\rho}_{f1} \, \bar{\phi}_1 \tag{62}$$

Substituting Eqs. (50), (51) and (53) into Eqs. (14) and (15) gives

$$\frac{d\bar{u}_{11}}{d\bar{z}} = \frac{d\bar{u}_{12}}{d\bar{z}} \triangleq \bar{u}_{1c} \qquad \text{at } \bar{z} = \frac{b}{a} \tag{63}$$

and

$$\frac{d\bar{u}_{11}}{d\bar{z}} + \nu \; \bar{w}_{11} = 0 \qquad \text{at } \bar{z} = \frac{H}{\bar{a}} \tag{64}$$

where $\bar{u}_{1c}$ is a constant to be determined later.

From Eq. (56) we find

$$\bar{w}_{11} = - \frac{\nu}{1 - \kappa_1 \omega^2} \frac{d\bar{u}_{11}}{d\bar{z}} + \frac{1}{1 - \kappa_1 \omega^2} \bar{p}_1 \tag{65}$$

Substituting Eq. (65) into Eq. (55) gives

$$\frac{d^2\bar{u}_{11}}{d\bar{z}^2} + g_{11} \bar{u}_{11} = - g_{12} \frac{d\bar{p}_1}{d\bar{z}} \tag{66}$$

where

$$\left.\begin{aligned} g_{11} &= \frac{\kappa_1 \omega^2 \left(1 - \kappa_1 \omega^2\right)}{1 - \nu^2 - \kappa_1 \omega^2} \\ g_{12} &= \frac{\nu}{1 - \nu^2 - \kappa_1 \omega^2} \end{aligned}\right\} \tag{67}$$

The homogeneous solution of Eq. (66) is

$$\bar{u}_{11H} = C_{u1} \cos\left(\sqrt{g_{11}}\; \bar{z}\right) + C_{u2} \sin\left(\sqrt{g_{11}}\; \bar{z}\right) \tag{68}$$

where C_{u1} and C_{u2} are arbitrary constants.

The particular solution of Eq. (66) is assumed as

$$\bar{u}_{11P} = K_{11}(\bar{z}) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) + K_{12}(\bar{z}) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \tag{69}$$

where

$$K_{11}(\bar{z}) = - \int^{\bar{z}} \frac{f_1(\bar{z})\ U_{12}(\bar{z})}{W_1\left[U_{11}(\bar{z}),\ U_{12}(\bar{z})\right]}\ d\bar{z}$$

$$K_{12}(\bar{z}) = \int^{\bar{z}} \frac{f_1(\bar{z})\ U_{11}(\bar{z})}{W_1\left[U_{11}(\bar{z}),\ U_{12}(\bar{z})\right]}\ d\bar{z}\ .$$

$$f_1(\bar{z}) = - g_{12} \frac{d\ \bar{p}_1}{d\bar{z}}$$

$$U_{11}(\bar{z}) = \cos\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$U_{12}(\bar{z}) = \sin\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$W_1\left[U_{11}(\bar{z}),\ U_{12}(\bar{z})\right] = \begin{vmatrix} U_{11}(\bar{z}) & U_{12}(\bar{z}) \\ U_{11}'(\bar{z}) & U_{12}'(\bar{z}) \end{vmatrix} \tag{70}$$

The Wronskian, W_1, is

$$W_1\left[U_{11}(\bar{z}),\ U_{12}(\bar{z})\right] = \begin{vmatrix} \cos\left(\sqrt{g_{11}}\ \bar{z}\right) & \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \\ -\sqrt{g_{11}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right) & \sqrt{g_{11}} \cos\left(\sqrt{g_{11}}\ \bar{z}\right) \end{vmatrix}$$

$$= \sqrt{g_{11}}\left[\cos^2\left(\sqrt{g_{11}}\ \bar{z}\right) + \sin^2\left(\sqrt{g_{11}}\ \bar{z}\right)\right] = \sqrt{g_{11}} \tag{71}$$

Thus, from Eqs. (70), we obtain

$$\left.\begin{aligned} K_{11}(\bar{z}) &= \frac{g_{12}}{\sqrt{g_{11}}}\ \bar{p}_1(z)\ \sin\left(\sqrt{g_{11}}\ \bar{z}\right) - g_{12} \int^{\bar{z}} \bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\ \bar{z}\right)d\bar{z} \\ K_{12}(\bar{z}) &= -\frac{g_{12}}{\sqrt{g_{11}}}\ \bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\ \bar{z}\right) - g_{12} \int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\ \bar{z}\right)d\bar{z} \end{aligned}\right\} \quad (72)$$

Substituting Eqs. (72) into Eq. (69) and adding the result to Eq. (68), gives the general solution of Eq. (66) as

$$\begin{aligned} \bar{u}_{11} = {} & C_{u1}\ \cos\left(\sqrt{g_{11}}\ \bar{z}\right) + C_{u2}\ \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \\ & - g_{12}\Bigg[\cos\left(\sqrt{g_{11}}\ \bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\ \cos\left(\sqrt{g_{11}}\ \bar{z}\right)d\bar{z} \\ & + \sin\left(\sqrt{g_{11}}\ \bar{z}\right)\ \int^{\bar{z}} \bar{p}_1(\bar{z})\ \sin\left(\sqrt{g_{11}}\ \bar{z}\right)d\bar{z}\Bigg] \end{aligned} \quad (73)$$

Differentiating Eq. (73) with respect to $\bar{z}$ gives

$$\frac{d\bar{u}_{11}}{d\bar{z}} = - C_{u1}\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right) + C_{u2}\sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$+ g_{12}\sqrt{g_{11}}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right.$$

$$\left. - \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right] - g_{12}\,\bar{p}_1(\bar{z}) \tag{74}$$

Substituting Eq. (74) into Eq. (65) gives

$$\bar{w}_{11} = \frac{\nu\, C_{u1}}{1 - \kappa_1\omega^2}\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right) - \frac{\nu\, C_{u2}}{1 - \kappa_1\omega^2}\sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$- \frac{\nu\, g_{12}\sqrt{g_{11}}}{1 - \kappa_1\omega^2}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right.$$

$$\left. - \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right]$$

$$+ \frac{\nu\, g_{12}}{1 - \kappa_1\omega^2}\,\bar{p}_1(\bar{z}) + \frac{1}{1 - \kappa_1\omega^2}\,\bar{p}_1(\bar{z}) \tag{75}$$

From Eqs. (74) and (75),

$$\frac{d\bar{u}_{11}}{d\bar{z}} + \nu\,\bar{w}_{11} = -\left(1 - \frac{\nu^2}{1 - \kappa_1\omega^2}\right) C_{u1}\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$+\left(1 - \frac{\nu^2}{1 - \kappa_1\omega^2}\right) C_{u2}\sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$+\left(1 - \frac{\nu^2}{1 - \kappa_1\omega^2}\right) g_{12}\sqrt{g_{11}}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right.$$

$$\left. - \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right]$$

$$-\left(1 - \frac{\nu^2}{1 - \kappa_1\omega^2}\right) g_{12}\,\bar{p}_1(z) + \frac{\nu}{1 - \kappa_1\omega^2}\,\bar{p}_1(\bar{z}) \tag{76}$$

By the boundary condition, Eq. (64),

$$- \left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right)\sqrt{g_{11}}\,\sin\left(\frac{\sqrt{g_{11}}\,H}{a}\right)C_{u1} + \left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right)\sqrt{g_{11}}\,\cos\left(\frac{\sqrt{g_{11}}\,H}{a}\right)C_{u2}$$

$$= -\left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12}\sqrt{g_{11}}\left[\sin\left(\frac{\sqrt{g_{11}}\,H}{a}\right)\left|\int^{\bar{z}} \bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=\frac{H}{a}}\right.$$

$$\left. - \cos\left(\frac{\sqrt{g_{11}}\,H}{a}\right)\left|\int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=\frac{H}{a}}\right]$$

$$+\left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12}\,\bar{p}_1\left(\frac{H}{a}\right) - \frac{\nu}{1-\kappa_1\omega^2}\,\bar{p}_1\left(\frac{H}{a}\right) \tag{77}$$

By the boundary condition, Eq. (63),

$$- \sqrt{g_{11}}\,\sin\left(\frac{\sqrt{g_{11}}\,b}{a}\right)C_{u1} + \sqrt{g_{11}}\,\cos\left(\frac{\sqrt{g_{11}}\,b}{a}\right)C_{u2}$$

$$= - g_{12}\sqrt{g_{11}}\left[\sin\left(\frac{\sqrt{g_{11}}\,b}{a}\right)\left|\int^{\bar{z}} \bar{p}_1(\bar{z})\cos\sqrt{g_{11}}\,\bar{z}\,d\bar{z}\right|_{\bar{z}=\frac{b}{a}}\right.$$

$$\left. - \cos\left(\frac{\sqrt{g_{11}}\,b}{a}\right)\left|\int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=\frac{b}{a}}\right]$$

$$+ g_{12}\,\bar{p}_1\left(\frac{b}{a}\right) + \bar{u}_{1c} \tag{78}$$

$$
\left.
\begin{aligned}
\text{Let}\quad & -\left(1-\frac{\nu^2}{1-\kappa_1\omega^2}\right)\sqrt{g_{11}}\,\sin\left(\frac{\sqrt{g_{11}}\,H}{a}\right) = \ell_{11} \\
& \left(1-\frac{\nu^2}{1-\kappa_1\omega^2}\right)\sqrt{g_{11}}\,\cos\left(\frac{\sqrt{g_{11}}\,H}{a}\right) = \ell_{12} \\
& -\left(1-\frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12}\sqrt{g_{11}}\left[\sin\left(\frac{\sqrt{g_{11}}\,H}{a}\right)\left|\int^{\bar z}\bar p_1(\bar z)\cos\left(\sqrt{g_{11}}\,\bar z\right)d\bar z\right|_{\bar z=\frac{H}{a}}\right. \\
& \left. -\cos\left(\frac{\sqrt{g_{11}}\,H}{a}\right)\left|\int^{\bar z}\bar p_1(\bar z)\sin\left(\sqrt{g_{11}}\,\bar z\right)d\bar z\right|_{\bar z=\frac{H}{a}}\right] \\
& +\left(1-\frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12}\,\bar p_1\left(\frac{H}{a}\right) - \frac{\nu}{1-\kappa_1\omega^2}\,\bar p_1\left(\frac{H}{a}\right) = h_{11} \\
& -\sqrt{g_{11}}\,\sin\left(\frac{\sqrt{g_{11}}\,b}{a}\right) = \ell_{13} \\
& \sqrt{g_{11}}\,\cos\left(\frac{\sqrt{g_{11}}\,b}{a}\right) = \ell_{14} \\
& -g_{12}\sqrt{g_{11}}\left[\sin\left(\frac{\sqrt{g_{11}}\,b}{a}\right)\left|\int^{\bar z}\bar p_1(\bar z)\cos\left(\sqrt{g_{11}}\,\bar z\right)d\bar z\right|_{\bar z=\frac{b}{a}}\right. \\
& \left. -\cos\left(\frac{\sqrt{g_{11}}\,b}{a}\right)\left|\int^{\bar z}\bar p_1(\bar z)\sin\left(\sqrt{g_{11}}\,\bar z\right)d\bar z\right|_{\bar z=\frac{b}{a}}\right] \\
& +g_{12}\,\bar p_1\left(\frac{b}{a}\right) = h_{12}
\end{aligned}
\right\} \quad (79)
$$

Eqs. (77) and (78) are then written as

$$\left.\begin{aligned} \ell_{11}C_{u1} + \ell_{12}C_{u2} &= h_{11} \\ \ell_{13}C_{u1} + \ell_{14}C_{u2} &= h_{12} + \bar{u}_{1c} \end{aligned}\right\} \tag{80}$$

from which

$$\left.\begin{aligned} C_{u1} &= \frac{\ell_{14}h_{11} - \ell_{12}h_{12} - \ell_{12}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}} \\ C_{u2} &= \frac{\ell_{11}h_{12} - \ell_{13}h_{11} + \ell_{11}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}} \end{aligned}\right\} \tag{81}$$

Substituting Eqs. (81) into Eqs. (73) through (75) gives

$$\begin{aligned} \bar{u}_{11} = {} & \frac{\ell_{14}h_{11} - \ell_{12}h_{12} - \ell_{12}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}} \cos\left(\sqrt{g_{11}}\,\bar{z}\right) \\ & + \frac{\ell_{11}h_{12} - \ell_{13}h_{11} + \ell_{11}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}} \sin\left(\sqrt{g_{11}}\,\bar{z}\right) \\ & - g_{12}\left[\cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right. \\ & \left. + \sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right] \end{aligned} \tag{82}$$

$$\frac{d\,\bar{u}_{11}}{d\bar{z}} = -\frac{\ell_{14}h_{11} - \ell_{12}h_{12} - \ell_{12}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$+\frac{\ell_{11}h_{12} - \ell_{13}h_{11} + \ell_{11}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}\sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$+ g_{12}\sqrt{g_{11}}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}}\bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right.$$

$$\left.- \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}}\bar{p}_1(\bar{z})\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right] - g_{12}\,\bar{p}_1(\bar{z}) \tag{83}$$

$$\bar{w}_{11} = \frac{\nu\sqrt{g_{11}}}{1-\kappa_1\omega^2}\,\frac{\ell_{14}h_{11} - \ell_{12}h_{12} - \ell_{12}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$-\frac{\nu\sqrt{g_{11}}}{1-\kappa_1\omega^2}\,\frac{\ell_{11}h_{12} - \ell_{13}h_{11} + \ell_{11}\bar{u}_{1c}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$-\frac{\nu\, g_{12}\sqrt{g_{11}}}{1-\kappa_1\omega^2}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}}\bar{p}_1(\bar{z})\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right.$$

$$\left.- \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}}\bar{p}_1(\bar{z})\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right]$$

$$+\frac{\nu\, g_{12}}{1-\kappa_1\omega^2}\,\bar{p}_1(\bar{z}) + \frac{1}{1-\kappa_1\omega^2}\,\bar{p}_1(\bar{z}) \tag{84}$$

4. Circular Conical Shell

Let the nondimensionalized displacement components $\bar{u}_2$ and $\bar{w}_2$, pressure $\bar{P}_{2\alpha}$ and generatrix $\bar{s}$ be denoted by the relations:

$$u_2 = \frac{\rho_2 a^4 \omega^2}{D_2} \bar{u}_2 \sin\omega t \tag{85}$$

$$w_2 = \frac{\rho_2 a^4 \omega^2}{D_2} \bar{w}_2 \sin\omega t \tag{86}$$

$$p_{2\alpha} = \rho_2 a^2 \omega^2 \bar{p}_{2\alpha} \sin\omega t \tag{87}$$

$$s = a\,\bar{s} \tag{88}$$

where $P_{2\alpha}$ is the pressure acting at the conical shell surface. Substituting these expressions into Eqs. (27) and (28) gives, respectively,

$$\bar{s}\frac{d^2\bar{u}_2}{d\bar{s}^2} + \frac{d\bar{u}_2}{d\bar{s}} - \frac{\bar{u}_2}{\bar{s}} + \nu\frac{d\bar{w}_2}{d\bar{s}}\cot\alpha - \frac{\bar{w}_2}{\bar{s}}\cot\alpha = -\kappa_2\omega^2\,\bar{s}\,\bar{u}_2 \tag{89}$$

$$\nu\frac{d\bar{u}_2}{d\bar{s}} + \frac{\bar{u}_2}{\bar{s}} + \frac{\bar{w}_2}{\bar{s}}\cot\alpha = \kappa_2\omega^2\,\bar{s}\,\bar{w}_2\tan\alpha - \bar{s}\,\bar{p}_{2\alpha}\tan\alpha \tag{90}$$

where

$$\kappa_2 = \frac{\rho_2 h_2 a^2}{D_2}$$

Let the nondimensionalized liquid density $\bar{\rho}_{f2}$ and velocity potentials $\bar{\phi}_2$ and $\bar{\Phi}_2$ be defined by the relations:

$$\rho_f = \rho_2 \bar{\rho}_{f2} \tag{91}$$

$$\phi_2 = a^2\omega \bar{\phi}_2 \tag{92}$$

$$\Phi_2 = \bar{\phi}_2 \cos\omega t \tag{93}$$

Substituting these expressions into Eq. (26) gives

$$p_2 = \rho_2 a^2 \omega^2 \bar{\rho}_{f2} \bar{\phi}_2 \sin\omega t \tag{94}$$

From Eqs. (87) and (94)

$$\bar{p}_2 = \bar{\rho}_{f2} \bar{\phi}_2 \tag{95}$$

Substituting Eqs. (85) and (86) into boundary conditions (29) and (30) gives

$$\bar{u}_2 = \text{finite} \qquad \text{at } \bar{s} = 0 \tag{96}$$

and

$$\bar{u}_2 - \bar{w}_2 \tan\alpha = 0 \qquad \text{at } \bar{s} = \frac{\ell}{a} \tag{97}$$

where

$$\ell = \sqrt{a^2 + b^2}.$$

It will be indicated below how the two Equations (89) and (90) involving two dependent variables, $\bar{u}_2$ and $\bar{w}_2$, can be reduced to a single differential equation of $\bar{u}_2$.

First, from Eq. (90)

$$\bar{w}_2 = - \frac{\nu\, \bar{s}\, \dfrac{d\bar{u}_2}{d\bar{s}} + \bar{u}_2 + \bar{s}^2\, \bar{p}_{2\alpha} \tan\alpha}{\cot\alpha - \kappa_2 \omega^2 \bar{s}^2 \tan\alpha} \tag{98}$$

Second, substitute Eq. (98) into Eq. (89),

$$\begin{aligned}
&\left[1 - \nu^2 - \left(2-\nu^2\right)\kappa_2\omega^2\bar{s}^2\tan^2\alpha + \kappa_2^{\,2}\omega^4\bar{s}^4\tan^4\alpha \right] \bar{s}^2 \frac{d^2\bar{u}_2}{d\bar{s}^2} \\
&+ \left[1 - \nu^2 - \left(2+\nu^2\right)\kappa_2\omega^2\bar{s}^2\tan^2\alpha + \kappa_2^{\,2}\omega^4\bar{s}^4\tan^4\alpha \right] \bar{s} \frac{d\bar{u}_2}{d\bar{s}} \\
&+ \Big[\left(1-2\nu + \cot^2\alpha\right)\kappa_2\omega^2\bar{s}^2\tan^2\alpha - \left(2 + \tan^2\alpha\right)\kappa_2^{\,2}\omega^4\bar{s}^4\tan^2\alpha \\
&+ \kappa_2^{\,3}\omega^6\bar{s}^6\tan^4\alpha \Big] \bar{u}_2 \\
&= \left(- \bar{s}^2 + \kappa_2\omega^2\bar{s}^4\tan^2\alpha + 2\nu\bar{s}^2\right)\bar{p}_{2\alpha}\tan\alpha \\
&\qquad + \left(1 - \kappa_2\omega^2\bar{s}^2\tan^2\alpha\right)\nu\, \bar{s}^3 \frac{d\bar{p}_{2\alpha}}{d\bar{s}} \tan\alpha
\end{aligned} \tag{99}$$

Simplifying Eq. (99) for the case $\kappa_2\omega^2\bar{s}^2\tan^2\alpha \ll 1$ gives

$$\frac{d^2\bar{u}_2}{d\bar{s}^2} + \frac{1}{\bar{s}}\frac{d\bar{u}_2}{d\bar{s}} - \lambda^2\bar{u}_2 = \bar{f}_2(\bar{s}) \tag{100}$$

where

$$\left.\begin{aligned} \lambda^2 &= \frac{1 - 2\nu + \cot^2\alpha}{1 - \nu^2}\,\kappa_2\omega^2\tan^2\alpha \\ \bar{f}_2(\bar{s}) &= -\frac{1 - 2\nu}{1 - \nu^2}\,\bar{p}_{2\alpha}\tan\alpha + \frac{\nu}{1 - \nu^2}\,\bar{s}\,\frac{d\bar{p}_{2\alpha}}{d\bar{s}}\tan\alpha \end{aligned}\right\} \tag{101}$$

Multiplying both sides of Eq. (100) by $\bar{s}J_0(\mu_i\bar{s})$, and integrating from 0 to $\bar{\ell}$ gives

$$\int_0^{\bar{\ell}} \bar{s}\left(\frac{d^2\bar{u}_2}{d\bar{s}^2} + \frac{1}{\bar{s}}\frac{d\bar{u}_2}{d\bar{s}}\right)J_0(\mu_i\bar{s})\,d\bar{s} + \lambda^2\bar{U}_2 = \bar{F}_2 \tag{102}$$

where $\bar{U}_2$ and $\bar{F}_2$ are the finite Hankel transforms of $\bar{u}_2$ and $\bar{f}_2$, respectively, and defined as

$$\bar{U}_2(\mu_i) = \int_0^{\bar{\ell}} \bar{s}\,\bar{u}_2(\bar{s})\,J_0(\mu_i\bar{s})\,d\bar{s} \tag{103}$$

$$\bar{F}_2(\mu_i) = \int_0^{\bar{\ell}} \bar{s}\,f_2(\bar{s})\,J_0(\mu_i\bar{s})\,d\bar{s} \tag{104}$$

where

$$\bar{\ell} = \sqrt{1 + \left(\frac{b}{a}\right)^2} \tag{105}$$

The first term on the left-hand side of Eq. (102) is written as

$$\int_0^{\bar{\ell}} \bar{s}\left(\frac{d^2\bar{u}_2}{d\bar{s}^2} + \frac{1}{\bar{s}}\frac{d\bar{u}_2}{d\bar{s}}\right) J_0\left(\mu_i\bar{s}\right) d\bar{s}$$

$$= \int_0^{\bar{\ell}} \frac{d}{d\bar{s}}\left(\bar{s}\frac{d\bar{u}_2}{d\bar{s}}\right) J_0\left(\mu_i\bar{s}\right) d\bar{s}$$

$$= J_0\left(\mu_i\bar{s}\right)\bar{s}\frac{d\bar{u}_2}{d\bar{s}}\Bigg]_0^{\bar{\ell}} + \mu_i\int_0^{\bar{\ell}} \bar{s}\, J_1\left(\mu_i\bar{s}\right)\frac{d\bar{u}_2}{d\bar{s}}\, d\bar{s} \tag{106}$$

Recognizing that $d\bar{u}_2/d\bar{s}$ must be finite from physical considerations, the first term on the right-hand side of Eq. (106) becomes

$$J_0\left(\mu_i\bar{s}\right)\bar{s}\frac{d\bar{u}_2}{d\bar{s}}\Bigg|_0^{\bar{\ell}} = J_0\left(\mu_i\bar{\ell}\right)\bar{\ell}\frac{d\bar{u}_2}{d\bar{s}}\Bigg]_{\bar{\ell}} \tag{107}$$

The second term on the right-hand side of Eq. (106) can be again integrated by parts as

$$\mu_i\int_0^{\bar{\ell}} \bar{s}\, J_1\left(\mu_i\bar{s}\right)\frac{d\bar{u}_2}{d\bar{s}}\, d\bar{s}$$

$$= \mu_i\bar{u}_2\,\bar{s}\, J_1\left(\mu_i\bar{s}\right)\Bigg|_0^{\bar{\ell}} - \mu_i^2\int_0^{\bar{\ell}} \bar{s}\, J_0\left(\mu_i\bar{s}\right)\bar{u}_2\, d\bar{s} \tag{108}$$

From the boundary condition, Eq. (29), the first term on the right-hand side of Eq. (108) becomes

$$\mu_i\,\bar{u}_2\,\bar{s}\, J_1\left(\mu_i\bar{s}\right)\Bigg|_0^{\bar{\ell}} = \mu_i\, J_1\left(\mu_i\bar{\ell}\right)\bar{\ell}\,\bar{u}_2\Bigg]_{\bar{\ell}} \tag{109}$$

Substituting Eqs. (107) through (109) into Eq. (106) and using the definition, Eq. (103), gives

$$\int_0^{\bar{\ell}} \bar{s}\left(\frac{d^2\bar{u}_2}{d\bar{s}^2} + \frac{1}{\bar{s}}\frac{d\bar{u}_2}{d\bar{s}}\right) J_0\left(\mu_i \bar{s}\right) d\bar{s}$$

$$= J_0\left(\mu_i \bar{\ell}\right)\bar{\ell}\left.\frac{d\bar{u}_2}{d\bar{s}}\right]_{\bar{\ell}} + \mu_i J_1\left(\mu_i \bar{\ell}\right)\bar{\ell}\left.\bar{u}_2\right]_{\bar{\ell}} - \mu_i^2 \bar{U}_2 \tag{110}$$

By choosing the μ_i so that

$$J_0\left(\mu_i \bar{\ell}\right)\bar{\ell}\left.\frac{d\bar{u}_2}{d\bar{s}}\right]_{\bar{\ell}} + \mu_i J_1\left(\mu_i \bar{\ell}\right)\bar{\ell}\left.\bar{u}_2\right]_{\bar{\ell}} = 0 \tag{111}$$

Eq. (110) may be written as

$$\int_0^{\bar{\ell}} \bar{s}\left(\frac{d^2\bar{u}_2}{d\bar{s}^2} + \frac{1}{\bar{s}}\frac{d\bar{u}_2}{d\bar{s}}\right) J_0\left(\mu_i \bar{s}\right) d\bar{s} = -\mu_i^2 \bar{U}_2 \tag{112}$$

Substituting this expression into Eq. (102) gives

$$\bar{U}_2 = \frac{1}{\lambda^2 - \mu_i^2}\bar{F}_2 \tag{113}$$

By applying an inversion formula of finite Hankel transform to Eq. (113),

$$\bar{u}_2 = \frac{2}{\bar{\ell}^2}\sum_i \frac{\mu_i^2 \bar{F}_2}{\left(\sigma^2 + \mu_i^2\right)\left(\lambda^2 - \mu_i^2\right)} \bullet \frac{J_0\left(\mu_i \bar{s}\right)}{\left[J_0\left(\mu_i \bar{\ell}\right)\right]^2} \tag{114}$$

where

$$\sigma = - \frac{\left.\dfrac{d\bar{u}_2}{d\bar{s}}\right]_{\bar{\ell}}}{\left.\bar{u}_2\right]_{\bar{\ell}}} \tag{115}$$

and the sum is taken over all the positive roots of Eq. (111).

The numerical value of σ can be determined by an iteration scheme. An estimated value of σ is first chosen. The numerical value of $\left[d\bar{u}_2/d\bar{s}\right]\Big/\bar{u}_2$ at $\bar{s} = \bar{\ell}$ is evaluated next. This process is repeated for successive values of σ until Eq. (115) is satisfied. An approximate value of σ is, however, determined by assuming that the resultant of the hydrostatic pressure acting on the conical shell surface in its tangential direction is negligibly small compared with the resultant of the dynamic pressure. For this case, the stress resultant, N_s of Eq. (22) can be approximately put equal to zero at $\bar{s} = \bar{\ell}$, i.e.,

$$\left.\frac{d\bar{u}_2}{d\bar{s}}\right]_{\bar{\ell}} + \frac{\nu}{\bar{\ell}}\left(\left.u_2\right]_{\bar{\ell}} + \left.\bar{w}_2\right]_{\bar{\ell}} \cot\alpha\right) = 0 \tag{116}$$

Substituting the boundary condition, Eq. (97) into the above equation gives

$$\left.\frac{d\bar{u}_2}{d\bar{s}}\right]_{\bar{\ell}} + \frac{\nu}{\bar{\ell}} \left.\bar{u}_2\right]_{\bar{\ell}} \operatorname{cosec}^2\alpha = 0 \tag{117}$$

from which

$$\frac{\left.\dfrac{d\bar{u}_2}{d\bar{s}}\right]_{\bar{\ell}}}{\left.\bar{u}_2\right]_{\bar{\ell}}} = - \frac{\nu}{\bar{\ell}} \operatorname{cosec}^2\alpha \tag{118}$$

Substituting this expression into Eq. (115) gives

$$\sigma = \frac{\nu}{\bar{\ell}} \operatorname{cosec}^2\alpha \tag{119}$$

Substituting Eq. (114) into Eq. (98) gives

$$\bar{w}_2 = \frac{2\tan\alpha}{\bar{\ell}^2} \sum_i \frac{\mu_i^{\,2}\bar{F}_2}{\left(\sigma^2 + \mu_i^{\,2}\right)\left(\lambda^2 - \mu_i^{\,2}\right)} \cdot \frac{\nu\bar{s}\mu_i\, J_1\left(\mu_i\bar{s}\right) - J_0\left(\mu_i\bar{s}\right)}{\left[J_0\left(\mu_i\,\bar{\ell}\right)\right]^2}$$

$$- \bar{s}^2\, \bar{p}_{2\alpha}\tan^2\alpha \tag{120}$$

where

$$\left.\begin{aligned} \bar{F}_2 &= \int_0^{\bar{\ell}} \bar{s}\; \bar{f}_2(\bar{s})\; J_0\left(\mu_i\bar{s}\right)ds \\ \bar{p}_{2\alpha} &= \bar{\rho}_{f2}\left[C_0 + \sum_{n=1}^{\infty} C_n\, \bar{s}^n\, P_n(\mu)\right] \end{aligned}\right\} \tag{121}$$

Substituting the expression, $\bar{p}_{2\alpha}$ of Eqs. (121) into the second equation of Eqs. (101) gives

$$\begin{aligned} \bar{f}_2(\bar{s}) = &- \frac{1 - 2\nu}{1 - \nu^2}\,\bar{\rho}_{f2}\, C_0\tan\alpha \\ &- \frac{1 - 2\nu}{1 - \nu^2}\,\bar{\rho}_{f2} \sum_{n=1}^{\infty} C_n\, \bar{s}^n\, P_n(\mu)\tan\alpha \\ &+ \frac{\nu}{1 - \nu^2}\,\bar{\rho}_{f2} \sum_{n=1}^{\infty} n\, C_n\, \bar{s}^n\, P_n(\mu)\tan\alpha \end{aligned} \tag{122}$$

5. Circular Cylindrical Shell Filled with Liquid L_2

Referring to Eqs. (73) and (75), the general solutions of two simultaneous equations, Eqs. (31) and (32) are obtained as

$$\bar{u}_{12} = C_{u3} \cos\left(\sqrt{g_{11}}\,\bar{z}\right) + C_{u4} \sin\left(\sqrt{g_{11}}\,\bar{z}\right) - g_{12}\left[\cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_2(\bar{z})\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z} + \sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_2(\bar{z}) \sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right] \tag{123}$$

$$\bar{w}_{12} = \frac{\nu C_{u3}}{1-\kappa_1\omega^2}\sqrt{g_{11}}\sin\left(\sqrt{g_{11}}\,\bar{z}\right) - \frac{\nu C_{u4}}{1-\kappa_1\omega^2}\sqrt{g_{11}}\cos\left(\sqrt{g_{11}}\,\bar{z}\right) - \frac{\nu g_{12}\sqrt{g_{11}}}{1-\kappa_1\omega^2}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_2(\bar{z}) \cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z} - \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_2(\bar{z}) \sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right] + \frac{\nu g_{12}}{1-\kappa_1\omega^2}\bar{p}_2(\bar{z}) + \frac{1}{1-\kappa_1\omega^2}\bar{p}_2(\bar{z}) \tag{124}$$

where $\bar{p}_2(\bar{z})$ is obtained from Eqs. (49) and (95) and the coordinate transformation as

$$\left.\begin{aligned} \bar{p}_2(z) &= \bar{\rho}_{f2}\left\{ C_0 + \sum_{n=1}^{\infty} C_n\left[1 + (\bar{b}-\bar{z})^2\right]^{\frac{n}{2}} P_n\left(\mu_{\bar{z}}\right)\right\} \\ \mu_{\bar{z}} &= \frac{\bar{b} - \bar{z}}{\sqrt{1 + (\bar{b} - \bar{z})^2}} \end{aligned}\right\} \tag{125}$$

The boundary conditions are

$$\frac{d\bar{u}_{12}}{d\bar{z}} = \bar{u}_{1c} \qquad \text{at} \qquad \bar{z} = \frac{b}{a} \tag{126}$$

$$\bar{u}_{12} = 0 \qquad \text{at} \qquad \bar{z} = 0 \tag{127}$$

From Eq. (123),

$$\begin{aligned} \frac{d\bar{u}_{12}}{d\bar{z}} = &- C_{u3}\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right) + C_{u4}\sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right) \\ &+ g_{12}\sqrt{g_{11}}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}}\bar{p}_2(\bar{z})\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right. \\ &\left. - \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}}\bar{p}_2(\bar{z})\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right] - g_{12}\bar{p}_2(\bar{z}) \end{aligned} \tag{128}$$

By the boundary condition, Eq. (126)

$$- C_{u3}\sqrt{g_{11}}\,\sin\left(\frac{b\sqrt{g_{11}}}{a}\right) + C_{u4}\sqrt{g_{11}}\cos\left(\frac{b\sqrt{g_{11}}}{a}\right)$$

$$+ g_{12}\sqrt{g_{11}}\left[\sin\left(\frac{b\sqrt{g_{11}}}{a}\right)\left|\int^{\bar{z}} \bar{p}_2(\bar{z})\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=\frac{b}{a}}\right.$$

$$\left. - \cos\left(\frac{b\sqrt{g_{11}}}{a}\right)\left|\int^{\bar{z}} \bar{p}_2(\bar{z})\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=\frac{b}{a}}\right]$$

$$- g_{12}\,\bar{p}_2\left(\frac{b}{a}\right) = \bar{u}_{1c} \tag{129}$$

By the boundary condition, Eq. (127)

$$C_{u3} = g_{12}\left|\int^{\bar{z}} \bar{p}_2(\bar{z})\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=0} \tag{130}$$

Let

$$
\left.
\begin{aligned}
& -\sqrt{g_{11}}\,\sin\left(\frac{b\sqrt{g_{11}}}{a}\right) = \ell_{23} \\
& \sqrt{g_{11}}\,\cos\left(\frac{b\sqrt{g_{11}}}{a}\right) = \ell_{24} \\
& g_{12}\left|\int^{\bar{z}} \bar{p}_2(\bar{z})\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=0} = h_{21} \\
& -g_{12}\sqrt{g_{11}}\left[\sin\left(\frac{b\sqrt{g_{11}}}{a}\right)\left|\int^{\bar{z}} \bar{p}_2(\bar{z})\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=\frac{b}{a}}\right. \\
& \left. -\cos\left(\frac{b\sqrt{g_{11}}}{a}\right)\left|\int^{\bar{z}} \bar{p}_2(\bar{z})\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right|_{\bar{z}=\frac{b}{a}}\right] + g_{12}\bar{p}_2\left(\frac{b}{a}\right) = h_{22}
\end{aligned}
\right\} \quad (131)
$$

From Eqs. (129) through (131),

$$
\left.
\begin{aligned}
C_{u3} &= h_{21} \\
C_{u4} &= \frac{h_{22} - \ell_{23}h_{21} + \bar{u}_{1c}}{\ell_{24}}
\end{aligned}
\right\} \quad (132)
$$

Substituting Eqs. (132) into Eqs. (123), (124), and (127) gives, respectively,

$$\bar{u}_{12} = h_{21} \cos\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{h_{22} - \ell_{23}h_{21} + \bar{u}_{1c}}{\ell_{24}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$- g_{12}\left[\cos\left(\sqrt{g_{11}}\ \bar{z}\right) \int^{\bar{z}} \bar{p}_2(\bar{z}) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z}\right.$$

$$\left. + \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \int^{\bar{z}} \bar{p}_2(\bar{z}) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z}\right] \tag{133}$$

$$\bar{w}_{12} = \frac{\nu}{1 - \kappa_1\omega^2} h_{21} \sqrt{g_{11}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$- \frac{\nu}{1 - \kappa_1\omega^2} \frac{h_{22} - \ell_{23}h_{21} + \bar{u}_{1c}}{\ell_{24}} \sqrt{g_{11}} \cos\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$- \frac{\nu\, g_{12}\sqrt{g_{11}}}{1 - \kappa_1\omega^2}\left[\sin\left(\sqrt{g_{11}}\ \bar{z}\right) \int^{\bar{z}} \bar{p}_2(\bar{z}) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z}\right.$$

$$\left. - \cos\left(\sqrt{g_{11}}\ \bar{z}\right) \int^{\bar{z}} \bar{p}_2(\bar{z}) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z}\right]$$

$$+ \frac{\nu\, g_{12}}{1 - \kappa_1\omega^2} \bar{p}_2(\bar{z}) + \frac{1}{1 - \kappa_1\omega^2} \bar{p}_2(\bar{z}) \tag{134}$$

$$\frac{d\bar{u}_2}{d\bar{z}} = - h_{21}\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right) + \frac{h_{22} - \ell_{23}h_{21} + \bar{u}_{1c}}{\ell_{24}}\sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$+ g_{12}\sqrt{g_{11}}\left[\sin\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_2(\bar{z})\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right.$$

$$\left. - \cos\left(\sqrt{g_{11}}\,\bar{z}\right)\int^{\bar{z}} \bar{p}_2(\bar{z})\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z}\right] - g_{12}\bar{p}_2(\bar{z}) \qquad (135)$$

From the solutions Eqs. (82) and (133) and the first boundary condition of Eqs. (14),

$$\bar{u}_{1c}\cot\sqrt{g_{11}}\,(\bar{H}-\bar{b}) - \bar{u}_{1c}\tan\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$= \frac{h_{11}}{\left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right)\sin\sqrt{g_{11}}\,(\bar{H}-\bar{b})} - h_{12}\cot\sqrt{g_{11}}\,(\bar{H}-\bar{b}) + \frac{\sqrt{g_{11}}\,h_{21}}{\cos\left(\sqrt{g_{11}}\,\bar{b}\right)}$$

$$+ h_{22}\tan\left(\sqrt{g_{11}}\,\bar{b}\right) + g_{12}\sqrt{g_{11}}\left\{\left[f_{p11}(\bar{b}) - f_{p21}(\bar{b})\right]\cos\left(\sqrt{g_{11}}\,\bar{b}\right)\right.$$

$$\left. + \left[f_{p12}(\bar{b}) - f_{p22}(\bar{b})\right]\sin\left(\sqrt{g_{11}}\,\bar{b}\right)\right\} \qquad (136)$$

where f_{p11}, f_{p12}, f_{p21}, and f_{p22} are defined, respectively, as

$$\left.\begin{aligned}
f_{p11}(\bar{b}) &= f_{p11}(\bar{z})\Big]_{\bar{z}=\bar{b}} = \int^{\bar{z}} \bar{p}_1(\bar{z}) \cos\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z}\,\Big]_{\bar{z}=\bar{b}} \\
f_{p12}(\bar{b}) &= f_{p12}(\bar{z})\Big]_{\bar{z}=\bar{b}} = \int^{\bar{z}} \bar{p}_1(\bar{z}) \sin\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z}\,\Big]_{\bar{z}=\bar{b}} \\
f_{p21}(\bar{b}) &= f_{p21}(\bar{z})\Big]_{\bar{z}=\bar{b}} = \int^{\bar{z}} \bar{p}_2(\bar{z}) \cos\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z}\,\Big]_{\bar{z}=\bar{b}} \\
f_{p22}(\bar{b}) &= f_{p22}(\bar{z})\Big]_{\bar{z}=\bar{b}} = \int^{\bar{z}} \bar{p}_2(\bar{z}) \sin\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z}\,\Big]_{\bar{z}=\bar{b}}
\end{aligned}\right\} \quad (137)$$

From which

$$\begin{aligned}
\bar{u}_{1c} = {} & \frac{\sin\sqrt{g_{11}}\,(\bar{H}-\bar{b})\,\cos\left(\sqrt{g_{11}}\,\bar{b}\right)}{\cos\left(\sqrt{g_{11}}\,\bar{H}\right)} \\
& \times \Bigg[\frac{h_{11}}{\left(1 - \dfrac{\nu^2}{1-\kappa_1\omega^2}\right)\sin\sqrt{g_{11}}\,(H-b)} - h_{12}\cot\sqrt{g_{11}}(\bar{H}-\bar{b}) \\
& + \frac{\sqrt{g_{11}}\,h_{21}}{\cos\left(\sqrt{g_{11}}\,\bar{b}\right)} + h_{22}\tan\left(\sqrt{g_{11}}\,\bar{b}\right) \\
& + g_{12}\sqrt{g_{11}}\,\Big\{ \left[f_{p11}(\bar{b}) - f_{p21}(\bar{b}) \right] \cos\left(\sqrt{g_{11}}\,\bar{b}\right) \\
& + \left[f_{p12}(\bar{b}) - f_{p22}(\bar{b}) \right] \sin\left(\sqrt{g_{11}}\,\bar{b}\right) \Big\} \Bigg]
\end{aligned} \quad (138)$$

IV. MATRIX EIGENVALUE PROBLEM

The eigenvalue problem for determining the natural frequencies and corresponding mode shapes of the coupled liquid-shell system is formulated using five equations which were obtained as a result of imposing the necessary boundary conditions. These five equations are Eqs. (4), (5), (6), (8) and (9), Other than the trivial solution in which all the unknown constants are zero, it may be seen that the series in these five equations cannot be satisfied term by term. Hence, some approximate numerical method of solution must be used. The approximate method used herein to formulate the eigenvalue problem is based on the least squared error technique. Although the series representations for the velocity potentials, Eqs. (44) and (49), would represent an exact solution if an infinite number of terms were used, as a practical matter, the series must be truncated. Since the chosen velocity potentials satisfy the differential equations and non-interface boundary conditions exactly term by term, it is the five interface conditions that suffer. These conditions can be satisfied only approximately. The functional errors can then be defined as:

$$\bar{\varepsilon}_1 = \left.\frac{\partial \bar{\phi}_1}{\partial \bar{r}}\right]_{\bar{r}=1} - \beta_{K1}\bar{w}_{11} \tag{139}$$

$$\bar{\varepsilon}_2 = \left.\frac{\partial \bar{\phi}_1}{\partial \bar{z}}\right]_{\bar{z}=\frac{b}{a}} - \left.\frac{1}{\bar{s}}\frac{\partial \bar{\phi}_2}{\partial \theta}\right]_{\theta=\frac{\pi}{2}} \tag{140}$$

$$\bar{\varepsilon}_3 = \left.\bar{\phi}_1\right]_{\bar{z}=\frac{b}{a}} - \left.\bar{\phi}_2\right]_{\theta=\frac{\pi}{2}} \tag{141}$$

$$\bar{\varepsilon}_4 = \left.\frac{1}{\bar{s}}\frac{\partial \bar{\phi}_2}{\partial \theta}\right]_{\theta=\alpha} - \beta_{K2}\bar{w}_2 \tag{142}$$

$$\bar{\varepsilon}_5 = \left[\frac{1}{\sqrt{1 + \left(\frac{b}{a} - \bar{z}\right)^2}} \frac{\partial \bar{\phi}_2}{\partial \bar{s}} + \frac{\frac{b}{a} - \bar{z}}{1 + \left(\frac{b}{a} - \bar{z}\right)^2} \frac{\partial \bar{\phi}_2}{\partial \theta} \right]_{\substack{\bar{s} = \sqrt{1 + \left(\frac{b}{a} - \bar{z}\right)^2} \\ \theta = \tan^{-1} \frac{1}{\frac{b}{a} - \bar{z}}}} - \beta_{K1} \bar{w}_{12} \tag{143}$$

where $\bar{\varepsilon}_k$ (k = 1, 2, 3, 4, and 5) is defined by the relation:

$$\left.\begin{aligned} \varepsilon_k &= a\,\omega\,\bar{\varepsilon}_k \cos\omega t \qquad (k=1,2,4,5) \\ \varepsilon_3 &= -\,a^2\omega^2\,\bar{\varepsilon}_3 \sin\omega t \end{aligned}\right\} \tag{144}$$

and

$$\left.\begin{aligned} \beta_{K1} &= \frac{\rho_1 a^3 \omega^2}{D_1} \\ \beta_{K2} &= \frac{\rho_2 a^3 \omega^2}{D_2} \end{aligned}\right\} \tag{145}$$

With these expressions, the total integrated squared error, S_T, over the boundaries involved can be expressed as

$$S_T = S_1 + S_2 + S_3 + S_4 + S_5 \tag{146}$$

where

where

$$\left.\begin{aligned}
S_1 &= 2\pi a^2 \int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1^{\,2}\left(A_n,\ B_n,\ C_{\hat{n}},\ C_{\bar{n}},\ \bar{z}\right) d\bar{z} \\
S_2 &= 2\pi a^2 \int_0^1 \bar{\varepsilon}_2^{\,2}\left(A_n,\ C_{\bar{n}},\ \bar{r}\right) \bar{r}\, d\bar{r} \\
S_3 &= 2\pi a^2 \int_0^1 \bar{\varepsilon}_3^{\,2}\left(A_n,\ B_n,\ C_{\hat{n}},\ \bar{r}\right) \bar{r}\, d\bar{r} \\
S_4 &= 2\pi a^2 \int_0^{\bar{\ell}} \bar{\varepsilon}_4^{\,2}\left(C_{\hat{n}},\ C_{\bar{n}},\ \bar{s}\right) \bar{s}\, \sin\alpha\, d\bar{s} \\
S_5 &= 2\pi a^2 \int_0^{\bar{b}} \bar{\varepsilon}_5^{\,2}\left(A_n,\ B_n,\ C_{\hat{n}},\ C_{\bar{n}},\ \bar{z}\right) d\bar{z}
\end{aligned}\right\} \quad \begin{aligned} n &= 0,1,2,---,N \\ \hat{n} &= 2n \\ \bar{n} &= 2n + 1 \end{aligned} \tag{147}$$

but $B_0 = 0$, and $\bar{H} = H/a$, $\bar{\ell} = \ell/a$, and $\bar{b} = b/a$.

The frequency, ω and constants, A_n, B_n, $C_{\hat{n}}$, and $C_{\bar{n}}$ are then determined by minimizing the total integrated squared error S_T. The conditions for this minimum are

$$\left.\begin{aligned}
\frac{\partial S_T}{\partial A_m} &= 0 \\
\frac{\partial S_T}{\partial B_m} &= 0 \\
\frac{\partial S_T}{\partial C_{\hat{m}}} &= 0 \\
\frac{\partial S_T}{\partial C_{\bar{m}}} &= 0
\end{aligned}\right\} \quad \begin{aligned} m &= 0,1,2,---,N \\ \hat{m} &= 2m \\ \bar{m} &= 2m + 1 \end{aligned} \tag{148}$$

Substituting Eq. (146) into Eqs. (148) gives

$$\left.\begin{aligned}
\frac{\partial S_1}{\partial A_m} + \frac{\partial S_2}{\partial A_m} + \frac{\partial S_3}{\partial A_m} + \frac{\partial S_5}{\partial A_m} &= 0 \\
\frac{\partial S_1}{\partial B_m} + \frac{\partial S_3}{\partial B_m} + \frac{\partial S_5}{\partial B_m} &= 0 \\
\frac{\partial S_1}{\partial C_{\hat{m}}} + \frac{\partial S_3}{\partial C_{\hat{m}}} + \frac{\partial S_4}{\partial C_{\hat{m}}} + \frac{\partial S_5}{\partial C_{\hat{m}}} &= 0 \\
\frac{\partial S_1}{\partial C_{\bar{m}}} + \frac{\partial S_2}{\partial C_{\bar{m}}} + \frac{\partial S_4}{\partial C_{\bar{m}}} + \frac{\partial S_5}{\partial C_{\bar{m}}} &= 0
\end{aligned}\right\} \tag{149}$$

Substituting Eqs. (147) into Eqs. (149) gives

$$\left.\begin{aligned}
&\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial \bar{\varepsilon}_1}{\partial A_m} d\bar{z} + \int_0^1 \bar{\varepsilon}_2 \frac{\partial \bar{\varepsilon}_2}{\partial A_m} \bar{r} d\bar{r} + \int_0^1 \bar{\varepsilon}_3 \frac{\partial \bar{\varepsilon}_3}{\partial A_m} \bar{r} d\bar{r} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial \bar{\varepsilon}_5}{\partial A_m} d\bar{z} = 0 \\
&\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial \bar{\varepsilon}_1}{\partial B_m} d\bar{z} + \int_0^1 \bar{\varepsilon}_3 \frac{\partial \bar{\varepsilon}_3}{\partial B_m} \bar{r} d\bar{r} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial \bar{\varepsilon}_5}{\partial B_m} d\bar{z} = 0 \\
&\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial \bar{\varepsilon}_1}{\partial C_{\hat{m}}} d\bar{z} + \int_0^1 \bar{\varepsilon}_3 \frac{\partial \bar{\varepsilon}_3}{\partial C_{\hat{m}}} \bar{r} d\bar{r} + \int_0^{\bar{\ell}} \bar{\varepsilon}_4 \frac{\partial \bar{\varepsilon}_4}{\partial C_{\hat{m}}} \bar{s} \sin\alpha \, d\bar{s} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial \bar{\varepsilon}_5}{\partial C_{\hat{m}}} d\bar{z} = 0 \\
&\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial \bar{\varepsilon}_1}{\partial C_{\bar{m}}} d\bar{z} + \int_0^1 \bar{\varepsilon}_2 \frac{\partial \bar{\varepsilon}_2}{\partial C_{\bar{m}}} \bar{r} d\bar{r} + \int_0^{\bar{\ell}} \bar{\varepsilon}_4 \frac{\partial \bar{\varepsilon}_4}{\partial C_{\bar{m}}} \bar{s} \sin\alpha \, d\bar{s} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial \bar{\varepsilon}_5}{\partial C_{\bar{m}}} d\bar{z} = 0
\end{aligned}\right\} \tag{150}$$

Eqs. (150) are a set of homogeneous algebraic equations for the constants A_n, B_n, $C_{\hat{n}}$ and $C_{\bar{n}}$ with ω as a parameter. These equations can be put into the form of a matrix eigenvalue problem.

The substitution of solutions for the error functions, Eqs. (139) through (143), together with solutions for the velocity potentials, Eqs. (45) and (49), and the displacement components, Eqs. (82), (120) and (134), into Eqs. (150) yields the following matrix form:

$$\begin{bmatrix} I_{11mn} & I_{12mn} & I_{13m\hat{n}} & I_{13m\bar{n}} \\ I_{21mn} & I_{22mn} & I_{23m\hat{n}} & I_{23m\bar{n}} \\ I_{31\hat{m}n} & I_{32\hat{m}n} & I_{33\hat{m}\hat{n}} & I_{33\hat{m}\bar{n}} \\ I_{31\bar{m}n} & I_{32\bar{m}n} & I_{33\bar{m}\hat{n}} & I_{33\bar{m}\bar{n}} \end{bmatrix} \begin{Bmatrix} A_m \\ B_m \\ C_{\hat{m}} \\ C_{\bar{m}} \end{Bmatrix} = 0 \qquad (151)$$

where I_{ijmn} $(i,j = 1,2,3)$ is a function of ω.

This linear algebraic system of $4 \times (N+1)$ homogeneous simultaneous equations has a non-trivial solution only if the determinant is zero.

The matrix eigenvalue problem formulated above can be solved only with the aid of a digital computer. The method of solution consists of searching for values of ω that make the determinant of a set of simultaneous equations, Eqs. (150) vanish. An estimated value of the frequency ω is first chosen. Numerical values of the coefficients, A_n B_n, C_n, and $C_{\bar{n}}$ are evaluated next. This process is repeated for successive successive values of ω until a zero value of the determinant is found to a desired degree of accuracy.

V. PROCEDURE FOR NUMERICAL CALCULATIONS

The eigenvalue problem for determining the natural frequencies, ω, and the corresponding mode shapes, A_n, B_n, $C_{\hat{n}}$, and $C_{\bar{n}}$ has been formulated as a linear algebraic system of $4 \times (N+1)$ homogeneous simultaneous equations. Such $4 \times (N+1)$ algebraic equations with respect to $4 \times (N+1)$ unknowns, A_n, B_n, $C_{\hat{n}}$ and $C_{\bar{n}}$ written from Eqs. (151) as)

$$\left.\begin{aligned}
&\sum_{n=0}^{N} \left(I_{11mn}A_n + I_{12mn}B_n + I_{13m\hat{n}}C_{\hat{n}} + I_{13mn}C_{\bar{n}}\right) = 0 \\
&\sum_{n=0}^{N} \left(I_{21mn}A_n + I_{22mn}B_n + I_{23m\hat{n}}C_{\hat{n}} + I_{23m\bar{n}}C_{\bar{n}}\right) = 0 \\
&\sum_{n=0}^{N} \left(I_{31\hat{m}n}A_n + I_{32\hat{m}n}B_n + I_{33\hat{m}\hat{n}}C_{\hat{n}} + I_{33\hat{m}\bar{n}}C_{\bar{n}}\right) = 0 \\
&\sum_{n=0}^{N} \left(I_{31\bar{m}n}A_n + I_{32\bar{m}n}B_n + I_{33\bar{m}\hat{n}}C_{\hat{n}} + I_{33\bar{m}\bar{n}}C_{\bar{n}}\right) = 0
\end{aligned}\right\} \tag{152}$$

$$\hat{m} = 2m, \ \bar{m} = 2m + 1; \ m = 0, 1, 2, \cdots, N$$

where I_{ijmn} (i and j = 1,2,3; m and n = 0,1,2,---,N) is a function of ω.

Obviously, $A_n = B_n = C_{\hat{n}} = C_{\bar{n}} = 0$ for $n = 0,1,2, \cdots, N$ is a solution of Eqs. (1). There exist, however, other solutions of the determinant of Eqs. (152) is zero. Thus, from the condition that the determinant of Eqs. (152) is zero, the frequencies ω are found. Hence.

$$\begin{vmatrix} I_{11mn} & I_{12mn} & I_{13m\hat{n}} & I_{13m\bar{n}} \\ I_{21mn} & I_{22mn} & I_{23m\hat{n}} & I_{23m\bar{n}} \\ I_{31\hat{m}n} & I_{32\hat{m}n} & I_{33\hat{m}\hat{n}} & I_{33\hat{m}\bar{n}} \\ I_{31\bar{m}n} & I_{32\bar{m}n} & I_{33\bar{m}\hat{n}} & I_{33\bar{m}\bar{n}} \end{vmatrix} = 0 \qquad (153)$$

Let $A_0 = k$, and solve any $4 \times (N+1) - 1$ of Eqs. (152) for the unknowns A_n $(n = 1, 2, ---, N)$, B_n, $C_{\hat{n}}$ and $C_{\bar{n}}$. For an example, solve the following $4 \times (N+1) - 1$ simultaneous equations in $4 \times (N+1) - 1$ unknowns:

$$\left.\begin{aligned} &\left.\begin{aligned} &\sum_{n=1}^{N} I_{11mn}\bar{A}_n + \sum_{n=0}^{N}\left(I_{12mn}\bar{B}_n + I_{13m\hat{n}}\bar{C}_{\hat{n}} + I_{13m\bar{n}}\bar{C}_{\bar{n}}\right) = -I_{11m0} \\ &\sum_{n=1}^{N} I_{21mn}\bar{A}_n + \sum_{n=0}^{N}\left(I_{22mn}\bar{B}_n + I_{23m\hat{n}}\bar{C}_{\hat{n}} + I_{23m\bar{n}}\bar{C}_{\bar{n}}\right) = -I_{21m0} \\ &\sum_{n=1}^{N} I_{31\hat{m}n}\bar{A}_n + \sum_{n=0}^{N}\left(I_{32\hat{m}n}\bar{B}_n + I_{33\hat{m}\hat{n}}\bar{C}_{\hat{n}} + I_{33\hat{m}\bar{n}}\bar{C}_{\bar{n}}\right) = -I_{31\hat{m}0} \end{aligned}\right] \quad (m = 0,1,2,---N) \\ &\left.\sum_{n=1}^{N} I_{31\bar{m}n}\bar{A}_n + \sum_{n=0}^{N}\left(I_{32\bar{m}n}\bar{B}_n + I_{33\bar{m}\hat{n}}\bar{C}_{\hat{n}} + I_{33\bar{m}\bar{n}}\bar{C}_{\bar{n}}\right) = -I_{31\bar{m}0}\right] \quad (m = 1,2,---N) \end{aligned}\right\} \qquad (154)$$

where $\bar{A}_n = A_n/k$, $\bar{B}_n = B_n/k$, $\bar{C}_{\hat{n}} = C_{\hat{n}}/k$ and $\bar{C}_{\bar{n}} = C_{\bar{n}}/k$.

Eqs. (3) can be solved for $\bar{A}_n$, $\bar{B}_n$, $\bar{C}_{\hat{n}}$ and $\bar{C}_{\bar{n}}$ by the aid of a digital computer. Known coefficients, I_{ijmn} of Eqs. (3) are calculated in Appendices A and B.

CONCLUDING REMARKS AND RECOMMENDATIONS

The solutions given by Equations (152) have been programmed and computed for the first three terms ($n = 0, 1, 2$) of the velocity potentials. This computer program has solved for the natural frequencies and the corresponding mode shapes. The results of the numerical computation are shown in Figures 2 through 4.

In the numerical calculation of the natural frequencies of the system shown in Figure 1, some difficulties have been encountered with the computer program to find zeros of the eigenvalue determinant accurately. With the present method of analysis, however, the zeros of the eigenvalue determinant are not necessarily points where the determinant changes its sign but are the minimum points where the determinant approaches zero. Thus, refining the computer program to improve the search for the minimum values of ω, three natural frequencies: $\omega = 14$, 25, and 39 rad/sec, have been found. From the solutions for the free vibration of a complete circular ring (Reference 8) and those of a circular cylinder with rigid bottom (Reference 9), the natural frequencies of the present system ($a/h_1 = 2000$) can be roughly estimated. Their numerical results show that the natural frequencies for the first three modes range from 13 to 45 rad/sec., approximately. Thus, the present method of analysis yields fairly good results.

In the numerical calculation of the mode shapes of the circular cylindrical shell, it has been found that the normal displacement does not satisfy a continuity condition at the boundary where a cylinder of liquid and an inverted cone of liquid match. This unsatisfactory result is obviously due to the fact that there is no boundary condition which can be applied to the normal displacement because with bending stiffness neglected the differential equation of motion, (Equation (12), is of zero order in the normal displa ement. To satisfy this continuity condition, a constraint, $\left|w_{11} - w_{12}\right|_{\bar{z}=\bar{b}} = 0$, has been imposed on the matrix eigenvalue equation, Equation (151), by introducing a Lagrange multiplier λ_1. On the upper edge of the circular cylindrical shell, $z = H$, one more constraint, $w_{11}\big|_{\bar{z}=\bar{H}} = 0$, The corresponding Lagrange multiplier is denoted by λ_2. With these two constraints, the matrix eigenvalue equation, Equation (151) has been replaced for the present numerical calculation by

$$
\left|\begin{array}{cccc|cc}
 & & I_{ijmn} & & q_{i1bn} & q_{11Hn} \\
 & & & & q_{i2bn} & q_{12Hn} \\
 & & & & q_{i3b\hat{n}} & q_{13H\hat{n}} \\
 & & & & q_{i3b\bar{n}} & q_{13H\bar{n}} \\
\hline
q_{i1bn} & q_{i2bn} & q_{i3b\hat{n}} & q_{i3b\bar{n}} & 0 & 0 \\
\hline
q_{11Hn} & q_{12Hn} & q_{13H\hat{n}} & q_{13H\bar{n}} & 0 & 0
\end{array}\right|
\left\{\begin{array}{c}
A_n \\ B_n \\ C_{\hat{n}} \\ C_{\bar{n}} \\ \hline \lambda_1 \\ \hline \lambda_2
\end{array}\right\} = 0
$$

where

$$
\left\{\begin{array}{l}
q_{i1bn} = q_{11bn} - q_{31bn} \\
q_{i2bn} = q_{12bn} - q_{32bn} \\
q_{i3b\hat{n}} = q_{13b\hat{n}} - q_{33b\hat{n}} \\
q_{i3b\bar{n}} = q_{13b\bar{n}} - q_{33b\bar{n}}
\end{array}\right.
$$

The continuity condition for the normal displacement at $\bar{z}=\bar{b}$ are then well satisfied as shown in Figures 2 through 4. Computing the unknown constants A_n, B_n, C_n and C_n and two Lagrange multipliers λ_1 and λ_2 for the natural frequencies ω = 14, 25, and 39, the three corresponding mode shapes have been found and shown in Figures 2 through 4.

The numerical results show that the present method of analysis yields reasonable results for both the frequencies and the mode shapes. An advantage of the present method is that the solutions do not necessarily have to satisfy the boundary conditions exactly but they need to satisfy the functional errors derived at the boundaries approximately. By introducing such functional errors, the present method can thus be developed to a problem with the more complicated boundary conditions. A tank partially filled with a liquid is known to be a very difficult problem because of the complicated nature of its boundary conditions. As the next problem, therefore, it is recommended that the longitudinal oscillation of an elastic cylindrical tank with a flexible conical bulkhead, partially filled with liquid, be investigated by the method of analysis presented here.

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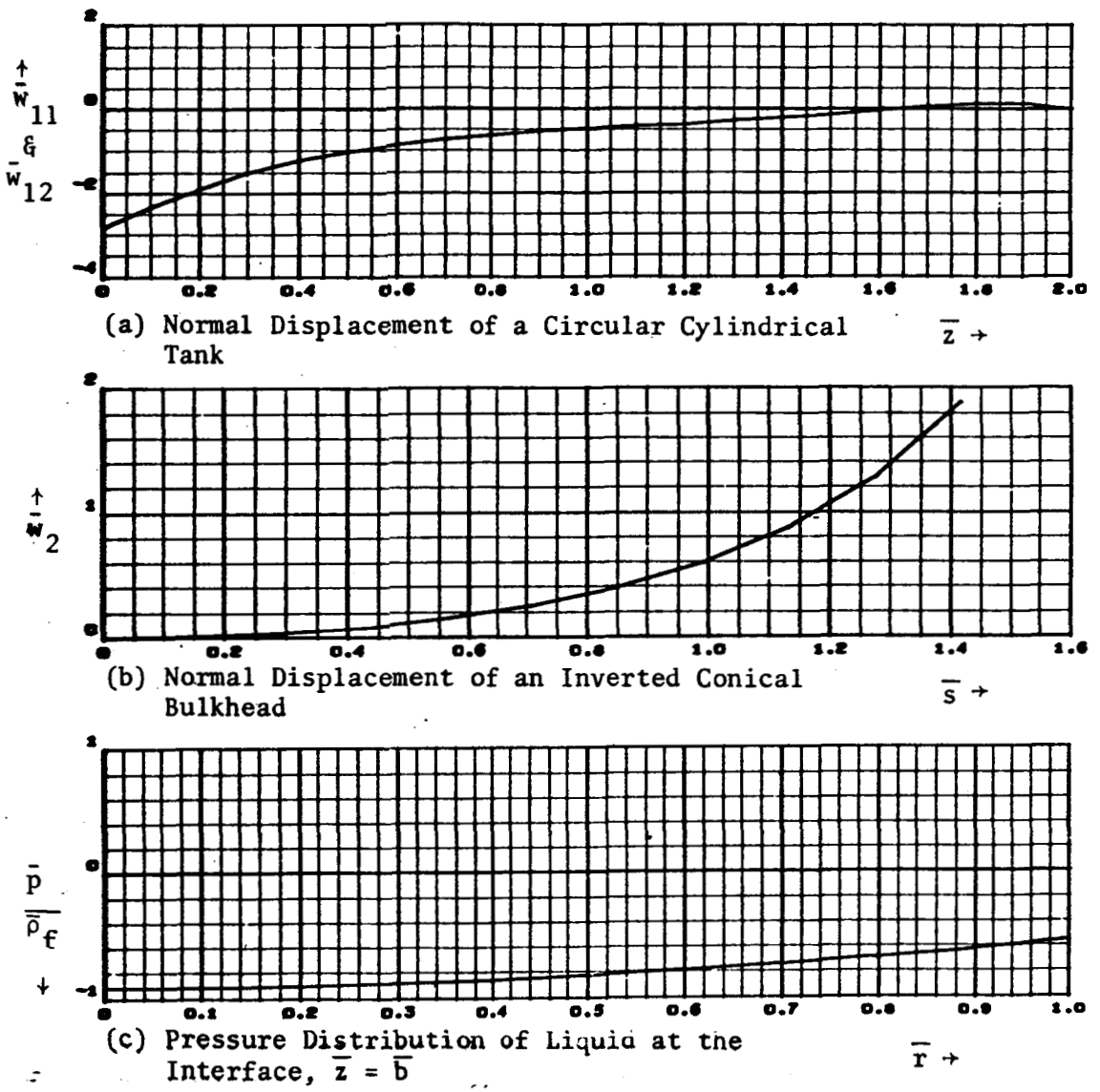
TABLE 1

Physical Characteristics of A Liquid-Filled Circular Cylindrical Tank with An Inverted Conical Bulkhead

Constants	Numerical Values	Units
a	200	In.
b	200	In.
g	32.2×12	In/sec^2
$h_i (i = 1,2)$	0.1	In.
ℓ	141.4×2	In.
D	1.099×10^6	$L_{bf}/In.$
E	10^7	$L_{bf}/In.^2$
H	400	In.
ν	0.3	
$\rho_i (i = 1,2)$	2.59×10^{-4}	$L_{bf}Sec^2/In.^4$
$\rho_{fi} (i = 1,2)$	1.06×10^{-4}	$L_{bf}Sec^2/In.^4$
α	$\pi/4$	

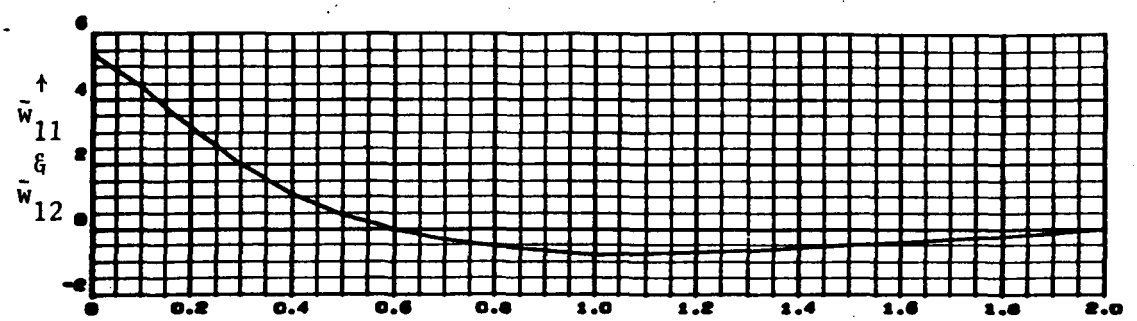


(a) Normal Displacement of a Circular Cylindrical Tank

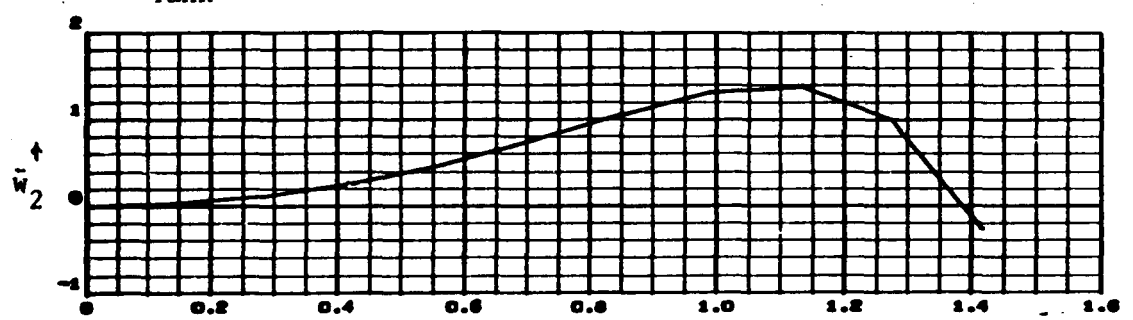
(b) Normal Displacement of an Inverted Conical Bulkhead

(c) Pressure Distribution of Liquid at the Interface, $\bar{z} = \bar{b}$

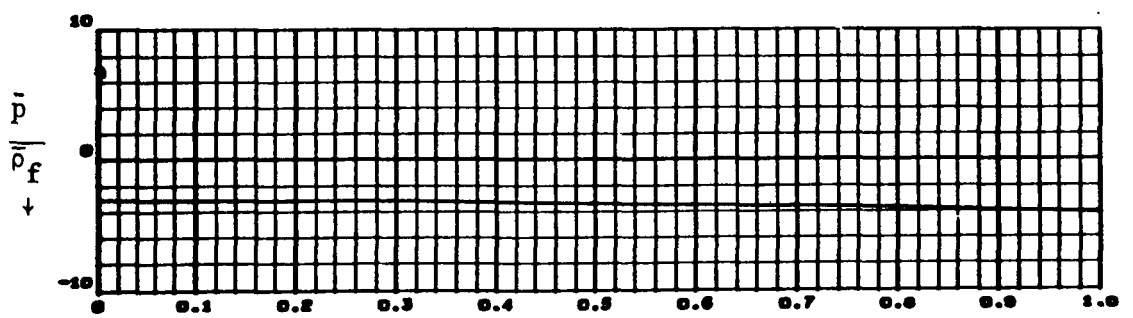
Figure 2. Axisymmetric Mode Shapes for a Liquid-Filled Circular Cylindrical Tank with an Inverted Conical Bulkhead, f = 2.23 cps



(a) Normal Displacement of a Circular Cylindrical Tank

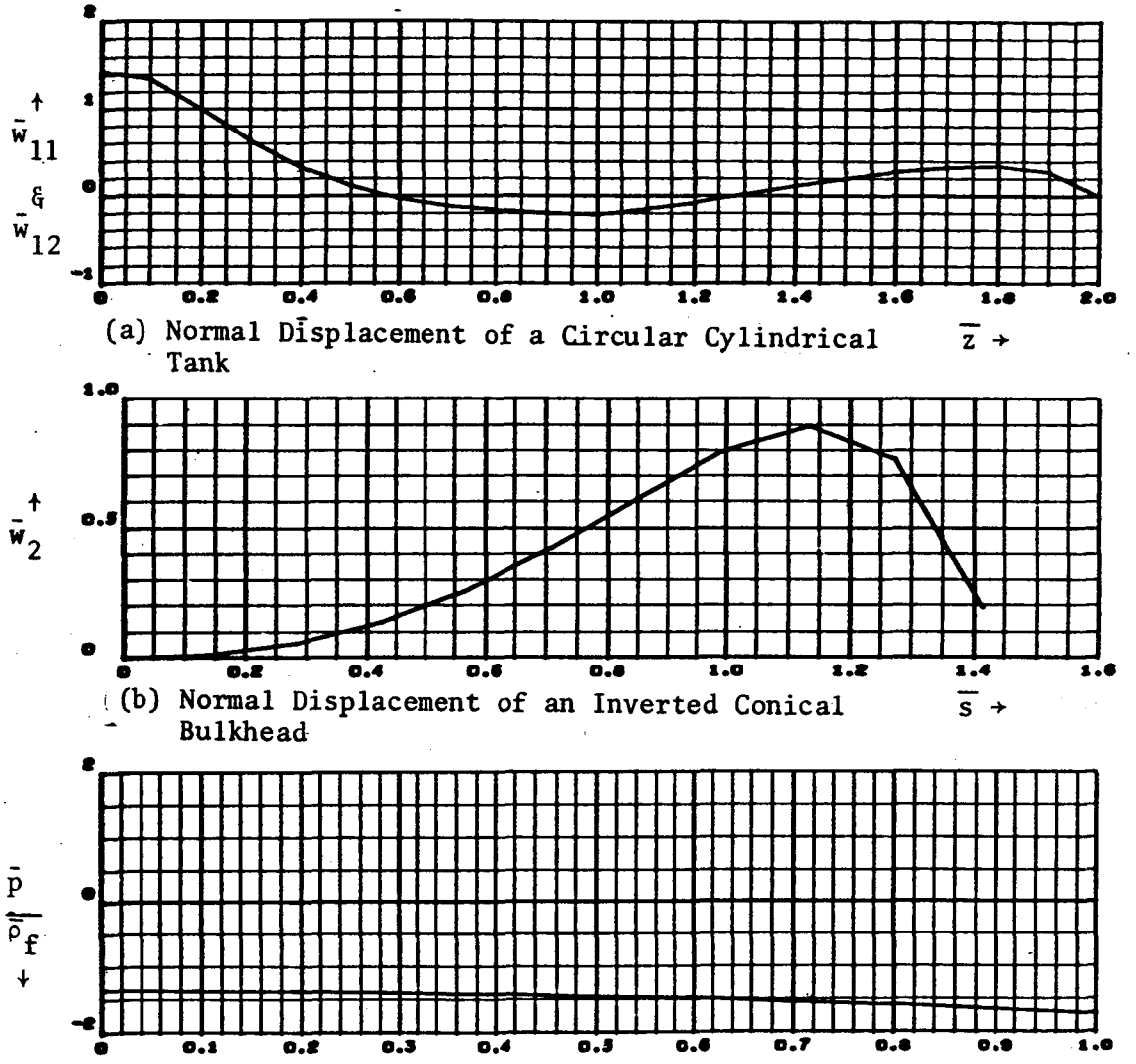


(b) Normal Displacement of an Inverted Conical Bulkhead



(c) Pressure Distribution of Liquid at the Interface, $\bar{z} = \bar{b}$

Figure 3. Axisymmetric Mode Shapes for a Liquid-Filled Circular Cylindrical Tank with an Inverted Conical Bulkhead, f = 3.98 cps



(a) Normal Displacement of a Circular Cylindrical Tank

(b) Normal Displacement of an Inverted Conical Bulkhead

(c) Pressure Distribution of Liquid at the Interface, $\bar{z} = \bar{b}$

Figure 4. Axisymmetric Mode Shapes for a Liquid-Filled Circular Cylindrical Tank with an Inverted Conical Bulkhead, f = 6.21 cps

APPENDICES

A. Liquid Velocity Potentials $\bar{\phi}_1(\bar{r},\bar{z})$ and $\bar{\phi}_2(\bar{s},\mu)$ and Shell Displacement Components $\bar{w}_{11}(\bar{z})$, $\bar{w}_{12}(\bar{z})$ and $\bar{w}_2(\bar{s})$

The liquid velocity potentials, $\bar{\phi}_1(\bar{r},\bar{z})$ and $\bar{\phi}_2(\bar{s},\mu)$ and the shell displacement components, $\bar{w}_{11}(\bar{z})$, $\bar{w}_2(\bar{s})$, and $\bar{w}_{12}(\bar{z})$ have been obtained and expressed by Eqs. (44), (49), (84), (120), and (124), respectively. These are put in the following forms:

$$\left.\begin{aligned}
\bar{\phi}_1(r,z) &= \sum_{n=0}^{\infty}\left(q_{4rzn}A_n + q_{5rzn}B_n\right)\\
\bar{\phi}_2(\bar{s},\mu) &= \sum_{n=0}^{\infty}\left(q_{6s\mu\hat{n}}C_{\hat{n}} + q_{6s\mu\bar{n}}C_{\bar{n}}\right)\\
\bar{w}_{11}(\bar{z}) &= \sum_{n=0}^{\infty}\left(q_{11zn}A_n + q_{12zn}B_n + q_{13z\hat{n}}C_{\hat{n}} + q_{13z\bar{n}}C_{\bar{n}}\right)\\
\bar{w}_2(\bar{s}) &= \sum_{n=0}^{\infty}\left(q_{21s\hat{n}}C_{\hat{n}} + q_{21s\bar{n}}C_{\bar{n}}\right)\\
\bar{w}_{12}(\bar{z}) &= \sum_{n=0}^{\infty}\left(q_{31zn}A_n + q_{32zn}B_n + q_{33z\hat{n}}C_{\hat{n}} + q_{33z\bar{n}}C_{\bar{n}}\right)
\end{aligned}\right\} \qquad \text{(A-1)}$$

where q_{4rzn}, q_{5rzn}, ---, $q_{33z\bar{n}}$ are the function of independent variables, $\bar{r},\bar{z},\bar{s}$ and μ which are shown by the subscripts, r, z, s, and μ, respectively. These functions will be derived herein from Eqs. (44), (49), (84), (120), and (124) by integrating the pressure terms given in Eqs. (121) and (137).

One of the pressure terms involved in the variables, $q_{21s\hat{n}}$ and $q_{21s\bar{n}}$ is $\bar{F}_2(\mu_i)$. Thus, substituting the second equation of Eqs. (101) into the first equation of Eqs. (121) gives

$$\bar{F}_2(\mu_i) = -\frac{1-2\nu}{1-\nu^2}\,\bar{\rho}_{f2}\,C_0 \tan\alpha \int_0^{\bar{\ell}} \bar{s}\,J_0(\mu_i \bar{s})\,d\bar{s}$$

$$-\sum_{n=1}^{\infty} \frac{1-2\nu-n\nu}{1-\nu^2}\,\bar{\rho}_{f2}\,C_n P_n(\mu_\alpha) \tan\alpha \int_0^{\bar{\ell}} \bar{s}^{n+1}\,J_0(\mu_i \bar{s})\,d\bar{s} \qquad \text{(A-2)}$$

The first and second integrations of Eq. (A-2) are, respectively,

$$\left.\begin{aligned} &\int_0^{\bar{\ell}} \bar{s}\,J_0(\mu_i \bar{s})\,d\bar{s} = \frac{\bar{\ell}}{\mu_i} J_1(\mu_i \bar{\ell}) \\ \text{and}\quad &\int_0^{\bar{\ell}} \bar{s}^{n+1} J_0(\mu_i \bar{s})\,d\bar{s} = \frac{1}{\mu_i^{n+2}} \Bigg[n\mu_i \bar{\ell}\, J_0(\mu_i \bar{\ell}) \\ &+ S_{n,-1}(\mu_i \bar{\ell}) + \mu_i \bar{\ell}\, J_1(\mu_i \bar{\ell}) S_{n+1,0}(\mu_i \bar{\ell}) + 2^{n+1} \frac{\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} \Bigg] \end{aligned}\right\} \qquad \text{(A-3)}$$

$$n = 1, 2, ---, \infty$$

where $S_{n,-1}$ and $S_{n+1,0}$ are Lommel functions and expressed, respectively, as:

$$S_{n,-1}(\mu_i \bar{\ell}) = (\mu_i \bar{\ell})^{n-1} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{\mu_i \bar{\ell}}{2}\right)^{2m+2} \Gamma\left(1+\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(2+\frac{n}{2}+m\right)\Gamma\left(1+\frac{n}{2}+m\right)}$$

$$+ 2^{n-1}\,\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right) \left\{ \sin\left[\tfrac{1}{2}(n+1)\pi\right] J_{-1}(\mu_i \bar{\ell}) - \cos\left[\tfrac{1}{2}(n+1)\pi\right] N_{-1}(\mu_i \bar{\ell}) \right\}$$

$$n = 2, 4, 6, ---$$

$$S_{n,-1}\left(\mu_i\bar{\ell}\right) = \left(\mu_i\bar{\ell}\right)^{n-1}\sum_{m=0}^{P-1}\frac{(-1)^m\,\Gamma\left(-\frac{1}{2}n+m\right)\Gamma\left(1-\frac{1}{2}n+m\right)}{\left(\frac{\mu_i\bar{\ell}}{2}\right)^m\Gamma\left(-\frac{1}{2}n\right)\Gamma\left(1-\frac{1}{2}n\right)}$$

$$+\,0\left[\left(\mu_i\bar{\ell}\right)^{n-2P}\right] \quad n = 1,\ 3,\ 5,\ ---$$

$N_{-1}\left(\mu_i\bar{\ell}\right)$ [Bessel functions of the second kind]

$$= -\frac{2}{\pi}\left[\left(\ell n\,\frac{\mu_i\bar{\ell}}{2} + 0.5772157---\right)J_1\left(\mu_i\bar{\ell}\right) - \frac{1}{\mu_i\bar{\ell}} - \frac{\mu_i\bar{\ell}}{4} + \frac{1+\left(1+\frac{1}{2}\right)}{2}\,\frac{\left(\mu_i\bar{\ell}\right)^3}{2^3 2!} - ---\right]$$

$$S_{n+1,0}\left(\mu_i\bar{\ell}\right) = \left(\mu_i\bar{\ell}\right)^n\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\mu_i\bar{\ell}}{2}\right)^{2m+2}\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(2+\frac{n}{2}+m\right)\Gamma\left(2+\frac{n}{2}+m\right)}$$

$$+\,2^n\,\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(1+\frac{n}{2}\right)\left\{\sin\left[\frac{1}{2}(n+1)\pi\right]J_0\left(\mu_i\bar{\ell}\right) - \cos\left[\frac{1}{2}(n+1)\pi\right]N_0\left(\mu_i\bar{\ell}\right)\right\}$$

$$n = 2,\ 4,\ 6,\ ---$$

$$N_0\left(\mu_i\bar{\ell}\right) = \frac{2}{\pi}\left\{\left(\ell n\,\frac{\mu_i\bar{\ell}}{2} + 0.5772157---\right)J_0\left(\mu_i\bar{\ell}\right) + \left[\frac{\left(\mu_i\bar{\ell}\right)^2}{2^2} - \frac{\left(\mu_i\bar{\ell}\right)^4}{2^4(2!)^2}\left(1+\frac{1}{2}\right) + ---\right]\right\}$$

$$S_{n+1,0}\left(\mu_i\bar{\ell}\right) = \left(\mu_i\bar{\ell}\right)^n \sum_{m=0}^{p-1} \frac{(-1)^m\,\Gamma\left(-\frac{1}{2}n+m\right)\Gamma\left(-\frac{1}{2}n+m\right)}{\left(\frac{\mu_i\bar{\ell}}{2}\right)^m \Gamma\left(-\frac{1}{2}n\right)\Gamma\left(-\frac{1}{2}n\right)} + 0\left[\left(\mu_i\bar{\ell}\right)^{n+1-2p}\right]$$

$$n = 1,\ 3,\ 5,\ ---$$

Substituting Eqs. (A-3) into Eqs. (A-2) and letting

$$\left.\begin{aligned} &-\frac{1-2\nu}{1-\nu^2}\,\frac{\bar{\ell}}{\mu_i}\,J_1\left(\mu_i\bar{\ell}\right)\tan\alpha = f_{Ji0}\left(\mu_i\right), \\ &-\sum_{n=1}^{\infty}\left\{\frac{1-2\nu}{1-\nu^2}\,P_n\left(\mu_\alpha\right)\,\frac{1}{\mu_i^{\,n+2}}\tan\alpha\left[n\,\mu_i\bar{\ell}\,J_0\left(\mu_i\bar{\ell}\right)\right.\right. \\ &\left.\left.+\,S_{n,-1}\left(\mu_i\bar{\ell}\right) - \mu_i\bar{\ell}\,J_{-1}\left(\mu_i\bar{\ell}\right)S_{n+1,0}\left(\mu_i\bar{\ell}\right) + 2^{n+1}\,\frac{\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)}\right]\right\} \\ &\qquad = f_{Jin}\left(\mu_i\right), \end{aligned}\right\} \quad \text{(A-4)}$$

the pressure term $\bar{F}_2\left(\mu_i\right)$ of Eq. (A-2) may be written in the form:

$$\bar{F}_2\left(\mu_i\right) = \bar{\rho}_{f2}\sum_{n=0}^{\infty}\left[f_{Ji\hat{n}}\left(\mu_i\right)C_{\hat{n}} + f_{Ji\bar{n}}\left(\mu_i\right)C_{\bar{n}}\right] \qquad \text{(A-5)}$$

The other pressure terms are defined by Eqs. (137) and they are written as follows:

$$\left.\begin{aligned}
f_{p11}(\bar{z}) &= \int^{\bar{z}} \bar{p}_1(\bar{z}) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z} \\
f_{p12}(\bar{z}) &= \int^{\bar{z}} \bar{p}_1(\bar{z}) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z} \\
f_{p21}(\bar{z}) &= \int^{\bar{z}} \bar{p}_2(\bar{z}) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z} \\
f_{p22}(\bar{z}) &= \int^{\bar{z}} \bar{p}_2(\bar{z}) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) d\bar{z}
\end{aligned}\right\} \qquad \text{(A-6)}$$

Substituting the pressure, $\bar{p}_1(\bar{z})$ from Eq. (62), together with Eq. (44) into the first two equations of Eqs. (A-6) gives:

$$\begin{aligned}
f_{p11}(\bar{z}) = \bar{\rho}_{f1} \Bigg[& \left[\frac{\bar{e}_{10}}{\sqrt{g_{11}}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{1}{g_{11}} \cos\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{\bar{z}}{\sqrt{g_{11}}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \right] A_0 \\
& + \sum_{n=1}^{\infty} A_n \Bigg[\frac{a_n \alpha_n}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\ z\right) \\
& + \frac{a_n \sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{\alpha_n}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) \\
& + \frac{\sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \Bigg] J_0\left(\alpha_n\right) \\
& + \sum_{n=1}^{\infty} \frac{B_n I_0\left(\beta_n\right)}{\cos\left(\beta_n \bar{b}\right)} \left\{ \frac{\sin\left[\left(\beta_n - \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n - \sqrt{g_{11}}\right)} \right. \\
& \left. + \frac{\sin\left[\left(\beta_n + \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n + \sqrt{g_{11}}\right)} \right\} \Bigg]
\end{aligned} \qquad \text{(A-7)}$$

$$
\begin{aligned}
f_{p12}(\bar{z}) = \bar{\rho}_{f1} \Bigg[& \Bigg[-\frac{\bar{e}_{10}}{\sqrt{g_{11}}} \cos\left(\sqrt{g_{11}}\,\bar{z}\right) + \frac{1}{g_{11}} \sin\left(\sqrt{g_{11}}\,\bar{z}\right) \\
& - \frac{\bar{z}}{\sqrt{g_{11}}} \cos\left(\sqrt{g_{11}}\,\bar{z}\right) \Bigg] A_0 \\
& + \sum_{n=1}^{\infty} A_n \Bigg[\frac{a_n \alpha_n}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\,\bar{z}\right) - \frac{a_n \sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\,\bar{z}\right) \\
& + \frac{\alpha_n}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\,\bar{z}\right) - \frac{\sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\,\bar{z}\right) \Bigg] J_0\left(\alpha_n\right) \\
& - \sum_{n=1}^{\infty} \frac{B_n I_0\left(\beta_n\right)}{\cos\left(\beta_n \bar{b}\right)} \Bigg\{ \frac{\cos\left[\left(\beta_n - \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n - \sqrt{g_{11}}\right)} \\
& + \frac{\cos\left[\left(\beta_n + \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n + \sqrt{g_{11}}\right)} \Bigg\} \Bigg]
\end{aligned}
\tag{A-8}
$$

For $\beta_n = \sqrt{g_{11}}$, however, the last terms of Eqs. (A-7) and (A-8) become, respectively,

$$
+ \sum_{n=1}^{\infty} \frac{B_n I_0\left(\beta_n\right)}{\cos\left(\beta_n \bar{b}\right)} \left[\frac{1}{2} \bar{z} \cos\left(\beta_n \bar{b}\right) + \frac{1}{4\beta_n} \sin\beta_n (2\bar{z} - \bar{b}) \right] \quad \text{and}
$$

$$
- \sum_{n=1}^{\infty} \frac{B_n I_0\left(\beta_n\right)}{\cos\left(\beta_n \bar{b}\right)} \left[\frac{1}{2} \bar{z} \sin\left(\beta_n \bar{b}\right) - \frac{1}{4\beta_n} \cos \beta_n (2\bar{z} - \bar{b}) \right].
$$

Similarly, substituting the pressure, $\bar{p}_2(\bar{z})$ from Eq. (95) together with Eq. (49) into the last two equations of Eqs. (A-6) gives:

$$f_{p21}(\bar{z}) = \bar{\rho}_{f2} \left\{ C_0 \int^{\bar{z}} \cos\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z} + \sum_{n=1}^{\infty} C_n \int^{\bar{z}} \left[1 + (\bar{b}-\bar{z})^2\right]^{\frac{n}{2}} P_n\left(\mu_{\bar{z}}\right) \cos\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z} \right\} \tag{A-9}$$

$$f_{p22}(\bar{z}) = \bar{\rho}_{f2} \left\{ C_0 \int^{\bar{z}} \sin\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z} + \sum_{n=1}^{\infty} C_n \int^{\bar{z}} \left[1 + (\bar{b}-\bar{z})^2\right]^{\frac{n}{2}} P_n\left(\mu_{\bar{z}}\right) \sin\left(\sqrt{g_{11}}\,\bar{z}\right) d\bar{z} \right\} \tag{A-10}$$

where

$$\mu_{\bar{z}} = \mu(z) = \frac{\bar{b} - \bar{z}}{\sqrt{1 + (b-z)^2}} \tag{A-11}$$

If the semivertex angle of cone, α is greater than $\pi/6$, the following approximation holds for the present system where the tank is fully filled with a liquid:

$$\cos\left(\sqrt{g_{11}}\,\bar{z}\right) \cong \cos\left(\sqrt{g_{11}}\,\bar{b}\right) - \frac{g_{11}}{2} \frac{\mu^2}{1-\mu^2} \cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$+ \sqrt{g_{11}} \frac{\mu}{\sqrt{1-\mu^2}} \sin\left(\sqrt{g_{11}}\,\bar{b}\right) - \frac{g_{11}^{\frac{3}{2}}}{6} \frac{\mu^3}{\left(1-\mu^2\right)^{\frac{3}{2}}} \sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$\sin\left(\sqrt{g_{11}}\,\bar{z}\right) \cong \sin\left(\sqrt{g_{11}}\,\bar{b}\right) - \frac{g_{11}}{2}\,\frac{\mu^2}{1-\mu^2}\sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$- \sqrt{g_{11}}\,\frac{\mu}{\sqrt{1-\mu^2}}\cos\left(\sqrt{g_{11}}\,\bar{b}\right) + \frac{g_{11}^{\frac{3}{2}}}{6}\,\frac{\mu^3}{\left(1-\mu^2\right)^{\frac{3}{2}}}\cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

For this approximation, the second integrals of Eqs. (A-9) and (A-10) become, respectively,

$$\left.\begin{aligned}
&\int^{\bar{z}}\left[1+(\bar{b}-\bar{z})^2\right]^{\frac{n}{2}} P_n\left(\mu_{\bar{z}}\right)\cos\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z} \\
&\qquad = \sum_{i=0}^{5} H_{c_i}\int^{\mu}\frac{\mu^i}{\left(1-\mu^2\right)^{\frac{n+i+1}{2}}}\,P_n(\mu)\,d\mu \\
&\text{and} \\
&\int^{\bar{z}}\left[1+(\bar{b}-\bar{z})^2\right]^{\frac{n}{2}} P_n\left(\mu_{\bar{z}}\right)\sin\left(\sqrt{g_{11}}\,\bar{z}\right)d\bar{z} \\
&\qquad = \sum_{i=0}^{5} H_{s_i}\int^{\mu}\frac{\mu^i}{\left(1-\mu^2\right)^{\frac{n+i+1}{2}}}\,P_n(\mu)\,d\mu
\end{aligned}\right\} \qquad \text{(A-12)}$$

where

$$H_{c0} = -\cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{c1} = -\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{c2} = -\left(1 - \frac{1}{2}g_{11}\right)\cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{c3} = -\left(\sqrt{g_{11}} - \frac{1}{6}g_{11}^{\frac{3}{2}}\right)\sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{c4} = \frac{1}{2}g_{11}\cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{c5} = \frac{1}{6}g_{11}^{\frac{3}{2}}\sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{s0} = -\sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{s1} = \sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{s2} = -\left(1 - \frac{1}{2}g_{11}\right)\sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{s3} = \left(g_{11} - \frac{1}{6}\,g_{11}^{\frac{3}{2}}\right)\cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{s4} = \frac{1}{2}g_{11}\sin\left(\sqrt{g_{11}}\,\bar{b}\right)$$

$$H_{s5} = -\frac{1}{6}g_{11}^{\frac{3}{2}}\cos\left(\sqrt{g_{11}}\,\bar{b}\right)$$

The general expression for the Legendre polynomials is expressed as

$$P_n(\mu) = \sum_{k=0}^{N} \frac{(-1)^k (2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!}\,\mu^{n-2k} \qquad \text{(A-13)}$$

$$\left.\begin{aligned}
&\int^{\bar{z}} \left[1 + (\bar{b}-\bar{z})^2\right]^{\frac{n}{2}} P_n\left(\mu_{\bar{z}}\right)\cos\left(\sqrt{g_{11}}\ \bar{z}\right)d\bar{z} \\
&\qquad = \sum_{i=0}^{5}\sum_{k=0}^{N} \frac{(-1)^k (2n-2k)!\,H_{ci}}{2^n k!\,(n-k)!\,(n-2k)!} \int^{\mu} \frac{\mu^{i+n-2k}}{\left(1-\mu^2\right)^{\frac{n+i+1}{2}}}\, d\mu \\
&\int^{\bar{z}} \left[1 + (\bar{b}-\bar{z})^2\right]^{\frac{n}{2}} P_n\left(\mu_{\bar{z}}\right)\sin\left(\sqrt{g_{11}}\ \bar{z}\right)d\bar{z} \\
&\qquad = \sum_{i=0}^{5}\sum_{k=0}^{N} \frac{(-1)^k (2n-2k)!\,H_{si}}{2^n k!\,(n-k)!\,(n-2k)!} \int^{\mu} \frac{\mu^{i+n-2k}}{\left(1-\mu^2\right)^{\frac{n+i+1}{2}}}\, d\mu
\end{aligned}\right\} \qquad \text{(A-14)}$$

From the integrated results: Eqs. (A-7), (A-8), (A-9), and (A-10), together with Eqs. (A-14), the pressure terms: $f_{p11}(\bar{z})$, $f_{p12}(\bar{z})$, $f_{p21}(\bar{z})$ and $f_{p22}(\bar{z})$ of Eqs. (A-6) are written, respectively, in the forms:

$$\left.\begin{aligned}
f_{p11}(\bar{z}) &= \bar{\rho}_{f1} \sum_{n=0}^{\infty} \left[f_{1zn}(\bar{z})A_n + f_{2zn}(\bar{z})B_n\right] \\
f_{p12}(\bar{z}) &= \bar{\rho}_{f1} \sum_{n=0}^{\infty} \left[f_{3zn}(\bar{z})A_n + f_{4zn}(\bar{z})B_n\right] \\
f_{p21}(\bar{z}) &= \bar{\rho}_{f2} \sum_{n=0}^{\infty} \left[f_{5z\hat{n}}(\bar{z})C_{\hat{n}} + f_{5z\bar{n}}(\bar{z})C_{\bar{n}}\right] \\
f_{p22}(\bar{z}) &= \bar{\rho}_{f2} \sum_{n=0}^{\infty} \left[f_{6z\hat{n}}(\bar{z})C_{\hat{n}} + f_{6z\bar{n}}(\bar{z})C_{\bar{n}}\right]
\end{aligned}\right\} \qquad \text{(A-15)}$$

where

$$f_{1z0}(\bar{z}) = \frac{\bar{e}_{10}}{\sqrt{g_{11}}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{1}{g_{11}} \cos\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{\bar{z}}{\sqrt{g_{11}}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$f_{1zn}(\bar{z}) = \left[\frac{a_n \alpha_n}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{a_n \sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \right.$$

$$+ \frac{\alpha_n}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$\left. + \frac{\sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) \right] J_0\left(\alpha_n\right) \qquad (n \geq 1)$$

$$f_{2z0}(\bar{z}) = 0$$

$$f_{2zn}(\bar{z}) = \frac{I_0\left(\beta_n\right)}{\cos\left(\beta_n \bar{b}\right)} \left\{ \frac{\sin\left[\left(\beta_n - \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n - \sqrt{g_{11}}\right)} \right.$$

$$\left. + \frac{\sin\left[\left(\beta_n + \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n + \sqrt{g_{11}}\right)} \right\} \qquad (n \geq 1)$$

$$f_{3z0}(\bar{z}) = -\frac{\bar{e}_{10}}{\sqrt{g_{11}}} \cos\left(\sqrt{g_{11}}\ \bar{z}\right) + \frac{1}{g_{11}} \sin\left(\sqrt{g_{11}}\ \bar{z}\right) - \frac{\bar{z}}{\sqrt{g_{11}}} \cos\left(\sqrt{g_{11}}\ \bar{z}\right)$$

$$f_{3zn}(\bar{z}) = \left[\frac{a_n \alpha_n}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) - \frac{a_n \sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) \right.$$

$$\left. + \frac{\alpha_n}{\alpha_n^2 + g_{11}} \sinh\left(\alpha_n \bar{z}\right) \sin\left(\sqrt{g_{11}}\ \bar{z}\right) - \frac{\sqrt{g_{11}}}{\alpha_n^2 + g_{11}} \cosh\left(\alpha_n \bar{z}\right) \cos\left(\sqrt{g_{11}}\ \bar{z}\right) \right] J_0\left(\alpha_n\right)$$

$$(n \geq 1)$$

$$f_{4z0}(\bar{z}) = 0$$

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$$f_{4zn}(\bar{z}) = -\frac{I_0(\beta_n)}{\cos(\beta_n \bar{b})}\left\{\frac{\cos\left[\left(\beta_n - \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n - \sqrt{g_{11}}\right)} + \frac{\cos\left[\left(\beta_n + \sqrt{g_{11}}\right)\bar{z} - \beta_n \bar{b}\right]}{2\left(\beta_n + \sqrt{g_{11}}\right)}, \quad (n \geq 1)\right\}$$

$$f_{5z\hat{n}}(\bar{z}) = \sum_{i=0}^{5}\sum_{k=0}^{N} \frac{(-1)^k (2\hat{n}-2k)!\,H_{ci}}{2^{\hat{n}} k!\,(\hat{n}-k)!\,(\hat{n}-2k)!} \int^{\mu} \frac{\mu^{i+\hat{n}-2k}}{\left(1-\mu^2\right)^{\frac{\hat{n}+i+1}{2}}}\, d\mu$$

$$f_{5z\bar{n}}(\bar{z}) = \sum_{i=0}^{5}\sum_{k=0}^{N} \frac{(-1)^k (2\bar{n}-2k)!\,H_{ci}}{2^{\bar{n}} k!\,(\bar{n}-k)!\,(\bar{n}-2k)!} \int^{\mu} \frac{\mu^{i+\bar{n}-2k}}{\left(1-\mu^2\right)^{\frac{\bar{n}+i+1}{2}}}\, d\mu$$

$$f_{6z\hat{n}}(\bar{z}) = \sum_{i=0}^{5}\sum_{k=0}^{N} \frac{(-1)^k (2\hat{n}-2k)!\,H_{si}}{2^{\hat{n}} k'\,(\hat{n}-k)!\,(\hat{n}-2k)!} \int^{\mu} \frac{\mu^{i+\hat{n}-2k}}{\left(1-\mu^2\right)^{\frac{\hat{n}+i+1}{2}}}\, d\mu$$

$$f_{6z\bar{n}}(\bar{z}) = \sum_{i=0}^{5}\sum_{k=0}^{N} \frac{(-1)^k (2\bar{n}-2k)!\,H_{si}}{2^{\bar{n}} k!\,(\bar{n}-k)!\,(\bar{n}-2k)!} \int^{\mu} \frac{\mu^{i+\bar{n}-2k}}{\left(1-\mu^2\right)^{\frac{\bar{n}+i+1}{2}}}\, d\mu \qquad \text{(A-16)}$$

The substitution of the expressions for $F_2(\mu_i)$, $f_{p11}(\bar{z})$, $f_{p12}(\bar{z})$, $f_{p21}(\bar{z})$, and $f_{p22}(\bar{z})$ from Eqs. (A-5) and (A-15) into Eqs. (44), (49), (84), (120) and (124) yields, after the lengthy mathematical manipulation, the expressions for q_{4rzn}, q_{5rzn}, ---, $q_{33z\bar{n}}$, which are the functional coefficients of A_n, B_n, $C_{\hat{n}}$ and $C_{\bar{n}}$, as follows:

$$q_{4rz0} = \bar{e}_{10} + \bar{z}$$

$$q_{4rzn} = \left[a_n \sinh\left(\alpha_n \bar{z}\right) + \cosh\left(\alpha_n \bar{z}\right)\right] J_0\left(\alpha_n \bar{r}\right), \qquad (n \geq 1) \qquad \text{(A-17)}$$

$$q_{5rz0} = 0$$

$$q_{5rzn} = \frac{\cos\left[\beta_n(\bar{z}-\bar{b})\right]}{\cos\left(\beta_n \bar{b}\right)} I_0\left(\beta_n \bar{r}\right), \qquad (n \geq 1) \qquad \text{(A-18)}$$

$$q_{6s\mu\hat{n}} = \bar{s}^{\hat{n}} P_{\hat{n}}(\mu) \qquad \text{(A-19)}$$

$$q_{6s\mu\bar{n}} = \bar{s}^{\bar{n}} P_{\bar{n}}(\mu) \qquad \text{(A-20)}$$

$$q_{11zn} = \bar{\rho}_{f1} \left[\frac{\nu\, \ell_{k0}}{1-\kappa_1 \omega^2} \left(\ell_{k11} f_{h1n} + \ell_{k12} f_{h3n} + \ell_{k12} \ell_{11} f_{h1n} + \ell_{k12} f_{h3n} \right) + \frac{\nu\, g_{12}}{1-\kappa_1 \omega^2} \left(\ell_{13} f_{1zn} + \ell_{14} f_{3zn} \right) + \frac{1+\nu\, g_{12}}{1-\kappa_1 \omega^2} f_{p1zn} \right] \qquad \text{(A-21)}$$

where

$$\ell_{k0} = -\ell_{c2}\ell_{13} + \ell_{c1}\ell_{14} - \ell_{c5}\ell_{14}$$

$$\ell_{k11} = \frac{-\ell_{c4}\ell_{13} + \ell_{c3}\ell_{14}}{\ell_{k0}}, \quad \ell_{k12} = \frac{\ell_{c2}\ell_{13} - \ell_{c1}\ell_{14}}{\ell_{k0}}$$

$$\ell_{k21} = \frac{\ell_{13} - \ell_{c6\ 14}}{\ell_{k0}}, \quad \ell_{k22} = \frac{\ell_{c5}\ell_{14}}{\ell_{k0}}$$

$$\ell_{c1} = \frac{\ell_{11}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}, \quad \ell_{c2} = \frac{\ell_{12}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}, \quad \ell_{c5} = \frac{1}{\ell_{24}}$$

$$\ell_{c3} = \frac{\ell_{13}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}, \quad \ell_{c4} = \frac{\ell_{14}}{\ell_{11}\ell_{14} - \ell_{12}\ell_{13}}, \quad \ell_{c6} = \frac{\ell_{23}}{\ell_{24}}$$

$$f_{h1n} = -\left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12}\sqrt{g_{11}}\left[f_{1Hn}\sin\left(\sqrt{g_{11}}\,\bar{H}\right) - f_{3Hn}\cos\left(\sqrt{g_{11}}\,\bar{H}\right)\right]$$

$$+\left\{\left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12} - \frac{\nu}{1-\kappa_1\omega^2}\right\} f_{p1Hn}$$

$$f_{h2n} = -\left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12}\sqrt{g_{11}}\left[f_{2Hn}\sin\left(\sqrt{g_{11}}\,\bar{H}\right) - f_{4Hn}\cos\left(\sqrt{g_{11}}\,\bar{H}\right)\right]$$

$$+\left\{\left(1 - \frac{\nu^2}{1-\kappa_1\omega^2}\right) g_{12} - \frac{\nu}{1-\kappa_1\omega^2}\right\} f_{p2Hn}$$

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$$f_{h3n} = - g_{12}\sqrt{g_{11}}\left[f_{1bn} \sin\left(\sqrt{g_{11}}\,\bar{b}\right) - f_{3bn}\cos\left(\sqrt{g_{11}}\,\bar{b}\right)\right] + g_{12}f_{p1bn}$$

$$f_{h4n} = - g_{12}\sqrt{g_{11}}\left[f_{2bn}\sin\left(\sqrt{g_{11}}\,\bar{b}\right) - f_{4bn}\cos\left(\sqrt{g_{11}}\,\bar{b}\right)\right] + g_{12}f_{p2bn}$$

$$f_{h5\hat{n}} = g_{12}f_{50\hat{n}}$$

$$f_{h5\bar{n}} = g_{12}f_{50\bar{n}}$$

$$f_{h6\hat{n}} = - g_{12}\sqrt{g_{11}}\left[f_{5b\hat{n}}\sin\left(\sqrt{g_{11}}\,\bar{b}\right) - f_{6b\hat{n}}\cos\left(\sqrt{g_{11}}\,\bar{b}\right)\right] + g_{12}f_{p3b\hat{n}}$$

$$f_{h6\bar{n}} = - g_{12}\sqrt{g_{11}}\left[f_{5b\bar{n}}\sin\left(\sqrt{g_{11}}\,\bar{b}\right) - f_{6b\bar{n}}\cos\left(\sqrt{g_{11}}\,\bar{b}\right)\right] + g_{12}f_{p3b\bar{n}}$$

$$f_{p1z0} = \bar{e}_{10} + \bar{z}$$

$$f_{p1zn} = \left[a_n\sinh\left(\alpha_n\bar{z}\right) + \cosh\left(\alpha_n\bar{z}\right)\right] J_0\left(\alpha_n\right) \qquad (n \geq 1)$$

$$f_{p2z0} = 0$$

$$f_{p2zn} = \frac{\cos\left[\beta_n(\bar{z}-\bar{b})\right]}{\cos\left(\beta_n\,\bar{b}\right)}\; I_0\left(\beta_n\right) \qquad (n \geq 1)$$

$$f_{p3z\hat{n}} = \left[1 + (\bar{b}-\bar{z})^2\right]^{\frac{\hat{n}}{2}} P_{\hat{n}}\left(\mu_{\bar{z}}\right)$$

$$f_{p3z\bar{n}} = \left[1 + (\bar{b}-\bar{z})^2\right]^{\frac{\bar{n}}{2}} P_{\bar{n}}\left(\mu_{\bar{z}}\right)$$

$$f_{Ji\hat{n}} = f_{Jin} \qquad \text{for} \qquad n = 0, 2, 4, ---$$

$$f_{Ji\bar{n}} = f_{Jin} \qquad \text{for} \qquad n = 1, 3, 5, ---$$

$$\mu_\alpha = \cos\alpha$$

$$\ell_{z13} = -\sqrt{g_{11}}\,\sin\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$\ell_{z14} = \sqrt{g_{11}}\,\cos\left(\sqrt{g_{11}}\,\bar{z}\right)$$

$$q_{12zn} = \bar{\rho}_{f1}\left[\frac{\nu\,\ell_{k0}}{1-\kappa_1\omega^2}\left(\ell_{k11}f_{h2n} + \ell_{k12}f_{h4n} + \ell_{k12}\ell_{k11}f_{h2n} + \ell_{k12}^2\,f_{h4n}\right)\right.$$

$$\left. + \frac{\nu\,g_{12}}{1-\kappa_1\omega^2}\left(\ell_{13}f_{2zn} + \ell_{14}f_{4zn}\right) + \frac{1+\nu g_{12}}{1-\kappa_1\omega^2}\,f_{p2zn}\right] \tag{A-22}$$

$$q_{13z\hat{n}} = \bar{\rho}_{f1}\,\frac{\nu\,\ell_{k0}}{1-\kappa_1\omega^2}\,\bar{\rho}_{fc}\ell_{k12}\left(\ell_{k21}f_{h5\hat{n}} + \ell_{k22}f_{h6\hat{n}}\right) \tag{A-23}$$

$$q_{13\bar{n}} = \bar{\rho}_{f1}\,\frac{\nu\,\ell_{k0}}{1-\kappa_1\omega^2}\,\bar{\rho}_{fc}\ell_{k12}\left(\ell_{k21}f_{h5\hat{n}} + \ell_{k22}f_{h6\bar{n}}\right) \tag{A-24}$$

$$q_{21s\hat{n}} = \sum_i\left\{\frac{2\tan\alpha}{\ell^{-2}}\,\bar{\rho}_{f2}\,\frac{\mu_i^{\,2}}{\left(\sigma^2+\mu_i^2\right)\left(\lambda^2-\mu_i^2\right)}\,\frac{\nu\bar{s}\mu_i J_1\left(\mu_i\bar{s}\right) - J_0\left(\mu_i\bar{s}\right)}{\left[J_0\left(\mu_i\,\ell\right)\right]^2}\right\}f_{Ji\hat{n}}$$

$$- \bar{\rho}_{f2}\,\bar{s}^{\hat{n}+2}\,P_{\hat{n}}\left(\mu_\alpha\right)\tan^2\alpha \tag{A-25}$$

$$q_{21s\bar{n}} = \sum_i \left\{ \frac{2\tan\alpha}{\ell^2} \bar{\rho}_{f2} \frac{\mu_i^2}{\left(\sigma^2+\mu_i^2\right)\left(\lambda^2-\mu_i^2\right)} \frac{\nu\bar{s}\mu_i J_1\left(\mu_i\bar{s}\right) - J_0\left(\mu_i\bar{s}\right)}{\left[J_0\left(\mu_i\ell\right)\right]^2} \right\} f_{Ji\bar{n}}$$

$$- \bar{\rho}_{f2}\bar{s}^{n+2} P_n\left(\mu_\alpha\right)\tan^2\alpha \tag{A-26}$$

$$q_{31zn} = - \frac{\nu}{1-\kappa_1\omega^2} \bar{\rho}_{f1} \ell_{c5}\ell_{z14}\left(\ell_{k11}f_{h1n} + \ell_{k12}f_{h3n}\right) \tag{A-27}$$

$$q_{32zn} = - \frac{\nu}{1-\kappa_1\omega^2} \bar{\rho}_{f1} \ell_{c5}\ell_{z14}\left(\ell_{k11}f_{h2n} + \ell_{k12}f_{h4n}\right) \tag{A-28}$$

$$q_{33z\hat{n}} = - \frac{\nu}{1-\kappa_1\omega^2} \bar{\rho}_{f2} \Big[\left(\ell_{z13} - \ell_{c6}\ell_{z14}\right)f_{h5\hat{n}} + \ell_{c5}\ell_{z14}f_{h6\hat{n}} +$$

$$+ \ell_{c5}\ell_{z14}\left(\ell_{k21}f_{h5\hat{n}} + \ell_{k22}f_{h6\hat{n}}\right)\Big] + \frac{\nu\, g_{12}}{1-\kappa_1\omega^2} \bar{\rho}_{f2}\left(\ell_{z13}f_{5z\hat{n}}\right.$$

$$\left. + \ell_{z14}f_{6z\hat{n}}\right) + \frac{1+\nu g_{12}}{1-\kappa_1\omega^2} \bar{\rho}_{f2} f_{p3z\hat{n}} \tag{A-29}$$

$$q_{33z\bar{n}} = - \frac{\nu}{1-\kappa_1\omega^2} \bar{\rho}_{f2} \Big[\left(\ell_{z13} - \ell_{c6}\ell_{z14}\right)f_{h5\hat{n}} + \ell_{c5}\ell_{z14}f_{h6\bar{n}}$$

$$+ \ell_{c5}\ell_{z14}\left(\ell_{k21}f_{h5\bar{n}} + \ell_{k22}f_{h6\bar{n}}\right)\Big] + \frac{\nu\, g_{12}}{1-\kappa_1\omega^2} \bar{\rho}_{f2}\left(\ell_{z13}f_{5z\bar{n}}\right. \tag{A-30}$$

$$\left. + \ell_{z14}f_{6z\bar{n}}\right) + \frac{1+\nu g_{12}}{1-\kappa_1\omega^2} \bar{\rho}_{f2}f_{p3z\bar{n}} \tag{A-30}$$

B. Functional Errors $\bar{\varepsilon}_i$ (i = 1, 2, ---5)

The functional errors, $\bar{\varepsilon}_i$, (i = 1, 2, ---5) defined by Eqs. (139) through (143) are derived in a series form in this appendix. The element, I_{ijmn}, of the matrix eigenvalue problem is then calculated in terms of the functional coefficient which is derived in a series form of each functional error, $\bar{\varepsilon}_i$.

From the expression of $\bar{\phi}_1(r,z)$ and $\bar{\phi}_2(s,)$ given in Eqs. (A-1), the derivatives of these two variables are obtained as

$$\left.\begin{aligned}
\frac{\partial\bar{\phi}_1}{\partial\bar{r}} &= \sum_{n=0}^{\infty}\left(\gamma_{1rzn}A_n + \gamma_{2rzn}B_n\right)\\
\frac{\partial\bar{\phi}_1}{\partial\bar{z}} &= \sum_{n=0}^{\infty}\left(\gamma_{3rzn}A_n + \gamma_{4rzn}B_n\right)\\
\frac{\partial\bar{\phi}_2}{\partial\bar{s}} &= \sum_{n=0}^{\infty}\left(\gamma_{5s\mu\hat{n}}C_{\hat{n}} + \gamma_{5s\mu\bar{n}}C_{\bar{n}}\right)\\
\frac{1}{\bar{s}}\frac{\partial\phi}{\partial\theta} &= \sum_{n=0}^{\infty}\left(\gamma_{6s\mu\hat{n}}C_{\hat{n}} + \gamma_{6s\mu\bar{n}}C_{\bar{n}}\right)
\end{aligned}\right\} \qquad \text{(B-1)}$$

where

$$\gamma_{1rzn} = -\alpha_n\left[a_n\sinh\left(\alpha_n\bar{z}\right) + \cosh\left(\alpha_n\bar{z}\right)\right]J_1\left(\alpha_n\bar{r}\right) \qquad (n \geq 1)$$

$$\gamma_{2rzn} = \beta_n\frac{\cos\left[\beta_n(z-b)\right]}{\cos\left(\beta_n\bar{b}\right)}I_1\left(\beta_n\bar{r}\right) \qquad (n \geq 1)$$

$$\gamma_{3rzn} = \alpha_n \left[a_n \cosh(\alpha_n \bar{z}) + \sinh(\alpha_n \bar{z})\right] J_0(\alpha_n \bar{r}) \qquad (n \geq 1)$$

$$\gamma_{4rzn} = -\beta_n \frac{\sin\left[\beta_n(\bar{z}-\bar{b})\right]}{\cos(\beta_n \bar{b})} I_0(\beta_n \bar{r}) \qquad (n \geq 1)$$

$$\gamma_{5s\mu\hat{n}} = \hat{n}\, \bar{s}^{\hat{n}-1} P'_{\hat{n}}(\mu)$$

$$\gamma_{5s\mu\bar{n}} = \bar{n}\, \bar{s}^{\bar{n}-1} P_{\bar{n}}(\mu)$$

$$\gamma_{6s\mu\hat{n}} = -\frac{1}{\sqrt{1-\mu^2}} \hat{n}\, \bar{s}^{\hat{n}-1} \left[P_{\hat{n}-1}(\mu) - \mu P_{\hat{n}}(\mu)\right]$$

$$\gamma_{6s\mu\bar{n}} = -\frac{1}{\sqrt{1-\mu^2}} \bar{n}\, \bar{s}^{\bar{n}-1} \left[P_{\bar{n}-1}(\mu) - \mu P_{\bar{n}}(\mu)\right]$$

$$\gamma_{1rz0} \triangleq 0, \qquad \gamma_{2rz0} \triangleq 0$$

$$\gamma_{3rz0} \triangleq 1, \qquad \gamma_{4rz0} \triangleq 0$$

Substituting Eqs. (A-1) and (B-1) into the functional errors, Eqs. (139) through (143) gives

$$\left.\begin{aligned}
\bar{\varepsilon}_1 &= \sum_{n=0}^{\infty} \left(e_{11zn}A_n + e_{12zn}B_n + e_{13z\hat{n}}C_{\hat{n}} + e_{13z\bar{n}}C_{\bar{n}}\right) \\
\bar{\varepsilon}_2 &= \sum_{n=0}^{\infty} \left(e_{21rn}A_n + e_{224n}B_n + e_{23r\bar{n}}C_{\bar{n}}\right) \\
\bar{\varepsilon}_3 &= \sum_{n=0}^{\infty} \left(e_{31rn}A_n + e_{32rn}B_n + e_{33r\hat{n}}C_{\hat{n}}\right) \\
\bar{\varepsilon}_4 &= \sum_{n=0}^{\infty} \left(e_{43s\hat{n}}C_{\hat{n}} + e_{43s\bar{n}}C_{\bar{n}}\right) \\
\bar{\varepsilon}_5 &= \sum_{n=0}^{\infty} \left(e_{51zn}A_n + e_{52zn}B_n + e_{53z\hat{n}}C_{\hat{n}} + e_{53z\bar{n}}C_{\bar{n}}\right)
\end{aligned}\right\} \qquad \text{(B-2)}$$

where

$$e_{11zn} = \gamma_{11zn} - \beta_{K1} q_{11zn}$$

$$e_{12zn} = \gamma_{21zn} - \beta_{K1} q_{12zn}$$

$$e_{13z\hat{n}} = - \beta_{K1} q_{13z\hat{n}}$$

$$e_{13z\bar{n}} = - \beta_{K1} q_{13z\bar{n}}$$

$$e_{21rn} = \gamma_{3rbn}$$

$$e_{22rn} = \gamma_{4rbn}$$

$$e_{23s\bar{n}} = e_{23r\bar{n}} = - \gamma_{6s0\bar{n}}$$

$$e_{31rn} = q_{4rbn}$$

$$e_{32rn} = q_{5rbn}$$

$$e_{33s\hat{n}} = e_{33r\hat{n}} = - q_{6s0\hat{n}}$$

$$e_{43s\hat{n}} = \gamma_{6s\alpha\hat{n}} - \beta_{K2} q_{21s\hat{n}}$$

$$e_{43s\bar{n}} = \gamma_{6s\alpha\bar{n}} - \beta_{K2} q_{21s\bar{n}}$$

$$e_{51zn} = - \beta_{K1} q_{31zn}$$

$$e_{52zn} = - \beta_{K1} q_{32zn}$$

$$e_{53zn} = K_{sz}\gamma_{5z\hat{n}} + K_{\theta z}\gamma_{6z\hat{n}} - \beta_{K1}q_{33z\hat{n}}$$

$$e_{53z\bar{n}} = K_{sz}\gamma_{5z\bar{n}} + K_{\theta z}\gamma_{6z\bar{n}} - \beta_{K1}q_{33z\bar{n}}$$

The conditions for the minimum of the total integrated squared error are obtained by substituting Eqs. (B-2) into Eqs. (147) and then Eqs. (148). It follows four simultaneous equations:

$$\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial\bar{\varepsilon}_1}{\partial A_m} d\bar{z} + \int_0^1 \bar{\varepsilon}_2 \frac{\partial\bar{\varepsilon}_2}{\partial A_m} \bar{r}d\bar{r} + \int_0^1 \bar{\varepsilon}_3 \frac{\partial\bar{\varepsilon}_3}{\partial A_m} \bar{r}d\bar{r} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial\bar{\varepsilon}_5}{\partial A_m} d\bar{z} = 0$$

$$\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial\bar{\varepsilon}_1}{\partial B_m} d\bar{z} + \int_0^1 \bar{\varepsilon}_2 \frac{\partial\bar{\varepsilon}_2}{\partial B_m} \bar{r}d\bar{r} + \int_0^1 \bar{\varepsilon}_3 \frac{\partial\bar{\varepsilon}_3}{\partial B_m} \bar{r}d\bar{r} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial\bar{\varepsilon}_5}{\partial B_m} d\bar{z} = 0$$

$$\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial\bar{\varepsilon}_1}{\partial C_{\hat{m}}} d\bar{z} + \int_0^1 \bar{\varepsilon}_3 \frac{\partial\bar{\varepsilon}_3}{\partial C_{\hat{m}}} \bar{r}d\bar{r} + \int_0^1 \bar{\varepsilon}_4 \frac{\partial\bar{\varepsilon}_4}{\partial C_{\hat{m}}} \bar{s}\sin\alpha d\bar{s} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial\bar{\varepsilon}_5}{\partial C_{\hat{m}}} d\bar{z} = 0$$

$$\int_{\bar{b}}^{\bar{H}} \bar{\varepsilon}_1 \frac{\partial\bar{\varepsilon}_1}{\partial C_{\bar{m}}} d\bar{z} + \int_0^1 \bar{\varepsilon}_2 \frac{\partial\bar{\varepsilon}_2}{\partial C_{\bar{m}}} \bar{r}d\bar{r} + \int_0^{\bar{\ell}} \bar{\varepsilon}_4 \frac{\partial\bar{\varepsilon}_4}{\partial C_{\bar{m}}} \bar{s}\sin\alpha d\bar{s} + \int_0^{\bar{b}} \bar{\varepsilon}_5 \frac{\partial\bar{\varepsilon}_5}{\partial C_{\bar{m}}} d\bar{z} = 0$$

(B-3)

The substitution of Eqs. (B-2) into Eqs. (B-3) yields a desired linear algebraic system of $4 \times (N+1)$ homogeneous simultenaous equations for $4 \times (N+1)$ unknowns A_n, B_n, $C_{\hat{n}}$ and $C_{\bar{n}}$ $(n = 0, 1, 2, ---, N)$ as follows:

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty}\left(I_{11mn}A_n + I_{12mn}B_n + I_{13m\hat{n}}C_{\hat{n}} + I_{13m\bar{n}}C_{\bar{n}}\right) = 0\\
&\sum_{n=0}^{\infty}\left(I_{21mn}A_n + I_{22mn}B_n + I_{23m\hat{n}}C_{\hat{n}} + I_{23m\bar{n}}C_{\bar{n}}\right) = 0\\
&\sum_{n=0}^{\infty}\left(I_{31\hat{m}n}A_n + I_{32\hat{m}n}B_n + I_{33\hat{m}\hat{n}}C_{\hat{n}} + I_{33\hat{m}\bar{n}}C_{\bar{n}}\right) = 0\\
&\sum_{n=0}^{\infty}\left(I_{31\bar{m}n}A_n + I_{32\bar{m}n}B_n + I_{33\bar{m}\hat{n}}C_{\hat{n}} + I_{33\bar{m}\bar{n}}C_{\bar{n}}\right) = 0
\end{aligned}\right\} \qquad \text{(B-4)}$$

where

$$I_{11mn} = \int_{\bar{b}}^{\bar{H}} e_{11zm}e_{11zn}\, d\bar{z} + \int_0^1 e_{21rm}e_{21rn}\, \bar{r}d\bar{r}$$

$$+ \int_0^1 e_{31rm}e_{31rn}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{51zn}e_{51zn}\, d\bar{z}$$

$$I_{12mn} = \int_{\bar{b}}^{\bar{H}} e_{11zm}e_{12zn}\, d\bar{z} + \int_0^1 e_{21rm}e_{22rn}\, \bar{r}d\bar{r}$$

$$+ \int_0^1 e_{31rm}e_{32rn}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{51zm}e_{52zn}\, d\bar{z}$$

$$I_{13m\hat{n}} = \int_{\bar{b}}^{\bar{H}} e_{11zm}e_{13z\hat{n}}\, d\bar{z} + \int_0^1 e_{31rm}e_{33r\hat{n}}\, \bar{r}d\bar{r}$$

$$+ \int_0^{\bar{b}} e_{51zm}e_{53z\hat{n}}\, d\bar{z}$$

$$I_{13m\bar{n}} = \int_{\bar{b}}^{\bar{H}} e_{11zm}e_{13z\bar{n}}\, d\bar{z} + \int_0^1 e_{21rm}e_{23r\bar{n}}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{51zm}e_{53z\bar{n}}\, d\bar{z}$$

$$I_{21mn} = \int_{\bar{b}}^{\bar{H}} e_{12zm}e_{11zn}\, d\bar{z} + \int_0^1 e_{22rm}e_{21rn}\, \bar{r}d\bar{r} + \int_0^1 e_{32rm}e_{31rn}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{52zm}e_{51zn}\, d\bar{z}$$

$$I_{22mn} = \int_{\bar{b}}^{\bar{H}} e_{12zm}e_{12zn}\, d\bar{z} + \int_0^1 e_{22rm}e_{22rn}\, \bar{r}d\bar{r} + \int_0^1 e_{32rm}e_{32rn}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{52zm}e_{52zn}\, d\bar{z}$$

$$I_{23m\hat{n}} = \int^{\bar{H}} e_{12zm}e_{13z\hat{n}}\, d\bar{z} + \int_0^1 e_{32rm}e_{33r\hat{n}}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{52zm}e_{53z\hat{n}}\, d\bar{z}$$

$$I_{23m\bar{n}} = \int_{\bar{b}}^{\bar{H}} e_{12zm}e_{13z\bar{n}}\, d\bar{z} + \int_0^1 e_{22rm}e_{23r\bar{n}}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{52zm}e_{53z\bar{n}}\, d\bar{z}$$

$$I_{31\hat{m}n} = \int_{\bar{b}}^{\bar{H}} e_{13z\hat{m}}e_{11zn}\, d\bar{z} + \int_0^1 e_{33r\hat{m}}e_{31rn}\, \bar{r}d\bar{r} + \int_0^{\bar{b}} e_{53z\hat{m}}e_{51zn}\, d\bar{z}$$

$$I_{32\hat{m}n} = \int_{\bar{b}}^{\bar{H}} e_{13z\hat{m}} e_{12zn} \, d\bar{z} + \int_{0}^{1} e_{33r\hat{m}} e_{32rn} \, \bar{r} d\bar{r} + \int_{0}^{\bar{b}} e_{53z\hat{m}} e_{52zn} \, d\bar{z}$$

$$I_{33\hat{m}\hat{n}} = \int_{\bar{b}}^{\bar{H}} e_{13z\hat{m}} e_{13z\hat{n}} \, d\bar{z} + \int_{0}^{1} e_{33r\hat{m}} e_{33r\hat{n}} \, \bar{r} d\bar{r} + \int_{0}^{\bar{\ell}} e_{43s\hat{m}} e_{43s\hat{n}} \, \bar{s} \sin\alpha \, d\bar{s} + \int_{0}^{\bar{b}} e_{53z\hat{m}} e_{53z\hat{n}} \, d\bar{z}$$

$$I_{33\hat{m}\bar{n}} = \int_{\bar{b}}^{\bar{H}} e_{13z\hat{m}} e_{13z\bar{n}} \, d\bar{z} + \int_{0}^{\bar{\ell}} e_{43s\hat{m}} e_{43s\bar{n}} \, \bar{s} \sin\alpha \, d\bar{s} + \int_{0}^{\bar{b}} e_{53z\hat{m}} e_{53z\bar{n}} \, d\bar{z}$$

$$I_{31\bar{m}n} = \int_{\bar{b}}^{\bar{H}} e_{13z\bar{m}} e_{11zn} \, d\bar{z} + \int_{0}^{1} e_{23r\bar{m}} e_{21rn} \bar{r} d\bar{r} + \int_{0}^{\bar{b}} e_{53z\bar{m}} e_{51zn} \, d\bar{z}$$

$$I_{32\bar{m}n} = \int_{\bar{b}}^{\bar{H}} e_{13z\bar{m}} e_{12zn}\, d\bar{z} + \int_0^1 e_{23r\bar{m}} e_{22rn}\, \bar{r}d\bar{r}$$

$$+ \int_0^{\bar{b}} e_{53z\bar{m}} e_{52zn}\, d\bar{z}$$

$$I_{33\bar{m}\hat{n}} = \int_{\bar{b}}^{\bar{H}} e_{13z\bar{m}} e_{13z\hat{n}} d\bar{z} + \int_0^{\bar{\ell}} e_{43s\bar{m}} e_{43s\hat{n}}\, \bar{s}\, \sin\alpha\, d\bar{s}$$

$$+ \int_0^{\bar{b}} e_{53z\bar{m}} e_{53z\hat{n}}\, d\bar{z}$$

$$I_{33\bar{m}\bar{n}} = \int_{\bar{b}}^{\bar{H}} e_{13z\bar{m}} e_{13z\bar{n}}\, d\bar{z} + \int_0^1 e_{23r\bar{m}} e_{23r\bar{n}}\, \bar{r}d\bar{r}$$

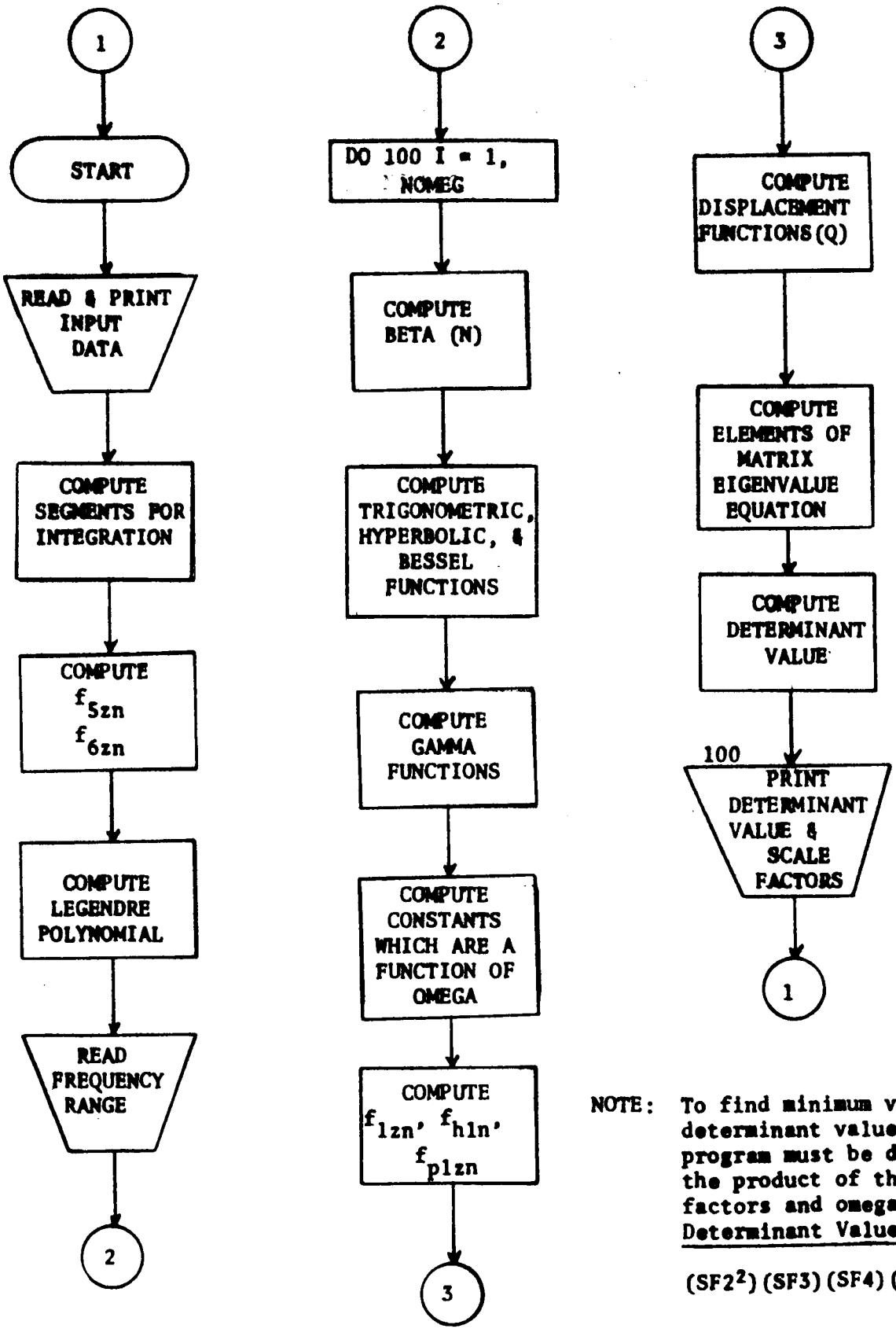
$$+ \int_0^{\bar{\ell}} e_{43s\bar{m}} e_{43s\bar{n}}\, \bar{s}\, \sin\alpha\, d\bar{s} + \int_0^{\bar{b}} e_{53z\bar{m}} e_{53z\bar{n}}\, d\bar{z}$$

C. COMPUTER PROGRAM DESCRIPTION

A matrix eigenvalue problem of the longitudinal oscillation of a liquid-filled elastic cylindrical tank with a flexible inverted conical bulkhead is formulated in Chapter IV and a procedure for the numerical calculation is presented in Chapter V. The FORTRAN computer program for this matrix eigenvalue problem is described in this appendix.

The purpose of the program is to determine the natural frequencies and the corresponding mode shapes of the system described above. A numerical example is worked out with the use of the first three terms of the series of the velocity potentials. The input data for this numerical example are shown in Table 1. The flow chart of the program is also shown in Figure 5.

Figure 5. Frequencies and Modes Program Logic



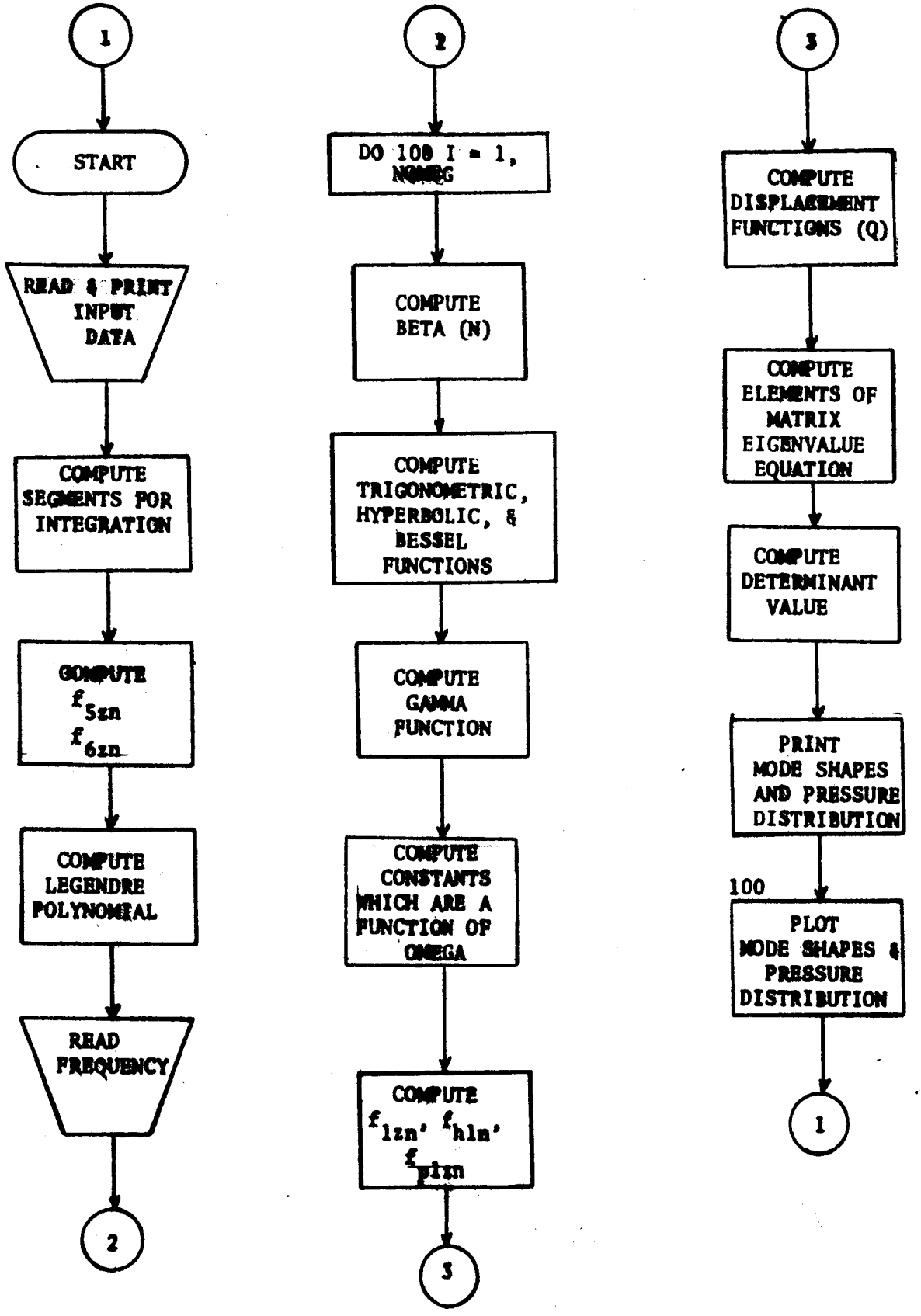
NOTE: To find minimum values, the determinant values from the program must be divided by the product of the scale factors and omega^{10}, i.e.

<u>Determinant Value</u>

$(SF2^2)(SF3)(SF4)(\omega^{10})$

(a) Flow Diagram for Evaluation of Frequencies

Figure 5. Frequencies and Modes Program Logic



(b) C
(b) Flow Diagram for Evaluation of Mode Shapes

```
$IBFTC OSCI                                                          OSC900 0
      COMMON/DAT2/ NITER,N,NSEG                                      OSCI0020
      COMMON/DMAT/ CMAT(14,14)                                       OSCI0030
      COMMON/DAT3/ ALPHAN(5), B(5),U(5)                              OSCI0040
      COMMON/Q/ Q1(12,11), Q2(12,11),Q3(12,11), Q4(6,11)             OSCI0042
      COMMON/SH/ BMAT(14,14)                                         OSCI0045
      DIMENSION  AMAT(14,14), OMEGA(50)                              OSCI0050
      LOGICAL ERR                                                    OSCI0060
    1 CALL INPUT                                                     OSCI0070
      CALL CONST                                                     OSCI0080
      CALL F5ZN                                                      OSCI0090
      CALL POLY                                                      OSCI0100
      READ(5,10) NOMEG,(OMEGA(I),I=1,NOMEG)                          OSCI0110
   10 FORMAT(I12/(6E12.8))                                           OSCI0120
      DO 100 I=1,NOMEG                                               OSCI0130
      F = 1.                                                         OSCI0124
      OMEG = OMEGA(I)                                                OSCI0140
      CALL BETAN(OMEG,ERR)                                           OSCI0150
      IF(ERR) GO TO 90                                               OSCI0160
      CALL TRANSF                                                    OSCI0170
      CALL GAMAN                                                     OSCI0180
      CALL COEF(OMEG)                                                OSCI0190
      CALL F1ZN                                                      OSCI0200
      CALL FH1N(OMEG)                                                OSCI0210
      CALL FP1ZN                                                     OSCI0220
      CALL QFUNCT                                                    OSCI0230
      CALL EPSLON                                                    OSCI0232
      CALL CMATX                                                     OSCI0233
C ***  DELETE ROW-4 AND COL.-4 ***                                   OSCI0251
      DO 15 J=1,3                                                    OSCI0254
      DO 15 K=1,14                                                   OSCI0252
   15 AMAT(K,J) = CMAT(K,J)                                          OSCI0256
      DO 20 K=1,14                                                   OSCI0270
      DO 20 J=5,14                                                   OSCI0280
   20 AMAT(K,J-1) = CMAT(K,J)                                        OSCI0290
      DO 30 K=5,14                                                   OSCI0300
      DO 30 J=1,13                                                   OSCI0310
   30 AMAT(K-1,J) = AMAT(K,J)                                        OSCI0320
      WRITE(6,40)                                                    OSCI0344
   40 FORMAT(1H1,45X,18HDETERMINANT MATRIX)                          OSCI0345
      DO 45 K=1,13                                                   OSCI0350
   45 WRITE(6,50) K, (AMAT(K,J),J=1,13)                              OSCI0360
   50 FORMAT(1H0,1X,4HROW=, I2, 2X,1P6E16.4/(10X,1P6E16.4))          OSCI0370
```

```
      CW = (5./OMEG)**6                                                 OS390372
      XX = CW**4                                                        OSCI0373
      SF3 = 1.0E-6 *CW                                                  OSCI0374
      SF2 = 1.0E-4 * CW                                                 OSCI0376
      SF4 = 1.0E-3 * CW                                                 OSCI0378
      DO 52 J=1,13                                                      OSCI0380
      AMAT(2,J) = SF2 * AMAT(2,J)                                       OSCI0382
      AMAT(3,J) = SF3 * AMAT(3,J)                                       OSCI0384
      AMAT(4,J) = SF4 * AMAT(4,J)                                       OSCI0386
   52 AMAT(5,J) = SF2 * AMAT(5,J)                                       OSCI0388
      DO 54 K = 1,10                                                    OSCI0389
      DO 54 J = 1,13                                                    OSCI0390
   54 BMAT(K,J) = AMAT(K,J)                                             OSCI0391
      DO 55 K=12,13                                                     OSCI0392
      DO 55 J = 1,13                                                    OSCI0393
   55 BMAT(K-1,J) = AMAT(K,J)                                           OSCI0394
      KK = IDETRM(14,13,AMAT,F)                                         OSCI0397
      IF(KK.EQ.2) GO TO 110                                             OSCI0398
      WRITE(6,60) OMEG,F                                                OSCI0410
   60 FORMAT(1H-,10X, 8HOMEGA = ,1PE13.6, 10X,20HDETERMINANT VALUE = ,1POSCI0420
     1E13.6)                                                            OSCI0430
      WRITE(6,65) SF2, SF3, SF4, XX                                     OSCI0434
   65 FORMAT(1H-,2X,6HSF2 = ,1PE13.6,5X,6HSF3 = ,1PE13.6,  5X, 6HSF4 = ,OSCI0436
     1 1PE13.6,3X,8HCW**4 = , 1PE13.6)                                  OSCI0438
      GO TO 100                                                         OSCI0440
   90 WRITE(6,95)                                                       OSCI0490
   95 FORMAT(1H-,10X,32H*** NO SOLUTIONS FOR BETA(I) ***)               OSCI0500
  100 CONTINUE                                                          OSCI0510
      GO TO 1                                                           OSCI0520
  110 WRITE(6,120)                                                      OSCI0525
  120 FORMAT(1H-,5X,40HOVERFLOW OCCURED CALCULATING DETERMINANT)        OSCI0526
      GO TO 1                                                           OSCI0527
      END                                                               OSCI0530
$*                                                                      OSCI0540
```

```
$IBFTC OSCI                                                          OSC900 0
      COMMON/DAT2/ NITER,N,NSEG                                      OSCI0020
      COMMON/DMAT/ CMAT(14,14)                                       OSCI0030
      COMMON/DAT3/ ALPHAN(5), B(5),U(5)                              OSCI0040
      COMMON/Q/ Q1(12,11), Q2(12,11),Q3(12,11), Q4(6,11)             OSCI0042
      COMMON/SH/ BMAT(14,14)                                         OSCI0045
      DIMENSION  AMAT(14,14), OMEGA(50)                              OSCI0050
      LOGICAL ERR                                                    OSCI0060
    1 CALL INPUT                                                     OSCI0070
      CALL CONST                                                     OSCI0080
      CALL F5ZN                                                      OSCI0090
      CALL POLY                                                      OSCI0100
      READ(5,10) NOMEG,(OMEGA(I),I=1,NOMEG)                          OSCI0110
   10 FORMAT(I12/(6E12.8))                                           OSCI0120
      DO 100 I=1,NOMEG                                               OSCI0130
      F = 1.                                                         OSCI0124
      OMEG = OMEGA(I)                                                OSCI0140
      CALL BETAN(OMEG,ERR)                                           OSCI0150
      IF(ERR) GO TO 90                                               OSCI0160
      CALL TRANSF                                                    OSCI0170
      CALL GAMAN                                                     OSCI0180
      CALL COEF(OMEG)                                                OSCI0190
      CALL F1ZN                                                      OSCI0200
      CALL FH1N(OMEG)                                                OSCI0210
      CALL FP1ZN                                                     OSCI0220
      CALL QFUNCT                                                    OSCI0230
      CALL FPSLON                                                    OSCI0232
      CALL CMATX                                                     OSCI0233
C ***  DELETE ROW-4 AND COL.-4 ***                                   OSCI0251
      DO 15 J=1,3                                                    OSCI0254
      DO 15 K=1,14                                                   OSCI0252
   15 AMAT(K,J) = CMAT(K,J)                                          OSCI0256
      DO 20 K=1,14                                                   OSCI0270
      DO 20 J=5,14                                                   OSCI0280
   20 AMAT(K,J-1) = CMAT(K,J)                                        OSCI0290
      DO 30 K=5,14                                                   OSCI0300
      DO 30 J=1,13                                                   OSCI0310
   30 AMAT(K-1,J) = AMAT(K,J)                                        OSCI0320
      WRITE(6,40)                                                    OSCI0344
   40 FORMAT(1H1,45X,18HDETERMINANT MATRIX)                          OSCI0345
      DO 45 K=1,13                                                   OSCI0350
   45 WRITE(6,50) K, (AMAT(K,J),J=1,13)                              OSCI0360
   50 FORMAT(1H0,1X,4HROW=, I2, 2X,1P6E16.4/(10X,1P6E16.4))          OSCI0370
```

```
      CW = (5./OMEG)**6                                                 OS390372
      XX = CW**4                                                        OSCI0373
      SF3 = 1.0E-6 *CW                                                  OSCI0374
      SF2 = 1.0E-4 * CW                                                 OSCI0376
      SF4 = 1.0E-3 * CW                                                 OSCI0378
      DO 52 J=1,13                                                      OSCI0380
      AMAT(2,J) = SF2 * AMAT(2,J)                                       OSCI0382
      AMAT(3,J) = SF3 * AMAT(3,J)                                       OSCI0384
      AMAT(4,J) = SF4 * AMAT(4,J)                                       OSCI0386
   52 AMAT(5,J) = SF2 * AMAT(5,J)                                       OSCI0388
      DO 54 K = 1,10                                                    OSCI0389
      DO 54 J = 1,13                                                    OSCI0390
   54 BMAT(K,J) = AMAT(K,J)                                             OSCI0391
      DO 55 K=12,13                                                     OSCI0392
      DO 55 J = 1,13                                                    OSCI0393
   55 BMAT(K-1,J) = AMAT(K,J)                                           OSCI0394
      KK = IDETRM(14,13,AMAT,F)                                         OSCI0397
      IF(KK.EQ.2) GO TO 110                                             OSCI0398
      WRITE(6,60) OMEG,F                                                OSCI0410
   60 FORMAT(1H-,10X, 8HOMEGA = ,1PE13.6, 10X,20HDETERMINANT VALUE = ,1POSCI0420
     1E13.6)                                                            OSCI0430
      WRITE(6,65) SF2, SF3, SF4, XX                                     OSCI0434
   65 FORMAT(1H-,2X,6HSF2 = ,1PE13.6,5X,6HSF3 = ,1PE13.6,  5X, 6HSF4 = ,OSCI0436
     1 1PE13.6,3X,8HCW**4 = , 1PE13.6)                                  OSCI0438
      CALL SHAPE(OMEG)                                                  OSCI0439
      GO TO 100                                                         OSCI0440
   90 WRITE(6,95)                                                       OSCI0490
   95 FORMAT(1H-,10X,32H*** NO SOLUTIONS FOR BETA(I) ***)               OSCI0500
  100 CONTINUE                                                          OSCI0510
      GO TO 1                                                           OSCI0520
  110 WRITE(6,120)                                                      OSCI0525
  120 FORMAT(1H-,5X,40HOVERFLOW OCCURED CALCULATING DETERMINANT)        OSCI0526
      GO TO 1                                                           OSCI0527
      END                                                               OSCI0530
$*                                                                      OSCI0540
```

```
$IBFTC QFUNC.  SDD                                                      QFUNC000
C **** COMPUTE Q FUNCTION ***                                           QFUNC005
      SUBROUTINE QFUNCT                                                 QFUNC010
      COMMON/DAT1/RADIUS,HTCON,AL,YOUNG,HTCY,RHO,RHOPRO,POISON,ALPHA,D, QFUNC020
     1 THICK                                                            QFUNC030
      COMMON/TRIG/ SINZ(5,11),CSINZ(5,11), SINHZ(5,11), CSINHZ(5,11),   QFUNC032
     1 CSINBN(5), CSINHB(5), SINHB(5)                                   QFUNC034
      COMMON/BESSEL/ J1AN(5), J0AN(5), I0BN(5),J0ANR(5,11)              QFUNC036
      COMMON/EL/ AL11,AL12,AL13,AL14,ALK11,ALK12,ALK21,ALK22,WK1,ALK0,  QFUNC040
     1ALC5,ALC6                                                         QFUNC042
      COMMON/BAR/HBAR, BBAR,CAP,G11,E10,G12,RHOBAR                      QFUNC050
      COMMON/DAT2/ NITER,N,NSEG                                         QFUNC060
      COMMON/F1/ F1Z(24,11)                                             QFUNC070
      COMMON/H1/ FH1(8,6)                                               QFUNC080
      COMMON/FPZ/ FP1Z(24,11)                                           QFUNC090
      COMMON/F56Z/ F5Z(6,11), F6Z(6,11)                                 QFUNC100
      COMMON/DAT3/ ALPHAN(5), B(5),U(5)                                 QFUNC105
      REAL J0ANR                                                        QFUNC117
      COMMON/Q/ Q1(12,11), Q2(12,11),Q3(12,11), Q4(6,11)                QFUNC110
      COMMON/PLDR/ PO(6),PA(6),PZ(6,11)                                 QFUNC112
      COMMON/DELTA/ DZBH(11),DZS(11),  DSL(11),DR(11),DZB(11),DS(11)    QFUNC113
      DIMENSION P(10), SBAR(11), BESI(5,11)                             QFUNC115
      IF(ALPHA.GT.89.) GO TO 2                                          QFUNC118
      BX = (SIND(ALPHA) / COSD(ALPHA))**2                               QFUNC119
      GO TO 4                                                           QFUNC120
    2 BX = 0.                                                           QFUNC124
    4 N2 = N+1                                                          QFUNC126
      NSEG2 = NSEG + 1                                                  QFUNC130
      DENOM = 1. - WK1                                                  QFUNC140
      A =  POISON * ALK0  / DENOM                                       QFUNC150
      B = POISON * G12 / DENOM                                          QFUNC160
      C = (1. + POISON * G12) / DENOM                                   QFUNC170
      QX = POISON * RHOBAR / DENOM                                      QFUNC171
      DELZ = BBAR / FLOAT(NSEG)                                         QFUNC172
      DZ = 0.                                                           QFUNC174
      DO 25 J=1,NSEG2                                                   QFUNC176
      RAD = G11 * DZ                                                    QFUNC178
      Q3(1,J) = -G11 * SIN(RAD)                                         QFUNC180
      Q3(2,J) = G11 * COS(RAD)                                          QFUNC182
   25 DZ = DZ + DELZ                                                    QFUNC184
      K = 1                                                             QFUNC188
      K1 = 1                                                            QFUNC189
      DO 50 I=1,N2                                                      QFUNC190
```

```
      X = ALK11 * FH1(1,I)                                              QFUNC200
      Y = ALK12 * FH1(3,I)                                              QFUNC210
      Q11 = A * ( X + Y + ALK12 * X +  ALK12 * Y)                       QFUNC220
      X2 = ALK11 * FH1(2,I)                                             QFUNC230
      Y2 = ALK12 * FH1(4,I)                                             QFUNC240
      Q12 = A * (X2 + Y2 + ALK12 * X2 + ALK12 * Y2)                     QFUNC250
      Q15 = RHOBAR * A * ALK12                                          QFUNC260
      Q13 = ALK21 * FH1(5,I) + ALK22 * FH1(7,I)                         QFUNC280
      Q14 = ALK21 * FH1(6,I) + ALK22 * FH1(8,I)                         QFUNC290
      DO 40 J =1,NSEG2                                                  QFUNC300
      Q1(K,J) = RHOBAR * (Q11 + B * (AL13 * F1Z(K,J) + AL14 * F1Z(K+2,J)QFUNC310
     1)+ C * FP1Z(K,J))                                                 QFUNC320
      Q1(K+1,J) = RHOBAR * (Q12 + B * (AL13 * F1Z(K+1,J) + AL14 * F1Z(K+QFUNC330
     13,J)) + C * FP1Z(K+1,J))                                          QFUNC340
      Q1(K+2,J) = Q13 * Q15                                             QFUNC350
      Q1(K+3,J) = Q14 * Q15                                             QFUNC360
      X5 = ALC5 * Q3(2,J)                                               QFUNC370
      X6 = ALC6 * Q3(2,J)                                               QFUNC371
      Q2(K,J) = -QX * X5 * (X + Y)                                      QFUNC380
      Q2(K+1,J) = -QX * X5 * (X2 + Y2)                                  QFUNC390
      Q2(K+2,J) = -QX *((Q3(1,J) - X6) * FH1(5,I) + X5 * FH1(7,I) + X5 *QFUNC400
     1 Q13) + B * RHOBAR * (Q3(1,J) *F5Z(K1,J) + Q3(2,J)*F6Z(K1,J)*G11)+QFUNC410
     2 C * RHOBAR * FP1Z(K+2,J)                                         QFUNC420
   40 Q2(K+3,J) = -QX * ((Q3(1,J) - X6) * FH1(5,I) + X5 * FH1(8,I) + X5 QFUNC430
     1* Q14) + B * RHOBAR * (Q3(1,J)*F5Z(K1+1,J)+ Q3(2,J)*F6Z(K1+1,J)*G1QFUNC440
     21)+C*RHOBAR * FP1Z(K+3,J)                                         QFUNC450
      K1 = K1 + 2                                                       QFUNC455
   50 K = K + 4                                                         QFUNC460
      DO 90 I=1,N                                                       QFUNC510
      DO 90 J=1,NSEG2                                                   QFUNC520
      X = B(I) * DR(J)                                                  QFUNC530
   90 BFSI(I,J) = BFSFL(X,0,2)                                          QFUNC540
      DO 95 J=1,NSEG2                                                   QFUNC550
      Q3(1,J) = E10 + BBAR                                              QFUNC560
      Q3(2,J) = 0.                                                      QFUNC570
      Q3(3,J) = 1.                                                      QFUNC580
      P(1) = -.5                                                        QFUNC586
      P(2) = .375                                                       QFUNC587
   95 Q3(4,J) = 0.                                                      QFUNC590
      K = 5                                                             QFUNC600
      DO 100 I=1,N                                                      QFUNC610
      NCAP = 2 * I                                                      QFUNC615
      DO 97 J=1,NSEG2                                                   QFUNC620
```

```
      Q3(K,J) =(SINHB(I) + CSINHB(I)) * JOANB(I,J)                      QFUNC630
      Q3(K+1,J) = BESI(I,J) / CSINBN(I)                                 QFUNC640
      Q3(K+2,J) = DR(J)**NCAP * P(I)                                    QFUNC650
   97 Q3(K+3,J) = 0.                                                    QFUNC660
  100 K = K + 4                                                         QFUNC670
      RBX = RHOBAR * BX                                                 QFUNC672
      DO 102 J=1,NSEG2                                                  QFUNC674
      Q4(1,J) = -RBX * PA(1) * DSL(J)**2                                QFUNC676
  102 Q4(2,J) = -RBX * PA(2) * DSL(J)**3                                QFUNC678
      K = 3                                                             QFUNC680
      DO 110 I=1,N                                                      QFUNC690
      NCAP = 2 * I                                                      QFUNC700
      NBAR = 2 * I + 1                                                  QFUNC710
      DO 105 J=1,NSEG2                                                  QFUNC720
      Q4(K,J) = -RBX * PA(NBAR) * DSL(J)**(NCAP+2)                      QFUNC730
  105 Q4(K+1,J) = -RBX * PA(NBAR+1) *DSL(J)**(NBAR+2)                   QFUNC740
  110 K = K + 2                                                         QFUNC750
      RETURN                                                            QFUNC800
      END                                                               QFUNC802
$*                                                                      QFUNC999
```

```
$IBFTC COEF.   SDD                                                      COEF0001
      SUBROUTINE COEF(OMEGA)                                            COEF0002
      COMMON/DAT1/ RADIUS,HTCON,AL,YOUNG,HTCY,RHO,RHOPRO,POISON,ALPHA,D,COEF0005
     1 THICK                                                            COEF0010
      COMMON/EL/ AL11,AL12,AL13,AL14,ALK11,ALK12,ALK21,ALK22,WK1,ALK0,  COEF0015
     1 ALC5, ALC6                                                       COEF0016
      COMMON/BAR/ HBAR,BBAR,CAP,G11,E10,G12, RHOBAR                     COEF0017
      COMMON/BK/ BK1                                                    COEF0018
      OMEG2= OMEGA**2                                                   COEF0019
      E10 = (386.089/OMEG2 - HTCY) / RADIUS                             COEF0020
      BK1 = RHO * OMEG2 * RADIUS**3 /D                                  COEF0022
      WK1 = CAP * OMEG2                                                 COEF0025
      DENOM = 1. - POISON**2 - WK1                                      COEF0030
      G11 =SQRT((WK1 - WK1**2) / DENOM)                                 COEF0035
      G12 = POISON / DENOM                                              COEF0040
      X = G11 * HBAR                                                    COEF0045
      X2 =(1. - POISON**2/(1.- WK1)) * G11                              CO56005
      AL11 = -X2 * SIN(X)                                               COEF0055
      AL12 =  X2 * COS(X)                                               COEF0060
      X = G11 * BBAR                                                    COEF0065
      AL13 = -G11 * SIN(X)                                              COEF0070
      AL14 = G11 * COS(X)                                               COEF0075
      DENOM = AL11 * AL14 - AL12 * AL13                                 COEF0080
      ALC1 = AL11/ DENOM                                                COEF0085
      ALC2 = AL12 / DENOM                                               COEF0090
      ALC3 = AL13 / DENOM                                               COEF0095
      ALC4 = AL14 / DENOM                                               COEF0100
      ALC5 = 1./ AL14                                                   COEF0105
      ALC6 = AL13 / AL14                                                COEF0110
      ALK0 = 1.                                                         COEF0120
      X1 = COS(G11 * BBAR)                                              COEF0122
      X2 = COS(G11 * HBAR)                                              COEF0124
      X = G11 * (HBAR-BBAR)                                             COEF0126
      X3 = COS(X)                                                       COEF0128
      X4 = SIN(X)                                                       COEF0130
      DN = 1. - POISON**2 / (1. - WK1)                                  COEF0132
      ALK11 = X1 / (DN * X2)                                            COEF0134
      ALK12 = -X1 * X3 / X2                                             COEF0136
      ALK21 = G11 * X4 / X2                                             COEF0138
      ALK22 = SIN(G11*BBAR) * X4 / X2                                   COEF0140
      RETURN                                                            COEF0150
      END                                                               COEF0160
$*
```

```
$IBFTC FP1ZN,  SDD                                                      FP1ZN000
      SUBROUTINE FP1ZN                                                  FP1ZN005
      COMMON/BAR/ HBAR,BBAR,CAP,G11,E10,G12, RHOBAR                     FP1ZN010
      COMMON/DAT2/ NITER, N, NSEG                                       FP1ZN015
      COMMON/FPZ/ FP1Z(24,11)                                           FP1ZN020
      REAL J1AN, J0AN, I0BN, I1BN,J0ANR                                 FP1ZN024
      COMMON/TRIG/ SINZ(5,11), CSINZ(5,11),SINHZ(5,11), CSINHZ(5,11),   FP1ZN025
     1 CSINBN(5), CSINHB(5), SINHB(5)                                   FP1ZN026
      COMMON/BESSEL/ J1AN(5),J0AN(5),  I0BN(5),I1BN(5),J0ANR(5,11)      FP1ZN027
      COMMON/PLDR/ PC(6),PA(6),PZ(6,11)                                 FP1ZN030
      COMMON/DELTA/DZBH(11),DZS(11),   DSL(11), DR(11),DZB(11),DS(11)   FP1ZN035
      COMMON/LIMIT/N2,NSEG2                                             FP1ZN040
      DO 10 I=1,NSEG2                                                   FP1ZN082
      FP1Z(1,I) = E10 + DZBH(I)                                         FP1ZN086
      FP1Z(2,I) = 0.                                                    FP1ZN086
      FP1Z(3,I) = PZ(1,I)                                               FP1ZN088
   10 FP1Z(4,I) = PZ(2,I) * DZS(I)**2                                   FP1ZN090
      K = 1                                                             FP1ZN100
      DO 20 I=1,N                                                       FP1ZN110
      NCAP = 2 * I                                                      FP1ZN120
      NBAR = 2 * I + 1                                                  FP1ZN130
      K = K + 4                                                         FP1ZN140
      A = I0BN(I) / CSINBN(I)                                           FP1ZN150
      DO 20 J=1,NSEG2                                                   FP1ZN160
      FP1Z(K,J) = J0AN(I) * (SINHZ(I,J) + CSINHZ(I,J))                  FP1ZN170
      FP1Z(K+1,J) = A * CSINZ(I,J)                                      FP1ZN180
      XX = DZS(J)**2                                                    FP1ZN185
      FP1Z(K+2,J) = PZ(I+1,J) * XX**NCAP                                FP1ZN190
   20 FP1Z(K+3,J) = PZ(I+2,J) * XX**NBAR                                FP1ZN200
      RETURN                                                            FP1ZN210
      END                                                               FP1ZN220
$*                                                                      FP1ZN999
```

```
$IBFTC EPSLN    SDD                                                     EPSLN000
      SUBROUTINE EPSLON                                                 EPSLN005
      COMMON/DAT2/ NITER, N, NSEG                                       EPSLN010
      COMMON/GAMA/ GAMA1(9,11), GAMA2(12,11), GAMA3(9,11)               EPSLN015
      COMMON/Q/ Q1(12,11), Q2(12,11),  Q3(12,11), Q4(6,11)              EPSLN020
      COMMON/BK/ BK1                                                    EPSLN025
      COMMON/BAR/ HBAR, BBAR,CAP, G11, E10, G12,RHOBAR                  EPSLN030
      COMMON/EPS/ E11(5,11), E12(5,11), E13(5,11),E13B(5,11), E21(5,11),EPSLN035
     1 E23(5,11),E31(5,11),  E32(5,11), E33(5,11), E43(5,11),E43B(5,11),EPSLN040
     2 E51(5,11), E52(5,11),E53(5,11),  E53B(5,11)                      EPSLN045
      COMMON/SK/SKZ(11), OKZ(11)                                        EPSLN050
      NSEG2 = NSEG + 1                                                  EPSLN060
      DELZ = BBAR / FLOAT(NSEG)                                         EPSLN070
      DZ = 0.                                                           EPSLN080
      DO 10 I=1,NSEG2                                                   EPSLN090
      X = BBAR - DZ                                                     EPSLN100
      SKZ(I) = 1. / SQRT(1. + X**2)                                     EPSLN110
      OKZ(I) = X * SKZ(I)                                               EPSLN120
   10 DZ = DZ + DELZ                                                    EPSLN130
      K2 = 1                                                            EPSLN140
      K3 = 1                                                            EPSLN145
      K = 1                                                             EPSLN150
      N2 = N + 1                                                        EPSLN160
      DO 30 I=1,N2                                                      EPSLN170
      DO 20 J=1,NSEG2                                                   EPSLN180
      E11(I,J) = GAMA1(K,J) - BK1 * Q1(K2,J)                            EPSLN190
      E12(I,J) = GAMA1(K+1,J) - BK1 *  Q1(K2+1,J)                       EPSLN200
      E13(I,J) =  -BK1* Q1(K2+2,J)                                      EPSLN210
      E21(I,J) = GAMA1(K+2,J)                                           EPSLN215
      E13B(I,J) = -BK1 * Q1(K2+3,J)                                     EPSLN220
      E23(I,J) = -GAMA3(K+2,J)                                          EPSLN230
      E31(I,J)=  Q3(K2,J)                                               EPSLN240
      E32(I,J) = Q3(K2+1,J)                                             EPSLN250
      E33(I,J) = -Q3(K2+2,J)                                            EPSLN260
      E51(I,J) = -BK1 * Q2(K2,J)                                        EPSLN270
      E52(I,J) = -BK1 * Q2(K2+1,J)                                      EPSLN280
      E53(I,J) = SKZ(J) * GAMA2(K2,J) + OKZ(J) * GAMA2(K2+2,J) - BK1 *  EPSLN290
     1 Q2(K2+2,J)                                                       EPSLN300
      E53B(I,J) = SKZ(J) * GAMA2(K2+1,J) + OKZ(J) * GAMA2(K2+3,J) - BK1 EPSLN310
     1 * Q2(K2+3,J)                                                     EPSLN320
      E43(I,J) = GAMA3(K,J) - BK1 * Q4(K3,J)                            EPSLN330
   20 E43B(I,J) = GAMA3(K+1,J) - BK1 * Q4(K3+1,J)                       EPSLN340
      K = K+3                                                           EPSLN350
```

```
      K2 = K2+4                                          EPSLN360
   30 K3 = K3+2                                          EPSLN370
      RETURN                                             EPSLN400
      END                                                EPSLN500
```

```
$IBFTC SHAPE.                                                           SHAPE001
C***   ROUTINE CALCULATE AND PLOTS MODESHAPES ***                       SHAPE010
      SUBROUTINE SHAPE(OMEG)                                            SHAPE020
      COMMON/DMAT/ CMAT(14,14)                                          SHAPE022
      COMMON/SK/ SKZ(11),OKZ(11)                                        SHAPE030
      COMMON/GAMA/ GAMA1(9,11), GAMA2(12,11), GAMA3(9,11)               SHAPE030
      COMMON/BK/ BK1                                                    SHAPE032
      COMMON/LIMIT/ N2,NSEG2                                            SHAPE040
      COMMON/DAT3/ ALPHAN(5),B(5),U(5)                                  SHAPE041
      COMMON/PLDR/ PO(6),PA(6),PZ(6,11)                                 SHAPE042
      COMMON/DELTA/ DZBH(11),DZS(11),DSL(11),DR(11),DZB(11), DS(11)     SHAPE043
      COMMON/TRIG/ SINZ(5,11),CSINZ(5,11), SINHZ(5,11), CSINHZ(5,11),   SHAPE044
     1 CSINBN(5), CSINHB(5), SINHB(5)                                   SHAPE045
      COMMON/BESSEL/ J1AN(5),J0AN(5),I0BN(5), I1BN(5), J0ANR(5,11)      SHAPE046
      COMMON/BAR/ HBAR,BBAR,CAP,G11,F10,G12,RHOBAR                      SHAPE047
      COMMON/DAT1/RADIUS,HTCON,AL,YOUNG, HTCY, RHO,RHOPRO,POISON, ALPHA,SHAPE048
     1 D, THICK                                                         SHAPE049
      COMMON/Q/ Q1(12,11), Q2(12,11),  Q3(12,11), Q4(6,11)              SHAPE050
      COMMON/SH/ BMAT(14,14)                                            SHAPE055
      CALL SETMIV(48,48,48,48)                                          SHAPE057
      DIMENSION Y(30), X(30), DPHI(30)                                  SHAPE058
      DIMENSION  C(12),XA(12), A(12,12)                                 SHAPE059
      REAL J1AN, J0AN, I0BN,J0ANR                                       SHAPE061
      CPS = OMEG / 6.283185                                             SHAPE062
      WRITE(6,5) OMEG, CPS                                              SHAPE064
    5 FORMAT(1H1,10X,12HFREQUENCY = ,1PE13.6,10H RAD./SEC.,10X,12HFREQUESHAPE066
     1NCY = ,1PE13.6,7H C.P.S.)                                         SHAPE068
      NN = 4 * N2                                                       SHAPE070
      DO 10 I=1,NN                                                      SHAPE080
      C(I) = -BMAT(I,1)                                                 SHAPE090
      DO 10 J=1,NN                                                      SHAPE095
   10 A(I,J) = BMAT(I,J+1)                                              SHAPE099
C                                                                       SHAPE110
C     COMPUTE A(N),B(N),C(N)                                            SHAPE120
      K = ISIMEQ(12,NN,1,A,C,0,XA)                                      SHAPE130
      IF(K-2) 20,100,120                                                SHAPE140
C***   STORE A(N), B(N), C(NCAP), C(NBAR), IN CMAT BY COLUMNS  ***      SHAPE150
   20 CMAT(1,1) = 1.                                                    SHAPE160
      CMAT(1,2) = 0.                                                    SHAPE170
      N = N2 - 1                                                        SHAPE180
      DO 25 I=1,N                                                       SHAPE190
      CMAT(I+1,1) = A(I,1)                                              SHAPE200
      K = N + I                                                         SHAPE210
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   25 CMAT(I+1,2) = A(K,1)                                              SHAPE220
      K = 2 * N                                                         SHAPE230
      K1 = K + N2                                                       SHAPE240
      D O 30 I=1,N2                                                     SHAPE250
      J = K + I                                                         SHAPE260
      CMAT(I,3) = A(J,1)                                                SHAPE270
      J = K1 + I                                                        SHAPE280
   30 CMAT(I,4) = A(J,1)                                                SHAPE290
C****  STORE MODE SHAPE W12 IN FIRST ROW OF MATRIX A ****              SHAPE300
C                                                                       SHAPE305
C ***  STORE MODE SHAPES OF W11 IN THE SECOND ROW OF MATRIX A ***       SHAPE307
      DO 50 J=1,NSEG2                                                   SHAPE310
      K = 1                                                             SHAPE310
      SUM = 0.                                                          SHAPE320
      SUM2 = 0.                                                         SHAPE330
      DO 40 I=1,N2                                                      SHAPE340
      SUM = Q2(K,J) * CMAT(I,1) + Q2(K+1,J) * CMAT(I,2) + Q2(K+2,J) * CMSHAPE350
     1AT(I,3) + Q2(K+3,J) * CMAT(I,4) + SUM                             SHAPE360
      SUM2 = Q1(K,J) * CMAT(I,1) + Q1(K+1,J) * CMAT(I,2) + Q1(K+2,J) * CSHAPE370
     1MAT(I,3) + Q1(K+3,J) * CMAT(I,4) + SUM2                           SHAPE380
   40 K = K + 4                                                         SHAPE390
      A(1,J) = SUM                                                      SHAPE400
   50 A(2,J) = SUM2                                                     SHAPE410
C                                                                       SHAPE420
C ***  STORE MODE SHAPES OF W2 IN THE THIRD ROW OF MATRIX A ***         SHAPE430
      DO 70 J=1,NSEG2                                                   SHAPE440
      K = 1                                                             SHAPE450
      SUM = 0.                                                          SHAPE460
      DO 60 I=1,N2                                                      SHAPE470
      SUM = Q4(K,J) * CMAT(I,3) + Q4(K+1,J) * CMAT(I,4) + SUM           SHAPE480
   60 K = K+2                                                           SHAPE490
   70 A(3,J) = SUM                                                      SHAPE500
      WRITE(6,75) (I,CMAT(I,1),I,CMAT(I,2),I,CMAT(I,3),I,CMAT(I,4),I=1,NSHAPE510
     12)                                                                SHAPE520
   75 FORMAT(1H-/ (1H0,10X,2HA(,I1,4H) = ,1PE13.6/ 1H0,10X,2HB(,I1,4H) =SHAPE530
     1 ,1PE13.6/ 1H0,10X,2HC(,I1,4H) = ,1PE13.6/ 1H0,10X,5HCBAR(,I1,4H) SHAPE540
     2= ,1PE13.6))                                                      SHAPE550
      WRITE(6,80) (A(1,J),J=1,NSEG2)                                    SHAPE560
   80 FORMAT(1H1,45X,14HW12 MODE SHAPE/ (1H0,3X,1P6E17.5))              SHAPE570
      WRITE(6,85) (A(2,J),J=1,NSEG2)                                    SHAPE580
   85 FORMAT(1H-,45X,14HW11 MODE SHAPE/ (1H0,3X,1P6E17.5))              SHAPE590
      WRITE(6,90) (A(3,J),J=1,NSEG2)                                    SHAPE600
   90 FORMAT(1H-,45X,13HW2 MODE SHAPE/ (1H0,3X,1P6E17.5))               SHAPE601
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```
      DO 92 J= 1,NSEG2                                                  SHAPE602
      C(J) = A(3,J)                                                     SHAPE603
      X(J) = DZB(J)                                                     SHAPE604
      DPHI(J)=((GAMA2(3,J) * CMAT(1,3) + GAMA2(4,J) * CMAT(1,4) + GAMA2(SHAPE605
     17,J) * CMAT(2,3) + GAMA2(8,J) * CMAT(2,4) + GAMA2(11,J) * CMAT(3,3SHAPE606
     2) + GAMA2(12,J) * CMAT(3,4)) * OKZ(J) + (GAMA2(1,J) * CMAT(1,3)   SHAPE607
     3+ GAMA2(2,J) * CMAT(1,4) + GAMA2(5,J) * CMAT(2,3) + GAMA2(6,J) * CSHAPE608
     4MAT(2,4) + GAMA2(9,J) * CMAT(3,3) + GAMA2(10,J) * CMAT(3,4))* SKZ(SHAPE609
     5J)) / BK1                                                         SHAPE610
   92 Y(J) = A(1,J)                                                     SHAPE612
      K = NSEG2                                                         SHAPE614
      DO 93 J=2,NSEG2                                                   SHAPE616
      K = K + 1                                                         SHAPE618
      X(K) = DZBH(J)                                                    SHAPE620
      DPHI(K) =(GAMA1(1,J) + GAMA1(4,J) * CMAT(2,1) + GAMA1(7,J) * CMAT(SHAPE622
     13,1) + GAMA1(5,J) * CMAT(2,2) +  GAMA1(8,J) * CMAT(3,2)) / BK1    SHAPE623
   93 Y(K) = A(2,J)                                                     SHAPE624
      WRITE(6,94) (DPHI(I),I=1,21)                                      SHAPE626
   94 FORMAT(1H1,40X, 5HPHI/R/ (1H0,3X,6E17.5))                         SHAPE628
      CALL GRAPH(3,42,-21,X,Y,1H ,1H ,1H )                              SHAPE642
      DO 99 J=1,NSEG2                                                   SHAPE650
   99 DPHI(J) = (GAMA3(1,J) * CMAT(1,3) + GAMA3(2,J) * CMAT(1,4) + GAMA3SHAPE655
     1(4,J) * CMAT(2,3) + GAMA3(5,J) * CMAT(2,4) + GAMA3(7,J) * CMAT(3,3SHAPE660
     2) +GAMA3(8,J) * CMAT(3,4)) / BK1                                  SHAPE665
      CALL GRAPH(3,42,-11,DSL,C,1H ,1H  ,1H )                           SHAPE710
C ***  COMPUTE PRESSURE DISTRIBUTION AND PLOT ON CRT ***                SHAPE822
      X1 = CMAT(2,1) * (SINHB(1) + CSINHB(1))                           SHAPE824
      X2 = CMAT(3,1) * (SINHB(2) + CSINHB(2))                           SHAPE825
      GAW = 386.089 / (RADIUS * OMEG**2) - HBAR + BBAR                  SHAPE826
      DO 95 I=1,11                                                      SHAPE827
      Y(I) = X1 * JOANR(1,I) + X2 * JOANR(2,I) + CMAT(2,2) * BESEL(B(1)*SHAPE828
     1DR(I),0,2) / CSINBN(1)  + CMAT(3,2) * BESEL(B(2)*DR(I),0,2) / CSINSHAPE829
     2BN(2) + GAW                                                       SHAPE830
   95 C(I) = CMAT(1,3) + CMAT(2,3) * PO(3) *  DR(I)**2                  SHAPE831
      CALL GRAPH(3,42,-11,DR,C,1H ,1H ,1H )                             SHAPE839
      WRITE(6,96) (Y(I),I=1,11)                                         SHAPE840
   96 FORMAT(1H1,37X,30HTHETA(1) PRESSURE DISTRIBUTION/ (1H0,3X,1P6E17.5SHAPE841
     1))                                                                SHAPE842
      WRITE(6,97) (C(I),I=1,11)                                         SHAPE844
   97 FORMAT(1H-,37X,30HTHETA(2) PRESSURE DISTRIBUTION/ (1H0,3X,1P6E17.5SHAPE846
     1))                                                                SHAPE848
      RETURN                                                            SHAPE850
  100 WRITE(6,110)                                                      SHAPE852
```

```
110 FORMAT(1H-,10X,53H*** ILL CONDITIONED MATRIX - EXECUTION TERMINATESHAPE854
   1D ***)                                                            SHAPE856
    STOP                                                              SHAPE860
120 WRITE(6,125)                                                      SHAPE870
125 FORMAT(1H-, 10X,46H** MATRIX IS SINGULAR - EXECUTION TERMINATED**)SHAPE880
    STOP                                                              SHAPE890
    END                                                               SHAPE900
```

```
$IBFTC ROOTD*  SDD
C     Z E R O S   O F   A N   A R B I T R A R Y   F U N C T I O N
C     ROOT SEARCHES FOR ZEROS BY SAMPLING F(X) AND FITTING L(X) TO
C     THE SAMPLED VALUES.  L(X) IS USED TO PREDICT F(X).  IF THE
C     PREDICTION IS BAD (SEE REL), DX IS DECREASED.  IF THE PREDICTION
C     IS VERY GOOD, DX IS INCREASED.  WHEN A PREDICTION IS GOOD, X IS
C     INCREASED BY DX.  WHEN F(X) CHANGES SIGN, A ROOT HAS BEEN
C     ISOLATED, AND X IS INTERPOLATED TO AN ACCURACY OF TOL.

C     INPUT   -- X      = START OF SEARCH.
C                DX     = INCREMENT IN X (+ OR -).
C                F      = NAME OF THE ARBITRARY FUNCTION.
C                TOL    = RELATIVE ERROR IN THE ANSWER (X).
C                REL    = MAXIMUM RELATIVE ERROR IN /F(X)-L(X)/.
C                LIMIT  = MAXIMUM NUMBER OF ITERATIONS TRIED.

C     OUTPUT  -- NTRY   = 1, EXACT ROOT FOUND.
C                       = 2, APPROXIMATE ROOT FOUND.
C                       = 3, LOCAL /MINIMUM/ FOUND.
C                       = 4, DISCONTINUITY FOUND.
C                       = 5, LIMIT REACHED (NO ROOT FOUND).
C                X      = ROOT (OR CURRENT VALUE).
C                DX     = ADJUSTED (CURRENT) VALUE.
C                Y      = F(X).
C*********************************************************************
      SUBROUTINE ROOT (NTRY,X,DX,Y,F,TOL,REL,LIMIT)

      EXTERNAL   F
      INTEGER    NTRY,LIMIT
      REAL       TOL,REL,L
C-------
      LOGICAL    SLOW,MED,FAST,BAD,FIRST
C---------
C     LAGRANGIAN POLYNOMIAL

                 L(X) = (X-X1)/(X0-X1)*Y0 + (X-X0)/(X1-X0)*Y1
C*********************************************************************
C     INITIALIZATION

      OLDDX = 0
      RATIO = 2.0
```

```
      FIRST = .TRUE.
      ITER = 1
      X0 = X
      Y0 = F(X0)
      IF (Y0 .EQ. 0.0) GO TO 60
C-------
    9 IF (.NOT.FIRST) GO TO 20
   10 IF (X0+DX .EQ. X0) GO TO 80
      IF (ITER .GE. LIMIT) GO TO 90
      ITER = ITER+1
      X1 = X0+DX
      Y1 = F(X1)
      IF (X1 .EQ. X0) GO TO 80
      DX = X1-X0
      IF (SIGN(1.0,Y0) .EQ. SIGN(1.0,Y1) .AND. Y1 .NE. 0.0) GO TO 20
      DX = DX/4.0
      GO TO 10
C-------
C     SEARCH

   20 IF (X1+DX .EQ. X1) GO TO 80
      IF (ITER .GE. LIMIT) GO TO 90
      ITER = ITER+1
      X = X1+DX
      Y = F(X)
      IF (X .EQ. X1) GO TO 80
      DX = X-X1
   30 ERROR =  ABS(Y-L(X))
      IF (ERROR .NE. 0.0) GO TO 31
      DX = DX*2.0
      GO TO 32
   31 D = (X0-X1)/2.0
      R =  ABS(Y)*(X -X0)*(X -X1)/ERROR
      DX = AMAX1(DX/4.000,AMIN1(DX*2.000,D+ SQRT(D*D+R*REL/2.000)))
      IF (ERROR .GT. REL* ABS(Y)) GO TO 9
   32 IF (Y .EQ. 0.0  ) GO TO 60
      IF (SIGN(1.0,Y) .NE. SIGN(1.0,Y1)) GO TO 46
      IF ( ABS(Y1) .GE.  ABS(Y0) .OR.  ABS(Y) .LT.  ABS(Y1)) GO TO 35
C-------
      IF (OLDDX .EQ. 0.0  ) OLDDX = X-X1
      IF ( ABS(X-X1) .LT. TOL* ABS(X)) GO TO 70
      DX = (X-X1)/4.0
      GO TO 10
```

```
C-------
   35 X0 = X1
      Y0 = Y1
      X1 = X
      Y1 = Y
      FIRST = .FALSE.
      GO TO 20
C-------
C     INTERPOLATE

   40 SLOW = RATIO .LT. 2.0
      RATIO = -SNGL(YY/Y)
      MED = .NOT. SLOW .AND. RATIO .LE. 4.0
      FAST = .NOT. SLOW .AND. RATIO .GT. 4.0
      IF (FAST) X = XX+(X0-X1)*((Y1+Y0)/(Y1-Y0))
      IF (MED) X = XX+(X0-X1)*(YY/(Y1-Y0))
      BAD = X .LE. AMIN1(X0,X1) .OR. X .GE. AMAX1(X0,X1)
      IF (SLOW .OR. BAD) X = (X0+X1)/2.0
      Y = F(X)
      IF (Y .EQ. 0.0  ) GO TO 60
      IF (SIGN(1.0,Y) .EQ. SIGN(1.0,Y0)) GO TO 45
C-------
      RATIO = SNGL(Y1/Y)
      X1 = X
      Y1 = Y
      XX = X0
      YY = Y0
      GO TO 50
C-------
   45 RATIO = SNGL(Y0/Y)
   46 X0 = X
      Y0 = Y
      XX = X1
      YY = Y1
C-------
   50 IF ( ABS(X0-X1) .GT. TOL* ABS(X)) GO TO 40
C-------
      NTRY = 2
      RETURN
   60 NTRY = 1
      RETURN
   70 DX = OLDDX
      NTRY = 3
```

```
      RETURN                                                    00000553
   80 NTRY = 4                                                  00000560
      RETURN                                                    00000570
   90 NTRY = 5                                                  00000580
      RETURN                                                    00000590
      END                                                       00000600
```

```
$IBFTC CMAT.                                                            CMAT0000
      SUBROUTINE CMATX                                                  CMAT0005
      COMMON/Q/ Q1(12,11), Q2(12,11), Q3(12,11), Q4(6,11)               CMAT0009
      COMMON/DMAT/ CMAT(14,14)                                          CMAT0010
      COMMON/DAT1/ RADIUS, HTCON, AL,YOUNG, HTCY, RHO, RHOPRO, POISON,  CMAT0015
     1 ALPHA, D, THICK                                                  CMAT0020
      COMMON/DAT2/ NITER,N, NSEG                                        CMAT0025
      COMMON/FPS/ E11(5,11), E12(5,11), E13(5,11), E13B(5,11),E21(5,11),CMAT0030
     1 E23(5,11),E31(5,11), E32(5,11),  E33(5,11), E43(5,11),E43B(5,11),CMAT0040
     2 E51(5,11), E52(5,11),E53(5,11),  E53B(5,11)                      CMAT0050
      COMMON/BAR/ HBAR,BBAR, CAP, G11, E10, G12, RHOBAR                 CMAT0060
      COMMON/LIMIT/ N2,NSEG2                                            CMAT0067
      COMMON/DELTA/ DZBH(11), DZS(11), DSL(11),DR(11),DZB(11),DS(11)    CMAT0070
      COMMON/DEL/A,B,C,X                                                CMAT0072
      COMMON/TERM/ T(4,11), ISEG,KSEG, NN                               CMAT0065
      KSEG = NSEG2                                                      CMAT0100
      ISEG = NSEG                                                       CMAT0122
      NN = NSEG2 - 3                                                    CMAT0170
C ***  COMPUTE I11 MATRIX ***                                           CMAT0175
      DO 20 I=1,N2                                                      CMAT0180
      DO 20 J=I,N2                                                      CMAT0190
      DO 15 K=1,NSEG2                                                   CMAT0200
      T(1,K) = E11(I,K) * E11(J,K)                                      CMAT0210
      T(2,K) = E21(I,K) * E21(J,K) * DR(K)                              CMAT0220
      T(3,K) = E31(I,K) * E31(J,K) * DR(K)                              CMAT0230
   15 T(4,K) = E51(I,K) * E51(J,K)                                      CMAT0240
   20 CMAT(I,J) = SRULE(4,A,X,X,B)                                      CMAT0250
C ***  COMPUTE I12 MATRIX ***                                           CMAT0255
      DO 30 I=1,N2                                                      CMAT0260
      DO 30 J=1,N2                                                      CMAT0270
      L = N2 + J                                                        CMAT0280
      DO 25 K=1,NSEG2                                                   CMAT0290
      T(1,K) = E11(I,K) * E12(J,K)                                      CMAT0300
      T(2,K) = E31(I,K) * E32(J,K) * DR(K)                              CMAT0310
   25 T(3,K) = E51(I,K) * E52(J,K)                                      CMAT0320
   30 CMAT(I,L) = SRULE(3,A,X,B)                                        CMAT0330
C ***  COMPUTE I13(CAP N) MATRIX ***                                    CMAT0340
      N4 = 2 * N2                                                       CMAT0350
      DO 40 I=1,N2                                                      CMAT0360
      DO 40 J=1,N2                                                      CMAT0370
      L = N4 + J                                                        CMAT0380
      DO 35 K=1,NSEG2                                                   CMAT0390
      T(1,K) = E11(I,K) * E13(J,K)                                      CMAT0400
```

```
      T(2,K) = E31(I,K) * E33(J,K) * DR(K)                              CMAT0410
   35 T(3,K) = E51(I,K) * E53(J,K)                                      CMAT0420
   40 CMAT(I,L) = SRULE(3,A,X,B)                                        CMAT0430
C ***  COMPUTE I13(BAR N) MATRIX ***                                    CMAT0440
      N5 = N4 + N2                                                      CMAT0450
      DO 50 I=1,N2                                                      CMAT0460
      DO 50 J=1,N2                                                      CMAT0470
      L = N5 + J                                                        CMAT0480
      DO 45 K = 1,NSEG2                                                 CMAT0490
      T(1,K) = E11(I,K) * E13B(J,K)                                     CMAT0500
      T(2,K) = E21(I,K) * E23(J,K) * DR(K)                              CMAT0510
   45 T(3,K) = E51(I,K) * E53B(J,K)                                     CMAT0520
   50 CMAT(I,L) = SRULE(3,A,X,B)                                        CMAT0530
C *** COMPUTE I22 MATRIX ***                                            CMAT0540
      DO 60 I=1,N2                                                      CMAT0550
      L1 = N2 + I                                                       CMAT0560
      DO 60 J=I,N2                                                      CMAT0570
      L2 = N2 + J                                                       CMAT0580
      DO 55 K=1,NSEG2                                                   CMAT0590
      T(1,K) = E12(I,K) * E12(J,K)                                      CMAT0600
      T(2,K) = E32(I,K) * E32(J,K) * DR(K)                              CMAT0610
   55 T(3,K) = E52(I,K) * E52(J,K)                                      CMAT0620
   60 CMAT(L1,L2) = SRULE(3,A,X,B)                                      CMAT0630
C ***  COMPUTE I23(CAP N) MATRIX                                        CMAT0640
      DO 70 I=1,N2                                                      CMAT0650
      L1 = N2 + I                                                       CMAT0660
      DO 70 J=1,N2                                                      CMAT0670
      L2 = N4 + J                                                       CMAT0680
      DO 65 K=1,NSEG2                                                   CMAT0690
      T(1,K) = E12(I,K) * E13(J,K)                                      CMAT0700
      T(2,K) = E32(I,K) * E33(J,K) * DR(K)                              CMAT0710
   65 T(3,K) = E52(I,K) * E53(J,K)                                      CMAT0720
   70 CMAT(L1,L2) = SRULE(3,A,X,B)                                      CMAT0730
C ***  COMPUTE I23(BAR N) MATRIX ***                                    CMAT0740
      DO 80 I=1,N2                                                      CMAT0750
      L1 = N2 + I                                                       CMAT0760
      DO 80 J=1,N2                                                      CMAT0770
      L2 = N5 + J                                                       CMAT0780
      DO 75 K =1,NSEG2                                                  CMAT0790
      T(1,K) = E12(I,K) * E13B(J,K)                                     CMAT0800
      T(2,K) = E52(I,K) * E53B(J,K)                                     CMAT0810
   75 T(3,K) = 0.                                                       CMAT0820
   80 CMAT(L1,L2) = SRULE(3,A,X,B)                                      CMAT0830
```

```
C ***  COMPUTE I33(CAP N) MATRIX ***                                CMAT0840
      DO 90 I=1,N2                                                  CMAT0850
      L1 = N4 + I                                                   CMAT0860
      DO 90 J=I,N2                                                  CMAT0870
      L2 = N4 + J                                                   CMAT0880
      DO 85 K=1,NSEG2                                               CMAT0890
      T(1,K) = E13(I,K) * E13(J,K)                                  CMAT0900
      T(2,K) = E33(I,K) * E33(J,K) * DR(K)                          CMAT0910
      T(3,K) = E43(I,K) * E43(J,K) * DS(K)                          CMAT0920
   85 T(4,K) = E53(I,K) * E53(J,K)                                  CMAT0930
   90 CMAT(L1,L2) = SRULE(4,A,X,C,B)                                CMAT0940
C ***  COMPUTE I33(BAR N) MATRIX ***                                CMAT0950
      DO 100 I=1,N2                                                 CMAT0960
      L1 = N4 + I                                                   CMAT0970
      DO 100 J=1,N2                                                 CMAT0980
      L2 = N5 + J                                                   CMAT0990
      DO 95  K=1,NSEG2                                              CMAT0995
      T(1,K) = E13(I,K) * E13B(J,K)                                 CMAT1000
      T(2,K) = E43(I,K) * E43B(J,K) *  DS(K)                        CMAT1010
   95 T(3,K) = E53(I,K) * E53B(J,K)                                 CMAT1020
  100 CMAT(L1,L2) = SRULE(3,A,C,B)                                  CMAT1030
C ***  COMPUTE I33(BAR N) MATRIX ***                                CMAT1035
      DO 110 I=1,N2                                                 CMAT1040
      L1 = N5 + I                                                   CMAT1050
      DO 110 J=I,N2                                                 CMAT1060
      L2 = N5 + J                                                   CMAT1070
      DO 105 K=1,NSEG2                                              CMAT1080
      T(1,K) = E13B(I,K) * E13B(J,K)                                CMAT1090
      T(2,K) = E23(I,K) * E23(J,K) * DR(K)                          CMAT1100
      T(3,K) = E43B(I,K) * E43B(J,K) *  DS(K)                       CMAT1110
  105 T(4,K) = E53B(I,K) * E53B(J,K)                                CMAT1120
  110 CMAT(L1,L2) = SRULE(4,A,X,C,B)                                CMAT1130
      NE = 4 * (N+1)                                                CMAT1140
      DO 120 I=1,NE                                                 CMAT1150
      DO 120 J=I,NE                                                 CMAT1160
  120 CMAT(J,I) = CMAT(I,J)                                         CMAT1170
      K = 0.                                                        CMAT1235
      DO 53 IN=1,4                                                  CMAT1236
      L = IN                                                        CMAT1237
      DO 53 J=1,3                                                   CMAT1238
      K = K + 1                                                     CMAT1239
      CMAT(K,13) = Q1(L,11)                                         CMAT1240
      CMAT(K,14) =(Q1(L,1) - Q2(L,11))                              CMAT1241
```

```
      CMAT(13,K) = CMAT(K,13)                                   CMAT1242
      CMAT(14,K) = CMAT(K,14)                                   CMAT1243
   53 L = L + 4                                                 CMAT1244
      CMAT(13,13) = 0.                                          CMAT1245
      CMAT(13,14) = 0.                                          CMAT1246
      CMAT(14,14) = 0.                                          CMAT1247
      CMAT(14,13) = 0.                                          CMAT1248
      RETURN                                                    CMAT2180
      END                                                       CMAT2190
```

```
$IBFTC INPUT.                                                           INPUT003
      SUBROUTINE INPUT                                                  INPUT004
      COMMON/DAT1/ RADIUS,HTCON,AL,YOUNG,HTCY,RHO,RHOPRO,POISON,ALPHA,D,INPUT005
     1 THICK                                                            INPUT010
      COMMON/DAT2/ NITER,N,NSEG                                         INPUT015
      COMMON/BAR/ HBAR,BBAR,CAP,G11,E10, G12, RHOBAR                    INPUT017
      COMMON/DAT3/ ALPHAN(5), B(5),U(5)                                 INPUT020
      READ(5,10) RADIUS,HTCY,THICK,HTCON,AL,YOUNG,RHO,RHOPRO,ALPHA,D,   INPUT025
     1POISON                                                            INPUT030
   10 FORMAT(6E12.8)                                                    INPUT035
      READ(5,15) N,NSEG,NITER                                           INPUT040
      READ(5,10) (ALPHAN(I),I=1,5)                                      INPUT042
      READ(5,10) (B(I),I=1,5)                                           INPUT043
      READ(5,10) (U(I),I=1,5)                                           INPUT044
   15 FORMAT(6I12)                                                      INPUT045
      HBAR = HTCY / RADIUS                                              INPUT050
      BBAR = HTCON / RADIUS                                             INPUT055
      CAP  = (RHO * THICK * RADIUS**2) / D                              INPUT060
      RHOBAR = RHOPRO / RHO                                             INPUT062
      WRITE(6,20)                                                       INPUT070
   20 FORMAT(1H1,24X,60HLONGITUDINAL OSCILLATION OF A LIQUID-FILLED CYLIINPUT080
     1NDRICAL TANK/38X,33HWITH AN INVERTED CONICAL BULKHEAD)            INPUT090
      WRITE(6,30) RADIUS,THICK,HTCY,HTCON,AL, ALPHA                     INPUT095
   30 FORMAT(1H0,40X,21HRADIUS OF CYLINDER = ,F7.2/ 38X,24HTHICKNESS OF INPUT097
     1CYLINDER = ,F7.2/ 41X,21HHEIGHT OF CYLINDER = ,F7.2/1H0,43X,18HHEIINPUT100
     2GHT OF CONIC = ,F7.2/58X,4HL = ,F7.2/ 54X, 8HALPHA = ,F7.2)       INPUT110
      WRITE(6,40) RHO,RHOPRO,POISON,YOUNG                               INPUT115
   40 FORMAT(1H0,51X,10HDENSITY = ,1PE13.6/ 41X,21HPROPELLANT DENSITY = INPUT120
     1,1PE13.6/ 46X,16HPOISONS RATIO = ,1PE13.6/ 45X,17HYOUNGS MODULUS =INPUT130
     2 ,1PE13.6)                                                        INPUT140
      WRITE(6,50) N,NSEG                                                INPUT150
   50 FORMAT(1H0,53X,8HDEGREE =,I3/ 45X,17HNO. OF SEGMENTS =,I3)        INPUT160
      RETURN                                                            INPUT170
      END                                                               INPUT180
```

```
$IBFTC TRANS    SDD                                                     TRANS001
C **THIS ROUTINE COMPUTES SIN, COSIN,  SINH, COSH, AND BESSEL FUNCTIONS*TRANS002
      SUBROUTINE TRANSF                                                 TRANS005
      REAL J1AN, J0AN, I0BN, I1BN,J0ANR                                 TRANS010
      COMMON/BAR/ HBAR,BBAR,CAP,G11,E10,G12, RHOBAR                     TRANS015
      COMMON/DAT2/ NITER,N,NSEG                                         TRANS017
      COMMON/DAT3/ ALPHAN(5),B(5),U(5)                                  TRANS020
      COMMON/TRIG/ SINZ(5,11),CSINZ(5,11),SINHZ(5,11), CSINHZ(5,11),    TRANS025
     1CSINBN(5), CSINHB(5), SINHB(5)                                    TRANS030
      COMMON/BESSEL/ J1AN(5), J0AN(5), I0BN(5),I1BN(5),J0ANR(5,11)      TRANS035
      DEL = (HBAR - BBAR) / FLOAT(NSEG)                                 TRANS040
      N2 = NSEG + 1                                                     TRANS050
      DEL2 = 1. / FLOAT(NSEG)                                           TRANS060
      DELTAR = 0.                                                       TRANS070
      DELTAZ = BBAR                                                     TRANS080
      DO 20 I=1,N                                                       TRANS090
      DO 10 J=1,N2                                                      TRANS100
      X2 =ALPHAN(I) * DELTAZ                                            TRANS110
      X3 = B(I) * (DELTAZ - BBAR)                                       TRANS120
      X4 = ALPHAN(I) * DELTAR                                           TRANS130
      SINZ(I,J) = SIN(X3)                                               TRANS140
      CSINZ(I,J) = COS(X3)                                              TRANS150
      SINHZ(I,J) = SINH(X2)                                             TRANS160
      CSINHZ(I,J) = COSH(X2)                                            TRANS170
      J0ANR(I,J) = BESEL(X4,0,1)                                        TRANS180
      DELTAZ = DELTAZ + DEL                                             TRANS190
   10 DELTAR = DELTAR + DEL2                                            TRANS200
      DELTAZ = BBAR                                                     TRANS210
   20 DELTAR = 0.                                                       TRANS220
      DO 30 I=1,N                                                       TRANS230
      X =ALPHAN(I) * BBAR                                               TRANS240
      SINHB(I) = SINH(X)                                                TRANS250
      CSINHB(I)= COSH(X)                                                TRANS260
      CSINBN(I) = COS(B(I)*BBAR)                                        TRANS270
      J1AN(I) = BESEL(ALPHAN(I),1,1)                                    TRANS280
      J0AN(I) = BESEL(ALPHAN(I),0,1)                                    TRANS290
      I0BN(I) = BESEL(B(I),0,2)                                         TRANS300
   30 I1BN(I) = BESEL(B(I),1,2)                                         TRANS310
      RETURN                                                            TRANS320
      END                                                               TRANS330
```

```
$IBFTC F5ZN.   SDD                                                      F5ZN0000
      SUBROUTINE F5ZN                                                   F5ZN0005
      COMMON/BAR/ HBAR, BBAR,CAP,G11,E10,G12, RHOBAR                    F5ZN0010
      COMMON/DAT2/ NITER, N, NSEG                                       F5ZN0015
      COMMON/F56Z/ F5Z(6,11), F6Z(6,11)                                 F5ZN0020
      DEL = 0.                                                          F5ZN0025
      DELZ = BBAR / FLOAT(NSEG)                                         F5ZN0030
      N2 = N + 1                                                        F5ZN0035
      NSEG2 = NSEG + 1                                                  F5ZN0040
      A6 = 1./6.                                                        F5ZN0045
      A8 = 35./8.                                                       F5ZN0050
      A9 = 21./8.                                                       F5ZN0055
      A16 = 63./16.                                                     F5ZN0060
      A17 = 35./16.                                                     F5ZN0062
      A18 = 35./ 6.                                                     F5ZN0063
      A31 = 106./30.                                                    F5ZN0064
      A24 = 11./24.                                                     F5ZN0065
      A25 = 245./24.                                                    F5ZN0070
      A30 = 181./30.                                                    F5ZN0075
      A56 = 55./56.                                                     F5ZN0076
      DO 30 J=1,NSEG2                                                   F5ZN0085
      A = BBAR - DEL                                                    F5ZN0090
      IF(J.EQ.NSEG2) A = 0.                                             F5ZN0092
      U = A / SQRT(1. + A**2)                                           F5ZN0095
      EP = SQRT(1. - U**2)                                              F5ZN0100
      UE = U / EP                                                       F5ZN0105
      UE3 = U**3 / EP**3                                                F5ZN0110
      UE5 = U / EP**5                                                   F5ZN0115
      E1 = 1. / EP**2                                                   F5ZN0120
      E2 = 1. / EP**4                                                   F5ZN0125
      F5Z(1,J) = -UE                                                    F5ZN0130
      F5Z(2,J) = -.5 * E1                                               F5ZN0135
      F5Z(3,J) = .5 * UE - UE3/3.                                       F5ZN0140
      F5Z(4,J) = 1.25 * E1 - .25 * E2                                   F5ZN0145
      F5Z(5,J) = 4. * UE - A24 * UE3 - A8 * UE5 + A25 * UE5 * U**2 - A30F5ZN0150
     1 * UE3 * E1 * U**2                                                F5ZN0155
      F5Z(6,J) = -A16 * E1 - A9 * E2 - A6 * E1 * E2                     F5ZN0160
   30 DEL = DEL + DELZ                                                  F5ZN0170
      ZBAR = 0.                                                         F5ZN0180
      DO 40 J=1,NSEG2                                                   F5ZN0190
      BZ = BBAR - ZBAR                                                  F5ZN0200
      IF(J.EQ.NSEG2) BZ = 0.                                            F5ZN0205
      BZ2 = BZ**2                                                       F5ZN0210
```

```
      BZ3 = BZ2 * BZ                                                    F5ZN0220
      BZ5 = BZ3 * BZ2                                                   F5ZN0230
      B1Z = 1. + BZ2                                                    F5ZN0240
      B2Z = B1Z**2                                                      F5ZN0250
      B4Z = B2Z**2                                                      F5ZN0260
      F6Z(1,J) = -BBAR * BZ + .5 * B1Z                                  F5ZN0270
      F6Z(2,J) =-BBAR * .5 * B2Z + BZ3/3.                               F5ZN0280
      F6Z(3,J) =  BBAR * (.5 * BZ - BZ3/3.) - .75 * B1Z + .25 * B2Z     F5ZN0290
      F6Z(4,J) = BBAR * (1.25 * B2Z - .25 * B4Z) - 2.5 *  BZ + BZ3/3. + F5ZN0300
     12.5 * BZ * B2Z - A18 * BZ3 * B1Z + A31 * BZ5                      F5ZN0310
      F6Z(5,J) = BBAR * (4.* BZ - A24 *BZ3 - A8 * BZ * B4Z + A25 * BZ3 *F5ZN0320
     1 B2Z - A30 * BZ5) + A18 * B1Z -1.25 * B2Z + A6 * B1Z * B2Z        F5ZN0330
      F6Z(6,J) = BBAR * (-A16 * B2Z -A9 * B4Z - A6 * B2Z * B4Z) - 12.15 F5ZN0340
     1* BZ + 7.875 * BZ * B1Z + 1.45 * BZ3 + 3.15 * BZ * B2Z - 18.375 * F5ZN0350
     2BZ3 * B2Z + 11.3 * BZ5 - A56 * BZ5 * BZ2                          F5ZN0360
   40 ZBAR = ZBAR + DELZ                                                F5ZN0370
      RETURN                                                            F5ZN0380
      END                                                               F5ZN0390
```

```
$IBFTC CONST.                                                          CONST000
      SUBROUTINE CONST                                                 CONST005
      COMMON/BAR/ HBAR, BBAR,CAP, G11, E10,G12,RHOBAR                  CONST010
      COMMON/DAT2/ NITER,N, NSEG                                       CONST015
      COMMON/DEL/DEL1,DEL2,DEL3,DEL4                                   CONST017
      COMMON/DELTA/ DZBH(11),DZS(11),  DSL(11), DR(11), DZB(11),DS(11) CONST020
      COMMON/LIMIT/ N2,NSEG2                                           CONST025
      COMMON/DAT1/RADIUS,HTCON,AL,YOUNG,HTCY,RHO, RHOPRO,POISON,ALPHA, CONST027
     1 D, THICK                                                        CONST028
      N2 = N+1                                                         CONST030
      NSEG2 = NSEG+1                                                   CONST035
      DEL1  =  (HBAR-BBAR) / FLOAT(NSEG)                               CONST040
      DEL2  = BBAR / FLOAT(NSEG)                                       CONST045
      BARL = SQRT(1.+BBAR**2)                                          CONST050
      DEL3  = BARL / FLOAT(NSEG)                                       CONST055
      DEL4 = 1./ FLOAT(NSEG)                                           CONST056
      ZBAR = 0.                                                        CONST060
      DZBH(1) = BBAR                                                   CONST070
      DZS(1) = SQRT(1.+BBAR**2)                                        CONST080
      DSL(1) = 0.                                                      CONST090
      DR(1) = 0.                                                       CONST100
      DZB  = 0.                                                        CONST110
      DO 10 I=2,NSEG2                                                  CONST120
      DZBH(I) = DZBH(I-1) + DEL1                                       CONST130
      ZBAR = ZBAR + DEL2                                               CONST140
      DZS(I) = SQRT(1.+(BBAR-ZBAR)**2)                                 CONST150
      DSL(I) = DSL(I-1) + DEL3                                         CONST160
      DR(I) = DR(I-1) + DEL4                                           CONST170
   10 DZB(I) = DZB(I-1) + DEL2                                         CONST180
      A = SIND(ALPHA)                                                  CONST190
      DO 20 I=1,NSEG2                                                  CONST200
   20 DS(I) = DSL(I) * A                                               CONST210
      RETURN                                                           CONST220
      END                                                              CONST230
$*                                                                     CONST999
```

```
$IBFTC F1ZN.    SDD                                                     F1ZN0000
      SUBROUTINE F1ZN                                                   F1ZN0002
C                                                                       F1ZN0005
      COMMON/BAR/ HBAR,BBAR,CAP,G11,E10,G12,RHOBAR                      F1ZN0010
      COMMON/DAT2/ NITER,N,NSEG                                         F1ZN0015
      COMMON/DAT3/ ALPHAN(5), B(5), U(5)                                F1ZN0017
      COMMON/F1/F1Z(24,11)                                              F1ZN0020
      N2 = N + 1                                                        F1ZN0025
      N3= NSEG + 1                                                      F1ZN0030
      DELTA = ( HBAR - BBAR) / FLOAT(NSEG)                              F1ZN0035
      DEL = BBAR                                                        F1ZN0036
      K2 = 1                                                            F1ZN0038
      DO 10 K=1,N3                                                      F1ZN0050
      F1Z(1,K)  = 1. / G11**2                                           F1ZN0055
      F1Z(2,K) = 0.                                                     F1ZN0060
      F1Z(3,K) = -E10 / G11                                             F1ZN0070
   10 F1Z(4,K) = 0.                                                     F1ZN0080
      DO 40 I=1,N                                                       F1ZN0090
      K2 = K2 + 4                                                       F1ZN0100
      X = BESEL(B(I),0,2) /  (B(I) * COS(B(I)*BBAR))                    F1ZN0105
      DO 40 K = 1,N3                                                    F1ZN0110
      A = B(I) * DEL - BBAR                                             F1ZN0115
      F1Z(K2,K) = 0.                                                    F1ZN0120
      F1Z(K2+1,K) = X * SIN(A)                                          F1ZN0125
      F1Z(K2+2,K) = 0.                                                  F1ZN0130
      F1Z(K2+3,K) = X * COS(A)                                          F1ZN0140
   40 DEL = DEL + DELTA                                                 F1ZN0150
      RETURN                                                            F1ZN0160
      END                                                               F1ZN0170
$*                                                                      FH1N0999
```

```
$IBFTC FH1N.   SDD                                                      FH1N0000
      SUBROUTINE FH1N(OMEGA)                                            FH1N0005
      COMMON/BAR/HBAR,BBAR,CAP,G11,E10,G12, RHOBAR                      FH1N0010
      COMMON/DAT2/ NITER,N,NSEG                                         FH1N0015
      COMMON/H1/ FH1(8,6)                                               FH1N0020
      COMMON/DAT1/RADIUS,HTCON,AL,YOUNG, HTCY,RHO,RHOPRO,POISON,ALPHA,D,FH1N0025
     1 THICK                                                            FH1N0030
      COMMON/DAT3/ALPHAN(5),B(5),U(5)                                   FH1N0032
      X = G11 * HBAR                                                    FH1N0035
      SIGN = SIN(X)                                                     FH1N0040
      COSIGN = COS(X)                                                   FH1N0045
      WK1 = 1. - CAP * OMEGA**2                                         FH1N0050
      WK2 = 1. - POISON**2 / WK1                                        FH1N0055
      WK3 = 1. - POISON / WK1                                           FH1N0060
      GG = G12 * G11                                                    FH1N0065
      X2 = G11 * BBAR                                                   FH1N0070
      SIGN2 = SIN(X2)                                                   FH1N0080
      COSN2 = COS(X2)                                                   FH1N0090
      FH1(1,1) = -WK2 * GG*(SIGN / G11**2 + COSIGN * E10 / G11)         FH1N0100
      FH1(2,1) = 0.                                                     FH1N0110
      FH1(3,1) = -GG * (SIGN2 / G11**2 + COSN2 * E10 / G11)             FH1N0120
      FH1(4,1) = 0.                                                     FH1N0130
      Y2 = HBAR - BBAR                                                  FH1N0135
      DO 40 I=1,N                                                       FH1N0140
      XA =ALPHAN(I) * HBAR                                              FH1N0150
      XX = EXP(XA)                                                      FH1N0160
      HYSIN = (XX - 1. / XX ) / 2.                                      FH1N0170
      HYCOS = (XX + 1. / XX ) / 2.                                      FH1N0180
      BES = (BESEL(B(I),0,2))/ (B(I) * COS(B(I)*BBAR))                  FH1N0190
      FH1(1,I+1) = ( WK2 * G12 - POISON / WK1) * (HYSIN + HYCOS) * BESELFH1N0210
     1(ALPHAN(I),0,1)                                                   FH1N0220
      FH1(2,I+1) = -WK2 * GG * (SIGN * BES * SIN(B(I)*Y2) - COSIGN * BESFH1N0230
     1 * COS(B(I)*Y2)) +(WK2 * G12 - POISON / WK1) * COS(B(I)*Y2) / COS(FH1N0235
     2B(I)*BBAR) * BESEL(B(I),0,2)                                      FH1N0240
      FH1(3,I+1) = G12 *(SINH(ALPHAN(I)*BBAR) +COSH(ALPHAN(I)*BBAR)) *  FH1N0250
     1BESEL(ALPHAN(I),0,1)                                              FH1N0260
   40 FH1(4,I+1) =  GG * COSH2 * BES * COS(B(I)) + G12 * BESEL(B(I),0,2)FH1N0270
     1 / COS(B(I)* BBAR)                                                FH1N0280
      U1 = BBAR / SQRT(1. + BBAR**2)                                    FH1N0290
      EP = SQRT(1. - U1**2)                                             FH1N0300
      UE = U1 / EP                                                      FH1N0310
      UE3 = UE**3                                                       FH1N0320
      FH1(5,1) = -UE                                                    FH1N0330
```

```
      FH1(6,1) = -.5 / EP**2                                            FH1N0340
      FH1(5,2) = -.5 * FH1(5,1) - UE3 / 3.                              FH1N0350
      FH1(6,2) = 1.25 / EP**2 - .25 / EP**4                             FH1N0360
      FH1(5,3) = 4. * UE - 11. * UE3 / 24. - 4.375 * UE / EP**4 + 245. *FH1N0370
     1 UE3 / (24. * EP**2) - 181. * UE**5 / 30.                         FH1N0380
      FH1(6,3) = -63. /(16.* EP**2) - 21. / (8. * EP**4) - 1. / (6. * EPFH1N0390
     1**6)                                                              FH1N0400
      FH1(7,1) = -GG * (-G11/2. * COSN2) + G12                          FH1N0410
      FH1(8,1) = -GG * (-.5 * SIGN2 - G11 * (-BBAR/2.))                 FH1N0420
      FH1(7,2) = -GG * G11 * 0.5  * COSN2 - .5 * G12                    FH1N0430
      FH1(8,2) = -GG *(SIGN2 - G11 * BBAR)                              FH1N0440
      FH1(7,3) = -GG * (-53./48. * G11 * COSN2) + .375 * G12            FH1N0450
      FH1(8,3) = -GG * (-323./48. * SIGN2 + 323./48. * BBAR * G11)      FH1N0460
      RETURN                                                            FH1N0480
      END                                                               FH1N0500
```

```
$IBFTC POLY.                                                            POLY0000
C                                                                       POLY0005
C *** ROUTINE COMPUTES THE FIRST SIX LEGENDRE POLYNOMIAL **             POLY0010
C                                                                       POLY0015
      SUBROUTINE POLY                                                   POLY0020
      COMMON/BAR/ HBAR, BBAR, CAP, G11,E10, G12, RHOBAR                 POLY0025
      COMMON/DAT2/ NITER,N,NSEG                                         POLY0030
      COMMON/DAT1/ RADIUS,HTCON, AL, YOUNG, HTCY, RHO, RHOPRO, POISON,  POLY0035
     1 ALPHA, D, THICK                                                  POLY0040
      COMMON/PLDR/ PO(6), PA(6), PZ(6,11)                               POLY0045
      U = COSD(ALPHA)                                                   POLY0050
      U2 = U**2                                                         POLY0060
      U3 = U2 * U                                                       POLY0065
      U4 = U2**2                                                        POLY0070
      U5 = U2 * U3                                                      POLY0075
      PA(1) = 1.                                                        POLY0080
      PA(2) = U                                                         POLY0085
      PA(3) = 0.5 * (3.*U2 - 1.)                                        POLY0090
      PA(4) = 0.5 * (5.*U3 - 3.*U)                                      POLY0095
      PA(5) = 0.125 * (35.*U4 - 30.*U2 + 3.)                            POLY0100
      PA(6)= 0.125 * (63.*U5 - 70.*U3 + 15.*U)                          POLY0105
      PO(1) = 1.                                                        POLY0110
      PO(2) = 0.                                                        POLY0115
      PO(3) = -.5                                                       POLY0120
      PO(4) = 0.                                                        POLY0125
      PO(5) = .375                                                      POLY0130
      PO(6) = 0.                                                        POLY0135
      DEL = 0.                                                          POLY0140
      DZ = BBAR / FLOAT(NSEG)                                           POLY0145
      NSEG2 = NSEG+1                                                    POLY0150
      DO 10 J=1,NSEG2                                                   POLY0155
      B1 = BBAR - DEL                                                   POLY0160
      IF(J.EQ.NSEG2) B1 = 0.                                            POLY0162
      U = B1 / SQRT(1. + B1**2)                                         POLY0165
      U2 = U**2                                                         POLY0170
      U3 = U**3                                                         POLY0175
      PZ(1,J) = 1.                                                      POLY0180
      PZ(2,J) = U                                                       POLY0185
      PZ(3,J) = 0.5 * (3.*U2 - 1.)                                      POLY0190
      PZ(4,J) = 0.5 * (5.*U3 - 3.*U)                                    POLY0195
      PZ(5,J) = 0.125 * (35.*U2**2 - 30. * U2 + 3.)                     POLY0200
      PZ(6,J) = 0.125 * (63.*U3*U2 - 70. *U3 + 15.*U)                   POLY0210
   10 DEL = DEL + DZ                                                    POLY0215
```

```
RETURN                                                    POLY0220
END                                                       POLY0225
```

```
$IBFTC GAMA.                                                            GAMMA000
C******* COMPUTE GAMMA FUNCTION ************                            GAMMA001
      SUBROUTINE GAMAN                                                  GAMMA002
      COMMON/GAMA/ GAMA1(9,11),GAMA2(12,11),GAMA3(9,11)                 GAMMA010
      COMMON/BAR/ HBAR, BBAR, CAP, G11,E10, G12, RHOBAR                 GAMMA015
      COMMON/DAT1/ RADIUS,HTCON,AL,YOUNG,HTCY,RHO,RHOPRO,POISON,ALPHA,D,GAMMA018
     1 THICK                                                            GAMMA019
      COMMON/TRIG/ SINZ(5,11), CSINZ(5,11),SINHZ(5,11),CSINHZ(5,11),    GAMMA020
     1CSINBN(5), CSINHB(5), SINHB(5)                                    GAMMA025
      COMMON/BFSSEL/ J1AN(5),J0AN(5),    I0BN(5), I1BN(5), J0ANR(5,11)  GAMMA030
      REAL J1AN, J0AN, I0BN,I1BN,J0ANR,LBAR                             GAMMA035
      COMMON/DAT2/ NITER,N, NSEG                                        GAMMA040
      COMMON/DAT3/ ALPHAN(5),B(5),U(5)                                  GAMMA050
      LBAR = SQRT(1. + BBAR**2)                                         GAMMA055
      DEL1 = BBAR                                                       GAMMA060
      N2 = N + 1                                                        GAMMA060
      DEL2 = 0.                                                         GAMMA070
      NSEG2 = NSEG + 1                                                  GAMMA070
      DZ =(HBAR - BBAR) / FLOAT(NSEG)                                   GAMMA080
      DR = 1. / FLOAT(NSEG)                                             GAMMA090
      K = 4                                                             GAMMA100
      DO 20 I=1,N                                                       GAMMA110
      A = B(I) * I1BN(I) / CSINBN(I)                                    GAMMA120
      A2 = ALPHAN(I) * (CSINHB(I) + SINHB(I))                           GAMMA122
      DO 10 J=1,NSEG2                                                   GAMMA130
      GAMA1(K,J) = 0.                                                   GAMMA140
      GAMA1(K+1,J) = A * CSINZ(I,J)                                     GAMMA150
   10 GAMA1(K+2,J) = A2 * J0ANR(I,J)                                    GAMMA160
   20 K = K + 3                                                         GAMMA170
      DO 25 J=1,NSEG2                                                   GAMMA172
      GAMA1(1,J) = 0.                                                   GAMMA174
      GAMA1(2,J) = 0.                                                   GAMMA176
   25 GAMA1(3,J) = 1.                                                   GAMMA178
C  **** COMPUTE LEGENDRE POLYNOMIAL ***                                 GAMMA179
      DIMENSION  P(6,11)                                                GAMMA180
      DEL = BBAR / FLOAT(NSEG)                                          GAMMA190
      DZ = 0.                                                           GAMMA200
      DO 30 J=1,NSEG2                                                   GAMMA210
      B1= BBAR - DZ                                                     GAMMA220
      U1= B1/ SQRT(1. +B1**2)                                           GAMMA230
      P(1,J) = 1.                                                       GAMMA250
      P(2,J) = U1                                                       GAMMA260
      U2 =U1**2                                                         GAMMA270
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      U3 = U2 * U1                                                      GAMMA280
      P(3,J) = .5 * (3. * U2 - 1.)                                      GAMMA290
      P(4,J) = .5 * (5. * U3 - 3. * U1)                                 GAMMA300
      P(5,J) = .125 * (35. * U2**2 - 30. * U2 + 3.)                     GAMMA310
      P(6,J) =.125*(63. *U1**5 - 70. * U3 + 15. * U1)                   GAMMA320
   30 DZ = DZ + DEL                                                     GAMMA330
      DZ = 0.                                                           GAMMA340
      DO 40 J=1,NSEG2                                                   GAMMA360
      B1 = BBAR - DZ                                                    GAMMA370
      GAMA3(1,J) = SQRT(1. + B1**2)                                     GAMMA380
      GAMA3(2,J) = B1 / GAMA3(1,J)                                      GAMMA390
      GAMA3(3,J) = 1. / SQRT(1. - GAMA3(2,J) * GAMA3(2,J))              GAMMA400
   40 DZ = DZ + DEL                                                     GAMMA410
      K = 1                                                             GAMMA420
      DO 55 J=1,NSEG2                                                   GAMMA430
      GAMA2(1,J) = 0.                                                   GAMMA440
      GAMA2(2,J) = P(2,J)                                               GAMMA450
      GAMA2(3,J) = 0.                                                   GAMMA460
   55 GAMA2(4,J) = -GAMA3(3,J) *(P(1,J)- GAMA3(2,J) * P(2,J))           GAMMA470
      K = 5                                                             GAMMA480
      DO 70 I=1,N                                                       GAMMA490
      NCAP = 2 * I                                                      GAMMA500
      NBAR = 2 * I + 1                                                  GAMMA510
      A = FLOAT(NCAP)                                                   GAMMA520
      AX= FLOAT(NBAR)                                                   GAMMA530
      DO 65 J=1,NSEG2                                                   GAMMA540
      GAMA2(K,J) = A * GAMA3(1,J)**(NCAP-1) * P(NCAP+1,J)               GAMMA550
      GAMA2(K+1,J) = AX * GAMA3(1,J)**NCAP * P(NBAR+1,J)                GAMMA560
      GAMA2(K+2,J) = -GAMA3(3,J) * A * GAMA3(1,J)**(NCAP-1) * (P(NCAP,J)GAMMA570
     1 - GAMA3(2,J) * P(NCAP+1,J))                                      GAMMA580
   65 GAMA2(K+3,J) = -GAMA3(3,J) * AX * GAMA3(1,J)**NCAP * (P(NBAR,J) - GAMMA590
     1GAMA3(2,J) * P(NBAR+1,J))                                         GAMMA600
   70 K = K + 4                                                         GAMMA610
      U1 = COSD(ALPHA)                                                  GAMMA630
      P(1,1) = 1.                                                       GAMMA640
      P(1,2) = U1                                                       GAMMA650
      U2 = U1**2                                                        GAMMA660
      P(1,3) = .5 * (3. * U2 - 1.)                                      GAMMA670
      U3 = U2 * U1                                                      GAMMA680
      P(1,4) = .5 * (5. * U3 - 3. *U1)                                  GAMMA690
      P(1,5) = .125 * (35. * U2**2 - 30. * U2 + 3.)                     GAMMA700
      P(1,6) = .125 * (63. * U3 * U2 - 70. * U3 + 15. * U1)             GAMMA710
      P(2,1) = 1.                                                       GAMMA720
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      P(2,2) = 0.                                                       GAMMA725
      P(2,3) = -.5                                                      GAMMA730
      P(2,4) = 0.                                                       GAMMA740
      P(2,5) = .375                                                     GAMMA750
      P(2,6) = 0.                                                       GAMMA760
      DEL = 0.                                                          GAMMA770
      A =-1. / SQRT(1. - U1**2)                                         GAMMA780
      DELL = LBAR / FLOAT(NSEG)                                         GAMMA790
      DEL2 = 0.                                                         GAMMA800
      DELR = 1. / FLOAT(NSEG)                                           GAMMA810
      DO 85 J=1,NSEG2                                                   GAMMA820
      P(3,J) = DEL                                                      GAMMA830
      P(4,J) = DEL2                                                     GAMMA840
      DEL = DEL + DELL                                                  GAMMA850
   85 DEL2 = DEL2 + DELR                                                GAMMA860
      DO 90 J=1,NSEG2                                                   GAMMA880
      GAMA3(1,J) = 0.                                                   GAMMA890
      GAMA3(2,J) = A * (P(1,1) - U1 * P(1,2))                           GAMMA900
   90 GAMA3(3,J) = -P(2,1)                                              GAMMA910
      GAMA3(2,1) = 0.                                                   GAMMA915
      GAMA3(3,1) = 0.                                                   GAMMA916
      K = 4                                                             GAMMA920
      DO 100 I=1,N                                                      GAMMA930
      NCAP = 2 * I                                                      GAMMA940
      NBAR = NCAP + 1                                                   GAMMA950
      NB = NBAR + 1                                                     GAMMA952
      NC = NCAP - 1                                                     GAMMA960
      DO 95  J=1,NSEG2                                                  GAMMA970
      GAMA3(K,J) = A * FLOAT(NCAP) * P(3,J)**NC * (P(1,NCAP) - U1 * P(1,GAMMA980
     1NBAR))                                                            GAMMA982
      GAMA3(K+1,J) = A * FLOAT(NBAR) * P(3,J)**NCAP * (P(1,NBAR) - U1 * GAMMA984
     1P(1,NB))                                                          GAMMA986
   95 GAMA3(K+2,J)=  - FLOAT(NBAR) * P(4,J)**NCAP * (P(2,NBAR))         GAMMA990
  100 K = K + 3                                                         GAMMA994
      RETURN                                                            GAMMA995
      END                                                               GAMMA999
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$IBFTC BETAN.                                                           BETAN000
C***   SUBROUTINE TO CALCULATE BETA(N)   ***                            BETAN005
      SUBROUTINE BETAN(OMEG,ERR)                                        BETAN010
      COMMON/BAR/ HBAR, BBAR, CAP,G11, EIO,G12, RHOBAR                  BETAN020
      COMMON/DAT3/ ALPHAN(5),B(5),U(5)                                  BETAN030
      COMMON/BET/ OMEG2,GA, BH                                          BETAN040
      COMMON/DAT1/ RADIUS,HTCON, AL,YOUNG, HTCY,RHO, RHOPRO, POISON,    BETAN050
     1ALPHA,D, THICK                                                    BETAN060
      EXTERNAL BETA                                                     BETAN070
      LOGICAL ERR                                                       BETAN075
      ERR = .FALSE.                                                     BETAN077
      WRITE(6,5)                                                        BETAN080
    5 FORMAT(1H1)                                                       BETAN090
      GA = 386.089 / RADIUS                                             BETAN100
      BH = HBAR - BBAR                                                  BETAN110
      TOL = .001                                                        BETAN120
      REL = 1.                                                          BETAN130
      NITER = 80                                                        BETAN150
      OMEG2 = OMEG**2                                                   BETAN160
      K = 1                                                             BETAN170
      DELBN = .5                                                        BETAN180
      BN = 1.58 / BH                                                    BETAN190
   11 BETN = GA * BN * TAN(BN * BH) + OMEG2                             BETAN191
      IF(BETN.LT.0.) GO TO 12                                           BETAN192
      B(K) = BN                                                         BETAN193
      BN = BN + (3.14159 / BH)                                          BETAN194
      IF(K.EQ.4) GO TO50                                                BETAN195
      K = K + 1                                                         BETAN196
      GO TO 11                                                          BETAN197
   12 CALL ROOT(NTRY,BN,DELBN,RESID,BETA,TOL,REL,NITER)                 BETAN200
      GO TO (15,15,30,30,35), NTRY                                      BETAN210
   15 B(K) = BN                                                         BETAN220
      IF(K.EQ.4) GO TO 50                                               BETAN230
      K = K+1                                                           BETAN240
      BN = BN + (3.14159 / BH)                                          BETAN260
      GO TO 11                                                          BETAN270
   30 WRITE(6,32) OMEG                                                  BETAN280
   32 FORMAT(1H0,10X,29H*** NO SOLUTIONS FOR OMEGA = ,1PE13.6)          BETAN290
      ERR = .TRUE.                                                      BETAN300
      GO TO 50                                                          BETAN310
   35 WRITE(6,37) OMEG                                                  BETAN320
   37 FORMAT(1H0,10X,70H*** EXCEEDED MAXIMUM NO. OF ITERATIONS WITHOUT ABETAN330
     1 SOLUTION FOR OMEGA = ,1PE13.6)                                   BETAN340
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      ERR = .TRUE.                                                      BETAN350
   50 WRITE(6,55) OMEG,(I,B(I),I=1,K)                                   BETAN360
   55 FORMAT(1H-,10X, 8HOMEGA = ,1PE13.6, 10H RAD./SEC./ (5X,2HB(,I1,4H)BETAN370
     1 = ,1PE13.6))                                                     BETAN380
      RETURN                                                            BETAN390
      END                                                               BETAN400
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$IBFTC BETA.                                                    BETA0000
      FUNCTION BETA(BN)                                         BETA0002
      COMMON/BET/ OMEG2, GA, BH                                 BETA0004
      X = BN * BH                                               BETA0006
      BETA = GA * BN * TAN(X) + OMEG2                           BETA0008
      RETURN                                                    BETA0010
      END                                                       BETA0012
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$IBFTC SRULF.                                                          SRULE001
      FUNCTION SRULE(N,A,B,C,D)                                        SRULE005
C ***  INTEGRATION BY SIMPSON'S RULE ***                               SRULE002
C       N = NO. OF INTEGRALS                                           SRULE003
      COMMON/TERM/ T(4,11),ISEG,NSEG2, NN                              SRULE010
      SUM4 = 0.                                                        SRULE012
      SUM1 = T(1,1) + T(1,NSEG2) + 4. * T(1,ISEG)                      SRULE015
      SUM2 = T(2,1) + T(2,NSEG2) + 4. * T(2,ISEG)                      SRULE020
      SUM3 = T(3,1) + T(3,NSEG2) + 4. * T(3,ISEG)                      SRULE025
      IF(N.EQ.3) GO TO 10                                              SRULE030
      SUM4 = T(4,1) + T(4,NSEG2) + 4. * T(4,ISEG)                      SRULE035
      DO 5 I=2,NN,2                                                    SRULE040
    5 SUM4 = SUM4 + 4. * T(4,I) + 2. * T(4,I+1)                        SRULE045
   10 DO 20 I=2,NN,2                                                   SRULE050
      SUM1 = SUM1 + 4. * T(1,I) + 2. * T(1,I+1)                        SRULE055
      SUM2 = SUM2 + 4. * T(2,I) + 2. * T(2,I+1)                        SRULE060
   20 SUM3 = SUM3 + 4. * T(3,I) + 2. * T(3,I+1)                        SRULE065
      SRULE = A*SUM1/3. + B*SUM2/3. + C*SUM3/3. + D*SUM4/3.            SRULE070
      RETURN                                                           SRULE075
      END                                                              SRULE080
$*                                                                     SRULE999
```