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OPTIMIZATION PROBLEMS SOLVED BY LIE-SERIES: SOFT LANDING ON THE MOON WITH FUEL MINIMIZATION

by

PRICES SUBJECT TO CHANGE

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Abstract

The problem of soft landing on the moon with the additional requirement that fuel consumption during the deceleration of the rocket should be minimized is solved formally with the help of Lie series. A corresponding one-dimensional problem having no solution is briefly reviewed.

1. Introduction:

The enduring effort to improve technology - generally speaking as well as, in our case, the technology of space craft - has given rise to the concept of optimization; optimum control systems are, therefore, gaining more and more importance in space flight. Optimization may be carried out with respect to various parameters, as, e.g., time of flight or consumption of propellant. In this report, we shall consider the problem of soft landing on the surface of the moon under the additional condition of minimizing the fuel needed to operate the decelerating rockets. The mathematical method of solving the Euler-Lagrangean equations of the optimization integral and the equations of motion appearing as the constraint conditions is based on the use of Lie-series; based on considerations by Groebner and his coworkers /1-30/ of the Mathematics Department of the Innsbruck University, Lie series have proved to be an efficient tool for solving differential equations in the last few years; they have proved their efficiency in treating problems of celestial mechanics /28/, /29/, reactor theo-

ry /22/, /23/ as well as other fields of mathematical physics (as, e.g., Hamilton-Jacobi theory, stability investigation /2/ and van der Pol equations /31/). Another paper to be published in the framework of this contract deals with the application of this method to the calculation of particle orbits in circular accelerators /32/. - Optimization problems were for the first time solved with the help of Lie-series by Groebner /33/ and Dotzauer /34/; the present report is closely related to their considerations. - The equations used by us show an intimate resemblance to those used by other authors /35/. Before stating and formally solving our specific problem we shall present the general formalism based on considerations of /33/, /34/.

2. General Formalism:

Let p functions $x_i(t)$ $(i=1, .., p)$ specifying, e.g., the positions and momenta of a spacecraft and q functions $y_j(t)$ $(j=1, ...,q)$ representing control forces be given. The equations to be solved are of the form

$$x_i = G_i(x,y) \quad (i = 1, ..., p) \tag{1}$$

They serve as the constraint conditions supplementing the equations stemming from a minimization of the integral:

$$I(y) = \int_0^T \left\{F(x, y, \dot{y}) + \lambda_i G_i(x, y)\right\} dt \tag{2}$$

To secure uniqueness, p initial or final conditions of the form

$$x_i(o) = a_i \tag{3a}$$

or

$$x_i(T) = c_i \tag{3b}$$

respectively, must be given. As will be shown below, Lie series formalism provides a convenient method of transforming final conditions to initial ones.

Briefly, we shall have to find the 2p + q functions $x_i(t)$, $y_j(t)$ and $\lambda_K(t)$ - the Lagrange multiplications - from (1) and the p equations:

$$\dot{\lambda}_K = - \quad \lambda_i \frac{\partial G_i}{\partial x_K} + \frac{\partial F}{\partial x_K} \tag{4}$$

and the q equations:

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{y}_j} = - \quad \lambda_i \frac{\partial G_i}{\partial y_j} + \frac{\partial F}{\partial y_j} \tag{5}$$

together with boundary conditions.

We shall now pass to the calculation of the corresponding new initial conditions from the final conditions, making use of Lie series formalism. Redefining our variables in a straightforward way appropriate to obtaining Lie solutions our system reads /1/:

$$\dot{Z}_i = \quad {}_i(z) \qquad (i = 1, \ldots, 2p + q) \tag{6}$$

Let, e.g. 2p + q - 1 initial conditions

$$(Z_i)_{t=o} = a_i \qquad (i = 1, \ldots, 2p + q - 1) \tag{7}$$

and one final condition

$$F(Z_1, \ldots\ldots, Z_n)_{t=T} = 0 \tag{8}$$

be given. The solution of this system is given by:

$$Z_i = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu a_i \equiv e^{tD} a_i \tag{9}$$

where the D-operator is composed of the right-handsides of (6) in the well-known manner (see, e.g. /1/). Using the well-known commutation theorem /1/, we have:

$$F(Z_1, \ldots\ldots Z_{2p+q}) = e^{tD}.F(a_i) \equiv \Phi(t; a_i) \tag{10}$$

The function Φ defined in this way may be used to reexpress the final condition:

$$\Phi(T; a_1, \ldots\ldots a_{n-1}, \xi) = 0 \tag{11}$$

where ξ is considered variable such that

$$\xi = b \equiv z_n(0) \tag{12}$$

The initial condition representing the final condition reads:

$$\Phi(0; a_1, \ldots\ldots a_{n-1}, \xi) = F(a_1, \ldots, a_{n-1}; \xi) = 0 \tag{13}$$

where the value of $\xi_{t=T}$ is connected with $b \equiv \xi_{t=o}$ by

$$\xi_{t=T} = (e^{TD_1} b)_{\tau=o} \tag{14}$$

with

$$D_1 = \frac{\partial}{\delta\tau} - \frac{\Phi_\tau(\tau; a_1, \ldots, a_{n-1}, \xi)}{\Phi_\xi(\tau; a_1, \ldots, a_{n-1}, \xi)} \frac{\partial}{\partial\xi} \tag{15}$$

$$(\Phi_\tau = \frac{\partial\Phi}{\partial\tau}, \quad \Phi_\xi = \frac{\partial\Phi}{\partial\xi})$$

This statement (13) is easily proved as follows:

With

$$D_1\tau = 1, \; D_1^{\vee}\tau = 0 \tag{16}$$

and

$$D_1^{\vee}\Phi(\tau;\, a_1, \ldots, a_{n-1}, \xi) = 0 \tag{17}$$

as well as

$$T = (e^{TD_1}\tau)_{\xi=b,\ \tau=0} \tag{18}$$

we obtain, using again the commutation theorem /1/:

$$o\,(T, \xi;\, a_1, \ldots, a_{n-1}) = (e^{TD_1}\Phi(\tau, \xi;\, a_1, \ldots a_{n-1})_{\substack{\xi=b\\ \tau=0}} =$$

$$= \Phi(a, a_1, \ldots, a_{n-1}, b) = 0 \tag{19}$$

q.e.d.

3. The Problem of Soft Landing on the Surface of the Moon with Fuel Optimization.

Our specific problem to be solved by Lie series formalism is a two-body problem, i.e., the decelerated motion of a spacecraft in the neighbourhood of the moon subject to the condition of soft landing ($v_{t=T}$ = 2-3 m/sec) as well as of minimum propellant consumption during the action of the decelerating rockets. The equations of motion to be employed read:

$$\dot{\bar{x}}_v = \bar{v}_v \tag{20a}$$

$$m_v(t)\dot{\vec{v}}(t) = \frac{\gamma m_m m_v \vec{x}_v}{r_v^3} + \vec{y}(t) \tag{20b}$$

where $m_v(t)$ is the mass of the vehicle, $\vec{x}_v(t)$ the position of the vehicle in the moon's coordinate system, $\vec{v}_v$ its velocity, r_v its distance from the center of the moon and γ the gravitational constant. $\vec{y}(t)$ is defined by:

$$- c \frac{dm_v}{dt} = \sqrt{\vec{y}^2} \tag{21}$$

with the optimization integral:

$$\int_0^T \frac{dm_v}{dt} dt = - \frac{1}{c} \int_0^T \sqrt{\vec{y}^2} \, dt = \text{extr.} \tag{22}$$

where c is the constant exhaust velocity of the vehicle.

Additionally, the following boundary conditions are to be satisfied:

$$\dot{\vec{x}}_v(0) = \vec{v}_v(0), \; \dot{\vec{x}}_v(T) = \vec{v}_v(T) = 2 - 3 \text{ m sec}^{-1} \tag{23}$$

$$\vec{x}_v(0) = \vec{x}_v^o, \quad \vec{x}_v(T) = \vec{x}_{vT}$$

if we assume the initial and final points of the vehicle's path to be given.

Rewriting our equations to be solved we obtain:

$$\begin{aligned} i &= 1, \; (\dot{\vec{x}}_v)_x \equiv \dot{x}_1 = G_1 \equiv x_4; \\ i &= 2, \; (\dot{\vec{x}}_v)_y \equiv \dot{x}_2 = G_2 \equiv x_5; \\ i &= 3, \; (\dot{\vec{x}}_v)_z \equiv \dot{x}_3 = G_3 = x_6; \\ i &= 4, \; (\vec{v}_v)_x \equiv x_4, \end{aligned} \tag{20c}$$

$$i = 5, \quad (\vec{v}_v)_y \equiv x_5$$
$$i = 6, \quad (\vec{v}_v)_z \equiv x_6 \tag{20c}$$

and

$$\begin{pmatrix} G_4 \\ G_5 \\ G_6 \end{pmatrix} \equiv \frac{\gamma m_m}{r^3_v} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \frac{1}{y_4} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{24}$$

with

$$\vec{x}_v = (x_1, x_2, x_3) \tag{25a}$$

and

$$\vec{y} = (y_1, y_2, y_3) \tag{25b}$$

as well as

$$m_v(t) \equiv y_4 \tag{25c}$$

such that we have to solve six equations of the type:

$$\dot{x}_i = G_i(x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4) \tag{20d}$$

which are supplemented by the Eulerian equations stemming from the optimization integral:

$$\dot{\lambda}_K = - \sum_i \lambda_i \frac{\partial G_i}{\partial x_k} + \frac{\partial F}{\partial x_k} \tag{4}$$

$$\frac{d}{dt} \frac{\partial F}{\partial y_j} = - \sum_i \lambda_i \frac{\partial G_i}{\partial y_j} + \frac{\partial F}{\partial y_j} \tag{5}$$

where in our case:

$$F = - \frac{1}{c} \sqrt{y_1^2 + y_2^2 + y_3^2} \tag{26}$$

such that (4) becomes

$$\dot{\lambda}_K = -\lambda_4 \frac{\partial G_4}{\partial x_K} - \lambda_5 \frac{\partial G_5}{\partial x_K} - \lambda_6 \frac{\partial G_6}{\partial x_K} \quad (K=1,2,3) \tag{4a}$$

and

$$\dot{\lambda}_4 = -\lambda_1, \quad \lambda_5 = -\lambda_2, \quad \lambda_6 = -\lambda_3 \tag{4b}$$

since the G_i($i = 1,2,3$) are of the form $G_1 \equiv x_4$, $G_2 \equiv x_5$, $G_3 \equiv x_6$ and the F are independent of x_i. The equations (5) yield:

$$\frac{d}{dt}\frac{\partial F}{\partial y_j} = 0 = \lambda_4 \frac{\partial G_4}{\partial y_j} + \lambda_5 \frac{\partial G_5}{\partial y_j} + \lambda_6 \frac{\partial G_6}{\partial y_j} + \frac{\partial F}{\partial y_j} \tag{5a}$$

$$(j = 1,2,3,4)$$

such that we have 16 equations for 16 functions from which 12 equations for 12 functions (i.e., corresponding to the number of boundary conditions) can be deduced by elimination. The final conditions $\dot{\vec{x}}_v(T) = \vec{v}(T)$ and $\vec{x}_v(T) = \vec{x}_{vT}$ are transformed to initial ones by the process given above.

It was not possible to carry out detailled or numerical calculations since NASA stopped its financial contribution.

The one-dimensional problem of soft landing is treated in many papers /36-41/.

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