ON THE IONIZED LAMINAR BOUNDARY LAYER EQUATIONS

R.J. TABACZYNSKI AND M.E. GOLDSTEIN*

Department of Mechanical Engineering
Massachusetts Institute of Technology,
Cambridge, Massachusetts

The results of previous work on the multi-component laminar boundary layer problem are extended to include two temperature ionized gases and calculations are performed for the equilibrium thermal conductivity of partially ionized argon.

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* Present address: NASA Lewis Research Center, Cleveland, Ohio.

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In a previous paper by one of the authors[1], a new formulation of the multicomponent laminar boundary layer problem, which is in a convenient form for numerical calculation, was developed by a modification of the usual Chapman-Enskog procedure. It was stated in the paper that this procedure could be extended to two temperature ionized gases and to higher orders in the Sonine polynomial expansion (which is necessary to secure accurate results when charged species are present)[2]. This extension is carried out below and the results are listed for convenient reference. The equations given below can then be used with those given in Reference 1 to obtain numerical solutions to two temperature ionized boundary layer problems. In addition to this, the use of these equations to calculate the equilibrium thermal conductivity of partially ionized argon is shown and the results compared with those of De Voto[2].

To develop the electron transport equations, we begin with Eq. (20) of Chmielewski and Ferziger[3] and use the technique developed in Reference 1. After neglecting the isotropic terms which give no contribution to the transport properties, and generalizing to the case where many heavy species (ions and neutrals) are present, Eq. (20) yields:

$$f_e^{\,o}\left[(W_e^{\,2} - 5/2)\,\underset{\sim}{V}_e \cdot \frac{\partial \ln T_e}{\partial \underset{\sim}{r}} + 2\,\underset{\sim}{W}_e^{\,o}\underset{\sim}{W}_e : \frac{\partial \underset{\sim}{V}_o}{\partial \underset{\sim}{r}} + \frac{n}{n_e}\,\underset{\sim}{d}_e \cdot \underset{\sim}{V}_e\right] =$$

$$- n_e \sum_{j \in H} n_j I'_{ej}(\phi_e) - n_e^{\,2} I_e(\phi_e) \qquad (1)$$

Where I'_{ej} and I_e are given by Eqs. (3) and (4) of section (4.4) of Chapman and Cowling[4], and $j \in H$ implies a sum over all the heavy species present. Defining

$$\theta_i = \phi_i - B_i : \frac{\partial V_o}{\partial r}$$

and following the development of section 2 of Reference 1 we find:

$$\theta_e = W_e \cdot \sum_{n=0}^{\infty} \alpha_e^{\,n} S^n(W_e^{\,2})$$

by defining

$$\nu_{ej} = n_j \iint (1-\cos\chi)\, g_{ej}\, b\,db\,d\varepsilon \qquad j \in H$$

where χ is the usual deflection angle,

$$\xi_e^{\,n} = n_e \sqrt{2kT_e m_e}\, \alpha_e^{\,n}$$

where $\xi_e^{\,o} = 2\, j_e$, and

$$Z_{e,H}^{n,r} = \sum_{j \in H} \frac{1}{n_e}(\pi)^{-3/2} \int \exp.\,(-W_e^2)\, W_e^2\, \nu_{ej}\, S^n(W_e^{\,2}) S^r(W_e^{\,2})\, dW_e$$

equation (1) can be put in the form

$$3\, kT_e \left\{ \delta_{ro}\, d_e - \delta_{r1} \frac{5}{2} \frac{\partial}{\partial r} \ln T_e \right\} = - \sum_{n=0}^{\infty} \xi_e^{\,n}\, Z_{e,H}^{n,r}$$

$$- \sum_{n=0}^{\infty} \xi_e^{\,n} \left[S^r(W_e^{\,2}) W_e \,;\, S^n(W_e^{\,2})\, W_e + S^n\,(W_{e'}^{\,2}) W_{e'} \right]_{ee'} \qquad (2)$$

where Eq. (9) Sec. 9.3 of reference 4 has also been used. Using the

expression for the bracketed Sonine polynomials given in References 5 and 6 and

$$\underset{\sim}{d}_e \equiv \left(\frac{e\underset{\sim}{E}}{kT_e} + \frac{1}{n_e kT_e} \nabla P_e\right)$$

we can now express the electron transport equation in the desired form:

$$3e\left(\underset{\sim}{E} + \frac{1}{en_e}\nabla P_e\right) = -2\underset{\sim}{j}_e Z^{o,o}_{e,H} - \sum_{n=1}^{3} \underset{\sim}{\xi}_e^{\,n} Z^{1,n}_{e,H}$$

$$5/2 \frac{\partial \ln T_e}{\partial \underset{\sim}{r}} = 2\underset{\sim}{j}_e Z^{*}_{o,o} + \sum_{n=1}^{3} B_n^{\,e} \underset{\sim}{\xi}_e^{\,n} \tag{3}$$

$$2\underset{\sim}{j}_e Z^{*}_{o,2} + \sum_{n=1}^{3} A^{e}_{1,n} \underset{\sim}{\xi}_e^{\,n} = 0$$

$$2\underset{\sim}{j}_e Z^{*}_{o,3} + \sum_{n=1}^{3} A^{e}_{2,n} \underset{\sim}{\xi}_e^{\,n} = 0$$

where the coefficients are defined by

$$Z^{*}_{ij} \equiv Z^{i,j}_{e,H} / 3kT_e$$

$$B^e_1 \equiv (4\Omega^{2,2}_{e,e} + Z^{1,1}_{e,H}) / 3kT_e \tag{4}$$

$$B^e_2 \equiv (7\Omega^{2,2}_{e,e} - 2\Omega^{2,3}_{e,e} + Z^{2,1}_{e,H}) / 3kT_e$$

$$B^e_3 \equiv \left(\frac{63}{8}\Omega^{2,2}_{e,e} - \frac{9}{2}\Omega^{2,3}_{e,e} + \frac{1}{2}\Omega^{2,4}_{e,e} + Z^{3,1}_{e,H}\right) / 3kT_e$$

$$A^{e}_{1,1} \equiv (7\Omega^{2,2}_{e,e} - 2\Omega^{2,3}_{e,e} + Z^{1,2}_{e,H}) \; /3kT_e$$

$$A^{e}_{1,2} \equiv (\frac{77}{4}\,\Omega^{2,2}_{e,e} - 7\Omega^{2,3}_{e,e} + \Omega^{2,4}_{e,e} + Z^{2,2}_{e,H}) \; /3kT_e$$

$$A^{e}_{1,3} \equiv (\frac{945}{32}\,\Omega^{2,2}_{e,e} - \frac{261}{16}\,\Omega^{2,3}_{e,e} + \frac{25}{8}\,\Omega^{2,4}_{e,e} - \frac{1}{4}\,\Omega^{2,5}_{e,e} + Z^{3,2}_{e,H}) \; /3kT_e$$

$$A^{e}_{2,1} \equiv (\frac{63}{8}\,\Omega^{2,2}_{e,e} - \frac{9}{2}\,\Omega^{2,3}_{e,e} + \frac{1}{2}\,\Omega^{2,4}_{e,e} + Z^{1,3}_{e,H}) \; /3kT_e \qquad (5)$$

$$A^{e}_{2,2} \equiv (\frac{945}{32}\,\Omega^{2,2}_{e,e} - \frac{261}{16}\,\Omega^{2,3}_{e,e} + \frac{25}{8}\,\Omega^{2,4}_{e,e} - \frac{1}{4}\,\Omega^{2,5}_{e,e} + Z^{2,3}_{e,H})/3kT_e$$

$$A^{e}_{2,3} \equiv (\frac{14553}{256}\,\Omega^{2,2}_{e,e} - \frac{1215}{32}\,\Omega^{2,3}_{e,e} + \frac{313}{32}\,\Omega^{2,4}_{e,e} - \frac{9}{8}\,\Omega^{2,5}_{e,e}$$

$$+ \frac{1}{16}\,\Omega^{2,5}_{e,e} + \frac{1}{6}\,\Omega^{4,4}_{e,e} + Z^{3,3}_{e,H}) \; /3kT_e$$

Since the heavy particle transport equations are unaffected by the light species collisions these remain the same as those given in Reference 1.

In order to test this set of equations, the equilibrium thermal conductivity of partially ionized argon was calculated by considering the one dimensional heat conduction problem of stationary ionized argon gas in local thermodynamic equilibrium between two parallel plates maintained at different temperatures. In addition to the above equations charge neutrality and zero net electric current were assumed. The Saha equation was used to determine the concentrations of the charged species and their change with temperature. This provides a complete set of equations. For simplicity, the ratio of electronic partition functions was taken as 1/12 in the Saha equation and the cross sections used in determining the transport

coefficients were taken from References 2 and 6.

For this problem the equations simplify to one ordinary differential equation (involving the derivitive of the temperature in the direction perpendicular to the plates) plus a set of algebraic equations (which are linear in the fluxes[1]). The energy equation shows that the total heat flux is the same at each cross section between the plates. After specifying the total heat flux and the temperature of one of the plates the solutions for the temperature profiles, concentration profiles etc. were obtained by a forward integration of the differential equation in conjunction with the simultaneous solution of the algebraic equations. This procedure was carried out numerically on an I.B.M. 360 computer and the results were used to calculate the equilibrium thermal conductivity which is presented in Figure 1. In this figure the thermal conductivity of argon calculated by De Voto[2] is also shown. Although the maximum discrepency is about 4% it is felt that a more careful integration of the electron-atom cross section would reduce this discrepency to approximately 2%.

It should also be pointed out that this calculation differs from De Voto's[2] in two respects. First, De Voto took into account the electronic state of the atoms and ions by adding the electronic specific heat to the translational contribution and by using the proper values for the electronic partition functions in the Saha equation, whereas this was not done in these calculations. Secondly, De Voto neglected certain terms in evaluating the reactive thermal conductivity but the full expression is retained here.

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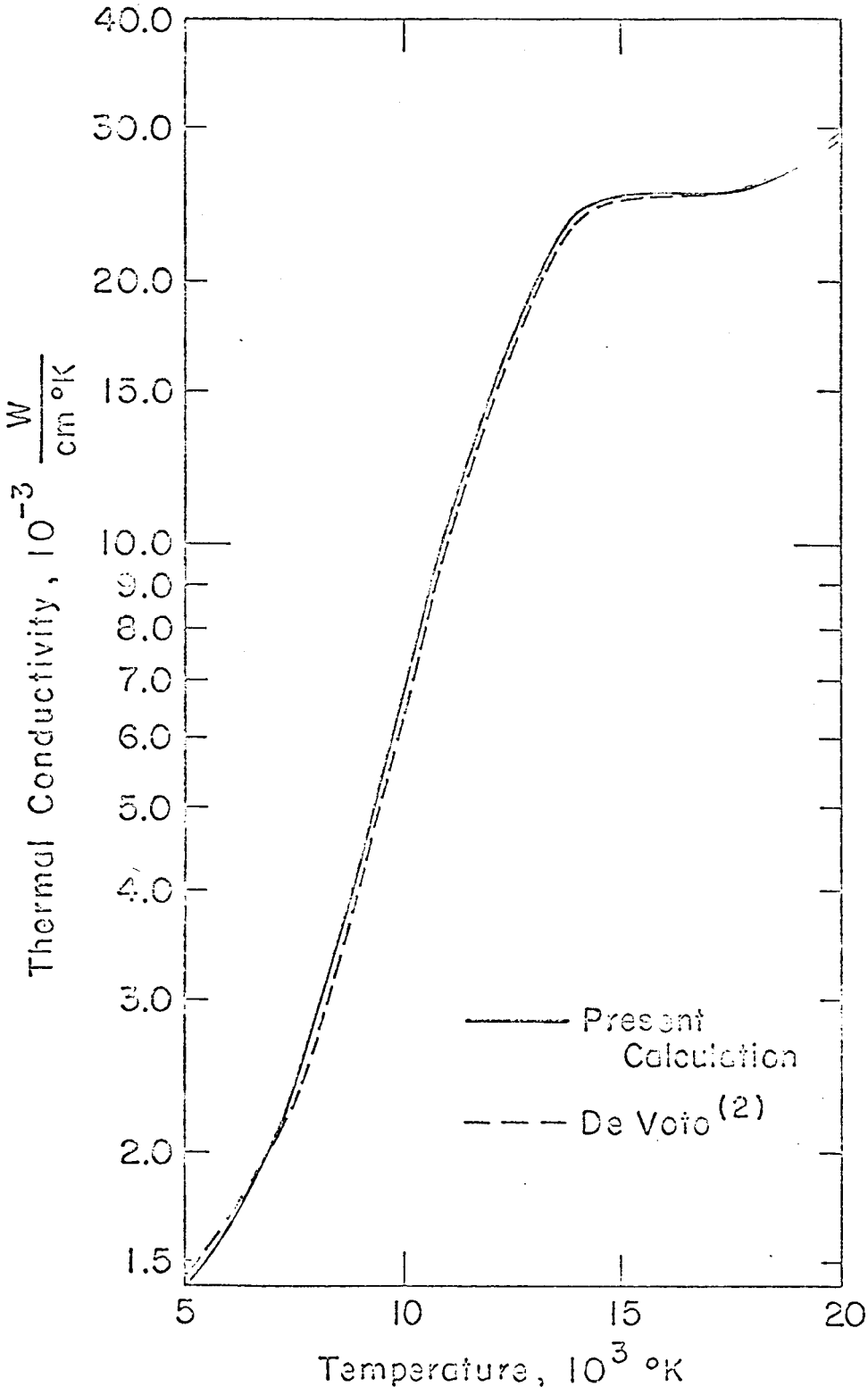

Figure 1. Total thermal conductivity of Argon at 1 atm compared with the theoretical results of De Voto