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Instabilities are important in many branches of physics. Although the question of potential instability of a configuration can be discussed by linearized theory, the question of what happens when an instability actually occurs can be answered only by non-linear analysis. Certain concepts and methods have been developed for treating this question which are well known in the theory of rotating gravitating liquids[1] and of vacuum-tube oscillators[2]. These ideas, which are discussed extensively by Minorski[3], show that instabilities (or rather, the onsets of instability) may be classified into two categories, of which only one can give rise to explosive physical phenomena.

These ideas are very relevant to the analysis of certain astrophysical phenomena which have an explosive nature, such as solar flares[4], galactic explosions[5,6], and Jupiter's decametric radio bursts[7]. As we shall see, an onset of instability which is intrinsically explosive can also be triggered. It is well known that one solar flare can be triggered by a preceding solar flare[4]. Helliwell[8] has documented the triggering of whistlers by natural and artificial disturbances. The triggering of a plasma instability in a laboratory experiment has been observed by Post[9]. [redacted]

In most physical problems, one is concerned not with the stability of a single equilibrium state, but with the stability of a sequence of states which may be enumerated by a parameter μ; such a sequence is known as a "linear series" [10,11]. The case of interest is that a particular linear series, LS1, is stable for values of μ up to a critical value μ_c and unstable for $\mu > \mu_c$. If the instability is not degenerate, i. e. if only one degree of freedom (described in linear theory by a pair of normal modes) becomes unstable at $\mu = \mu_c$, it is permissible to confine our attention to this one degree of freedom.

We wish to know what happens to the system if μ increases slowly and reaches the value μ_c. The crucial point to note, in order to answer this question, is that an even number of linear series branch off from LS1 at the critical point at which $\mu = \mu_c$. If the instability is not degenerate, just two new linear series, LS2 and LS3, branch off from LS1.

We may now make an important distinction. The two new linear series (1) both branch off in directions for which $\mu < \mu_c$; (2) both branch off in directions for which $\mu > \mu_c$. The third possibility, (3) that one branch extends in one direction and the other branch in the other direction, seems to occur only rarely, and will be discussed briefly later on.

It is clear that each equilibrium state may be determined to be either stable or unstable by linearized analysis. It may be shown that

in case (1) the two additional linear series are both unstable; in case (2) the two additional linear series are both stable. These cases are shown schematically in Fig. 1a, and Fig. 2a. These rules may be derived very easily for a simple dynamical system characterized by a positive-definite kinetic energy and a potential energy[12]. The method of Poincaré indices[13,14] provides a more general demonstration.

Suppose now that the system is taken through the sequence of equilibrium states LS1 in case (1) by a slow variation of the parameter μ. When μ attains the value μ_c, the equilibrium states of LS1 become unstable. Furthermore, there is no series of stable states branching off from LS1. Hence, if the system is subject to some slight disturbance, its state will depart from the series LS1. Linear theory does not allow one to understand how the change of state could be catastrophic or "explosive". As Dungey[15] has pointed out in connection with solar flares, one would expect the instability to develop significantly when the linear growth rate is comparable with the rate of change of the states due to the changing parameter μ. One then has the paradox that the instability will never lead to more rapid changes in the system than changes in the parameter μ itself. The resolution of this paradox lies in the fact that, for the topology of equilibrium states of Fig. 1a, the nonlinear terms in the equations of motion will accelerate the instability so that a perturbation will in fact grow "faster than exponentially." Hence, once μ has passed the value μ_c, there will be a drastic change of state so that the onset of instability may be termed "explosive".

For the case of a Hamiltonian system, the change in the dynamical properties of the system may be exemplified by the sequence of diagrams giving the phase-space trajectories. These are shown in Figs. 1b, 1c. Consideration of these diagrams shows that the instability may be precipitated if the system is subjected to a perturbation large enough to put it on a trajectory outside the set of closed trajectories. This shows that, <u>if a system exhibits an explosive onset of instability, this instability may be "triggered"</u>.

In case (2), LS1 becomes unstable at $\mu = \mu_c$, but the branch series LS2 and LS3 are stable. One will therefore expect the state of the system to follow one of the two new branch series. This is known as an "exchange of stabilities". Such an onset of instability may be termed "non-explosive", since there is no finite change of state when μ passes the value μ_c.

It is to be noted that one may determine an onset of instability to be explosive or non-explosive by consideration only of equilibrium states; it is not necessary to actually consider the time development of an instability. This is a valuable simplification. For instance, in studying an extensive medium for which it is appropriate to determine modes which are periodic in one (or more) spatial variable x, it is sufficient to consider solutions of the equation which are also periodic in time. This means that one need consider only "traveling wave" solutions in which all variables are functions of $x-vt$, v being a variable parameter. Hence, although the equations to be solved are

nonlinear, it has been possible to replace a set of partial differential by ordinary differential equations in the single variable x-vt. Friedberg[16] has recently carried out such an analysis of the two-stream electrostatic plasma instability. For a particular wave number, the system becomes unstable when the relative velocity is decreased below a critical value; if one examines Friedberg's results in the light of the present rules, one finds that the onset of instability is non-explosive.

One may also note that the required classification of onsets of instability does not require a rigorously nonlinear analysis. It is possible to classify an onset of instability by including just enough nonlinear terms in the equations of motion to determine the direction in which the new series branch off from the original series. The classification may therefore be discussed in the framework of quasi-linear theory[17].

The central point of this note is the following proposition: <u>if a physical system, which is slowly evolving by a known mechanism, suddenly exhibits a catastrophic change of state, one may assert that there has been an explosive onset of instability in the sense already defined</u>. It is suggested that solar flares, galactic explosions and Jupiter's decametric radio bursts are examples of explosive onsets of instability. This criterion has already been used in appraising models of solar flares[18]; it is proposed that it may also be used to appraise theories of other catastrophic physical phenomena.

The concept of "free energy" has recently been introduced into discussions of instability[19,20]. This enables one to assert the following

simple rule (which has been used elsewhere[21]): <u>If the free energy</u> (in the sense of Gardner) <u>is zero at the onset of instability, the onset is non-explosive</u>.

In the case that a system is composed of many regions or parts, each of which is close to explosive instability, the precipitation of instability in one part will trigger instability in all the other parts, as is seen to occur in a solar flare[22]. Should the instability have the character of an "overstability", the trigger mechanism may lead to a "phase-locking" so that the system develops a <u>coherent</u> pattern of oscillation. This process can explain the remarkable spatial coherence of the source of Jupiter's decametric bursts, inferred by Warwick[23] from the fine-scale structure of these bursts.

In case (3), which was mentioned only briefly, it seems that a system may exhibit either a transfer of stability (non-explosive behavior, determined by the stable series) or a loss of stability (explosive behavior, determined by the unstable series) at the onset of instability, depending on the relative magnitudes of the rate of change of the parameter μ and the amplitude of oscillation of the system before the onset of instability.

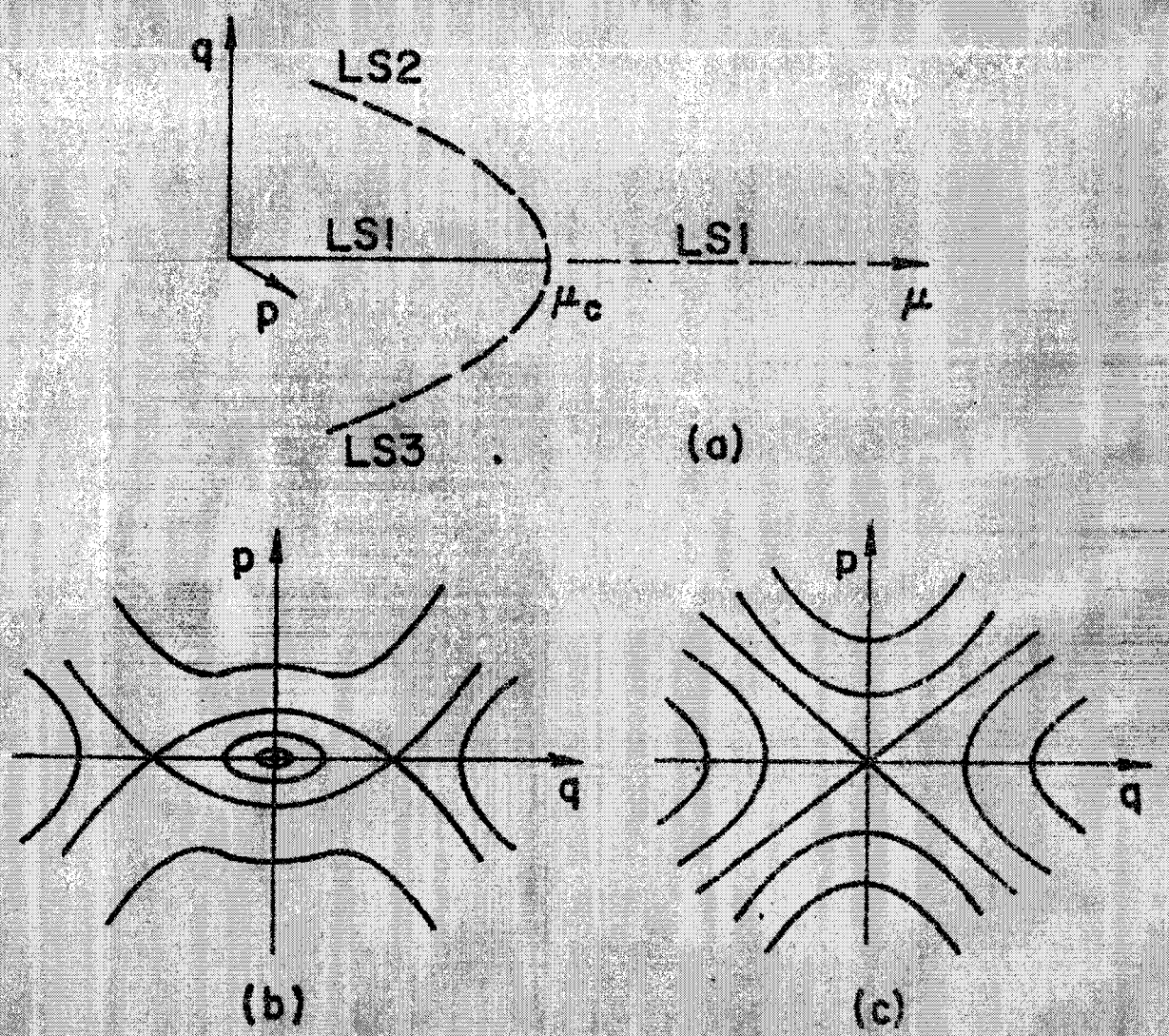

Fig. 1 (a) Linear series in neighborhood of critical point of L/1 for case (I). Solid line, stable equilibria; broken line, unstable equilibria.

(b) Phase-plane curves for conservative system, $\mu < \mu_c$.

(c) Similar curves for $\mu > \mu_c$.

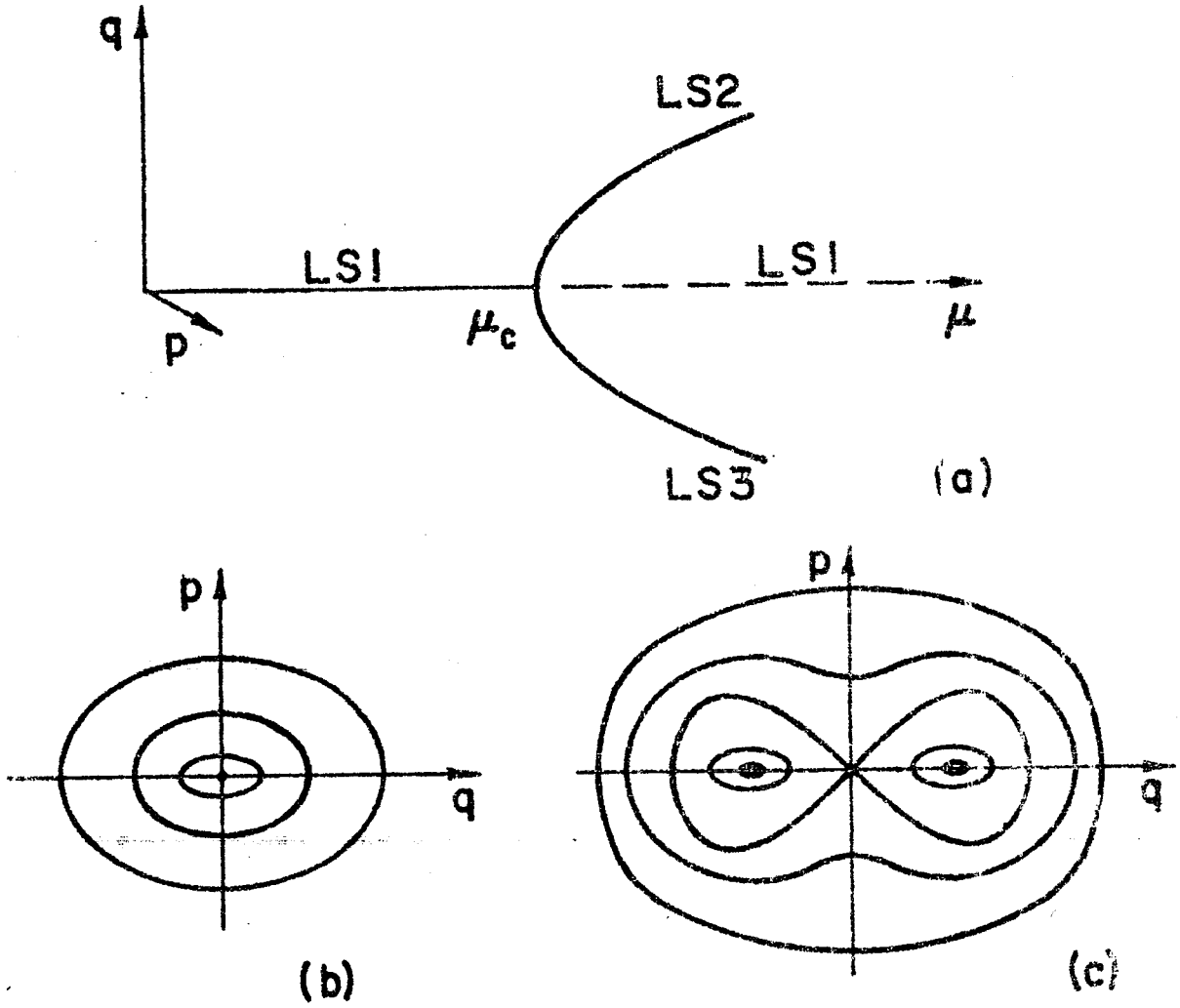

Fig. 2 (a) Linear series in neighbourhood of critical point $\mu = \mu_c$ of LS1 for case (2). Solid line, stable equilibrium; broken line, unstable equilibrium.

(b) Phase-plane curves for conservative system, $\mu < \mu_c$

(c) Similar curves for $\mu > \mu_c$

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