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Blade Temperature Response of the Rotor Entry Vehicle

by

Latif M. Jiji

PREPARED UNDER GRANT NO. NGR 33-013-026
BY THE CITY COLLEGE OF THE CITY UNIVERSITY
OF NEW YORK, NEW YORK

FOR

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

AUGUST 1967

CASE FILE
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NOMENCLATURE

A	defined in equation (45)
$\bar{A}$	defined in equation (25)
a	thickness of refractory layer
B	defined in equation (49)
$\bar{B}$	defined in equation (26)
b	thickness of structural layer
D	defined in equation (51)
E	defined in equation (52)
E_n	defined in equation (14)
F_o	constant surface heat flux
$F(\xi)$	defined in equation (37)
$G_n(\xi)$	defined in equation (38)
H	defined in equation (53)
K	thermal conductivity
$Q(\tau)$	defined in equation (50)
$q_c(o)$	aerodynamic heat flux for a rotating blade at $t = o$
$q_c(t)$	aerodynamic heat flux for a rotating blade at any time t, defined in equation (1)
$q_o(o)$	aerodynamic heat flux for a non-rotating blade at $t = o$
$q_o(t)$	aerodynamic heat flux for a non-rotating blade at any time t
q_R	re-radiation heat flux
R	scale factor
T	temperature
T_e	earth temperature
t	time

t_a	time limit of stage I
t_b	" " " " II
t_c	" " " " III
t_d	" " " " IV
t_m	time at which maximum aerodynamic heating takes place
u	temperature distribution corresponding to a constant heat input, F_o
x	distance
α	thermal diffusivity
β	defined in equation (15)
Γ	defined in equation (24)
γ	defined in equation (27)
δ	amplitude parameter for the aerodynamic heat flux oscillation
ϵ	emissivity
η	variable of integration
θ	dimensionless temperature, defined in equation (17)
Λ	defined in equation (23)
λ_n	eigen-values, roots of equation (13)

μ	dimensionless parameter, defined in equation (15)
ξ	dimensionless distance, defined in equation (15)
σ	dimensionless parameter, defined in equation (15)
$\bar{\sigma}$	Stefan - Boltzmann constant
τ	dimensionless time, defined in equation (15)
τ_α	defined in equation (41)
τ_b	defined in equation (42)
τ_c	defined in equation (47)
τ_d	defined in equation (48)
τ_m	defined in equation (44)
Φ	defined in equation (39)
φ	constant obtained from the aerodynamic heating curve
Ω	dimensionless rotor angular velocity, defined in equation (18)
ω	rotor angular velocity

Subscripts

i	initial
1	refractory material
2	structural material

Superscripts

I	stage I
II	stage II
III	stage III
IV	stage IV

ABSTRACT

An analysis is made of the temperature response of refractory coated blades of the Rotor Entry Vehicle during atmospheric entry. The mathematical model used in the analysis consists of a composite slab insulated on one side and exposed to a time dependent oscillating heat flux on the other side. An exact analytical solution for the temperature response is obtained for a non re-radiating surface. A numerical solution is also obtained which incorporates the effect of re-radiation on the temperature behavior. The effect of re-radiation on the temperature level during entry is found to be significant and favorable.

Oscillation in the aerodynamic heat flux, which is due to blade rotation, results in temperature oscillation during entry. The amplitude of temperature oscillation is negligible for rotor angular velocities which are of practical interest in the Rotor Entry Vehicle.

I. INTRODUCTION

Figure 1 is a diagram of a Rotor Entry Vehicle in hypersonic flight. Because of blade rotation, the stagnation point moves over the blade surface, alternating between leading edge, tip and trailing edge. The resulting aerodynamic heating at any given point on the blade surface becomes a steady periodic function of time for the case of constant free stream conditions and a transient periodic function of time for variable free stream conditions. The latter case characterizes the aerodynamic heating during entry of the Rotor Entry Vehicle. The heating pattern is further complicated by shock impingment. Furthermore, the location of boundary-layer shock interaction may traverse the blade surface. Depending on vehicle orientation and flight angle, the blades may be free of shock impingment during a portion of the cycle, as illustrated in Figure 1.

Part of the energy which is convected to the rotor is reradiated to the surroundings while the rest is conducted through the blades. To provide thermal protection, the blades are coated with a thin layer of refractory material. A typical blade cross-section is shown in Figure 2.

This report deals with the theoretical prediction of the temperature response of the blades' composite structure. Of interest is the identification of the various governing parameters involving thermal properties, rotor angular velocity, and structure geometry, as well as the examination of their effects on the temperature behavior.

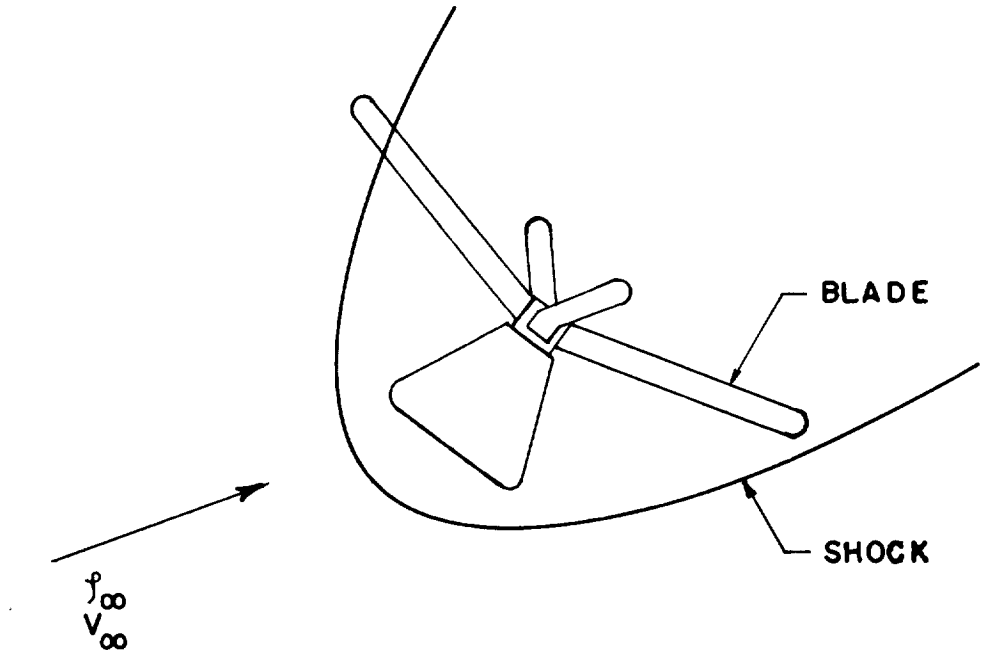


Figure 1.- Rotor Entry Vehicle

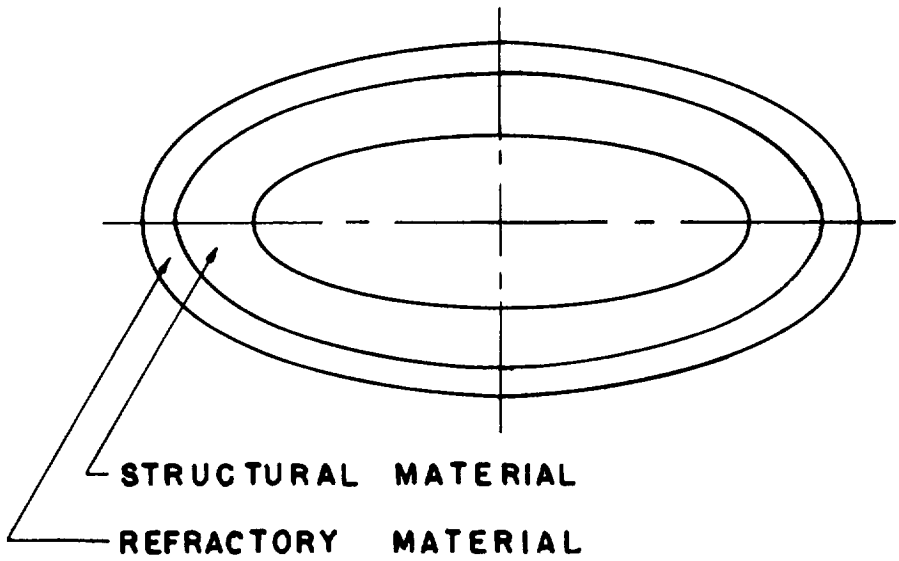


Figure 2.- Cross - sectional area of rotor blade.

The analysis is based on one-dimensional conduction in a composite slab insulated on one side and heated on the other. The convective aerodynamic heat flux, $q_c(t)$, is assumed to vary according to

$$q_c(t) = q_o(t)\, R\,(1 + \delta \sin \omega t) \tag{1}$$

where $q_o(t)$ is the heat flux for a non-rotating blade, R is a scale factor which takes into consideration the change in the mean heat flux as a result of rotation, $\delta R\, q_o(t)$ is the amplitude of heat flux oscillation, and ω is the rotor angular velocity. Figure 3 is a representation of the model used in the analysis. This simplified model does not take into consideration the effect of shock impingment.

Two cases are of interest:

(i) Constant free stream conditions (constant vehicle speed and altitude)

(ii) Variable free stream conditions (entry case)

In case (i) the complications which result from the variations in free stream conditions are eliminated to provide a simplified model which is suitable for a parametric study of the transient response of the blade. Here $q_o(t)$ in equation (1) is treated as constant.

Of more interest is case (ii) where free stream conditions vary during entry according to vehicle speed and altitude, causing $q_o(t)$ to be time dependent. Figure 4 shows a typical stagnation point heating curve for a sphere during entry.

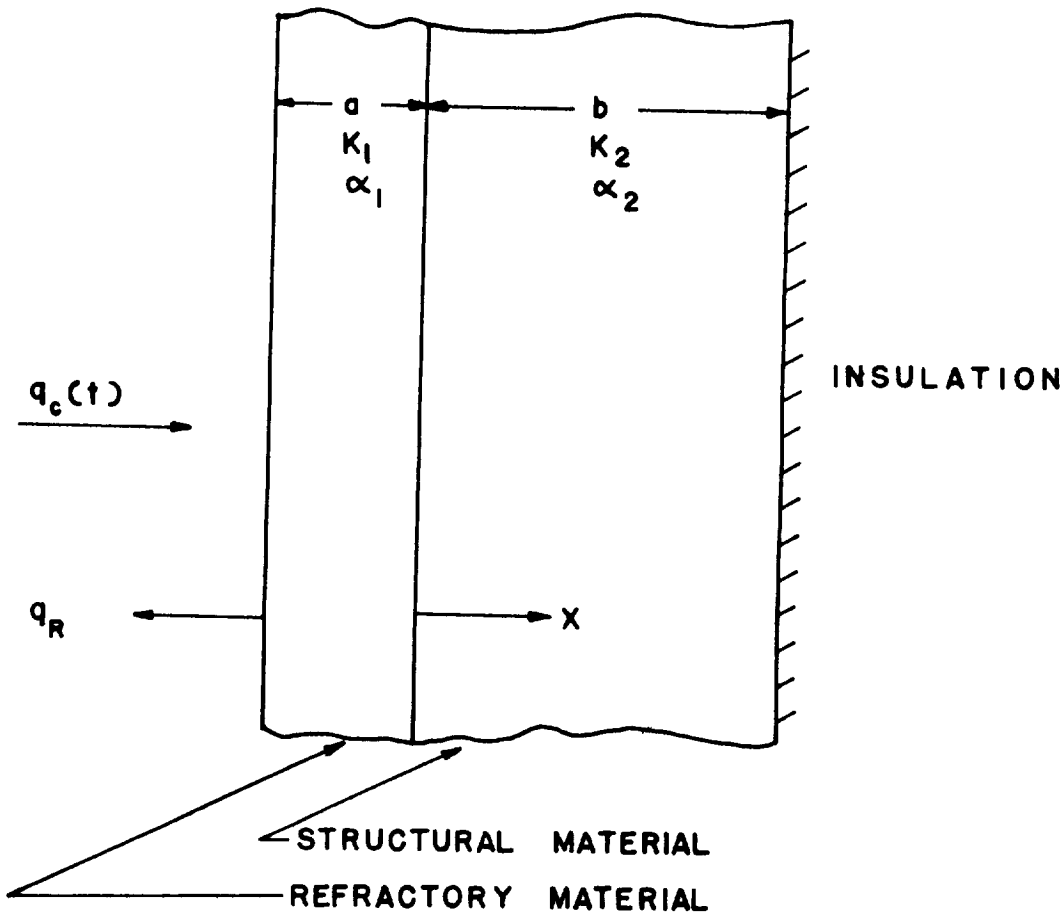


Figure 3.- Analytical model for rotor blade.

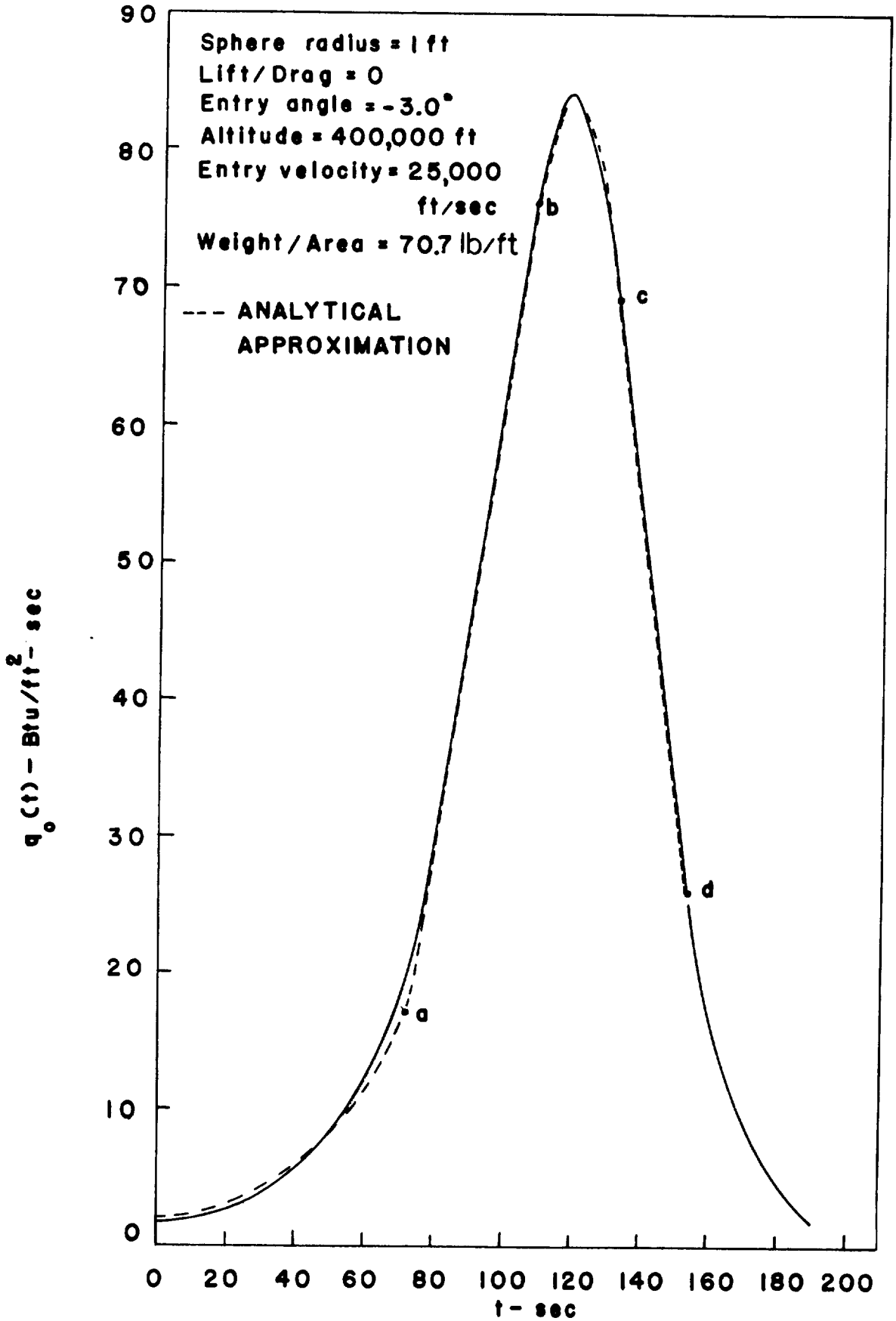


Figure 4.- Typical Stagnation point aerodynamic heating for a sphere during entry.

II. ANALYSIS

FORMULATION

The model used in the analysis of the temperature behavior in the composite structure of the blade is shown in figure 3. The governing equations for the temperature distribution, based on constant thermal properties, are

$$\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial^2 T_1}{\partial x^2} \tag{2}$$

$$\frac{\partial T_2}{\partial t} = \alpha_2 \frac{\partial^2 T_2}{\partial x^2} \tag{3}$$

where subscripts 1 and 2 refer to the refractory and structural materials respectively.

The initial and boundary conditions are

$$T_1(x,0) = T_2(x,0) = T_i \tag{4}$$

$$T_1(0,t) = T_2(0,t) \tag{5}$$

$$K_1 \frac{\partial T_1(0,t)}{\partial x} = K_2 \frac{\partial T_2(0,t)}{\partial x} \tag{6}$$

$$\frac{\partial T_2(b,t)}{\partial x} = 0 \tag{7}$$

$$-K_1 \frac{\partial T_1(-a,t)}{\partial x} = q_c(t) - q_R \tag{8}$$

In boundary condition (8) $q_c(t)$ is the aerodynamic heat flux described by equation (1) and q_R is the re-radiation heat flux given by

$$q_R = \epsilon_1 \sigma \left[T_1^4(-a,t) - T_e^4 \right] \tag{9}$$

SOLUTION

Because of the non linear nature of boundary condition (8), an analytical solution which takes into consideration the effect of re-radiation is not attempted. However, an analytical solution is presented for cases (i) and (ii) for $q_R = 0$. To examine the effect of re-radiation, a numerical solution is obtained.

Analytical Solution

The solution to equations (2) and (3) for any time dependent heat flux, $q_c(t)$, in boundary condition (8) can be obtained from the corresponding solution of constant heat flux by using Duhamel's Integral Equation (Reference 1):

$$T(x,t) - T_i = q_c(0)\,\frac{u(x,t) - u_i}{F_o} + \int_0^t \frac{dq_c(\eta)}{d\eta}\,\frac{u(x,t-\eta) - u_i}{F_o}\,d\eta \quad , \tag{10}$$

where $u(x,t)$ is the solution to the corresponding problem with $q_c(t) = F_o$ = constant. Equation (10) is valid for any function $q_c(t)$ which is piece-wise continuous in the interval $0 < t < \infty$. The temperature response of a composite slab under constant surface heat flux, F_o, is given by (Reference 2)

$$\frac{u_1(x,t) - u_i}{\frac{a F_0}{K_1}} = \frac{1}{6(1+\mu\sigma)}\left[6\tau + 3\xi^2 + 2\sigma^2 - 6\mu\sigma\xi - 1 - 2\frac{(\sigma+\mu)}{(1+\mu\sigma)}\right]$$

$$- \sum_{n=1}^{\infty}\left\{\frac{(1+\mu)\cos[\lambda_n(\sigma-\xi)] + (1-\mu)\cos[\lambda_n(\sigma+\xi)]}{2E_n}\right\} e^{-\lambda_n^2\tau} \tag{11}$$

and

$$\frac{u_2(x,t) - u_i}{\frac{a F_0}{K_1}} = \frac{1}{6(1+\mu\sigma)}\left[6\tau + 3\left(\frac{\sigma}{\beta}\right)^2\xi^2 + 2\sigma^2 - 6\frac{\sigma^2}{\beta}\xi - 1 - 2\sigma\frac{(\sigma+\mu)}{(1+\mu\sigma)}\right]$$

$$- \sum_{n=1}^{\infty} \frac{\cos\left[\lambda_n \frac{\sigma}{\beta}(\xi-\beta)\right]}{E_n} \quad e^{-\lambda_n^2\tau} \tag{12}$$

where λ_n are the roots of

$$(1+\mu)\sin[\lambda_n(1+\sigma)] + (1-\mu)\sin[\lambda_n(1-\sigma)] = 0 \tag{13}$$

and

$$E_n = \frac{\lambda_n^2}{4}\left\{(1+\mu)(1+\sigma)\cos[\lambda_n(1+\sigma)] + (1-\mu)(1-\sigma)\cos[\lambda_n(1-\sigma)]\right\} \tag{14}$$

and

$$\left.\begin{aligned} \tau &= \frac{\alpha_1 t}{a^2} \\ \xi &= \frac{x}{a} \\ \mu &= \frac{k_2}{k_1}\sqrt{\frac{\alpha_1}{\alpha_2}} \\ \sigma &= \frac{b}{a}\sqrt{\frac{\alpha_1}{\alpha_2}} \\ \beta &= \frac{b}{a} \end{aligned}\right\} \tag{15}$$

<u>Case (i) - Constant Free Stream Conditions</u>:

This case represents a simplified model of a rotating blade under constant free stream flow conditions. The aerodynamic heating, $q_c(t)$, in boundary condition (8), is given by equation (1) with $q_o(t) = q_o(0)$ = constant. The purpose of examing this model is to obtain a simplified solution which is useful in making a parametric study of the problem.

Substituting equations (1) and (11) into equation (10) and carrying out the integration, the temperature response of the refractory layer for a periodic heat input is obtained.

$$\theta_1(\xi,\tau) = \frac{1}{1+\mu\sigma}\left[\tau + \frac{\delta}{\Omega}(1-\cos\Omega\tau)\right] + \frac{1}{6(1+\mu\sigma)}\left[3\xi^2\right.$$

$$\left. + 2\sigma^2 - 6\mu\sigma\xi - 1 - 2\sigma\frac{\sigma+\mu}{1+\mu\sigma}\right](1+\delta\sin\Omega\tau)$$

$$- \sum_{n=1}^{\infty}\frac{(1+\mu)\cos[\lambda_n(\sigma-\xi)] + (1-\mu)\cos[\lambda_n(\sigma-\xi)]}{2E_n}\left\{\right.$$

$$1 + \frac{\delta}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\left[\left(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau + \sin\Omega\tau\right)e^{\lambda_n^2\tau}\right.$$

$$\left.\left. - \frac{\lambda_n^2}{\Omega}\right]\right\} e^{-\lambda_n^2\tau} \tag{16}$$

where

$$\theta_1(\xi,\tau) = \frac{T_1(\xi,\tau) - T_i}{\frac{aR}{k_1}q_s(0)} \tag{17}$$

and

$$\Omega = \frac{a^2\omega}{\alpha_1} \tag{18}$$

Similarily, by substituting equations (1) and (12) into equation (10) the temperature response, $\theta_2(\xi,\tau)$, of the structural material may be obtained. However, this will not be carried out here since the interface temperature, which is of primary interest, can be obtained from equation (16) by setting $\xi = 0$.

Case (ii) - Variable Free Stream Conditions: Entry Case

Because of variations in vehicle speed and altitude, the heat flux, $q_o(t)$, in equation (1) is time dependent. A typical plot of $q_o(t)$ during atmospheric entry is shown in figure 4. To evaluate the integral in equation (10), $q_c(t)$ must be known. To obtain an analytical solution, the heating curve of figure 4 is divided into four stages in which the heat flux is approximated by exponential, linear, parabolic, and linear functions, respectively, as follows:

$$q_o^{I}(t) = q_o(0)\, e^{\varphi t} \qquad 0 < t < t_a \tag{19}$$

$$q_o^{II}(t) = q_o(0)\, e^{\varphi t_a}\left[1 + \frac{t - t_a}{t_b - t_a}(\Lambda - 1)\right] \qquad t_a < t < t_b \tag{20}$$

$$q_o^{III}(t) = q_o(0)\left[\Gamma - \bar{A}(t - t_m)^2\right] \qquad t_b < t < t_c \tag{21}$$

$$q_o^{IV}(t) = q_o(0)\, \bar{B}\left[1 + \frac{t - t_c}{t_d - t_c}(\gamma - 1)\right] \qquad t_c < t < t_d \tag{22}$$

where t_a, t_b, t_c and t_d are the limits of stages I, II, III and IV respectively and

$$\Lambda = \frac{q_o^{II}(t_b)}{q_o^{I}(t_a)} \tag{23}$$

$$\Gamma = \frac{q(t_m)}{q_o(o)} \tag{24}$$

$$\bar{A} = (\Gamma - \Lambda e^{\varphi t_a}) \frac{1}{(t_b - t_m)^2} \tag{25}$$

$$\bar{B} = \Gamma - (\Gamma - \Lambda e^{\varphi t_a}) \frac{(t_c - t_m)^2}{(t_b - t_m)^2} \tag{26}$$

$$\gamma = \frac{q_o^{IV}(t_d)}{q_o^{III}(t_c)} \tag{27}$$

The quantities $q_o(o)$, t_a, t_b, t_c, t_d, t_m and φ are known for any given entry heating curve.

Substituting (19), (20), (21) and (22) into equation (1), the aerodynamic heating for a rotating blade during the four stages is obtained.

$$q_c^{I}(t) = q_o(o) R(1 + \delta \sin \omega t) e^{\varphi t} \qquad o < t < t_a \tag{28}$$

$$q_c^{II}(t) = q_o(o) R(1 + \delta \sin \omega t) e^{\varphi t_a} \left[1 + \frac{t - t_a}{t_b - t_a} (\Lambda - 1)\right] \qquad t_a < t < t_b \tag{29}$$

$$q_c^{III}(t) = q_o(o) R(1 + \delta \sin \omega t) \left[\Gamma - \bar{A}(t - t_m)^2\right] \qquad t_b < t < t_c \tag{30}$$

$$q_c^{IV}(t) = q_o(o) R(1 + \delta \sin \omega t) \bar{B} \left[1 + \frac{t - t_c}{t_d - t_c} (\gamma - 1)\right] \qquad t_c < t < t_d \tag{31}$$

The corresponding solutions $T^{I}(x,t)$, $T^{II}(x,t)$, $T^{III}(x,t)$ and $T^{IV}(x,t)$ are obtained from equation (10) by breaking up the limits of integration into appropriate stages. Thus for the four stages, equation (10) gives:

$$T^{I}(x,t)-T_i = q_c(0)\,\frac{u(x,t)-u_i}{F_0} + \int_0^t \frac{dq_c^{I}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta \tag{32}$$

$$T^{II}(x,t)-T_i = q_c(0)\,\frac{u(x,t)-u_i}{F_0} + \int_0^{t_a} \frac{dq_c^{I}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta + \int_{t_a}^{t} \frac{dq_c^{II}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta \tag{33}$$

$$T^{III}(x,t)-T_i = q_c(0)\,\frac{u(x,t)-u_i}{F_0} + \int_0^{t_a} \frac{dq_c^{I}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta + \int_{t_a}^{t_b} \frac{dq_c^{II}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta + \int_{t_b}^{t} \frac{dq_c^{III}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta \tag{34}$$

$$T^{IV}(x,t)-T_i = q_c(0)\,\frac{u(x,t)-u_i}{F_0} + \int_0^{t_a} \frac{dq_c^{I}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta + \int_{t_a}^{t_b} \frac{dq_c^{II}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta + \int_{t_b}^{t_c} \frac{dq_c^{III}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta + \int_{t_c}^{t} \frac{dq_c^{IV}(\eta)}{d\eta}\left[\frac{u(x,t-\eta)-u_i}{F_0}\right]d\eta \tag{35}$$

Solution for Stage I: $0 < \tau < \tau_\alpha$

Substituting equations (11) and (28) into (32), the temperature response $\theta_1^I(\xi,\tau)$ in the refractory layer is obtained. The result, expressed in dimensionless form, is given by:

$$\theta_1^I(\xi,\tau) = \frac{1}{1+\mu\sigma}\,\frac{1}{\Phi}\left[e^{\Phi\tau}-1\right] + F(\xi)e^{\Phi\tau}\left[1+\delta\sin\Omega\tau\right]$$

$$-\sum_{n=1}^{\infty} G_n(\xi)\left\{e^{-\lambda_n^2\tau} + \frac{\Phi}{(\lambda_n^2+\Phi)}\left[e^{\Phi\tau}-e^{-\lambda_n^2\tau}\right]\right\}$$

$$+\frac{1}{1+\mu\sigma}\,\frac{e^{\Phi\tau}}{\left[\left(\frac{\Phi}{\Omega}\right)^2+1\right]}\,\frac{\delta}{\Omega}\left[\frac{\Phi}{\Omega}\sin\Omega\tau - \cos\Omega\tau\right.$$

$$\left.+\,e^{-\Phi\tau}\right] - \sum_{n=1}^{\infty} G_n(\xi)\,\frac{\delta e^{\Phi\tau}}{\left[\frac{\lambda_n^2}{\Omega}+\frac{\Phi}{\Omega}\right]^2+1}\left\{\left[\right.\right.$$

$$\left.\frac{\Phi}{\Omega}\left(\frac{\lambda_n^2}{\Omega}+\frac{\Phi}{\Omega}\right)+1\right]\sin\Omega\tau + \frac{\lambda_n^2}{\Omega}\cos\Omega\tau$$

$$\left.-\frac{\lambda_n^2}{\Omega}e^{-(\lambda_n^2+\Phi)\tau}\right\} \qquad (36)$$

where

$$F(\xi) = \frac{1}{6(1+\mu\sigma)}\left[3\xi^2 + 2\sigma^2 - 6\mu\sigma\xi - 1 - 2\sigma\frac{\sigma+\mu}{1+\mu\sigma}\right] \tag{37}$$

$$G_n(\xi) = \frac{(1+\mu)\cos[\lambda_n(\sigma-\xi)] + (1-\mu)\cos[\lambda_n(\sigma+\xi)]}{2E_n} \tag{38}$$

$$\Phi = \frac{a^2\varphi}{\alpha_1} \tag{39}$$

<u>Solution for Stage II: $\tau_a < \tau < \tau_b$</u>

Using (11), (29) and (33), the solution for stage II is obtained.

$$\theta_1^{II}(\xi,\tau) = \frac{1}{1+\mu\sigma}\left\{(\tau-\tau_a)e^{\Phi\tau_a} + \frac{1}{\Phi}(e^{\Phi\tau_a}-1)\right\}$$

$$+ \frac{1}{1+\mu\sigma}\frac{\delta}{\Omega}\frac{e^{\Phi\tau_a}}{\left(\frac{\Phi}{\Omega}\right)^2+1}\left[\frac{\Phi}{\Omega}\sin\Omega\tau_a - \cos\Omega\tau_a + e^{-\Phi\tau_a}\right]$$

$$-\sum_{n=1}^{\infty}\frac{G_n(\xi)}{\left(\frac{\lambda_n^2}{\Phi}+1\right)}e^{-\lambda_n^2(\tau-\tau_a)}\left[e^{\Phi\tau_a} + \frac{\lambda_n^2}{\Phi}e^{-\lambda_n^2\tau_a}\right]$$

$$-\delta\sum_{n=1}^{\infty}G_n(\xi)\frac{e^{-\lambda_n^2(\tau-\tau_a)}}{\left(\frac{\Phi}{\Omega}+\frac{\lambda_n^2}{\Omega}\right)^2+1}\left\{\left[\frac{\Phi}{\Omega}\left(\frac{\Phi}{\Omega}+\frac{\lambda_n^2}{\Omega}\right)\sin\Omega\tau_a\right.\right.$$

$$\left.\left.+\sin\Omega\tau_a + \frac{\lambda_n^2}{\Omega}\cos\Omega\tau_a\right]e^{\Phi\tau_a} - \frac{\lambda_n^2}{\Omega}e^{-\lambda_n^2\tau_a}\right\}$$

$$\frac{1}{1+\mu\sigma} e^{\Phi\tau_a}\Big\{\frac{\delta}{\Omega}(\cos\Omega\tau_a - \cos\Omega\tau) + \frac{\delta}{\Omega^2}\frac{(\Lambda-1)}{(\tau_b-\tau_a)}\Big[-\Omega(\tau-\tau_a)\cos\Omega\tau$$

$$+ \sin\Omega\tau - \sin\Omega\tau_a\Big] + \frac{(\Lambda-1)}{(\tau_b-\tau_a)}\frac{(\tau-\tau_a)^2}{2}\Big\} + e^{\Phi\tau_a} F(\xi)\Big(1$$

$$+ \delta\sin\Omega\tau\Big)\Big[1 + \frac{(\Lambda-1)}{(\tau_b-\tau_a)}(\tau-\tau_a)\Big] - e^{\Phi\tau_a}\sum_{n=1}^{\infty} G_n(\xi)\Big[\Big[$$

$$\frac{\delta}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Big\{\frac{\lambda_n^2}{\Omega}\cos\Omega\tau + \sin\Omega\tau - \Big(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_a$$

$$+ \sin\Omega\tau_a\Big) e^{-\lambda_n^2(\tau-\tau_a)}\Big\} + \frac{(\Lambda-1)}{(\tau_b-\tau_a)}\delta\Big\{\frac{(\tau-\tau_a)}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Big(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau$$

$$+ \sin\Omega\tau\Big) - \frac{1}{\Omega}\frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2+1\right]^2}\Big[\Big(\frac{\lambda_n^2}{\Omega}\Big)^2\cos\Omega\tau + 2\frac{\lambda_n^2}{\Omega}\sin\Omega\tau$$

$$- \cos\Omega\tau - \Big\langle\Big(\frac{\lambda_n^2}{\Omega}\Big)^2\cos\Omega\tau_a + 2\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_a$$

$$- \cos\Omega\tau_a\Big\rangle e^{-\lambda_n^2(\tau-\tau_a)}\Big]\Big]\Big\} + \frac{(\Lambda-1)}{(\tau_b-\tau_a)}\frac{1}{\lambda_n^2}\Big\{1$$

$$- e^{-\lambda_n^2(\tau-\tau_a)}\Big\} + \frac{(\Lambda-1)}{(\tau_b-\tau_a)}\frac{\delta}{\Omega}\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Big[\frac{\lambda_n^2}{\Omega}\sin\Omega\tau$$

$$- \cos\Omega\tau - \Big(\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_a - \cos\Omega\tau_a\Big)e^{-\lambda_n^2(\tau-\tau_a)}\Big]\Big]\Big] \qquad (40)$$

where

$$\tau_a = \frac{\alpha_1 t_a}{a^2} \tag{41}$$

$$\tau_b = \frac{\alpha_1 t_b}{a^2} \tag{42}$$

<u>Solution for Stage III: $\tau_b < \tau < \tau_c$</u>

Substituting equations (11) and (30) into (34), the solution for stage III is obtained

$$\theta_1^{III}(\xi,\tau) = \frac{1}{1+\mu\sigma}\left\{(\tau-\tau_a)(1+\delta \sin\Omega\tau_a)e^{\Phi\tau_a} + \frac{1}{\Phi}(e^{\Phi\tau_a}-1)\right\}$$

$$+ \frac{1}{1+\mu\sigma}\frac{\delta}{\Omega}\frac{e^{\Phi\tau_a}}{\left(\frac{\Phi}{\Omega}\right)^2+1}\left[\frac{\Phi}{\Omega}\sin\Omega\tau_a - \cos\Omega\tau_a + e^{-\Phi\tau_a}\right]$$

$$+ F(\xi)(1+\delta \sin\Omega\tau_a)e^{\Phi\tau_a} - \sum_{n=1}^{\infty}\frac{G_n(\xi)}{\left(\frac{\lambda_n^2}{\Phi}+1\right)}e^{-\lambda_n^2(\tau-\tau_a)}\Bigg[$$

$$e^{\Phi\tau_a} + \frac{\lambda_n^2}{\Phi}e^{-\lambda_n^2\tau_a}\Bigg] - \delta\sum_{n=1}^{\infty}\frac{G_n(\xi)}{\left(\frac{\Phi}{\Omega}+\frac{\lambda_n^2}{\Omega}\right)^2+1}e^{-\lambda_n^2(\tau-\tau_a)}\Bigg\{\Bigg[$$

$$\frac{\Phi}{\Omega}\left(\frac{\Phi}{\Omega}+\frac{\lambda_n^2}{\Omega}\right)\sin\Omega\tau_a + \sin\Omega\tau_a + \frac{\lambda_n^2}{\Omega}\cos\Omega\tau_a\Bigg]e^{\Phi\tau_a}$$

$$-\frac{\lambda_n^2}{\Omega}e^{-\lambda_n^2\tau_a}\Bigg\} + \frac{e^{\Phi\tau_a}}{1+\mu\sigma}\Bigg[\frac{\delta}{\Omega}(\Omega\tau\sin\Omega\tau_b - \cos\Omega\tau_b$$

$$-\Omega\tau_b\sin\Omega\tau_b - \Omega\tau\sin\Omega\tau_a + \cos\Omega\tau_a + \Omega\tau_a\sin\Omega\tau_a)$$

$$+\frac{(\Lambda-1)}{(\tau_b-\tau_a)}\frac{\delta}{\Omega^2}(\sin\Omega\tau_b-\sin\Omega\tau_a)-(\Lambda-1)\frac{\delta}{\Omega}(\cos\Omega\tau_b$$

$$+\Omega\tau_b\sin\Omega\tau_b-\Omega\tau\sin\Omega\tau_b)+(\Lambda-1)\left[\tau-\tfrac{1}{2}(\tau_b+\tau_a)\right]\Bigg]$$

$$+e^{\Phi\tau_a}F(\xi)\left[\delta(\sin\Omega\tau_b-\sin\Omega\tau_a)+(\Lambda-1)(1+\delta\sin\Omega\tau_b)\right]$$

$$-e^{\Phi\tau_a}\sum_{n=1}^{\infty}G_n(\xi)e^{-\lambda_n^2(\tau-\tau_b)}\Bigg[\frac{\delta}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\left\{\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_b\right.$$

$$\left.+\sin\Omega\tau_b-\left(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_a+\sin\Omega\tau_a\right)e^{-\lambda_n^2(\tau_b-\tau_a)}\right\}$$

$$+\frac{(\Lambda-1)}{(\tau_b-\tau_a)}\delta\left\{\frac{\tau_b-\tau_a}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\left(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_b+\sin\Omega\tau_b\right)\right.$$

$$-\frac{1}{\Omega}\frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2+1\right]^2}\left\{\left(\frac{\lambda_n^2}{\Omega}\right)^2\cos\Omega\tau_b+2\left(\frac{\lambda_n^2}{\Omega}\right)\sin\Omega\tau_b-\cos\Omega\tau_b\right.$$

$$\left.\left.-\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2\cos\Omega\tau_a+2\left(\frac{\lambda_n^2}{\Omega}\right)\sin\Omega\tau_a-\cos\Omega\tau_a\right]e^{-\lambda_n^2(\tau_b-\tau_a)}\right\}\right\}$$

$$+\frac{(\Lambda-1)}{\tau_b-\tau_a}\frac{1}{\lambda_n^2}\left(1-e^{-\lambda_n^2(\tau_b-\tau_a)}\right)+\frac{(\Lambda-1)}{\tau_b-\tau_a}\frac{\delta}{\Omega}\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\left\{\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_b\right.$$

$$\left.-\cos\Omega\tau_b-\left(\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_a-\cos\Omega\tau_a\right)e^{-\lambda_n^2(\tau_b-\tau_a)}\right\}\Bigg]$$

$$+\frac{1}{1+\mu\sigma}\Bigg[-A\left\{\frac{1}{3}\tau^3-\tau\tau_b^2+\frac{2}{3}\tau_b^3-\tau_m(\tau-\tau_b)^2\right\}$$

$$+ \frac{\delta}{\Omega}(\Gamma - A\tau_m^2)\left[-\cos\Omega\tau - \Omega\tau\sin\Omega\tau_b + \cos\Omega\tau_b + \Omega\tau_b\sin\Omega\tau_b\right]$$

$$- 2A\frac{\delta}{\Omega^3}\left[-\Omega\tau\sin\Omega\tau - 2\cos\Omega\tau - \Omega\tau\sin\Omega\tau_b + \Omega^2\tau\tau_b\cos\Omega\tau_b\right.$$

$$\left. + 2\Omega\tau_b\sin\Omega\tau_b - (\Omega^2\tau_b^2 - 2)\cos\Omega\tau_b\right] + 2A\tau_m\frac{\delta}{\Omega^2}\left[-\sin\Omega\tau\right.$$

$$\left. + \sin\Omega\tau_b + \Omega(\tau - \tau_b)\cos\Omega\tau_b\right] + 2A\tau_m\frac{\delta}{\Omega^2}\left[-\Omega\tau\cos\Omega\tau\right.$$

$$+ 2\sin\Omega\tau - \Omega\tau\cos\Omega\tau_b - \Omega^2\tau\tau_b\sin\Omega\tau_b + 2\Omega\tau_b\cos\Omega\tau_b$$

$$\left. + \Omega^2\tau_b^2\sin\Omega\tau_b - 2\sin\Omega\tau_b\right] - A\frac{\delta}{\Omega^3}\left\{-\Omega^2\tau^2\cos\Omega\tau\right.$$

$$- 2\Omega\tau\sin\Omega\tau + 6\cos\Omega\tau + 6\Omega\tau\sin\Omega\tau - 2\Omega^2\tau\tau_b\cos\Omega\tau_b$$

$$- \Omega\tau(\Omega^2\tau_b^2 - 2)\sin\Omega\tau_b + (3\Omega^2\tau_b^2 - 6)\cos\Omega\tau_b + \Omega^3\tau_b^3\sin\Omega\tau_b$$

$$\left.\left. - 6\Omega\tau_b\sin\Omega\tau_b\right\}\right] + F(\xi)\left\{-A(\tau^2 - \tau_b^2) + 2A\tau_m(\tau - \tau_b)\right.$$

$$+ \delta(\Gamma - A\tau_m^2)(\sin\Omega\tau - \sin\Omega\tau_b) - 2A\frac{\delta}{\Omega^2}(\sin\Omega\tau - \Omega\tau\cos\Omega\tau$$

$$- \sin\Omega\tau_b + \Omega\tau_b\cos\Omega\tau_b) + 2A\frac{\delta}{\Omega}\tau_m(\cos\Omega\tau_b + \Omega\tau\sin\Omega\tau$$

$$- \cos\Omega\tau_b - \Omega\tau_b\sin\Omega\tau_b) - A\frac{\delta}{\Omega^2}(2\Omega\tau\cos\Omega\tau + \Omega^2\tau^2\sin\Omega\tau$$

$$- 2 \sin \Omega\tau - 2\Omega\tau_b \cos \Omega\tau_b - \Omega^2 \tau_b^2 \sin \Omega\tau_b + 2 \sin \Omega\tau_b) \Big\}$$

$$+ \sum_{n=1}^{\infty} G_n(\xi) \Big[\Big[2A \left(\frac{1}{\lambda_n^2}\right)^2 \Big\{ (\lambda_n^2 \tau - 1) - (\lambda_n^2 \tau_b - 1) e^{-\lambda_n^2 (\tau - \tau_b)} \Big\}$$

$$- 2A \frac{\tau_m}{\lambda_n^2} (1 - e^{-\lambda_n^2 (\tau - \tau_b)}) - \delta(\Gamma - A\tau_m^2) \frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1} \Big\{ \frac{\lambda_n^2}{\Omega} \cos \Omega\tau$$

$$+ \sin \Omega\tau - \left(\frac{\lambda_n^2}{\Omega} \cos \Omega\tau_b + \sin \Omega\tau_b \right) e^{-\lambda_n^2 (\tau - \tau_b)} \Big\}$$

$$+ 2A\delta \Big[\frac{1}{\Omega} \frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1} \Big\{ \tau \left(\frac{\lambda_n^2}{\Omega} \sin \Omega\tau - \cos \Omega\tau \right) - \tau_b \left(\frac{\lambda_n^2}{\Omega} \sin \Omega\tau_b$$

$$- \cos \Omega\tau_b \right) e^{-\lambda_n^2 (\tau - \tau_b)} \Big\} - \frac{1}{\Omega^2} \frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1 \right]^2} \Big\{ \left(\frac{\lambda_n^2}{\Omega}\right)^2 \sin \Omega\tau$$

$$- 2\left(\frac{\lambda_n^2}{\Omega}\right) \cos \Omega\tau - \sin \Omega\tau - \Big\{ \left(\frac{\lambda_n^2}{\Omega}\right)^2 \sin \Omega\tau_b - 2 \frac{\lambda_n^2}{\Omega} \cos \Omega\tau_b$$

$$- \sin \Omega\tau_b \Big\} e^{-\lambda_n^2 (\tau - \tau_b)} \Big\} \Big] - 2A\delta\tau_m \Big[\frac{1}{\Omega} \frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1} \Big\{ \frac{\lambda_n^2}{\Omega} \sin \Omega\tau$$

$$- \cos \Omega\tau - \left(\frac{\lambda_n^2}{\Omega} \sin \Omega\tau_b - \cos \Omega\tau_b \right) e^{-\lambda_n^2 (\tau - \tau_b)} \Big\} \Big]$$

$$- 2A\tau_m \delta \Big[\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1} \Big\{ \tau \left(\frac{\lambda_n^2}{\Omega} \cos \Omega\tau + \sin \Omega\tau \right) - \tau_b \left(\frac{\lambda_n^2}{\Omega} \cos \Omega\tau_b$$

$$+ \sin \Omega\tau_b \right) e^{-\lambda_n^2 (\tau - \tau_b)} \Big\} - \frac{1}{\Omega} \frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1 \right]^2} \Big\{ \left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega\tau$$

$$+ 2\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega\tau - \cos \Omega\tau - \Big\{ \left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega\tau_b + 2\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega\tau_b$$

$$- \cos \Omega \tau_b \Big\} e^{-\lambda_n^2(\tau-\tau_b)} \Big\} \Big] + A\delta \Big[\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1} \Big\{ \tau^2 \Big(\frac{\lambda_n^2}{\Omega} \cos \Omega \tau$$

$$+ \sin \Omega \tau \Big) - \tau_b^2 \Big(\frac{\lambda_n^2}{\Omega} \cos \Omega \tau_b + \sin \Omega \tau_b \Big) e^{-\lambda_n^2(\tau-\tau_b)} \Big\}$$

$$- \frac{1}{\Omega} \frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2+1\right]^2} \Big\{ \tau \Big\{ 2\left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega \tau + 4\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega \tau$$

$$- 2 \cos \Omega \tau \Big\} - \tau_b \Big\{ 2\left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega \tau_b + 4\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega \tau_b$$

$$- 2 \cos \Omega \tau_b \Big\} e^{-\lambda_n^2(\tau-\tau_b)} \Big\} + \frac{1}{\Omega^3} \frac{2}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2+1\right]^3} \Big\{ \left(\frac{\lambda_n^2}{\Omega}\right)^3 \cos \Omega \tau$$

$$+ 3\left(\frac{\lambda_n^2}{\Omega}\right)^2 \sin \Omega \tau - 3\left(\frac{\lambda_n^2}{\Omega}\right) \cos \Omega \tau - \sin \Omega \tau - \Big\{ \left(\frac{\lambda_n^2}{\Omega}\right)^3 \cos \Omega \tau_b$$

$$+ 3\left(\frac{\lambda_n^2}{\Omega}\right)^2 \sin \Omega \tau_b - 3\left(\frac{\lambda_n^2}{\Omega}\right) \cos \Omega \tau_b$$

$$- \sin \Omega \tau_b \Big\} e^{-\lambda_n^2(\tau-\tau_b)} \Big\} \Big] \Big] \tag{43}$$

where

$$\tau_m = \frac{\alpha_1 t_m}{a^2} \tag{44}$$

$$A = \bar{A}\left(\frac{a^2}{\alpha_1}\right)^2 = \left(\Gamma - \Lambda e^{\Phi\tau_a}\right)\frac{1}{(\tau_b - \tau_m)^2} \tag{45}$$

<u>Solution for Stage IV: $\tau_c < \tau < \tau_d$</u>

Using (11), (31) and (35), the solution for stage IV is obtained

$$\theta_1^{IV}(\xi,\tau) = \frac{1}{1+\mu\sigma}\left\{(\tau-\tau_a)(1+\delta\sin\Omega\tau_a)e^{\Phi\tau_a} + \frac{1}{\Phi}(e^{\Phi\tau_a}-1)\right\}$$

$$+ \frac{1}{1+\mu\sigma}\frac{\delta}{\Omega}e^{\Phi\tau_a}\frac{1}{\left(\frac{\Phi}{\Omega}\right)^2+1}\left[\frac{\Phi}{\Omega}\sin\Omega\tau_a - \cos\Omega\tau_a + e^{-\Phi\tau_a}\right] + F(\xi)(1$$

$$+ \delta\sin\Omega\tau_a)e^{\Phi\tau_a} - \sum_{n=1}^{\infty}\frac{G_n(\xi)}{\left(\frac{\lambda_n^2}{\Phi}\right)+1}e^{-\lambda_n^2(\tau-\tau_a)}\left[e^{\Phi\tau_a} + \frac{\lambda_n^2}{\Phi}e^{-\lambda_n^2\tau_a}\right]$$

$$- \delta\sum_{n=1}^{\infty}\frac{G_n(\xi)}{\left(\frac{\Phi}{\Omega}+\frac{\lambda_n^2}{\Omega}\right)^2+1}e^{-\lambda_n^2(\tau-\tau_a)}\left[\left\{\frac{\Phi}{\Omega}\left(\frac{\Phi}{\Omega}+\frac{\lambda_n^2}{\Omega}\right)\sin\Omega\tau_a\right.\right.$$

$$\left.\left. + \sin\Omega\tau_a + \frac{\lambda_n^2}{\Omega}\cos\Omega\tau_a\right\}e^{\Phi\tau_a} - \frac{\lambda_n^2}{\Omega}e^{-\lambda_n^2\tau_a}\right]$$

$$+ e^{\Phi\tau_a}\frac{1}{1+\mu\sigma}\left\{\frac{\delta}{\Omega}(\Omega\tau\sin\Omega\tau_b - \cos\Omega\tau_b - \Omega\tau_b\sin\Omega\tau_b\right.$$

$$- \Omega\tau\sin\Omega\tau_a + \cos\Omega\tau_a + \Omega\tau_a\sin\Omega\tau_a) + \frac{(A-1)}{\tau_b-\tau_a}\frac{\delta}{\Omega^2}(\sin\Omega\tau_b$$

$$- \sin \Omega\tau_a) - (\Lambda - 1)\frac{\delta}{\Omega}(\cos \Omega\tau_b + \Omega\tau_b \sin \Omega\tau_b - \Omega\tau \sin \Omega\tau_b)$$

$$+ (\Lambda - 1)\left[\tau - \frac{1}{2}(\tau_b + \tau_a)\right]\Big\} + e^{\Phi\tau_a} F(\xi)\Big\{\delta(\sin \Omega\tau_b - \sin \Omega\tau_a)$$

$$+ (\Lambda - 1)(1 + \delta \sin \Omega\tau_b)\Big\} - e^{\Phi\tau_a} \sum_{n=1}^{\infty} G_n(\xi) e^{-\lambda_n^2(\tau - \tau_b)} \Big[$$

$$\frac{\delta}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1}\left\{\frac{\lambda_n^2}{\Omega}\cos \Omega\tau_b + \sin \Omega\tau_b - \left(\frac{\lambda_n^2}{\Omega}\cos \Omega\tau_a + \sin \Omega\tau_a\right)e^{-\lambda_n^2(\tau_b - \tau_a)}\right\}$$

$$+ \frac{(\Lambda - 1)}{\tau_b - \tau_a}\delta \Big\{\frac{\tau_b - \tau_a}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1}\left(\frac{\lambda_n^2}{\Omega}\cos \Omega\tau_b + \sin \Omega\tau_b\right)$$

$$- \frac{1}{\Omega}\frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1\right]^2}\Big\{\left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega\tau_b + 2\left(\frac{\lambda_n^2}{\Omega}\right)\sin \Omega\tau_b - \cos \Omega\tau_b$$

$$- \left[\left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega\tau_a + 2\frac{\lambda_n^2}{\Omega}\sin \Omega\tau_a - \cos \Omega\tau_a\right]e^{-\lambda_n^2(\tau_b - \tau_a)}\Big\}\Big\}$$

$$+ \frac{(\Lambda - 1)}{(\tau_b - \tau_a)}\frac{1}{\lambda_n^2}\left(1 - e^{-\lambda_n^2(\tau_b - \tau_a)}\right) + \frac{(\Lambda - 1)}{(\tau_b - \tau_a)}\frac{\delta}{\Omega}\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1}\Big\{\frac{\lambda_n^2}{\Omega}\sin \Omega\tau_b$$

$$- \cos \Omega\tau_b - \left(\frac{\lambda_n^2}{\Omega}\sin \Omega\tau_a - \cos \Omega\tau_a\right)e^{-\lambda_n^2(\tau_b - \tau_a)}\Big\}\Big]$$

$$+ \frac{1}{1 + \mu\sigma}\Big[-A\tau\tau_c^2 + \frac{2}{3}A\tau_c^3 + A\tau\tau_b^2 - \frac{2}{3}A\tau_b^3 + 2A\tau_m(\tau\tau_c$$

$$- \frac{1}{2}\tau_c^2 - \tau\tau_b + \frac{1}{2}\tau_b^2) + \frac{\delta}{\Omega}(\Gamma - A\tau_m^2)(\Omega\tau \sin \Omega\tau_c - \cos \Omega\tau_c$$

$$- \Omega\tau_c \sin\Omega\tau_c - \Omega\tau \sin\Omega\tau_b + \cos\Omega\tau_b + \Omega\tau_b \sin\Omega\tau_b) - 2A\frac{\delta}{\Omega^3}\Big\{\Omega\tau \sin\Omega\tau_c$$

$$- \Omega^2\tau\tau_c \cos\Omega\tau_c - 2\Omega\tau_c \sin\Omega\tau_c + (\Omega^2\tau_c^2 - 2)\cos\Omega\tau_c - \Omega\tau \sin\Omega\tau_b$$

$$+ \Omega^2\tau\tau_b \cos\Omega\tau_b + 2\Omega\tau_b \sin\Omega\tau_b - (\Omega^2\tau_b^2 - 2)\cos\Omega\tau_b\Big\}$$

$$+ 2A\delta\tau_m \frac{1}{\Omega^2}\big(-\Omega\tau \cos\Omega\tau_c - \sin\Omega\tau_c + \Omega\tau_c \cos\Omega\tau_c + \Omega\tau \cos\Omega\tau_b$$

$$+ \sin\Omega\tau_b - \Omega\tau_b \cos\Omega\tau_b\big) + 2A\tau_m \delta \frac{1}{\Omega^2}\Big\{\Omega\tau \cos\Omega\tau_c$$

$$+ \Omega^2\tau\tau_c \sin\Omega\tau_c - 2\Omega\tau_c \cos\Omega\tau_c - (\Omega^2\tau_c^2 - 2)\sin\Omega\tau_c$$

$$- \Omega\tau \cos\Omega\tau_b - \Omega^2\tau\tau_b \sin\Omega\tau_b + 2\Omega\tau_b \cos\Omega\tau_b + (\Omega^2\tau_b^2 - 2)\sin\Omega\tau_b\Big\}$$

$$- A\delta \frac{1}{\Omega^3}\Big\{2\Omega^2\tau\tau_c \cos\Omega\tau_c + \Omega\tau(\Omega^2\tau_c^2 - 2)\sin\Omega\tau_c$$

$$- (3\Omega^2\tau_c^2 - 6)\cos\Omega\tau_c - (\Omega^3\tau_c^3 - 6\Omega\tau_c)\sin\Omega\tau_c - 2\Omega^2\tau\tau_b \cos\Omega\tau_b$$

$$- \Omega\tau(\Omega^2\tau_b^2 - 2)\sin\Omega\tau_b + (3\Omega^2\tau_b^2 - 6)\cos\Omega\tau_b + (\Omega^3\tau_b^3 - 6\Omega\tau_b)\sin\Omega\tau_b\Big\}\Big]$$

$$+ F(s)\Big[-A(\tau_c^2 - \tau_b^2) + 2A\tau_m(\tau_c - \tau_b) + \delta(\Gamma - A\tau_m^2)(\sin\Omega\tau_c - \sin\Omega\tau_b)$$

$$- 2A\frac{\delta}{\Omega^2}\Big\{\sin\Omega\tau_c - \Omega\tau_c \cos\Omega\tau_c - \sin\Omega\tau_b + \Omega\tau_b \cos\Omega\tau_b\Big\}$$

$$+ 2A\tau_m \frac{\delta}{\Omega}(\cos\Omega\tau_b - \cos\Omega\tau_c) + 2A\tau_m \frac{\delta}{\Omega}(\cos\Omega\tau_c + \Omega\tau_c \sin\Omega\tau_c$$

$$- \cos\Omega\tau_b - \Omega\tau_b \sin\Omega\tau_b) - A\frac{\delta}{\Omega^2}(2\Omega\tau_c \cos\Omega\tau_c + \Omega^2\tau_c^2 \sin\Omega\tau_c$$

$$- 2\sin\Omega\tau_c - 2\Omega\tau_b \cos\Omega\tau_b - \Omega^2\tau_b^2 \sin\Omega\tau_b + 2\sin\Omega\tau_b)\Big]$$

$$+\sum_{n=1}^{\infty} G_n(\xi)\, e^{-\lambda_n^2(\tau-\tau_c)} \Bigg[\Bigg[2A \frac{1}{(\lambda_n^2)^2}\left\{(\lambda_n^2\tau_c-1)-(\lambda_n^2\tau_b-1)e^{-\lambda_n^2(\tau_c-\tau_b)}\right\}$$

$$-2A\tau_m \frac{1}{\lambda_n^2}\left(1-e^{-\lambda_n^2(\tau_c-\tau_b)}\right)-\delta(\Gamma-A\tau_m^2)\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Bigg[\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_c$$

$$+\sin\Omega\tau_c-\left(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_b+\sin\Omega\tau_b\right)e^{-\lambda_n^2(\tau_c-\tau_b)}\Bigg]$$

$$+2A\delta\Bigg[\frac{1}{\Omega}\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Bigg\{\tau_c\left(\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_c-\cos\Omega\tau_c\right)-\tau_b\left(\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_b$$

$$-\cos\Omega\tau_b\right)e^{-\lambda_n^2(\tau_c-\tau_b)}\Bigg\}\Bigg]-\frac{1}{\Omega^2}\frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2+1\right]^2}\Bigg\{\left(\frac{\lambda_n^2}{\Omega}\right)^2\sin\Omega\tau_c$$

$$-2\left(\frac{\lambda_n^2}{\Omega}\right)\cos\Omega\tau_c-\sin\Omega\tau_c-\Bigg\{\left(\frac{\lambda_n^2}{\Omega}\right)^2\sin\Omega\tau_b-2\left(\frac{\lambda_n^2}{\Omega}\right)\cos\Omega\tau_b$$

$$-\sin\Omega\tau_b\Bigg\}e^{-\lambda_n^2(\tau_c-\tau_b)}\Bigg\}\Bigg]-2A\delta\tau_m\Bigg[\frac{1}{\Omega}\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Bigg\{\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_c$$

$$-\cos\Omega\tau_c-\left(\frac{\lambda_n^2}{\Omega}\sin\Omega\tau_b-\cos\Omega\tau_b\right)e^{-\lambda_n^2(\tau_c-\tau_b)}\Bigg\}\Bigg]$$

$$-2A\tau_m\delta\Bigg[\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Bigg\{\tau_c\left(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_c+\sin\Omega\tau_c\right)-\tau_b\left(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_b$$

$$+\sin\Omega\tau_b\right)e^{-\lambda_n^2(\tau_c-\tau_b)}\Bigg\}-\frac{1}{\Omega}\frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2+1\right]^2}\Bigg\{\left(\frac{\lambda_n^2}{\Omega}\right)^2\cos\Omega\tau_c$$

$$+2\left(\frac{\lambda_n^2}{\Omega}\right)\sin\Omega\tau_c-\cos\Omega\tau_c-\Bigg\{\left(\frac{\lambda_n^2}{\Omega}\right)^2\cos\Omega\tau_b+2\left(\frac{\lambda_n^2}{\Omega}\right)\sin\Omega\tau_b$$

$$-\cos\Omega\tau_b\Bigg\}e^{-\lambda_n^2(\tau_c-\tau_b)}\Bigg\}\Bigg]+A\delta\Bigg[\frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1}\Bigg\{\tau_c^2\left(\frac{\lambda_n^2}{\Omega}\cos\Omega\tau_c\right.$$

$$+ \sin \Omega \tau_c) - \tau_b^2 \left(\frac{\lambda_n^2}{\Omega} \cos \Omega \tau_b + \sin \Omega \tau_b\right) e^{-\lambda_n^2(\tau_c - \tau_b)} \Bigg\}$$

$$- \frac{1}{\Omega} \frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1\right]^2} \Bigg\{ \tau_c \left\{ 2\left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega \tau_c + 4\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega \tau_c - 2 \cos \Omega \tau_c \right\}$$

$$- \tau_b \left\{ 2\left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega \tau_b + 4\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega \tau_b - 2 \cos \Omega \tau_b \right\} e^{-\lambda_n^2(\tau_c - \tau_b)} \Bigg\}$$

$$+ \frac{1}{\Omega^2} \frac{2}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1\right]^3} \Bigg\{ \left(\frac{\lambda_n^2}{\Omega}\right)^3 \cos \Omega \tau_c + 3\left(\frac{\lambda_n^2}{\Omega}\right)^2 \sin \Omega \tau_c - 3\left(\frac{\lambda_n^2}{\Omega}\right) \cos \Omega \tau_c$$

$$- \sin \Omega \tau_c - \left\{ \left(\frac{\lambda_n^2}{\Omega}\right)^3 \cos \Omega \tau_b + 3\left(\frac{\lambda_n^2}{\Omega}\right)^2 \sin \Omega \tau_b - 3\left(\frac{\lambda_n^2}{\Omega}\right) \cos \Omega \tau_b \right.$$

$$\left. - \sin \Omega \tau_b \right\} e^{-\lambda_n^2(\tau_c - \tau_b)} \Bigg\} \Bigg] \Bigg] \Bigg] + \frac{1}{1 + \mu\sigma} B \Bigg[\frac{(\gamma - 1)}{2(\tau_d - \tau_c)} (\tau - \tau_c)^2$$

$$+ \delta \frac{(\gamma - 1)}{\tau_d - \tau_c} \left\{ \frac{1}{\Omega^2} (\sin \Omega \tau_c - \sin \Omega \tau) + \frac{1}{\Omega} (\tau - \tau_c) \cos \Omega \tau_c \right\}$$

$$+ \delta \left\{ \frac{1}{\Omega} (\cos \Omega \tau_c - \cos \Omega \tau) - (\tau - \tau_c) \sin \Omega \tau_c \right\} + \frac{\gamma - 1}{\tau_d - \tau_c} \frac{\delta}{\Omega^2} \Big\{ \Omega(\tau + \tau_c)(\cos \Omega \tau$$

$$+ \Omega \tau \sin \Omega \tau - \cos \Omega \tau_c - \Omega \tau_c \sin \Omega \tau_c) - 2 \Omega \tau \cos \Omega \tau - (\Omega^2 \tau^2$$

$$- 2) \sin \Omega \tau + 2 \Omega \tau_c \cos \Omega \tau_c + (\Omega^2 \tau_c^2 - 2) \sin \Omega \tau_c - \Omega^2 \tau \tau_c (\sin \Omega \tau - \sin \Omega \tau_c) \Big\} \Bigg]$$

$$+ F(\xi) B \left[\frac{\gamma - 1}{\tau_d - \tau_c} (\tau - \tau_c)(1 + \delta \sin \Omega \tau) + \delta (\sin \Omega \tau - \sin \Omega \tau_c) \right]$$

$$- \sum_{n=1}^{\infty} G_n(\xi) B \Bigg[\frac{(\gamma - 1)}{(\tau_d - \tau_c)} \frac{1}{\lambda_n^2} \left(1 - e^{-\lambda_n^2(\tau - \tau_c)}\right) + \frac{\delta}{\Omega} \frac{(\gamma - 1)}{(\tau_d - \tau_c)} \frac{1}{\left(\frac{\lambda_n^2}{\Omega}\right)^2 + 1} \Bigg\{ \frac{\lambda_n^2}{\Omega} \sin \Omega \tau$$

$$- \cos \Omega\tau - \left(\frac{\lambda_n^2}{\Omega} \sin \Omega\tau_c - \cos \Omega\tau_c \right) e^{-\lambda_n^2(\tau-\tau_c)} \Big\} + \frac{\delta}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1} \Big\{ \frac{\lambda_n^2}{\Omega} \cos \Omega\tau$$

$$+ \sin \Omega\tau - \left(\frac{\lambda_n^2}{\Omega} \cos \Omega\tau_c + \sin \Omega\tau_c \right) e^{-\lambda_n^2(\tau-\tau_c)} \Big\}$$

$$+ \frac{(\gamma-1)}{(\tau_d-\tau_c)} \delta \Bigg[\frac{(\tau-\tau_c)}{\left(\frac{\lambda_n^2}{\Omega}\right)^2+1} \left(\frac{\lambda_n^2}{\Omega} \cos \Omega\tau + \sin \Omega\tau \right) - \frac{1}{\Omega} \frac{1}{\left[\left(\frac{\lambda_n^2}{\Omega}\right)^2+1\right]^2} \Big\{$$

$$\left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega\tau + 2\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega\tau - \cos \Omega\tau - \Big\langle \left(\frac{\lambda_n^2}{\Omega}\right)^2 \cos \Omega\tau_c$$

$$+ 2\left(\frac{\lambda_n^2}{\Omega}\right) \sin \Omega\tau_c - \cos \Omega\tau_c \Big\rangle e^{-\lambda_n^2(\tau-\tau_c)} \Big\} \Bigg] \Bigg] \tag{46}$$

where

$$\tau_c = \frac{\alpha_1 t_c}{a^2} \tag{47}$$

$$\tau_d = \frac{\alpha_1 t_d}{a^2} \tag{48}$$

$$B = \Gamma - (\Gamma - \Lambda e^{\Phi\tau_a}) \frac{(\tau_c - \tau_m)^2}{(\tau_d - \tau_m)^2} \tag{49}$$

Numerical Solution:

Since the analytical solution is limited to the case of a non re-radiating surface, a numerical solution is presented which takes into consideration the re-radiation effect.

The governing equations and boundary conditions are first non-dimensionalized by introducing the following dimensionless parameters:

$$Q(\tau) = \frac{q_o(t)}{q_o(0)} \tag{50}$$

$$D = \frac{\epsilon_1 \bar{\sigma} T_i^4}{q_o(0) R} \tag{51}$$

$$E = \frac{q_o(0) R a}{K_1 T_i} \tag{52}$$

$$H = \left(\frac{T_a}{T_i}\right)^4 \tag{53}$$

The complete governing equations and boundary conditions written in dimensionless from become:

$$\frac{\partial \theta_1(\xi,\tau)}{\partial \tau} = \frac{\partial^2 \theta_1(\xi,\tau)}{\partial \xi^2} \tag{54}$$

$$\frac{\partial \theta_2(\xi,\tau)}{\partial \tau} = \left(\frac{\beta}{\sigma}\right)^2 \frac{\partial^2 \theta_2(\xi,\tau)}{\partial \xi^2} \tag{55}$$

$$\theta_1(\xi,0) = \theta_2(\xi,0) = 0 \tag{56}$$

$$\theta_1(0,\tau) = \theta_2(0,\tau) \tag{57}$$

$$\frac{\partial\theta_1(0,\tau)}{\partial\xi} = \mu\,\frac{\beta}{\sigma}\,\frac{\partial\theta_2(0,\tau)}{\partial\xi} \tag{58}$$

$$\frac{\partial\theta_2(\beta,\tau)}{\partial\xi} = 0 \tag{59}$$

$$-\frac{\partial\theta_1(-1,\tau)}{\partial\xi} = Q(\tau)\left[1 + \delta \sin \Omega\tau\right] - D\left\{\left[1 + E\,\theta_1(-1,\tau)\right]^4 - H\right\} \tag{60}$$

By setting D = 0, the numerical solution gives the case of a non re-radiating surface. By setting $Q(\tau) = 1$ the solution to case (i) is obtained.

Details of the formulation of the numerical solution is given in Appendix A. The computer programs of the numerical solution for cases (i) and (ii) are presented in Appendices B and C respectively.

III. RESULTS

Case (i) - Constant Free Stream Conditions

The transient temperature behavior in the refractory material, $\theta_1(\xi,\tau)$, for a non re-radiating surface under constant free stream condition is described by equation (16). This simple analytical solution of the idealized model is obtained for the purpose of making a parametric study of the problem. Equation (16) contains four parameters, μ, σ, δ and Ω, which govern the temperature behavior. The eigenvalues, λ_n, are obtained from equation (13). Computer programs for determining λ_n and for evaluating $\theta_1(\xi,\tau)$ are presented in Appendices D and E respectively.

Figure 5 gives the temperature response at the exposed surface of the refractory material, $\xi = -1$, and at the interface, $\xi = 0$, for various values of the governing parameters. The effect of Ω on the temperature behavior during the initial period is shown in figure 5(a). At the exposed surface, values of Ω greater than 100 have a negligible effect on the temperature level. For low values of Ω the effect is significant. However, at the interface, temperature oscillation is damped out for $\Omega \geq 100$ and negligible for $\Omega = 10$.

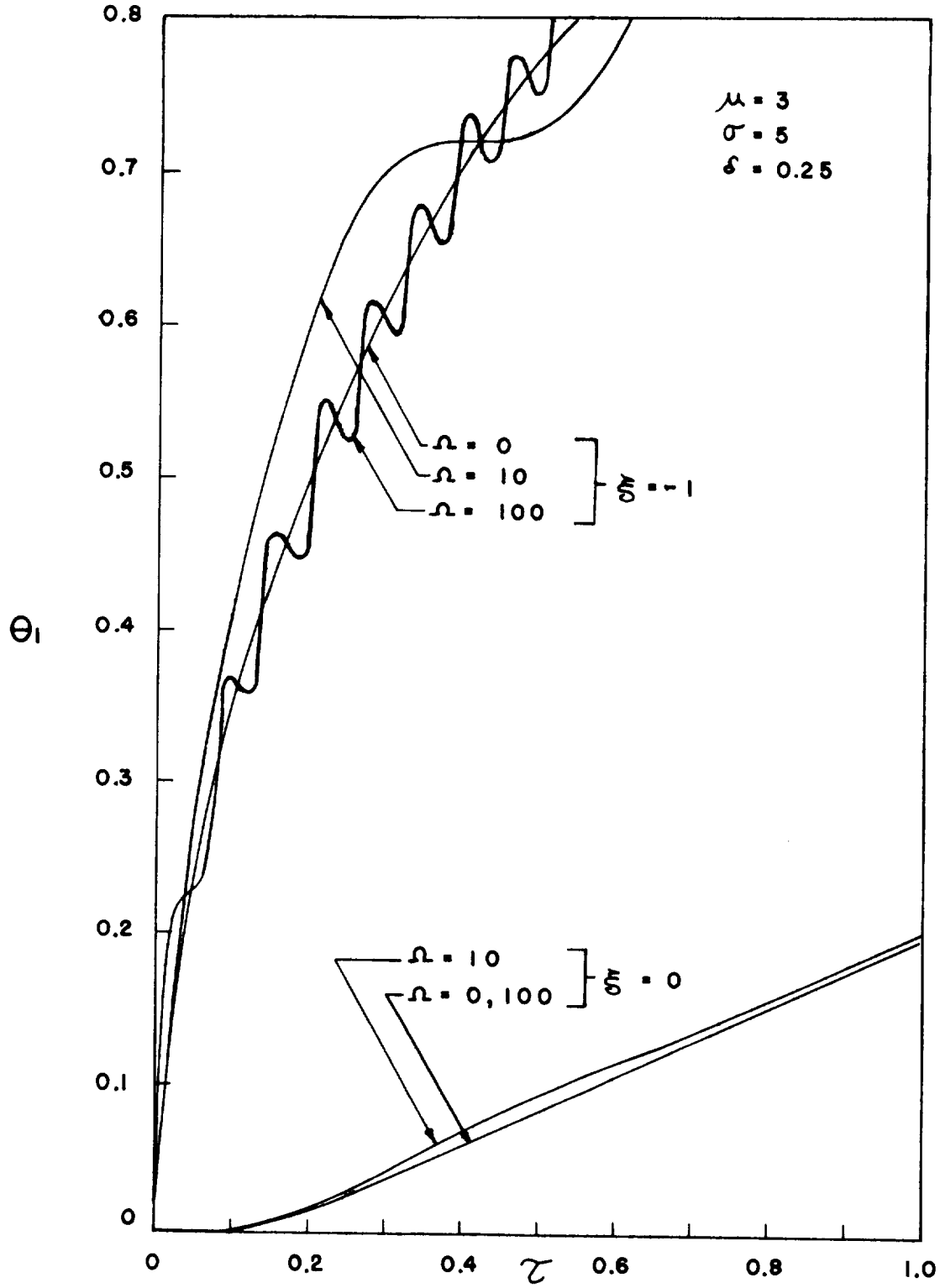
A long time solution is presented in figure 5(b). Temperature oscillation at the interface, $\xi = 0$, is not observable, while at the exposed surface, $\xi = -1$, the temperature oscillates about the curve $\Omega = 0$. Because of the time scale used in figure 5(b), individual cycles are not distinguishable. Instead, the locus of amplitudes of temperature oscillation for $\Omega = 10$ is shown.

Since δ is a measure of the amplitude of heat flux oscillation at the exposed surface, figures 5(c) and 5(d) are presented to illustrate its effect on the temperature behavior. For values of δ as high as 0.75, no effect is observed at the interface for $\Omega = 100$. At the exposed surface a short time solution, figure 5(c), shows that the amplitude of temperature oscillation increases with increasing δ.

The parameter μ, which is defined as $\frac{k_2}{k_1}\sqrt{\frac{\alpha_1}{\alpha_2}}$ has a significant effect on the temperature level. While this effect is not pronounced at small values of τ (figure 5(e)), it becomes appreciable at large values of τ (figure 5(f)).

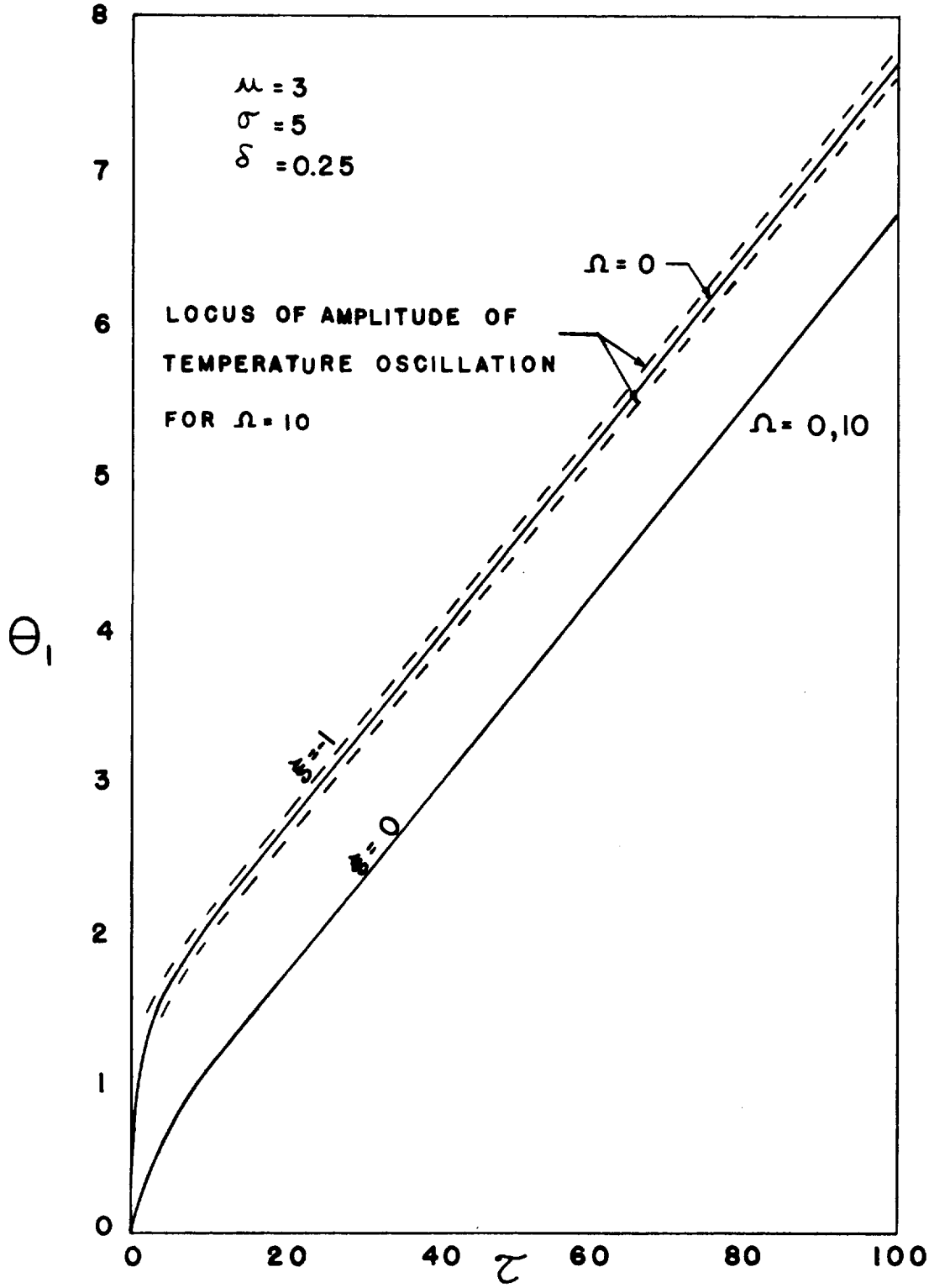
Figures 5(g) and 5(h) give the temperature response for two values of σ. The temperature level is observed to decrease as σ is increased.

Figure 5(g), which gives a short time solution, indicates that for $\sigma = 50$ the solution does not converge to the initial value, i.e. $\theta_1(\xi,0) = 0$, for $\tau = 0$. This indicates that additional terms in the series appearing in equation (16) should be taken. In constructing the charts for figure 5 the series was truncated at n = 32.



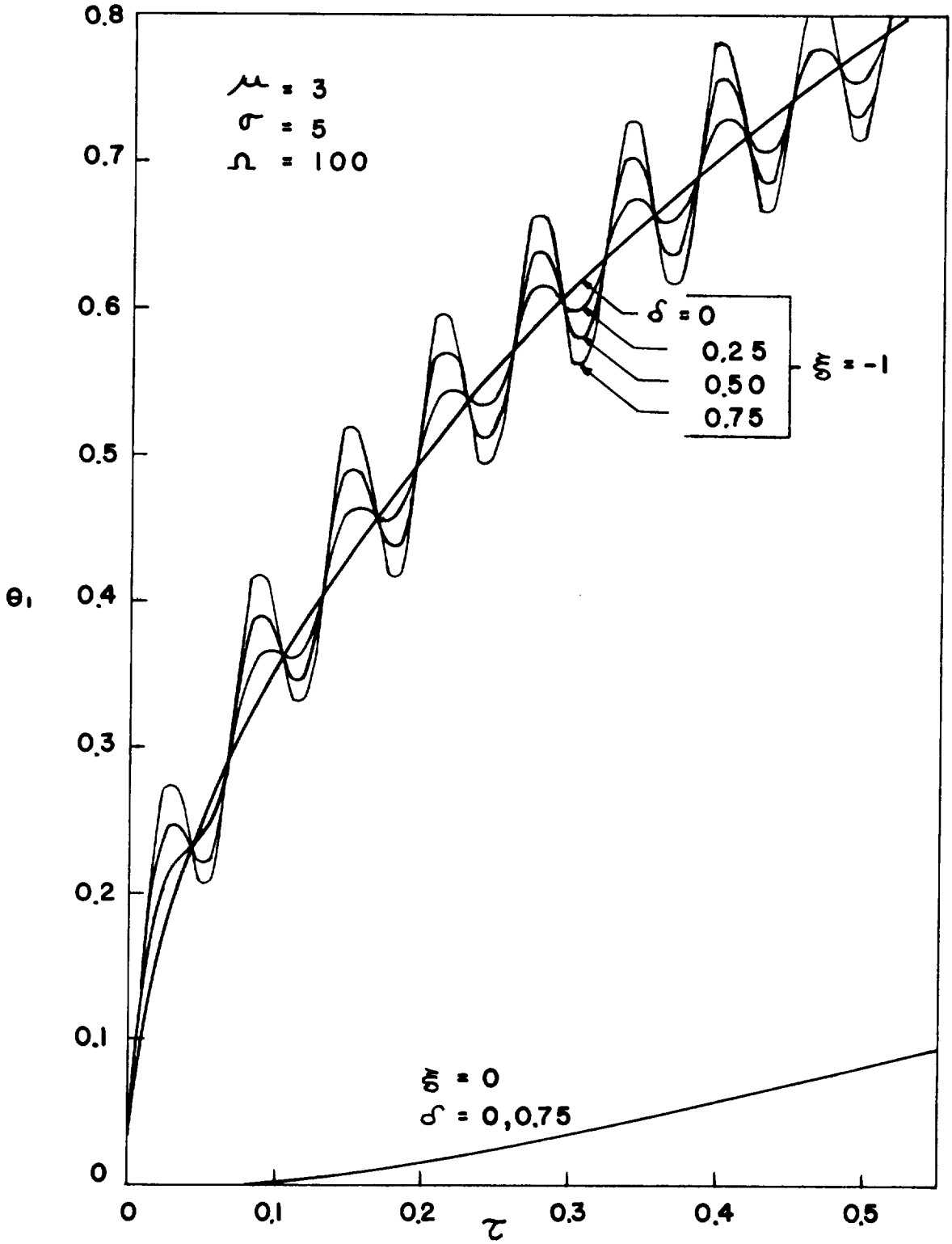
(a) Effect of Ω- short time

Figure 5.- Temperature response for case (i): Constant free stream conditions.



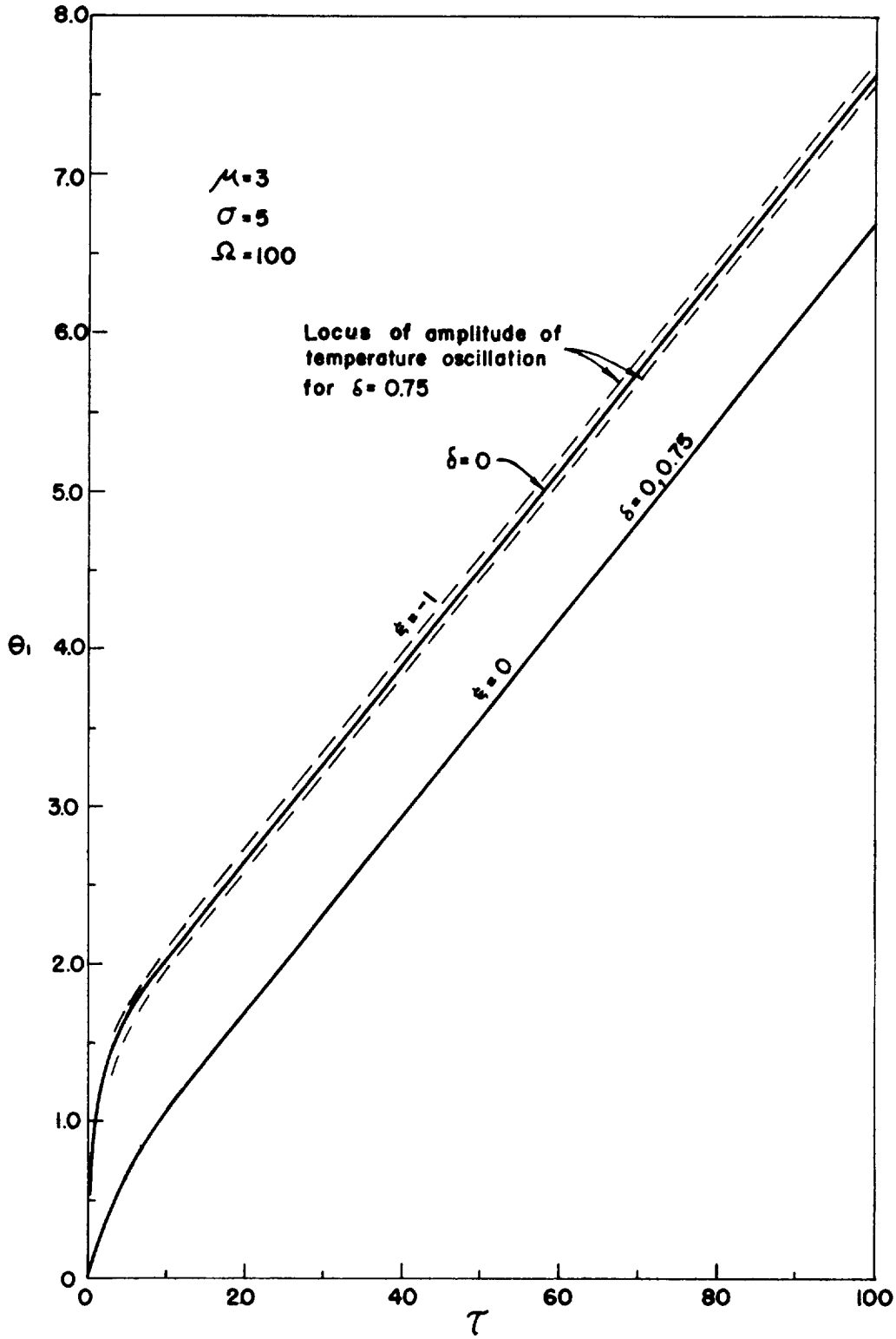
(b) Effect of Ω-Long time

Figure 5.- Continued.



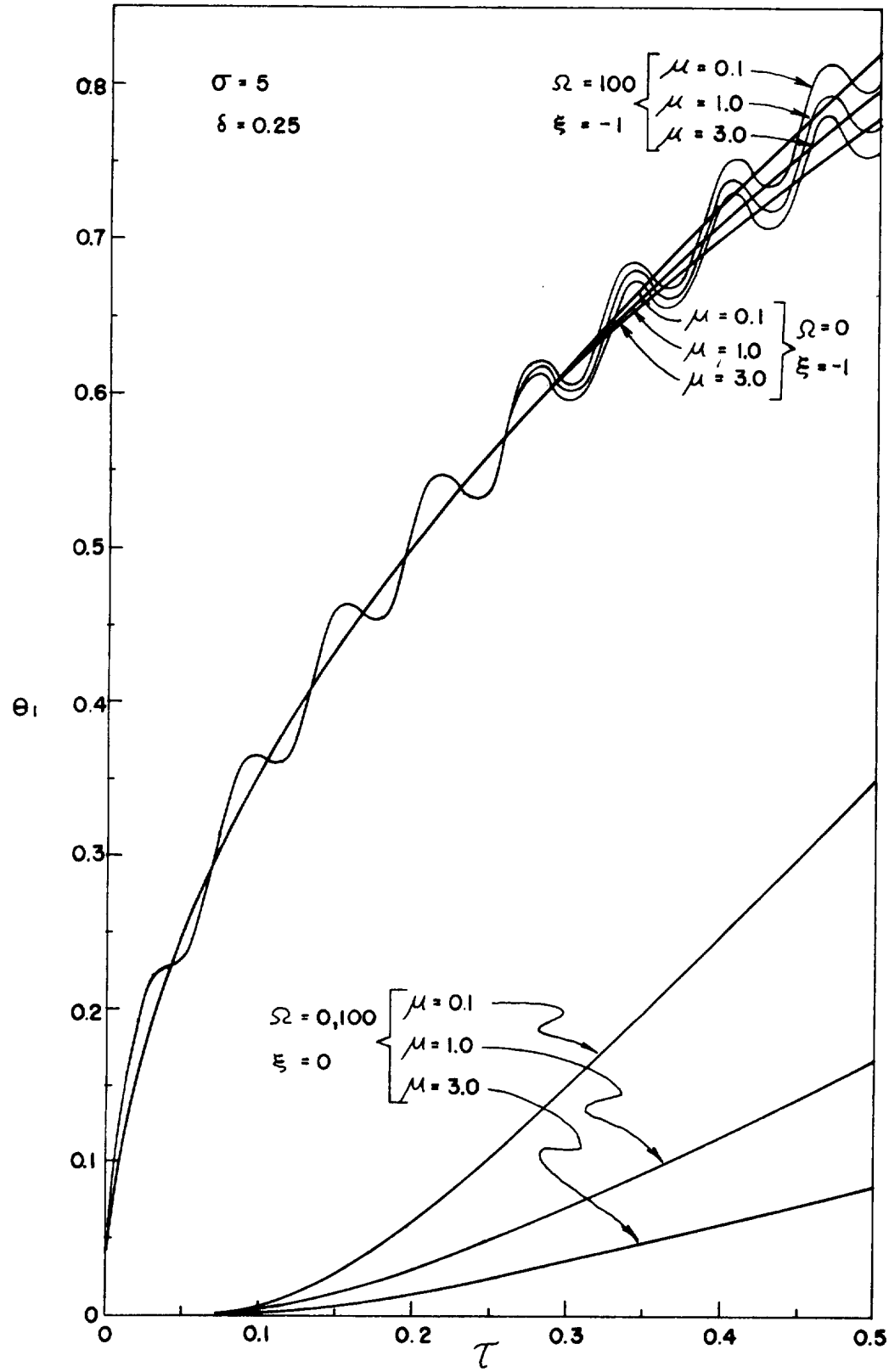
(c) Effect of δ - Short time

Figure 5.- Continued.



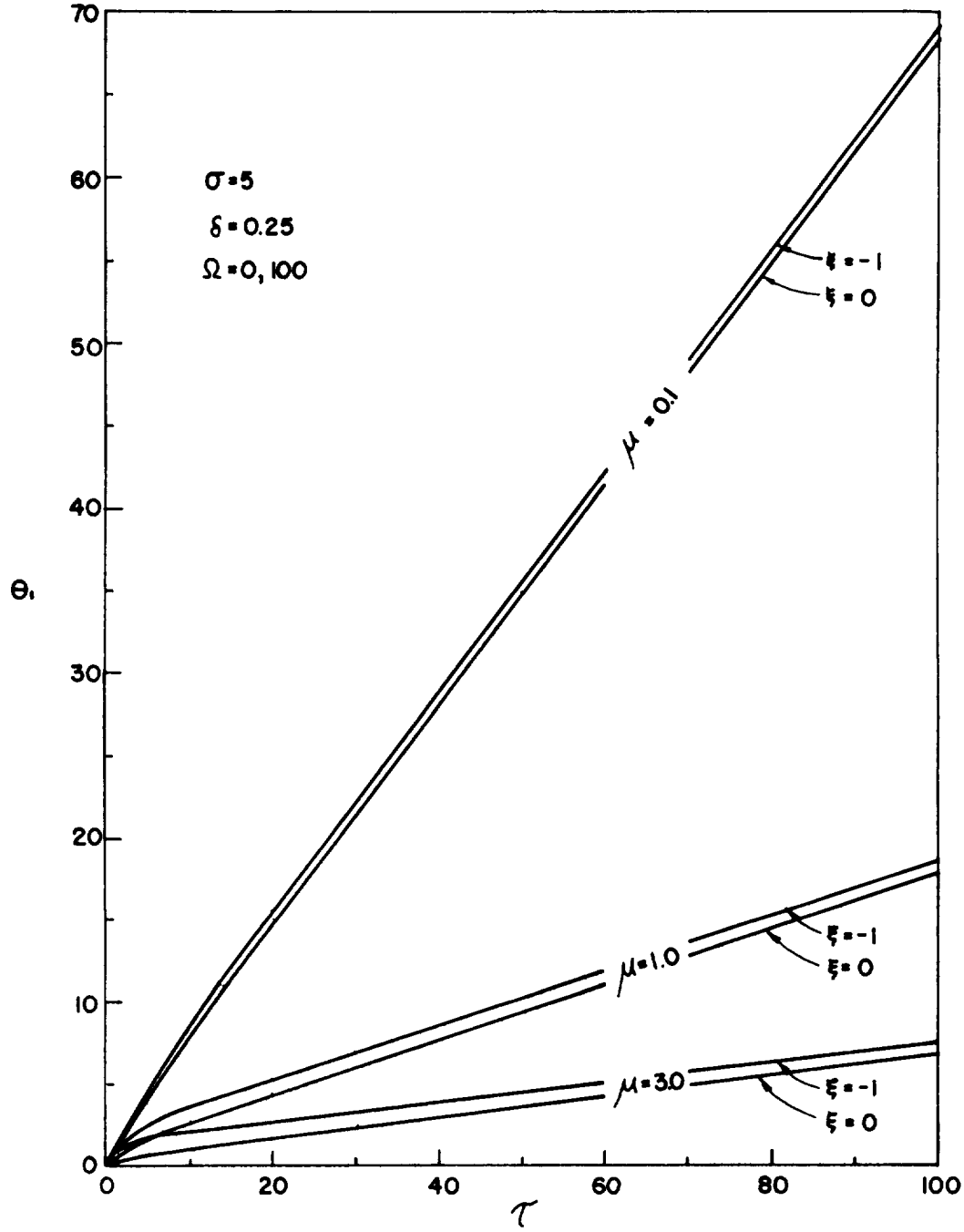
(d) Effect of δ – long time

Figure 5. – continued.



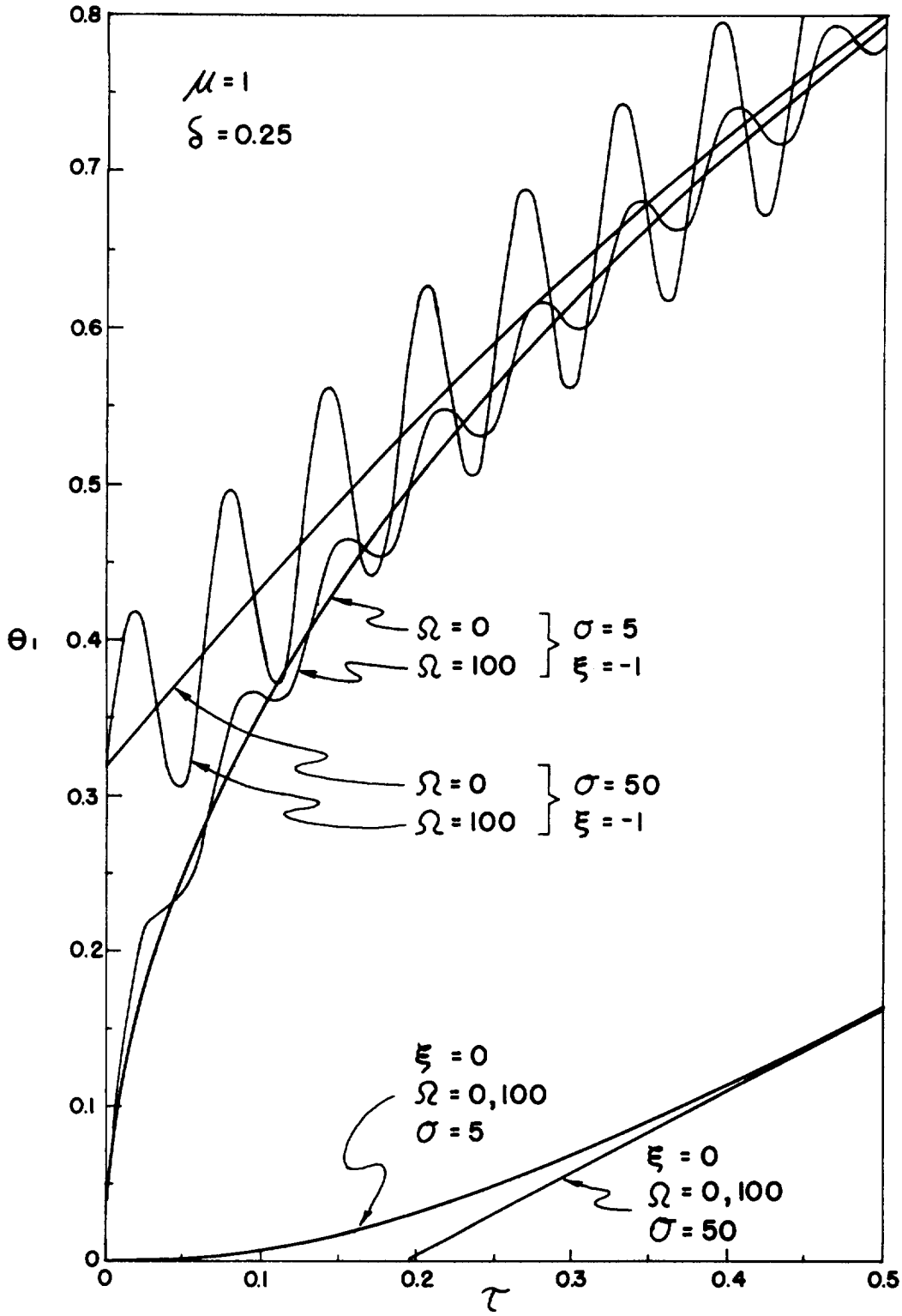
(e) Effect of μ — short time

Figure 5.- continued.



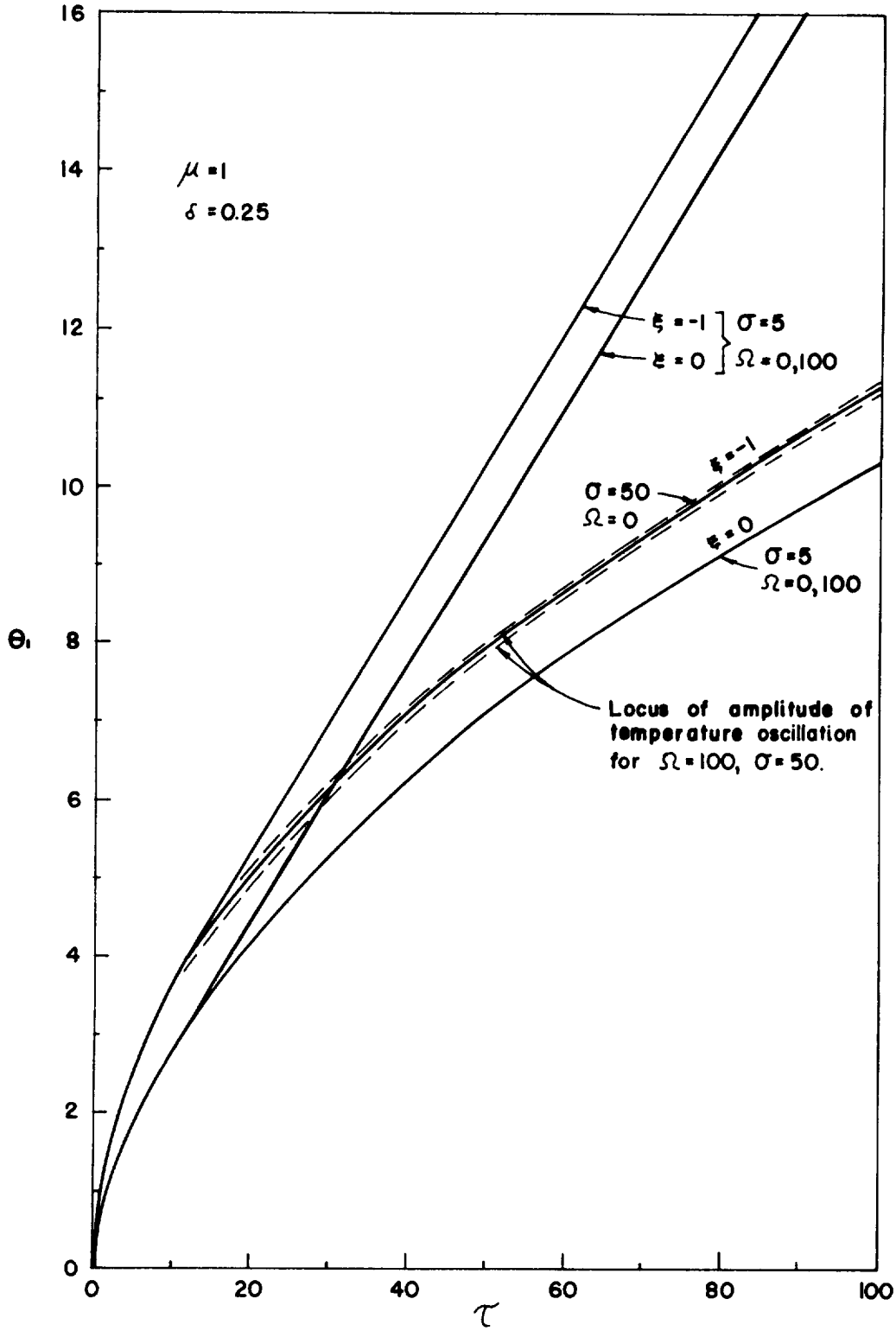
(f) Effect of μ – long time

Figure 5.– Continued.



(g) Effect of σ - short time

Figure 5.- Continued.



(h) Effect of σ – long time

Figure 5. – Concluded.

Case (ii) - Variable Free Stream Conditions: Entry Case:

The temperature response of the blade's composite structure during entry is determined analytically and numerically for a non re-radiating surface, and numerically for a re-radiating surface. The computer program for the analytical solution for the four stages (equations (36), (40), (43), and (46)) is presented in Appendix F. The program for the numerical solution is given in Appendix C.

To obtain the temperature response for this case the aerodynamic heating curve during entry must be specified. A heating curve for a sphere of one foot radius in a non-lifting trajectory is selected as an example (figure 4). The following data is obtained from this curve and is used to evaluate analytically the temperature response during the four stages of this trajectory:

$$\bar{A} = 0.0506 \ \frac{1}{sec^2}$$

$$B = 44.576$$

$$t_a = 72.5 \ sec$$

$$t_b = 108.5 \ sec$$

$$t_c = 133 \ sec$$

$$t_d = 154 \ sec$$

$$t_m = 119 \ sec$$

$$q_o(o) = 1.5451 \ Btu/ft^2\text{-}sec$$

$$\Gamma = 54.494$$

$$\gamma = 0.3775$$

$$\Lambda = 4.471$$

$$\varphi = 0.033 \ 1/sec$$

The blade construction is assumed to consist of 0.07 inch of stainless steel slab covered with 0.02 inch alumina. The thermal properties for the structural and refractory materials, evaluated at 2000°F, are

$$K_1 = 0.903 \times 10^{-3} \text{ Btu/sec-ft-}^{\circ}\text{F}$$

$$K_2 = 0.458 \times 10^{-2} \text{ Btu/sec-ft-}^{\circ}\text{F}$$

$$\alpha_1 = 1.18 \times 10^{-5} \text{ ft}^2\text{/sec}$$

$$\alpha_2 = 5.7 \times 10^{-5} \text{ ft}^2\text{/sec}$$

For such a structure the dimensionless parameters μ and σ are 2.31 and 1.59 respectively. For a rotor angular velocity ω = 4000 RPM the dimensionless angular velocity, Ω, is 100.

Using the above input data, the computer program (Appendix F) for equations (36), (40), (43), and (46) was used to calculate the temperature response at the exposed surface, $\xi = -1$, and at the interface, $\xi = 0$, during the four stages. Figure 6(a) gives the variation of the dimensionless temperature with dimensionless time. The heating curve of figure 4 is non dimensionalized and plotted on the same graph. The temperature at the exposed surface is seen to continue to rise even after the aerodynamic heating decreases during entry. This is expected for a non re-radiating model which is insulated at the back side.

To check this analytical solution, the same problem is solved numerically using the computer program given in Appendix C. The result is plotted in figure 6(a). Excellent agreement is indicated between the analytical and numerical solutions.

To examine the effect of re-radiation, a numerical solution is obtained using the following re-radiation data:

$$D = 0.0126$$

$$E = 0.01355$$

$$H = 0.397$$

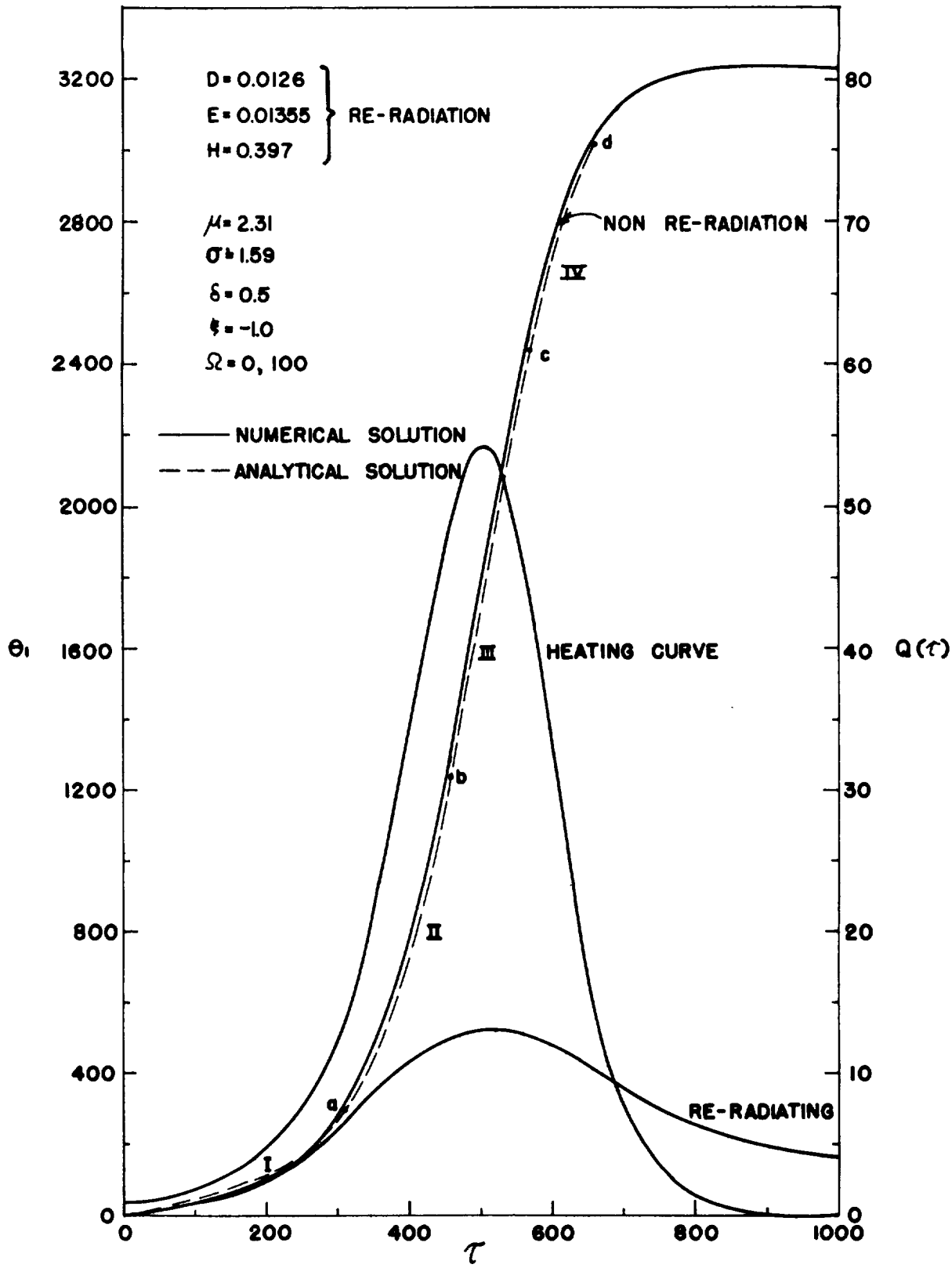
which is based on $\epsilon_1 = 0.78$, $T_i = 630^{\circ}R$, $T_e = 500^{\circ}R$ and $R = 3$. The result plotted in figure 6(a) indicates that the maximum surface temperature peaks at approximately the time the maximum heat flux takes place. It is noted that re-radiation has a significant effect on the temperature level during entry. For the example selected, re-radiation decreases the maximum temperature by a factor of six.

The dimensionless temperature curve of figure 6(a) is presented in a dimensional form in figure 6(b). The non re-radiating model results in a maximum temperature of $28,200^{\circ}R$ while the re-radiating solution gives a maximum temperature of $5000^{\circ}R$.

Figure 6(c) shows the temperature drop across the refractory material during entry for a re-radiating and a non re-radiating surface. For the re-radiating surface the temperature drop becomes negative after the maximum aerodynamic heating has been reached. That is, as the aerodynamic heating begins to decrease during entry, the interface temperature becomes higher than the exposed surface temperature.

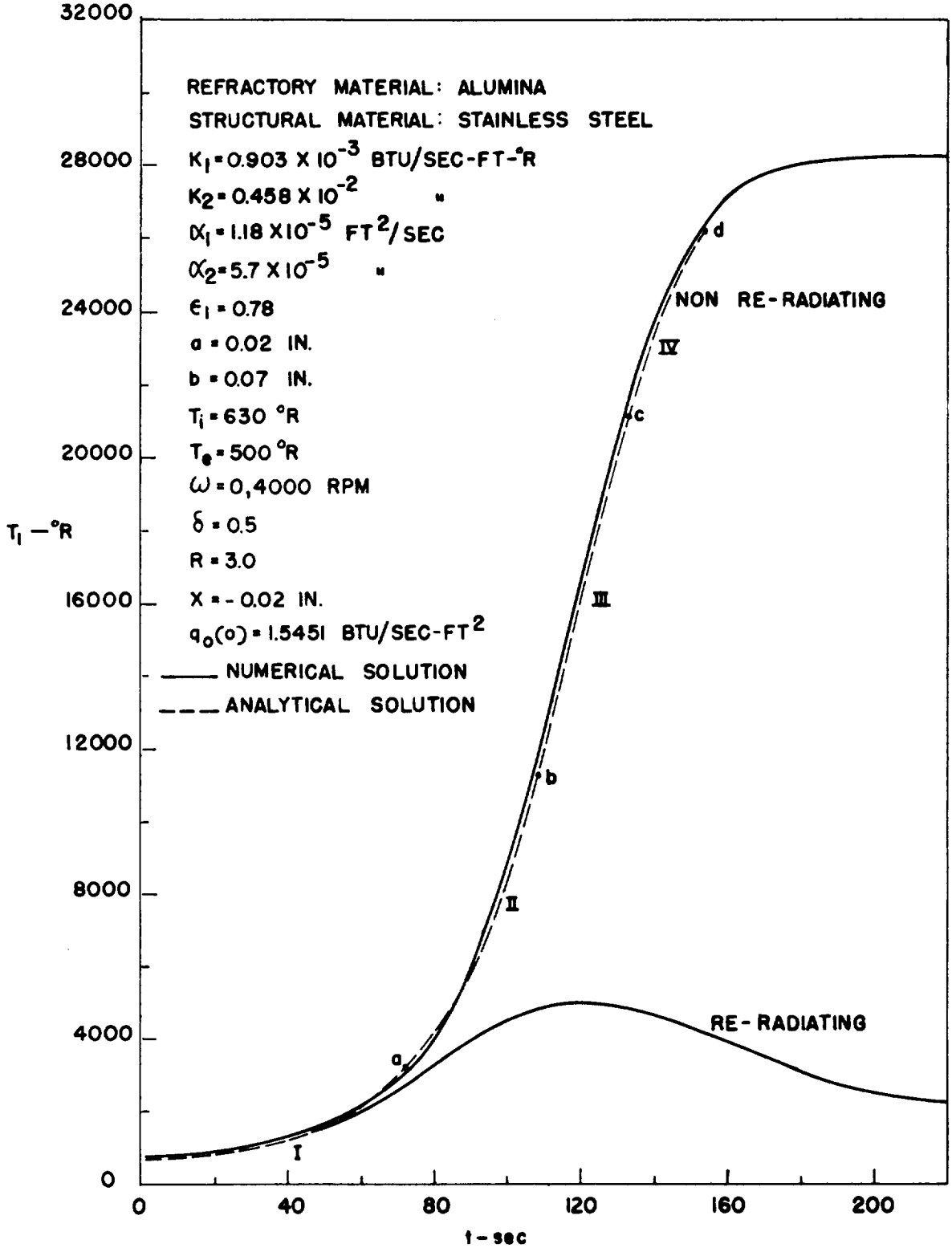
Because of the scale chosen for figure 6(a), the temperature oscillation is not detectable on the graph. Figures 7 and 8 show enlarged portions of figure 6(a) at the four stages of the heating curve during entry for non re-radiating and re-radiating surfaces respectively. No temperature oscillation is detected at the interface for $\Omega = 100$. Temperature oscillation becomes quasi-periodic soon after entry with an amplitude which is small

compared to the temperature level for a non-rotating blade (Ω = o). The analytical solution for a non-rotating blade is obtained by setting δ = 0 rather than Ω = 0. This is necessary because some terms in the solution are divided by Ω .



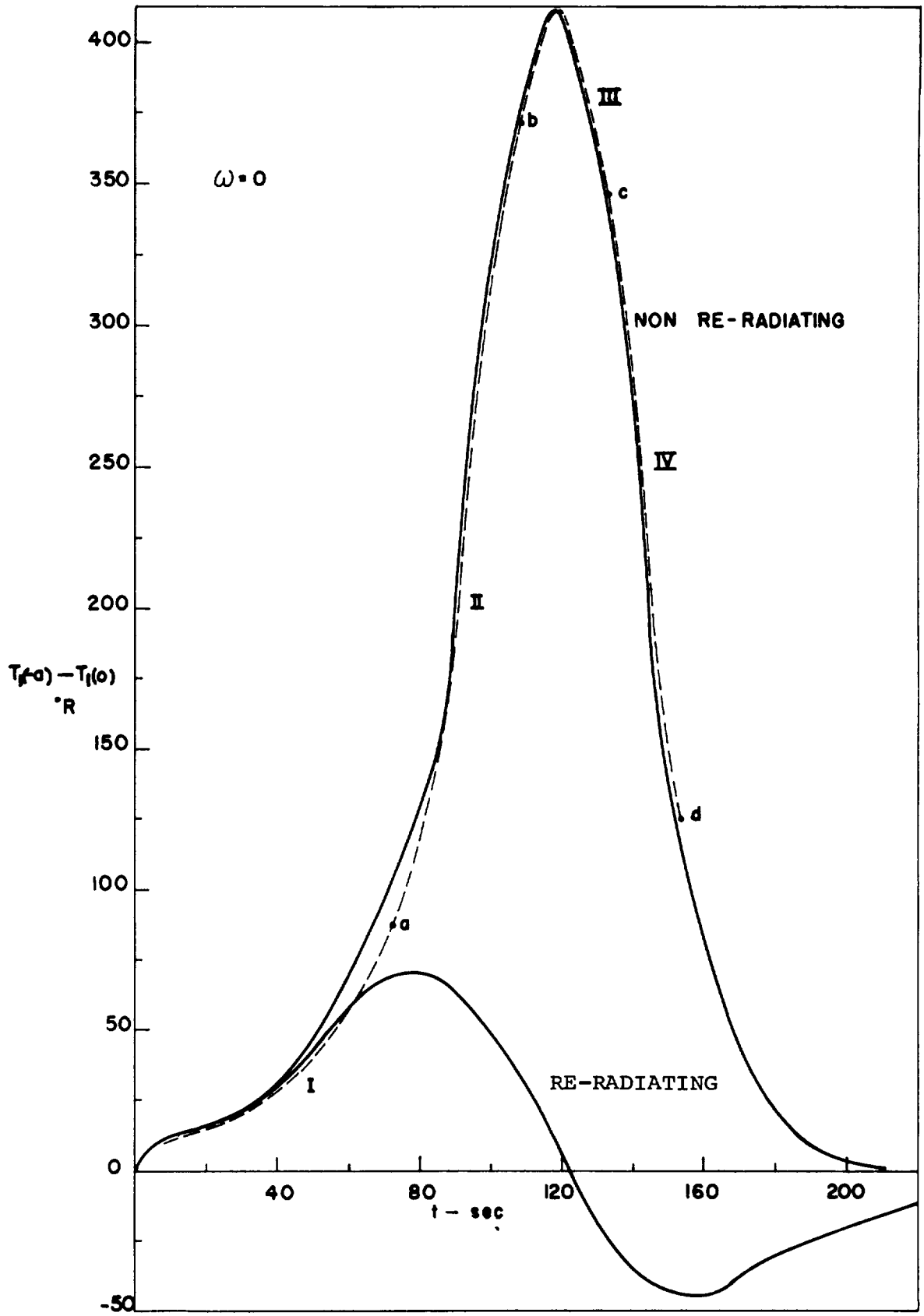
(a) Nondimensional

Figure 6.– Temperature response during entry. Case (ii): Variable free stream conditions.



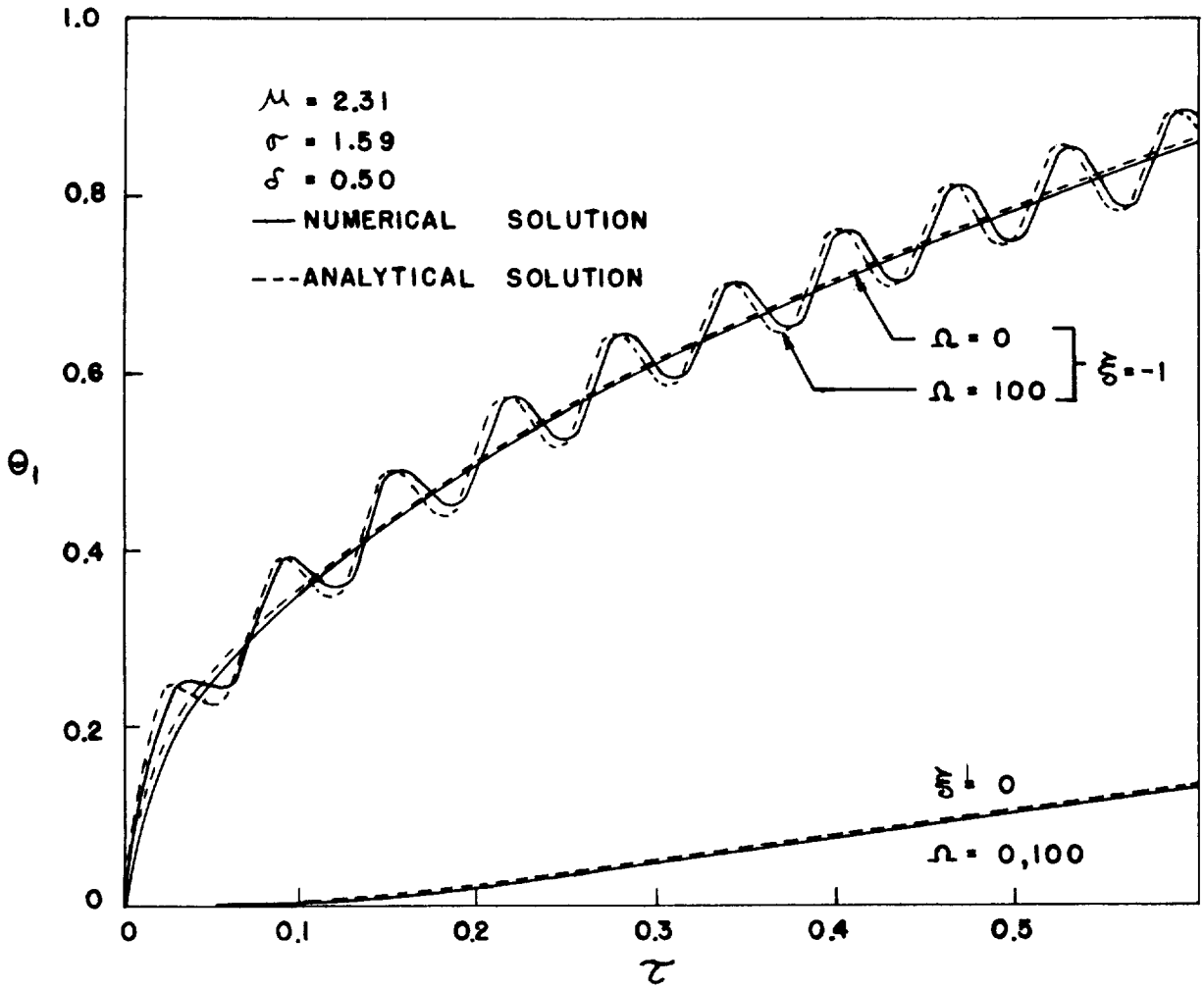
(b) Dimensional

Figure 6.- Continued.



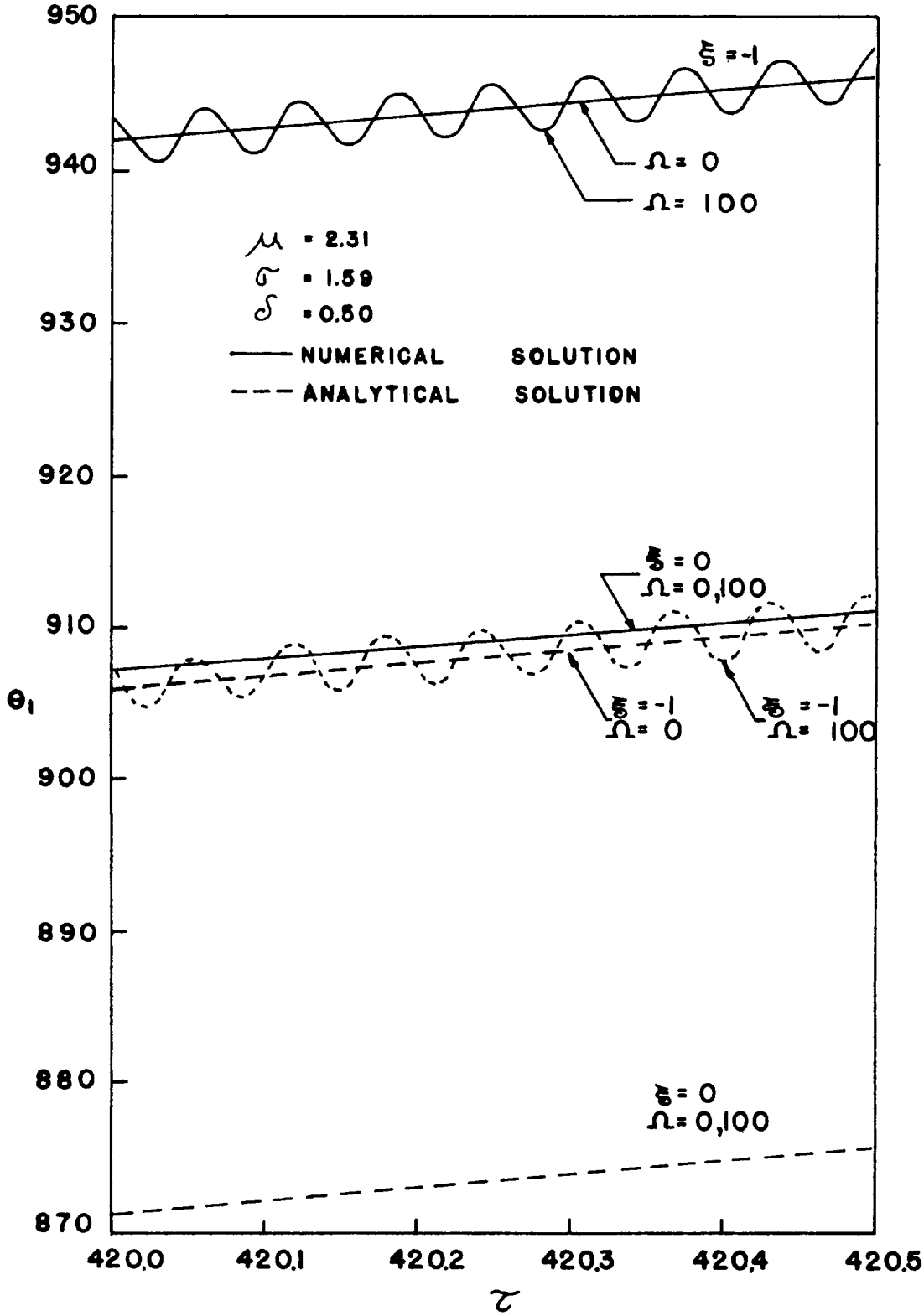
(c) Refractory temperature drop

Figure 6.- Concluded.



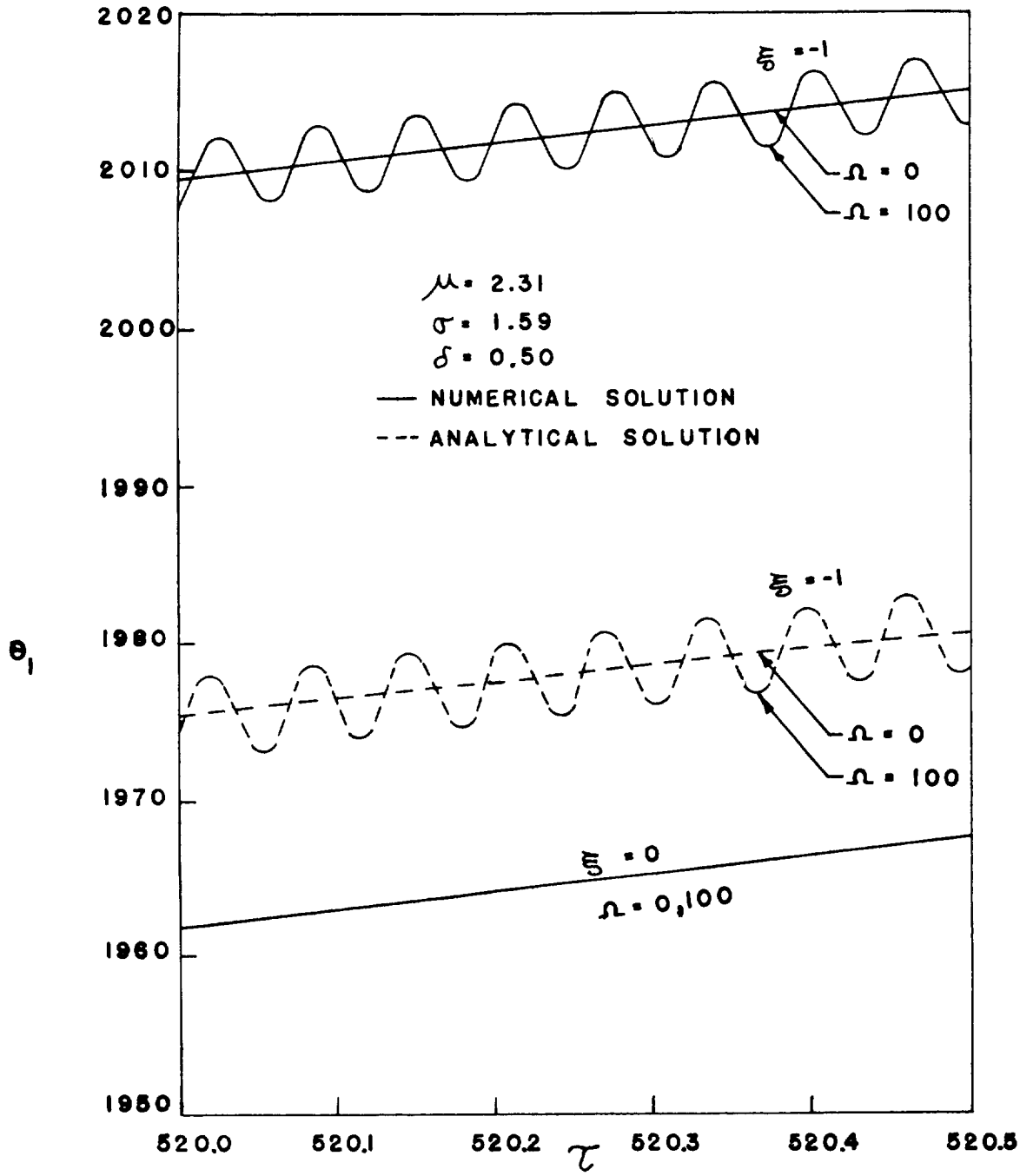
(a) Stage I

Figure 7.- Temperature oscillation during entry for a non re-radiating surface Case (ii): Variable free stream conditions.



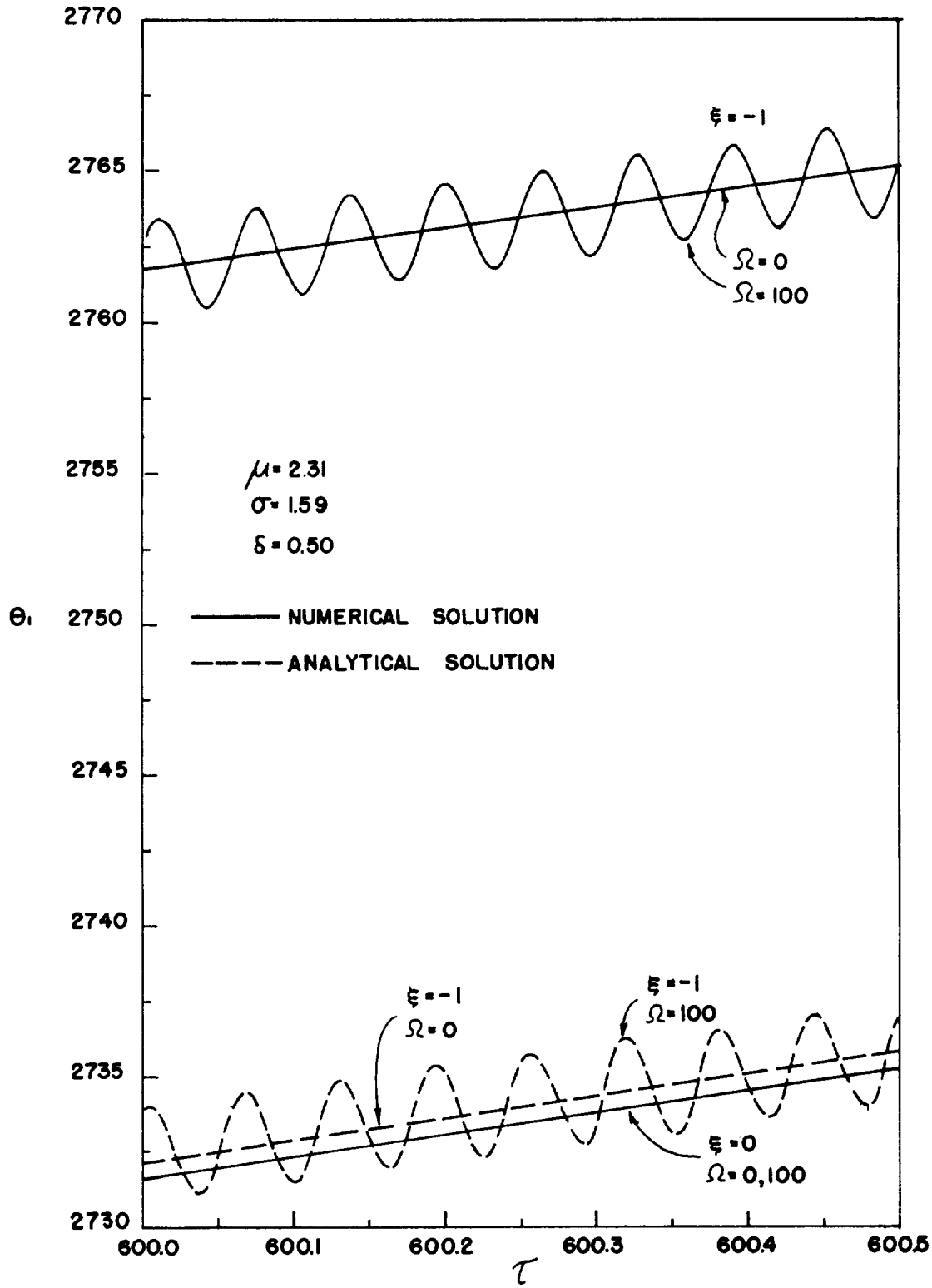
(b) Stage II

Figure 7.- Continued.



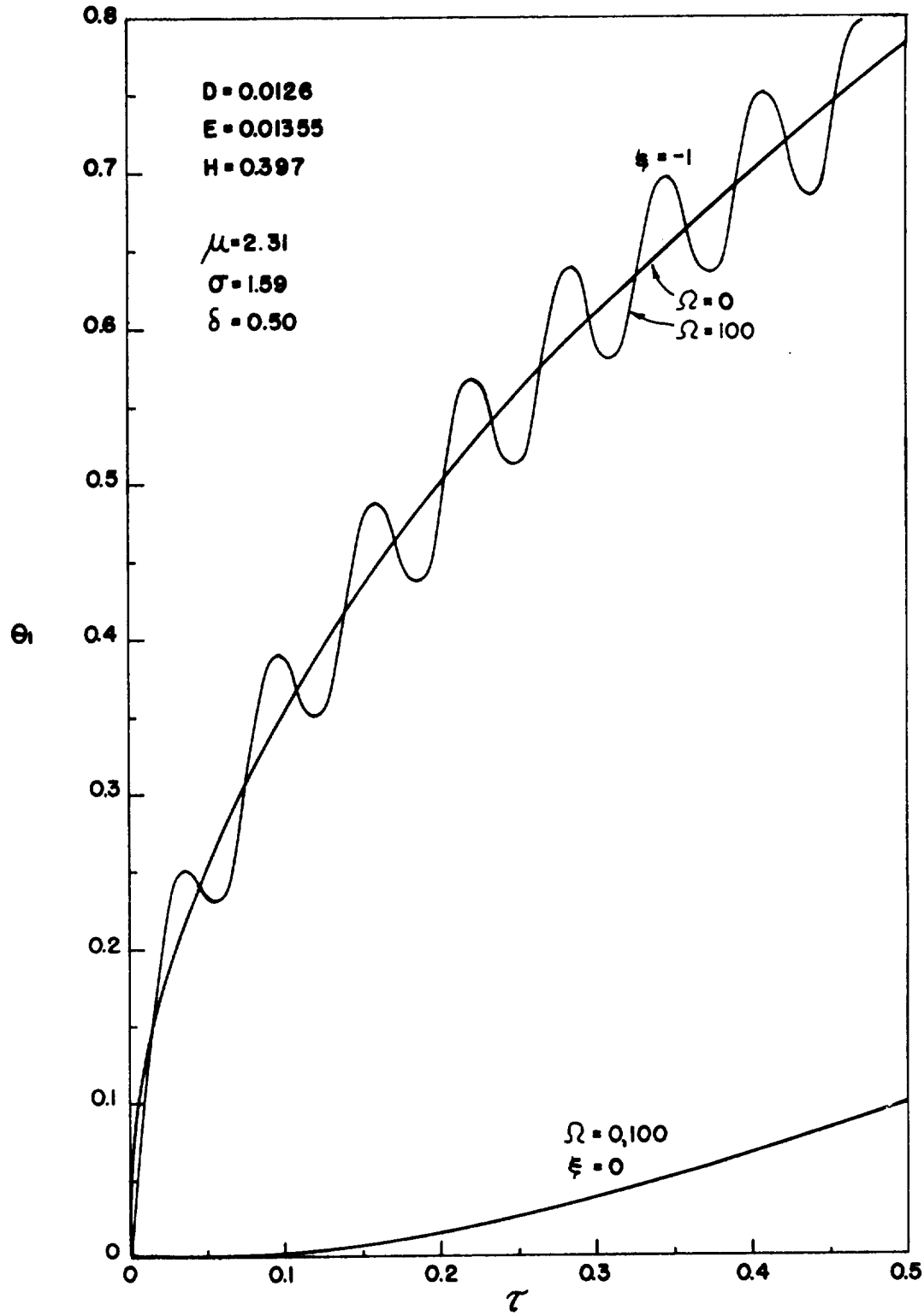
(c) Stage III

Figure 7.– Continued.



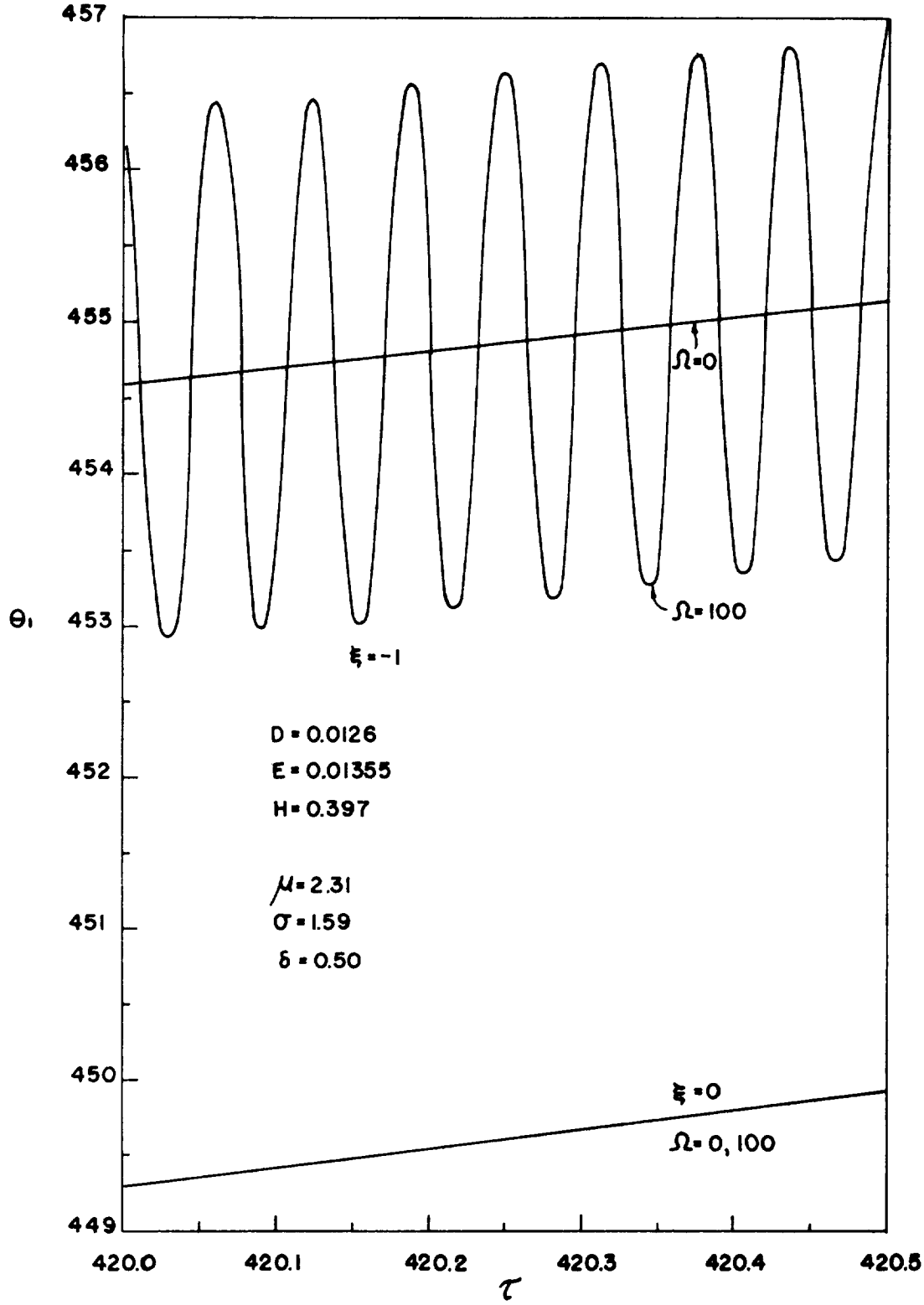
(d) Stage IV

Figure 7.— Concluded.

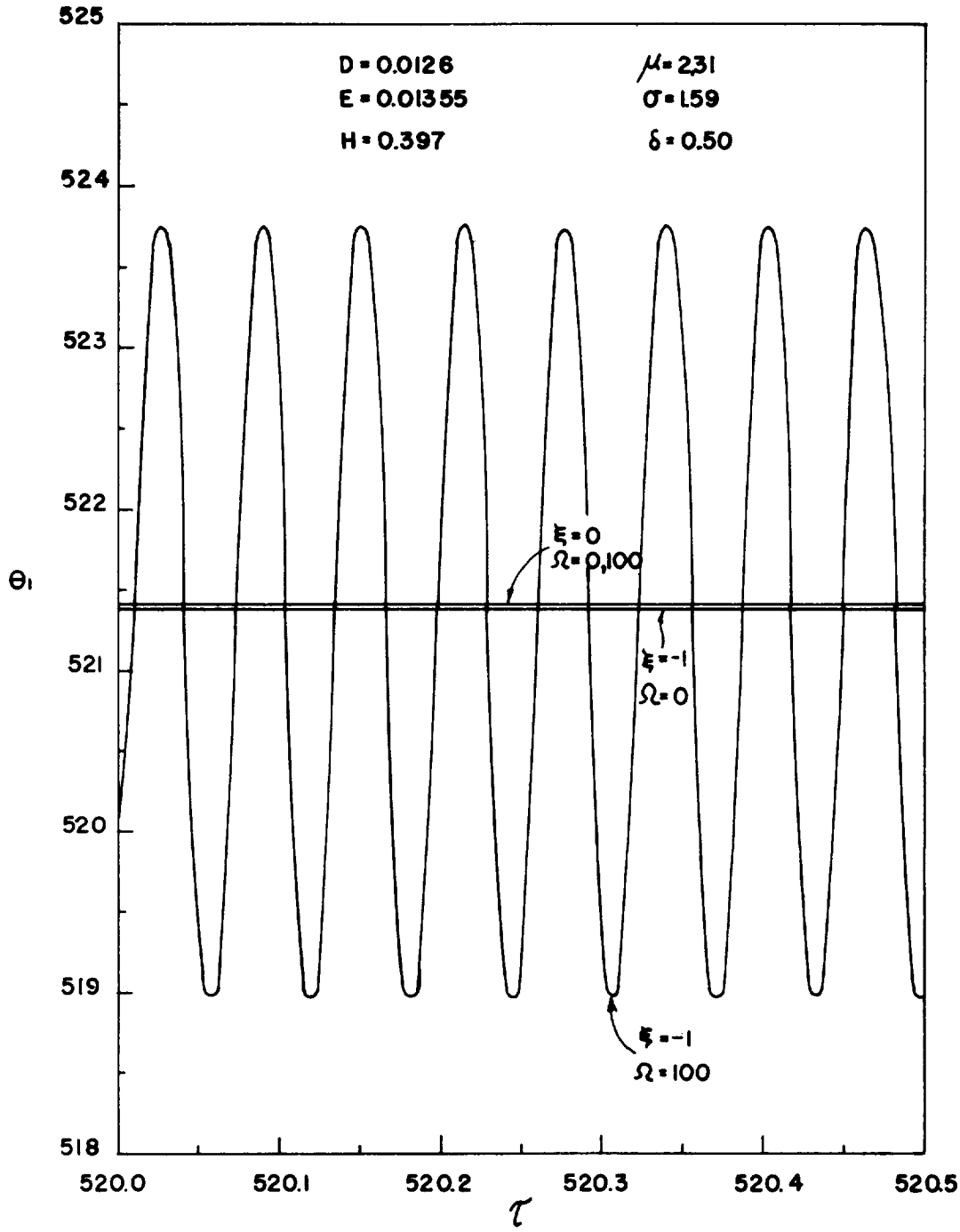


(a) Stage I

Figure 8.– Temperature oscillation during entry for a re-radiating surface. Case (ii): Variable free stream conditions.

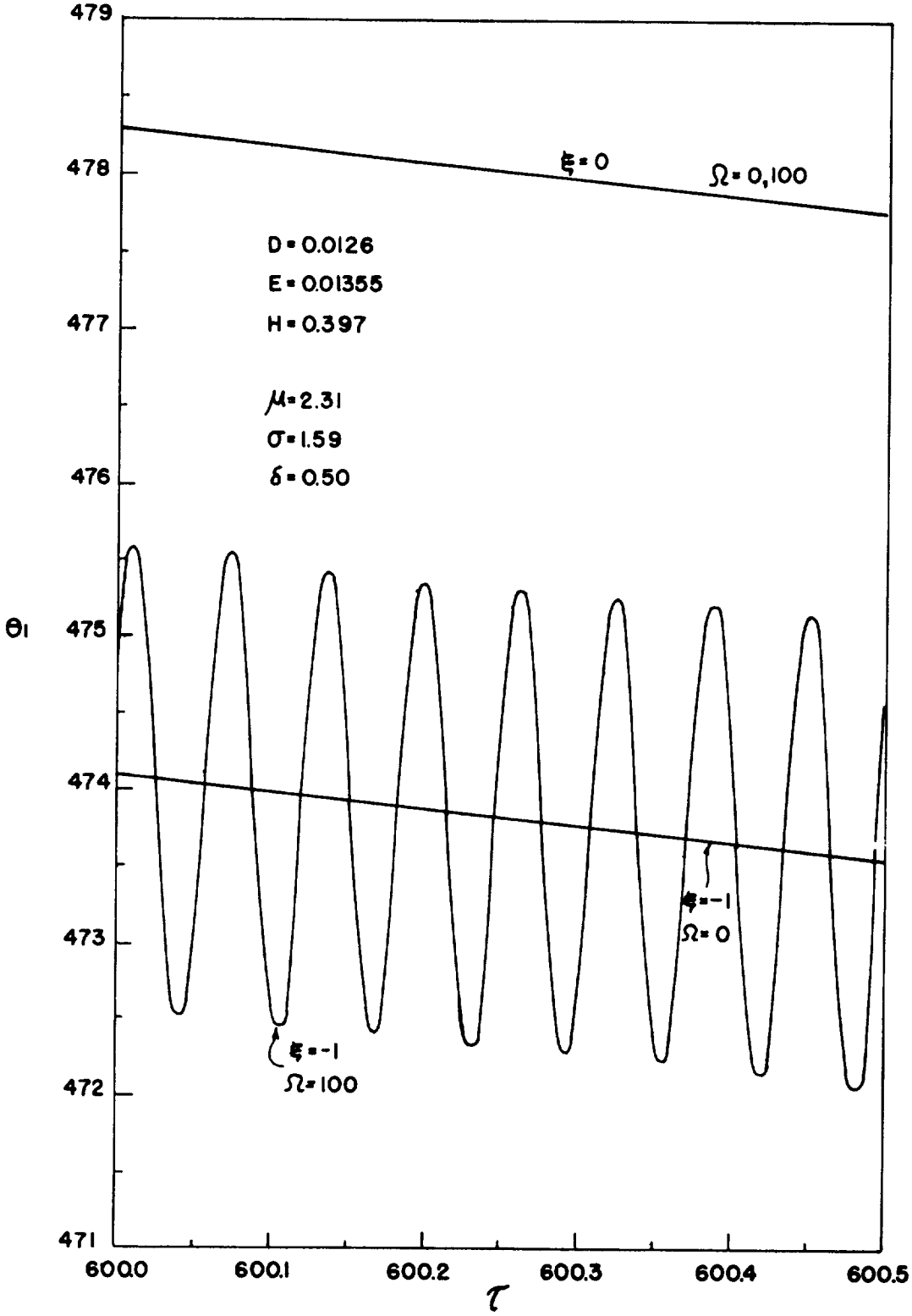


(b) Stage II
Figure 8. – Continued.



(c) Stage III

Figure 8.— Continued.



(d) Stage IV

Figure 8.— Concluded.

IV. CONCLUDING REMARKS

1. While blade rotation may have an appreciable effect on the aerodynamic heat flux, oscillation in this flux due to rotation results in temperature amplitudes which are small compared to the temperature level for an equal but non oscillating flux ($\Omega = 0$ or $\delta = 0$). This is particularly true for high angular velocities and after the initial transient time. In the entry example considered, an angular velocity of 4000 RPM results in a negligible effect on the temperature level at the exposed surface, while a velocity of 400 RPM affects the temperature only during the first fraction of a second of entry time. At the interface no oscillation in temperature is observed for an angular velocity of 400 RPM. Unless there is specific interest in the temperature oscillation at the exposed surface during the initial time of entry for a slowly rotating blade, adequate results are obtained by setting $\Omega = 0$ (or $\delta = 0$) in the solution. The aerodynamic heat flux must, however, still be adjusted to take into consideration the effect of rotation. In the analytical model used, the factor R is introduced for that purpose.

2. Re-radiation has a significant and favorable effect on the temperature response. Excessively high temperatures result when re-radiation is neglected.

For the entry example considered the maximum surface temperature occurs at approximately the time the aerodynamic heating reaches its maximum value. During the portion of entry where the heat flux is decreasing, the temperature of the exposed surface lags behing that of the interface.

This suggests that the temperature profile throughout the composite structure be monitored during entry.

3. Because re-radiation is important, the analytical solution for case (ii) (entry), which is restricted to a non re-radiating surface, is of limited practical use. The numerical solution, on the other hand, may require excessive computer time depending on stability requirements and angular velocity. Stability considerations limit the maximum value of the time increment used in advancing the numerical solution. Furthermore, in order to obtain sufficient points during a temperature cycle, the time increment must be a fraction of $\frac{2\pi}{\Omega}$. This limit may be a more severe than stability requirements.

4. Since the parametric charts of case (i) (constant free stream conditions), figure 5, are for a non re-radiating surface, they may be used only to give trends in temperature behavior as the various governing parameters are changed.

REFERENCES

1. Schapker, R.L. : On the Use of Duhamel's Equation for Heat Conduction Problems. Journal of the Aerospace Sciences, vol.29, July 1962, p. 883.

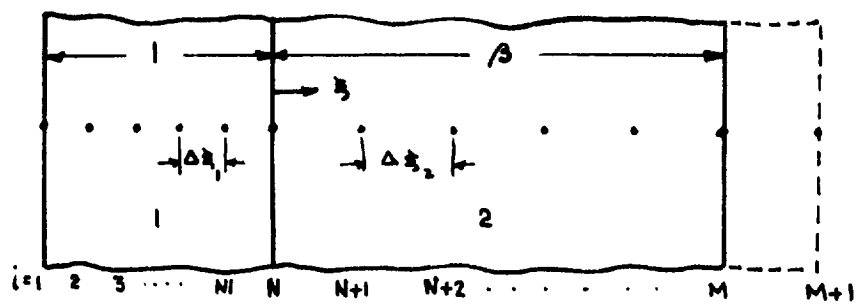
2. Giere, A.C., and Franklin, M.E. : Analysis of Heat Transfer in a Two-Layer Slab: Constant Flux on One Surface and Zero Flux on Other Surface. NAVWEPS Report 8005, December 1964.

APPENDIX A

FORMULATION OF THE NUMERICAL SOLUTION

FINITE DIFFERENCE FORMULATION

Governing Equations:



For material 1 (i = 1,2 ....N1), equation (54) is approximated by

$$\frac{\theta_i' - \theta_i}{\Delta\tau} = \frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta\xi_1^2}$$

or

$$\theta_i' = \theta_i + \frac{\Delta\tau}{\Delta\xi_1^2}\left(\theta_{i-1} - 2\theta_i + \theta_{i+1}\right) \tag{A1}$$

where

$$\Delta\xi_1 = \frac{1}{N1}$$

N1 being the number of strips in material 1.

For material 2(i = N+1, N+2,...M), equation (55) is approximated by

$$\frac{\theta_i' - \theta_i}{\Delta\tau} = \left(\frac{\beta}{\sigma}\right)^2 \left(\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta\xi_2^2}\right)$$

or

$$\theta_i' = \theta_i + \left(\frac{\beta}{\sigma}\right)^2 \frac{\Delta\tau}{\Delta\xi_2^2}\left(\theta_{i-1} - 2\theta_i + \theta_{i+1}\right) \tag{A2}$$

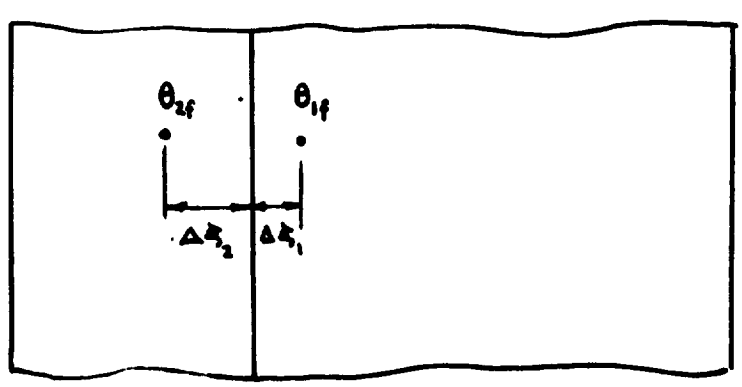
where

$$\Delta\xi_2 = \frac{\beta}{N2}$$

N2 being the number of strips in material 2.

Boundary Conditions

To formulate the boundary conditions at the interface (i = N1 = n) the fictitious temperatures θ_{1f} at a distance $\Delta\xi_1$, to the right of the interface, and θ_{2f} at a distance $\Delta\xi_2$ to the left of interface are introduced. Then, by equation (A1)



$$\theta'_{(1)n} = \theta_n + \frac{\Delta\tau}{\Delta\xi_1^2}\left(\theta_{n-1} - 2\theta_n + \theta_{1f}\right) \tag{A3}$$

and by equation (A2)

$$\theta'_{(2)n} = \theta_n + \left(\frac{\beta}{\sigma}\right)^2 \frac{\Delta\tau}{\Delta\xi_2^2}\left(\theta_{2f} - 2\theta_n + \theta_{n+1}\right) \tag{A4}$$

To eliminate the fictitious temperatures θ_{1f} and θ_{2f}, boundary conditions (57) and (58) are used. Equation (57) gives

$$\theta_{(1)n} = \theta_{(2)n} \tag{A5}$$

Boundary condition (58) is approximated by

$$\frac{\theta_{1f} - \theta_{n-1}}{\Delta\xi_1} = \mu\,\frac{\beta}{\sigma}\left(\frac{\theta_{n+1} - \theta_{2f}}{\Delta\xi_2}\right)$$

or

$$\theta_{1f} = \theta_{n-1} + \frac{\mu\beta}{\sigma}\,\frac{\Delta\xi_1}{\Delta\xi_2}\left(\theta_{n+1} - \theta_{2f}\right) \tag{A6}$$

Equations (A3), (A4), (A5) and (A6) give

$$\theta'_n = \theta_n + C_3\,\Delta\tau\left(\theta_{n-1} - \theta_n\right) + C_4\,\Delta\tau\left(\theta_{n+1} - \theta_n\right) \tag{A7}$$

where

$$C_3 = \frac{\frac{2\beta}{\sigma}\,\frac{1}{\Delta\xi_1^2}}{\frac{\beta}{\sigma} + \mu\,\frac{\Delta\xi_2}{\Delta\xi_1}} \tag{A8}$$

$$C_4 = \frac{2\mu\left(\frac{\beta}{\sigma}\right)^2 \frac{1}{\Delta\xi_1\,\Delta\xi_2}}{\frac{\beta}{\sigma} + \mu\,\frac{\Delta\xi_2}{\Delta\xi_1}} \tag{A9}$$

For the boundary condition at $\xi = -1$ (i = 1) a fictitious temperature θ_0 at a distance $\Delta\xi_1$ to the left of the boundary at $\xi = -1$ is introduced. Equation (A1) gives

$$\theta_1' = \theta_1 + \frac{\Delta\tau}{\Delta\xi_1^2}(\theta_0 - 2\theta_1 + \theta_2) \tag{A10}$$

To eliminate the fictitious temperature θ_0, boundary condition (60) is employed in finite difference form.

$$-\frac{(\theta_2 - \theta_0)}{2\Delta\xi_1} = Q(\tau)\left[1 + \delta \sin \Omega\tau\right] - D\left\{\left[1 + E\theta_1\right]^4 - H\right\}$$

or

$$\theta_0 = \theta_2 + 2\Delta\xi_1\left\{Q(\tau)\left[1 + \delta \sin \Omega\tau\right] - D\left[(1 + E\theta_1)^4 - H\right]\right\} \tag{A11}$$

(A10) and (A11) give

$$\theta_1' = \theta_1 + \frac{2\Delta\tau}{\Delta\xi_1^2}\left\{\theta_2 - \theta_1 + \Delta\xi_1\left[Q(\tau)(1 + \delta \sin \Omega\tau)\right] - D\,\Delta\xi_1\left[(1 + E\theta_1)^4 - H\right]\right\} \tag{A12}$$

The adiabatic condition at $\xi = \beta$ is handled by setting

$$\theta_{m+1} = \theta_{m-1} \tag{A13}$$

where m is node at $\xi = \beta$.

STABILITY

To establish the stability requirement in material 1, equation (A1) is rewritten as

$$\theta_i' = \frac{\Delta\tau}{\Delta\xi_1^2}\left(\theta_{i-1} + \theta_{i+1}\right) + \left(1 - 2\frac{\Delta\tau}{\Delta\xi_1^2}\right)\theta_i \tag{A14}$$

stability consideration requires that

$$1 - 2\frac{\Delta\tau}{\Delta\xi_1^2} \geq 0$$

or

$$\Delta\tau \leq \frac{\Delta\xi_1^2}{2} \tag{A15}$$

For material 2, equation (A2) is rewritten as

$$\theta_i = \left(\frac{\beta}{\sigma}\right)^2 \frac{\Delta\tau}{\Delta\xi_2^2}\left(\theta_{i-1} + \theta_{i+1}\right) + \left[1 - 2\left(\frac{\beta}{\sigma}\right)^2 \frac{\Delta\tau}{\Delta\xi_2^2}\right]\theta_i \tag{A16}$$

stability requires that

$$1 - 2\left(\frac{\beta}{\sigma}\right)^2 \frac{\Delta\tau}{\Delta\xi_2^2} \geq 0$$

or

$$\Delta\tau \leq \left(\frac{\sigma}{\beta}\right)^2 \frac{\Delta\xi_2^2}{2} \tag{A17}$$

To determine the stability condition at the interface, equation (A7) is rewritten as

$$\theta_n' = C_3 \Delta\tau\, \theta_{n-1} + C_4 \Delta\tau\, \theta_{n+1} + (1 - C_3 \Delta\tau - C_4 \Delta\tau)\theta_n \tag{A18}$$

stability requires that

$$1 - C_3 \Delta\tau - C_4 \Delta\tau \geq 0$$

or

$$\Delta\tau \leq \frac{1}{C_3 + C_4} \tag{A19}$$

To establish the stability requirement at $\xi=-1$, equation (A12) is rewritten as

$$\theta_1' = \frac{2\Delta\tau}{\Delta\xi_1^2}\left\{\theta_2 + \Delta\xi_1\left[Q(\tau)(1+\delta\sin\Omega\tau) + DH\right]\right\}$$

$$+ \theta_1 + \frac{2\Delta\tau}{\Delta\xi_1}\left[-\theta_1 - \Delta\xi_1 D(1+E\theta_1)^4\right] \tag{A20}$$

or

$$\theta_1' = \frac{2\Delta\tau}{\Delta\xi_1^2}\left\{\theta_2 + \Delta\xi_1\left[Q(\tau)(1+\delta\sin\Omega\tau) + D(H-1)\right]\right\}$$

$$+\left\{1 - \frac{2\Delta\tau}{\Delta\xi_1^2}\left[1 + \Delta\xi_1 D(4E + 6E^2\theta_1 + 4E^3\theta_1^2\right.\right.$$

$$\left.\left. + E^4\theta_1^3)\right]\right\}\theta_1 \tag{A21}$$

stability is insured if

$$1 - \frac{2\Delta\tau}{\Delta\xi_1^2}\left[1 + \Delta\xi_1 D(4E + 6E^2\theta_1 + 4E^3\theta_1^2 + E^4\theta_1^3\right] \geq 0$$

or

$$\Delta\tau \leq \frac{\frac{\Delta\xi_1^2}{2}}{\left[1 + \Delta\xi_1 D(4E + 6E^2\theta_1 + 4E^3\theta_1^2 + E^4\theta_1^3)\right]} \tag{A22}$$

APPENDIX B

COMPUTER PROGRAM FOR THE NUMERICAL SOLUTION OF CASE (i) - CONSTANT FREE STREAM CONDITIONS.

Definitions of Input Quantities

FORTRAN NAME	SYMBOL OR MEANING
NAME (I)	Title of run under consideration. Two cards to be used.
BETA	β
SIGMA	σ
U	μ
DELTA	δ
D	D
E	E
H	H
OMEGA	Ω
N1	N1 (Number of strips in material 1)
N2	N2 (Number of strips in material 2)
STAU	Starting time of solution
DTAU	$\Delta\tau$
PTAU1	Print out interval in range 1
PTAU2	Print out interval in range 2
CHANGE	Value of τ at end of range 1
FTAU	Final dimensionless time
Q	$Q(\tau) = 1.0$
T(I)	Initial temperature distribution at all nodes, dimensionless

```
$IBJOB KRAMER  DECK,MAP
$IBFTC ROBERT  DECK,FULIST

      DIMENSION NAME (32)
      DIMENSION T(1000), TP(1000)
C
    1 READ (5,97) (NAME(I),I=1,32)
   97 FORMAT (16A5)
      WRITE (6,99) (NAME(I),I=1,32)
   99 FORMAT (1H1,24A5/1X,24A5)
      READ(5,2) BETA,SIGMA,U,DELTA,D,E,H,OMEGA
  2   FORMAT( 8E10.0)
      WRITE(6,5)BETA,SIGMA,U,DELTA,D,E,H,OMEGA
  5   FORMAT(///1H , 39X,  8H   BETA=E14.7, /
     1               40X,  8H  SIGMA=E14.7, /
     2               40X,  8H      U=E14.7, /
     3               40X,  8H  DELTA=E14.7, /
     4               40X,  8H      D=E14.7, /
     5               40X,  8H      E=E14.7, /
     6               40X,  8H      H=E14.7, /
     7               40X,  8H  OMEGA=E14.7  )
C
      READ(5,10)N1,N2,STAU,DTAU,PTAU1,PTAU2,CHANGE,FTAU,Q
   10 FORMAT (2I5,7E10.0)
      WRITE(6,11)N1,N2,STAU,DTAU,PTAU1,PTAU2,CHANGE,FTAU,Q
 11   FORMAT(         40X,  8H     N1=I2   , /
     1               40X,  8H     N2=I2   , /
     2               40X,  8H   STAU=E14.7, /
     3               40X,  8H   DTAU=E14.7, /
     4               40X,  8H  PTAU1=E14.7, /
     5               40X,  8H  PTAU2=E14.7, /
     6               40X,  8H CHANGE=E14.7, /
     7               40X,  8H   FTAU=E14.7, /
     8               40X,  8H      Q=E14.7  )
      WRITE (6,99) (NAME(I),I=1,32)
C
C....M=TOTAL NO. OF POINTS
      M=N1+N2+1
C
      IF(STAU.EQ.0.) GO TO 30
C....READ INITIAL TEMPERATURE DISTRIBUTION
      READ(5,25)(T(I),I=1,M)
 25   FORMAT(8E10.0)
      GO TO 40
C
C....INITIALIZE ALL TEMPERATURES TO ZERO
 30   DO 31 I=1,M
 31   T(I)=0.
C
```

```
C....CALCULATE ALL NECESSARY CONSTANTS
 40     FN=N1
        DX1=1./FN
        FN=N2
        DX2=BETA/FN
        C1=1./(DX1**2)
        C12=2.*C1
        BOS=BETA/SIGMA
        C2=(BOS/DX2)**2
        DEN=BOS+(U*DX2)/DX1
        C3=(2.*BOS*C1)/DEN
        C4=(2.*U*BOS**2)/(DEN*DX1*DX2)
        N=N1+1
        NP1=N+1
        MM1=M-1
        MP1=M+1
        PRTIME=STAU-DTAU/10.
        TAU=STAU
C
C....CHECK FOR STABILITY
C....MATERIAL 1
        DTAUP=1./C12
 50     IF(DTAU.LE.DTAUP) GO TO 60
        DTAU=DTAU/2.
        GO TO 50
C....MATERIAL 2
 60     DTAUP=1./(2.*C2)
 65     IF(DTAU.LE.DTAUP) GO TO 70
        DTAU=DTAU/2.
        GO TO 65
C....INTERFACE
 70     DTAUP=1./(C3+C4)
 75     IF(DTAU.LE.DTAUP) GO TO 80
        DTAU=DTAU/2.
        GO TO 75
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C....BOUNDARY
 80     DTAUP=1./(C12*(1.+DX1*D*E*(4.+E*T(1)*(6.+E*T(1)*(4.+E*T(1))))))
 85     IF(DTAU.LE.DTAUP) GO TO 90
        DTAU=DTAU/2.
        GO TO 85
C....ADIABATIC BOUNDARY CONDITION
 90     T(MP1)=T(MM1)
C
        IF(TAU.LT.PRTIME) GO TO 120
        IF (PRTIME.GT.CHANGE) GO TO 91
        PRTIME=PRTIME+PTAU1
        GO TO 92
   91 PRTIME=PRTIME + PTAU2
C....PRINT TEMPERATURE DISTRIBUTION
   92 IF (PRINT.LE.4.) GO TO 94
        PRINT = 0.0
        WRITE (6,99) (NAME(I),I=1,32)
   94 WRITE(6,100) TAU,DTAU
100     FORMAT(// 36X, 4HTAU=E12.5, 5X,5HDTAU=E12.5)
        WRITE(6,101) (I,T(I),I=1,M)
101     FORMAT( 5(4X, 2HT(, I2,2H)=, E12.5) )
        PRINT =PRINT+1.0
```

```
C
120    IF(TAU.GE.FTAU) GO TO 1
C
       W1=Q*(1.+DELTA*SIN(OMEGA*TAU))
       W2=D*((1.+E*T(1))**4 -H)
C
C....CALCULATE TEMPERATURES AT TAU+DTAU
C....BOUNDARY
       TP(1)=T(1)+C12*DTAU*(T(2)-T(1)+DX1*(W1-W2))
C....MATERIAL 1
       DO 130 I=2,N1
130    TP(I)=T(I)+C1*DTAU*(T(I-1)-2.*T(I)+T(I+1))
C....INTERFACE
       TP(N)=T(N)+DTAU*(C3*(T(N1)-T(N))+C4*(T(NP1)-T(N)))
C....MATERIAL 2
       DO 140 I=NP1,M
140    TP(I)=T(I)+C2*DTAU*(T(I-1)-2.*T(I)+T(I+1))
C
C....INCREMENT TAU
       TAU=TAU+DTAU
C....RESET
       DO 150 I=1,M
150    T(I)=TP(I)
       GO TO 80
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
       END

$ENTRY
```

APPENDIX C

COMPUTER PROGRAM FOR THE NUMERICAL SOLUTION OF CASE (ii) - VARIABLE FREE STREAM CONDITIONS: ENTRY CASE

Definitions of Input Quantities

FORTRAN NAME	SYMBOL
NAME (I)	Title of run under consideration. Two cards are to be used.
BETA	β
SIGMA	σ
U	μ
DELTA	δ
D	D
E	E
H	H
OMEGA	Ω
N1	N1 (Number of strips in material 1)
N2	N2 (Number of strips in material 2)
STAU	Starting dimensionless time
DTAU	$\Delta\tau$
PTAU1	Print out dimensionless time interval in range 1
PTAU2	Print out dimensionless time interval in range 2
CHANGE	Vale of τ at end of range 1
FTAU	Final dimensionless time
DIFFUS	α_1 (ft^2/hr)
AA	a (in.)
NOPTS	Number of points in the table of $q_o(t)$ vs. t for entry heating.
J	1 for new table of $q_o(t)$ vs. t 2 for table of $q_o(t)$ vs. t of previous case.

FORTRAN NAME	SYMBOL
TIME (I)	Set of time values t, in seconds, at which values of $q_o(t)$ are tabulated
QQ(I)	Set of entry heat flux, $q_o(t)$, in Btu/ft^2 - sec
T(I)	Initial temperature distribution at all nodes, dimensionless

```
ISN        SOURCE STATEMENT

  0 $IBFTC ROBERT  DECK
  1        DIMENSION NAME (32)
  2        DIMENSION T(100), TP(100)
  3        DIMENSION TIME(50),QQ(50),TTAU(50),QT(50),ZTAU(50)
    C
  4      1 READ (5,97) (NAME(I),I=1,32)
 11     97 FORMAT (16A5)
 12        WRITE (6,99) (NAME(I),I=1,32)
 17     99 FORMAT (1H1,24A5/1X,24A5)
 20        READ(5,2) BETA,SIGMA,U,DELTA,D,E,H,OMEGA
 21    2   FORMAT( 8E10.0)
 22        WRITE(6,5)BETA,SIGMA,U,DELTA,D,E,H,OMEGA
 23    5   FORMAT(///1H , 39X,  8H   BETA=E14.7, /
          1                40X,  8H  SIGMA=E14.7, /
          2                40X,  8H      U=E14.7, /
          3                40X,  8H  DELTA=E14.7, /
          4                40X,  8H      D=E14.7, /
          5                40X,  8H      E=E14.7, /
          6                40X,  8H      H=E14.7, /
          7                40X,  8H  OMEGA=E14.7  )
    C
 24        READ(5,10)N1,N2,STAU,DTAU,PTAU1,PTAU2,CHANGE,FTAU
 27     10 FORMAT (2I5,6E10.0)
 30        WRITE(6,11)N1,N2,STAU,DTAU,PTAU1,PTAU2,CHANGE,FTAU
 31   11   FORMAT(          40X,  8H     N1=I2   , /
          1                40X,  8H     N2=I2   , /
          2                40X,  8H   STAU=E14.7, /
          3                40X,  8H   DTAU=E14.7, /
          4                40X,  8H  PTAU1=E14.7, /
          5                40X,  8H  PTAU2=E14.7, /
          6                40X,  8H CHANGE=E14.7, /
          7                40X,  8H   FTAU=E14.7  )
    C
 32        READ (5,7) DIFFUS,AA,NCPTS,J
 35      7 FORMAT(2F10.6,2I10)
    C      DIFFUS=THERMAL DIFFUSIVITY IN FCCT SQUARE PER HOUR
    C      AA=THICKNESS OF REFRACTORY MATERIAL IN INCHES
    C      NOPTS IS THE NUMBER OF DATA PCINTS USED FCR TIME(I) OR CQ(I)
 36        WRITE (6,33) DIFFUS, AA, NCPTS
 37     33 FORMAT(          40X,  8H DIFFUS=E14.7, /
          1                40X,  8H     AA=E14.7, /
          2                40X,  8H  NCPTS=I3     )
    C
 40        WRITE (6,34)
 41     34 FORMAT (//  25X, 10H       D = ,
          162HEPSILON * STEFAN * T(I)**4 / (CQ(0) * R)
          2             25X, 10H       E = ,
          362HQC(C) * R * AA/ (K1 * T(I))
          4             25X, 10H       H = ,
          562H(TE/T(I))**4
          6             25X, 10H  PTAU1 = ,
          762HPRINT INTERVAL UP TC VALUE CF CHANGE
          8             25X, 10H  PTAU2 = ,
          962HPRINT INTERVAL AFTER VALUE OF CHANGE
 42        WRITE (6,36)
```

```
43     36 FORMAT (     25X, 1CH DIFFUS = ,
         162HTHERMAL DIFFUSIVITY IN FOOT SQUARE PER HOUR
         2              25X, 1CH      AA = ,
         362HTHICKNESS OF REFRACTORY MATERIAL IN INCHES
         4              25X, 1CH   NOPTS = ,
         562HNUMBER OF POINTS USED FOR INPUT DATA OF TIME VS QCOOT
   C
44        WRITE (6,99) (NAME(I),I=1,32)
   C
51        IF (J.EQ.2) GO TO 111
   C....IF J=2 USE THE SAME TABLE OF Q VS TAU
   C....READ IN TABLE OF Q VS. TAU
54        READ(5,3)(TIME(I),I=1,NOPTS)
61        READ(5,3)                          (QQ(I),I=1,NOPTS)
66      3 FORMAT(5E16.8)
   C      TIME(I) IS THE SET OF SELECTED VALUES CALLED TIME WHICH ARE USED
   C      AS DATA FOR THIS PROGRAM
   C      QQ(I) IS THE SET OF SELECTED VALUES CALLED QCOOT WHICH ARE USED
   C      AS DATA FOR THIS PROGRAM
   C      TIME(1)=0.0
   C      QQ(1)  =Q(0)
67 111    CAPPA=DIFFUS*144.0/(3600.0*AA*AA)
   C      CAPPA=(SQ.FT./HR)(SQ.IN./SQ.FT.)/(SEC/HR)-(SQ.IN))=1/SECONDS
   C....WRITE OUT TABLE OF Q VS. TAU
70        WRITE(6,9)
71      9 FORMAT(// 36X, 35HTABLE OF TAU VS Q AND TIME VS QCOOT///31X,1HN,
         1 7X, 6HTAU(N), 9X, 4HQ(N), 9X, 7HTIME(N), 6X, 8HQCOOT(N)/)
72        DO 6 II=1,NOPTS
73        TTAU(II)=TIME(II)*CAPPA
74        QT(II)=QQ(II)/QQ(1)
75      6 WRITE(6,4)II,TTAU(II),QT(II),TIME(II),QQ(II)
77      4 FORMAT( 29X, I3, 4( 3X, F11.6 ))
   C
100       WRITE (6,99) (NAME(I),I=1,32)
   C
   C....M=TOTAL NO. OF POINTS
105       M=N1+N2+1
   C
106       IF(STAU.EQ.0.) GO TO 30
   C....READ INITIAL TEMPERATURE DISTRIBUTION
111       READ(5,25)(T(I),I=1,M)
116   25  FORMAT(8E10.0)
117       GO TO 40
   C
   C....INITIALIZE ALL TEMPERATURES TO ZERO
120   30  DO 31 I=1,M
121   31  T(I)=0.
   C
   C....CALCULATE ALL NECESSARY CONSTANTS
123   40  FN=N1
124       DX1=1./FN
125       FN=N2
126       DX2=BETA/FN
127       C1=1./(DX1**2)
130       C12=2.*C1
```

```
131          BOS=BETA/SIGMA
132          C2=(BOS/DX2)**2
133          DEN=BOS+(U*DX2)/DX1
134          C3=(2.*BOS*C1)/DEN
135          C4=(2.*U*BOS**2)/(DEN*DX1*DX2)
136          N=N1+1
137          NP1=N+1
140          MM1=M-1
141          MP1=M+1
142          PRTIME=STAU-DTAU/10.
143          TAU=STAU
144          KCOUNT=C
     C
     C....CHECK FOR STABILITY
     C....MATERIAL 1
145          DTAUP=1./C12
146   50     IF(DTAU.LE.DTAUP) GO TO 60
151          DTAU=DTAU/2.
152          GO TO 50
     C....MATERIAL 2
153   60     DTAUP=1./(2.*C2)
154   65     IF(DTAU.LE.DTAUP) GO TO 70
157          DTAU=DTAU/2.
160          GO TO 65
     C....INTERFACE
161   70     DTAUP=1./(C3+C4)
162   75     IF(DTAU.LE.DTAUP) GO TO 80
165          DTAU=DTAU/2.
166          GO TO 75
     C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
     C....BOUNDARY
167   80     DTAUP=1./(C12*(1.+DX1*D*E*(4.+E*T(1)*(6.+E*T(1)*(4.+E*T(1))))))
170   85     IF(DTAU.LE.DTAUP) GO TO 90
173          DTAU=DTAU/2.
174          GO TO 85
     C
     C....ADIABATIC BOUNDARY CONDITICN
175   90     T(MP1)=T(MM1)
     C
176          IF(TAU.LT.PRTIME) GO TO 120
201          IF (PRTIME.GE.CHANGE-0.01) GO TC 91
204          PRTIME=PRTIME+PTAU1
205          GO TO 92
206      91  PRTIME=PRTIME + PTAU2
     C
     C....PRINT TEMPERATURE DISTRIBUTION
207      92  IF (PRINT.LE.4.) GO TO 94
212          PRINT = 0.0
213          WRITE (6,99) (NAME(I),I=1,32)
220      94  WRITE(6,100) TAU,DTAU
221  100     FORMAT(// 36X, 4HTAU=E12.5, 5X,5HDTAU=E12.5)
222          WRITE(6,101) (I,T(I),I=1,M)
227  101     FORMAT( 5(4X, 2HT(, I2,2H)=, E12.5) )
230          PRINT =PRINT+1.0
     C
```

```
231 120    IF(TAU.GE.FTAU) GO TO 1
    C
    C....LOOK-UP Q
234        DO 8 K=1,NOPTS
235        IF(TAU-TTAU(K))12,13,8
236      8 CONTINUE
240        KCOUNT=KCOUNT+1
241        Q4TAU=QT(NOPTS)+(TAU-TTAU(NOPTS))*(QT(NOPTS)-QT(NOPTS-1))/(TTAU(NO
          1PTS)-TTAU(NOPTS-1))
242        IF (KCOUNT.GT.1) GO TO 15
245        WRITE(6,16)
246 16     FORMAT ( 1X, 27HIN ALL FURTHER CALCULATIONS,
          162HTHE LARGEST VALUE OF TAU IN THE TABLE IS SMALLER THAN TAU REQU,
          240HIRED.  THEREFORE                          / 1X, 10HTHE VALUE ,
          362HOF Q USED IS THE RESULT OF A LINEAR EXTRAPOLATION BASED ON THE,
          44CH LAST TWO TABULATED VALUES.                )
247        GO TO 15
250     12 IF(K.GT.1) GO TO 14
253        WRITE (6,18)
254     18 FORMAT(//5X,54HRESUBMIT THE DATA WITH THE FIRST ENTRY FOR TIME = 0
          1.0.)
255        GO TO 1
256     14 Q4TAU=QT(K-1)+(QT(K)-QT(K-1))*(TAU-TTAU(K-1))/(TTAU(K)-TTAU(K-1))
257        GO TO 15
260     13 Q4TAU=QT(K)
261     15 Q=Q4TAU
    C
262        W1=Q*(1.+DELTA*SIN(OMEGA*TAU))
263        W2=D*((1.+E*T(1))**4 -H)
    C
    C....CALCULATE TEMPERATURES AT TAU+DTAU
    C....BOUNDARY
264        TP(1)=T(1)+C12*DTAU*(T(2)-T(1)+DX1*(W1-W2))
    C....MATERIAL 1
265        DO 130 I=2,N1
266 130    TP(I)=T(I)+C1*DTAU*(T(I-1)-2.*T(I)+T(I+1))
    C....INTERFACE
270        TP(N)=T(N)+DTAU*(C3*(T(N1)-T(N))+C4*(T(NP1)-T(N)))
    C....MATERIAL 2
271        DO 140 I=NP1,M
272 140    TP(I)=T(I)+C2*DTAU*(T(I-1)-2.*T(I)+T(I+1))
    C
    C....INCREMENT TAU
274        TAU=TAU+DTAU
    C....RESET
275        DO 150 I=1,M
276 150    T(I)=TP(I)
300        GO TO 80
    C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
301        END
```

APPENDIX D

COMPUTER PROGRAM FOR THE EIGEN-VALUES

Definitions of Input Quantities

FORTRAN NAME	SYMBOL OR MEANING
U	μ
S	σ
N	A number greater than or equal to the value of the last Lambda divided by DL
DL	Width of interval checked for change in sign (root)
STOP	Number of λ's desired (value of n)

```
ISN        SOURCE STATEMENT

  0 $IBFTC VALUES  DECK
  1        DIMENSICN F(3000),ZLAM(3000)
  2        CCMMON A,B,C,D
  3        EXTERNAL FUN
  4 1      WRITE(6,2)
  5 2      FCRMAT(1H1,40X,26HCOMPOSITE SLAB EIGENVALUES/////)
  6        READ(5,3) U,S,N,DL,STOP
 10 3      FCRMAT (2E 10.0,I10,2E10.0)
 11        WRITE (6,4) S,U,DL
 12 4      FCRMAT (3X,6HSIGMA=F12.8,6X,3HMU=F12.8,6X,13HDELTA LAMBDA=F12.8///
          1//)
 13        WRITE (6,51)
 14 51     FCRMAT (6X,1HN,8X,9HLAMBDA(N)//)
    C
    C....DEFINITONS FOR SYMBOLS
    C
    C         U = GREEK LETTER MU
    C         S = GREEK LETTER SIGMA
    C        DL = WIDTH OF INTERVAL CHECKED FOR CHANGE IN SIGN (ROOT) OF F
    C      STOP = NUMBER OF LAMBDAS DESIRED
    C         N = A NUMBER GREATER THAN OR EQUAL TO THE VALUE OF THE LAST
    C             LAMBDA DIVIDED BY DL
    C
 15        A=1.+U
 16        B=1.+S
 17        C=1.-U
 20        D=1.-S
 21        E1=1.0E-6
 22        E2=1.0E-6
 23        DO 5 I=1,N
 24        ZLAM(I)=DL*FLOAT(I)
 25        G=A*SIN(B*ZLAM(I))
 26        H=C*SIN(D*ZLAM(I))
 27 5      F(I)=G+H
 31        NCOUNT=0
 32 7      DC 11 L=2,N
 33        IF ((F(L-1)/F(L)).GT.0.) GO TO 11
 36        BL=ZLAM(L-1)
 37        BU=ZLAM(L)
 40 9      GUESS=0.5*(ZLAM(L-1)+ZLAM(L))
 41        CALL ROOT (FUN,T,GUESS,BL,BU,E1,E2,I)
 42        IF (I.NE.1) GO TO 13
 45        NCOUNT=NCOUNT+1
 46        WRITE (6,10) NCOUNT,T
 47 10     FCRMAT (4X,I3,5X,F13.8)
 50        KSTOP=STOP
 51        IF(NCOUNT.GT.KSTOP) GO TO 1
 54        GC TO 11
 55 13     WRITE (6,14)
 56 14     FORMAT (6X,34HROOT HAS NOT GIVEN A NORMAL RETURN)
 57 11     CCNTINUE
 61        WRITE (6,12)
 62 12     FCRMAT(1H1)
 63        GC TO 1
 64        END
```

```
ISN      -    SOURCE STATEMENT

  0 $IBFTC FUNCTN
  1        FUNCTION FUN(T)
  2        COMMON A,B,C,D
  3        FUN=A*SIN(B*T)+C*SIN(D*T)
  4        RETURN
  5        END
```

```
ISN        SCURCE STATEMENT

  0 $IBFTC RCOTSL
  1        SUBROUTINE RCOT (FUN,R,A,BL,BU,EP,E2,I)
    C
    C      SUBROUTINE ROOT IS USED TO APPROXIMATE AND HOME IN ON THE DESIRED
    C      RCOT
    C
    C      ARGUMENT LIST--
    C                FUN-FUNCTION SUBPROGRAM NAME
    C                R  -ARGUMENT OF SUBPROGRAM AND VALUE OF ROOT
    C                A  -INITIAL GUESS AT ROOT
    C                BL -LOWER BOUND ON ROOT
    C                BU -UPPER BOUND ON ROOT
    C                EP -EPSILON TEST FOR FUNCTION - IF FUN(R)(ABSOLUTE
    C                         VALUE) IS LESS THAN OR EQUAL TO EP, I=1 AND
    C                         RETURN
    C                E2 -EPSILON TEST ON RCOT APPROXIMATION. IF RATIO OF DIFF-
    C                         ERENCE OF SUCCESSIVE APPROXIMATIONS TO CURRENT
    C                         APPROXIMATION IS LESS THAN E2 IN ABSOLUTE VALUE,
    C                         I=1 AND RETURN.
    C                I  -ERROR CODE =1 FOR NORMAL RETURN
    C
    C
  2        CIMENSICN X(3),FX(3)
  3        G=A
  4        ISP=0
    C      TEST FOR LOWER BOUND ON ROOT LESS THAN UPPER BOUND
  5        IF (BL-BU) 1,2,2
    C      IF NOT , ERROR CODE I=3. RETURN.
  6 2      I=3
  7        RETURN
    C      TEST FOR GUESS OUTSIDE BOUNDS.
 10 1      IF (G-BL) 9,7,8
 11 8      IF (G-BU)6,7,9
    C      IF YES, ERROR CODE I=1 AND RETURN
 12 9      I=2
 13        RETURN
    C      IF GUESS=LOWER BOUND, COMPUTE GUESS=0.5*(BU+BL)
 14 7      G=(BL+BU)/2.0
 15 6      GL=BL+0.1*(G-BL)
 16        GU=BU-   0.1*(BU-G)
 17 80     R=G
 20        FG=FUN(G)
 21 .      IF (ABS(FG)-EP) 13,13,32
 22 10     IF (ABS (FX(11))-EP) 13,13,17
 23 13     I=1
 24        RETURN
 25 17     I1=I1
 26        IF (FX(I1)) 18,18,19
 27 18     IF(FX(I2)) 20,20,21
 30 20     IM=1
 31        GC TO 23
 32 19     IF (FX(I2)) 24,24,22
 33 22     IM=0
 34        GC TO 23
 35 21     IP=I2
```

```
 36          IM=I1
 37          GC TO 25
 40 24       IP=I1
 41          IM=I2
 42          GC TO 25
 43 23       DO 26 IC1=1,20
    C        RCOT HAS NOT BEEN BRACKETED. USE SUCCESSIVE APPROXIMATIONS.
 44          T=FX(I2)-FX(I1)
 45          IF (ABS(T)-0.01*EP) 27,27,28
 46 27       IF (X(I1)-X(I2)) 14,12,14
 47 12       I=4
 50          RETURN
 51 14       X(IF)=0.5*(X(I1)+X(I2))
 52          R=X(IF)
 53          GC TO 33
 54 28       X(IF)=X(I2)-(X(I2)-X(I1))/T*FX(I2)
 55          R=X(IF)
 56          IF (X(IF)) 63,64,63
 57 63       IF (ABS((X(IF)-X(I2))/X(IF))-E2    ) 34,34,65
 60 64       IF (ABS(X(IF)-X(I2))-E2          ) 34,34,65
 61 34       FX(IF)=FUN(X(IF))
 62          GC TO 13
 63 65       IF (IB-1) 29,29,33
 64 29       IF (BL-X(IF)) 31,31,32
 65 31       IF (X(IF)-BU) 33,33,32
 66 33       FX(IF)=FUN (X(IF))
 67          IF (ABS(FX(IF))-EP) 13,13,35
 70 35       IF (FX(IF)) 36,13,37
 71 36       IF (IM) 38,38,39
 72 38       IP=I2
 73          IM=IF
 74          IF=I1
 75          GC TO 25
 76 37       IF (IM) 39,39,40
 77 40       IP=IF
100          IM=I2
101          IF =I1
102          GC TO 25
103 39       I1=I1+1
104          I2=I2+1
105          IF=IF+1
106          GC TO (42,42,43,44),IF
107 42       I2=1
110          GC TO 26
111 43       I1=1
112          GC TO 26
113 44       IF=1
114 26       CCNTINUE
116 32       ISP=ISP+1
117          I1=1
120          I2=2
121          IF=3
122          X(I2)=G
123          FX(I2)=FG
124          GC TO (47,48,49,50,51,69,70),ISP
```

```
125 47      X(I1)=GL
126         R=GL
127         FGL=FUN(R)
130         FX(I1)=FGL
131         GO TO 10
132 48      X(I1)=GU
133         R=GU
134         FGU=FUN(R)
135         FX(I1)=FGU
136         GO TO 10
137 49      X(I2)=GL
140         FX(I2)=FGL
141         G2=0.5*(G+BL)
142 71      X(I1)=G2
143         R=G2
144         FG2=FUN(R)
145         FX(I1)=FG2
146         GO TO 10
147 50      X(I1)=G2
150         FX(I1)=FG2
151         GO TO 17
152 51      G2=0.5*(G+BU)
153         GO TO 71
154 69      X(I1)=GU
155         FX(I1)=FGU
156         X(I2)=G2
157         FX(I2)=FG2
160         GO TO 17
161 70      I=5
162         RETURN
163 25      DO 52 IC=1,50
    C       ROOT HAS BEEN BRACKETED. USE APPROXIMATIONS ON EACH SIDE.
164         X(IF)=X(IP)-(X(IP)-X(IM))/(FX(IP)-FX(IM))*FX(IP)
165         R=X(IF)
166         IF (X(IF)) 61,62,61
167 61      IF (ABS((X(IF)-X(IP))/X(IF))-E2     ) 34,34,67
170 67      IF (ABS((X(IF)-X(IM))/X(IF))-E2     ) 34,34,53
171 62      IF (ABS(X(IF)-X(IP))         -E2) 34,34,68
172 68      IF (ABS(X(IF)-X(IM))         -E2) 34,34,53
173 53      IF (IB-1) 54,54,55
174 54      IF (X(IF)-BL) 32,56,56
175 56      IF (X(IF)-BU) 55,55,32
176 55      FX(IF)=FUN(X(IF))
177         IF (ABS(FX(IF))-EP) 13,13,57
200 57      IF (FX(IF)) 58,13,59
201 58      IT=IM
202         IM=IF
203         GO TO 60
204 59      IT=IP
205         IP=IF
206 60      IF=IT
207 52      CONTINUE
211         GO TO 32
212         END
```

APPENDIX E

COMPUTER PROGRAM FOR THE ANALYTICAL SOLUTION OF CASE (i) - CONSTANT FREE STREAM CONDITIONS

Definitions of Input Quantities

FORTRAN NAME	SYMBOL OR MEANING
TI	Initial time
TF	Final time
DT	$\Delta\tau$
OM	Ω
MU	μ
S	σ
XI	ξ
D	δ
IC	1 for new set of λ_n 2 use set of λ_n from previous case
N	n, number of λ's
LAM(I)	λ_n

```
ISN        SOURCE STATEMENT

  0 $IBFTC SLAB2T  DECK
  1        REAL MU,LAM,LAM2
  2        DIMENSION LAM(100),LAM2(100),E(100),H1(100),H2(100),H3(100)
  3 15     READ (5,2) TI,TF,DT,OM,MU,S,XI,D
  4 2      FORMAT (8E10.0)
  5        READ (5,3) IC,N
 10 3      FORMAT (2I5)
 11        IF (IC.EQ.2)  GO TO 19
 14        READ (5,4)  (LAM(I),I=1,N)
 21 4      FORMAT (5E16.8)
 22 19     WRITE (6,1)
 23 1      FORMAT (1H1,20X,78HNO RADIATION COMPOSITE SLAB TEMPERATURE----FOR
          1CONSTANT FREE STREAM CONDITIONS////)
 24 16     WRITE (6,5)
 25 5      FORMAT (45X,37HINPUT PARAMETERS FOR THIS CASE ARE    )
 26        WRITE (6,6) TI,TF,DT,S,D,OM,XI,MU,N
 27 6      FORMAT (///    45X,    15H TAU INITIAL = F12.8,/
          1               45X,    15H   TAU FINAL = F12.8,/
          2               45X,    15H   DELTA TAU = F12.8,/
          3               45X,    15H       SIGMA = F12.8,/
          4               45X,    15H       DELTA = F12.8,/
          5               45X,    15H       OMEGA = F12.8,/
          6               45X,    15H          XI = F12.8,/
          7               45X,    15H          MU = F12.8,/
          8               45X,    15H           N = I3    ,/)
 30        WRITE(6,7)
 31 7      FORMAT (1H0,40X,52HTHE VALUES OF LAMBDA ARE, READING ACROSS FROM 1
          1 TO N///)
 32        WRITE (6,8) (LAM(I),I=1,N)
 37 8      FORMAT ( 5 (1X,F20.8))
 40        WRITE (6,12)
 41 12     FORMAT(1H0,14X,3HTAU,30X,13HTHETA/(ARQ/K),25X,20HABS(SUM N/SUM (N-
          11)))
    C
    C....DEFINITONS FOR SYMBOLS
    C
    C      TF = TAU FINAL
    C      TI = TAU INITIAL
    C      DT = DELTA TAU
    C      OM = GREEK LETTER CAPITAL OMEGA
    C       S = GREEK LETTER SIGMA
    C      MU = GREEK LETTER MU
    C      XI = GREEK LETTER XI
    C       D = GREEK LETTER DELTA
    C      IC = CYCLE CHECK--READ IN 1 FOR NEW SET OF LAMBDAS---
    C                        READ IN 2 TO USE SET OF LAMBDAS FROM PREVIOUS CASE
    C       N = NUMBER OF LAMBDAS READ IN
    C
 42        A=1.+MU
 43        B=1.+S
 44        C=1.-MU
 45        P=1.-S
 46        F=1.+MU*S
 47        G=S+XI
 50        H=S-XI
```

```
 51          IF (IC.EQ.2) GO TO 17
 54          DO 18 I=1,N
 55 18       LAM2(I)=LAM(I)*LAM(I)
 57 17       DO 9 I=1,N
 60          E1=A*B*COS(B*LAM(I))
 61          E2=C*P*COS(P*LAM(I))
 62          E(I)=E1+E2
 63          H2(I)=(LAM2(I)/OM)**2+1.
 64          H3(I)=1./LAM2(I)-(D/OM)/H2(I)
 65 9        H1(I)=(A*COS(LAM(I)*H)+C*COS(LAM(I)*G))/E(I)
 67          TAU=TI
 70 10       G1=TAU+(D/OM)*(1.-COS(OM*TAU))
 71          G2=3.*XI*XI+2.*S*S-6.*MU*S*XI-1.-2.*S*(S+MU)/F
 72          G3=G2*(1.+D*SIN(OM*TAU))/6.
 73          G4=(G1+G3)/F
 74          SUM=0.0
 75          DO 11 K=1,N
 76          IF (LAM2(K)*TAU.GE.86.) GO TO 20
101          H4=H3(K)*EXP(-LAM2(K)*TAU)
102          GO TO 21
103 20       H4=0.0
104 21       H5=(COS(OM*TAU)/OM+SIN(OM*TAU)/LAM2(K))*D/H2(K)
105          SUM=SUM+H1(K)*(H4+H5)
106          IF ((N-K).EQ.1) ZI=SUM
111 11       CONTINUE
113          THETA=G4-2.*SUM
114          Z2=ABS(SUM/ZI)
115          WRITE (6,13) TAU,THETA,Z2
116 13       FORMAT (1H ,6X,F15.8,8X,F20.8, 20X,F20.8)
117          TAU=TAU+DT
120          IF (TAU.LE.TF) GO TO 10
123 14       GO TO 15
124          END
```

APPENDIX F

COMPUTER PROGRAM FOR THE ANALYTICAL SOLUTION OF CASE (ii) - VARIABLE FREE STREAM CONDITIONS: ENTRY CASE

Definitions of Input Quantities

FORTRAN NAME	SYMBOL OR MEANING
TITLE (I)	Title of run under consideration. Two cards are to be used.
U	μ
S	σ
D	δ
OM	Ω
XI	ξ
LC	Λ
GC	Γ
GL	γ
R1	Limit of τ in print out of range 1
R2	" " " " " " " " 2
R3	" " " " " " " " 3
R4	" " " " " " " " 4
DT1	$\Delta\tau$ in print out of range 1
DT2	" " " " " " 2
DT3	" " " " " " 3
DT4	" " " " " " 4
TIMEA	t_a, in seconds
TIMEB	t_b, " "
TIMEC	t_c, " "
TIMED	t_d, " "
TIMEM	t_m, " "

FORTRAN NAME	SYMBOL OR MEANING
TIMEI	Initial τ
TIMEF	Final τ
TEMPI	Initial temperature
ASMALL	a, ft.
ABAR	$\bar{A}$, $1/sec^2$
R	R
ALPHA	α_1, ft^2/sec
PHI	φ, 1/sec
CAPPA	K_1 Btu/ft-sec-^oF
QO	$q_o(o)$ Btu/ft^2-sec
B	$B = \bar{B}$
N	n, number of λ's
J	1 for new set of λ's 2 use set of λ's from previous case
L(I)	λ_n

```
ISN        SOURCE STATEMENT

  0 $IBFTC ROBERT  DECK
  1        REAL L,LL,LC,LCM1,LLOM,LLOM2
  2        DIMENSION L(100),LL(100),LLOM(100),LLOM2(100),E(100),G(100)
  3        DIMENSION EMBMA(100),EMTTB(100),EPPTA(100),GP(100),TITLE(32)
  4        DIMENSION EMTTC(100),EMCMB(100),EMTTA(100),EMTA(100)
  5 1      READ (5,2) (TITLE(I),I=1,32)
 12 2      FORMAT(16A5)
 13        READ (5,3) U,S,D,OM,XI,LC,GC,GL
 14        READ (5,3) R1,R2,R3,R4,DT1,DT2,DT3,DT4
 15        READ (5,3) TIMEA,TIMEB,TIMEC,TIMED,TIMEM,TIMEI,TIMEF,TEMPI
 16        READ (5,3) ASMALL,ABAR,R,ALPHA,PHI,CAPPA,QO,B
 17        READ (5,4) N,J
 22 3      FORMAT (8 E 10.0)
 23 4      FORMAT (2I5)
 24 6      WRITE (6,7) (TITLE(I),I=1,32)
 31        WRITE (6,8) U,TIMEA,R1,ASMALL,S,TIMEB,R2,ABAR,D,TIMEC,R3,ALPHA,OM,
          1TIMED,R4,PHI,XI,TIMEM,DT1,CAPPA,LC,TIMEI,DT2,TEMPI,GC,TIMEF,DT3,B,
          2GL,N,DT4,QO,J,R
 32        WRITE (6,9)
 33        WRITE (6,10)
 34        WRITE (6,11)
 35        WRITE (6,18)
 36        WRITE (6,19)
    C
    C....FORMAT STATEMENTS
    C
 37 7      FORMAT(1H1,24X,16A5/25X,16A5)
 40 8      FORMAT(//50X,20H* INPUT PARAMETERS *//
          A12X4H U =E14.8,9X7HTIMEA =E14.8,11X5H R1 =E14.8,8X8HASMALL =E14.8/
          B12X4H S =E14.8,9X7HTIMEB =E14.8,11X5H R2 =E14.8,8X8H  ABAR =E14.8/
          C12X4H D =E14.8,9X7HTIMEC =E14.8,11X5H R3 =E14.8,8X8H ALPHA =E14.8/
          D12X4HOM =E14.8,9X7HTIMED =E14.8,11X5H R4 =E14.8,8X8H   PHI =E14.8/
          E12X4HXI =E14.8,9X7HTIMEM =E14.8,11X5HDT1 =E14.8,8X8H CAPPA =E14.8/
          F12X4HLC =E14.8,9X7HTIMEI =E14.8,11X5HDT2 =E14.8,8X8H TEMPI =E14.8/
          G12X4HGC =E14.8,9X7HTIMEF =E14.8,11X5HDT3 =E14.8,8X8H     B =E14.8/
          H12X4HGL =E14.8,9X7H    N =I4,    21X5HDT4 =E14.8/
          I12X4HQO =E14.8,9X7H    J =I3,    22X5H  R =E14.8)
 41 9      FORMAT(//52X,15H* DEFINITIONS *//11X15HGREEK   FORTRAN,13X7HFORTRA
          1N,40X7HFORTRAN/10X13HLETTER   NAME,19X,14HNAME   MEANING,33X,14HNA
          2ME   MEANING//)
 42 10     FORMAT(
          A6X18H     ALPHA   ALPHA,17X,32HTIMEA   STAGE 1 UPPER LIMIT      ,
          B17X,35H R1   LIMIT OF TAU IN PRINT RANGE 1/
          C6X18H     GAMMA   GL    ,17X,32HTIMEB   STAGE 2 UPPER LIMIT      ,
          D17X,35H R2   LIMIT OF TAU IN PRINT RANGE 2/
          E6X18H CAP GAMMA   GC    ,17X,32HTIMEC   STAGE 3 UPPER LIMIT      ,
          F17X,35H R3   LIMIT OF TAU IN PRINT RANGE 3/
          G6X18H     DELTA   D     ,17X,32HTIMED   STAGE 4 UPPER LIMIT      ,
          H17X,35H R4   LIMIT OF TAU IN PRINT RANGE 4)
 43 11     FORMAT(
          I6X18H     KAPPA   CAPPA,17X,32HTIMEM   TIME OF MAX HEAT LOAD    ,
          L17X,35HDT1   DELTA TAU IN PRINT RANGE 1   /
          K6X18H    LAMBDA   L(I) ,17X,32HTIMEI   INITIAL TIME             ,
          L17X,35HDT2   DELTA TAU IN PRINT RANGE 2   /
          M6X18HCAP LAMBDA   LC    ,17X,32HTIMEF   FINAL TIME              ,
```

```
      L17X,35HDT3   DELTA TAU IN PRINT RANGE 3   /
      O6X18H        MU   U     ,17X,32HTEMPI   INITIAL TEMPERATURE      ,
      L17X,35HDT4   DELTA TAU IN PRINT RANGE 4   )
 44 18  FORMAT(
      Q6X18H CAP OMEGA   OM    ,17X,32H TEMP   TEMPERATURE, DEG R       ,
      R17X,35H  R   SCALE FACTOR                  /
      A6X18H       PHI   PHI   ,17X,34H    N   NUMBER OF LAMBDAS READ IN ,
      B12X,38HASMALL   CHARACTERISTIC LENGTH          /
      C6X18H   CAP PHI   P     ,17X,34H    J   CONTROL-1-USE NEW LAMBDAS ,
      D12X,38H  ABAR   GROUPED TERMS-SEE REPORT       /
      E6X18H     SIGMA   S     ,17X,34H                2-USE LAST LAMBDAS,
      212X,38H     B   CONSTANT TERM-SEE REPORT       )
 45 19  FORMAT(
      G6X18H     THETA   THETA/
      H6X18H        XI   XI    //)
    C
 46     IF (J.EQ.1) GO TO 12
 51     WRITE (6,13)
 52 13  FORMAT(/30X49HTHE LAMBDAS FROM THE PREVIOUS CASE ARE SPECIFIED.)
 53     GO TO 15
 54 12  READ (5,16) (L(I),I=1,N)
 61 16  FORMAT(5E16.8)
 62 15  WRITE(6,14) (I,L(I),I=1,N)
 67 14  FORMAT(40X,29HTHE EIGENVALUES (LAMBDAS) ARE//5(3X,2HL(,I2,2H)=,
      1E14.8))
    C
    C....CALCULATION OF CONSTANTS FOR INTERNAL USE
 70     P=ASMALL*ASMALL*PHI/ALPHA
 71     TA=ALPHA*TIMEA/(ASMALL*ASMALL)
 72     TB=ALPHA*TIMEB/(ASMALL*ASMALL)
 73     TC=ALPHA*TIMEC/(ASMALL*ASMALL)
 74     TD=ALPHA*TIMED/(ASMALL*ASMALL)
 75     TM=ALPHA*TIMEM/(ASMALL*ASMALL)
 76     TI=ALPHA*TIMEI/(ASMALL*ASMALL)
 77     TF=ALPHA*TIMEF/(ASMALL*ASMALL)
100     SOTA = SIN(OM*TA)
101     SOTB = SIN(OM*TB)
102     SOTC = SIN(OM*TC)
103     SOTD = SIN(OM*TD)
104     COTA = COS(OM*TA)
105     COTB = COS(OM*TB)
106     COTC = COS(OM*TC)
107     COTD = COS(OM*TD)
110     LCM1 = LC-1.
111     GLM1=GL-1.
112     DOM = D/OM
113     POM = P/OM
114     EPA = EXP(P*TA)
115     EPB = EXP(P*TB)
116     EPC = EXP(P*TC)
117     EPD = EXP(P*TD)
120     EMA = EXP(-P*TA)
121     EMB = EXP(-P*TB)
122     EMC = EXP(-P*TC)
123     EMD = EXP(-P*TD)
```

```
124          TBMTA = TB-TA
125          TCMTB = TC-TB
126          TDMTC = TD-TC
127          C = 1./(1.+U*S)
130          F = (C/6.)*(3.*XI*XI+2.*S*S-6.*U*S*XI-1.-2.*S*(S+U)*C)
131          DO 20 I=1,N
     C       L(I) = THE ITH EIGENVALUE (LAMBDA)
132          LL(I) = L(I)*L(I)
133          LLOM(I) = LL(I)/OM
134          LLOM2(I) = LLOM(I)*LLOM(I)
135          E(I) = 0.25*LL(I)*((1.+U)*(1.+S)*COS(L(I)*(1.+S))+(1.-U)*( 1.-S)*
            1COS(L(I)*(1.-S)))
136          G(I)=(0.5/E(I))*((1.+U)*COS(L(I)*(S-XI))+(1.-U)*COS(L(I)*(S+XI)))
137          IF (LL(I)*TA.GE.86.) GO TO 17
142          EMTA(I)=EXP(-LL(I)*TA)
143          GO TO 31
144 17       EMTA(I)=0.0
145 31       IF (LL(I)*TBMTA.GE.86.) GO TO 32
150          EMBMA(I)=EXP(-LL(I)*TBMTA)
151          GO TO 21
152 32       EMBMA(I)=0.0
153 21       IF (LL(I)*TCMTB.GE.86.) GO TO 22
156          EMCMB(I)= EXP(-LL(I)*TCMTB)
157          GO TO 20
160 22       EMCMB(I)=0.0
161 20       CONTINUE
163          ARQK=ASMALL*R*QO/CAPPA
     C       DIMENSIONLESS TEMP=CAP THETA=(TEMP-TEMP INITIAL)/ARQK
164          A=ABAR*(ASMALL*ASMALL/ALPHA)**2
165          T = TI
166          PT =TI
167          SUM = 0.0
170          THETA = 0.0
171          J1 = 1
172          J2 = 2
173          J3 = 3
174          J4 = 4
     C
     C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
     C
175          WRITE (6,101) (TITLE(I),I=1,32)
202 101      FORMAT(1H1,24X,16A5/25X,16A5//)
203          IF (T.LE.TA) GO TO 100
206          IF (T.LE.TB) GO TO 200
211          IF (T.LE.TC) GO TO 300
214          IF (T.LE.TD) GO TO 400
     C
     C ** STAGE 1 ***********************************************************
     C
217 100      WRITE (6,102) J1,TI,TA
220 102      FORMAT(/30X,36HTHE FOLLOWING RESULTS ARE FOR STAGE ,I1,12H,TAU BET
            1WEEN,F8.3,1X,3HAND,F8.3///22X13HDIMENSIONLESS,12X13HDIMENSIONLESS,
            216X4HTIME,19X11HTEMPERATURE/23X10HTIME (TAU),15X11HTEMPERATURE,17X
            35H(SEC),18X11H(DEGREES R)//)
221 105      THETA = 0.0
```

```
222          EP = EXP( P*T)
223          EM = EXP(-P*T)
224          SOT = SIN(OM*T)
225          COT = COS(OM*T)
226          DO 106 I=1,4
227 106      GP(I) = 0.0
231          GP(1)=C/P*(EP-1.)+F*EP*(1.+D*SOT)
232          DO 109 I=1,N
233          IF (LL(I)*T.GE.86.) GO TO 107
236          EX=EXP(-LL(I)*T)
237          GO TO 108
240 107      EX=0.0
241 108      SUM=G(I)*(EX+P/(LL(I)+P)*(EP-EX))
242 109      GP(2)=GP(2)-SUM
244          GP(3)=C*EP/(POM*POM+1.)*DOM*(POM*SOT-COT+EM)
245          DO 111 I=1,N
246          IF ((LL(I)+P)*T.GE.86.) GO TO 131
251          EZ = EXP(-(LL(I)+P)*T)
252          GO TO 132
253 131      EZ = 0.0
254 132      SUM=G(I)*D*EP/((LLOM(I)+POM)**2+1.)*((POM*(LLOM(I)+POM)+1.)*SOT
            1   +LLOM(I)*COT-LLOM(I)*EZ)
255 111      GP(4)=GP(4)-SUM
257          DO 112 I=1,4
260 112      THETA=THETA+GP(I)
262          TIME=ASMALL*ASMALL*T/ALPHA
263          TEMP=TEMPI+ARQK*THETA
    C....PRINT OUT TIME
264          WRITE (6,120) T,THETA,TIME,TEMP
265 120      FORMAT(10X,4(10X,F15.8))
266          IF (T.LE.R1) GO TO 125
271          IF (T.LE.R2) GO TO 130
274          IF (T.LE.R3) GO TO 135
277          IF (T.LE.R4) GO TO 140
302          WRITE (6,124)
303          GO TO 1
304 125      T = T + DT1
305          GO TO 110
306 130      T = T + DT2
307          GO TO 110
310 135      T = T + DT3
311          GO TO 110
312 140      T = T + DT4
313 110      IF (T.LE.TA) GO TO 105
316          IF (T.LE.TB) GO TO 200
321          WRITE (6,121)
322          IF (T.LE.TC) GO TO 300
325          WRITE (6,122)
326          IF (T.LE.TD) GO TO 400
331          WRITE (6,123)
332          GO TO 1
333 121      FORMAT(/1X,49H DT IS GREATER THAN (TB-TA).  HENCE SKIP STAGE 2.//)
334 122      FORMAT(/1X,49H DT IS GREATER THAN (TC-TA).  HENCE SKIP STAGE 3.//)
335 123      FORMAT(/1X,49H DT IS GREATER THAN (TD-TA).  HENCE SKIP STAGE 4.//)
336 124      FORMAT(/1X,49H TA IS GREATER THAN R4.  HENCE SKIP STAGES 2,3,4.//)
```

```
C
C ** STAGE 2 ***************************************************************
C
200   WRITE (6,102) J2,TA,TB
205   THETA = 0.0
      SOT=SIN(OM*T)
      COT=COS(OM*T)
      TMTB=T-TB
      TMTA=T-TA
      DO 206 I=1,100
206   GP(I)=0.0
      GP(1)=C*(TMTA*EPA+(EPA-1.)/P)
      GP(2)=C*DOM*EPA/(POM*POM+1.)*(POM*SOTA-COTA+EMA)
      DO 209 I=1,N
      IF (LL(I)*TMTA.GE.86.) GO TO 207
      EMTTA(I)=EXP(-LL(I)*TMTA)
      GO TO 208
207   EMTTA(I) =0.0
208   SUM=G(I)/(LL(I)/P+1.)*EMTTA(I)*(EPA+LL(I)/P*EMTA(I))
209   GP(3)=GP(3)-SUM
      DO 211 I=1,N
      SUM=G(I)*EMTTA(I)/((POM+LLOM(I))**2+1.)*((POM*(POM+LLOM(I))*SOTA
     A   +SOTA+LLOM(I)*COTA)*EPA-LLOM(I)*EMTA(I))
211   GP(4)=GP(4)-D*SUM
      GP(5)=C*EPA*(DOM*(COTA-COT)+DOM/OM*LCM1/TBMTA*(-OM*TMTA*COT+SOT
     1      -SOTA)+LCM1/TBMTA*TMTA*TMTA/2.)
      GP(6)=EPA*F*(1.+D*SOT)*(1.+LCM1/TBMTA*TMTA)
      DO 212 I=1,N
      Z=LLOM(I)
      ZZ=LLOM2(I)
      EML=EMTTA(I)
218   SUM=G(I)*(D/(ZZ+1.)*(Z*COT+SOT-(Z*COTA+SOTA)*EML)+LCM1/TBMTA*D
     1      *(TMTA/(ZZ+1.)*(Z*COT+SOT)-1./(OM*(ZZ+1.)**2)*(ZZ*COT+2.*Z
     2      *SOT-COT-(ZZ*COTA+2.*Z*SOTA-COTA)*EML))+LCM1/(TBMTA*LL(I))
     3      *(1.-EML)+LCM1/TBMTA*DOM/(ZZ+1.)*(Z*SOT-COT-(Z*SOTA-COTA)*EML
     4      ))
212   GP(7)=GP(7)-EPA*SUM
      DO 213 I=1,7
213   THETA=THETA+GP(I)
      TIME=ASMALL*ASMALL*T/ALPHA
      TEMP=TEMPI+ARQK*THETA
C....PRINT OUT TIME
      WRITE (6,120) T,THETA,TIME,TEMP
      IF (T.LE.R1) GO TO 225
      IF (T.LE.R2) GO TO 230
      IF (T.LE.R3) GO TO 235
      IF (T.LE.R4) GO TO 240
      WRITE (6,224)
      GO TO 1
225   T = T + DT1
      GO TO 210
230   T = T + DT2
      GO TO 210
235   T = T + DT3
      GO TO 210
```

```
433 240    T = T + DT4
434 210    IF(T.LE.TB) GO TO 205
437        IF(T.LE.TC) GO TO 300
442        WRITE(6,222)
443        IF(T.LE.TD) GO TO 400
446        WRITE(6,223)
447        GO TO 1
450 222    FORMAT(/1X,49H DT IS GREATER THAN (TC-TB).  HENCE SKIP STAGE 3.//)
451 223    FORMAT(/1X,49H DT IS GREATER THAN (TD-TB).  HENCE SKIP STAGE 4.//)
452 224    FORMAT(/1X,49H TB IS GREATER THAN R4.  HENCE SKIP STAGES 3, 4. //)
    C
    C ** STAGE 3 ************************************************************
    C
453 300    WRITE (6,102) J3,TB,TC
454 305    THETA = 0.0
455        SOT=SIN(OM*T)
456        COT=COS(OM*T)
457        TMTA=T-TA
460        TMTB=T-TB
461        DO 304 I=1,N
462 304    GP(I)=0.0
464        DO 306 I=1,N
465        IF (LL(I)*TMTA.GE.86.) GO TO 317
470        EMTTA(I)=EXP(-LL(I)*TMTA)
471        GO TO 318
472 317    EMTTA(I)=0.0
473 318    IF (LL(I)*TMTB.GE.86.) GO TO 315
476        EMTTB(I)=EXP(-LL(I)*TMTB)
477        GO TO 306
500 315    EMTTB(I)=0.0
501 306    CONTINUE
    C      EMBMA(I)=EXP(-LL(I)*TBMTA) WAS CALCULATED PREVIOUSLY
503        GP(1)=C*(TMTA*(1.+D*SOTA)*EPA+(EPA-1.)/P)
504        GP(2)=C*DOM*EPA/(POM*POM+1.)*(POM*SOTA-COTA+1./EPA)
505        GP(3)=F*(1.+D*SOTA)*EPA
506        DO 307 I=1,N
507        SUM=G(I)/(LL(I)/P+1.)*EMTTA(I)*(EPA+LL(I)/P*EMTA(I))
510 307    GP(4)=GP(4) -SUM
512        DO 308 I=1,N
513        SUM=G(I)*EMTTA(I)/((POM+LLOM(I))**2+1.)*((POM*(POM+LLOM(I))*SOTA
          A   +SOTA+LLOM(I)*COTA)*EPA-LLOM(I)*EMTA(I))
514 308    GP(5)=GP(5)-D*SUM
516        GP(6)=EPA*C*(DOM*(OM*T*SOTB-COTB-OM*TB*SOTB-OM*T*SOTA+COTA+OM*TA*
          1SOTA)+LCM1/TBMTA*DOM/OM*(SOTB-SOTA)-LCM1*DOM*(COTB+OM*TB*SOTB-OM*
          2T*SOTB)+LCM1*(T-0.5*(TB+TA)))
517        GP(7)=EPA*F*(D*(SOTB-SOTA)+LCM1*(1.+D*SOTB))
520        DO 309 I=1,N
521        SUM=G(I)*EMTTB(I)*(D/(LLOM2(I)+1.)*(LLOM(I)*COTB+SOTB-(LLOM(I)*
          1COTA+SOTA)*EMBMA(I))+LCM1/TBMTA*D*(TBMTA/(LLOM2(I)+1.)*(LLOM(I)*
          2 COTB+SOTB)-1./(OM*(LLOM2(I)+1.)**2)*(LLOM2(I)*COTB+2.*LLOM(I)*
          3SOTB-COTB-(LLOM2(I)*COTA+2.*LLOM(I)*SOTA-COTA)*EMBMA(I))) +
          4LCM1/(TBMTA*LL(I))*(1.-EMBMA(I))+LCM1/TBMTA*DOM/(LLOM2(I)+1.) *
          5(LLOM(I)*SOTB-COTB-(LLOM(I)*SOTA-COTA)*EMBMA(I)))
522 309    GP(8)=GP(8)-EPA*SUM
524        GP(9)=C*(-A*(T*T*T/3.-T*TB*TB+2.73.*TB*TB*TB-TM*TMTB*TMTB)+DOM*
```

```
          1(GC-A*TM*TM)*(-COT-OM*T*SOTB+COTB+OM*TB*SOTB)-2.*A*DOM/(OM*OM)*
          2(-OM*T*SOT-2.*COT-OM*T*SOTB+OM*OM*T*TB*COTB+2.*OM*TB*SOTB-COTB*
          3((OM*TB)**2-2.))+2.*A*TM*DOM/OM*(-SOT+SOTB+OM*TMTB*COTB)+2.*A*TM*
          4DOM/OM*(-OM*T*COT+2.*SOT-OM*T*COTB-OM*OM*T*TB*SOTB+2.*OM*TB*COTB
          5+OM*OM*TB*TB*SOTB-2.*SOTB) - A*DOM/OM/OM*(-OM*OM*T*T*COT-2.*OM*T*
          6SOT+6.*COT+6.*OM*T*SOT-2.*OM*OM*T*TB*COTB-OM*T*(OM*OM*TB*TB-2.)*
          7SOTB+(3.*OM*OM*TB*TB-6.)*COTB+(OM*TB)**3*SOTB-6.*OM*TB*SOTB))
525        GP(10)=F*(-A*(T*T-TB*TB)+2.*A*TM* TMTB +D*(GC-A*TM*TM)*(SOT-SOTB)
          1-2.*A*DOM/OM*(SOT-OM*T*COT-SOTB+OM*TB*COTB)+2.*A*TM*DOM*(COTB+OM*
          2T*SOT-COTB-OM*TB*SOTB)-A*DOM/OM*(2.*OM*T*COT+OM*OM*T*T*SOT-2.*SOT
          3-2.*OM*TB*COTB-OM*OM*TB*TB*SOTB+2.*SOTB))
526        DO 311 I=1,N
527        X1= 2.*A/(LL(I)*LL(I)) *((LL(I)*T-1.)-(LL(I)*TB-1.)*EMTTB(I))
530        X2=-2.*A*TM/LL(I)*(1.-EMTTB(I))-D*(GC-A*TM*TM)/(LLOM2(I)+1.)
          B    *(LLOM(I)*COT+SOT-(LLOM(I)*COTB+SOTB)*EMTTB(I))
531        X3= 2.*A*D*(1./OM
          C    /(LLOM2(I)+1.)*(T*(LLOM(I)*SOT-COT)-TB*(LLOM(I)*SOTB-COTB)
          D    *EMTTB(I))-1./(OM*(LLOM2(I)+1.))**2*(LLOM2(I)*SOT-2.*LLOM(I)
          E    *COT-SOT-(LLOM2(I)*SOTB-2.*LLOM(I)*COTB-SOTB)*EMTTB(I)))
532        Z=1./(LLOM2(I)+1.)
533        X4= -2.*A*D*TM*(Z/OM*(LLOM(I)*SOT-COT-(LLOM(I)*SOTB-COTB)
          A     *EMTTB(I)))
534        X5= -2.*A*D*TM*(Z*(T*(LLOM(I)*COT+SOT)-TB*(LLOM(I)*COTB+SOTB)
          A     *EMTTB(I))-Z*Z/OM*(LLOM2(I)*COT+2.*LLOM(I)*SOT-COT-(LLOM2(I)
          B     *COTB+2.*LLOM(I)*SOTB-COTB)*EMTTB(I)))
535        X6= Z*(T*T*(LLOM(I)*COT+SOT)-TB*TB*(LLOM(I)*COTB+SOTB)*EMTTB(I))
536        X7=-Z*Z/OM*(T*(2.*LLOM2(I)*COT+4.*LLOM(I)*SOT-2.*COT)-TB*(2.
          A     *LLOM2(I)*COTB+4.*LLOM(I)*SOTB-2.*COTB)*EMTTB(I))
537        X8= 2.*Z**3/OM/OM*(LLOM2(I)*LLOM(I)*COT+3.*LLOM2(I)*SOT-3.*LLOM(I)
          A     *COT-SOT-(LLOM2(I)*LLOM(I)*COTB+3.*LLOM2(I)*SOTB-3.*LLOM(I)
          B     *COTB-SOTB)*EMTTB(I))
540        SUM=X1+X2+X3+X4+X5+A*D*(X6+X7+X8)
541  311   GP(11)=GP(11)+G(I)*SUM
543        DO 312 I=1,11
544  312   THETA=THETA+GP(I)
546        TIME=ASMALL*ASMALL*T/ALPHA
547        TEMP=TEMPI+ARQK*THETA
     C....PRINT OUT TIME
550        WRITE (6,120) T,THETA,TIME,TEMP
551        IF (T.LE.R1) GO TO 325
554        IF (T.LE.R2) GO TO 330
557        IF (T.LE.R3) GO TO 335
562        IF (T.LE.R4) GO TO 340
565        WRITE (6,324)
566        GO TO 1
567  325   T = T + DT1
570        GO TO 310
571  330   T = T + DT2
572        GO TO 310
573  335   T = T + DT3
574        GO TO 310
575  340   T = T + DT4
576  310   IF (T.LE.TC) GO TO 305
601        IF (T.LE.TD) GO TO 400
604        WRITE (6,323)
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605          GO TO 1
606 323      FORMAT(/1X,49H DT IS GREATER THAN (TD-TC).  HENCE SKIP STAGE 4.//)
607 324      FORMAT(/1X,49H TC IS GREATER THAN R4.  HENCE SKIP STAGE 4.     //)
    C
    C ** STAGE 4 ***********************************************************
    C
610 400      WRITE (6,102) J4,TC,TD
611 405      THETA = 0.0
612          SOT=SIN(OM*T)
613          COT=COS(OM*T)
614          TMTA=T-TA
615          TMTB=T-TB
616          TMTC=T-TC
617          TMTD=T-TD
620          OT=OM*T
621          OTA=OM*TA
622          OTB=OM*TB
623          OTC=OM*TC
624          OTD=OM*TD
625          DO 406 I=1,N
626          IF (LL(I)*TMTA.GE.86.) GO TO 417
631          EMTTA(I)=EXP(-LL(I)*TMTA)
632          GO TO 418
633 417      EMTTA(I)=0.0
634 418      IF (LL(I)*TMTB.GE.86.) GO TO 407
637          EMTTB(I)= EXP(-LL(I)*TMTB)
640          GO TO 408
641 407      EMTTB(I)=0.0
642 408      IF (LL(I)*TMTC.GE.86.) GO TO 409
645          EMTTC(I)=EXP(-LL(I)*TMTC)
646          GO TO 406
647 409      EMTTC(I)=0.0
650 406      CONTINUE
    C
652          DO 450 I=1,100
653 450      GP(I)=0.0
655          GP(1)=C*(TMTA*(1.+D*SOTA)*EPA+(EPA-1.)/P)
656          GP(2)=C*DOM*EPA/(POM*POM+1.)*(POM*SOTA-COTA+1./EPA)
657          GP(3)=F*(1.+D*SOTA)*EPA
660          DO 451 I=1,N
661          SUM=G(I)*EMTTA(I)/(LL(I)/P+1.)*(EPA+LL(I)/P*EMTA(I))
662 451      GP(4)=GP(4)-SUM
664          DO 452 I=1,N
665          SUM=G(I)*EMTTA(I)/((POM+LLOM(I))**2+1.)*((POM*(POM+LLOM(I))*SOTA
            1   +SOTA+LLOM(I)*COTA)*EPA-LLOM(I)*EMTA(I))
666 452      GP(5)=GP(5)-D*SUM
670          GP(6)=EPA*C*(DOM*(OT*SOTB-COTB-OTB*SOTB-OT*SOTA+COTA+OTA*SOTA)
            A      +LCM1/TBMTA*DOM/OM*(SOTB-SOTA)-LCM1*DOM*(COTB+OTB*SOTB-OT
            B      *SOTB)+LCM1*(T-0.5*(TB+TA)))
671          GP(7)=EPA*F*(D*(SOTB-SOTA)+LCM1*(1.+D*SOTB))
672          DO 453 I=1,N
673          SUM=G(I)*EMTTB(I)*(D/(LLOM2(I)+1.)*(LLOM(I)*COTB+SOTB-(LLOM(I)
            1   *COTA+SOTA)*EMBMA(I)) + LCM1/TBMTA*D*(TBMTA/(LLOM2(I)+1.)
            2   *(LLOM(I)*COTB+SOTB)-1./(OM*((LLOM2(I)+1.)**2))*(LLOM2(I)*COTB
            3   +2.*LLOM(I)*SOTB-COTB-(LLOM2(I)*COTA+2.*LLOM(I)*SOTA-COTA)
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     4    *EMBMA(I))) + LCM1/TBMTA/LL(I)*(1.-EMBMA(I)) + LCM1/TBMTA
     5    *DOM/(LLOM2(I)+1.)*(LLOM(I)*SOTB-COTB-(LLOM(I)*SOTA-COTA)
     6    *EMBMA(I)))
453   GP(8)=GP(8)-EPA*SUM
      X1= -A*T*TC*TC+2./3.*A*TC*TC*TC+A*T*TB*TB-2./3.*A*TB*TB*TB
     A     +2.*A*TM*(T*TC-TC*TC/2.-T*TB+TB*TB/2.)+DOM*(GC-A*TM*TM)
     B     *(OT*SOTC-COTC-OTC*SOTC-OT*SOTB+COTB+OTB*SOTB)-2.*A*DOM
     C     /(OM*OM)*(OT*SOTC-OT*OTC*COTC-2.*OTC*SOTC+(OTC*OTC-2.)*COTC
     D     -OT*SOTB+OT*OTB*COTB+2.*OTB*SOTB-(OTB*OTB-2.)*COTB)+2.*A
     E     *DOM/OM*TM*(-OT*COTC-SOTC+OTC*COTC+OT*COTB+SOTB-OTB*COTB)
      X2= 2.*A*TM*DOM/OM*(OT*COTC+OT*OTC*SOTC-2.*OTC*COTC-(OTC*OTC-2.)
     A *SOTC-OT*COTB-OT*OTB*SOTB+2.*OTB*COTB+(OTB*OTB-2.)*SOTB)
     B     -A*DOM/(OM*OM)*(2.*OT*OTC*COTC+OT*(OTC*OTC-2.)*SOTC-(3.*OTC
     C     *OTC-6.)*COTC-(OTC**3     -6.*OTC)*SOTC-2.*OT*OTB*COTB-OT*
     D     (OTB*OTB-2.)*SOTB+(3.*OTB*OTB-6.)*COTB+(OTB**3-6.*OTB)*SOTB)
      GP(9)=C*(X1+X2)
      GP(10)=F*(-A*(TC*TC-TB*TB)+2.*A*TM*TCMTB+D*(GC-A*TM*TM)*(SOTC-
     A       SOTB)-2.*A*DOM/OM*(SOTC-OTC*COTC-SOTB+OTB*COTB)+2.*A*TM*DOM
     B       *(COTB-COTC)+2.*A*TM*DOM*(COTC+OTC*SOTC-COTB-OTB*SOTB)
     C       -A*DOM/OM*(2.*OTC*COTC+OTC*OTC*SOTC-2.*SOTC-2.*OTB*COTB
     D         - OTB*OTB*SOTB+2.*SOTB))
C
      DO 454 I=1,N
      X1=2.*A/(LL(I)*LL(I))*((LL(I)*TC-1.)-(LL(I)*TB-1.)*EMCMB(I))
     1  -2.*A*TM/LL(I)*(1.-EMCMB(I))
      Y=1./(LLOM2(I)+1.)
      X2=-D*(GC-A*TM*TM)*Y*(LLOM(I)*COTC+SOTC-(LLOM(I)*COTB+SOTB)
     1  *EMCMB(I))
      X3=2.*A*D*(Y/OM*(TC*(LLOM(I)*SOTC-COTC)-TB*(LLOM(I)*SOTB-COTB)
     1  *EMCMB(I))-Y*Y/(OM*OM)*(LLOM2(I)*SOTC-2.*LLOM(I)*COTC-SOTC
     2  -(LLOM2(I)*SOTB-2.*LLOM(I)*COTB-SOTB)*EMCMB(I)))
      X4=-2.*A*D*TM*(Y/OM*(LLOM(I)*SOTC-COTC-(LLOM(I)*SOTB-COTB)
     1  *EMCMB(I)))
      X5=-2.*A*D*TM*(Y*(TC*(LLOM(I)*COTC+SOTC)-TB*(LLOM(I)*COTB+SOTB)
     A  *EMCMB(I))-Y*Y/OM*(LLOM2(I)*COTC+2.*LLOM(I)*SOTC-COTC-(LLOM2(I)
     B  *COTB+2.*LLOM(I)*SOTB-COTB)*EMCMB(I)))
      X6=Y*(TC*TC*(LLOM(I)*COTC+SOTC)-TB*TB*(LLOM(I)*COTB+SOTB)
     1  *EMCMB(I))
      W=Y*Y
      X7=-W /OM*(TC*(2.*LLOM2(I)*COTC+4.*LLOM(I)*SOTC-2.*COTC)-TB*(2.
     1  *LLOM2(I)*COTB+4.*LLOM(I)*SOTB-2.*COTB)*EMCMB(I))
      X8=2.*Y*Y*Y/(OM*OM)*(LLOM2(I)*LLOM(I)*COTC+3.*LLOM2(I)*SOTC
     1   -3.*LLOM(I)*COTC-SOTC-(LLOM2(I)*LLOM(I)*COTB+3.*LLOM2(I)*SOTB
     2  -3.*LLOM(I)*COTB-SOTB)*EMCMB(I))
      SUM=X1+X2+X3+X4+X5+A*D*(X6+X7+X8)
454   GP(11)=GP(11)+G(I)*EMTTC(I)*SUM
      GP(12)=C*B*(GLM1   /(2.*TDMTC)*TMTC*TMTC+D*GLM1/TDMTC*(1./(OM*OM)
     1       *(SOTC-SOT)+COTC*TMTC/OM)+D*((COTC-COT)/OM-TMTC*SOTC)
     2       +GLM1/TDMTC*DOM/OM*(OM*(T+TC)*(COT+OT*SOT-COTC-OTC*SOTC)
     3       -2.*OT*COT-(OT*OT-2.)*SOT+2.*OTC*COTC+(OTC*OTC-2.)*SOTC
     4       -OT*OTC*(SOT-SOTC)))
      GP(13)=F*B*(GLM1/TDMTC*TMTC*(1.+D*SOT)+D*(SOT-SOTC))
      DO 455 I=1,N
      Y=1./(LLOM2(I)+1.)
      X1=GLM1/TDMTC/LL(I)*(1.-EMTTC(I))+DOM*GLM1/TDMTC*Y*(LLOM(I)*SOT
```

```
        1  -COT-(LLOM(I)*SOTC-COTC)*EMTTC(I))
725      X2=D*Y*(LLOM(I)*COT+SOT-(LLOM(I)*COTC+SOTC)*EMTTC(I))
726      X3=GLM1/TDMTC*D*(TMTC*Y*(LLOM(I)*COT+SOT)-Y*Y/OM*(LLOM2(I)*COT
        1  +2.*LLOM(I)*SOT-COT-(LLOM2(I)*COTC+2.*LLOM(I)*SOTC-COTC)
        2  *EMTTC(I)))
727      SUM=X1+X2+X3
730 455  GP(14)=GP(14)-G(I)*B*SUM
732      DO 456 I=1,14
733 456  THETA=THETA+GP(I)
735      TIME=ASMALL*ASMALL*T/ALPHA
736      TEMP=TEMPI+ARQK*THETA
    C....PRINT OUT TIME
737      WRITE (6,120) T,THETA,TIME,TEMP
740      IF (T.LE.R1) GO TO 425
743      IF (T.LE.R2) GO TO 430
746      IF (T.LE.R3) GO TO 435
751      IF (T.LE.R4) GO TO 440
754      WRITE (6,424)
755      GO TO 1
756 425  T = T + DT1
757      GO TO 410
760 430  T = T + DT2
761      GO TO 410
762 435  T = T + DT3
763      GO TO 410
764 440  T = T + DT4
765 410  IF (T.LE.TD) GO TO 405
770      WRITE (6,423)
771      GO TO 1
772 423  FORMAT(/1X,49H* T IS GREATER THAN TD.  THIS CASE IS COMPLETE. *//)
773 424  FORMAT(/1X,49H* T IS GREATER THAN R4.  THIS CASE IS COMPLETE. *//)
774      END
```