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THEORY OF DEFORMABLE BODIES

E. Cosserat, F. Cosserat

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PREFACE

This volume expands upon a summary of *Théorie de l'action euclidienne* /v
(Theory of Euclidian Action) which M. Appell wanted to include in the second edition of his *Traité de mecanique rationnelle (Treatise of Rational Mechanics)*. It is a copy of an appendix to the French edition of the *Traité de physique (Treatise of Physics)* of M. Chwolson, which explains certain editorial refinements and certain references which we shall make to an earlier work on the dynamics of points and material bodies. This was also added to the work of the Russian scientists. We have taken advantage of this new printing by correcting certain errors in our text.

We are not presently interested in deriving all the consequences of the general results which we have obtained; we would primarily like to find and clarify the classical laws. In order that this type of proof for the theory of Euclidian action be more complete, we had to establish the form which the equations of deformable bodies assume for each part of our discussion, when we consider only *infinitesimally close states*. However, we have already discussed this point in sufficient detail in the *Premier mémoire sur la Théorie de l'élasticité (First Memoir on the Theory of Elasticity)* which he wrote in 1896 *(Annals of the Faculty of Sciences of Toulouse, Vol. X)*. We shall also assume that the important lessons of M. G. Darboux regarding the *General Theory of Surfaces* are completely familiar to the reader.

Our investigations would not be fully valid unless we could illustrate the manner in which the same methods may be applied to the theories of heat and electricity. Two reports in Volumes III and IV in the treatise by M. Chwolson will deal with this subject. The *division*, if we use the language of pragmatism, seems to be a scientific necessity. However, it must be realized that this raises serious questions. We have attempted to give an idea of these difficulties in a note concerning *theory of thin bodies*, which was published in 1908 in the Comptes Rendus of the Academy of Sciences. The treatise by M. Appell also deals with this.

E. and F. Cosserat

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THEORY OF DEFORMABLE BODIES

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ABSTRACT. A historical introduction of the theory of elasticity is given and the concepts of the deformable line (one-dimensional elastic body), surface (two-dimensional elastic body) and medium (three-dimensional elastic body) are introduced. The equations of statics and dynamics of the deformable line, surface and medium are derived in the same way: The kinematical relationships are established. A triad which moves with the body is introduced at each point of the body. Expressions for the variations of the rates of translation and rotation of the triad are introduced formulations for the deformed state. Euclidian action of deformation, external force and moment, external stress and moment of deformation are introduced, as well as transformation relationships for them. The equations are written in terms of various coordinates and using several formulations of the theory of elasticity. These are compared with equations obtained by Lagrange, Poincaré, Varignon, M. Duhem, Saint-Guilhem and Poisson. The same quantities are introduced formulations for the deformed body in motion. The action formulation of Euclid, Euler and others are discussed. The independent variables of Lagrange and Euler and the auxiliary quantities introduced are treated. The equations of the statics and dynamics of the deformable medium are derived in terms of Eulerian variables from those written in terms of Lagrange variables. The conceptsof the calculus of variations are introduced. The action variable method expressed in terms of Euler and Lagrange variables is formulated. The concept of radiation energy of deformation and of motion is defined. A future work by the authors will treat the theories of heat and electricity, and will give a more detailed development of radiation energy of deformable bodies.

I. General Remarks /1*

1. Development of the Idea of a Continuous Medium. - The concept of a deformable body played an important role in the development of theoretical physics during the last century. Fresnel(1) must be regarded as one of the

* Numbers in the margin indicate pagination in the original foreign text.

(1) Fresnel. *OEuvres complètes (Complete Works)*, Paris, 1866 (see the Introduction by É. Verdet).

precursors of the present theory of elasticity. This holds true for Navier, Poisson and Cauchy[2]. Due to the influence of Newtonian ideas, only systems of discrete points were considered at this time by these scientists. The continuous point systems appeared in the outstanding works of G. Green[3]. Since that time, attempts have been made to expand upon the ideas advanced by Green. His ideas do not present the importance of the theory of light waves. Lord Kelvin[4], in particular, attempted to define continuous media at any point which can exercise a moment. The same tendency can be found in the work of Helmholtz[5]; the controversy with J. Bertrand[6] regarding the theory of magnetism is very characteristic. The beginning of this development can be traced, on the one hand, to the ideas introduced in the strength of materials /2
by Bernoulli and Euler[7], and, on the other hand, in the theory of couples of Poinsot[8]. It is thus natural to reunite the various concepts of deformable bodies which one encounters today in natural philosophy under the same geometric definition. *A deformable line is a one-parameter ensemble of triads; a deformable surface is an ensemble with two parameters; a deformable medium is an ensemble with three parameters* ρ_i; *when there is motion, the time t must be added to these geometric parameters* ρ_i. The mathematical continuity which is

(2) See Isaac Todhunter and Karl Pearson. A History of the Theory of Elasticity and of the Strength of Materials, from Gallileo to the present time. Vol. 1, Gallileo to Saint-Venant, 1886; Vol. 2, Part I and II, Saint-Venant to Lord Kelvin, 1893. This remarkable work contains a very complete and very precise analysis of the foundations of the theory of elasticity.

(3) G. Green. *Mathematical Papers*, edited by N. M. Ferrers, Facsimile Reprint, Paris, A. Hermann, 1903.

(4) Lord Kelvin. *Math. and Phys. Papers*, Volume I, 1882; Vol. II, 1884; Vol. III, 1890; *Reprint of Papers on Electrostatics and Magnetism*, 2nd edition, 1884; *Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light*, 1904; W. Thomson and P. G. Tait, *Treatise on Natural Philosophy*, 1st edition, Oxford, 1867; 2nd edition, Cambridge, 1879-1895.

(5) Helmoholtz. *Vorles. über die Dynamik diskreter Massenpunkte (Lectures on the Dynamics of Discrete Mass Points)*, Berlin, 1897; *Vorles. über die elektromagnetische Theorie des Lichtes (Lectures on the Electromagnetic Theory of Light)*, Leipzig, 1897; *Wiss. Abhandl.* 3rd Volume, Leipzig, 1892-1895.

(6) J. Bertrand. Comptes Rendus 73, p. 965; 75, p. 860; 77, p. 1049; see also H. Poincaré, *Electricité et Optique (Electricity and Optics)*, II, *Les théories de Helmholtz et les expériences de Hertz (The Theories of Helmholtz and the Experiments of Hertz)*, Paris, p. 51, 1891; 2nd edition, p. 275, 1901.

(7) See Todhunter and Pearson. Loc. Cit.

(8) Auguste Comte. *Course de Philosophie Positive (Course in Positive Philosophy)*, 5th Edition, Paris, Vol. I, p. 338, 1907: "No matter what the fundamental properties of the ideas advanced by M. Poinsot may be in reality, in relation to statics, it must nevertheless be realized, it seems to me, that they are in principle essentially destined to perfect dynamics. In this connection, I believe I can make the statement that this idea has not exerted its most important influence up to the present time."

assumed in such a definition makes it possible, as can be seen, for the trace of an invariable solid to exist at each point. From the point of view of mechanics, one can predict the emergence of moments, which have been well known from the study of the elastic line and elastic surface since Euler and Bernoulli, and which Lord Kelvin and Helmholtz attempted to discover in three-dimensional media.

2. Difficulties Encountered in the Application of the Inductive Method in Mechanics. The first form of mechanics is inductive; this is what is primarily observed in the theory of deformable bodies. This theory borrows theorems regarding the concept of static force from the mechanics of invariable bodies. These have been expressed in the principle of solidification. In addition, the relationship between stress and deformation was established in the form of a hypothesis (generalized Hooke's law), and *a posteriori* the condition for which energy was conserved was investigated (Green). Carnot[9] already pointed out the error entailed in this method a century ago, when he constantly used *a priori* concepts and where the path which is followed is not always a very reliable one. A static force does not always represent a constructive definition within the framework of classical mechanics, and the scope of the revision which Reech[10] proposed in this connection in 1852 has remained almost unknown up to the present day. Perhaps this is due to the incertitude which the elasticians have felt for a long period of time regarding /3
the rational foundation that lies at the basis of Hooke's law. Analogous hesitations have been manifestated in almost the same way in all other areas

(9) Carnot, in his *Essai de 1783 sur les machines en général (Essay on General Machines of 1783)*, which in 1803 became *The Fundamental Principles of Equilibrium and Motion*, attempted to reduce mechanics to principles and precise definitions completely void of any metaphysical property and void of any vague terms which philosophers debate, but for which they have no understanding among themselves. This reaction led Carnot somewhat far afield, because he went to the point of contesting the legality of the term force, which is an obscure idea. He wanted to substitute exclusively the idea of motion for it. For the same reason, he was unable to accept any of the known proofs of the parallelogram rule of forces. "The very existence of the word *force* in the statement of the proposition makes this proof impossible due to the nature of things." (Ch. Combres, Ed. Phillips and Ed. Collignon, *Exposé de la situation de la mécanique appliquée (Exposition of the Position of Applied Mechanics)*, Paris, 1867).

(10) F. Reech. *Cours de Mécanique, d'après la Nature Généralement Flexible et Élastique des Corps (Course in Mechanics According to the Generalized Flexible and Elastic Behavior of Bodies)*, Paris, 1852. This work was written by the brilliant mechanical engineer as a revision to the teaching procedure in mechanics at the Polytechnic School. His ideas have since been discussed by J. Andrade, *Lecons de Mécanique Physique (Lessons of Mechanical Physics)*, Paris, 1898, and by Marbec, Chief Naval Engineer, in his Elementary Textbook of Mechanics at the Lower Officers' School at Toulon (1906). See also J. Perrin, *Traité de Chimie physique, les Principes (Treatise on Physical Chemistry, Principles)*, Paris, 1903.

of physics[11].

In order to avoid the difficulties, Helmholtz attempted to construct what is called an energetic, which is based on the principle of least action and — also based on the idea of energy, or force, no matter what its nature may be. It thus becomes a secondary concept, and has a deductive origin. However, the principle of a minimum in natural phenomena[12], and the concept of energy[13] reveal to us the faults of the inductive method. Why is there a minimum and what should be the definition of energy in order not to have simply a physical theory, but instead a truly mechanical theory? Helmholtz seems to have left no answer to these questions. He has made a contribution by establishing more completely what had not been done previously -- namely, the distinction between two concepts, energy and action. We must start from this basis, it seems to us, in order to express the opinions of Helmholtz in a precise way and in order to give a completely deductive form to mechanics, or more generally, to physics.

3. Theory of Euclidian Action. When we are dealing with the motion of a point, the essential element in the definition of action is the Euclidian distance between two infinitesimally close positions of the moving point. We previously showed[14] that all the fundamental definitions of classical dynamics

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(11) The remark made by Lord Kelvin in his *Baltimore Lectures*, page 131, regarding the works of Blanchet is particularly interesting in this respect. It states that Poisson, Corolis and Sturm (C. R.* 7, p. 1143), as well as Cauchy, Liouville and Duhamel ** accepted without objection the 36 coefficients which Blanchet had introduced in the generalized Hooke's law. From the same point of view, Lord Kelvin also objected to the force-at-a-distance law of Weber, in the first edition of *Natural Philosophy*. More recently, the application of a static adiabatic law to the study of waves of finite amplitude was criticized for the same reasons by Lord Rayleigh. Also, it is known that Houginot proposed a dynamic adiabatic law.

(12) Maupertuis himself had sensed the danger of the principle which he introduced in mechanics, when he wrote in 1744: "We do not know sufficiently well *what is the aim* of nature, and we can be mistaken regarding the quantity which we must regard as the output in the production of its effects." Lagrange first had the intention of making the principle of least action the basis of his *Mecanique analytique (Analytical Mechanics)*, but later on he realized the superiority of the method which considers the virtual work.

(13) Hertz, *Die Prinzipien der Mechanik, etc. (The Principles of Mechanics, etc.)*, 1894; see the Introduction, in particular.

(14) *Note sur la Dynamique du Point et du Corps Invariable (Note on the Dynamics of a Point and an Invariable Body), in Traite' de Physique (Treatise on Physics)* of Chwolson, Vol. I, page 236.

* Translator's Note: C. R. designates Comptes Rendus.

** Translator's Note: Illegible in original foreign text.

can be derived from this single concept -- namely, those of momentum, force and energy. /4

At the present time, we shall establish the fact that we can follow an identical path in the study of static or dynamic deformation of discrete systems of points or continuous bodies. We thus arrive at the formulation of a *general theory of extended action and motion,* which encompasses everything which is directly subjected to the laws of mechanics in theoretical physics.

Here the action will also be the integral of a function of two infinitesimally close elements in time and in space of the medium under consideration. By introducing the condition of invariance in the group of Euclidian displacements and by defining the medium as we indicated in Section 1, *the action density at a point will have the same interesting form that we have already encountered in the dynamics of a point and of an invariable body.* According to the notation used in the *Lessons* of M. Darboux, let (ξ_i, η_i, ζ_i), (p_i, q_i, r_i) be the geometrical rates of translation and rotation of the elementary triad, and (ξ, η, ζ), (p, q, r) be analogous rates for motion relative to this triad. The action will be the integral

$$\int_{t_1}^{t_2}\int\cdots\int W(\rho_i, t; \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i; \xi, \eta, \zeta, p, q, r)\, d\rho_1, \ldots d\rho_i, \ldots dt.$$

It is sufficient to consider the variation of this action in order to obtain the definition of momentum, the strain and moment of deformation, external force and moment, and finally the definition of deformation energy and motion, by means of the concept of work.

In this theory, statics could become entirely autonomous, according to the views of Carnot and Reech. It would only be necessary to assume the work density W is independent of the velocities (ξ, η, ζ), (p, q, r) -- i.e., to consider a body *without inertia,* or a body with inertia under the condition that the deformation is regarded as a *reversible transformation,* in the sense of M. Duhem. On the other hand, making use of the concept of *hidden arguments,* we would again find the original mechanical concepts which are used in physics -- as well as the less rigorous definitions which have been proposed for the deformable line by D. Bernoulli and Euler up to Thomson and Tait. For the deformable surface, this was done by Sophie Germain and Lagrange up to Lord Rayleigh. For the deformable medium, it was done by Navier and Green up to Lord Kelvin and W. Voigt.

If we simultaneously consider deformation and movement, we arrive at the idea which contains the d'Alembert principle, using a strictly deductive method. This is applicable only to the case in which *the action of deformation can be completely separated from the kinetic action.* Finally, if we assume that the deformable body is not influenced in any way by the outside world, and if in the following we introduce the fundamental concept of an *isolated system* -- which M. Duhem(15) found to be necessary for the rational construction of /5

(15) P. Duhem. *Commentaire aux Principes de la Thermodynamique (Comment on the Principles of Thermodynamics),* 1892; *la Théorie Physique, son Objet et sa Structure (Theoretical Physics, Its Objectives and Its Structure),* 1906.

theoretical physics along with M. LeRoy[16] -- one is led in a natural way to the idea of the minimum which Helmholtz had assumed as a starting point. At the same time this idea belongs to the principle of the conservation of energy, which is the basis of our present scientific system.

Finally, it seems to me that we will thus avoid all of the difficulties and trials of inductive research which we have described previously.

4. Criticism of the Principles of Mechanics. As we have just outlined, the theory of Euclidian action is the first contribution to the criticism of the principles of mechanics.

Its generality makes it possible to predict that there are singular phenomena in the action of motion, as well as in the deformation of extended bodies -- for example, the behavior of solids in the plastic state or in the vicinity of rupture, and also the behavior of fluids for large strains[17]. Under ordinary circumstances, this generality can be reduced by considering the state which is infinitesimally close to the natural state. This is the point which we have already discussed in our previous note.

However, we may also assume that one or several of the dimensions of the deformable body can become infinitesimally small, and we may envision what is called a *thin body*[18]. This concept was developed by Poisson in 1828, and also by Cauchy a short time later. Their goal, as well as the goal of all elasticians who dealt with this difficult question at a later time, was to establish a transition between the distinct theories of bodies having one, two and three dimensions. It is known that a very important part of the work by Barré de Saint-Venant and by Kirchhoff deals with a discussion of the research carried out by Poisson and Cauchy. These scientists, as well as their disciples, did not really isolate the real difficulty of the question. This difficulty consists of the fact that, *in general, the zero value of the parameter which is introduced is not an ordinary point, as Poisson and Cauchy admitted, and not even a pole, but an essential singular point. This important fact justifies separate studies* of the line, of the surface, and of the medium which are to be found in the present work[19].

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(16) E. Le Roy. *La Science Positive et les Philosophies de la Liberté, Congrès int. de Philosophie (Positive Science and the Philosophies of Liberty, International Congress of Philosophy)*, Vol. I., 1900.

(17) E. and F. Cosserat. *Sur la Mécanique Générale (On General Mechanics)*, C. R. 145, p. 1139, 1907.

(18) E. and F. Cosserat. *Sur la Théorie des Corps Minces (On the Theory of Thin Bodies)*, C. R. 146, p. 169, 1908.

(19) The interest and importance of the theories of the deformable line and deformable surface are not appreciated today, because the *Encyclopédie des Sciences Mathématiques Pures et Appliquées (Encyclopedia of Pure Mathematical and Applied Sciences)*, which at present is being published in Germany, did not give any space to these problems. W. Thomson and Tait

As a final word in these preliminary observations, we would like to point out that the theory of Euclidian action is based on the concept of *the differential invariant*, expressed in its simplest form. If we enlarge this concept in such a way that we include the idea of a *differential parameter*, modern theoretical physics is found to be an immediate extension, *from the Eulerian point of view*, of mechanics proper, and one is naturally led to the principles of the theory of heat and the present electrical laws. This new field of research, which we are beginning to enter by deriving the idea of energy of radiation from the consideration of deformable bodies, will be examined more completely in a future work. In this way, we can introduce a new precision into the points of view of H. Lorentz(20) and H. Poincaré(21), which is called *the reaction principle* in mechanics. /6

II. Statics of the Deformable Line

5. Deformable Line. Natural State and Deformed State. Let us consider a curve (M_0) described by a point M_0, whose coordinates x_0, y_0, z_0 with respect to three rectangular, fixed axes Ox, Oy, Oz are functions of a single parameter, which we will assume in the following to be equal to the arc s_0 of the curve, measured from a specified origin and in a specified direction. Let us introduce at each point M_0 of the curve (M_0) an orthogonal triad whose axes $M_0x'_0$, $M_0y'_0$, $M_0z'_0$ have the direction cosines α_0, α'_0, α''_0; β_0, β'_0, β''_0; γ_0, γ'_0, γ''_0, respectively, with respect to the axes Ox, Oy, Oz, and which are functions of the same parameter s_0.

The continuous, one-dimensional ensemble of these triads $M_0x'_0y'_0z'_0$ shall be what we shall call a *deformable line*.

Let us give a displacement M_0M to the point M_0. Let x, y, z be the coordinates of the point M with respect to the fixed axes Ox, Oy, Oz. Let us subject, among other things, a rotation to the triad $M_0x'_0y'_0z'_0$, which will finally make its axes coincide with those of a triad Mx'y'z', which we will attach to the point M. We will define this rotation by specifying the direction

(19) (Continued): did not omit them in their *Natural Philosophy*, and they presented them *before* the treatment of three-dimensional elastic bodies. This also holds true for the work by P. Duhem, *Hydrodynamique, Elasticité, Acoustique (Hydrodynamics, Elasticity, Acoustics)*, Paris, 1891.

(20) H. Lorentz. *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern (Attempt of a Theory of Electrical and Optical Phenomena in Bodies in Motion)*, Leiden 1895; reprinted in Leipzig in 1906. *Abhandl. über theoretische Physik. (Treatise. On Theoretical Physics)*, 1907; *Encyklop. der Math. Wissenschaften (Encyclopedia of Mathematical Sciences)*, V.2, *Elektronen Theorie (Electron Theory)*, 1903.

(21) H. Poincaré. *Electricité et Optique (Electricity and Optics)*, 2nd Edition, p. 448, 1901.

cosines α, α', α''; β, β', β''; γ, γ', γ'' of the axes Mx', My', Mz' with respect to the fixed axes Ox, Oy, Oz.

The continuous, one-dimensional ensemble of triads $M_0x'_0y'_0z'_0$ will be what we shall call the *deformed state* of the deformable line under consideration which, in the initial state, which will be called the *natural state*.

6. Kinematic Elements for the States of the Deformable Line. Let us suppose that s_0 varies and that, for one instant, we make it play the role of the time. Using the notation by M. Darboux(1), we will call the projections of the velocity of the origin M_0 of the axes $M_0x'_0$, $M_0y'_0$, $M_0z'_0$ on these axes ξ_0, η_0, ζ_0. Terms p_0, q_0, r_0 are the projections on the same axes of the instantaneous rates of rotation of the triad $M_0x'_0y'_0z'_0$. We will call the analogous quantities for the triad $Mx'y'z'$, ξ, η, ζ and p, q, r when they are /7
referred to the fixed triad $Oxyz$, like the triad $M_0x'_0y'_0z'_0$.

The elements which we have introduced can be calculated in the ordinary way. In particular we have ——

$$(1)\qquad \begin{cases} \xi = \alpha \dfrac{dx}{ds_0} + \alpha' \dfrac{dy}{ds_0} + \alpha'' \dfrac{dz}{ds_0}, \\ \eta = \beta \dfrac{dx}{ds_0} + \beta' \dfrac{dy}{ds_0} + \beta'' \dfrac{dz}{ds_0}, \\ \zeta = \gamma \dfrac{dx}{ds_0} + \gamma' \dfrac{dy}{ds_0} + \gamma'' \dfrac{dz}{ds_0}, \end{cases} \qquad \begin{cases} p = \sum \gamma \dfrac{d\beta}{ds_0} = -\sum \beta \dfrac{d\gamma}{ds_0}, \\ q = \sum \alpha \dfrac{d\gamma}{ds_0} = -\sum \gamma \dfrac{d\alpha}{ds_0}, \\ r = \sum \beta \dfrac{d\alpha}{ds_0} = -\sum \alpha \dfrac{d\beta}{ds_0}. \end{cases} \qquad (2)$$

Using these quantities, the linear element ds of the curve described by the point M is defined by the equation

$$ds^2 = (\xi^2 + \eta^2 + \zeta^2)\, ds_0^2. \qquad (3)$$

Let x', y', z' be the projections of the segment OM on the axes Mx', My', Mz', in such a way that the coordinates of the *fixed point O* with respect to these axes will be $-x'$, $-y'$, $-z'$. We will obtain the known formulas:

$$\xi - \frac{dx'}{ds_0} - qz' + ry' = 0, \quad \eta - \frac{dy'}{ds_0} - rx' + pz' = 0, \quad \zeta - \frac{dz'}{ds_0} - py' + qx' = 0, \qquad (4)$$

which result in the new expressions for ξ, η, ζ.

7. Expressions for the Variations in the Rate of Translation and Rotation of the Triad Referred to the Deformed State. Let us assume that we have imparted an infinitesimally small displacement to each of the triads of the deformed state which can vary in a continuous manner for these triads. Let δx, δy, δz; $\delta x'$, $\delta y'$, $\delta z'$; δx, $\delta x'$, ... $\delta\gamma''$ be the respective variations of x, y, z; x', y', z'; α, α', ... γ''. The variations $\delta\alpha$, $\delta\alpha'$, ... $\delta\gamma''$ can be expressed by formulas of the following type:

$$\delta\alpha = \beta\delta\alpha' - \gamma\delta\beta', \qquad (5)$$

(1) G. Darboux. *Lecons sur la Théorie Générale des Surfaces (Lessons on the General Theory of Surfaces)*, Vol. I., Paris, 1887.

where use is made of three auxiliary quantities δ_I', δ_J' δ_K' which are the components, in the Mx', My', Mz' system, of the well known instantaneous rotation connected with the infinitesimally small displacement under consideration. The variations δx, δy, δz are the projections on Ox, Oy, Oz of the infinitely small displacements experienced by the point M. The projections $\delta' x$, $\delta' y$, $\delta' z$ of this displacement on Mx', My', Mz' can immediately be derived and have the following values:

$$\delta' x = \delta x' + z'\delta_J' - y'\delta_K', \quad \delta' y = \delta y' + x'\delta_K' - z'\delta_I', \quad \delta' z = \delta z' + y'\delta_I' - x'\delta_J'. \tag{6}$$

Let us now attempt to determine the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr experienced by ξ, η, ζ, p, q, r, respectively. From equations (2), we have

$$\delta p = \sum \left(\frac{d\beta}{ds_0}\delta\gamma + \gamma\frac{d\delta\beta}{ds_0}\right).$$
$$\delta q = \sum \left(\frac{d\gamma}{ds_0}\delta\alpha + \alpha\frac{d\delta\gamma}{ds_0}\right).$$
$$\delta r = \sum \left(\frac{d\alpha}{ds_0}\delta\beta + \beta\frac{d\delta\alpha}{ds_0}\right).$$

Let us replace $\delta\alpha$ by its value $\beta\delta K' - \gamma\delta J'$, and $\delta 2'$, ... $\delta\gamma''$ by the analogous values. We have /8

$$\delta p = \frac{d\delta_I'}{ds_0} + q\delta_K' - r\delta_J', \quad \delta q = \frac{d\delta_J'}{ds_0} + r\delta_I' - p\delta_K', \quad \delta r = \frac{d\delta_K'}{ds_0} + p\delta_J' - q\delta_I'. \tag{7}$$

Formulas (4) also give us three formulas, the first of which is given by:

$$\delta\xi = \frac{d\delta x'}{ds_0} + q\delta z' - r\delta y' + z'\delta q - y'\delta r;$$

let us replace δp, δq, δr by their values given by equations (7). We find

$$\begin{cases} \delta\xi = \eta\delta_K' - \zeta\delta_J' + \dfrac{d\delta' x}{ds_0} + q\delta' x - r\delta' y, \\ \delta\eta = \zeta\delta_I' - \xi\delta_K' + \dfrac{d\delta' y}{ds_0} + r\delta' y - p\delta' z, \\ \delta\zeta = \xi\delta_J' - \eta\delta_I' + \dfrac{d\delta' z}{ds_0} + p\delta' z - q\delta' x, \end{cases} \tag{8}$$

by introducing the three symbols $\delta' x$, $\delta' y$, $\delta' z$, in order to abbreviate the notation, which are defined by equations (6).

8. Euclidian Action of Deformation Over a Deformable Line. Let us consider a function W of *two infinitesimally close positions* of the triad Mx'y'z', i.e., a function of s_0, of x, y, z, α, α', ..., γ'' and their first derivatives with respect to s_0. Let us attempt to determine the form of W such that the integral

$$\int W ds_0,$$

extended over an arbitrary portion of the line (M_0) has the variation zero when the ensemble of all the triads of the deformable line, assumed in the deformed state, is subjected to *the same arbitrary infinitesimal transformation of the group of Euclidian displacements.*

More specifically, we are to determine W in such a way that we will have

$$\delta W = 0,$$

such that, on the one hand, the origin M of the triads Mx'y'z' is subjected to an infinitesimally small displacement whose projections δx, δy, δz on the axes Ox, Oy, Oz are:

$$\begin{cases} \delta x = (a_1 + \omega_2 z - \omega_3 y)\,\delta t, \\ \delta y = (a_2 + \omega_3 x - \omega_1 z)\,\delta t, \\ \delta z = (a_3 + \omega_1 y - \omega_2 x)\,\delta t, \end{cases} \tag{9}$$

where a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constants and δt is an infinitesimally small quantity independent of s_0, and, on the other hand, this triad Mx'y'z' carries out an infinitesimally small rotation whose components along the axes Ox, Oy, Oz are:

$$\omega_1 \delta t, \quad \omega_2 \delta t, \quad \omega_3 \delta t.$$

We should note that in the present case the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr of the six expressions ξ, η, ζ, p, q, r are zero, as follows from the well known theorem of the moving triad. Also, we can verify this immediately by means of equations (7) and (8) by replacing $\delta' x$, δ_1' by their present values: /9

$$\begin{cases} \delta' x = \alpha(a_1 + \omega_2 z - \omega_3 y)\,\delta t + \alpha'(a_2 + \omega_3 x - \omega_1 z)\,\delta t + \alpha''(a_3 + \omega_1 y - \omega_2 x)\,\delta t \\ \delta_1' = (\alpha\omega_1 + \alpha'\omega_2 + \alpha''\omega_3)\,\delta t, \end{cases} \tag{9'}$$

and $\delta' y$, $\delta' z$, $\delta J'$, $\delta K'$ by their analogous present values. As a result we obtain a solution to the question by assuming W to be an arbitrary function of s_0 and the six expressions ξ, η, ζ, p, q, r. We will now show that we may obtain the general solution(2) to the problem which we have posed ourselves.

In this connection let us note that the relationships (2) allow one to express the first derivatives of the nine cosines α, α, ..., γ'' with respect to s_0 in terms of these cosines and of p, q, r, using well known equations. Let us also note, on the other hand, that the equations (1) allow one to express the nine cosines α, α', ... γ'' in terms of ξ, η, ζ and the first derivatives of x, y, z with respect to s_0. Finally, we can express the function W which we desire as a function of s_0, of x, y, z and their first derivatives, and finally of ξ, η, ζ, p, q, r, which we will indicate by writing:

$$W = W\left(s_0, x, y, z, \frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \xi, \eta, \zeta, p, q, r\right).$$

Since the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr are zero in the present case, as we remarked a moment ago, we may finally write the following for the new form of W, which we obtain on the basis of equations (9) no matter what a_1, a_2, a_3, ω_1, ω_2, ω_3 are:

---

(2) We will assume in the following that the deformable line *can carry out all possible deformations, and, consequently, the deformed state can be assumed to be completely arbitrary;* this fact can be expressed by saying that the deformable line is *free.*

$$\frac{\partial W}{\partial x}\delta x+\frac{\partial W}{\partial y}\delta y+\frac{\partial W}{\partial z}\delta z+\frac{\partial W}{\partial \frac{dx}{ds_0}}\delta\frac{dx}{ds_0}+\frac{\partial W}{\partial \frac{dy}{ds_0}}\delta\frac{dy}{ds_0}+\frac{\partial W}{\partial \frac{dz}{ds_0}}\delta\frac{dz}{ds_0}=0.$$

Let us replace δx, δy, δz by their values (9) and $\delta\frac{dx}{ds_0}$, $\delta\frac{dy}{ds_0}$, $\delta\frac{dz}{ds_0}$ by the values one obtains by differentiation. Let us set the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3 equal to zero. We obtain the following six conditions:

$$\frac{\partial W}{\partial x}=0,\qquad \frac{\partial W}{\partial y}=0,\qquad \frac{\partial W}{\partial z}=0,$$

$$\frac{\partial W}{\partial \frac{dy}{ds_0}}\frac{dz}{ds_0}-\frac{\partial W}{\partial \frac{dz}{ds_0}}\frac{dy}{ds_0}=0,\qquad \frac{\partial W}{\partial \frac{dz}{ds_0}}\frac{dx}{ds_0}-\frac{\partial W}{\partial \frac{dx}{ds_0}}\frac{dz}{ds_0}=0,\qquad \frac{\partial W}{\partial \frac{dx}{ds_0}}\frac{dy}{ds_0}-\frac{\partial W}{\partial \frac{dy}{ds_0}}\frac{dx}{ds_0}=0.$$

As could have easily have been predicted, the first three tell us that W is independent of x, y, z. The three latter ones express the fact that W does not depend on $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$ except through the intermediary of the quantity: /10

$$\left(\frac{dx}{ds_0}\right)^2+\left(\frac{dy}{ds_0}\right)^2+\left(\frac{dz}{ds_0}\right)^2,$$

and since the latter one, according to equation (3), is equal to $\xi^2+\eta^2+\zeta^2$, we can finally see that the *desired function W has the following remarkable form:*

$$W(s_0, \xi, \eta, \zeta, p, q, r).$$

If we multiply W by ds_0, the product Wds_0 obtained is an invariant within the group of Euclidian displacements, which is analogous to the one which carries the name *linear element* and which arises from the consideration of the distance between two infinitely close points in the neighborhood of the curve (M) described by the point M.

In the same way the common value of the integrals

$$\int_{A_0}^{B_0}\frac{ds}{ds_0}ds_0,\qquad \int_A^B ds,$$

taken, respectively, between two points A_0 and B_0 of the curve (M_0) and between its corresponding points A and B of the curve (M), determines the *length* of the arc AB of this curve (M). In the same way, by associating in one's mind the concept of *action* with the transition from the natural state (M_0) to the deformed state (M), we can relate the function W to the elements which define the deformable line, and we will say that the integral:

$$\int_{A_0}^{B_0}Wds_0$$

is the *action of deformation* on the deformed line between the points A and B, which correspond to the points A_0 and B_0 of (M_0). In this definition and in the following, we will assume that the arcs s_0 and s are measured in the direction of A_0 to B_0 and in the direction from A to B. Conversely, we will assume that the nomenclature A_0, B_0, A, B is to be used for designating the

extremities of the line in the natural state and in the deformed state using this convention.

We will also say that W is the *density* of the action of deformation *at a point* of the deformed line referred to a unit of length of the non-deformed line. $W\frac{ds_0}{ds}$ will be the action density at a point referred to a unit of length of the deformed line.

9. The External Force and Moment; the External Deformation Stress and Moment, at a Point of a Deformed Line. Let us consider an *arbitrary* variation of the action of deformation between two points A and B of the line (M), that is:

$$\delta\int_{A_0}^{B_0} W ds_0 = \int_{A_0}^{B_0}\left(\frac{\partial W}{\partial \xi}\delta\xi + \frac{\partial W}{\partial \eta}\delta\eta + \frac{\partial W}{\partial \zeta}\delta\zeta + \frac{\partial W}{\partial p}\delta p + \frac{\partial W}{\partial q}\delta q + \frac{\partial W}{\partial r}\delta r\right)ds_0.$$

We might write, in view of equations (7) and (8) of Section 7, /11

$$\begin{aligned}\delta\int_{A_0}^{B_0} W ds_0 = \int_{A_0}^{B_0}\Bigg[&\frac{\partial W}{\partial \xi}\left(\eta\delta\kappa' - \zeta\delta\lambda' + \frac{d\delta' x}{ds_0} + q\delta' z - r\delta' y\right)\\ &+ \frac{\partial W}{\partial \eta}\left(\zeta\delta\lambda' - \xi\delta\kappa' + \frac{d\delta' y}{ds_0} + r\delta' x - p\delta' z\right)\\ &+ \frac{\partial W}{\partial \zeta}\left(\xi\delta\lambda' - \eta\delta\lambda' + \frac{d\delta' z}{ds_0} + p\delta' y - q\delta' x\right)\\ &+ \frac{\partial W}{\partial p}\left(\frac{d\delta\lambda'}{ds_0} + q\delta\kappa' - r\delta\lambda'\right) + \frac{\partial W}{\partial q}\left(\frac{d\delta\lambda'}{ds_0} + r\delta\lambda' - p\delta\kappa'\right)\\ &+ \frac{\partial W}{\partial r}\left(\frac{d\delta\kappa'}{ds_0} + p\delta\lambda' - q\delta\lambda'\right)\Bigg]ds_0.\end{aligned}$$

By integrating the six terms which explicitly contain a derivative with respect to s_0 by parts, we obtain

$$\begin{aligned}\delta\int_{A_0}^{B_0} W ds_0 = &\left[\frac{\partial W}{\partial \xi}\delta' x + \frac{\partial W}{\partial \eta}\delta' y + \frac{\partial W}{\partial \zeta}\delta' z + \frac{\partial W}{\partial p}\delta\lambda' + \frac{\partial W}{\partial q}\delta\lambda' + \frac{\partial W}{\partial r}\delta\kappa'\right]_{A_0}^{B_0}\\ &\int_{A_0}^{B_0}\Bigg[\left(\frac{d}{ds_0}\frac{\partial W}{\partial \xi} + q\frac{\partial W}{\partial \zeta} - r\frac{\partial W}{\partial \eta}\right)\delta' x + \left(\frac{d}{ds_0}\frac{\partial W}{\partial \eta} + r\frac{\partial W}{\partial \xi} - p\frac{\partial W}{\partial \zeta}\right)\delta' y\\ &+ \left(\frac{d}{ds_0}\frac{\partial W}{\partial \zeta} + p\frac{\partial W}{\partial \eta} - q\frac{\partial W}{\partial \xi}\right)\delta' z + \left(\frac{d}{ds_0}\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} + \eta\frac{\partial W}{\partial \zeta} - \zeta\frac{\partial W}{\partial \eta}\right)\delta\lambda'\\ &+ \left(\frac{d}{ds_0}\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial p} - p\frac{\partial W}{\partial r} + \zeta\frac{\partial W}{\partial \xi} - \xi\frac{\partial W}{\partial \zeta}\right)\delta\lambda'\\ &+ \left(\frac{d}{ds_0}\frac{\partial W}{\partial r} + p\frac{\partial W}{\partial q} - q\frac{\partial W}{\partial p} + \xi\frac{\partial W}{\partial \eta} - \eta\frac{\partial W}{\partial \xi}\right)\delta\kappa'\Bigg]ds_0.\end{aligned}$$

Let us set:

$$\left\{\begin{aligned}&F' = \frac{\partial W}{\partial \xi},\quad G' = \frac{\partial W}{\partial \eta},\quad H' = \frac{\partial W}{\partial \zeta},\quad I' = \frac{\partial W}{\partial p},\quad J' = \frac{\partial W}{\partial q},\quad K' = \frac{\partial W}{\partial r},\\ &X'_0 = \frac{d}{ds_0}\frac{\partial W}{\partial \xi} + q\frac{\partial W}{\partial \zeta} - r\frac{\partial W}{\partial \eta},\end{aligned}\right. \qquad (10)$$

$$
\left\{
\begin{aligned}
Y'_0 &= \frac{d}{ds_0}\frac{\partial W}{\partial \eta} + r\frac{\partial W}{\partial \xi} - p\frac{\partial W}{\partial \zeta},\\
Z'_0 &= \frac{d}{ds_0}\frac{\partial W}{\partial \zeta} + p\frac{\partial W}{\partial \eta} - q\frac{\partial W}{\partial \xi},\\
L'_0 &= \frac{d}{ds_0}\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} + \eta\frac{\partial W}{\partial \zeta} - \zeta\frac{\partial W}{\partial \eta},\\
M'_0 &= \frac{d}{ds_0}\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial p} - p\frac{\partial W}{\partial r} + \zeta\frac{\partial W}{\partial \xi} - \xi\frac{\partial W}{\partial \zeta},\\
N'_0 &= \frac{d}{ds_0}\frac{\partial W}{\partial r} + p\frac{\partial W}{\partial q} - q\frac{\partial W}{\partial p} + \xi\frac{\partial W}{\partial \eta} - \eta\frac{\partial W}{\partial \xi}.
\end{aligned}
\right. \tag{10}
$$

We will have: /12

$$
\delta\int_{A_0}^{B_0} W ds_0 = [F'\delta'x + G'\delta'y + H'\delta'z + I'\delta\iota' + J'\delta\lambda' + K'\delta\kappa']_{A_0}^{B_0}
$$
$$
- \int_{A_0}^{B_0} (X'_0\delta'x + Y'_0\delta'y + Z'_0\delta'z + L'_0\delta\iota' + M'_0\delta\lambda' + N'_0\delta\kappa')\, ds_0.
$$

Considering the integral which appears in the expression for $\delta \int_{A_0}^{B_0} W ds_0$, we will call the segments emanating from M, whose projections on the axes Mx', My', Mz' are X'_0, Y'_0, Z'_0 and L'_0, M'_0, N'_0, respectively, *the external force and external moment at the point M, referred to a unit of length of non-deformed line.* If we then consider the completely integrated part of $\delta \int_{A_0}^{B_0} W ds_0$ we will call the segments emanating from B and whose projections on the axes Mx', My', Mz' are the values $-F'_{B_0}$, $-G'_{B_0}$, $-H'_{B_0}$ and $-I'_{B_0}$, $-J'_{B_0}$, $-K'_{B_0}$, which the expressions -F', -G', -H', and -I', -J', -K' take on at the point B_0, the *external stress or external moment of deformation at the point B.* We will call the analogous segments formed with the values F'_{A_0}, G'_{A_0}, H'_{A_0}, and I'_{A_0}, J'_{A_0}, K'_{A_0}, which the expressions F', G', H' and I', J', K' take on at the point A_0, *the external stress and external moment of deformation at the point A.*

The points A and B are not treated in the same way here, conforming to the convention which makes a distinction between them and which was made in the way one measures the arc s_0.

Let us assume that the deformed line AB is cut at the point M and that the two parts AM and MB are separated in one's mind. We may regard the two segments (-F', -G', -H') and (-I', -J', -K') determined for the point M as the external stress and external moments of deformation of the part AM at the point M, and the two segments (F', G', H') and (I', J', K') determined for the same point M as the external stress and external moment of the part MB at the point M. It would be the same as though, instead of considering AM and MB, we consider two parts of the deformable line belonging to AM and MB, respectively, and which have an extremity at M. Due to these remarks, we will say that -F', -G',

-H' and -I', -J, -K' are the components *of the stress and moment of deformation at the point M exerted on AM and on any portion of AM ending at M* on the axes Mx', My', Mz'. We will also say that F', G', H' and I', J', K' are the components *of the stress and moment of deformation at the point M exerted on MB and on any part of MB which ends at M* on the axes Mx', My', Mz'.

It should be noted that if we replace the triad Mx'y'z' by a triad which is invariably related to it, we are led to conclusions which are identical to those which we have indicated previously(3).

10. Relationships Between the Elements Defined in the Preceding Section; Various Transformations of these Relationships. /13 The various elements introduced in the preceding section are related by the following relationships, which immediately result by comparing the formulas used to define them:

$$\begin{cases} \dfrac{dF'}{ds_0} + qH' - rG' - X'_0 = 0, & \dfrac{dI'}{ds_0} + qK' - rJ' + \eta H' - \zeta G' - L'_0 = 0 \\ \dfrac{dG'}{ds_0} + rF' - pH' - Y'_0 = 0, & \dfrac{dJ'}{ds_0} + rI' - pK' + \zeta F' - \xi H' - M'_0 = 0 \\ \dfrac{dH'}{ds_0} + pG' - qF' - Z'_0 = 0, & \dfrac{dK'}{ds_0} + pJ' - qI' + \xi G' - \eta F' - N'_0 = 0 \end{cases} \tag{11}$$

We may pose the problem of transforming these relationships which we have written down, *independently of the values calculated from W and the quantities which it contains*. These relationships in effect hold between the segments attached to the point M and which we have named. Instead of defining these segments by their projections on Mx', My', Mz', we can also define them by their projections on other axes. These latter projections will be related by relationships that are obtained by transforming the preceding ones.

The transformed relationships can immediately be obtained, if we note that the original formulas have an immediate and simple interpretation which is found by assigning to the moving axes such axes which are parallel and pass through the point O.

1. Let us first consider axes Ox, Oy, Oz. Let us call X_0, Y_0, Z_0 and L_0, M_0, N_0 the projections of the external force and moment on these axes at an arbitrary point M of the deformed line. F, G, H and I, J, K are called the projections of the stress and moment of deformation whose projections on the axes Mx', My', Mz' were F', G', H', and I', J', K'. The transformation of the preceding relationships is obviously given by:

$$\begin{aligned} &\frac{dF}{ds_0} - X_0 = 0, & &\frac{dI}{ds_0} + H\frac{dy}{ds_0} - G\frac{dz}{ds_0} - L_0 = 0, \\ &\frac{dG}{ds_0} - Y_0 = 0, & &\frac{dJ}{ds_0} + F\frac{dz}{ds_0} - H\frac{dx}{ds_0} - M_0 = 0, \\ &\frac{dH}{ds_0} - Z_0 = 0, & &\frac{dK}{ds_0} + G\frac{dx}{ds_0} - F\frac{dy}{ds_0} - N_0 = 0. \end{aligned}$$

(3) *Note sur la Dynamique du Point et du Corps Invariable (Note on the Dynamics of a Point and an Invariable Body)*, in Treatise of Physics of Chwolson, Volume I, pages 260 and 269.

We may consider the force X'_0, Y'_0, Z'_0 and the moment L'_0, M'_0, N'_0, or, if we wish, the force X_0, Y_0, Z_0 and the moment L_0, M_0, N_0 as distributed in a continuous way along the line, and this force and moment will here be referred to a unit of length of the non-deformed line. In order to obtain the force and moment referred to a unit of length of the deformed line, it is sufficient to multiply X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0 or X_0, Y_0, Z_0, L_0, M_0, N_0 by $\frac{ds_0}{ds}$. ds is the linear element of the deformed line which corresponds to the linear element ds_0 of the non-deformed line. Let us introduce the projections X, Y, Z, L, M, N of the external force and moment on the fixed axes Ox, Oy, Oz referred to a unit of length of the deformed line. We thus obtain the relationships /14

$$\begin{cases} \frac{dF}{ds} - X = 0, & \frac{dI}{ds} + H\frac{dy}{ds} - G\frac{dz}{ds} - L = 0, \\ \frac{dG}{ds} - Y = 0, & \frac{dJ}{ds} + F\frac{dz}{ds} - H\frac{dx}{ds} - M = 0, \\ \frac{dH}{ds} - Z = 0, & \frac{dK}{ds} + G\frac{dx}{ds} - F\frac{dy}{ds} - N = 0, \end{cases} \tag{12}$$

which are identical to those considered by several authors and in particular by Lord Kelvin and Tait(4). However, these latter authors obtain them by applying what in classical mechanics is called the principle of solidification, and by departing *a priori* from the concept of forces and couples which are then expressed *a posteriori* as a function of deformations using hypotheses. In these hypotheses only infinitely small deformations have been considered up to the present, whereas in the present work we will consider the more general case.

2. The equations referring to the fixed axes Ox, Oy, Oz can be given in a new form. We may express the nine cosines α, α', ... γ'' by means of three auxiliary quantities. Let λ_1, λ_2, λ_3 be these three auxiliary quantities. Let us set:

$$\sum \gamma d\beta = -\sum \beta d\gamma = \varpi'_1 d\lambda_1 + \varpi'_2 d\lambda_2 + \varpi'_3 d\lambda_3,$$
$$\sum \alpha d\gamma = -\sum \gamma d\alpha = \chi'_1 d\lambda_1 + \chi'_2 d\lambda_2 + \chi'_3 d\lambda_3,$$
$$\sum \beta d\alpha = -\sum \alpha d\beta = \sigma'_1 d\lambda_1 + \sigma'_2 d\lambda_2 + \sigma'_3 d\lambda_3,$$

The functions ϖ'_i, χ'_i, σ'_i of λ_1, λ_2, λ_3 defined in this way satisfy the relationships

$$\frac{\partial \varpi'_j}{\partial \lambda_i} - \frac{\partial \varpi'_i}{\partial \lambda_j} + \chi'_i \sigma'_j - \chi'_j \sigma'_i = 0,$$
$$\frac{\partial \chi'_j}{\partial \lambda_i} - \frac{\partial \chi'_i}{\partial \lambda_j} + \sigma'_i \varpi'_j - \sigma'_j \varpi'_i = 0, \quad (i, j = 1, 2, 3)$$
$$\frac{\partial \sigma'_j}{\partial \lambda_i} - \frac{\partial \sigma'_i}{\partial \lambda_j} + \varpi'_i \chi'_j - \varpi'_j \chi'_i = 0,$$

(4) Lord Kelvin and Tait. *Natural Philosophy*, Part II, Section 614.

and we have:

$$p = \varpi_1' \frac{d\lambda_1}{ds_0} + \varpi_2' \frac{d\lambda_2}{ds_0} + \varpi_3' \frac{d\lambda_3}{ds_0},$$
$$q = \chi_1' \frac{d\lambda_1}{ds_0} + \chi_2' \frac{d\lambda_2}{ds_0} + \chi_3' \frac{d\lambda_3}{ds_0},$$
$$r = \sigma_1' \frac{d\lambda_1}{ds_0} + \sigma_2' \frac{d\lambda_2}{ds_0} + \sigma_3' \frac{d\lambda_3}{ds_0},$$

Let us call $\bar{\omega}_i$, χ_i, σ_i the projections of the segment on the fixed axes Ox, Oy, Oz which has the projections $\bar{\omega}'_i$, χ'_i, σ'_i on the axes Mx', My', Mz'. /15
We will have

$$\alpha d\alpha' + \beta' d\beta'' + \gamma' d\gamma'' = -(\alpha'' d\alpha' + \beta'' d\beta' + \gamma'' d\gamma') = \sum \varpi_i d\lambda_i,$$
$$\alpha' d\alpha + \beta'' d\beta + \gamma'' d\gamma = -(\alpha d\alpha'' + \beta d\beta'' + \gamma d\gamma'') = \sum \chi_i d\lambda_i,$$
$$\alpha'' d\alpha' + \beta d\beta' + \gamma d\gamma' = -(\alpha' d\alpha + \beta' d\beta + \gamma' d\gamma) = \sum \sigma_i d\lambda_i.$$

from which it follows that(5) the new functions $\bar{\omega}_i$, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy the relationships

$$\frac{\partial \varpi_j}{\partial \lambda_i} - \frac{\partial \varpi_i}{\partial \lambda_j} = \chi_i \sigma_j - \chi_j \sigma_i$$
$$\frac{\partial \chi_j}{\partial \lambda_i} - \frac{\partial \chi_i}{\partial \lambda_j} = \sigma_i \varpi_j - \sigma_j \varpi_i \qquad (i, j = 1, 2, 3).$$
$$\frac{\partial \sigma_j}{\partial \lambda_i} - \frac{\partial \sigma_i}{\partial \lambda_j} = \varpi_i \chi_j - \varpi_j \chi_i:$$

We would also like to note that if $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ represent the variations of λ_1, λ_2, λ_3 corresponding to the variations $\delta\alpha$, $\delta\alpha'$,... $\delta\gamma''$ of α, α', ... γ'', we have:

$$\delta_I' = \varpi_1' \delta\lambda_1 + \varpi_2' \delta\lambda_2 + \varpi_3' \delta\lambda_3.$$
$$\delta_J' = \chi_1' \delta\lambda_1 + \chi_2' \delta\lambda_2 + \chi_3' \delta\lambda_3.$$
$$\delta_K' = \sigma_1' \delta\lambda_1 + \sigma_2' \delta\lambda_2 + \sigma_3' \delta\lambda_3.$$
$$\delta_I = \alpha \delta_I' + \beta \delta_J' + \gamma \delta_K' = \varpi_1 \delta\lambda_1 + \varpi_2 \delta\lambda_2 + \varpi_3 \delta\lambda_3.$$
$$\delta_J = \alpha' \delta_I' + \beta' \delta_J' + \gamma' \delta_K' = \chi_1 \delta\lambda_1 + \chi_2 \delta\lambda_2 + \chi_3 \delta\lambda_3$$
$$\delta_K = \alpha'' \delta_I' + \beta'' \delta_J' + \gamma'' \delta_K' = \sigma_1 \delta\lambda_1 + \sigma_2 \delta\lambda_2 + \sigma_3 \delta\lambda_3$$

where δ_I, δ_J, δ_K are the projections on fixed axes of the segment whose projections on Mx', My', Mz' are δ_I', δ_J', δ_K'. These relationships will be of

---

(5) These formulas can be used for the direct definition of the functions $\bar{\omega}_i$, χ_i, σ_i and can be substituted into

$$\varpi_i = \alpha m_i' + \beta \chi_i' + \gamma \sigma_i'.$$
$$\chi_i = \alpha' m_i' + \beta' \chi_i' + \gamma' \sigma_i'. \qquad (i = 1, 2, 3).$$
$$\sigma_i = \alpha'' \varpi_i' + \beta'' \chi_i' + \gamma'' \sigma_i'.$$

value later on.

Let us now set

$$
\begin{aligned}
\mathfrak{I} &= \varpi_1' I' + \chi_1' J' + \sigma_1' K' = \varpi_1 I + \chi_1 J + \sigma_1 K \\
\mathfrak{J} &= \varpi_2' I' + \chi_2' J' + \sigma_2' K' = \varpi_2 I + \chi_2 J + \sigma_2 K \\
\mathfrak{K} &= \varpi_3' I' + \chi_3' J' + \sigma_3' K' = \varpi_3 I + \chi_3 J + \sigma_3 K \\
\mathfrak{L}_0 &= \varpi_1' L_0' + \chi_1' M_0' + \sigma_3 N_0' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0 \\
\mathfrak{M}_0 &= \varpi_2' L_0' + \chi_2' M_0' + \sigma_2' N_0' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0 \\
\mathfrak{N}_0 &= \varpi_3' L_0' + \chi_3' M_0' + \sigma_3' N_0' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0,
\end{aligned}
$$

and we thus obtain the equation: /16

$$
\begin{aligned}
\frac{d\mathfrak{I}}{ds_0} - I'\left(\frac{d\varpi_1'}{ds_0} + q\sigma_1' - r\chi_1'\right) - J'\left(\frac{d\chi_1'}{ds_0} + r\varpi_1' - p\sigma_1'\right) - K'\left(\frac{d\sigma_1'}{ds_0} + p\chi_1' - q\varpi_1'\right) \\
+ F'(\chi_1'\zeta - \sigma_1'\eta) + G'(\sigma_1'\xi - \varpi_1'\zeta) + H'(\varpi_1'\eta - \chi_1'\xi) - \mathfrak{L}_0 = 0.
\end{aligned}
$$

and two analogous equations. If we note that the functions ξ, η, ζ, p, q, r of λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}$, $\frac{d\lambda_2}{ds_0}$, $\frac{d\lambda_3}{ds_0}$ result in equations

$$
\begin{aligned}
\frac{\partial\xi}{\partial\lambda_i} + \chi_i'\zeta - \sigma_i'\eta &= 0, & \frac{\partial p}{\partial\lambda_i} &= \frac{d\varpi_i'}{ds_0} + q\sigma_i' - r\chi_i' \\
\frac{\partial\eta}{\partial\lambda_i} + \sigma_i'\xi - \varpi_i'\zeta &= 0, & \frac{\partial q}{\partial\lambda_i} &= \frac{d\chi_i'}{ds_0} + r\varpi_i' - p\sigma_i' \\
\frac{\partial\zeta}{\partial\lambda_i} + \varpi_i'\eta - \chi_i'\xi &= 0, & \frac{\partial r}{\partial\lambda_i} &= \frac{d\sigma_i'}{ds_0} + p\chi_i' - q\varpi_i'.
\end{aligned}
$$

which result from the defining relationships of the functions ϖ_i, χ'_i, σ'_i and the nine identities which they satisfy, we can pose the preceding equation in the following new form:

$$
\frac{d\mathfrak{I}}{ds_0} - F'\frac{\partial\xi}{\partial\lambda_1} - G'\frac{\partial\eta}{\partial\lambda_1} - H'\frac{\partial\zeta}{\partial\lambda_1} - I'\frac{\partial p}{\partial\lambda_1} - J'\frac{\partial q}{\partial\lambda_1} - K'\frac{\partial r}{\partial\lambda_1} - \mathfrak{L}_0 = 0.
$$

and two analogous equations.

By setting

$$
\begin{aligned}
\mathfrak{I}' &= \varpi_1'(I' + y'H' - z'G') + \chi_1'(J' + z'F' - x'H') + \sigma_1'(K' + x'G' - y'F'), \\
\mathfrak{L}_0' &= \varpi_1'(L_0' + y'Z_0' - z'Y_0') + \chi_1'(M_0' + z'X_0' - x'Z_0') + \sigma_1'(N_0' + x'Y_0' - y'X_0'),
\end{aligned}
$$

with analogous equations for $\mathfrak{J}'$, $\mathfrak{K}'$, $\mathfrak{M}_0'$, $\mathfrak{N}_0'$, we find the following form of the equation

$$
\begin{aligned}
\frac{d\mathfrak{I}'}{ds_0} - (I' + y'H' - z'G')\frac{\partial p}{\partial\lambda_1} - (J' + z'F' - x'H')\frac{\partial q}{\partial\lambda_1} \\
- (K' + x'G' - y'F')\frac{\partial r}{\partial\lambda_1} - \mathfrak{L}_0' = 0.
\end{aligned}
$$

and two analogous equations.

We will soon apply the transformations we have just derived. For the moment, we will be satisfied to remark that *the expressions* δ_I', δ_J', δ_K' and δ_I, δ_J, δ_K *are not exact differentials.*

3. Instead of referring the elements corresponding to the point M to the fixed axes Oxyz, let us consider a trirectangular triad $Mx'_1y'_1z'_1$ which moves with M and whose axis Mx'_1 is constrained to be directed along the tangent to the curve (M) in the direction of increasing arc, in order to define these elements. In order to define this triad $Mx'_1y'_1z'_1$, let us relate it to the triad $Mx'y'z'$. Let l, l', l'' be the direction cosines of Mx'_1 with respect to the axes of the latter. Let m, m', m'' be those of My'_1. Let n, n', n'' be those of Mz'_1. The cosines l, l', l'' will be defined by the equations:

$$l = \xi \frac{ds_0}{ds}, \quad l' = \eta \frac{ds_0}{ds}, \quad l'' = \zeta \frac{ds_0}{ds}.$$

i.e., by the following: /17

$$l = \frac{\xi}{\varepsilon}, \quad l' = \frac{\eta}{\varepsilon}, \quad l'' = \frac{\zeta}{\varepsilon},$$

By setting

$$\varepsilon = \sqrt{\xi^2 + \eta^2 + \zeta^2}.$$

we assume that the triad $Mx'_1y'_1z'_1$ has the same orientation as the others. We will not make the special hypothesis concerning the other cosines. They will simply be subjected to the following conditions:

$$m\xi + m'\eta + m''\zeta = 0,$$
$$n\xi + n'\eta + n''\zeta = 0.$$

according to their definition.

Let us assume that s_0 varies and that, for an instant, we let it play the role of time. In addition, let us relate the triad $Mx'_1y'_1z'_1$ to the fixed triad Oxyz and let us call p_1, q_1, r_1 the respective projections of the instantaneous rotation of the triad $Mx'_1y'_1z'_1$ on the axes Mx'_1, My'_1, Mz'_1, such that we will obtain three equations of the following type:

$$p_1 = lp + l'q + l''r + \sum n \frac{dm}{ds_0},$$

assuming the same orientation for the triads.

Finally, let us designate by X'_1, Y'_1, Z'_1 and by L'_1, M'_1, N'_1 the projections of the external force and moment on Mx'_1, My'_1, Mz'_1 at an arbitrary point M of the deformed line, referred to a unit of length of the non-deformed line. Let F'_1, G'_1, H'_1 and I'_1, J'_1, K'_1 be the projections of the deformation stress and moment. The transformed equations of the preceding section are obviously given by

$$\begin{cases} \frac{dF'_1}{ds_0} + q_1H'_1 - r_1G'_1 - X'_1 = 0, & \frac{dI'_1}{ds_0} + q_1K'_1 - r_1J'_1 - L'_1 = 0, \\ \frac{dG'_1}{ds_0} + r_1F'_1 - p_1H'_1 - Y'_1 = 0, & \frac{dJ'_1}{ds_0} + r_1I'_1 - p_1K'_1 - \varepsilon H'_1 - M'_1 = 0, \\ \frac{dH'_1}{ds_0} + p_1G'_1 - q_1F'_1 - Z'_1 = 0, & \frac{dK'_1}{ds_0} + p_1J'_1 - q_1I'_1 + \varepsilon G'_1 - N'_1 = 0. \end{cases} \quad (13)$$

In the theory of strength of materials, F_1' is called *tensile stress*. The components G_1', H_1' are *intersection stresses* in the plane normal to the deformed line. In the same way, the component I_1' of the deformation moment is a *torsion moment*. The components J_1', K_1' are called *flexure moments*.

If, in the fourth equation (13), we have $L'_1 = 0$ and $q_1 = 0$, we find

$$\frac{dI_1'}{ds_0} - r_1 J_1' = 0.$$

This results in the proposition that, for the case where $L'_1 = 0$, $M'_1 = 0$, $N'_1 = 0$, $q_1 = 0$, when $J'_1 = 0$ we have $I'_1 = \text{const}$. This was established by Poisson[6].

11. External Virtual Work. Theorem of Varignon. Remarks Concerning the Auxiliary Quantities Introduced in the Preceding Section. We will give the name *external virtual work* on the deformed line AB during an arbitrary virtual deformation to the following expression: /18

$$\delta\mathfrak{T}_e = -\left[F'\delta'x + G'\delta'y + H'\delta'z + I'\delta\iota' + J'\delta\jmath' + K'\delta\kappa'\right]_{A_0}^{B_0} + \int_{A_0}^{B_0} (X_0'\delta'x + Y_0'\delta'y + Z_0'\delta'z + L_0'\delta\iota' + M_0'\delta\jmath' + N_0'\delta\kappa')\, ds_0.$$

According to the preceding section, we can give this expression the following form, by setting $\mathcal{L} = \mathcal{L}_0 \frac{ds_0}{ds}$, ..., $\mathcal{L}' = \mathcal{L}_0' \frac{ds_0}{ds}$, ...,

$$\delta\mathfrak{T}_e = -\left[F\delta x + G\delta y + H\delta z + I\delta\iota + J\delta\jmath + K\delta\kappa\right]_A^B + \int_A^B (X\delta x + Y\delta y + Z\delta z + L\delta\iota + M\delta\jmath + N\delta\kappa)\, ds,$$

$$\delta\mathfrak{T}_e = -\left[F\delta x + G\delta y + H\delta z + \mathfrak{I}\delta\lambda_1 + \mathfrak{J}\delta\lambda_2 + \mathfrak{K}\delta\lambda_3\right]_A^B + \int_A^B (X\delta x + Y\delta y + Z\delta z + \mathfrak{L}\delta\lambda_1 + \mathfrak{M}\delta\lambda_2 + \mathfrak{N}\delta\lambda_3)\, ds,$$

$$\delta\mathfrak{T}_e = -\left[F'\delta x' + G'\delta y' + H'\delta z' + \mathfrak{I}'\delta\lambda_1 + \mathfrak{J}'\delta\lambda_2 + \mathfrak{K}'\delta\lambda_3\right]_A^B + \int_A^B (X'\delta x' + Y'\delta y' + Z'\delta z' + \mathfrak{L}'\delta\lambda_1 + \mathfrak{M}'\delta\lambda_2 + \mathfrak{N}'\delta\lambda_3)\, ds.$$

Further on we will make use of the two latter expressions. From the two first ones, we will derive a fundamental law of statics. The idea for it, but not the exact form, is due to Varignon. We have already encountered it in the interpretation of the relationships which relate external forces to momenta in dynamics, given by Saint-Guilhem. Let us represent the stress and the moment

(6) Poisson. *Sur les Lignes Élastiques à Double Courbure (Concerning Elastic Lines having Double Curvature)*. Correspondance sur l'Ecole Polytechnique, V. III, No. 3, p. 355-360, January 1816. The theorem of Poisson is independent of the equations which defined the stress and the moment of deformation in terms of W. Poisson established it by writing down equilibrium equations for one part of the line, using the principle of solidification. Bertrand gave a proof in a note contained in *Analytical Mechanics* by Lagrange, which we will speak about later.

of deformation at a point M of the line M by the resultant and the resulting moment of a system of vectors referred to the point M. Let $P\nu$, $P\sigma$ be the general resultant and the moment resultant relative to a point P of space. In the same way, let us represent the external force and moment at a point M, referred to the unit of length of (M), by the resultant and a resulting moment of a system of vectors related to the point M. Let PN and PS be the resultant and the resulting moment relative to a point P of space. The following theorem then holds:

If the arc s plays the part of the time, the velocities of the geometric points ν and σ *are equal and parallel to the segments* PN *and* PS, *respectively.*

This theorem is obviously the translation of the equations (12), which can be written as follows: /19

$$\begin{cases} \dfrac{dF}{ds} - X = 0, & \dfrac{d}{ds}(I + Hy - Gz) - (L + Zy - Yz) = 0, \\ \dfrac{dG}{ds} - Y = 0, & \dfrac{d}{ds}(J + Fz - Hx) - (M + Xz - Zx) = 0, \\ \dfrac{dH}{ds} - Z = 0, & \dfrac{d}{ds}(K + Gx - Fy) - (N + Yx - Xy) = 0. \end{cases} \tag{12'}$$

We can also arrive at this result in the following way. Let us start with

$$\int_A^M \delta W ds_0 = -\delta \mathfrak{T}_c,$$

where $\delta\mathfrak{T}_c$ is taken between A and M. Since δW must be identically zero, due to the invariance of W for the group of Euclidian displacements, when the expressions $\delta' x, \ldots, \delta_I' \ldots$ are given by equations (9'), or, which is the same thing, when δx, δy, δz are given by equations (9) and when $\delta_I = \omega_1 \delta t$, $\delta_J = \omega_2 \delta t$, $\delta_K = \omega_3 \delta t$, and when this holds true no matter what the values of the constants a_1, a_2, a_3, ω_1, ω_2, ω_3, are, we conclude that we have:

$$[F]_A^M - \int_A^M X ds = 0, \quad [G]_A^M - \int_A^M Y ds = 0, \quad [H]_A^M - \int_A^M Z ds = 0,$$

$$[I + yH - zG]_A^M - \int_A^M (L + yZ - zY)\, ds = 0,$$

and two analogous equations. *In these relationships, we can look upon M as a variable* and they are also equivalent to relationships (12'). It should be noted that these equations can be easily derived from ones that are usually written down using the principle of solidification. We will return to this further on in connection with the reasoning of Poisson and continued by Bertrand regarding the deformable line considered by Binet.

In addition to the expressions F', G', H', I', J', K' introduced here, we have considered other expressions which could be calculated. On the other hand, in this calculation it is possible to explicitly write down the functions which one wishes to obtain due to the nature of the problem, which, for example are x, y, z or x', y', z' and the three parameters λ_1, λ_2, λ_3 which

are used to define α, α', ... γ''(7).

If we introduce x, y, z and λ_1, λ_2, λ_3, we have:

$$F = \frac{\partial W}{\partial \frac{dx}{ds_0}}, \quad G = \frac{\partial W}{\partial \frac{dy}{ds_0}}, \quad H = \frac{\partial W}{\partial \frac{dz}{ds_0}},$$

$$\mathfrak{I} = \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}}, \quad \mathfrak{J} = \frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}}, \quad \mathfrak{K} = \frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}.$$

If we introduce x', y', z' and three parameters λ_1; λ_2, λ_3, we have /20

$$F' = \frac{\partial W}{\partial \frac{dx'}{ds_0}}, \quad G' = \frac{\partial W}{\partial \frac{dy'}{ds_0}}, \quad H' = \frac{\partial W}{\partial \frac{dz'}{ds_0}},$$

$$\mathfrak{I}' = \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}}, \quad \mathfrak{J}' = \frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}}, \quad \mathfrak{K}' = \frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}.$$

12. Concept of the Energy of Deformation. Let us consider the two states (M_0) or A_0B_0 and (M) or AB of the deformable line, and let us consider an arbitrary sequence of states beginning with (M_0) and which ends in (M). For this purpose it is sufficient to consider the functions x, y, z; α, α', ... γ'' of s_0 and of one variable h which, for the value zero of h, reduces to x_0, y_0, z_0; α'_0, α'_0, ... γ''_0, respectively. For the value h of h, they are reduced to the values x, y, z; α, α', ... γ'', respectively, relative to (M).

By letting the parameter h vary from o to h in a continuous way, we will obtain a continuous deformation which allows us to pass from the state A_0B_0 to the state AB. Let us consider *the total work* during this continuous deformation which is performed by the external forces and moments applied to the different elements of the line and also performed by the stresses and moments of deformation applied to the extremities of the line. In order to obtain this total work, it is sufficient to integrate the differential obtained, from 0 to h by departing from one of the expressions for $\delta\mathcal{T}_e$ of the preceding section and by substituting the partial differentials corresponding to the increase dh of h in the variations of x, y, z; α, α', ... γ''. The equation

$$\delta\mathcal{T}_e = -\int_{s_0}^{s_{0_1}} \delta W ds_0$$

results in the expression $\int_{A_0}^{B_0} \frac{\partial W}{\partial h} dh ds_0$ for the present value of $\delta\mathcal{T}_e$. We obtain the following for the total work:

(7) As the auxiliary quantities λ_1, λ_2, λ_3 we can, for example, choose the components of rotation which make the fixed axes Ox, Oy, Oz become parallel to Mx', My', Mz', respectively.

$$-\int_0^h \left(\int_{s_0}^{s_0} \frac{\partial W}{\partial h} ds_0\right) dh =$$

$$-\int_{s_0}^{s_0} [W(s_0, \xi, \eta, \zeta, p, q, r) - W(s_0, \xi_0, \eta_0, \zeta_0, p_0, q_0, r_0)] ds_0.$$

The work considered is independent of the intermediary states and only depends on the extreme states (M_0) and (M).

This leads to the introduction of the concept of *deformation energy* which must be distinguished from the action considered previously. We will say that -W is *density of the energy of deformation* referred to the unit of length of the non-deformed line.

13. Natural State of the Deformable Line. General Remarks Regarding the Problems Encountered when Considering this Line. In the preceding we started with a *natural* state of the deformable line, and we defined a state called the /21 *deformed state*. We indicated the equations which allow one to calculate the external force and the analogous elements when the function W representing the action of deformation at a point is introduced for the deformable line.

Let us briefly consider the concept of the *natural state*. In the preceding, the natural state is a state which has not experienced a deformation. We should understand that the functions x, y, z, ... which determine the deformed state depend on one parameter and that, for a particular value of this parameter, we will obtain the natural state. The latter appears to us to be a particular case of the deformed state, and we may attempt to apply the concepts of the deformed state to the natural state.

Without changing the values of the parameters defined by equations (10), we can replace the function W by this function increased by an arbitrary *known* function of s_0. If we recall the idea of the *action*, which we associate with the transformation from a natural state (M_0) to the deformed state (M), we may assume, if we want, that the *function of* s_0 defined by the expression

$$W(s_0, \xi_0, \eta_0, \zeta_0, p_0, q_0, r_0)$$

is identically zero. The values obtained for the external force and for the analogous parameters, regarding the natural state, will not necessarily be zero. We will say that they define the external force and the analogous parameters for the natural state(8).

In the preceding we have said that the natural state manifests itself as the initial state of a sequence of deformed states, as a state from which we

(8) We may speak of external force and moment, stress and moment of deformation because we regard the natural as the limit of a series of states, for which we know what the external force and moment and the stress and moment of deformation are. The external force and moment and the stress and moment of deformation have not been defined by us up to the present, except where there is a deformation which makes it possible for them to become noticeable and measureable.

may depart to study the deformation. One is then led to ask oneself whether it is not possible to have one of the deformed states play the role which we have assigned to the natural state, in such a way that the parameters defined in Section 9 (external force, external stress, ...), calculated for the other deformed states, would have the same value, if we relate the first of these parameters to the unit of length of the deformed line. This question can only be answered if we introduce and precisely formulate the concept of action corresponding to the transition from a deformed state to another deformed state.

The simplest hypothesis consists of postulating that this latter action is obtained by subtracting the action corresponding to the transition from the natural state (M_0) to the first deformed state $(M_{(0)})$ from the action corresponding to the transition from the natural state (M_0) to the second deformed state (M). If we use $s_{(0)}$ to designate the arc of $(M_{(0)})$ and $\xi_{(0)}$, $\eta_{(0)}$, $\zeta_{(0)}$, $p_{(0)}$, $q_{(0)}$, $r_{(0)}$ the analogous quantities for ξ, η, ζ, p, q, r, one is therefore led to adopt the following expression for the action of deformation, relative to the transition from the state $(M_{(0)})$ to the state (M) /22

$$\int_{A_0}^{B_0}\left[W(s_0, \xi, \eta, \zeta, p, q, r) - W(s_0, \xi_{(0)}, \eta_{(0)}, \zeta_{(0)}, p_{(0)}, q_{(0)}, r_{(0)})\right] ds_0. \tag{14}$$

Let us introduce $s_{(0)}$ as a variable in the place of s_0, and let us define $\xi^{(0)}$, $\eta^{(0)}$, $\zeta^{(0)}$, $p^{(0)}$, $q^{(0)}$, $r^{(0)}$ as the values of ξ, η, ζ, p, q, r when $s_{(0)}$ plays the same role as s_0. We will obtain relationships of the following type:

$$\xi = \xi^{(0)} \frac{ds_{(0)}}{ds_0},$$

and, by calling $A_{(0)}$, $B_{(0)}$ the points of $(M_{(0)})$ corresponding to the points A_0, B_0 of (M_0), the expression (14) becomes:

$$\int_{A_{(0)}}^{B_{(0)}} W_0^{(0)}(s_{(0)}, \xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)})\, ds_{(0)}, \tag{15}$$

$W_0^{(0)}(s_{(0)}, \xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)})$ designates the expression

$$\left[W\left(s_0, \xi^{(0)} \frac{ds_{(0)}}{ds_0}, \eta^{(0)} \frac{ds_{(0)}}{ds_0}, \ldots, r^{(0)} \frac{ds_{(0)}}{ds_0}\right) - W(s_0, \xi_{(0)}, \eta_{(0)}, \ldots, r_{(0)})\right] \frac{ds_0}{ds_{(0)}},$$

in which s_0, $\frac{ds_0}{ds_{(0)}}$, $\frac{ds_{(0)}}{ds_0}$ are replaced as a function of $s_{(0)}$.

According to the remark made at the beginning of this section, if one wishes one may substitute the following for expression (15):

$$\int_{A_{(0)}}^{B_{(0)}} W^{(0)}(s_{(0)}, \xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)})\, ds_{(0)} \tag{15'}$$

where the function $W^{(0)}(s_{(0)}, \xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)})$ is the expression

$$W\left(s_0, \xi^{(0)} \frac{ds_{(0)}}{ds_0}, \ldots, r^{(0)} \frac{ds_{(0)}}{ds_0}\right),$$

in which s_0, $\frac{ds_{(0)}}{ds_0}$, $\frac{ds_{(0)}}{ds_0}$ are replaced as a function of $s_{(0)}$.

It is immediately seen that the application of the equations of Section 9 to expression (15) or to (15') results in -- starting from $(M_{(0)})$ as the natural state -- the same values of external force and moment with respect to the state (M) referred to the unit of length of (M), as well as the same values of the stress and moment of deformation.

In this way, we can look upon (M) as a deformed state where $(M_{(0)})$ would be the natural state, provided that the function W associated with the state (M) will now be[9] $W_{(0)}$ and $W_0^{(0)}$.

Let us now make certain general remarks regarding the problems which can be encountered by considering the deformable line. /23

In the preceding, as we have already pointed out, we gave equations which determined the external force and analogous parameters assuming the functions x, y, z, ... of s_0, which define the deformed state are known.

We may now consider the problem of defining, completely or in part, the functions x, y, z, ... when a certain number of elements such as the external force and moment as well as certain additional data concerning the unknowns are known.

Let us say that if we specify x, y, z, ... and if we calculate, as an example, X'_0, Y'_0, Z'_0, we will obtain specific functions of s_0 after all calculations have been carried out. However, in view of the equations which define x, y, z, ... as a function of s_0, one can obviously express X'_0, Y'_0, Z'_0 by means of s_0, of x, y, z, ... and their derivatives up to any order one desires. If we consider the problem in which, for example, X, Y'_0, Z'_0 are among the given quantities, we may then assume that these expressions are given as a function of s_0. However, we can just as well assume that they contain x, y, z, ... and its derivatives with respect to s_0.

Let us now consider a problem in which the knowns include the projections of the external force and moment either along the axes Ox, Oy, Oz or on the axes Mx', My', Mz'. For the purposes of argument, let us assume that these projections are given functions of s_0, of x, y, z, α, α', ... γ'' and their derivatives of the first and second order. In addition, let us assume that the external force and moment are expressed with respect to the unit length of (M_0) and that x_0, y_0, z_0 are given functions of s_0. It is clear that under

(9) This allows one, as we stated at the beginning of this section, to generalize the concept of the natural state that we have introduced. In order to make this word simply correspond to the idea of a particular state, we can, in a more general way, make it correspond to the idea of an arbitrary state from which we will study the deformation.

these conditions, the equations of Section 9, which were used to define X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0 become six differential equations between the unknowns x, y, z, λ_1, λ_2, λ_3, where these three latter quantities are three auxiliary quantities, with which the nine cosines α, α', ... γ'' are expressed. These differential equations do not contain derivatives higher than the second order using the hypotheses that have been made above regarding the external force and moment.

If the problem posed is deterministic or if it at least does not contain an indeterminacy as large as the one which results from the differential equations just discussed, it is necessary to take into account complementary knowns in order to complete the investigation of the unknowns. These complementary knowns can be limiting conditions, i.e., conditions satisfied by the unknowns at the extreme points A_0 and B_0. For example, we could specify, at A_0 and B_0, the values of a certain number of the expressions x, y, z, λ_1, λ_2, λ_3 and expressions such as F'_0, G'_0, H'_0, I'_0, J'_0, K'_0 which specify the stress and moment of deformation, or even functions, linear in most cases, of x, y, z, λ_1, λ_2, λ_3 and of F'_0, G'_0, H'_0, I'_0, J'_0, K'_0.

Using particular examples and particular hypotheses, we have shown how differential equations and complementary conditions can correspond to various problems. However, it is possible to vary the question. /24

If the arc s is explicitly contained in the knowns, we will consider s as an additional variable and will add the relationship

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

Often we are chiefly interested in the deformed line (M), where the line (M_0) has a less important role in a certain sense. If we assume that the expression for W as a function of s_0, ξ, η, ζ, p, q, r is given and does not require specifying (M_0) in order to determine it explicitly, the function W will in the final analysis be a function of s_0, its first derivatives with respect to x, y, z, of λ_1, λ_2, λ_3 and the first derivatives of λ_1, λ_2, λ_3. If the external force and moment are also given explicitly by means of s_0, of x, y, z, λ_1, λ_2, λ_3 and their derivatives, it is obvious that the problem can be considered to consist, on the one hand, of the determination of the state (M) using a variable related to this state, s for example or one of the letters x, y, z. On the other hand, it consists of the determination of the relationship between s_0 and s. Using the hypotheses which we have just made, s_0 can appear explicitly and, among other things, (10) will appear through its

(10) If we specify the external force and moment relative to the unit of length of (M), and, more generally, if we specify these elements as a function of s_0, s, x, ... and the first derivatives with respect to one of these letters.

differential $\frac{ds}{ds_0}$ or, if one wishes, through the expression $\frac{ds_0}{ds}$, or through the inverse expression $\frac{ds}{ds_0}$. Let us note that the concept of the quotient $\frac{ds}{ds_0}$, which specifies the derivative of s with respect to s_0, corresponds to the linear dilatation experienced by the line element ds_0 emenating from the point M_0 of (M_0) and which has become the element ds emenating from the point M of (M) corresponding to the point M_0. We will return to the discussion of this dilatation which Lamé in particular introduced in the discussion of the particular deformable line that he studied(11).

We will develop another kind of problem later on, when we will attempt to relate the deformable line defined up to the present, i.e., the *free line*(12), which can carry out all possible deformations, to the very special lines considered by geometricians. The latter have also been concerned with this subject before and considered the study of these lines as the study of particular deformations of the free line.

14. Normal Form of the Equations of the Deformable Line, when the External Force and Moment are Given as Simple Functions of s_0 and of the Elements which Specify the Position of the Triad Mx'y'z'. Principle of Minimum Work of Castigliano. Let us assume that in accordance with the assumptions of the preceding section, the external force and moment are given in terms of simple functions of s_0 and the elements which fix the position of the triad Mx'y'z'. /25

Let us also assume that the natural state is given. We may look upon the equations of Section 9 as differential equations with respect to the unknowns x, y, z and containing three parameters λ_1, λ_2, λ_3 which are used to specify α, α', ... γ'', or in terms of the unknowns x', y', z' and containing three parameters λ_1, λ_2, λ_3, which corresponds to a change of variables. These two points of view are the ones which come about naturally. In the first case, the expressions ξ, η, ζ, p, q, r are functions of $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}$, $\frac{d\lambda_2}{ds_0}$, $\frac{d\lambda_3}{ds_0}$, which can be computed using equations (1) and (2). In the second case, these will be functions of x', y', z', $\frac{dx'}{ds_0}$, ... , λ_1, ... $\frac{d\lambda_1}{ds_0}$, ... , which will be calculated using equations (2) and (4).

The first case is the most interesting one due to the analogy which exists between the problem under consideration and the dynamics of a particle, a triad

(11) Lamé. *Lecons sur la Théorie Mathématique de l'Élasticité des Corps Solides (Lessons on the Mathematical Theory of Elasticity of Solid Bodies)*, 2nd Edition, p. 98-99 (8th lesson, Section 41, titled *Dilatation of a Thread)*.

(12) The expression *free* means that the theory is developed based on a function W which is based on elements which result from the consideration of this line alone, and which can be subjected to all possible variations.

and an invariable body. Let us examine it.

1. Let us assume that X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0, or, which amounts to the same thing, X_0, Y_0, Z_0, L_0, M_0, N_0, are given functions of s_0, x, y, z, λ_1, λ_2, λ_3. The expression W is a determined function of s_0, $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}$, $\frac{d\lambda_2}{ds_0}$, $\frac{d\lambda_3}{ds_0}$, after substitution of the values of ξ, η, ζ, p, q, r taken from equations (1) and (2), which we will continue to call W. The equations of the problem can be written as:

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} - X_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} - \frac{\partial W}{\partial \lambda_1} - \mathcal{L}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} - Y_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}} - \frac{\partial W}{\partial \lambda_2} - \mathcal{M}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} - Z_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}} - \frac{\partial W}{\partial \lambda_3} - \mathcal{N}_0 = 0,$$

$\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are functions of s_0 x, y, z, λ_1, λ_2, λ_3 which are obtained from the functions of Section 10.

This result follows immediately either from the equations of the preceding sections or, in a more direct way, due to the fact that the defining equations for X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, F, G, H, $\mathcal{E}$, $\mathcal{J}$, $\mathcal{K}$ can be summarized by the relationship:

$$\delta\int_{s_0}^{s_0} W ds_0 + \delta\mathcal{E}_0 = 0,$$

i.e., by the relationship: /26

$$\delta\int_{s_0}^{s_0} W ds_0 = [F\delta x + G\delta y + H\delta z + \mathcal{J}\delta\lambda_1 + \mathcal{J}\delta\lambda_2 + \mathcal{K}\delta\lambda_3]_{s_0}^{s_0}$$
$$-\int_{s_0}^{s_0}(X_0\delta x + Y_0\delta y + Z_0\delta z + \mathcal{L}_0\delta\lambda_1 + \mathcal{M}_0\delta\lambda_2 + \mathcal{N}_0\delta\lambda_3)ds_0.$$

We may replace the preceding system by a system of equations of the first order, by introducing six auxiliary unknowns. Instead of the first-order derivatives of x, y, z, λ_1, λ_2, λ_3, we will choose the following six expressions previously considered:

$$F = \frac{\partial W}{\partial \frac{dx}{ds_0}}, \quad G = \frac{\partial W}{\partial \frac{dy}{ds_0}}, \quad H = \frac{\partial W}{\partial \frac{dz}{ds_0}},$$
$$\mathcal{J} = \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}}, \quad \mathcal{J} = \frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}}, \quad \mathcal{K} = \frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}.$$

Assuming that the Hessian of W with respect to $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, $\frac{d\lambda_1}{ds_0}$, $\frac{d\lambda_2}{ds_0}$,

$\frac{d\lambda_3}{ds_0}$ is not zero (or by assuming that the Hessian of the function W with respect to ξ, η, ζ, p, q, r is not zero) from these six relationships we may obtain values for the six latter derivatives $\frac{dx}{ds_0}$, ... , $\frac{d\lambda_3}{ds_0}$ as a function of F, G, H, $\mathfrak{J}$, $\mathfrak{J}$, $\mathfrak{K}$. Let us introduce these values into the expression

$$\mathcal{E} = \frac{dx}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} + \frac{dy}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} + \frac{dz}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} + \sum \frac{d\lambda_i}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_i}{ds_0}} - W$$

which is nothing else than the expression:

$$\xi\frac{\partial W}{\partial \xi} + \eta\frac{\partial W}{\partial \eta} + \zeta\frac{\partial W}{\partial \zeta} + p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial r} - W,$$

as a function of s_0, $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, λ_1 ... , $\frac{d\lambda_1}{ds_0}$ ... After substitution, we obtain a function of s_0, λ_1, λ_2, λ_3, F, G, H, $\mathfrak{J}$, $\mathfrak{J}$, $\mathfrak{K}$, , which we will continue to specify by the letter $\mathcal{E}$. Thus, the total differential of this latter function is obviously:

$$\frac{dx}{ds_0}\, d\frac{\partial W}{\partial \frac{dx}{ds_0}} + \dots + \frac{d\lambda_1}{ds_0}\, d\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} + \dots - \frac{\partial W}{\partial s_0}\, ds_0 - \sum \frac{\partial W}{\partial \lambda_i}\, d\lambda_i$$

where

$$\frac{dx}{ds_0}\, dF + \frac{dy}{ds_0}\, dG + \frac{dz}{ds_0}\, dH + \frac{d\lambda_1}{ds_0}\, d\mathfrak{J} + \frac{d\lambda_2}{ds_0}\, d\mathfrak{J} + \frac{d\lambda_3}{ds_0}\, d\mathfrak{K} - \frac{\partial W}{\partial s_0}\, ds_0 - \sum \frac{\partial W}{\partial \lambda_i}\, d\lambda_i,$$

and the following defining system for x, y, z, λ_1, λ_2, λ_3, F, G, H, $\mathfrak{J}$, $\mathfrak{J}$, $\mathfrak{K}$ /27
is obtained:

$$\frac{dx}{ds_0} = \frac{\partial \mathcal{E}}{\partial F},\ \frac{dy}{ds_0} = \frac{\partial \mathcal{E}}{\partial G},\ \frac{dz}{ds_0} = \frac{\partial \mathcal{E}}{\partial H},\ \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathfrak{J}},\ \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathfrak{J}},\ \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathfrak{K}},$$

$$\frac{dF}{ds_0} - X_0 = 0,\ \frac{dG}{ds_0} - Y_0 = 0,\ \frac{dH}{ds_0} - Z_0 = 0,\ \frac{d\mathfrak{J}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_1} - \mathfrak{L}_0 = 0,$$

$$\frac{d\mathfrak{J}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_2} - \mathfrak{M}_0 = 0,\ \frac{d\mathfrak{K}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_3} - \mathfrak{N}_0 = 0.$$

We have assumed that due to the defining equations x, y, z, λ_1, λ_2, λ_3 as a function of s_0, we may express X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ as a function of s_0, x, y, z, λ_1, λ_2, λ_3. This is possible in infinitely many ways, and we may always choose the new forms of X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ in such a way that they are, respectively, the partial derivatives $\frac{\partial \mathfrak{U}}{\partial x}$, $\frac{\partial \mathfrak{U}}{\partial y}$, $\frac{\partial \mathfrak{U}}{\partial z}$, $\frac{\partial \mathfrak{U}}{\partial \lambda_1}$, $\frac{\partial \mathfrak{U}}{\partial \lambda_2}$, $\frac{\partial \mathfrak{U}}{\partial \lambda_3}$ of one and the same function $\mathfrak{U}$ with opposite sign. It may or may not depend on s_0. Let us assume that this is the case, and let us designate by $\mathfrak{V}$ the function of x, y, z, λ_1, λ_2, λ_3 (and possibly of s_0) defined by the equation:

$$\mathfrak{V} = \mathcal{E} + \mathfrak{U};$$

The preceding system takes on the form:

$$\frac{dx}{ds_0}=\frac{\partial\mathfrak{V}}{\partial F},\ \frac{dy}{ds_0}=\frac{\partial\mathfrak{V}}{\partial G},\ \frac{dz}{ds_0}=\frac{\partial\mathfrak{V}}{\partial H},\ \frac{d\lambda_1}{ds_0}=\frac{\partial\mathfrak{V}}{\partial\mathfrak{J}},\ \frac{d\lambda_2}{ds_0}=\frac{\partial\mathfrak{V}}{\partial\mathfrak{J}},\ \frac{d\lambda_3}{ds_0}=\frac{\partial\mathfrak{V}}{\partial\mathfrak{K}},$$

$$\frac{dF}{ds_0}=-\frac{\partial\mathfrak{V}}{\partial x},\ \frac{dG}{ds_0}=-\frac{\partial\mathfrak{V}}{\partial y},\ \frac{dH}{ds_0}=-\frac{\partial\mathfrak{V}}{\partial z},\ \frac{d\mathfrak{J}}{ds_0}=-\frac{\partial\mathfrak{V}}{\partial\lambda_1},$$

$$\frac{d\mathfrak{J}}{ds_0}=-\frac{\partial\mathfrak{V}}{\partial\lambda_2},\ \frac{d\mathfrak{K}}{ds_0}=-\frac{\partial\mathfrak{V}}{\partial\lambda_3}.$$

We have here a set of equations which are presented in the form of the Hamilton equations of dynamics. If in particular we assume that the new forms of X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are selected in such a way that s_0 is not contained in them, which is always possible, and that they are the partial derivatives of a function -- $\mathfrak{U}$ of x, y, z, λ_1, λ_2, λ_3, and if we also assume that $W(s_0, \xi, \eta, \zeta, p, q, r)$ does not depend on[13] s_0, we will obtain a system of canonical equations in this particular case.

Let us now consider the functions x', y', z', and let us assume that the functions α, α' ... γ'' are expressed by means of the three auxiliary quantities λ_1, λ_2, λ_3. Let us assume that X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0 are given functions of s_0, x', y', z', λ_1, λ_2, λ_3. The expression W is a determined function of s_0, x', y', z', $\frac{dx'}{ds_0}$, $\frac{dy'}{ds_0}$, $\frac{dz'}{ds_0}$, λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}$, $\frac{d\lambda_2}{ds_0}$, $\frac{d\lambda_3}{ds_0}$, after substitution of the values of ξ, η, ζ, p, q, r obtained from equations (2) and (4), which we will continue to designate by W. The equations of the problem may then be written as /28

$$\frac{d}{ds_0}\frac{\partial W}{\partial\frac{dx'}{ds_0}}-\frac{\partial W}{\partial x'}-X'_0=0 \qquad \frac{d}{ds_0}\frac{\partial W}{\partial\frac{d\lambda_1}{ds_0}}-\frac{\partial W}{\partial\lambda_1}-\mathfrak{L}'_0=0$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial\frac{dy'}{ds_0}}-\frac{\partial W}{\partial y'}-Y'_0=0 \qquad \frac{d}{ds_0}\frac{\partial W}{\partial\frac{d\lambda_2}{ds_0}}-\frac{\partial W}{\partial\lambda_2}-\mathfrak{M}'_0=0$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial\frac{dz'}{ds_0}}-\frac{\partial W}{\partial z'}-Z'_0=0 \qquad \frac{d}{ds_0}\frac{\partial W}{\partial\frac{d\lambda_3}{ds_0}}-\frac{\partial W}{\partial\lambda_3}-\mathfrak{N}'_0=0$$

$\mathfrak{L}'_0$, $\mathfrak{M}'_0$, $\mathfrak{N}'_0$ are functions of s_0, x', y', z', λ_1, λ_2, λ_3 which result from Section 10.

We may replace the preceding system by a system of equations of the first order, introducing six auxiliary unknowns. Instead of the derivatives of the first order with respect to x', y', z', λ_1, λ_2, λ_3, we will select the auxiliary unknowns as the six expressions already considered before:

$$F'=\frac{\partial W}{\partial\frac{dx'}{ds_0}},\qquad G'=\frac{\partial W}{\partial\frac{dy'}{ds_0}},\qquad H'=\frac{\partial W}{\partial\frac{dz'}{ds_0}}.$$

(13) We may say that in this case the line is *homogeneous* by definition, in order to express this hypothesis.

$$\mathfrak{I}' = \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}}, \quad \mathfrak{J}' = \frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}}, \quad \mathfrak{K}' = \frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}.$$

Using these six relationships, and assuming that the Hessian of W with respect to $\frac{dx'}{ds_0}$, $\frac{dy'}{ds_0}$, $\frac{dz'}{ds_0}$, $\frac{d\lambda_1}{ds_0}$, $\frac{d\lambda_2}{ds_0}$, $\frac{d\lambda_3}{ds_0}$ is not zero, we may obtain values for the six latter derivatives as a function of F', G' H', $\mathfrak{I}'$, $\mathfrak{J}'$, $\mathfrak{K}'$. Let us introduce these values into the expression:

$$\mathcal{E}' = \frac{dx'}{ds_0}\frac{\partial W}{\partial \frac{dx'}{ds_0}} + \frac{dy'}{ds_0}\frac{\partial W}{\partial \frac{dy'}{ds_0}} + \frac{dz'}{ds_0}\frac{\partial W}{\partial \frac{dz'}{ds_0}} + \sum \frac{d\lambda_i}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_i}{ds_0}} - W.$$

After substitution we obtain a function of s_0, x', y', z', λ_1, λ_2, λ_3, F', G', H', $\mathfrak{I}'$, $\mathfrak{J}'$, $\mathfrak{K}'$ which we will continue to designate by the letter $\mathcal{E}'$. Thus, the total differential of this latter function is obviously

$$\frac{dx'}{ds_0}dF' + \frac{dy'}{ds_0}dG' + \frac{dz'}{ds_0}dH' + \frac{d\lambda_1}{ds_0}d\mathfrak{I}' + \frac{d\lambda_2}{ds_0}d\mathfrak{J}' + \frac{d\lambda_3}{ds_0}d\mathfrak{K}' - \frac{\partial W}{\partial s_0}ds_0$$
$$- \frac{\partial W}{\partial x'}dx' - \frac{\partial W}{\partial y'}dy' - \frac{\partial W}{\partial z'}dz' - \sum \frac{\partial W}{\partial \lambda_i}d\lambda_i.$$

We thus obtain the following defining system for x', y', z', λ_1, λ_2, λ_3, F', G', H', $\mathfrak{I}'$, $\mathfrak{J}'$, $\mathfrak{K}'$

$$\frac{dx'}{ds_0} = \frac{\partial \mathcal{E}'}{\partial F'}, \quad \frac{dy'}{ds_0} = \frac{\partial \mathcal{E}'}{\partial G'}, \quad \frac{dz'}{ds_0} = \frac{\partial \mathcal{E}'}{\partial H'}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{E}'}{\partial \mathfrak{I}'}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{E}'}{\partial \mathfrak{J}'}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{E}'}{\partial \mathfrak{K}'},$$
$$\frac{dF'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial x'} - X'_0 = 0, \quad \frac{dG'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial y'} - Y'_0 = 0, \quad \frac{dH'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial z'} - Z'_0 = 0,$$
$$\frac{d\mathfrak{I}'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial \lambda_1} - \mathfrak{L}'_0 = 0, \quad \frac{d\mathfrak{J}'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial \lambda_2} - \mathfrak{M}'_0 = 0, \quad \frac{d\mathfrak{K}'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial \lambda_3} - \mathfrak{N}'_0 = 0.$$

We have assumed that, due to the defining equations for x', y', z', λ_1, λ_2, λ_3 as a function of s_0, X'_0, Y'_0, Z'_0, $\mathfrak{L}'_0$, $\mathfrak{M}'_0$, $\mathfrak{N}'_0$ can be expressed as a function of s_0, x', y', z', λ_1, λ_2, λ_3. This is possible in infinitely many ways and we can always choose the new forms of X'_0, Y'_0, Z'_0, $\mathfrak{L}'_0$, $\mathfrak{M}'_0$, $\mathfrak{N}'_0$ in such a way that they are, respectively, the partial derivatives $\frac{\partial \mathfrak{U}'}{\partial x'}$, $\frac{\partial \mathfrak{U}'}{\partial y'}$, $\frac{\partial \mathfrak{U}'}{\partial z'}$, $\frac{\partial \mathfrak{U}'}{\partial \lambda_1}$, $\frac{\partial \mathfrak{U}'}{\partial \lambda_2}$, $\frac{\partial \mathfrak{U}'}{\partial \lambda_3}$ of one and the same function $\mathfrak{U}'$ with the sign reversed, which may or may not be independent of s_0. Let us assume that this is the case and let us designate by $\mathfrak{V}'$ the function of x', y', z, λ_1, λ_2, λ_3 (and possibly of s_0) defined by the equation: /29

$$\mathfrak{V}' = \mathcal{E}' + \mathfrak{U}'.$$

The preceding system takes on the form:

$$\frac{dx'}{ds_0} = \frac{\partial \mathfrak{V}'}{\partial F'}, \quad \frac{dy'}{ds_0} = \frac{\partial \mathfrak{V}'}{\partial G'}, \quad \frac{dz'}{ds_0} = \frac{\partial \mathfrak{V}'}{\partial H'}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathfrak{V}'}{\partial \mathfrak{I}'}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathfrak{V}'}{\partial \mathfrak{J}'}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathfrak{V}'}{\partial \mathfrak{K}'},$$
$$\frac{dF'}{ds_0} = -\frac{\partial \mathfrak{V}'}{\partial x'}, \quad \frac{dG'}{ds_0} = -\frac{\partial \mathfrak{V}'}{\partial y'}, \quad \frac{dH'}{ds_0} = -\frac{\partial \mathfrak{V}'}{\partial z'}, \quad \frac{d\mathfrak{I}'}{ds_0} = -\frac{\partial \mathfrak{V}'}{\partial \lambda_1}.$$

$$\frac{d\mathfrak{Y}'}{ds_0} = -\frac{\partial \mathfrak{V}'}{\partial \lambda_3}, \quad \frac{d\mathfrak{Z}'}{ds_0} = -\frac{\partial \mathfrak{V}''}{\partial \lambda_3}$$

The equation

$$\delta \int W ds_0 + \delta \mathfrak{C}_0 = 0$$

in the case where the external forces and moments are zero corresponds to the *principle of minimum work of Castigliano*(14), already discussed by Véne, Cournot, Menabrea and others. —

Let us consider equations having the normal form:

$$\frac{dx}{ds_0} = \frac{\partial \mathcal{E}}{\partial F}, \ldots\ldots, \frac{dF}{ds_0} - X_0 = 0, \ldots\ldots$$

By integrating from A_0 to B_0, we obtain:

$$x_{B_0} - x_{A_0} = \int_{A_0}^{B_0} \frac{\partial \mathcal{E}}{\partial F} ds_0, \ldots, F_{B_0} - F_{A_0} = \int_{A_0}^{B_0} X_0 ds_0, \ldots$$

If we assume, for example, X_0, Y_0, Z_0 to be zero, we have F = const. = $= F_{B_0} = F_{A_0}$, G = const., H = const. in the three equations such that:

$$x_{B_0} - x_{A_0} = \int_{A_0}^{B_0} \frac{\partial \mathcal{E}}{\partial F} ds_0,$$

F, G, H are independent of s_0, and we may write /30

$$x_{B_0} - x_{A_0} = \frac{\partial}{\partial F} \int_{A_0}^{B_0} \mathcal{E} ds_0.$$

If $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are zero and if $\frac{\partial \mathcal{E}}{\partial \lambda_1} = \frac{\partial \mathcal{E}}{\partial \lambda_2} = \frac{\partial \mathcal{E}}{\partial \lambda_3} = 0$, we obtain analogous theorems with respect to λ_1, λ_2, λ_3. We are thus led in a very direct and natural way to what is called the theorems of Castigliano in the strength of materials. In general, one considers the simple case of infinitely small deformations. W is a quadratic form, as well as $\mathcal{E}$ which can be derived from W as an adjoint form.

15. Concept of the Concealed Triad and Concealed W. In the study of the deformable line, it is natural to give particular attention to the curve described by this line. This amounts to considering x, y, z separately and to consider α, α', ... γ'' as simple auxiliary variables. We may also express this fact by stating that we will ignore the existence of the triads which determine the deformable line and that we only know the apexes of these triads. If we take this point of view, for the purposes of formulating the appropriate differential equations, we can introduce the concept of the *concealed triad*, and we are led to a classification depending on the various ways the elimination

(14) Castigliano. *Théorie de l'Équilibre des Systèmes Élastiques et ses Applications (Theory of the Equilibrium of Elastic Systems and Applications)*, Turin , 1879. See also Müller-Breslau, *Die Neueren Methoden der Festigkeitslehre (Newer Methods in Strength of Materials)*, 3rd Edition, Leipzig, 1904.

of α, α', ... γ'' can be carried out.

We may first carry out a study of the reductions which can come about due to the elimination of α, α', ... γ''. In the corresponding particular cases, where our attention is almost exclusively focused on the curve traced by the deformed line (M), it is sometimes possible to even make an abstraction of (M_0) and, correspondingly, of the deformation which allows one to transfer from (M_0) to (M). Using this latter point of view, we encounter the line which is called flexible and inextensible in rational mechanics.

The triad may also be considered in another way. We may make particular hypotheses for it, as well as for the curve (M). This is the same as considering particular deformations of the free deformable line. If the relationships we will stipulate are simple relationships between ξ, η, ζ, p, q, r, as will be the case in the applications which we will study, we may take into account these relationships in the calculation of W and may derive more special functions from W. The interesting question which will develop will be the one of introducing these particular forms in a simple way and of considering the general function W (used as a point of departure) as concealed, in a certain sense. We will thus obtain a *theory corresponding to special deformations specified by given relationships between* ξ, η, ζ, p, q, r.

In this way we can see that, using the theory of the free deformable line, for particular cases we may formulate equations which are all derived from the same origin and which are the result of special problems which are encountered in the usual development of rational mechanics and in the classical theory of elasticity.

In these theories, one is often placed in the position that the consideration of the deformations is avoided. In reality, it is necessary to complete these theories. We will carry this out in the discussion of the practical applications by considering infinitesimally small deformations. /31

Let us treat the case where the external force and moment at the most contain only the first derivatives of the unknowns x, y, z and λ_1, λ_2, λ_3. The second derivatives of these unknowns will not be introduced in the differential equations except through W. The derivatives of x, y, z do not appear in ξ, η, ζ and those of λ_1, λ_2, λ_3 do not appear in p, q, r. We therefore see that if W depends only on ξ, η, ζ or only on p, q, r, a reduction in the orders of the derivatives contained in the system of differential equations will occur. Consequently there will also be a reduction in the system obtained due to elimination of p, q, r. We will begin by considering these first two cases.

16. Case Where W Only Depends on s_0, ξ, η, ζ. The Method of Finding the Equations for the Flexible and Extensible Line of Lagrange. Let us assume that W only depends on s_0, ξ, η, ζ. The equations of Section 14 are then reduced to the following

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} - X_0 = 0, \qquad \frac{\partial W}{\partial \lambda_i} + \mathfrak{L}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} - Y_0 = 0, \qquad \frac{\partial W}{\partial \lambda_2} + \mathfrak{M}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} - Z_0 = 0, \qquad \frac{\partial W}{\partial \lambda_3} + \mathfrak{N}_0 = 0,$$

in which W only depends on s_0, $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, λ_1, λ_2, λ_3. This shows us that, if we assume the simple case where X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are given functions(15) of s_0, x, y, z, $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, λ_1, λ_2, λ_3, the three equations on the right can be solved with respect to λ_1, λ_2, λ_3 and one obtains three differential equations which only contain s_0, x, y, z and their first and second derivatives.

Let us now consider the particular case where the given functions $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are zero. This will also hold true for the corresponding values of the functions of any one of the systems (L', M', N'), (L_0, M_0, N_0), (L, M, N). We then obtain the equations:

$$\frac{\partial W}{\partial \lambda_1} = 0, \qquad \frac{\partial W}{\partial \lambda_2} = 0, \qquad \frac{\partial W}{\partial \lambda_3} = 0,$$

which agree with /32

$$\frac{F}{\frac{dx}{ds}} = \frac{G}{\frac{dy}{ds}} = \frac{H}{\frac{dz}{ds}},$$

and, by setting the common value of these ratios as -T, the equations, which are supplemented by those containing λ_1, λ_2, λ_3 can be written as

$$\frac{d}{ds_0}\left(T\frac{dx}{ds}\right) + X_0 = 0, \quad \frac{d}{ds_0}\left(T\frac{dy}{ds}\right) + Y_0 = 0, \quad \frac{d}{ds_0}\left(T\frac{dz}{ds}\right) + Z_0 = 0.$$

or, if one desires,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + X = 0, \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \quad \frac{d}{ds}\left(T\frac{dz}{ds}\right) + Z = 0.$$

In this case stress is reduced to a *tensile stress* T.

If we assume this to be the case, we should note that if we start from two positions (M_0) and (M), which are assumed *given*, and if we derive the functions $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$, as was done in Sections 9 and 10, it can happen -- in the case where these three functions are zero -- that the result is obtained by accident, i.e., for a certain ensemble of particular deformations. However, this can be the case no matter what the deformation (M) may be, that is, it is a consequence of the nature of (M), i.e., the form of W.

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(15) We assume that X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ do not contain the derivatives of λ_1, λ_2, λ_3, in order to simplify the derivation and in order to more conveniently express what we wish to say.

Let us consider the latter case which is of particular interest. W is then a simple function of s_0 and of $\xi^2 + \eta^2 + \zeta^2$ or, which amounts to the same thing[16], of s_0 and of $\frac{ds_0}{ds}$. The equations $\frac{\partial W}{\partial \lambda_i} = 0$ (i = 1, 2, 3) are reduced to identities[17], and we simply obtain the equations:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + X = 0, \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \quad \frac{d}{ds}\left(T\frac{dz}{ds}\right) + Z = 0,$$

where we have:

$$T = -\frac{\partial W}{\partial \mu},$$

if we assume that W is expressed in terms of s_0 and of $\mu = \frac{ds_0}{ds} = 1$ (μ is the linear dilatation at the point M).

Let us assume that the function W is known and we specify X, Y, Z or X_0, Y_0, Z_0, as a function of s_0, s, x, y, z and the derivatives of four of these latter quantities[18] with respect to one of them. The preceding equations and the equation

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

result in four differential equations which define four of the variables s_0, s, /33
x, y, z in terms of the fifth one.

If s is not contained explicitly in them, we may eliminate ds using the relationship we have just written down, and three differential equations which define the three unknowns x, y, z as a function of s_0 will remain.

Let us now consider the special case in which W only depends on μ and where s_0 is not contained explicitly. We then obtain the equations proposed by Lagrange[19] for studying the line which he qualifies as follows: "flexible thread which is extensible and contractible at the same time". We should note that the explanations given by Lagrange in the second of his works which he devoted to the present question (No. 43), must be modified as follows: If we consider W as a given function of μ, this also holds true for T (which corresponds to the assertion by Lagrange stipulating that [using his

(16) We may also say that W is a function of s_0 and the linear dilatation $\mu = \frac{ds}{ds} - 1$ at the point M, which was considered by Lamé in his *Lessons on the Mathematical Theory of Elasticity of Solid Bodies*, page 98, 99 of the 2nd Edition.

(17) The *triad* is completely concealed; we may therefore also assume that we are dealing with a *point-like* line.

(18) We may assume that derivatives of order higher than the first have been introduced.

notation] F is a given function of $\frac{ds}{d\sigma}$). We may substitute the unknown T for the unknown μ, since the knowledge of one as a function of s implies the knowledge of the other. Finally, we are led to the analysis of four functions T, x, y, z of s using the four preceding equations (and additional conditions, if the problem is deterministic). As Lagrange probably assumed, it should be noted that the given expressions of X, Y, Z do not contain s explicitly, and we may confine ourselves to the consideration of the three first equations and the three variables x, y, z, where the differential of s is eliminated using the fourth equation.

In the first of the works to which we have referred (No. 43), Lagrange remarks that he is led to the same equations as he obtained for the thread he considered previously in his investigation, and which he called the flexible and inextensible thread. In No. 44 he discusses tension. It seems to us that there is a certain confusion regarding the concept of force in the treatise by Lagrange (a confusion which was explicitly described by J. Bertrand from the point of view of dynamics alone, in the note which he wrote in regard to No. 44). It is clear that Lagrange took the point of view of dynamics and that the word *equilibrium* is equivalent to the word *rest* in his treatise. By introducing "the force F due to which each element ds of the curve of the thread tends to become contracted" at the beginning of No. 44, Lagrange introduces a force concept which does not conform with the definition he gives at the beginning of his treatise (page 1), and which is not a kinetic force but a force which we must qualify as a *static force*, measured from the deformations.

17. The Flexible and Inextensible Thread. How can we introduce and define the concept of the *flexible inextensible thread* and remain within that /34
domain of statics where forces are measured from deformations? In order to give the definition of a flexible and inextensible thread, it is sufficient for us to follow the course usually taken, *but in the reverse order*, i.e., by being guided by what one calls the *principle of solidification*(20).

(19) Lagrange. *Mécanique Analytique (Analytical Mechanics)*, 1st Part, Section V, Section 11, No. 42-43, 4th Edition, p. 156-158. The same question was treated again by Lamé in his *Lecons sur la Théorie Mathématique de l'Élasticité des Corps Solides (Lessons on the Mathematical Theory of Elasticity of Solid Bodies)*, 2nd Edition, 8th Lesson, and then by M. Duhem, in Volume II of his work *Hydrodynamique, Elasticité, Acoustique (Hydrodynamics, Elasticity, Acoustics)*, p. 1 on. The treatment by Lamé, as Todhunter and Pearson noted, on page 235 of Volume 1 of their *History* etc., is a duplication of the one given by Poisson on pages 422 on in his *Mémoire sur le Mouvement des Corps Élastiques (Memoir on the Motion of Elastic Bodies)*, which appeared in 1829 in Volume VIII of his *Mémoires de l'Institut de France (Memoirs of the Institute of France)*.

(20) Appell. 1st Edition, Volume 1, No. 132, p. 165; in the 2nd Edition, Volume I, No. 120, p. 161, the expression *principle of solidification* is omitted; this is also true in Thomson and Tait, *Treatise on Natural Philosophy*, Volume 1, Part II, Section 564, p. 110.

In a general way let us now consider the deformable line of Section 5, its natural state (M_0) and its deformed state (M). Let us now assume that we will impose conditions on the deformations of this line, which is defined as in Section 5, i.e., by a *correspondence* between the points (M_0) and those of a deformation (M), and let the condition[21] state that an arbitrary part of (M) has the same length as the *corresponding* part. This is the same as stating that x, y, z is subjected to the condition

$$ds = ds_0,$$

assuming that ds and ds_0 have the same sign, as we assumed previously. Assume that we wish to define the elements for such a line: external force, etc. We will consider a deformable line of the type considered up to the present, and, instead of considering an arbitrary deformation (M) of the natural state (M_0), we will focus our attention on the deformations (M) for which we have $ds = ds_0$. These deformations coincide with the given deformations of the inextensible line, both with regard to position of the points as well as the associated triads. We will assume the equations previously adopted for a certain deformable line to define external forces, etc. acting on the inextensible line. We will apply them to the positions of this line, which coincides with those of the given inextensible line.

If, in particular, we consider the flexible and inextensible line, we can derive the definition of the external forces with respect to this line from those of the line considered previously and for which W is a simple function of s and of μ, *considering the deformations of the latter for which the function* μ *is reduced to zero.* If we only retain the letters s, X, Y, Z (since $s = s_0$, $X = X_0$, $Y = Y_0$, $Z = Z_0$), we are led to the system

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) - X = 0, \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \quad \frac{d}{ds}\left(T\frac{dz}{ds}\right) + Z = 0,$$

in which $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$ and where T represents the function of s defined by the equation $T = -\left(\frac{\partial W}{\partial \mu}\right)_{\mu = 0}$.

In order to obtain a deterministic problem, it will be necessary to assume that the function T is known. It will be sufficient to add the proper conditions at the limits.

18. <u>The Case where W only Depends on s_0, ξ, η, ζ and where $\mathcal{L}_0$ $\mathcal{M}_0$ $\mathcal{N}_0$ are not Zero</u>. Let us now consider the general case where $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are not all zero. The equations /35

$$\frac{\partial W}{\partial \lambda_1} + \mathcal{L}_0 = 0, \quad \frac{\partial W}{\partial \lambda_2} + \mathcal{M}_0 = 0, \quad \frac{\partial W}{\partial \lambda_3} + \mathcal{N}_0 = 0$$

(21) What we are about to say can be repeated under different analogous circumstances, where we will connect what we will later on call *internal bonds* with the system we studied at the beginning.

agree with the following relationships, by introducing the auxiliary quantities F, G, H

$$H\frac{dy}{ds} - G\frac{dz}{ds} - L = 0$$
$$F\frac{dz}{ds} - H\frac{dx}{ds} - M = 0$$
$$G\frac{dx}{ds} - F\frac{dy}{ds} - N = 0,$$

in such a way that, in the present case, the stress compent tangent to the line, which one can call *tensile stress*, the component of stress normal to the line, which we can call the *shear stress* (as is used in the strength of materials) and finally the vector (L, M, N) determine an orthogonal triad.

As an auxiliary quantity, let us introduce the tensile stress

$$T = -\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right);$$

We obtain

$$-F = T\frac{dx}{ds} + N\frac{dy}{ds} - M\frac{dz}{ds}$$
$$-G = T\frac{dy}{dz} + L\frac{dz}{ds} - N\frac{dx}{ds}$$
$$-H = T\frac{dz}{ds} + M\frac{dx}{ds} - L\frac{dy}{ds}.$$
$$L\frac{dx}{ds} + M\frac{dy}{ds} + N\frac{dz}{ds} = 0.$$

Thus, if X, Y, Z, L, M, N are given as a function of s, of x, y, z and of their first derivatives, we find three equations of the following type:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + X + \frac{d}{ds}\left(N\frac{dy}{ds} - M\frac{dz}{ds}\right) = 0,$$

to which we must add:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1, \qquad L\frac{dx}{ds} + M\frac{dy}{ds} + N\frac{dz}{ds} = 0,$$

in such a way that, *for the problem under consideration*, we have *five* differential equations containing *four* unknowns, which are x, y, z and the auxiliary unknown T.

19. The Case where W only Depends on s_0, p, q, r. Let us assume that W only depends on s_0, p, q, r. Then the equations of Section 14, which can be /36 reduced to the following:

$$X_0 = 0, \quad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} - \frac{\partial W}{\partial \lambda_1} - \mathfrak{L}_0 = 0,$$

$$Y_0 = 0, \quad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}} - \frac{\partial W}{\partial \lambda_2} - \mathfrak{M}_0 = 0,$$

$$Z_0 = 0, \quad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}} - \frac{\partial W}{\partial \lambda_3} - \mathfrak{N}_0 = 0.$$

in which W only depends on s_0, λ_1, λ_2, λ_3 $\frac{d\lambda_1}{ds_0}$, $\frac{d\lambda_2}{ds_0}$, $\frac{d\lambda_3}{ds_0}$, tells us that if we consider the simple case where X_0, Y_0, Z_0 do not contain the derivatives of x, y, z, we may eliminate x, y, z from the left-hand equations and substitute their values into the right-hand equations, i.e., in $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$. If these three latter equations do not contain the derivatives of x, y, z of order higher than the first, we will find three equations of the second order for determining λ_1, λ_2, λ_3, provided that X_0, Y_0, Z_0 contains only s_0, x, y, z, λ_i, $\frac{d\lambda_i}{ds_0}$, and $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ -- only s_0, x, y, z, $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, λ_i, $\frac{d\lambda_i}{ds_0}$, $\frac{d^2\lambda_i}{ds_0^2}$.

The particular case in which the given functions X_0, Y_0, Z_0 are identically zero is of particular interest. We then simply have the three equations on the right which, if $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ only depend on λ_1, λ_2, λ_3 and their derivatives, constitute three differential equations for determining λ_1, λ_2, λ_3.

20. <u>The Case where W is a Function of s_0, ξ, η, ζ, p, q, r, which Depends on ξ, η, ζ only through the Intermediary $\xi^2 + \eta^2 + \zeta^2$, or, which is the Same Thing, the Intermediary $\mu = \frac{ds}{ds_0} - 1$.</u> Let us consider the stress at a point of the deformed line and let us assume that, for *all the deformations* it is reduced to a tensile stress. This is the same as saying that the function W of s_0, ξ, η, ζ, p, q, r satisfies the identities

$$\frac{\frac{\partial W}{\partial \xi}}{\xi} = \frac{\frac{\partial W}{\partial \eta}}{\eta} = \frac{\frac{\partial W}{\partial \zeta}}{\zeta},$$

i.e., that it only depends on ξ, η, ζ through the intermediary of the quantity $\xi^2 + \eta^2 + \zeta^2$, or, which is the same thing, on $\mu = \frac{ds}{ds_0} - 1$.

We will then have:

$$\frac{F}{\frac{dx}{ds}} = \frac{G}{\frac{dy}{ds}} = \frac{H}{\frac{dz}{ds}},$$

and by introducing the common value -T of these ratios defined by the equation: /37

$$T = -\frac{\partial W}{\partial \mu},$$

we can present the system in the following form:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + X = 0, \quad (1+\mu)\frac{d}{ds}\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} - \frac{\partial W}{\partial \lambda_1} - \mathcal{L}_0 = 0,$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \quad (1+\mu)\frac{d}{ds}\frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}} - \frac{\partial W}{\partial \lambda_2} - \mathcal{M}_0 = 0,$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) + Z = 0, \quad (1+\mu)\frac{d}{ds}\frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}} - \frac{\partial W}{\partial \lambda_3} - \mathcal{N}_0 = 0,$$

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1,$$

by which x, y, z, λ_1, λ_2, λ_3 and s_0 are defined (where μ designates $\frac{1}{\frac{ds_0}{ds}}-1$ as a function of s.

If we consider the case in which X, Y, Z are given functions of only the quantities s, x, y, z, where we wish to consider a specific case, we see that we are able to determine x, y, z and the auxiliary quantity T separately, using the following system of differential equations:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+X=0,\qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+Y=0,\qquad \frac{d}{ds}\left(T\frac{dz}{ds}\right)+Z=0,$$
$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1.$$

We again find the system which we encountered in connection with the flexible and extensible thread of Lagrange as well as for the inextensible flexible thread.

21. The Deformable Line Obtained Assuming that Mx' is the Tangent to M at (M). With regard to the general case and the case of *particular arbitrary deformations*, we may repeat what we have said regarding the flexible extensible thread of Lagrange for the flexible inextensible thread of rational mechanics. Let us now consider the following case, which is important in the strength of materials and which will later on lead us to the deformable line studied, in particular, by Lord Kelvin and Tait, but only from the point of view of infinitesimally small deformations(22), as we have already stated.

Let us consider the deformable line of Section 5 and assume that we have defined the external force, etc., as we did in Section 9. Let us assume that we will only focus our attention on the deformations (M) of (M_0) for which the axis Mx' is tangent to the curve (M) at each point, and let us also assume that these deformations can consist of continuous sequences beginning with (M_0) and that the latter is such that M_0x_0' is the tangent at M_0. By appropriately choosing the way in which s_0 and s are measured, this amounts to assuming that we have: /38

$$\begin{cases} \alpha_0=\frac{dx_0}{ds_0}, & \alpha_0'=\frac{dy_0}{ds_0}, & \alpha_0''=\frac{dz_0}{ds_0}, \\ \alpha=\frac{dx}{ds}, & \alpha'=\frac{dy}{ds}, & \alpha''=\frac{dz}{ds}. \end{cases} \tag{14}$$

or that

$$\eta=\eta_0=0,\qquad \zeta=\zeta_0=0,\qquad \xi_0=1,\qquad \xi=\frac{ds}{ds_0}. \tag{15}$$

(22) W. Thomson and Tait. *Treatise on Natural Philosophy*, Volume 1, Part II, Edition 1883, Section 588 on. Page 130 on.

The application of the definitions will give us deterministic expressions for the external, etc. We may say that the study of these expressions and the problems which may devlop out of them constitutes the object of the line subjected to the conditions defined by equation (14) and (15), by repeating everything that has been said.

Limiting the deformations of (M_0) to those of the deformations (M) which satisfy the conditions (14) or (15), or introducing a new concept for a line which can only undergo deformations satisfying the preceding conditions may be regarded as identical here, with respect to the calculations which can define elements such as the external force, etc. This manner of regarding the problem absolutely conforms with the principle called *solidification*, which is introduced by the authors in the *inverse order* to the one we are following in a certain sense.

Before considering the form assumed by equations of Section 9, let us establish certain equations for the triad Mx'y'z' under the particular conditions which apply here. Let us assume that we take the principal normal Mn and the binormal Mb of the curve (M) at M. If they constitute a triad Mx'nb with respect to Mx' with the same orientation as the triad Mx'y'z', we may call the direction cosines of Mn and of Mb with respect to the axes Mx', My', Mz' by o, cos ω, sin ω and par o, - sin ω, cos ω, respectively, which is the same as stating that we have:

$$\begin{cases} \beta = \beta_1 \cos\omega - \gamma_1 \sin\omega, & \gamma = \beta_1 \sin\omega + \gamma_1 \cos\omega, \\ \beta' = \beta'_1 \cos\omega - \gamma'_1 \sin\omega, & \gamma' = \beta'_1 \sin\omega + \gamma'_1 \cos\omega, \\ \beta'' = \beta''_1 \cos\omega - \gamma''_1 \sin\omega, & \gamma'' = \beta''_1 \sin\omega + \gamma''_1 \cos\omega, \end{cases} \tag{16}$$

β_1, β'_1, β''_1 are the direction cosines of Mn with respect to the fixed axes Ox, Oy, Oz, and γ_1, γ'_1, γ''_1 are those of Mb with respect to the same axes. An auxiliary quantity ω has been introduced, which is the angle between My' and Mn, measured in an appropriate direction.

We may now determine ω using the expressions already introduced. The principal normal is the tangent to the indicatrix of P. Serret. Let us now consider the point whose coordinates are 1, 0, 0 with respect to the triad whose origin O is fixed and whose axes are parallel to those of Mx'y'z'. The projections of the displacement of this point on the axes of the moving triad or on those of Mx'y'z' are given by: /39

$$o, \quad r ds_0, \quad -q ds_0,$$

and we have:

$$\frac{\cos\omega}{r} = -\frac{\sin\omega}{q}.$$

It is possible to obtain more complete equations by replacing the cosines β, β', ... γ'' by their expressions (16) in equations (2) of Section 6. We find

$$p\frac{ds_0}{ds} = \sum \gamma \frac{d\beta}{ds} = \sum \gamma_1 \frac{d\beta_1}{ds} - \frac{d\omega}{ds},$$

$$q\frac{ds_0}{ds} = \sum \alpha \frac{d\gamma}{ds} = \cos\omega \sum \alpha \frac{d\gamma_1}{ds} + \sin\omega \sum \alpha \frac{d\beta_1}{ds},$$

$$r\frac{ds_0}{ds}=\sum\beta\frac{d\alpha}{ds}=\cos\omega\sum\beta_1\frac{d\alpha}{ds}-\sin\omega\sum\gamma_1\frac{d\alpha}{ds},$$

i.e.:

$$\begin{cases} p\dfrac{ds_0}{ds}=\dfrac{1}{\tau}-\dfrac{d\omega}{ds}, \\ q\dfrac{ds_0}{ds}=-\dfrac{\sin\omega}{\rho}, \\ r\dfrac{ds_0}{ds}=\dfrac{\cos\omega}{\rho}, \end{cases} \tag{17}$$

by setting

$$\frac{1}{\rho}=\sum\beta_1\frac{d\alpha}{ds},\qquad \frac{1}{\tau}=\sum\gamma_1\frac{d\beta_1}{ds},$$

and remembering that $\Sigma\ \alpha\ \frac{d\gamma 1}{ds} = 0$. The absolute value of the expressions $\frac{1}{\rho}$ and $\frac{1}{\tau}$ are equal to the curvature and the torsion (term *Cambrure* of Barré de Saint-Venant, *tortuosity* of Thomson and Tait) of the curve (M) at M. The two latter equations (17) correspond to the remarks made by Thomson and Tait(23).

Let us reconsider the equations of Section 9. For an instant let us designate by W_1 the expression that W becomes when we take conditions (15) into account, i.e., let us set:

$$W_1=[W(s_0,\xi,\eta,\zeta,p,q,r)]_{\eta=0,\zeta=0}=W(s_0,\xi,0,0,p,q,r),$$

or, noting that according to equations (14) /40

$$\xi=\frac{ds}{ds_0}=1+\mu,$$

let us set:

$$W_2=W(s_0,1+\mu,0,0,p,q,r).$$

We have

$$F'=\left[\frac{\partial W}{\partial\xi}\right]_{\eta=0,\zeta=0}=\frac{\partial W_1}{\partial\xi}=\frac{\partial W_2}{\partial\mu},\qquad G'=\left[\frac{\partial W}{\partial\eta}\right]_{\eta=0,\zeta=0},$$

$$H'=\left[\frac{\partial W}{\partial\zeta}\right]_{\eta=0,\zeta=0},\qquad I'=\left[\frac{\partial W}{\partial p}\right]_{\eta=0,\zeta=0}=\frac{\partial W_1}{\partial p},$$

$$J'=\left[\frac{\partial W}{\partial q}\right]_{\eta=0,\zeta=0}=\frac{\partial W_1}{\partial q},\qquad K'=\left[\frac{\partial W}{\partial r}\right]_{\eta=0,\zeta=0}=\frac{\partial W_1}{\partial r}.$$

Thus, if we only want to introduce the function W_1, i.e., the value of W when $\eta = \zeta = 0$, and if we assume that the values at the derivatives $\frac{\partial W}{\partial\eta}$, $\frac{\partial W}{\partial\zeta}$ take on for $\eta = \zeta = 0$ are not given, we obtain six expressions of which only four F', I', J', K' may be considered to be given, and two of which G', H' remain to be determined. Expressed differently, the knowledge of W_1 only leads

(23) Thomson and Tait. *Treatise on Natural Philosophy*, Volume I, Part II, Section 590, p. 131.

to a knowledge of the tensile stress F' and deformation moment (I', J', K')[24].

If we introduce the expressions F, G, H, I, J, K, we may say that the first three are three auxiliary quantities of which we simply know that[25]:

$$F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}=\frac{\partial W}{\partial \mu}. \tag{18}$$

and the three latter ones can be calculated using one of the systems:

$$\begin{cases}\alpha I+\alpha' J+\alpha'' K=\dfrac{\partial W}{\partial p},\\ \beta I+\beta' J+\beta'' K=\dfrac{\partial W}{\partial q},\\ \gamma I+\gamma' J+\gamma'' K=\dfrac{\partial W}{\partial r}.\end{cases} \tag{19}$$

$$\begin{cases}I=\alpha\dfrac{\partial W}{\partial p}+\beta\dfrac{\partial W}{\partial q}+\gamma\dfrac{\partial W}{\partial r},\\ J=\alpha'\dfrac{\partial W}{\partial p}+\beta'\dfrac{\partial W}{\partial q}+\gamma'\dfrac{\partial W}{\partial r},\\ K=\alpha''\dfrac{\partial W}{\partial p}+\beta''\dfrac{\partial W}{\partial q}+\gamma''\dfrac{\partial W}{\partial r}.\end{cases} \tag{19'}$$

α, α', α'' ... γ'' are defined by equations (14) and (16).

The external force and moment result depending on the way in which they can be determined if W_1 alone is given, using equations of Sections 9 and 10. /41

Let us now assume that the external force and moment are given. The equations:

$$\begin{cases}\dfrac{dF}{ds}-X=0, & \dfrac{dI}{ds}+H\dfrac{dy}{ds}-G\dfrac{dz}{ds}-L=0,\\ \dfrac{dG}{ds}-Y=0, & \dfrac{dJ}{ds}+F\dfrac{dz}{ds}-H\dfrac{dx}{ds}-M=0,\\ \dfrac{dH}{ds}-Z=0, & \dfrac{dK}{ds}+G\dfrac{dx}{ds}-F\dfrac{dy}{ds}-N=0,\end{cases} \tag{20}$$

combined with equations (18) and (19) and the relationship:

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1, \tag{21}$$

result in a system of eight differential equations for five of these quantities (as a function of the sixth) and of F, G, H, when X, Y, Z, L, M, N or X_0, Y_0, L_0, M_0, N_0 are given functions of s_0, s, x, y, z, ω and the derivatives of these quantities with respect to one of them.

If s does not appear explicitly in the given functions, (21) can be used to eliminate ds and, taking s_0 as an independent variable, for example, we find

(24) Assuming that we only know the function W_1, we can assume that we can ignore the existence of the function W which we have used since the beginning, and that this function is in a certain sense *concealed, as well as the positions of the triad* Mx'y'z', *for which* Mx' *is not tangent to the curve* (M).

(25) From now on we will call W the function W_1 of s_0, μ, p, q, r.

a system of seven differential equations defining the seven unknowns x, y, z, ω and F, G, H.

In the present case, where the function W of the initial point is concealed, the expressions F, G, H, are simply auxiliary quantities defined by the differential equations we have just discussed. We can attempt to eliminate them. This elimination is easy, because they appear linearly and their derivatives do not appear in the expression (18) and in the three expressions (20) on the right. These four relationships result in:

$$\begin{cases} F = -T\frac{dx}{ds} + \left(\frac{dK}{ds} - N\right)\frac{dy}{ds} - \left(\frac{dJ}{ds} - M\right)\frac{dz}{ds}, \\ G = -T\frac{dy}{ds} + \left(\frac{dI}{ds} - L\right)\frac{dz}{ds} - \left(\frac{dK}{ds} - N\right)\frac{dx}{ds}, \\ H = -T\frac{dz}{ds} + \left(\frac{dJ}{ds} - M\right)\frac{dx}{ds} - \left(\frac{dI}{ds} - L\right)\frac{dy}{ds}, \\ \left(\frac{dI}{ds} - L\right)\frac{dx}{ds} + \left(\frac{dJ}{ds} - M\right)\frac{dy}{ds} + \left(\frac{dK}{ds} - N\right)\frac{dz}{ds} = 0, \end{cases} \tag{22}$$

where we have used the following notation for abbreviation purposes

$$T = -\frac{\partial W}{\partial \mu}. \tag{23}$$

The following system of four equations results by elimination of F, G, H: /42

$$\begin{cases} \frac{d}{ds}\left[-T\frac{dx}{ds} + \left(\frac{dK}{ds} - N\right)\frac{dy}{ds} - \left(\frac{dJ}{ds} - M\right)\frac{dz}{ds}\right] - X = 0, \\ \frac{d}{ds}\left[-T\frac{dy}{ds} + \left(\frac{dI}{ds} - L\right)\frac{dz}{ds} - \left(\frac{dK}{ds} - N\right)\frac{dx}{ds}\right] - Y = 0, \\ \frac{d}{ds}\left[-T\frac{dz}{ds} + \left(\frac{dJ}{ds} - M\right)\frac{dx}{ds} - \left(\frac{dI}{ds} - L\right)\frac{dy}{ds}\right] - Z = 0, \end{cases} \tag{24}$$

$$\left(\frac{dI}{ds} - L\right)\frac{dx}{ds} + \left(\frac{dJ}{ds} - M\right)\frac{dy}{ds} + \left(\frac{dK}{ds} - N\right)\frac{dz}{ds} = 0, \tag{25}$$

in which I, J, K, T have been replaced by their values (19') and (23), and which, together with (21) form a system of five differential equations connecting five of the quantities s_0, s, x, y, z, ω to one of them. If s is not contained explicitly in the known quantities, (21) can be used to eliminate ds, and the relationships (24) and (25) result in four differential equations which define x, y, z, ω as a function s_0.

22. Reduction of the System of the Preceding Section to a Form which can be Derived from the Calculus of Variations. As a final result of the preceding section, we found a function W which, among other things, depends on s_0, on ω, $\frac{d\omega}{ds_0}$, $\frac{dx}{ds_0}$, ..., $\frac{d^3x}{ds_0^3}$, by means of the quantities μ, p, q, r.

Let us note that equation (25) can be written by writing down the latter arguments:

$$\frac{d}{ds_0}\left(\frac{\partial W}{\partial \frac{d\omega}{ds_0}}\right) - \frac{\partial W}{\partial \omega} + \left(L_0\frac{dx}{ds} + M_0\frac{dy}{ds} + N_0\frac{dz}{ds}\right) = 0.$$

Let us examine whether, by successively combining each of equations (24) with equation (25), it is possible to obtain three equations which may be directly derived from the calculus of variations, i.e., such expressions as the following one:

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3x}{ds_0^3}} - \frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2x}{ds_0^2}} + \frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} - X_0 + \ldots = 0,$$

where the missing terms only depend on external moments.

We should note that the equations under consideration contain only derivatives which are at most of the fifth order. It can be seen that we must attempt to introduce the third derivatives of equation (25), which can be written in the form:

$$\frac{d}{ds_0}\left(\frac{\partial W}{\partial p}\right) + q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} - \left(L_0\frac{dx}{ds} + M_0\frac{dy}{ds} + N_0\frac{dz}{ds}\right) = 0,$$

or /43

$$V = \frac{d}{ds_0}\left(\frac{\partial W}{\partial p}\right) + q\frac{\partial W}{\partial r} - r\frac{\partial W}{\partial q} - L_0' = 0,$$

using the notation of Section 9.

Let us consider the first equation (24). It can be written as:

$$\frac{d}{ds_0}\left[-Tx + \frac{ds_0}{ds}\left(\frac{dK}{ds_0} - N_0\right)x' - \frac{ds_0}{ds}\left(\frac{dJ}{ds_0} - M_0\right)x''\right] - X_0 = 0,$$

i.e.:

$$U_1 = \frac{d}{ds_0}\left[-Tx + \frac{ds_0}{ds}\left(\gamma\frac{d}{ds_0}\frac{\partial W}{\partial q} - \beta\frac{d}{ds_0}\frac{\partial W}{\partial r} + \frac{\gamma_1}{\rho}\frac{ds}{ds_0}\frac{\partial W}{\partial p} - p\beta\frac{\partial W}{\partial q} - p\gamma\frac{\partial W}{\partial r}\right)\right.$$
$$\left. - \frac{ds_0}{ds}(x'N_0 - x''M_0)\right] - X_0 = 0.$$

If we formulate the first term $\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3x}{ds_0^3}} + \ldots$, by carrying out a calculation which we shall not describe in detail here, we may readily see that the combination:

$$U_1 + \frac{d^2}{ds_0^2}\left\{\frac{\gamma_1\rho}{\left(\frac{ds}{ds_0}\right)^2}V\right\} + \frac{d}{ds_0}\left\{\frac{\gamma_1\rho\frac{d^2s}{ds_0^2}}{\left(\frac{ds}{ds_0}\right)^3}V\right\}$$

results in the various terms of the expression to be formulated, except for terms which vanish with the external forces.

If we thus set:

$$\mathfrak{N}_0 = N_0 + \frac{d^2}{ds_0^2}\left\{\frac{\gamma_1\rho}{\left(\frac{ds}{ds_0}\right)^2}L_0'\right\} + \frac{d}{ds_0}\left\{\frac{\gamma_1\rho\frac{d^2s}{ds_0^2}}{\left(\frac{ds}{ds_0}\right)^3}L_0'\right\} + \frac{d}{ds_0}\left[\frac{ds_0}{ds}(x'N_0 - x''M_0)\right]$$

and if we designate by $\mathcal{Y}_0, \mathcal{Z}_0$ the analogous expressions obtained by replacing X_0, Y_1, respectively by Y_0, Y'_1, and then by Z_0, Y''_1, and by performing the permutations in the last term, we obtain the following form of the system:

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3x}{ds_0^3}} - \frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2x}{ds_0^2}} + \frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} - \mathcal{X}_0 = 0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3y}{ds_0^3}} - \frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2y}{ds_0^2}} + \frac{d}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} - \mathcal{Y}_0 = 0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3z}{ds_0^3}} - \frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2z}{ds_0^2}} + \frac{d}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} - \mathcal{Z}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\omega}{ds_0}} - \frac{\partial W}{\partial \omega} + \left(L_0\frac{dx}{ds} + M_0\frac{dy}{ds} + N_0\frac{dz}{ds}\right) = 0.$$

which can be summarized by the equation: /44

$$\int_{s_0^0}^{s_0'} (\delta W + \mathcal{X}_0\delta x + \mathcal{Y}_0\delta y + \mathcal{Z}_0\delta z - L_0'\delta\omega)\, ds_0 = 0$$

where we have only considered terms which finally appear under the simple integral sign(26).

This summarized form, which we have just obtained and which must be treated using the rules of the calculus of variations, is particularly convenient for carrying out changes of variables.

Assuming that the expressions $\mathcal{X}_0, \mathcal{Y}_0, \mathcal{Z}_0, L_0'$ have a particular form, we will obtain the equations of the extremals of a problem of the calculus of variations.

Let us consider the case in which U designates a function of x, y, z, $\alpha = \frac{1}{\xi}\frac{dx}{ds_0}$, $\alpha' = \frac{1}{\xi}\frac{dy}{ds_0}$, $\alpha'' = \frac{1}{\xi}\frac{dz}{ds_0}$. We have:

$$X_0 = \frac{\partial U}{\partial x}, \qquad Y_0 = \frac{\partial U}{\partial y}, \qquad Z_0 = \frac{\partial U}{\partial z},$$

$$\frac{1}{\xi}\left(M_0\frac{dz}{ds} - N_0\frac{dy}{ds}\right) = \frac{\partial U}{\partial \frac{dx}{ds_0}} = \frac{1}{\xi}\left[\frac{\partial U}{\partial x} - \frac{\alpha}{\xi}\left(\frac{\partial U}{\partial x}\frac{dx}{ds_0} + \frac{\partial U}{\partial x'}\frac{dy}{ds_0} + \frac{\partial U}{\partial x''}\frac{dz}{ds_0}\right)\right],$$

$$\frac{1}{\xi}\left(N_0\frac{dx}{ds} - L_0\frac{dz}{ds}\right) = \frac{\partial U}{\partial \frac{dy}{ds_0}} = \frac{1}{\xi}\left[\frac{\partial U}{\partial x'} - \frac{\alpha'}{\xi}\left(\frac{\partial U}{\partial x}\frac{dx}{ds_0} + \frac{\partial U}{\partial x'}\frac{dy}{ds_0} + \frac{\partial U}{\partial x''}\frac{dz}{ds_0}\right)\right],$$

$$\frac{1}{\xi}\left(L_0\frac{dy}{ds} - M_0\frac{dx}{ds}\right) = \frac{\partial U}{\partial \frac{dz}{ds_0}} = \frac{1}{\xi}\left[\frac{\partial U}{\partial x''} - \frac{\alpha''}{\xi}\left(\frac{\partial U}{\partial x}\frac{dx}{ds_0} + \frac{\partial U}{\partial x'}\frac{dy}{ds_0} + \frac{\partial U}{\partial x''}\frac{dz}{ds_0}\right)\right],$$

(26) One obtains a form analogous to $\int_{t_0}^{t_1} (\delta T + U')\, dt = 0$ which Tisserand gives for the Hamilton principle, p. 4 of Volume 1 of his *Traité de Mécanique Céleste (Treatise of Celestial Mechanics)*.

$$L_0 \frac{dx}{ds_0} + M_0 \frac{dy}{ds_0} + N_0 \frac{dz}{ds_0} = 0$$

or, which is the same thing:

$$X_0 = \frac{\partial U}{\partial x}, \quad Y_0 = \frac{\partial U}{\partial y}, \quad Z_0 = \frac{\partial U}{\partial z},$$

$$L_0 = \left(\alpha' \frac{\partial U}{\partial x''} - \alpha'' \frac{\partial U}{\partial x'}\right), \; M_0 = \left(\alpha'' \frac{\partial U}{\partial x} - \alpha \frac{\partial U}{\partial x''}\right), \; N_0 = \left(\alpha \frac{\partial U}{\partial x'} - \alpha' \frac{\partial U}{\partial x}\right).$$

We thus have:

$$\mathfrak{X}_0 = \frac{\partial U}{\partial x} - \frac{d}{ds_0} \frac{\partial U}{\partial \frac{dx}{ds_0}}, \; \mathfrak{Y}_0 = \frac{\partial U}{\partial y} - \frac{d}{ds_0} \frac{\partial U}{\partial \frac{dy}{ds_0}}, \; \mathfrak{Z}_0 = \frac{\partial U}{\partial z} - \frac{d}{ds_0} \frac{\partial U}{\partial \frac{dz}{ds_0}},$$

and the equations of the extremals with respect to the integral are:

$$\int (W + U)\, ds_0.$$

Another special case which may be combined with the preceding one is the one in which W has the form $Bp + \phi(q^2 + r^2, \xi)$, where B is a constant. W can then be written as: /45

$$Bp + \psi(s_0, \xi, \rho).$$

If, among other things, we assume that $L_0' = 0$, the four equations are reduced to three and the fourth one is reduced to an identity. The case under consideration includes, as a special case, the one where W has the form

$$A\frac{1}{\rho^2} + C,$$

where A and C are constants. This leads us to the case considered by D. Bernoulli and later on by Euler. This is the case which inspired Sophie Germain and Poisson to carry out their investigations of elastic surfaces.

23. Deformable, Inextensible Line where Mx' is the Tangent to (M) at M. Instead of simply assuming, as in the preceding case, that the conditions (14) and (15) are introduced, we must also assume that the line is inextensible, which amounts to stating that

$$\xi = 1.$$

due to equation (14).

Let us assume that we only know the value of the function $W(s_0, \xi, \eta, \zeta, p, q, r)$ for $\xi = 1$, $\eta = 0$, $\zeta = 0$, or that, starting with the line of the preceding section where we have imposed the condition $\mu = 0$, we assume that we only know the value of the function W for $\mu = 0$. It can be seen that F, G, H all become indeterminate. We then have either equations (20), where I, J, K must be replaced by the values (19'), in which W designates $W(s_0, 1, 0, 0, p, q, r)$ or $(W_1)_{\mu = 0}$. Together with equation (21) it represents a system of seven differential equations which define the unknowns x, y, z, ω, F, G, H as a function of $s = s_0$, or equations (24) and (25), where I, J, K must be replaced by the same values (19') and which, together with expression (21), form

a system of five differential equations which define the unknowns x, y, z, ω, T as a function of $s = s_0$.

The system obtained coincides with that introduced by Thomson and Tait[27] if we assume that $W(s_0, 1, 0, 0, p, q, r)$ is obtained by substitution of the values of p_0, q_0, r_0 as a function of s_0 in a quadratic form (with constant coefficients) with respect to the expressions $p - p_0$, $q - q_0$, $r - r_0$. This occurs, for example, if we assume that the expression W_1 introduced at the beginning of the preceding section is obtained by substituting for p_0, q_0, r_0 their values as a function of s_0 in a quadratic form with respect to $p(1 + \mu) - p_0$, $q(1 + \mu) - q_0$, $r(1 + \mu) - r_0$.

Let us note that in the applications of Thomson and Tait of the material in Section 14, for example in the application carried out in Section 616, they consider the case of an infinitesimally small deformation. Thus, we find these applications quite naturally by departing from a generalized function W and by considering the infinitesimally small deformation. /46

We may develop arguments which are analogous to those for the line discussed previously. The only difference is that we must add the condition:

$$\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2 = 1.$$

We then obtain the equation:

$$\int_{s_0^1}^{s_0^2} (\delta W + \mathfrak{X}_0 \delta x + \mathfrak{Y}_0 \delta y + \mathfrak{Z}_0 \delta z - L_0' \delta\omega)\, ds_0 = 0,$$

which must hold *due to:*

$$\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2 = 1.$$

and where $\mathfrak{X}_0$, $\mathfrak{Y}_0$, $\mathfrak{Z}_0$ have a meaning which we will now define.

The system of equilibrium equations is equivalent to the following:

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3x}{ds_0^3}} - \frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2x}{ds_0^2}} + \frac{d}{ds_0}\left[\frac{\partial W}{\partial \frac{dx}{ds_0}} - T\frac{dx}{ds_0}\right] - \mathfrak{X}_0 = 0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3y}{ds_0^3}} - \frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2y}{ds_0^2}} + \frac{d}{ds_0}\left[\frac{\partial W}{\partial \frac{dy}{ds_0}} - T\frac{dy}{ds_0}\right] - \mathfrak{Y}_0 = 0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3z}{ds_0^3}} - \frac{d^2}{ds_0}\frac{\partial W}{\partial \frac{d^2z}{ds_0^2}} + \frac{d}{ds_0}\left[\frac{\partial W}{\partial \frac{dz}{ds_0}} - T\frac{dz}{ds_0}\right] - \mathfrak{Z}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\omega}{ds_0}} - \frac{\partial W}{\partial \omega} + L_0\frac{dx}{ds} + M_0\frac{dy}{ds} + N_0\frac{dz}{ds} = 0.$$

(27) Thomson and Tait. *Treatise on Natural Philosophy*, Volume 1, Part II, Section 614, p. 152-155.

where we must set

$$\mathfrak{X}_0 = X_0 + \frac{d^2}{ds_0^2}(\gamma_1 \rho L_0') + \frac{d}{ds_0}(\alpha' N_0 - \alpha'' M_0),$$
$$\mathfrak{Y}_0 = Y_0 + \frac{d^2}{ds_0^2}(\gamma'_1 \rho L_0') + \frac{d}{ds_0}(\alpha'' N_0 - \alpha M_0),$$
$$\mathfrak{Z}_0 = Z_0 + \frac{d^2}{ds_0^2}(\gamma''_1 \rho L_0') + \frac{d}{ds_0}(\alpha N_0 - \alpha' M_0).$$

24. The Case where the External Forces and Moments are Zero. Special Form of W Leading to the Equations Discussed by Binet and Wantzel. Instead of using equations (24) and (25), it may be more convenient to come back to equations used at the beginning. It may also be useful to have recourse to a geometrical interpretation. /47

Let us assume, for example, that X_0, Y_0, Z_0 are zero. It follows that F, G, H are constants equal to F_{A_0}, G_{A_0}, H_{A_0} at one of the extremities A_0. We find the three equations:

$$\frac{dI}{ds_0} + H_{A_0}\frac{dy}{ds_0} - G_{A_0}\frac{dz}{ds_0} - L_0 = 0,$$
$$\frac{dJ}{ds_0} + F_{A_0}\frac{dz}{ds_0} - H_{A_0}\frac{dx}{ds_0} - M_0 = 0,$$
$$\frac{dK}{ds_0} + G_{A_0}\frac{dx}{ds_0} - F_{A_0}\frac{dy}{ds_0} - N_0 = 0,$$

which are the three original equations and which now result in the elimination of T between (24) and (25).

If, among other things, we have L_0, M_0, N_0 equal to zero, i.e., if the deformation (M) is subjected only to forces applied at the two extremities, we find:

$$I + H_{A_0}y - G_{A_0}z = const.,$$
$$J + F_{A_0}z - H_{A_0}x = const.,$$
$$K + G_{A_0}x - F_{A_0}y = const.,$$

These are relationships which are obtained by a geometric interpretation of the equations, or, which is the same thing, by consideration of equations such as(28)

(28) It should be noted that the reasoning of Bertrand [*Sur l'Équilibre d'une Ligne Élastique (On the Equilibrium of an Elastic Line)*, Note III of the *Mécanique Analytique (Analytical Mechanics)* of Lagrange, p. 460-464 of Volume XI of the Works of Lagrange] is reduced to the use of these equations, or more exactly, to the equivalent equations such as:

$$I_{M_0} - I_{A_0} = G_{A_0}z_M - H_{A_0}y_M - G_{A_0}z_A + H_{A_0}y_A$$
$$-\left[\int_{A_0}^{M_0}(Y_0 z - Z_0 y - L_0)\,ds_0 + y_M\int_{A_0}^{M_0} Z_0\,ds_0 - z_M\int_{A_0}^{M_0} Y_0\,ds_0\right];$$

It is sufficient to refer to Section 9, where we said that the deformation stress and moment at A_0 are $(F'_{A_0}, G'_{A_0}, H'_{A_0})$, $(I'_{A_0}, J'_{A_0}, K'_{A_0})$, i.e., the values of (F', G', H'), (I', J', K') at A_0.

$$I_{M_0} + H_{M_0} y_M - G_{M_0} z_M = I_{A_0} + H_{A_0} y_A - G_{A_0} z_A - \int_{A_0}^{M_0} (Y_0 z - Z_0 y - L_0)\, ds_0.$$

With these remarks, let us consider the case where the function W of s_0, p, q, r is of the form:(29)

$$\frac{1}{2} A (q^2 + r^2) + Bp + C,$$

where A, B, C are constants. We then have:

$$I' = B, \quad J' = Aq, \quad K' = Ar;$$

where the vector (I', J', K') or (I, J, K) is the resultant of a constant vector/48 equal to B directed along the tangent Mx', and of a vector directed along the binormal which has the absolute value $\frac{A}{\rho}$. The three equations

$$I + H_{A_0} y - G_{A_0} z = const., \quad J + F_{A_0} z - H_{A_0} x = const., \quad K + G_{A_0} x - F_{A_0} y = const.,$$

are identical to the equations:

$$p \frac{dy d^2 z - dz d^2 y}{ds^3} = \theta \frac{dx}{ds} + cy - bz + a_1,$$

$$p \frac{dz d^2 x - dx d^2 z}{ds^3} = \theta \frac{dy}{ds} + az - cx + b_1,$$

$$p \frac{dx d^2 y - dy d^2 x}{ds^3} = \theta \frac{dz}{ds} + bx - ay + c_1,$$

considered by Binet(30), Wantzel(31), Hermite(32), and where p, θ, a, b, c, a_1, b_1, c_1 are constants.

In the note already mentioned, J. Bertrand dealt with the case where the three equations

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(29) If W is obtained by replacing p_0, q_0, r_0 by their values in a function of $p - p_0$, $q - q_0$, $r - r_0$, we assume that $p_0 = q_0 = r_0 = 0$, in such way that $(q_0)^2 + (r_0)^2 = 0$ and the curve (M_0) is a straight line.

(30) J. Binet. *Mémoire sur l'Intégration des Équations de la Courbe Élastique à Double Courbure (Memoir on the Integration of the Equations of an Elastic Curve with Double Curvature)*, (Excerpt), C. R. 18, p. 1115-1119, June 17, 1844. *Réflexions sur l'Intégration des Formules de la Tige Élastique à Double Courbure (Reflections on the Integration of the Equations for the Elastic Rod with Double Curvature)*, C. R., 19, p. 1-3, July 1, 1844.

(31) Wantzel. *Note sur l'Intégration des Équations de la Courbe Élastique à Double Courbure (Note on the Integration of the Equations of the Elastic Curve with Double Curvature)*, C. R., 18, p. 1197-1201, June 24, 1844.

(32) Ch. Hermite. *Sur Quelques Applications des Fonctions Elliptiques (Certain Applications of Elliptic Functions)*, C. R., 90, p. 478, March 8, 1880; also in the work with the same title which appeared in 1885 (see Section 35).

$$cy - bz + a_1 = 0, \quad az - cx + b_1 = 0, \quad bx - ay + c_1 = 0,$$

represent a straight line, following Wantzel. He put this at the end of the Analytical Mechanics of Lagrange, where he states that he imitated a method developed by Poisson in the article mentioned in Section 10 and which will be treated again in the following section. Let this line be defined by

$$H_A(y - y_A) - G_A(z - z_A) = I_A,$$
$$F_A(z - z_A) - H_A(x - x_A) = J_A,$$
$$G_A(x - x_A) - F_A(y - y_A) = K_A.$$

The preceding hypothesis becomes

$$F_A I_A + G_A J_A + H_A K_A = 0,$$

and this amounts to the assumption that the couple (I_A, J_A, K_A) and the force (F_A, G_A, H_A) are reduced to a single force. According to the relationship (2) on page 463 of Lagrange, this line under discussion must not intersect the curve (M). This remark was made by J. Bertrand in the case he discussed. It may seem strange that at the top of page 462 there is a hypothesis from which it follows that $\theta = 0$, according to the note on page 463.

Assuming that the constant θ of Binet is zero, i.e., by setting $B = 0$ in our notations, we have the special curve considered by Lagrange. /49

Let us note that, in the present case, the unknown which we have designated by ω does not appear in the equations. However, the three equations:

$$\frac{dI}{ds_0} + H\frac{dy}{ds_0} - G\frac{dz}{ds_0} = 0,$$
$$\frac{dJ}{ds_0} + F\frac{dz}{ds_0} - H\frac{dx}{ds_0} = 0,$$
$$\frac{dK}{ds_0} + G\frac{dx}{ds_0} - F\frac{dy}{ds_0} = 0,$$

are reduced to two, because by multiplying them respectively by $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$ and adding them, we obtain the result zero, taking into account the special form of I, J, K considered previously.

We will encounter the line discussed before in the following section. This means we can present the preceding results as follows.

Let us investigate under what conditions the deformation stress of the line of the preceding section is perpendicular to the principal normal.

We have the condition

$$r\frac{\partial W}{\partial q} - q\frac{\partial W}{\partial r} = 0.$$

If we assume that this condition is due to the nature of the line, i.e.,

the form of its W, we find that this condition is a partial differential equation satisfied by W, according to which W does not depend on q and r except through the intermediary $q^2 + r^2$. If this condition is satisfied, the equations of the problem imply that

$$I' = const.$$

according to the remark of Poisson noted in Section 10.

If we assume that this conclusion is due to the nature of the line, i.e., due to the form of its W, this results in the condition

$$\frac{\partial W}{\partial p} = B,$$

where B is a constant, and we find

$$W = Bp + \varphi,$$

where ϕ is a function of $q^2 + r^2 = \frac{1}{\rho^2}$. Assuming that ϕ is of the first order with respect to $q^2 + r^2$, we find again the W which we used as a point of departure in this section.

25. The Deformable Line where the Plane Mx'y' is the Osculating Plane of (M) at M. The Case where the Line is Inextensible Among Other Things; the Line Considered by Lagrange and its Generalization Given by Binet and Studied by Poisson. We may go even farther in the hypotheses regarding the deformations /50 of a deformable line. Instead of simply assuming that Mx' is tangent to the curve (M), we may assume that the plane Mx'y' is the osculating plane of the curve (M).

1. Let us first disregard the hypothesis of inextensibility. Let us assume that the relationships (14) or (15) hold, and that in addition

$$q = q_0 = 0.$$

If, for an instant, we let W_2 designate the function obtained by setting $\eta = \zeta = q = 0$, where $q = 0$ in W_1, we will have

$$F' = \frac{\partial W_2}{\partial \mu}, \quad I' = \frac{\partial W_2}{\partial p}, \quad K' = \frac{\partial W_2}{\partial r};$$

G', H', J' cannot be calculated if W_2 is only given, and these quantities must be considered as three auxiliary quantities which must be defined by means of equations.

In the present case, equations (20) together with the relationships (18), (21) and the following:

$$\begin{cases} I = \alpha \frac{\partial W}{\partial p} + \beta J' + \gamma \frac{\partial W}{\partial r}, \\ J = \alpha' \frac{\partial W}{\partial p} + \beta' J' + \gamma' \frac{\partial W}{\partial r}, \\ K = \alpha'' \frac{\partial W}{\partial p} + \beta'' J' + \gamma'' \frac{\partial W}{\partial r}, \end{cases} \tag{26}$$

will result in a system of eight differential equations for four of these quantities (as a function of the fifth) and of F, G, H, J'. Here W designates the expression for W_2, when X, Y, Z, L, M, N or X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of s_0, s, x, y, z and the derivatives of these quantities with respect to one of them.

As was done previously, we can eliminate F, G, H, and the four equations (24) and (25) remain, in which I, J, K, T must be replaced by their values (26) and (23). With (21), they form a system of five differential equations which relate five of the quantities s, s_0, x, y, z, J' to one of them.

2. Let us, among other things, introduce the inextensibility by means of the relationships

$$\xi = \xi_0 = 1.$$

Let us continue to call W the function $W(s_0, 1, 0, 0, p, 0, r)$, and let us assume that only this function is known. We then simply have the relationships:

$$I' = \frac{\partial W}{\partial p}, \quad K' = \frac{\partial W}{\partial r}.$$

Consequently, if X, Y, Z, L, M, N or X_0, Y_0, Z_0, M_0, N_0 are given functions of s_0, s, x, y, z and the derivatives of these quantities with respect to one of them, we obtain the seven equations (20) and (21), where I, J, K must be replaced by their values (26) and which determine the seven unknowns x, y, z, F, G, H, J' as a function of s_0, for example. By eliminating F, G, H, we obtain the four equations (24) and (25), which define the four unknowns x, y, z, J' as a function of s_0. /51

From the preceding, it is easy to derive the case considered by Lagrange, Binet and Poisson.

Let us assume that given functions L, M, N are zero. The three equations (20) on the right form an equivalent system as follows:

$$\frac{dI'}{ds} - rJ' = 0,$$

$$\frac{dJ'}{ds} + rI' - pK' - H' = 0,$$

$$\frac{dK'}{ds} + pJ' + G' = 0,$$

which reduces to the one of Section 10. One or two of these three equations can even replace one or two of the right equations of (20), in general.

In particular, the relationship

$$\frac{dI'}{ds_0} - rJ' = 0, \tag{27}$$

obtained by adding the three equations (20) on the right, after multiplication

by $\alpha = \frac{dx}{ds}$, $\alpha' = \frac{dy}{ds}$, $\alpha'' = \frac{dz}{ds}$ can in general be substituted for any of the equations (20) on the right.

With these remarks, let us first assume that the function W of s_0, p, r, which is now contained in the relationships (26), does not depend on p. We will then have I' = 0, and the relationship (27) results in J' = 0 under the assumption that $r \neq 0$. Thus, in the present case, the deformation moment is directed along the binormal of the curve (M). In equations (20), I, J, K must be replaced by the values:

$$I = \gamma\frac{\partial W}{\partial r}, \quad J = \gamma'\frac{\partial W}{\partial r}, \quad K = \gamma''\frac{\partial W}{\partial r}.$$

The three equations on the right of (20) are reduced to two.

We thus obtain the case considered by Lagrange in No. 46 and the following one of Section III, first part, Section V of *Mécanique Analytique (Analytical Mechanics)* (p. 162 and following of Volume I of the last edition).

It is useful to show that our derivation is identical to that of Lagrange. We may set

$$\begin{aligned} I &= J_1(dy\,d^2z - dz\,d^2y), \\ J &= J_1(dz\,d^2x - dx\,d^2z), \\ K &= J_1(dx\,d^2y - dy\,d^2x), \end{aligned}$$

since the vector I, J, K is perpendicular to the osculating plane of (M).

The equations (20) on the right, which actually (L = M = N = 0) may be written as follows: /52

$$\begin{aligned} dy\,d(J_1d^2z) - dz\,d(J_1d^2y) &= -H\,dy + G\,dz \\ dz\,d(J_1d^2x) - dx\,d(J_1d^2z) &= -F\,dz + H\,dx \\ dx\,d(J_1d^2y) - dy\,d(J_1d^2x) &= -G\,dx + F\,dy, \end{aligned}$$

or

$$\frac{d(J_1d^2x) + F}{dx} = \frac{d(J_1d^2y) + G}{dy} = \frac{d(J_1d^2z) + H}{dz},$$

enable us to set

$$\begin{aligned} F &= \lambda\frac{dx}{ds} - d(J_1d^2x), \\ G &= \lambda\frac{dy}{ds} - d(J_1d^2y), \\ H &= \lambda\frac{dz}{ds} - d(J_1d^2z), \end{aligned}$$

by introducing an auxiliary term λ.

If we transfer these values into the three equations (20) on the left, we obtain the equations given by Lagrange at the beginning of his No. 48:

$$X\,ds - d\frac{\lambda\,dx}{ds} + d^2(J_1d^2x) = 0.$$

$$Y ds - d\frac{\lambda dy}{ds} + d^2(J_1 d^2 y) = 0,$$

$$Z ds - d\frac{\lambda dz}{ds} + d^2(J_1 d^2 z) = 0.$$

In the preceding theory developed by Lagrange, the moment of deformation is normal to the osculatory plane. Binet(33) suggested that the case be considered in which this moment of deformation is simply perpendicular to the principal normal. In addition, Binet assumed that the elements of the line are subjected to external forces, so that we are still dealing with a case in which $L = M = N = 0$. According to (27), the hypothesis $J' = 0$ advanced by Binet entails the following

$$I' = \text{const.}$$

This result, which we already presented in Section 10 in a general form which is independent of W and which was obtained by Poisson(34), may occur due to the specification of W or due to the specification of the force.

If we consider this latter case, we have— /53

$$W = \varphi(s_0, r) + mp,$$

where m is a constant. Consequently,

$$I' = m, \quad K' = \frac{\partial\varphi}{\partial r}.$$

With this hypothesis, it can be seen that if $r \neq 0$ condition (27) amounts to stating that the unknown J' equals zero and, consequently, I, J, K must be replaced by the following values in equations (20)

$$I = \alpha m + \gamma\frac{\partial\varphi}{\partial r},$$

$$J = \alpha' m + \gamma'\frac{\partial\varphi}{\partial r},$$

$$K = \alpha'' m + \gamma''\frac{\partial\varphi}{\partial r},$$

and the three equations (20) on the right must be reduced to two equations. In particular, if $\frac{\partial\phi}{\partial r}$ is deduced from an expression having the form $n(r - r_0)$, where n is constant, and if r_0 is replaced as a function of s_0, we have the hypothesis which was explicitly advanced by Binet and Poisson. In addition, if we assume that the curve (M_0) is a straight line and that the external forces

(33) J. Binet. *Mémoire sur l'Expression Analytique de l'Élasticité et de la Raideur des Courbes à Double Courbure (Report on the Analytical Expression of Elasticity and of the Rigidity of Double-Curvature Curves). (Bull. de la Soc. Philomatique*, 1814, p. 159-160; *Journ. de l'Ec. Polyt.*, 17th Issue, Volume X, p. 418-456, 1815).

(34) Poisson. *Sur les Lignes Élastiques à Double Courbure, Correspondance sur l'Ecole Polytechnique (On the Elastic Lines having Double Curvature. Report to the l'Ecole Polytechnique)*, Volume III, No. 3, p. 355-360, January 1816. This study was designed to complete the preceding study of Binet.

are zero, so that the change of (M_0) into (M) only arises from the forces and moments applied at the ends, we obtain the problem developed by Binet and Wantzel, which we already discussed previously.

If we set $m = 0$, we again encounter the Lagrange case in all the statements just presented.

26. Rectilinear Deformations of a Deformable Line. Let us assume that, since (M_0) is a straight line, our attention is directed to the deformations (M) which are also straight lines, and in addition let us assume that the axis Mx' is directed along the line (M) and $M_0x'_0$ is directed along the line (M_0).

1. If we first assume the extensible line, we have

$$\eta = \eta_0 = 0, \quad \zeta = \zeta_0 = 0, \quad q = q_0 = 0, \quad r = r_0 = 0.$$

If we continue to designate the function $W(s_0, 1 + \mu, 0, 0, p, 0, 0)$ by W, we shall have

$$F' = \frac{\partial W}{\partial \mu}, \quad I' = \frac{\partial W}{\partial p}.$$

With respect to G', H', J', K', they cannot be calculated by a knowledge of the function $W(s_0, 1 + \mu, 0, 0, p, 0, 0)$ alone. Therefore, if this function alone is given, we must assume that G', H', J', K' are four auxiliary quantities which will be defined by the equations.

In actuality, since X, Y, Z, L, M, N or X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of s_0, s, x, y, z and of the derivatives of these terms with respect to one of them, equations (20) combined with relationships (18) and (21) and with the following /54

$$\begin{cases} I = \alpha \dfrac{\partial W}{\partial p} + \beta J' + \gamma K', \\ J = \alpha' \dfrac{\partial W}{\partial p} + \beta' J' + \gamma' K', \\ K = \alpha'' \dfrac{\partial W}{\partial p} + \beta'' J' + \gamma'' K', \end{cases} \tag{28}$$

will provide a system of eight differential equations with respect to four of these terms (as a function of the fifth) and with respect to ω, F, G, H, J', K'. In addition, we have two equations of the first degree (whose coefficients must be determined) between x, y, z.

Just as previously, we may eliminate F, G, H.

A particular case is that in which (M) coincides with (M_0) in the point representation (coincidence of the triad vertices).

2. In addition, if we introduce the nonextensibility by the following relationships

$$\xi = \xi_0 = 1,$$

and if we continue to designate the function $W(s_0, 1, 0, 0, p, 0, 0)$ by W, we shall simply have the following relationship, assuming that this last function alone is known:

$$I' = \frac{\partial W}{\partial p}.$$

If X, Y, Z, L, M, N or X_0, Y_0, Z_0, L_0, M_0, N_0 are the given functions of s_0, s, x, y, z and of the derivatives of these terms with respect to one of them, we have the seven equations (20) and (21), where I, J, K must be replaced by their values (28). When combined with two relationships of the first degree containing x, y, z (with the coefficients to be determined by the auxiliary conditions), they determine the nine unknowns x, y, z, ω, F, G, H, J", K" as a function of s_0.

Just as previously, we may eliminate F, G, H.

27. The Deformable Line Obtained by the Addition of the Conditions $p = p_0$, $q = q_0$, $r = r_0$ and, in particular, $p = p_0 = 0$, $q = q_0 = 0$, $r = r_0 = 0$. This deformable line may be studied in different ways, either by considering the deformations (M) of the general deformable line which satisfy the indicated conditions, or by starting with general W and by defining a new line by considering the given conditions. This line may be also studied by starting with a W which is a function of s_0, ξ, η, ζ and by defining the line in conformance with these conditions.

Let us consider the first viewpoint. For the time being, let us designate that which W becomes by W_1, when the following conditions are taken into account

$$p = p_0, \quad q = q_0, \quad r = r_0,$$

i.e., let us set: /55

$$W_1 = [W(s_0, \xi, \eta, \zeta, p, q, r)]_{p=p_0, q=q_0, r=r_0} = W(s_0, \xi, \eta, \zeta, p_0, q_0, r_0).$$

We have:

$$F' = \left[\frac{\partial W}{\partial \xi}\right]_{p=p_0, q=q_0, r=r_0} = \frac{\partial W_1}{\partial \xi}, \quad I' = \left[\frac{\partial W}{\partial p}\right]_{p=p_0, q=q_0, r=r_0},$$

$$G' = \left[\frac{\partial W}{\partial \eta}\right]_{p=p_0, q=q_0, r=r_0} = \frac{\partial W_1}{\partial \eta}, \quad J' = \left[\frac{\partial W}{\partial q}\right]_{p=p_0, q=q_0, r=r_0},$$

$$H' = \left[\frac{\partial W}{\partial \zeta}\right]_{p=p_0, q=q_0, r=r_0} = \frac{\partial W_1}{\partial \zeta}, \quad K' = \left[\frac{\partial W}{\partial r}\right]_{p=p_0, q=q_0, r=r_0}.$$

Therefore, if we *only wish to introduce the function* W_1 of s_0, ξ, η, ζ -- i.e., the value which W assumes for $p = p_0$, $q = q_0$, $r = r_0$ -- and if we assume that we do not have the values which the derivatives $\frac{\partial W}{\partial p}$, $\frac{\partial W}{\partial q}$, $\frac{\partial W}{\partial r}$ assume for $p = p_0$, $q = q_0$, $r = r_0$, we find ourselves dealing with six expressions. Out of these six expressions, only three F', G', H' may be regarded as given, and

three I', J', K' must be determined.

The equations in question are then as follows:

$$\frac{d}{ds_0}\left(\frac{\partial W_1}{\partial \xi}\right) + q_0 \frac{\partial W_1}{\partial \zeta} - r_0 \frac{\partial W_1}{\partial \eta} - X_0' = 0,$$
$$\frac{d}{ds_0}\left(\frac{\partial W_1}{\partial \eta}\right) + r_0 \frac{\partial W_1}{\partial \xi} - p_0 \frac{\partial W_1}{\partial \zeta} - Y_0' = 0,$$
$$\frac{d}{ds_0}\left(\frac{\partial W_1}{\partial \zeta}\right) + p_0 \frac{\partial W_1}{\partial \eta} - q_0 \frac{\partial W_1}{\partial \xi} - Z_0' = 0,$$
$$\frac{dI'}{ds_0} + q_0 K' - r_0 J' + \eta \frac{\partial W_1}{\partial \zeta} - \zeta \frac{\partial W_1}{\partial \eta} - L_0' = 0,$$
$$\frac{dJ'}{ds_0} + r_0 I' - p_0 K' + \zeta \frac{\partial W_1}{\partial \xi} - \xi \frac{\partial W_1}{\partial \eta} - M_0' = 0,$$
$$\frac{dK'}{ds_0} + p_0 J' - q_0 I' + \xi \frac{\partial W_1}{\partial \eta} - \eta \frac{\partial W_1}{\partial \xi} - N_0' = 0.$$

We must add $p = p_0$, $q = q_0$, $r = r_0$ to these equations, which will give us a total of nine equations with respect to the nine unknowns x, y, z, λ_1, λ_2, λ_3, I', J', K'.

The last three formulas must be composed of those which Maxwell called magnetic induction within a magnet.

In the particular case $p = p_0 = 0$, $q = q_0 = 0$, $r = r_0 = 0$, the preceding formulas assume a very simple form.

28. Deformable Line Subjected to Constraints. Canonical Equations. In the preceding, we considered a deformable line which we qualified as being *free* -- i.e., its theory may be developed without introducing external elements, by means of a function W composed of elements of the line in the natural state /56 and in its deformed state.

By calling attention to certain deformations, by adding the concept of a *concealed* W we have been able to obtain the equations proposed for different lines by the authors.

Along with this discussion, we have been able to develop another one. Instead of considering the deformable line given in Sections 5 to 9, for deformations satisfying certain determined conditions, we have considered a *unique* deformable line, whose *definition has already taken into consideration* the specific conditions satisfied by the particular deformations of the preceding line.

In order to define the new line, and still maintain the same order as previously, we have proceeded as follows.

Let us first note that the conditions imposed on the functions x, y, z, α, α', ... γ" may be of two kinds: (1) conditions between these functions and their derivatives(35), no matter what s_0 may be; (2) conditions which are satisfied for

(35) Our discussion does not encompass the distinction between holonomic and non-holonomic relationships.

certain values of s_0.

Let us confine ourselves to the conditions of the first type.

As an example, let us set

$$f_1 = 0, \quad f_2 = 0,$$

as two conditions or *constraint equations*. Instead of formulating the expressions which were previously defined by means of the following identity:

$$\int_{s_0}^{s_0} \delta W ds_0 = [F'\delta'x + G'\delta'y + H'\delta'z + I'\delta\iota' + J'\delta\lambda' + K'\delta\kappa']_{s_0}^{s_0}$$
$$- \int_{s_0}^{s_0} (X'\delta'x + Y'\delta'y + Z'\delta'z + L'\delta\iota' + M'\delta\lambda' + N'\delta\kappa') ds_0,$$

as a function of s_0, where we introduce F', G', H', I', J', K'; X', Y', Z', L', M', N', by way of an example let us state that, by definition, the preceding identity must hold in view of the fact that

$$f_1 = 0, \quad f_2 = 0.$$

By definition, let us consider a deformable line whose theory results from the consideration of a function $W(s_0, \xi, \eta, \zeta, p, q, r)$ and of two auxiliary functions λ_1, λ_2 of s_0, by means of the identity:

$$\int_{s_0}^{s_0} (\delta W + \lambda_1 \delta f_1 + \lambda_2 \delta f_2) ds_0 = [F'\delta'x + G'\delta'y + H'\delta'y + I'\delta\iota' + J'\delta\lambda' + K'\delta\kappa']_{s_0}^{s_0}$$
$$- \int_{s_0}^{s_0} (X'\delta'x + Y'\delta'y + Z'\delta'z + L'\delta\iota' + M'\delta\lambda' + N'\delta\kappa') ds_0,$$

where all the variations are arbitrary this time. We must add the following *a posteriori*

$$f_1 = 0, \quad f_2 = 0.$$

We should point out that, in the case when some of the first terms f_1, f_2, ... /57
of the constraint equations only contain the arguments which appear in W, we could either follow the procedure which was just indicated, or we could introduce the particular given constraint equation in W *a priori*, by changing the auxiliary terms. This again brings us to the concept of a *concealed* W. We shall discuss this point in the different particular cases which follow, to which the present remarks apply.

(1) Flexible and inextensible Line. Let us start with a function W of $\mu = \frac{ds}{ds_0} - 1$ and of s_0, and let us add the condition $\mu = 0$. We may define the functions F', G', H', X', Y', Z' starting with

$$\int_{s_0}^{s_0} (\delta W + \lambda \delta\mu) ds = [F'\delta'x + G'\delta'y + H'\delta'z]_{s_0}^{s_0}$$
$$- \int_{s_0}^{s_0} (X_0'\delta'x + Y_0'\delta'y + Z_0'\delta'z) ds_0.$$

This amounts to replacing W by $W_1 = W + \lambda\mu$ in the preceding statements, and we obtain the following formulas

$$F = \frac{\partial W_1}{\partial \frac{dx}{ds_0}}, \quad G = \frac{\partial W_1}{\partial \frac{dy}{ds_0}}, \quad H = \frac{\partial W_1}{\partial \frac{dz}{ds_0}},$$

$$\frac{dF}{ds_0} - X_0 = 0, \quad \frac{dG}{ds_0} - Y_0 = 0, \quad \frac{dH}{ds_0} - Z_0 = 0,$$

in which we must take the fact into account that $\mu = 0$, and which then determine F, G, H, X_0, Y_0, Z_0.

It can be readily seen that we have again encountered the theory of the flexible and extensible line which is a generalization of the Lagrange theory, which corresponds to the function W_1 of s_0 and of μ. We shall confine ourselves to studying the deformations of this theory which correspond to $\mu = 0$. If we consider the case of *concealed* W_1, we shall then assume that we simply know the value $W_0(s_0)$ which W and W_1 assume simultaneously for $\mu = 0$, and we then have the classical system of mechanics.

In order to compile the flexible and inextensible line, if we take into account *a priori* the condition $\mu = 0$ in W, we are led to replace W by $\lambda\mu$ in the calculations with respect to the general deformable line. We thus arrive at formulas which then lead to the study of a flexible, extensible thread, and we shall confine ourselves to considering the deformations corresponding to $\mu = 0$. Assuming that λ *is not known*, these formulas lead us again to the classical system of mechanics.

Let us conclude by the following remarks. In view of the formulas which define the deformation, we shall assume that we have expressed X_0, Y_0, Z_0 as a function of s_0, x, y, z, so that $X_0dx + Y_0dy + Z_0dz$ is the total differential of a function ϕ of s_0, x, y, z with respect to x, y, z. In addition, let us assume that we are dealing with a case of a *concealed* W_1, or the case considered last, so that we are lead to the case of mechanics. Based on the preceding /58
statements, we again encounter the condition which served as the point of departure for Clebsch(36) -- namely, the equations in question which contain X_0, Y_0, Z_0 are none other than the extremal equations of the calculus of variations problem, which consist of determining an extremum of the integral

$$\int_{s_0}^{s_0} \varphi ds,$$

under the condition(37)

$$\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2 = 1.$$

(36) A. Clebsch. *Ueber die Gleichgewichtsfigur Eines Biegsamen Fadens (Equilibrium Figure of a Flexible Thread)*, Journ. für die Reine und Angewandte Math., Volume LVII, p. 93-116 [1859], 1860.

By setting:

$$\psi_1 = -\frac{1}{2}\left[\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2 - 1\right].$$

and applying the considerations developed by M. Jourdan(38), we may reduce the system to the canonical form. With λ_1 used in place of T, the system expresses the fact that the first variation of the following integral is cancelled

$$\int_{s_0}^{s_1} F ds_0,$$

by setting:

$$F = -(\varphi + \lambda_1 \psi_1).$$

The equations

$$\frac{\partial F}{\partial \frac{dx}{ds_0}} = p_1, \quad \frac{\partial F}{\partial \frac{dy}{ds_0}} = p_2, \quad \frac{\partial F}{\partial \frac{dz}{ds_0}} = p_3, \quad \psi_1 = 0$$

make it possible to express the variables $x' = \frac{dx}{ds_0}$, $y' = \frac{dy}{ds_0}$, $z' = \frac{dz}{dz_0}$, λ_1 as a function of the variables x, y, z, p_1 p_2 p_3 by the following formulas:

$$\lambda_1 = \sqrt{p_1^2 + p_2^2 + p_3^2}, \quad x' = \frac{p_1}{\lambda_1}, \quad y' = \frac{p_2}{\lambda_1}, \quad z' = \frac{p_3}{\lambda_1}.$$

Placing these values in

$$p_1 x' + p_2 y' + p_3 z' - F,$$

we obtain the function

$$\mathcal{H} = \varphi(s_0, x, y, z) + \sqrt{p_1^2 + p_2^2 + p_3^2}.$$

and we have the following equations (which are canonical if s_0 is not contained /59 in ϕ) for determining the variables x, y, z, p_1, p_2, p_3:

$$\frac{dq_\nu}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_\nu}, \quad \frac{dp_\nu}{ds} = -\frac{\partial \mathcal{H}}{\partial q_\nu},$$

designating the coordinates x, y, z by q_1, q_2, q_3, as was done by M. Appell(39).

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(37) We must draw a distinction between the present question and that discussed by M. Appell, *Traité de Mécanique Rationnelle (Treatise on Rational Mechanics)*, Volume 1, 1st Edition, Section 158, p. 205 on; 2nd Edition Section 146, p. 201 on.

(38) Jordan. *Cours d'Analyse de l'Ecole Polytechnique (Course on Analysis of the Polytechnic School)*, Volume III, 2nd Edition, No. 375, p. 501, 502.

(39) Appell. *Traité de Mécanique Rationnelle (Treatise on Rational Mechanics)*, 1st Edition, Volume II, Number 14, p. 48-49; 2nd Edition, Volume I, No. 14, p. 583-584.

As may be seen, we again encounter the results obtained by M. Appell[40] in the simple form which was ascribed to them first by M. Legoux[41], then by M. Marcolongo[42]. From these results we may derive the transition to the Jacobi method and the results first given by Clebsch in the reports already cited[43].

The preceding discussion may also be presented as we did for the dynamics of a point in our first reports, and for the general deformable line.

Let us start with the following equations

$$\frac{d}{ds_0}\left(T\frac{dx}{ds_0}\right)+X_0=0,\quad \frac{d}{ds_0}\left(T\frac{dy}{ds_0}\right)+Y_0=0,\quad \frac{d}{ds_0}\left(T\frac{dz}{ds_0}\right)+Z_0=0,$$

or rather from the system which produced them

$$F=-T\frac{dx}{ds_0},\quad G=-T\frac{dy}{ds_0},\quad H=-T\frac{dz}{ds_0},$$
$$\frac{dF}{ds_0}-X_0=0,\quad \frac{dG}{ds_0}-Y_0=0,\quad \frac{dH}{ds_0}-Z_0=0,$$

which may be regarded as defining the six unknowns x, y, z, F, G, H. Let us assume that X_0, Y_0, Z_0 are the given functions of s_0, x, y, z.

If we add the three equations of the first line, after having squared them, we find that T is defined as a function of F, G, H by the following relationship

$$T^2=F^2+G^2+H^2,$$

from which it follows that

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(40) Appell. *Réduction à la Forme Canonique des Équations d'un Fil Flexible et Inextensible (Reduction to the Canonical Form of the Equations of a Flexible and Inextensible Thread)*, C. R., 96, p. 688-691, March 12, 1883; *Traité de Mécanique Rationnelle (Treatise on Rational Mechanics)*, loc. cit.

(41) A. Legoux. *Equations Canoniques, Application à la Recherche de l'Équilibre des fils Flexibles et des Courbes Brachistochrones (Canonical Equations, Application to Research on Equilibrium of Flexible Threads, and Brachistochronic Curves)*. Mém. de l'Acad. des Sciences, Inscriptions et Belles Lettres de Toulouse, 8th Series, Volume VII, 2nd Semester, p. 159-184, 1885.

(42) R. Marcolongo. *Sull' Equilibrio di un filo Flessible ed Inestensible (On the Equilibrium of a Flexible and Inextensible Thread)*. Rend. dell' Accad. delle Scienze Fisiche e Matematiche (Sezione della Società Reale di Napoli), Series 2, Volume II, p. 363-368, 1888.

(43) See also Appell. *Sur l'Équilibre d'un fil Flexible et Inextensible (On the Equilibrium of a Flexible and Inextensible Thread)*. Ann. de la Fac. des Sc. de Toulouse, (1), 1, p. B_1-B_3, 1887.

$$\frac{F}{T}=\frac{\partial T}{\partial F},\quad \frac{G}{T}=\frac{\partial T}{\partial G},\quad \frac{H}{T}=\frac{\partial T}{\partial H}.$$

The normal form of the system under consideration is as follows: /60

$$\frac{dx}{ds_0}=-\frac{\partial T}{\partial F},\quad \frac{dy}{ds_0}=-\frac{\partial T}{\partial G},\quad \frac{dz}{ds_0}=-\frac{\partial T}{\partial H},$$
$$\frac{dF}{ds_0}=X_0,\quad \frac{dG}{ds_0}=Y_0,\quad \frac{dH}{ds_0}=-Z_0,$$

where T designates a function of F, G, H defined by the formula $T^2 = F^2 + G^2 + H^2$. If we assume that X_0, Y_0, Z_0 are partial derivatives $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $\frac{\partial U}{\partial z}$ with opposite sign of a function U of s_0, x, y, z, and if we set $\mathcal{H} = -T + U$, the preceding system assumes the following form

$$\frac{dx}{ds_0}=\frac{\partial\mathcal{H}}{\partial F},\quad \frac{dy}{ds_0}=\frac{\partial\mathcal{H}}{\partial G},\quad \frac{dz}{ds_0}=\frac{\partial\mathcal{H}}{\partial H},$$
$$\frac{dF}{ds_0}=-\frac{\partial\mathcal{H}}{\partial x},\quad \frac{dG}{ds_0}=-\frac{\partial\mathcal{H}}{\partial y},\quad \frac{dH}{ds_0}=-\frac{\partial\mathcal{H}}{\partial z}.$$

(2). Elastic Line of Lord Kelvin and Tait. We may repeat the statements which we just made for the inextensible flexible line for this line. Let us start with a function W of s_0, ξ, η, ζ, p, q, r, and let us add the conditions

$$\xi=\xi_0=1,\quad \eta=\eta_0=0,\quad \zeta=\zeta_0=0.$$

We may define the functions F', G', H', I', J', K'; X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0 by means of the identity

$$\int_{A_0}^{B_0}(\delta W+\mu_1\delta\xi+\mu_2\delta\eta+\mu_3\delta\zeta)ds_0=\left[F'\delta'x+G'\delta'y+\dots+K'\delta\kappa'\right]_{B_0}^{A_0}$$
$$-\int_{A_0}^{B_0}(X'_0\delta'x+Y'_0\delta'y+\dots+N'_0\delta\kappa')\,ds_0;$$

which amounts to replacing W by $W_1 + \mu_1(\xi - 1) + \mu_2\eta + \mu_3\zeta$ in the preceding section, and to adding the indicated conditions $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ to the formulas.

As may be seen, we again encounter the *theory of the deformable line corresponding to the function* W_1 *of* s_0, ξ, η, ζ, p, q, r, *for which we limit ourselves to studying the deformations corresponding to* $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. *If we are dealing with the case of concealed* W_1, we shall then assume that we simply know the function $W(s_0, 1, 0, 0, p, q, r)$, to which W and W_1 are simultaneously reduced for $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. We thus again encounter *the theory developed by* Lord Kelvin and Tait.

In order to compile the preceding line, if we take the three conditions $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ in W into account *a priori*, by changing the

auxiliary terms, we are led to replace W by $W(s_0, 1, 0, 0, p, q, r) + \mu_1(\xi - 1) + \mu_2 \eta + \mu_3 \zeta$ in the calculations pertaining to the general deformable line. We thus obtain formulas which may be reduced to studying a deformable line, /61
and we may confine ourselves to considering the deformations which correspond to the three conditions $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. By assuming that μ_1, μ_2, μ_3 *are not known*, these formulas again lead us to the theory of Lord Kelvin and Tait.

Let us assume that, using the equations which determine the deformation, we have expressed X_0, Y_0, Z_0, L_0, M_0, N_0 as a function of s_0, x, y, z, λ_1, λ_2, λ_3 in such a way that

$$X_0 dx + Y_0 dy + Z_0 dz + \mathfrak{L}_0 d\lambda_1 + \mathfrak{M}_0 d\lambda_2 + \mathfrak{N}_0 d\lambda_3,$$

which is the total differential of a function U of s_0, x, y, z, λ_1, λ_2, λ_3 simply considered with respect to x, y, z, λ_1, λ_2, λ_3. Among other things, let us also assume that we are dealing with the case of the concealed W, or with the case considered last. From the above, the equations under consideration in which X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ appear are none other than the equations for the extremals of the problem of the calculus of variations, which consist of determining an extremum of the integral

$$\int_{A_0}^{B_0} (W + U)\, ds_0,$$

where W is a given function of s_0, p, q, r. We assume that the six unknown functions x, y, z, λ_1, λ_2, λ_3 satisfy the following three differential equations:

$$\xi - 1 = 0, \quad \eta = 0, \quad \zeta = 0.$$

Setting $\psi_1 = \xi - 1$, $\psi_2 = \eta$, $\psi_2 = \zeta$ and applying the ideas developed by M. Jordan, we may reduce the system to the canonical form. This system expresses the fact that the first variation of the integral $\int_{A_0}^{B_0} \mathcal{F} ds_0$ vanishes by setting

$$\mathcal{F} = W + U + F'\psi_1 + G'\eta + H'\zeta.$$

if we write F', G', H' in the place of the terms λ_1, λ_2, λ_3 of M. Jordan.

The equations

$$\frac{\partial \mathcal{F}}{\partial \frac{dx}{ds_0}} = p_1, \quad \frac{\partial \mathcal{F}}{\partial \frac{dy}{ds_0}} = p_2, \quad \frac{\partial \mathcal{F}}{\partial \frac{dz}{ds_0}} = p_3, \quad \frac{\partial \mathcal{F}}{\partial \frac{d\lambda_1}{ds_0}} = p_4, \quad \frac{\partial \mathcal{F}}{\partial \frac{d\lambda_2}{ds_0}} = p_5, \quad \frac{\partial \mathcal{F}}{\partial \frac{d\lambda_3}{ds_0}} = p_6,$$

$$\psi_1 = 0, \quad \psi_2 = 0, \quad \psi_3 = 0,$$

allow one to express the nine variables $x' = \frac{dx}{ds_0}$, $y' = \frac{dy}{ds_0}$, $z' = \frac{dz}{ds_0}$,

$\lambda_1' = \frac{d\lambda_1}{ds_0}$, $\lambda'_2 = \frac{d\lambda_2}{ds_0}$, $\lambda'_3 = \frac{d\lambda_3}{ds_0}$, F', G', H' as a function of the twelve variables x, y, z, λ_1, λ_2, λ_3, p_1, p_2, ... p_6 by means of the equations

$$x' = \alpha, \quad y' = \alpha', \quad z' = \alpha'',$$

$$F' = \alpha p_1 + \alpha' p_2 + \alpha'' p_3, \quad G' = \beta p_1 + \beta' p_2 + \beta'' p_3, \quad H' = \gamma p_1 + \gamma' p_2 + \gamma'' p_3,$$

the equations /62

$$\begin{cases} p_1 = \frac{\partial W}{\partial p}\varpi_1' + \frac{\partial W}{\partial q}\chi_1' + \frac{\partial W}{\partial r}\sigma \\ p_2 = \frac{\partial W}{\partial p}\varpi_2' + \frac{\partial W}{\partial q}\chi_2' + \frac{\partial W}{\partial r}\sigma_2' \\ p_3 = \frac{\partial W}{\partial p}\varpi_3' + \frac{\partial W}{\partial q}\chi_3' + \frac{\partial W}{\partial r}\sigma_3' \end{cases} \tag{29}$$

must be solved using the same notation as in Section 10 for the time being.

Introducing these values in

$$p_1 x' + p_2 y' + p_3 z' + p_4 \lambda_1' + p_5 \lambda_2' + p_6 \lambda_3' - \mathcal{F},$$

we obtain the function $\mathcal{H}$ of s_0, x, y, z, λ_1, λ_2, λ_3, p_1, ... p_6 which is derived from

$$-W - U + \alpha p_1 + \alpha' p_2 + \alpha'' p_3 + p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial r},$$

by the p, q, r substitution of the values of the function of s_0, λ_1, λ_2, λ_3, p_4, p_5, p_6 which may be derived from equations (29).

For the determination of the twelve variables x, y, z, λ_1, λ_2, λ_3, p_1, ... p_6 we have the equations (they are canonical if s_0 is not contained explicitly):

$$\frac{dx}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_1}, \quad \frac{dy}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_2}, \quad \frac{dz}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_3}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_4}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_5}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_6},$$

$$\frac{dp_1}{ds_0} = -\frac{\partial \mathcal{H}}{\partial x}, \quad \frac{dp_2}{ds_0} = -\frac{\partial \mathcal{H}}{\partial y}, \quad \frac{dp_3}{ds_0} = -\frac{\partial \mathcal{H}}{\partial z}, \quad \frac{dp_4}{ds_0} = -\frac{\partial \mathcal{H}}{\partial \lambda_1},$$

$$\frac{dp_5}{ds_0} = -\frac{\partial \mathcal{H}}{\partial \lambda_2}, \quad \frac{dp_6}{ds_0} = -\frac{\partial \mathcal{H}}{\partial \lambda_3}.$$

From this we can conclude that the method of Jacobi has been applied to the line under study.

It is also possible to carry out the preceding development, which we have just finished, for the general deformable line, as well as for the dynamics of the point treated in our first note.

(3). Deformable Line Where Mx' is Tangent to (M) at M. Let us again start with a function W of s_0, ξ, η, ζ, p, q, r and let us impose the conditions $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. We define the functions F', G', H', I', J', K', X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0 by means of the identity

$$\int_{s_0}^{s_0}(\delta W + \mu_1\delta\eta + \mu_2\delta\zeta)ds = \left[F'\delta'x + G'\delta'y + \ldots + K'\delta z'\right]_{s_0}^{s_0}$$
$$- \int_{s_0}^{s_0}(X'_0\delta'x + Y'_0\delta'y + \ldots + N'_0\delta z')ds_0 .$$

This amounts to replacing W by $W_1 = W + \mu_1\eta + \mu_2\zeta$ in the preceding discussion and to adding the indicated conditions $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ to the equations.

It can be seen that we again find *the theory of the deformable line corresponding to the function* W_1 *of* s_0, ξ, η, ζ, p, q, r *and in which we limit ourselves to study deformations which correspond to* $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. *If we assume the-concealed* W_1 *case,* we will assume that we only know the function W (s_0, ξ, 0, 0, p, q, r), to which W and W_1 are reduced for $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. /63

If, in order to define the preceding line, we take into account *a priori* in W the two conditions $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ by means of a change in the auxiliary variables, we are led to replacing W by $W(s_0, \xi, 0, 0, p, q, r) + \mu_1\eta + \mu_2\zeta$ in the calculation of the general deformable line, and we obtain equations which lead to the study of a deformable line in which we are restricted to deformations which correspond to the two conditions $\eta = \eta_0$, $\zeta = \zeta_0$.

Let us assume that, due to the equations which determine the deformation, we have expressed X_0, Y_0, Z_0, L_0, M_0, N_0 as a function of s_0, x, y, z, λ_1, λ_2, λ_3 in such a way that

$$X_0dx + Y_0dy + Z_0dz + \mathfrak{L}_0d\lambda_1 + \mathfrak{M}_0d\lambda_2 + \mathfrak{N}_0d\lambda_3$$

which is the total differential of a function U of s_0, x, y, z, λ_1, λ_2, λ_3 simply considered as a function of x, y, z, λ_1, λ_2, λ_3. In addition, let us consider the case of the concealed W or the case considered last. From the above, the equations under consideration, which contain X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$, are none other than the equations of the extremals of the problem of the calculus of variations which consists of determining an extremum of the integral

$$\int_{s_0}^{s_0}(W + U)\,ds_0,$$

where W is a given function of s_0, ξ, η, ζ, p, q, r assuming that the six unknown functions x, y, z, λ_1, λ_2, λ_3 satisfy the two differential equations $\eta = 0$, $\zeta = 0$. The previous developments are therefore repeated and this is also true for all the other particular lines which we considered.

29. State Infinitesimally Close to the Natural State. Hooke Deformation

Moduli. Critical Values of General Moduli. Similarity with the Dynamics of a Triad. Let us again consider the general deformable line. Let us assume that in the natural state the action is zero, as well as the stress and moment of deformation and the external force and moment. In this case not only the function W must vanish identically, but also the six partial derivatives of W with respect to ξ, η, ζ, p, q, r for the values ξ_0, η_0, ζ_0, p_0, q_0, r_0 of these variables. In addition, let us assume that W can be developed according to whole, positive powers of $\xi - \xi_0$, $\eta - \eta_0$, ... $r - r_0$ in the vicinity of $\xi = \xi_0$, $\eta = \eta_0$, $\zeta = \zeta_0$, $p = p_0$, $q = q_0$, $r = r_0$. Under these conditions we will have

$$W = W_2 + W_3 + \dots,$$

in which W_2, W_3, ... are homogeneous polynomials of the differences $\xi - \xi_0$, /64
$\eta - \eta_0$, ..., $r - r_0$ of degree 2, 3, ...

Let the coordinates of a point M_0 of the line (M_0) in the natural state and the three parameters used to express the direction cosines of the axes of the triad associated with this point be given by x_0, y_0, z_0, λ_{10}, λ_{20}, λ_{30}, respectively. Let us assume that the coordinates x, y, z of the corresponding point M in the deformed state (M) and the parameters λ_1, λ_2, λ_3 with respect to the axes of the associated triad are functions of s_0 and of h, and that they can be developed in powers of h by means of the equations:

$$\begin{aligned} x &= x_0 + x_1 + \dots + x_i + \dots & \lambda_1 &= \lambda_{10} + \lambda_{11} + \dots + \lambda_{1i} + \dots, \\ y &= y_0 + y_1 + \dots + y_i + \dots, & \lambda_2 &= \lambda_{20} + \lambda_{21} + \dots + \lambda_{2i} + \dots, \\ z &= z_0 + z_1 + \dots + z_i + \dots & \lambda_3 &= \lambda_{30} + \lambda_{31} + \dots + \lambda_{3i} + \dots, \end{aligned}$$

in which x_i, y_i, z_i, λ_{1i}, λ_{2i}, λ_{3i} are terms which contain h^i as a factor. We introduce these developments in series in order to abbreviate the development, and we admit that they follow the ordinary rules of calculus. The equations of Section 14 allow us to calculate the developments of F, G, H, $\mathcal{J}$, $\mathcal{J}$, $\mathcal{K}$; X_0, Y_0, Z_0, $\mathcal{X}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ according to powers of h; the terms independent of h are zero and the terms F_1, G_1, H_1, $\mathcal{J}_1$, $\mathcal{J}_1$, $\mathcal{K}_1$; X_{01}, Y_{01}, Z_{01}, $\mathcal{X}_{01}$, $\mathcal{M}_{01}$, $\mathcal{N}_{01}$ are given by the equations

$$F_1 = \frac{\partial W_2}{\partial \frac{dx^{(1)}}{ds_0}}, G_1 = \frac{\partial W_2}{\partial \frac{dy^{(1)}}{ds_0}}, H_1 = \frac{\partial W_2}{\partial \frac{dz^{(1)}}{ds_0}}, \mathcal{J}_1 = \frac{\partial W_2}{\partial \frac{d\lambda_1^{(1)}}{ds_0}}, \mathcal{J}_1 = \frac{\partial W_2}{\partial \frac{d\lambda_2^{(1)}}{ds_0}}, \mathcal{K}_1 = \frac{\partial W_2}{\partial \frac{d\lambda_3^{(1)}}{ds_0}},$$

$$X_{01} = \frac{d}{ds_0} \frac{\partial W_2}{\partial \frac{dx^{(1)}}{ds_0}}, \quad Y_{01} = \frac{d}{ds_0} \frac{\partial W_2}{\partial \frac{dy^{(1)}}{ds_0}}, \quad Z_{01} = \frac{d}{ds_0} \frac{\partial W_2}{\partial \frac{dz^{(1)}}{ds_0}}$$

$$\mathcal{X}_{01} = \frac{d}{ds_0} \frac{\partial W_2}{\partial \frac{d\lambda_1^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_1^{(1)}}, \mathcal{M}_{01} = \frac{d}{ds_0} \frac{\partial W_2}{\partial \frac{d\lambda_2^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_2^{(1)}}, \mathcal{N}_{01} = \frac{d}{ds_0} \frac{\partial W_2}{\partial \frac{d\lambda_3^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_3^{(1)}},$$

where we have set

$$x^{(1)} = x_0 + x_1,\ y^{(1)} = y_0 + y_1,\ z^{(1)} = z_0 + z_1,\ \lambda_1^{(1)} = \lambda_{10} + \lambda_{11},$$
$$\lambda_2^{(1)} = \lambda_{20} + \lambda_{21},\ \lambda_3^{(1)} = \lambda_{30} + \lambda_{31}.$$

If we use the name *state of deformation infinitesimally close to the natural state* to designate the state (M) in which the point M has the coordinates $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ and where the parameters of the associated triad have the values $\lambda_1^{(1)}$, $\lambda_2^{(1)}$, $\lambda_3^{(1)}$, and if, on the other hand, we call *stress and moment of deformation, external force and moment*, relative to this state the vectors (F_1, G_1 H_1), (J_1, $\mathcal{J}_1$, $\mathcal{K}_1$), (X_{01}, Y_{01}, Z_{01}), (L_{01}, M_{01}, N_{01}), we arrive at the hypotheses usually made by the classical authors and where the two first vectors are linear functions of the elements which describe the deformed state under consideration. L_{01}, M_{01}, N_{01} are calculated from λ_{10}, λ_{20}, λ_{30}, $\mathcal{L}_{01}$, $\mathcal{M}_{01}$, $\mathcal{N}_{01}$ in the same way as L_0, M_0, N_0 are calculated by means of λ_1, λ_2, λ_3, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$. As a consequence we find what is called the *generalized Hooke's law* but subject to the limitation, if appropriate, that *the principle of the conservation of energy must be satisfied.* In order to satisfy this condition in the classical method, /65 it is necessary to follow a procedure which is opposite to that which we employed.

The coefficients which appear in the linear functions which express Hooke's law are the *deformation moduli* of the deformable line in a state infinitesimally close to the natural state. They are *invariable* at a given point of the line. This concept of the modulus can be generalized by considering the first and second derivatives of the function W. In addition to the case where the generalized moduli are defined and continuous, it is possible to consider the case where they have critical values.

The preceding considerations are usually repeated for various particular deformable lines. They can be related to those which we discussed in our first note. The dynamics of the triad is related to the preceding in a very direct manner. It is sufficient to look upon the arc s_0 as the *time t*, and the deformable line as a *trajectory*. This simple fact immediately explains the analogies which were found long ago between classical dynamics of a particle and of an invariable body and the statics of the deformable line.

We should note that there is a correspondence between the result obtained previously(44) (in the case of the invariable body) regarding the kinetic energy and a theorem valid for the deformable line which states that when W does not explicitly depend on s_0, equations (10) imply that the expression:

$$(\xi X_0' + \eta Y_0' + \zeta Z_0' + pL_0' + qM_0' + rN_0')ds_0,$$

which can be written in the form

$$X_0 dx + Y_0 dy + Z_0 dz + \mathcal{L}_0 d\lambda_1 + \mathcal{M}_0 d\lambda_2 + \mathcal{N}_0 d\lambda_3,$$

(44) *Note sur la Dynamique du Point et du Corps Invariable (Note on the Dynamics of a Point and of an Invariable Body)*, Volume I, p. 261.

is equal to the differential of the quantity:

$$\xi \frac{\partial W}{\partial \xi} + \eta \frac{\partial W}{\partial \eta} + \zeta \frac{\partial W}{\partial \zeta} + p \frac{\partial W}{\partial p} + q \frac{\partial W}{\partial q} + r \frac{\partial W}{\partial r} - W,$$

which was already presented in Section 14.

On the other hand, we should note that we may augment the considerations given in our first note by statements regarding the invariable body which are analogous to those which were developed in the present work for the deformable line and which deal with the constraints.

III. Statics of the Deformable Surface and Dynamics of the Deformable Line

30. Deformable Surface. Natural State and Deformed State. The developments which were carried out for the deformable line will, as we shall see, be almost completely reproduced in the theories of the deformable surface and of /66 the three-dimensional deformable medium. This repetition points to the fruitfullness of the concept of Euclidian action. It suggests numerous comparisons and opens up a large field of study, which the first scientists were only able to explore with a large amount of difficulty. However, today it is possible to approach it successfully, using the present state of the general theory of geometrical surfaces and curvilinear coordinates, as M. Darboux did in his large works(1).

Let us consider a surface (M_0) described by a point M_0 whose coordinates with respect to three rectangular axes Ox, Oy, Oz are x_0, y_0, z_0, which are functions of two parameters which are selected in an arbitrary way and which we will call ρ_1 and ρ_2. Let us attach an orthogonal triad to each point M_0 of the surface (M_0) whose axes M_0x_0', M_0y_0', M_0z_0' have the direction cosines α_0, α_0', α_0''; β_0, β_0', β_0''; γ_0, γ_0', γ_0'', respectively, with respect to the axes Ox, Oy, Oz and which are functions of the same parameters ρ_1 and ρ_2. The continuous, two-dimensional ensemble of these triads $M_0x_0'y_0'z_0'$ is what we shall call a *deformable surface*.

Let us impart a displacement M_0M to a point M_0 and let x, y, z be the coordinates of the point M with respect to the fixed axes Ox, Oy, Oz. Also, let us subject the triad $M_0x_0'y_0'z_0'$ to a rotation which will finally make its axes coincide with those of a triad Mx'y'z' which we will attach to the point M. We shall define this rotation by specifying the direction cosines α, α', α''; β, β', β''; γ, γ', γ'' of the axes Mx, My', Mz' with respect to the fixed axes Ox, Oy, Oz. The continuous, two-dimensional ensemble of triads Mx'y'z' is

(1) Gaston Darboux. *Lecons sur la Théorie Générale des Surfaces (Lessons on the General Theory of Surfaces)*, 4 Volumes, Pairs, 1887-1896; *Lesons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes (Lessons on Orthogonal Systems and Curvilinear Coordinates)*, Volume I, Paris, 1898.

what we shall call *the deformed state* of the deformable surface under consideration which, in its initial state, will be called the *natural state*.

31. Kinematic Elements for the States of the Deformable Surface. Let us call $\xi_i^{(0)}$, $\eta_i^{(0)}$, $\zeta_i^{(0)}$ *the components of the velocity of the origin* M_0 *of the axes* M_0x_0', M_0y_0', M_0z_0' *along these axes, when* ρ_i *only is varied and plays the part of time.* Also, let $p_i^{(0)}$, $q_i^{(0)}$, $r_i^{(0)}$ be the quantities which define the projections of the instantaneous rotation of the triad $M_0x_0'y_0'z_0'$ on these axes for the parameter ρ_i. We will call ζ_i, η_i, ξ_i and p_i, q_i, r_i the analogous quantities for the triad $Mx'y'z'$, when it is referred to the fixed triad $Oxyz$, as was done for the triad $M_0x_0'y_0'z_0'$.

The elements which have been introduced can be calculated in the usual manner. We have

$$(30)\quad \begin{cases} \xi_i = \alpha \dfrac{\partial x}{\partial \rho_i} + \alpha' \dfrac{\partial y}{\partial \rho_i} + \alpha'' \dfrac{\partial z}{\partial \rho_i} \\ \eta_i = \beta \dfrac{\partial x}{\partial \rho_i} + \beta' \dfrac{\partial y}{\partial \rho_i} + \beta'' \dfrac{\partial z}{\partial \rho_i} \\ \zeta_i = \gamma \dfrac{\partial x}{\partial \rho_i} + \gamma' \dfrac{\partial y}{\partial \rho_i} + \gamma'' \dfrac{\partial z}{\partial \rho_i} \end{cases} \qquad \begin{cases} p_i = \Sigma\gamma \dfrac{\partial \beta}{\partial \rho_i} = -\Sigma\beta \dfrac{\partial \gamma}{\partial \rho_i} \\ q_i = \Sigma\alpha \dfrac{\partial \gamma}{\partial \rho_i} = -\Sigma\gamma \dfrac{\partial \alpha}{\partial \rho_i} \\ r_i = \Sigma\beta \dfrac{\partial \alpha}{\partial \rho_i} = -\Sigma\alpha \dfrac{\partial \beta}{\partial \rho_i}. \end{cases} \quad (31)$$

The linear elements ds_0 and ds of the surface, in the natural and deformed /67 state, are defined by the equations:

$$ds_0^2 = \mathcal{E}_0 d\rho_1^2 + 2\mathcal{F}_0 d\rho_1 d\rho_2 + \mathcal{G}_0 d\rho_2^2, \quad ds^2 = \mathcal{E} d\rho_1^2 + 2\mathcal{F} d\rho_1 d\rho_2 + \mathcal{G} d\rho_2^2,$$

in which $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, are calculated by the double equations:

$$(32)\quad \begin{cases} \mathcal{E} = \left(\dfrac{\partial x}{\partial \rho_1}\right)^2 + \left(\dfrac{\partial y}{\partial \rho_1}\right)^2 + \left(\dfrac{\partial z}{\partial \rho_1}\right)^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2, \\ \mathcal{F} = \dfrac{\partial x}{\partial \rho_1}\dfrac{\partial x}{\partial \rho_2} + \dfrac{\partial y}{\partial \rho_1}\dfrac{\partial y}{\partial \rho_2} + \dfrac{\partial z}{\partial \rho_1}\dfrac{\partial z}{\partial \rho_2} = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2, \\ \mathcal{G} = \left(\dfrac{\partial x}{\partial \rho_2}\right)^2 + \left(\dfrac{\partial y}{\partial \rho_2}\right)^2 + \left(\dfrac{\partial z}{\partial \rho_2}\right)^2 = \xi_2^2 + \eta_2^2 + \zeta_2^2. \end{cases}$$

and where $\mathcal{E}_0$, $\mathcal{F}_0$, $\mathcal{G}_0$ are calculated by analogous equations.

Let x', y', z' be the projections of the segment OM on the axes Mx', My', Mz' such that the coordinates of the fixed point O with respect to these axes will be $-x'$, $-y'$, $-z'$. We obtain the known equations:

$$(33)\quad \begin{cases} \xi_i - \dfrac{\partial x'}{\partial \rho_i} - qz' + ry' = 0, \quad \eta_i - \dfrac{\partial y'}{\partial \rho_i} - rx' + pz' = 0, \\ \zeta_i - \dfrac{\partial z'}{\partial \rho_i} - py' + qx' = 0, \end{cases}$$

which result in new expressions for ξ_i, η_i, ζ_i.

32. Expressions for the Variations of the Rate of Translation and Rotation of the Triad for the Deformed State. Let us assume that we impart an

infinitesimally small displacement to each of the triads of the deformed state in a continuous way. Let us call δx, δy, δz; $\delta x'$, $\delta y'$, $\delta z'$; $\delta\alpha$, $\delta\alpha'$, ...$\delta\gamma''$ the respective variations of x, y, z; x', y', z'; α, α', ... γ''. The variations $\delta\alpha$, $\delta\alpha'$, ... $\delta\gamma$ can be expressed by equations of the following type:

$$\delta\alpha = \beta\delta_K' - \gamma\delta_J', \tag{34}$$

in terms of three auxiliary quantities δ_I', δ_J', δ_K' which are the well known components along Mx', My', Mz' of the instantaneous rotation associated with the infinitesimally small displacement under consideration. The variations δx, δy, δz are the projections of the infinitely small displacement experienced by the point M along Ox, Oy, Oz. The projections $\delta' x$, $\delta' y$, $\delta' z$ of this displacement along Mx', My', Mz' can be derived immediately and have the following values:

$$\begin{cases} \delta' y = \delta x' + z'\delta_J' - y'\delta_K', \quad \delta' y = \delta y' + x'\delta_K' - z'\delta_I', \\ \delta' z = \delta z' + y'\delta_I' - x'\delta_J'. \end{cases} \tag{35}$$

Let us attempt to determine the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i experienced, respectively, by ξ_i, η_i, ζ_i, p_i, q_i, r_i. From equations (31) we /68 have

$$\delta p_i = \sum\left(\frac{\partial\beta}{\partial\rho_i}\delta\gamma + \gamma\frac{\partial\delta\beta}{\partial\rho_i}\right),$$

$$\delta q_i = \sum\left(\frac{\partial\gamma}{\partial\rho_i}\delta\alpha + \alpha\frac{\partial\delta\gamma}{\partial\rho_i}\right),$$

$$\delta r_i = \sum\left(\frac{\partial\alpha}{\partial\rho_i}\delta\beta + \beta\frac{\partial\delta\alpha}{\partial\rho_i}\right).$$

Let us replace $\delta\alpha$ by its value $\beta\delta K' - \gamma\delta J'$ and $\delta\alpha'$, ... $\delta\gamma''$ by their analogous values. We have

$$\begin{cases} \delta p_i = \dfrac{\partial\delta_I'}{\partial\rho_i} + q_i\delta_K' - r_i\delta_J', \quad \delta q_i = \dfrac{\partial\delta_J'}{\partial\rho_i} + r_i\delta_I' - p_i\delta_K' \\ \delta r_i = \dfrac{\partial\delta_K'}{\partial\rho_i} + p_i\delta_J' - q_i\delta_I'. \end{cases} \tag{36}$$

Equations (33) result in three equations as well, and the first one is:

$$\delta\xi_i = \frac{\partial\delta x'}{\partial\rho_i} + q_i\delta z' - r_i\delta y' + z'\delta q_i - y'\delta r_i;$$

Let us replace δp_i, δq_i, δr_i by their values given by equations (36). We have:

$$\begin{cases} \delta\xi_i = \eta_i\delta_K' - \zeta_i\delta_J' + \dfrac{\partial\delta' x}{\partial\rho_i} + q_i\delta' x - r_i\delta' y \\ \delta\eta_i = \zeta_i\delta_I' - \xi_i\delta_K' + \dfrac{\partial\delta' y}{\partial\rho_i} + r_i\delta' y - p_i\delta' z, \\ \delta\zeta_i = \xi_i\delta_J' - \eta_i\delta_I' + \dfrac{\partial\delta' z}{\partial\rho_i} + p_i\delta' z - q_i\delta' x, \end{cases} \tag{37}$$

in which we have introduced, in order to abbreviate the notation, the three symbols $\delta' x$, $\delta' y$, $\delta' z$ defined by equations (35).

33. Euclidian Action of Deformation for a Deformable Surface. Let us consider a function W of *two infinitesimally close positions* of the triad Mx'y'z', i.e., a function of ρ_1, ρ_2 of x, y, z, α, α'; ... γ'' and their first

derivatives with respect to ρ_1 and ρ_2. Let us use the notation of Section 31 by setting:

$$\Delta_0 = \sqrt{\mathcal{E}_0 \mathcal{G}_0 - \mathcal{F}_0^2}.$$

Let us attempt to determine the form of W such that the integral

$$\iint W \Delta_0 d\rho_1 \delta\rho_2,$$

extended over an arbitrary part of the surface (M_0) has a zero variation when /69
the ensemble of all the triads of the deformable surface, taken in its deformed state, is subjected to the *same infinitesimal, arbitrary transformation of the group of Euclidian displacements.*

More precisely, we must determine W in such a way that

$$\delta W = 0,$$

when, on the one hand, the origin M of the triad Mx'y'z' is subjected to an infinitesimally displacement whose projections δx, δy, δz on the axes Ox, Oy, Oz are

$$\begin{cases} \delta x = (a_1 + \omega_2 z - \omega_3 y)\,\delta t \\ \delta y = (a_2 + \omega_3 x - \omega_1 z)\,\delta t \\ \delta z = (a_3 + \omega_1 y - \omega_2 x)\,\delta t. \end{cases} \tag{38}$$

and, on the other hand, when this triad Mx'y'z' is subjected to an infinitesimally small rotation whose components on the axes Ox, Oy, Oz are

$$\omega_1 \delta t, \quad \omega_2 \delta t, \quad \omega_3 \delta t.$$

a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constants and δt is an infinitesimally small quantity independent of ρ_1, ρ_2.

We should note that in the present case the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i; $\delta\xi_2$, $\xi\eta_2$, $\xi\zeta_2$, δp_2, δq_2, δr_2 of the twelve expressions ξ_i, η_i, ζ_i, p_i, q_i, r_i; ξ_2, η_2, ζ_2, p_2, q_2, r_2 are zero, which follows from the well-known theory of the mobile triad. This can immediately be verified by means of equations (36) and (37) by replacing $\delta'x$, $\delta'y$, $\delta'z$; $\delta I'$, $\delta J'$, $\delta K'$ by their present values. As a result we obtain a solution of the question by taking W to be an arbitrary function of ρ_1, ρ_2 and of the twelve expressions ξ_i, η_i, ζ_i, p_i, q_i, r_i,; ξ_2, η_2, ζ_2, p_2, q_2, r_2. We will now show that we can obtain the general solution(2) of the problem we have posed ourselves in this way.

In this regard we should note that the relationships (31) allow us, by

(2) In the following we will assume that the *deformable surface can carry out all possible deformations* and that, consequently, *the deformed state can be chosen completely arbitrarily.* We are expressing this fact when we say that *the surface is free.*

means of well known equations, to express the first derivatives of the nine cosines α, α' ... γ'' with respect to ρ_1 and ρ_2 by means of these cosines and of p_1, q_1, r_1, p_2, q_2, r_2. On the other hand, let us note that equations (30) make it possible to imagine that we can express the nine cosines α, α', ... γ'' by means of ξ_1, η_1, ζ_1 and the first derivatives of x, y, z with respect to ρ_1, or in terms of ξ_2, η_2, ζ_2 and the first derivatives of x, y, z with respect to ρ_2. In the latter case, it is therefore useless to make a hypothesis regarding the method of solution, because it is clear that we will not obtain a more general form than the one we started from, assuming finally that the function W which we are looking for is an arbitrary function of ρ_1, ρ_2 and of x, y, z and their first derivatives with respect to ρ_1, ρ_2, and finally of ξ_1, η_1, ζ_1, p_1, q_1, /70
r_1,; ξ_2, η_2, ζ_2, p_2, q_2, r_2. We indicate this fact by writing:

$$W = W\left(\rho_1, \rho_2, x, y, z, \frac{\partial x}{\partial \rho_1}, \frac{\partial y}{\partial \rho_1}, \frac{\partial z}{\partial \rho_1}, \frac{\partial x}{\partial \rho_2}, \ldots, \xi_1, \eta_1, \zeta_1, \xi_2 \ldots, p_1, q_1, r_1, p_2 \ldots\right),$$

The variations $\delta\xi_1$, ... δr_1, $\delta\xi_2$, ... δr_2 are zero in the present case, as we noted a short time ago, and we need only state that the new form of W is

$$\frac{\partial W}{\partial x}\delta x + \frac{\partial W}{\partial y}\delta y + \frac{\partial W}{\partial z}\delta z + \sum_i \left(\frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}}\delta\frac{\partial x}{\partial \rho_i} + \frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}}\delta\frac{\partial y}{\partial \rho_i} + \frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}}\delta\frac{\partial z}{\partial \rho_i}\right) = 0,$$

due to equations (38) and no matter what a_1, a_2, a_3, ω_1, ω_2, ω_3 may be.

Let us replace δx, δy, δz by their values (38) and $\delta \frac{\partial x}{\partial \rho_i}$, $\delta \frac{\partial y}{\partial \rho_i}$, $\delta \frac{\partial z}{\partial \rho_i}$ by values that are obtained by differentiation. Let us set the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3 equal to zero. We obtain the six following conditions:

$$\frac{\partial W}{\partial x} = 0, \quad \frac{\partial W}{\partial y} = 0, \quad \frac{\partial W}{\partial z} = 0,$$

$$\sum_i \left(\frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}}\frac{\partial z}{\partial \rho_i} - \frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}}\frac{\partial y}{\partial \rho_i}\right) = 0, \quad \sum_i \left(\frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}}\frac{\partial x}{\partial \rho_i} - \frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}}\frac{\partial z}{\partial \rho_i}\right) = 0,$$

$$\sum_i \left(\frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}}\frac{\partial y}{\partial \rho_i} - \frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}}\frac{\partial x}{\partial \rho_i}\right) = 0,$$

which are identities if we assume that the expressions which occur in W have been reduced to a minimum.

As could have easily been predicted, the first three show us that W is independent of x, y, z. The latter three express the fact that W only depends on the first derivatives of x, y, z through the intermediary of the quantities $\mathcal{E}, \mathcal{F}, \mathcal{G}$ defined by equations (32). We finally obtain the result that the *desired function W has the unusual form:*

$$W(\rho_1, \rho_2, \xi_1, \eta_1, \zeta_1; \xi_2, \eta_2, \zeta_2; p_1, q_1, r_1; p_2, q_2, r_2),$$

which is analogous to that encountered previously for the deformable line.

Let us call Δ the quantity analogous to Δ_0 and defined by the equation:

$$\Delta = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}.$$

If we multiply W by the area element $d\sigma_0 = \Delta_0 d\rho_1 d\rho_2$ of the surface (M_0), the product $W\Delta_0 d\rho_1 d\rho_2$ is, for the group of Euclidian displacements, an invariant analogous to the surface element of the surface (M). In the same way, /71 the value of the integral

$$\iint_{C_0} \frac{\Delta}{\Delta_0} \Delta_0 d\rho_1 d\rho_2 = \iint_{C_0} \Delta d\rho_1 d\rho_2,$$

taken over the interior of a contour C_0 of the surface (M_0), or the corresponding contour C of the surface (M), represents *the area* of the region limited by C on (M). In the same way, if we think of the concept of action for the transformation from the natural state (M_0) to the deformed state (M), we may relate the function W to the elements which define the deformable surface. We will say that the integral:

$$\iint_{C_0} W\Delta_0 d\rho_1 d\rho_2$$

is the *deformation action* along the deformed surface, in the interior of the contour C.

On the other hand, we will say that W is the *density* of the deformation action at a point of the deformed surface referred to a unit of area of the non-deformed surface. $W \frac{\Delta_0}{\Delta}$ will be this density at a point referred to a unit of area of the deformed surface.

34. <u>External Force and Moment; External Stress and Moment of Deformation; Stress and Moment of Deformation at a Point of a Deformed Surface</u>. Let us consider an *arbitrary* variation of the deformation action in the interior of a contour C of the surface (M), i.e.:

$$\delta \iint_{C_0} W\Delta_0 d\rho_1 d\rho_2 = \iint_{C_0} \sum_i \left(\frac{\partial W}{\partial \xi_i} \delta\xi_i + \frac{\partial W}{\partial \eta_i} \delta\eta_i + \frac{\partial W}{\partial \zeta_i} \delta\zeta_i + \frac{\partial W}{\partial p_i} \delta p_i + \frac{\partial W}{\partial q_i} \delta q_i + \frac{\partial W}{\partial r_i} \delta r_i \right) \Delta_0 d\rho_1 d\rho_2.$$

We may write this as follows, due to equations (36) and (37) of Section 32:

$$\delta \iint_{C_0} W\Delta_0 d\rho_1 d\rho_2 = \iint_{C_0} \sum_i \left[\frac{\partial W}{\partial \xi_i} \left(\eta_i \delta z' - \zeta_i \delta y' + \frac{\partial \delta' x}{\partial \rho_i} + q_i \delta' z - r_i \delta' y \right) \right.$$
$$+ \frac{\partial W}{\partial \eta_i} \left(\zeta_i \delta x' - \xi_i \delta z' + \frac{\partial \delta' y}{\partial \rho_i} + r_i \delta' x - p_i \delta' z \right)$$
$$+ \frac{\partial W}{\partial \zeta_i} \left(\xi_i \delta y' - \eta_i \delta x' + \frac{\partial \delta' z}{\partial \rho_i} + p_i \delta' y - q_i \delta' x \right)$$

$$+\frac{\partial W}{\partial p_i}\left(\frac{\partial \delta_1'}{\partial \rho_i}+q_i\delta_K'-r_i\delta_J'\right)+\frac{\partial W}{\partial q_i}\left(\frac{\partial \delta_J'}{\partial \rho_i}+r_i\delta_I'-p_i\delta_K'\right)$$
$$+\frac{\partial W}{\partial r_i}\left(\frac{\partial \delta_K'}{\partial \rho_i}+p_i\delta_J'-q_i\delta_I'\right)\Bigg]\Delta_0 d\rho_1 d\rho_2.$$

Let us apply Green's formula to the terms which explicitly contain a derivative with respect to ρ_1 or with respect to ρ_2. We have: /72

$$\delta\iint_{C_0} W\Delta_0 d\rho_1 d\rho_2=\int_{C_0}\Bigg[\left(\frac{\partial W}{\partial \xi_1}\delta' x+\frac{\partial W}{\partial \eta_1}\delta' y+\frac{\partial W}{\partial \zeta_1}\delta' z+\frac{\partial W}{\partial p_1}\delta_I'+\frac{\partial W}{\partial q_1}\delta_J'+\frac{\partial W}{\partial r_1}\delta_K'\right)\Delta_0 d\rho_1$$
$$-\left(\frac{\partial W}{\partial \xi_2}\delta' x+\frac{\partial W}{\partial \eta_2}\delta' y+\frac{\partial W}{\partial \zeta_2}\delta' z+\frac{\partial W}{\partial p_2}\delta_I'+\frac{\partial W}{\partial q_2}\delta_J'+\frac{\partial W}{\partial r_2}\delta_K'\right)\Bigg]\Delta_0 d\rho_1$$
$$-\iint_{C_0}\Bigg\{\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0\frac{\partial W}{\partial \xi_i}\right)+q_i\frac{\partial W}{\partial \zeta_i}-r_i\frac{\partial W}{\partial \eta_i}\right]\delta' x$$
$$+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0\frac{\partial W}{\partial \eta_i}\right)+r_i\frac{\partial W}{\partial \xi_i}-p_i\frac{\partial W}{\partial \zeta_i}\right]\delta' y$$
$$+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0\frac{\partial W}{\partial \zeta_i}\right)+p_i\frac{\partial W}{\partial \eta_i}-q_i\frac{\partial W}{\partial \xi_i}\right]\delta' z$$
$$+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0\frac{\partial W}{\partial p_i}\right)+q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial \zeta_i}-\zeta_i\frac{\partial W}{\partial \eta_i}\right]\delta_I'$$
$$+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0\frac{\partial W}{\partial q_i}\right)+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\zeta_i\frac{\partial W}{\partial \xi_i}-\xi_i\frac{\partial W}{\partial \zeta_i}\right]\delta_J'$$
$$+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0\frac{\partial W}{\partial r_i}\right)+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial \eta_i}-\eta_i\frac{\partial W}{\partial \xi_i}\right]\delta_K'\Bigg\}\Delta_0 d\rho_1 d\rho_2$$

The curvilinear integral which appears in the preceding formula must be defined more precisely by stating the direction in which it is traversed. As is known, this direction is defined by the rotation which brings the po itive part of the curve (ρ_2) -- i.e., the part which corresponds to the direction in which the coordinate ρ_1 increases and which varies along this curve -- to the side of the positive part of the curve (ρ_1). It is therefore possible to give a more precise meaning to this curvilinear integral, following the example of Beltrami, by giving it the form resulting from the application of the equations:

$$\iint_{C_0}\frac{\partial\varphi}{\partial \rho_1}d\rho_1 d\rho_2=\int_{C_0}\left(\mathcal{E}_0\frac{\partial \rho_1}{\partial n_0}+\mathcal{F}_0\frac{\partial \rho_2}{\partial n_0}\right)\frac{\varphi}{\Delta_0}ds_0,$$
$$\iint_{C_0}\frac{\partial\varphi}{\partial \rho_2}d\rho_1 d\rho_2=\int_{C_0}\left(\mathcal{F}_0\frac{\partial \rho_1}{\partial n_0}+\mathcal{G}_0\frac{\partial \rho_2}{\partial n_0}\right)\frac{\varphi}{\Delta_0}ds_0,$$

where ϕ is a function of ρ_1, ρ_2, where ds_0 is the absolute value of the linear element of the curve (C_0) and where n_0 indicates the direction of the normal to the contour (C_0) drawn in the plane tangent to the surface (M_0) and directed towards the exterior of the region delimited by this contour. In order to obtain the new form of the curvilinear integral, it is sufficient to replace /73
$d\rho_1$ and $d\rho_2$ under the integration sign in the first form obtained by the

following values:

$$-\left(\bar{\sigma}_0 \frac{\partial \rho_1}{\partial n_0} + \zeta_0 \frac{\partial \rho_2}{\partial n_0}\right) \frac{ds_0}{\Delta_0}, \qquad \left(\zeta_0 \frac{\partial \rho_1}{\partial n_0} + \bar{\sigma}_0 \frac{\partial \rho_2}{\partial n_0}\right) \frac{ds_0}{\Delta_0}.$$

If λ_0', μ_0', ν_0' are the direction cosines of the external normal of the contour C_0 with respect to the triad $M_0x_0'y_0'z_0'$ just considered, the two preceding expressions, which must be substituted for $d\rho_1$ and $d\rho_2$, can be given in the following forms (3):

$$-(\lambda_0'\xi_2^{(0)} + \mu_0'\eta_2^{(0)} + \nu_0'\zeta_2^{(0)}) \frac{ds_0}{\Delta_0}, \; (\lambda_0'\xi_1^{(0)} + \mu_0'\eta_1^{(0)} + \nu_0'\zeta_1^{(0)}) \frac{ds_0}{\Delta_0}, \tag{39}$$

This is because equations:

$$\lambda_0' = \xi_1^{(0)} \frac{\partial \rho_1}{\partial n_0} + \xi_2^{(0)} \frac{\partial \rho_2}{\partial n_0}, \; \mu_0' = \eta_1^{(0)} \frac{\partial \rho_1}{\partial n_0} + \eta_2^{(0)} \frac{\partial \rho_2}{\partial n_0}, \; \nu_0' = \zeta_1^{(0)} \frac{\partial \rho_1}{\partial n_0} + \zeta_2^{(0)} \frac{\partial \rho_2}{\partial n_0}.$$

determine λ_0', μ_0', ν_0'.

If ds_0 is the absolute value of the arc element of the contour C_0 drawn on the surface (M_0), let us set:

$$F_0' = \Delta_0 \left(\frac{\partial W}{\partial \xi_1} \frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial \xi_2} \frac{d\rho_1}{ds_0}\right), \qquad G_0' = \Delta_0 \left(\frac{\partial W}{\partial \eta_1} \frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial \eta_2} \frac{d\rho_1}{ds_0}\right),$$

$$H_0' = \Delta_0 \left(\frac{\partial W}{\partial \zeta_1} \frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial \zeta_2} \frac{d\rho_1}{ds_0}\right);$$

$$I_0' = \Delta_0 \left(\frac{\partial W}{\partial p_1} \frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial p_2} \frac{d\rho_1}{ds_0}\right), \qquad J_0' = \Delta_0 \left(\frac{\partial W}{\partial q_1} \frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial q_2} \frac{d\rho_1}{ds_0}\right),$$

$$K_0' = \Delta_0 \left(\frac{\partial W}{\partial r_1} \frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial r_2} \frac{d\rho_1}{ds_0}\right),$$

in which the signs of $d\rho_1$ and $d\rho_2$ are determined by the direction in which the curvilinear integral indicated above is traversed. In addition, the values of $d\rho_1$ and $d\rho_2$ are those just indicated and which contain the external normal to the contour C_0 located in the plane tangent to (M_0). In addition, let us set:

$$\sum_i \left[\frac{1}{\Delta_0} \frac{\partial}{\partial \rho_i} \left(\Delta_0 \frac{\partial W}{\partial \xi_i}\right) + q_i \frac{\partial W}{\partial \zeta_i} - r_i \frac{\partial W}{\partial \eta_i}\right] = X_0',$$

$$\sum_i \left[\frac{1}{\Delta_0} \frac{\partial}{\partial \rho_i} \left(\Delta_0 \frac{\partial W}{\partial \eta_i}\right) + r_i \frac{\partial W}{\partial \xi_i} - p_i \frac{\partial W}{\partial \zeta_i}\right] = Y_0'$$

$$\sum_i \left[\frac{1}{\Delta_0} \frac{\partial}{\partial \rho_i} \left(\Delta_0 \frac{\partial W}{\partial \zeta_i}\right) + p_i \frac{\partial W}{\partial \eta_i} - q_i \frac{\partial W}{\partial \xi_i}\right] = Z_0'$$

$$\sum_i \left[\frac{1}{\Delta_0} \frac{\partial}{\partial \rho_i} \left(\Delta_0 \frac{\partial W}{\partial p_i}\right) + q_i \frac{\partial W}{\partial r_i} - r_i \frac{\partial W}{\partial q_i} + \eta_i \frac{\partial W}{\partial \zeta_i} - \zeta_i \frac{\partial W}{\partial \eta_i}\right] = L_0'$$

$$\sum_i \left[\frac{1}{\Delta_0} \frac{\partial}{\partial \rho_i} \left(\Delta_0 \frac{\partial W}{\partial q_i}\right) + r_i \frac{\partial W}{\partial p_i} - p_i \frac{\partial W}{\partial r_i} + \zeta_i \frac{\partial W}{\partial \xi_i} - \xi_i \frac{\partial W}{\partial \zeta_i}\right] = M_0'$$

$$\sum_i \left[\frac{1}{\Delta_0} \frac{\partial}{\partial \rho_i} \left(\Delta_0 \frac{\partial W}{\partial r_i}\right) + p_i \frac{\partial W}{\partial q_i} - q_i \frac{\partial W}{\partial p_i} + \xi_i \frac{\partial W}{\partial \eta_i} - \eta_i \frac{\partial W}{\partial \xi_i}\right] = N_0',$$

(3) Naturally, analogous equations are obtained by introducing the direction cosines λ', μ', ν', with respect to the triad $Mx'y'z'$, of the external normal to the contour C which corresponds to C_0.

We have: /74

$$\delta \iint_{c_0} W\Delta_0 d\rho_1 d\rho_2 = \int_{c_0} (F_0'\delta' x + G_0'\delta' y + H_0'\delta' z + I_0'\delta\iota' + J_0'\delta\iota' + K_0'\delta\kappa') ds_0$$
$$- \iint_{c_0} (X_0'\delta' x + Y_0'\delta' y + Z_0'\delta' z + L_0'\delta\iota' + M_0'\delta\iota' + N_0'\delta\kappa')\Delta_0 d\rho_1 d\rho_2.$$

We will first consider the double integral which appears in the expression for $\delta \iint_{c_0} W\Delta_0 d\rho_1 d\rho_2$ and which we will call *external force and external moment at the point* M *referred to the unit of area of the non-deformed surface*. The segments have their origin at M and their projections along the axes Mx', My', Mz' are X_0', Y_0', Z_0' and L_0', M_0', N_0', respectively. We will then consider the curvilinear integral which appears in $\delta \iint_{c_0} W\Delta_0 d\rho_1 d\rho_2$ and we will call *external stress and external moment of deformation at the point* M *of the contour* C *of the deformed surface referred to the unit of length of the contour* C_0 the segments emanating from the point M and whose projections along the axes Mx', My', Mz' are $-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$, respectively.

As we have seen, these six latter quantities at a specific point M of C only depend on the direction of the exterior normal to the curve C_0 leading to the point M_0 in the plane tangent to (M_0). They remain invariable if, when varying the region of (M_0) under consideration, this direction of the external normal does not change, and they change sign if this direction is replaced by the opposite direction.

Let us assume that, in the interior of the deformed surface limited by the contour C, a line Σ is drawn which circumscribes either alone or with a part of the contour C a part (A) of the surface. Let (B) designate the remainder of the surface outside the part (A). Let Σ_0 be the curve of (M_0) which corresponds to the curve Σ of (M). Let (A_0) and (B_0) be the regions of (M_0) which correspond to the regions (A) and (B) of (M). In our thinking let us separate the two parts (A) and (B). We may regard the two segments $(-F_0', -G_0', -H_0')$ and $(-I_0', -J_0', -K_0')$, determined for the point M and for the direction of the normal directed towards Σ_0 in the plane tangent to (M_0) and exterior to (A_0), as the external stress and moment of deformation at the point M of the contour Σ of the region (A). In this same way, we may regard the two segments (F_0', G_0', H_0') and (I_0', J_0', K_0') as the external stress and moment of deformation at the point M of the contour Σ of the region (B). Due to this remark, we will say that $-F_0'$, $-G_0'$, $-H_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$ are the components along the axes Mx', My', Mz' *of the stress and moment of deformation exerted at* M *on the portion* (A) *of the surface* (M) and that F_0', G_0', H_0', I_0', J_0', K_0' are the components along the axes Mx', My', Mz' *of the stress and moment of deformation exerted at* M *by the portion* (B) *of the surface* (M).

The observation made at the end of Section 9 regarding the replacement of the triad Mx'y'z' by a triad which is invariably associated with it can be repeated without modification here.

35. Various Requirements Placed on the Stress and Moment of Deformation. /75

Let us set:

$$A_i' = \Delta_0 \frac{\partial W}{\partial \xi_i}, \quad B_i' = \Delta_0 \frac{\partial W}{\partial \eta_i}, \quad C_i' = \Delta_0 \frac{\partial W}{\partial \zeta_i},$$
$$P_i' = \Delta_0 \frac{\partial W}{\partial p_i}, \quad Q_i' = \Delta_0 \frac{\partial W}{\partial q_i}, \quad R_i' = \Delta_0 \frac{\partial W}{\partial r_i};$$

$\frac{1}{\sqrt{\mathcal{G}_0}} A_1'$, $\frac{1}{\sqrt{\mathcal{G}_0}} B_1'$, $\frac{1}{\sqrt{\mathcal{G}_0}} C_1'$ and $\frac{1}{\sqrt{\mathcal{G}_0}} P_1'$, $\frac{1}{\sqrt{\mathcal{G}_0}} Q_1'$, $\frac{1}{\sqrt{\mathcal{G}_0}} R_1'$ represent the projections along Mx', My', Mz' of the stress and moment of deformation, respectively, which are exerted at the point M of a curve which has the same tangent as ρ_1 = const. This stress and moment of deformation are *referred to the unit of length of the non-deformed contour*. Regarding ρ_2 = const., the stress and moment of deformation have the following projections $\frac{1}{\sqrt{\mathcal{E}_0}} A_2'$, $\frac{1}{\sqrt{\mathcal{E}_0}} B_2'$, $\frac{1}{\sqrt{\mathcal{E}_0}} C_2'$ and $\frac{1}{\sqrt{\mathcal{E}_0}} P_2'$, $\frac{1}{\sqrt{\mathcal{E}_0}} Q_2'$, $\frac{1}{\sqrt{\mathcal{E}_0}} R_2'$.

The new stresses and new moments of deformation which we have just defined are related to the elements introduced in the preceding section by the following relationships:

$$\sum_i \left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i'\right) = \Delta_0 X_0', \qquad F_0' = A_1' \frac{d\rho_2}{ds_0} - A_2' \frac{d\rho_1}{ds_0},$$
$$\sum_i \left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i'\right) = \Delta_0 Y_0', \qquad G_0' = B_1' \frac{d\rho_2}{ds_0} - B_2' \frac{d\rho_1}{ds_0},$$
$$\sum_i \left(\frac{\partial C_i'}{\partial \rho_i} + p_i B_i' - q_i A_i'\right) = \Delta_0 Z_0', \qquad H_0' = C_1' \frac{d\rho_2}{ds_0} - C_2' \frac{d\rho_1}{ds_0},$$
$$\sum_i \left(\frac{\partial P_i'}{\partial \rho_i} + q_i R_i' - r_i Q_i' + \eta_i C_i' - \zeta_i B_i'\right) = \Delta_0 L_0', \quad I_0' = P_1' \frac{d\rho_2}{ds_0} - P_2' \frac{d\rho_1}{ds_0},$$
$$\sum_i \left(\frac{\partial Q_i'}{\partial \rho_i} + r_i P_i' - p_i R_i' + \zeta_i A_i' - \xi_i C_i'\right) = \Delta_0 M_0', \quad J_0' = Q_1' \frac{d\rho_2}{ds_0} - Q_2' \frac{d\rho_1}{ds_0},$$
$$\sum_i \left(\frac{\partial R_i'}{\partial \rho_i} + p_i Q_i' - q_i P_i' + \xi_i B_i' - \eta_i A_i'\right) = \Delta_0 N_0', \quad K_0' = R_1' \frac{d\rho_2}{ds_0} - R_2' \frac{d\rho_1}{ds_0},$$

where, if desired, $d\rho_1$ and $d\rho_2$ may be replaced by the values (39) in the equations on the right.

Let us consider the transformation of these relationships in a way which is independent of the values of the quantities contained in them, which were calculated by means of W. These relationships hold for the segments attached to the point M and which we have named. Instead of defining these segments by their projections along Mx', My', Mz', we may also define them by their projections along other axes. These latter projections will be related by relationships which are the transforms of the preceding ones.

The transformed relationships can immediately be obtained if one notes that the basic equations have simple and obvious interpretations[4] if one associates axes parallel to the moving axes through the point O with the moving axes. /76

1. Let us first consider fixed axes Ox, Oy, Oz. Let X_0, Y_0, Z_0 and L_0, M_0, N_0 be the projections of the external force and external moment on these axes at an arbitrary point M of the deformed medium. Let F_0, G_0, H_0 and I_0, J_0, K_0 be the projections of the stress and moment of deformation taken in the direction ($d\rho_1$, $d\rho_2$) of the tangent to a curve C and referred to the unit of length of the non-deformed curve C_0, which was defined previously. Let A_i, B_i, C_i and P_i, Q_i, R_i be the projections of the stress (A_i', B_i', C_i') and the moment of deformation (P_i', Q_i', R_i'). The transforms of the preceding relationships are obviously:

$$\begin{aligned}
&\frac{\partial A_1}{\partial \rho_1}+\frac{\partial A_2}{\partial \rho_2}=\Delta_0 X_0, && F_0=A_1\frac{d\rho_2}{ds_0}-A_2\frac{d\rho_1}{ds_0},\\
&\frac{\partial B_1}{\partial \rho_1}+\frac{\partial B_2}{\partial \rho_2}=\Delta_0 Y_0, && G_0=B_1\frac{d\rho_2}{ds_0}-B_2\frac{d\rho_1}{ds_0},\\
&\frac{\partial C_1}{\partial \rho_1}+\frac{\partial C_2}{\partial \rho_2}=\Delta_0 Z_0, && H_0=C_1\frac{d\rho_2}{ds_0}-C_2\frac{d\rho_1}{ds_0},\\
&\frac{\partial P_1}{\partial \rho_1}+\frac{\partial P_2}{\partial \rho_2}+C_1\frac{\partial y}{\partial \rho_1}+C_2\frac{\partial y}{\partial \rho_2}-B_1\frac{\partial z}{\partial \rho_1}-B_2\frac{\partial z}{\partial \rho_2}=\Delta_0 L_0, && I_0=P_1\frac{d\rho_2}{ds_0}-P_2\frac{d\rho_1}{ds_0},\\
&\frac{\partial Q_1}{\partial \rho_1}+\frac{\partial Q_2}{\partial \rho_2}+A_1\frac{\partial z}{\partial \rho_1}+A_2\frac{\partial z}{\partial \rho_2}-C_1\frac{\partial x}{\partial \rho_1}-C_2\frac{\partial x}{\partial \rho_2}=\Delta_0 M_0, && J_0=Q_1\frac{d\rho_2}{ds_0}-Q_2\frac{d\rho_1}{ds_0},\\
&\frac{\partial R_1}{\partial \rho_1}+\frac{\partial R_2}{\partial \rho_2}+B_1\frac{\partial x}{\partial \rho_1}+B_2\frac{\partial x}{\partial \rho_2}-A_1\frac{\partial y}{\partial \rho_1}-A_2\frac{\partial y}{\partial \rho_2}=\Delta_0 N_0, && K_0=R_1\frac{d\rho_2}{ds_0}-R_2\frac{d\rho_1}{ds_0};
\end{aligned}$$

$\frac{d\rho_1}{ds_0}$ and $\frac{d\rho_2}{ds_0}$ must obviously be replaced by, respectively

$$-\frac{1}{\Delta_0}\left(\lambda_0\frac{\partial x_0}{\partial \rho_2}+\mu_0\frac{\partial y_0}{\partial \rho_2}+\nu_0\frac{\partial z_0}{\partial \rho_2}\right),\qquad \frac{1}{\Delta_0}\left(\lambda_0\frac{\partial x_0}{\partial \rho_1}+\mu_0\frac{\partial y_0}{\partial \rho_1}+\nu_0\frac{\partial z_0}{\partial \rho_1}\right),$$

and $\frac{d\rho_1}{ds}$, $\frac{d\rho_2}{ds}$ by

$$-\frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial \rho_2}+\mu\frac{\partial y}{\partial \rho_2}+\nu\frac{\partial z}{\partial \rho_2}\right),\qquad \frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial \rho_1}+\mu\frac{\partial y}{\partial \rho_1}+\nu\frac{\partial z}{\partial \rho_1}\right),$$

where λ_0, μ_0, ν_0 are the direction cosines with respect to fixed axes of the external normal to C_0. λ, μ, ν are those of the external normal to C.

In particular these equations result in the equations for the *infinitesimally small deformation of a plane surface* which was used by Lord Kelvin and Tait[5].

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(4) An interesting interpretation is analogous to the one given by Varignon in the case of statics and by P. Saint-Guilhem in the case of dynamics.

(5) *Treatise on Natural Philosophy*, Volume I, Part II, Section 644, p. 186-188.

2. The equations with respect to the fixed axes Ox, Oy, Oz can be given a new form. We can express the nine cosines α, α', ... γ'' in terms of the three /77 auxiliary quantities. Let λ_1, λ_2, λ_3 be the three auxiliary quantities. Let us set:

$$\sum \gamma d\beta = -\sum \beta d\gamma = \varpi_1' d\lambda_1 + \varpi_2' d\lambda_2 + \varpi_3' d\lambda_3,$$
$$\sum \alpha d\gamma = -\sum \gamma d\alpha = \chi_1' d\lambda_1 + \chi_2' d\lambda_2 + \chi_3' d\lambda_3,$$
$$\sum \beta d\alpha = -\sum \alpha d\beta = \sigma_1' d\lambda_1 + \sigma_2' d\lambda_2 + \sigma_3' d\lambda_3.$$

The functions ϖ_i', χ_i', σ_i' of λ_1, λ_2, λ_3 defined in this way satisfy the relationships:

$$\frac{\partial \varpi_j'}{\partial \lambda_i} - \frac{\partial \varpi_i'}{\partial \lambda_j} + \chi_i' \sigma_j' - \chi_j' \sigma_i' = 0,$$
$$\frac{\partial \chi_j'}{\partial \lambda_i} - \frac{\partial \chi_i'}{\partial \lambda_j} + \sigma_i' \varpi_j' - \sigma_j' \varpi_i' = 0, \quad (i, j = 1, 2, 3),$$
$$\frac{\partial \sigma_j'}{\partial \lambda_i} - \frac{\partial \sigma_i'}{\partial \lambda_j} + \varpi_i' \chi_j' - \varpi_j' \chi_i' = 0,$$

and we have:

$$p_i = \varpi_1' \frac{\partial \lambda_1}{\partial p_i} + \varpi_2' \frac{\partial \lambda_2}{\partial p_i} + \varpi_3' \frac{\partial \lambda_3}{\partial p_i},$$
$$q_i = \chi_1' \frac{\partial \lambda_1}{\partial p_i} + \chi_2' \frac{\partial \lambda_2}{\partial p_i} + \chi_3' \frac{\partial \lambda_3}{\partial p_i},$$
$$r_i = \sigma_1' \frac{\partial \lambda_1}{\partial p_i} + \sigma_2' \frac{\partial \lambda_2}{\partial p_i} + \sigma_3' \frac{\partial \lambda_3}{\partial p_i}.$$

Let us call ϖ_i, χ_i, σ_i the projections on fixed axes Ox, Oy, Oz of the segment whose projections on the axes Mx', My', Mz' are ϖ_i', χ_i', σ_i'. We will have:

$$\sum \alpha' d\alpha'' = -\sum \alpha'' d\alpha' = \varpi_1 d\lambda_1 + \varpi_2 d\lambda_2 + \varpi_3 d\lambda_3,$$
$$\sum \alpha'' d\alpha = -\sum \alpha d\alpha'' = \chi_1 d\lambda_1 + \chi_2 d\lambda_2 + \chi_3 d\lambda_3,$$
$$\sum \alpha d\alpha' = -\sum \alpha' d\alpha = \sigma_1 d\lambda_1 + \sigma_2 d\lambda_2 + \sigma_3 d\lambda_3.$$

from which it follows that (6) the new functions ω_i, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy the relationships:

$$\frac{\partial \varpi_j}{\partial \lambda_i} - \frac{\partial \varpi_i}{\partial \lambda_j} = \chi_i \sigma_j - \chi_j \sigma_i,$$
$$\frac{\partial \chi_j}{\partial \lambda_i} - \frac{\partial \chi_i}{\partial \lambda_j} = \sigma_i \varpi_j - \sigma_j \varpi_i, \quad (i, j = 1, 2, 3),$$
$$\frac{\partial \sigma_j}{\partial \lambda_i} - \frac{\partial \sigma_i}{\partial \lambda_j} = \varpi_i \chi_j - \varpi_j \chi_i.$$

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(6) These equations can be used to directly define the functions ϖ_i, χ_i, σ_i and can be substituted for

$$\varpi_i = \alpha \varpi_i' + \beta \chi_i' + \gamma \sigma_i'$$
$$\chi_i = \alpha' \varpi_i' + \beta' \chi_i' + \gamma' \sigma_i' \quad (i = 1, 2, 3)$$
$$\sigma_i = \alpha'' \varpi_i' + \beta'' \chi_i' + \gamma'' \sigma_i'.$$

We would also like to remark that if we designate by $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ the variations of λ_1, λ_2, λ_3 which correspond to the variations $\delta\alpha$, $\delta\alpha'$, ... $\delta\gamma''$ of α, α', ... γ'', we will have: /78

$$
\begin{aligned}
\delta\imath' &= \varpi_1'\delta\lambda_1 + \varpi_2'\delta\lambda_2 + \varpi_3'\delta\lambda_3,\\
\delta\jmath' &= \chi_1'\delta\lambda_1 + \chi_2'\delta\lambda_2 + \chi_3'\delta\lambda_3,\\
\delta\kappa' &= \sigma_1'\delta\lambda_1 + \sigma_2'\delta\lambda_2 + \sigma_3'\delta\lambda_3,\\
\delta\imath &= \alpha\delta\imath' + \beta\delta\jmath' + \gamma\delta\kappa' = \varpi_1\delta\lambda_1 + \varpi_2\delta\lambda_2 + \varpi_3\delta\lambda_3,\\
\delta\jmath &= \alpha'\delta\imath' + \beta'\delta\jmath' + \gamma'\delta\kappa' = \chi_1\delta\lambda_1 + \chi_2\delta\lambda_2 + \chi_3\delta\lambda_3,\\
\delta\kappa &= \alpha''\delta\imath' + \beta''\delta\jmath' + \gamma''\delta\kappa' = \sigma_1\delta\lambda_1 + \sigma_2\delta\lambda_2 + \sigma_3\delta\lambda_3,
\end{aligned}
$$

where δI, δJ, δK are the projections on fixed axes of the segment whose projections on Mx', My', Mz' are δ_I', δ_J', δ_K'. This will be of use to us later.

Let us now set:

$$
\begin{aligned}
\mathfrak{I}_0 &= \varpi_1'I_0' + \chi_1'J_0' + \sigma_1'K_0' = \varpi_1 I_0 + \chi_1 J_0 + \sigma_1 K_0,\\
\mathfrak{J}_0 &= \varpi_2'I_0' + \chi_2'J_0' + \sigma_2'K_0' = \varpi_2 I_0 + \chi_2 J_0 + \sigma_2 K_0,\\
\mathfrak{K}_0 &= \varpi_3'I_0' + \chi_3'J_0' + \sigma_3'K_0' = \varpi_3 I_0 + \chi_3 J_0 + \sigma_3 K_0,\\
\mathfrak{L}_0 &= \varpi_1'L_0' + \chi_1'M_0' + \sigma_1'N_0' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0,\\
\mathfrak{M}_0 &= \varpi_2'L_0' + \chi_2'M_0' + \sigma_2'N_0' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0,\\
\mathfrak{N}_0 &= \varpi_3'L_0' + \chi_3'M_0' + \sigma_3'N_0' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0.
\end{aligned}
$$

Also, let us introduce the following notation:

$$
\begin{aligned}
\Pi_i &= \varpi_1'P_i' + \chi_1'Q_i' + \sigma_1'R_i' = \varpi_1 P_i + \chi_1 Q_i + \sigma_1 R_i,\\
X_i &= \varpi_2'P_i' + \chi_2'Q_i' + \sigma_2'R_i' = \varpi_2 P_i + \chi_2 Q_i + \sigma_2 R_i,\\
\Sigma_i &= \varpi_3'P_i' + \chi_3'Q_i' + \sigma_3'R_i' = \varpi_3 P_i + \chi_3 Q_i + \sigma_3 R_i;
\end{aligned}
$$

Instead of the last system where P_i', Q_i', R_i' or P_i, Q_i, R_i appear, we will have the following:

$$
\begin{aligned}
\mathfrak{L}_0 = \sum_i \Big[&\frac{\partial \Pi_i}{\partial \rho_i} - A_i'(\sigma_1'\eta_i - \chi_1'\zeta_i) - B_i'(\varpi_1'\zeta_i - \sigma_1'\xi_i) - C_i'(\chi_1'\xi_i - \varpi_1'\eta_i)\\
&- P_i'\Big(\frac{\partial \varpi_1'}{\partial \rho_i} + q_i\sigma_1' - r_i\chi_1'\Big) - Q_i'\Big(\frac{\partial \chi_1'}{\partial \rho_i} + r_i\varpi_1' - p_i\sigma_1'\Big)\\
&- R_i'\Big(\frac{\partial \sigma_1'}{\partial \rho_i} + p_i\chi_1' - q_i\varpi_1'\Big)\Big]
\end{aligned}
$$

and two similar equations. If we note that the functions ξ_i, η_i, ζ_i, p_i, q_i, r_i, of λ_1, λ_2, λ_3 $\frac{\partial\lambda_1}{\partial\rho_i}$, $\frac{\partial\lambda_2}{\partial\rho_i}$, $\frac{\partial\lambda_3}{\partial\rho_i}$ give rise to the equations:

$$
\begin{aligned}
&\frac{\partial \xi_i}{\partial \lambda_j} + \chi_j'\zeta_i - \sigma_j'\eta_i = 0, &&\frac{\partial p_i}{\partial \lambda_j} = \frac{\partial \varpi_j'}{\partial \rho_i} + q_i\sigma_j' - r_i\chi_j',\\
&\frac{\partial \eta_i}{\partial \lambda_j} + \sigma_j'\xi_i - \varpi_j'\zeta_i = 0, &&\frac{\partial q_i}{\partial \lambda_j} = \frac{\partial \chi_j'}{\partial \rho_i} + r_i\varpi_j' - p_i\sigma_j',\\
&\frac{\partial \zeta_i}{\partial \lambda_j} + \varpi_j'\eta_i - \chi_j'\xi_i = 0, &&\frac{\partial r_i}{\partial \lambda_j} = \frac{\partial \sigma_j'}{\partial \sigma_i} + p_i\chi_j' - q_i\varpi_j'.
\end{aligned}
$$

which are the result of the defining relationships for the functions ϖ_i', χ_i', σ_i' and the nine identities which they satisfy, the preceding system can then /79

be written in the new form:

$$\mathfrak{L}_0 = \sum_i \left[\frac{\partial \Pi_i}{\partial \rho_i} - A_i' \frac{\partial \xi_i}{\partial \lambda_i} - B_i' \frac{\partial \eta_i}{\partial \lambda_i} - C_i' \frac{\partial \zeta_i}{\partial \lambda_i} - P_i' \frac{\partial p_i}{\partial \lambda_i} - Q_i' \frac{\partial q_i}{\partial \lambda_i} - R_i' \frac{\partial r_i}{\partial \lambda_i} \right],$$

and two analogous equations.

3. Instead of referring the elements for the point M with respect to the fixed axes Ox, Oy, Oz, let us consider an orthogonal triad $Mx_1'y_1'z_1'$ in motion with M and whose axis Mz_1' is normal to the surface (M) at M. For the definition of this triad $Mx_1'y_1'z_1'$, let us relate it to the triad Mx'y'z'. With respect to the axes of the latter, let l, l', l'' be the direction cosines of Mx_1'; m, m', m'' -- those of My_1'; and n, n', n'' -- those of Mz_1'.

We will define the cosines n, n', n'' in a rigorous way by means of the equations:

$$n = \frac{1}{\Delta}(\eta_1\zeta_2 - \eta_2\zeta_1), \quad n' = \frac{1}{\Delta}(\zeta_1\xi_2 - \zeta_2\xi_1), \quad n'' = \frac{1}{\Delta}(\xi_1\eta_2 - \xi_2\eta_1).$$

We will let the triad $Mx_1'y_1'z_1'$ have the same configuration as the others, and for the time being we will not make any special hypotheses regarding the other cosines.

Let $\xi_i^{(1)}$, $\eta_i^{(1)}$, $\zeta_i^{(1)}$ be the components of speed of the origin M of the axes Mx_1', My_1', Mz_1' along these axes when only ρ_i is varied and plays the role of time. Also, let $p_i^{(1)}$, $q_i^{(1)}$, $r_i^{(1)}$ be the projections along these axes of the instantaneous rotation of the triad $Mx_1'y'z_1'$ corresponding to the parameter ρ_i. In these definitions, the triad $Mx_1'y_1'z_1'$ is naturally referred to the fixed triad Oxyz. We will have:

$$\xi_i^{(1)} = l\xi_i + l'\eta_i + l''\zeta_i, \quad \eta_i^{(1)} = m\xi_i + m'\eta_i + m''\zeta_i, \quad \zeta_i^{(1)} = n\xi_i + n'\eta_i + n''\zeta_i = 0,$$

and three equations of the following type:

$$p_i^{(1)} = lp_i + l'q_i + l''r_i + \sum n \frac{\partial m}{\partial \rho_i},$$

where the triads under consideration have the same configuration.

Let X_0'', Y_0'', Z_0'' and L_0'', M_0'', N_0'', respectively, be the projections on Mx_1', My_1', Mz_1' of the external force and external moment at an arbitrary point M of the deformed surface referred to a unit of area of the non-deformed surface. Let F_0'', G_0'', H_0'' and L_0'', J_0'', K_0'' be the projections along the same axes of the stress (F_0, G_0, H_0) and the moment (I_0, J_0, K_0). Let A_i'', B_i'', C_i'' and P_i'', Q_i'', R_i'' be the projections of the stress (A_i', B_i', C_i') and the moment (P_i', Q_i', R_i') defined above.

The transformations of the preceding relationships (or of the initial /80

relationships) are obviously(7):

$$
\left\{
\begin{array}{ll}
\sum_i \left(\frac{\partial A_i''}{\partial \rho_i} + q_i^{(1)} C_i'' - r_i^{(1)} B_i''\right) = \Delta_0 X_0'', & F_0'' = A_1'' \frac{d\rho_2}{ds_0} - A_2'' \frac{d\rho_1}{ds_0}. \\
\sum_i \left(\frac{\partial B_i''}{\partial \rho_i} + r_i^{(1)} A_i'' - p_i^{(1)} C_i''\right) = \Delta_0 Y_0'', & G_0'' = B_1'' \frac{d\rho_2}{ds_0} - B_2'' \frac{d\rho_1}{ds_0}. \\
\sum_i \left(\frac{\partial C_i''}{\partial \rho_i} + p_i^{(1)} B_i'' - q_i^{(1)} A_i''\right) = \Delta_0 Z_0'', & H_0'' = C_1'' \frac{d\rho_2}{ds_0} - C_2'' \frac{d\rho_1}{ds_0}. \\
\sum_i \left(\frac{\partial P_i''}{\partial \rho_i} + q_i^{(1)} R_i'' - r_i^{(1)} Q_i'' + \eta_i^{(1)} C_i''\right) = \Delta_0 L_0'', & I_0'' = P_1'' \frac{d\rho_2}{ds_0} - P_2'' \frac{d\rho_1}{ds_0}. \\
\sum_i \left(\frac{\partial Q_i''}{\partial \rho_i} + r_i^{(1)} P_i'' - p_i^{(1)} R_i'' - \xi_i^{(1)} C_i''\right) = \Delta_0 M_0'', & J_0'' = Q_1'' \frac{d\rho_2}{ds_0} - Q_2'' \frac{d\rho_1}{ds_0}. \\
\sum_i \left(\frac{\partial R_i''}{\partial \rho_i} + p_i^{(1)} Q_i'' - q_i^{(1)} P_i'' + \xi_i^{(1)} B_i'' - \eta_i^{(1)} A_i''\right) = \Delta_0 N_0'' & K_0'' = R_1'' \frac{d\rho_2}{ds_0} - R_2'' \frac{d\rho_1}{ds_0}.
\end{array}
\right. \tag{40}
$$

Instead of replacing $d\rho_1$, $d\rho_2$ by the values (39) in the equations on the right or by analogous ones for (M), we may take the following values, where $(\lambda'', \mu'', 0)$ designate the direction cosines of the external normal to the contour C with respect to the triad $Mx_1'y_1'z_1'$:

$$-(\lambda''\xi_2^{(1)} + \mu''\eta_2^{(1)})\frac{ds}{\Delta}, \qquad (\lambda''\xi_1^{(1)} + \mu''\eta_1^{(1)})\frac{ds}{\Delta};$$

In this way we obtain:

$$
\left\{
\begin{array}{l}
F_0'' \frac{ds_0}{ds} = \lambda'' \frac{\xi_1^{(1)} A_1'' + \xi_2^{(1)} A_2''}{\Delta} + \mu'' \frac{\eta_1^{(1)} A_1'' + \eta_2^{(1)} A_2''}{\Delta}, \\
I_0'' \frac{ds_0}{ds} = \lambda'' \frac{\xi_1^{(1)} P_1'' + \xi_2^{(1)} P_2''}{\Delta} + \mu'' \frac{\eta_1^{(1)} P_1'' + \eta_2^{(1)} P_2''}{\Delta},
\end{array}
\right. \tag{41}
$$

and two systems of two similar equations.

These equations lead to the substitution of twelve new auxiliary quantities for the twelve auxiliary quantities A_i'', B_i'', C_i'', P_i'', Q_i'', R_i'', which are the coefficients of λ'' and μ'' in the preceding expressions for the stresses and moments referred to a unit of length of C, or which will be related to these coefficients in a simple way. We will set:

$$
\begin{array}{ll}
\frac{\xi_1^{(1)} A_1'' + \xi_2^{(1)} A_2''}{\Delta} = N_1, & \frac{\eta_1^{(1)} A_1'' + \eta_2^{(1)} A_2''}{\Delta} = T - S_3, \\
\frac{\xi_1^{(1)} B_1'' + \xi_2^{(1)} B_2''}{\Delta} = T + S_3, & \frac{\eta_1^{(1)} B_1'' + \eta_2^{(1)} B_2''}{\Delta} = N_2 \\
\frac{\xi_1^{(1)} C_1'' + \xi_2^{(1)} C_2''}{\Delta} = S_2, & \frac{\eta_1^{(1)} C_1'' + \eta_2^{(1)} C_2''}{\Delta} = S_1,
\end{array}
$$

by introducing six first auxiliary quantities N_1, N_2, T, S_1, S_2, S_3. In the /81

(7) It is sufficient to replace ξ_i, ... A_i', ... by $\xi_i^{(1)}$, ... A_i'', ... and to take into account the hypotheses $\zeta_i^{(1)} = 0$. For an *arbitrary* triad with an apex M, the calculations would be the same.

same way we have

$$\frac{\xi^{(1)}P_1'' + \xi^{(2)}P_2''}{\Delta} = \mathfrak{N}_1, \qquad \frac{\eta^{(1)}P_1'' + \eta^{(2)}P_2''}{\Delta} = \mathfrak{T} - \mathfrak{S}_3,$$

$$\frac{\xi^{(1)}Q_1'' + \xi^{(2)}Q_2''}{\Delta} = \mathfrak{T} + \mathfrak{S}_3, \qquad \frac{\eta^{(1)}Q_1'' + \eta^{(2)}Q_2''}{\Delta} = \mathfrak{N}_2,$$

$$\frac{\xi^{(1)}R_1'' + \xi^{(2)}R_2''}{\Delta} = \mathfrak{S}_2, \qquad \frac{\eta^{(1)}R_1'' + \eta^{(2)}R_2''}{\Delta} = \mathfrak{S}_1,$$

by introducing the six other auxiliary quantities $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{T}$, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$.

The twelve equations which we have written down can be solved immediately with respect to the first auxiliary quantities A_i'', B_i'', C_i'', P_i'', Q_i'', R_i''. Let us note that due to the hypothesis made regarding the common configuration of these triads, we have:

$$\begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix} = 1;$$

and the equations defining $\xi_i^{(1)}$, $\eta_i^{(1)}$ are therefore,

$$\xi_1^{(1)}\eta_2^{(1)} - \xi_2^{(1)}\eta_1^{(1)} = \Delta.$$

It then follows that

$$A_1'' = N_1\eta_1^{(1)} - (T - S_3)\xi_2^{(1)}, \qquad A_2'' = (T - S_3)\xi_1^{(1)} - N_1\eta_1^{(1)},$$

$$B_1'' = (T + S_3)\eta_2^{(1)} - N_2\xi_2^{(1)}, \qquad B_2'' = N_2\xi_1^{(1)} - (T + S_3)\eta_1^{(1)},$$

$$C_1'' = S_2\eta_2^{(1)} - S_1\xi_2^{(1)}, \qquad C_2'' = S_1\xi_1^{(1)} - S_2\eta_1^{(1)},$$

and six analogous equations for P_i'', Q_i'', R_i'', with italic letters in the second terms. Introducing these values into the relationships (40) and (41), we will obtain equations for the stresses and moments of deformation and for the external forces and moments in the form they assume when the new auxiliary quantities are used [8].

Obviously we may designate the components of the stress and moment of deformation in a way similar to the ones used for the deformable line. Thus, we may call the components N_1, N_2 of the stress the *tensile stresses*; the components $T - S_3$, $T + S_3$ are the *shear stresses* in the plane tangent to the deformed surface; the components S_1, S_2 are the *shear stresses* normal to the deformed surface. In the same way, the components $\mathfrak{N}_1$, $\mathfrak{N}_2$ of the moment of deformation can be regarded as *torsion moments*; the components $\mathfrak{T} - \mathfrak{S}_3$, $\mathfrak{T} + \mathfrak{S}_3$ have the character of *flexure moments*; the components $\mathfrak{S}_1$, $\mathfrak{S}_2$ may be called *geodesic flexure moments*.

36. Remarks Concerning the Components S_1, S_2, S_3 and $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$. For S_1, S_2, S_3 and the analogous quantities $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, let us explicitly make

---

(8) It should be noted that the coefficient of S_3 in the third of these equations is zero.

the statements used above for writing down the transformed equations.

Let us consider, in a general way, a segment whose projections along Ox, Oy, Oz are /82

$$\frac{\partial y}{\partial \rho_i} C_i - \frac{\partial z}{\partial \rho_i} B_i, \quad \frac{\partial z}{\partial \rho_i} A_i - \frac{\partial x}{\partial \rho_i} C_i, \quad \frac{\partial x}{\partial \rho_i} B_i - \frac{\partial y}{\partial \rho_i} A_i;$$

considering this segment as the moment of a vector (A_1, B_1, C_1) applied at the point $(\frac{\partial x}{\partial \rho_1}, \frac{\partial y}{\partial \rho_1}, \frac{\partial z}{\partial \rho_1})$. We see that its projections along Mx', My', Mz' will be

$$\eta_i C_i' - \zeta_i B_i', \quad \zeta_i A_i' - \xi_i C_i', \quad \xi_i B_i' - \eta_i A_i',$$

and along Mx_1', My_1', Mz_1', they are

$$\eta_i^{(1)} C_i', \quad -\xi_i^{(1)} C_i', \quad \xi_i^{(1)} B_i' - \eta_i^{(1)} A_i'.$$

It follows from this that the segment whose projections along Ox, Oy, Oz are

$$\sum_i \left(\frac{\partial y}{\partial \rho_i} C_i - \frac{\partial z}{\partial \rho_i} B_i\right), \quad \sum_i \left(\frac{\partial z}{\partial \rho_i} A_i - \frac{\partial x}{\partial \rho_i} C_i\right), \quad \sum_i \left(\frac{\partial x}{\partial \rho_i} B_i - \frac{\partial y}{\partial \rho_i} A_i\right),$$

will have the following projections on Mx', My', Mz',

$$\sum_i (\eta_i C_i' - \zeta_i B_i'), \quad \sum_i (\zeta_i A_i' - \xi_i C_i'), \quad \sum_i (\xi_i B_i' - \eta_i A_i'),$$

and on Mx_1', My_1', Mz_1' they will be:

$$\sum_i \eta_i^{(1)} C_i' = \Delta S_1, \quad -\sum_i \xi_i^{(1)} C_i' = -\Delta S_2, \quad \sum_i (\xi_i^{(1)} B_i' - \eta_i^{(1)} A_i') = 2\Delta S_3.$$

An identical law naturally holds for letters in italics.

From this it follows that the conditions:

$$S_1 = 0, \quad S_2 = 0, \quad S_3 = 0$$

lead to:

$$\sum_i (\eta_i C_i' - \zeta_i B_i') = 0, \quad \sum_i (\zeta_i A_i' - \xi_i C_i') = 0, \quad \sum_i (\xi_i B_i' - \eta_i A_i') = 0,$$

and that the conditions:

$$g_1 = 0, \quad g_2 = 0, \quad g_3 = 0$$

can be reduced to

$$\sum_i (\eta_i R_i' - \zeta_i Q_i') = 0, \quad \sum_i (\zeta_i P_i' - \xi_i R_i') = 0, \quad \sum_i (\xi_i Q_i' - \eta_i P_i') = 0.$$

In the two cases one arrives at a system of two conditions which do not depend on the choice of the triad $Mx_1'y_1'z_1'$.

If the conditions $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ are *conditions which result from the form of* W, then W satisfies the three partial differential equations: /83

$$\sum_i \left(\eta_i \frac{\partial W}{\partial \zeta_i} - \zeta_i \frac{\partial W}{\partial \eta_i}\right) = 0, \quad \sum_i \left(\zeta_i \frac{\partial W}{\partial \xi_i} - \xi_i \frac{\partial W}{\partial \zeta_i}\right) = 0, \quad \sum_i \left(\xi_i \frac{\partial W}{\partial \eta_i} - \eta_i \frac{\partial W}{\partial \xi_i}\right) = 0,$$

which means that W only depends on ξ_i, η_i, ζ_i through the intermediary of the expressions

$$\mathcal{E} = \xi_i^2 + \eta_i^2 + \zeta_i^2, \quad \mathcal{F} = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2, \quad \mathcal{G} = \xi_2^2 + \eta_2^2 + \zeta_2^2.$$

If the conditions are $\mathcal{Y}_1 = 0$, $\mathcal{Y}_2 = 0$, $\mathcal{Y}_3 = 0$ are *conditions which result from the form of* W, then W satisfies the three partial differential equations:

$$\sum_i \left(\eta_i \frac{\partial W}{\partial r_i} - \zeta_i \frac{\partial W}{\partial q_i}\right) = 0, \quad \sum_i \left(\zeta_i \frac{\partial W}{\partial p_i} - \xi_i \frac{\partial W}{\partial r_i}\right) = 0, \quad \sum_i \left(\xi_i \frac{\partial W}{\partial q_i} - \eta_i \frac{\partial W}{\partial p_i}\right) = 0,$$

which means that W only depends on p_i, q_i, r_i through the intermediary of the three expressions:

$$p_1\xi_1 + q_1\eta_1 + r_1\zeta_1, \quad p_1\xi_2 + q_1\eta_2 + r_1\zeta_2 + p_2\xi_1 + q_2\eta_1 + r_2\zeta_1, \quad p_2\xi_2 + q_2\eta_2 + r_2\zeta_2,$$

which are expressions which are reduced to the coefficients of $d\rho_1^2$, $d\rho_1$, $d\rho_2$ and of $d\rho_2^2$ in the equation for the lines of curvature of (M), for $\zeta_1 = \zeta_2 = 0$.

Let us also note that if we simply assume the conditions:

$$S_1 = 0, \quad S_2 = 0,$$

this is the same as stating that the segment whose projections along Mx', My', Mz' have values identical to those indicated on the preceding page is parallel to Mz_1', or that it is simultaneously perpendicular to the two vectors (ξ_1, η_1, ζ_2), which results in the conditions:

$$\xi_1(\eta_2 C_2' - \zeta_2 B_2') + \eta_1(\zeta_2 A_2' - \xi_2 C_2') + \zeta_1(\xi_2 B_2' - \eta_2 A_2') = 0,$$
$$\xi_2(\eta_1 C_1' - \zeta_1 B_1') + \eta_2(\zeta_1 A_1' - \xi_1 C_1') + \zeta_2(\xi_1 B_1' - \eta_1 A_1') = 0,$$

which can be written as

$$(\eta_1\zeta_2 - \zeta_1\eta_2)A_2' + (\zeta_1\xi_2 - \xi_1\zeta_2)B_2' + (\xi_1\eta_2 - \eta_1\xi_2)C_2' = 0,$$
$$(\eta_1\zeta_2 - \zeta_1\eta_2)A_1' + (\zeta_1\xi_2 - \xi_1\zeta_2)B_1' + (\xi_1\eta_2 - \eta_1\xi_2)C_1' = 0,$$

In this form, they state that the vectors (A_1', B_1', C_1') and (A_2', B_2', C_2') are perpendicular to the normal Mz_1'. In this way we find *two conditions which are independent of the choice of the triad* $Mx_1'y_1'z_1'$, and which can immediately be verified *a posteriori*, by referring to the meaning of the shear stresses S_1, S_2. If the conditions $S_1 = 0$, $S_2 = 0$ are conditions *which result from the form of* W, then W will satisfy the two partial differential equations

$$(\eta_1\zeta_2 - \zeta_1\eta_2)\frac{\partial W}{\partial \xi_1} + (\zeta_1\xi_2 - \xi_1\zeta_2)\frac{\partial W}{\partial \eta_1} + (\xi_1\eta_2 - \eta_1\xi_2)\frac{\partial W}{\partial \zeta_1} = 0,$$
$$(\eta_1\zeta_2 - \zeta_1\eta_2)\frac{\partial W}{\partial \xi_2} + (\zeta_1\xi_2 - \xi_1\zeta_2)\frac{\partial W}{\partial \eta_2} + (\xi_1\eta_2 - \eta_1\xi_2)\frac{\partial W}{\partial \zeta_2} = 0,$$

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which means that W is a function which only depends on ξ_i, η_i, ζ_i through the intermediary of the three expressions $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$.

The same reasoning proves that the conditions:

$$\mathcal{S}_1 = 0, \quad \mathcal{S}_2 = 0$$

amount to two conditions independent of the choice of the triad $Mx_1'y_1'z_1'$ and which may be written in the final form

$$(\eta_1\zeta_2 - \zeta_1\eta_2)P_1' + (\zeta_1\xi_2 - \xi_1\zeta_2)Q_1' + (\xi_1\eta_2 - \eta_1\xi_2)R_1' = 0,$$
$$(\eta_1\zeta_2 - \zeta_1\eta_2)P_2' + (\zeta_1\xi_2 - \xi_1\zeta_2)Q_2' + (\xi_1\eta_2 - \eta_1\xi_2)R_2' = 0.$$

If the conditions $\mathcal{S}_1 = 0$, $\mathcal{S}_2 = 0$ are conditions *which result from the form of* W, then W satisfies the following two partial differential equations:

$$(\eta_1\zeta_2 - \zeta_1\eta_2)\frac{\partial W}{\partial p_1} + (\zeta_1\xi_2 - \xi_1\zeta_2)\frac{\partial W}{\partial q_1} + (\xi_1\eta_2 - \eta_1\xi_2)\frac{\partial W}{\partial r_1} = 0,$$
$$(\eta_1\zeta_2 - \zeta_1\eta_2)\frac{\partial W}{\partial p_2} + (\zeta_1\xi_2 - \xi_1\zeta_2)\frac{\partial W}{\partial q_2} + (\xi_1\eta_2 - \eta_1\xi_2)\frac{\partial W}{\partial r_2} = 0,$$

which means that W is a function which does not depend on p_i, q_i, r_i except through the intermediary of the four expressions

$$p_1\xi_1 + q_1\eta_1 + r_1\zeta_1,\ p_1\xi_2 + q_1\eta_2 + r_1\zeta_2,\ p_2\xi_1 + q_2\eta_1 + r_2\zeta_1,\ p_2\xi_2 + q_2\eta_2 + r_2\zeta_2.$$

Let us consider the same condition

$$S_3 = 0;$$

It expresses the fact that the segment whose projections along Mx', My', Mz' have values identical to those given on page 1034 is perpendicular to Mz_1', which results in the condition:

$$(\eta_1\zeta_2 - \zeta_1\eta_2)\sum_i(\eta_i C_i' - \zeta_i B_i') + (\zeta_1\xi_2 - \xi_1\zeta_2)\sum_i(\zeta_i A_i' - \xi_i C_i')$$
$$+ (\xi_1\eta_2 - \eta_1\xi_2)\sum_i(\xi_i B_i' - \eta_i A_i') = 0,$$

which does not depend on the *choice of the triad* $Mx_1'y_1'z_1'$, and which leads to a partial differential equation satisfied by W when the condition $S_3 = 0$ is a result of the form of W. This equation is:

$$(\xi_2\mathcal{E} - \xi_1\mathcal{F})\frac{\partial W}{\partial \xi_1} + (\eta_2\mathcal{E} - \eta_1\mathcal{F})\frac{\partial W}{\partial \eta_1} + (\zeta_2\mathcal{E} - \zeta_1\mathcal{F})\frac{\partial W}{\partial \zeta_1}$$
$$+ (\xi_2\mathcal{F} - \xi_1\mathcal{G})\frac{\partial W}{\partial \xi_2} + (\eta_2\mathcal{F} - \eta_1\mathcal{G})\frac{\partial W}{\partial \eta_2} + (\zeta_2\mathcal{F} - \zeta_1\mathcal{G})\frac{\partial W}{\partial \zeta_2} = 0,$$

which may be easily integrated, because it has particular integrals formed with $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$.

The same reasoning may be applied to the condition

$$\mathcal{S}_3 = 0,$$

which also corresponds to a *condition independent of the choice of the triad* /85
$Mx_1'y_1'z_1'$, and which leads to the following partial differential equation when it is a result of the form of W

$$(\xi_2\mathcal{E} - \xi_1\mathcal{F})\frac{\partial W}{\partial p_1} + (\eta_2\mathcal{E} - \eta_1\mathcal{F})\frac{\partial W}{\partial q_1} + (\zeta_2\mathcal{E} - \zeta_1\mathcal{F})\frac{\partial W}{\partial r_1}$$
$$+ (\xi_2\mathcal{F} - \xi_1\mathcal{G})\frac{\partial W}{\partial p_2} + (\eta_2\mathcal{F} - \eta_1\mathcal{G})\frac{\partial W}{\partial q_2} + (\zeta_2\mathcal{F} - \zeta_1\mathcal{G})\frac{\partial W}{\partial r_2} = 0,$$

whose integration is straightforward.

37. Equations Obtained, Following the Example of Poisson, by Introducing the Coordinates x, y as Independent Variables in the Place of ρ_1, ρ_2. Let us attempt to formulate equations similar to those of Section 35 following a certain analogy with what we did for the three-dimensional, deformable medium, but in which the independent variables are x, y.

In order to abbreviate the notation, let us designate by $\mathcal{X}_0'$, $\mathcal{Y}_0'$, $\mathcal{Z}_0'$, $\mathcal{L}_0'$, $\mathcal{M}_0'$, $\mathcal{N}_0'$ the first terms of the relationships to be transformed for the time being, i.e., let us set:

$$\mathcal{X}_0' = \frac{\partial A_1}{\partial \rho_1} + \frac{\partial A_2}{\partial \rho_2} - \Delta_0 X_0, \quad \mathcal{Y}_0' = \frac{\partial B_1}{\partial \rho_1} + \frac{\partial B_2}{\partial \rho_2} - \Delta_0 Y_0, \quad \mathcal{Z}_0' = \frac{\partial C_1}{\partial \rho_1} + \frac{\partial C_2}{\partial \rho_2} - \Delta_0 Z_0,$$

and

$$\mathcal{L}_0' = \frac{\partial P_1}{\partial \rho_1} + \frac{\partial P_2}{\partial \rho_2} + C_1\frac{\partial y}{\partial \rho_1} + C_2\frac{\partial y}{\partial \rho_2} - B_1\frac{\partial z}{\partial \rho_1} - B_2\frac{\partial z}{\partial \rho_2} - \Delta_0 L_0,$$
$$\mathcal{M}_0' = \frac{\partial Q_1}{\partial \rho_1} + \frac{\partial Q_2}{\partial \rho_2} + A_1\frac{\partial z}{\partial \rho_1} + A_2\frac{\partial z}{\partial \rho_2} - C_1\frac{\partial x}{\partial \rho_1} - C_2\frac{\partial x}{\partial \rho_2} - \Delta_0 M_0,$$
$$\mathcal{N}_0' = \frac{\partial R_1}{\partial \rho_1} + \frac{\partial R_2}{\partial \rho_2} + B_1\frac{\partial x}{\partial \rho_1} + B_2\frac{\partial x}{\partial \rho_2} - A_1\frac{\partial y}{\partial \rho_1} - A_2\frac{\partial y}{\partial \rho_2} - \Delta_0 N_0.$$

We may again consider the twelve relationships of Section 35 in which the elements were referred to fixed axes. We have:

$$0 = \iint (\mathcal{X}_0'\lambda_1 + \mathcal{Y}_0'\lambda_2 + \mathcal{Z}_0'\lambda_3 + \mathcal{L}_0'\mu_1 + \mathcal{M}_0'\mu_2 + \mathcal{N}_0'\mu_3)\, d\rho_1 d\rho_2$$
$$+ \int \Big\{ \left(F_0 - A_1\frac{d\rho_2}{ds_0} + A_2\frac{d\rho_1}{ds_0}\right)\lambda_1 + \left(G_0 - B_1\frac{d\rho_2}{ds_0} + B_2\frac{d\rho_1}{ds_0}\right)\lambda_2$$
$$+ \left(H_0 - C_1\frac{d\rho_2}{ds_0} + C_2\frac{d\rho_1}{ds_0}\right)\lambda_3 + \left(I_0 - P_1\frac{d\rho_2}{ds_0} + P_2\frac{d\rho_1}{ds_0}\right)\mu_1$$
$$+ \left(J_0 - Q_1\frac{d\rho_2}{ds_0} + Q_2\frac{d\rho_1}{ds_0}\right)\mu_2 + \left(K_0 - R_1\frac{d\rho_2}{ds_0} + R_2\frac{d\rho_1}{ds_0}\right)\mu_3 \Big\}\, ds_0$$

where λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary functions and where the integrals are taken over the curve C_0 of the surface (M_0) and the region limited by it.

Let us apply Green's formula. The preceding relationship becomes the following one: /86

$$-\iint (X_0\lambda_1 + Y_0\lambda_2 + Z_0\lambda_3 + L_0\mu_1 + M_0\mu_2 + N_0\mu_3)\,\Delta_0\, d\rho_1\, d\rho_2$$
$$+ \int (F_0\lambda_1 + G_0\lambda_2 + H_0\lambda_3 + I_0\mu_1 + J_0\mu_2 + K_0\mu_3)\, ds_0$$
$$- \iint \left(A_1\frac{\partial\lambda_1}{\partial\rho_1} + A_2\frac{\partial\lambda_1}{\partial\rho_2} + B_1\frac{\partial\lambda_2}{\partial\rho_1} + B_2\frac{\partial\lambda_2}{\partial\rho_2} + C_1\frac{\partial\lambda_3}{\partial\rho_1} + C_2\frac{\partial\lambda_3}{\partial\rho_2}\right) d\rho_1 d\rho_2$$

$$-\iint\left(P_1\frac{\partial\mu_1}{\partial\rho_1}+P_2\frac{\partial\mu_1}{\partial\rho_2}+Q_1\frac{\partial\mu_2}{\partial\rho_1}+Q_2\frac{\partial\mu_2}{\partial\rho_2}+R_1\frac{\partial\mu_3}{\partial\rho_1}+R_2\frac{\partial\mu_3}{\partial\rho_2}\right)d\rho_1 d\rho_2$$
$$+\iint\left(\frac{\partial y}{\partial\rho_1}C_1+\frac{\partial y}{\partial\rho_2}C_2-\frac{\partial z}{\partial\rho_1}B_1-\frac{\partial z}{\partial\rho_2}B_2\right)\mu_1\,d\rho_1\,d\rho_2$$
$$+\iint\left(\frac{\partial z}{\partial\rho_1}A_1+\frac{\partial z}{\partial\rho_2}A_2-\frac{\partial x}{\partial\rho_1}C_1-\frac{\partial x}{\partial\rho_2}C_2\right)\mu_2\,d\rho_1\,d\rho_2$$
$$+\iint\left(\frac{\partial x}{\partial\rho_1}B_1+\frac{\partial x}{\partial\rho_2}B_2-\frac{\partial y}{\partial\rho_1}A_1-\frac{\partial y}{\partial\rho_2}A_2\right)\mu_3\,d\rho_1\,d\rho_2=0.$$

Let us find the transform of this latter relationship, when the new variables are taken to be functions x, y of ρ_1, ρ_2. If ϕ is an arbitrary function of ρ_1, ρ_2, which means it is a function of x, y, the elementary equations for the change of variables are given by:

$$\frac{\partial\varphi}{\partial\rho_1}=\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\rho_1}+\frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\rho_1},$$
$$\frac{\partial\varphi}{\partial\rho_2}=\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\rho_2}+\frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\rho_2}.$$

Let us apply these equations to the functions λ_1, λ_2, λ_3, μ_1, μ_2, μ_3. C is still the curve of (M) which corresponds to the curve (C_0) of (M_0). In addition, let us designate by X, Y, Z, L, M, N the projections on Ox, Oy, Oz of the external force and moment applied to the point M and referred to the area unit of the deformed surface (M). Let F, G, H, I, J, K be the projections along Ox, Oy, Oz of the stress and moment of deformation which are applied to the point M on C, referred to the unit of length of C. Finally, let us introduce twelve new auxiliary quantities $A_1^{(1)}$, $B_1^{(1)}$, $C_1^{(1)}$; $A_2^{(1)}$, $B_2^{(1)}$, $C_2^{(1)}$; $P_1^{(1)}$, $Q_1^{(1)}$, $R_1^{(1)}$, $P_2^{(1)}$, $Q_2^{(1)}$, $R_2^{(1)}$, by means of the equations:

$$\frac{\Delta}{\Delta_1}A_1^{(1)}=A_1\frac{\partial x}{\partial\rho_1}+A_2\frac{\partial x}{\partial\rho_2},\qquad \frac{\Delta}{\Delta_1}P_1^{(1)}=P_1\frac{\partial x}{\partial\rho_1}+P_2\frac{\partial x}{\partial\rho_2},$$
$$\frac{\Delta}{\Delta_1}A_2^{(1)}=A_1\frac{\partial y}{\partial\rho_1}+A_2\frac{\partial y}{\partial\rho_2},\qquad \frac{\Delta}{\Delta_1}P_2^{(1)}=P_1\frac{\partial y}{\partial\rho_1}+P_2\frac{\partial y}{\partial\rho_2},$$

and by means of analogous ones obtained by replacing

$$A_1,\ A_2,\ A_1^{(1)},\ A_2^{(1)},\ P_1,\ P_2,\ P_1^{(1)},\ P_2^{(1)}$$

by /87

$$B_1,\ B_2,\ B_1^{(1)},\ B_2^{(1)},\ Q_1,\ Q_2,\ Q_1^{(1)},\ Q_2^{(1)},$$

respectively, and then by

$$C_1,\ C_2,\ C_1^{(1)},\ C_2^{(1)},\ R_1,\ R_2,\ R_1^{(1)},\ R_2^{(1)}.$$

Δ_1 will be the quantity analogous to Δ. We will thus set

$$\Delta_1=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}.$$

We thus obtain the transformed relationship

$$-\iint(X\lambda_1+Y\lambda_2+Z\lambda_3\pm L\mu_1+M\mu_2+N\mu_3)\,\Delta_1\,dx\,dy$$

$$+\int (F\lambda_1+G\lambda_2+H\lambda_3+I\mu_1+J\mu_2+K\mu_3)\,ds$$
$$-\iint\left(A_1^{(1)}\frac{\partial\lambda_1}{\partial x}+A_2^{(1)}\frac{\partial\lambda_1}{\partial y}+B_1^{(1)}\frac{\partial\lambda_2}{\partial x}+B_2^{(1)}\frac{\partial\lambda_2}{\partial y}+C_1^{(1)}\frac{\partial\lambda_3}{\partial x}+C_2^{(1)}\frac{\partial\lambda_3}{\partial y}\right)dx\,dy$$
$$-\iint\left(P_1^{(1)}\frac{\partial\mu_1}{\partial x}+P_2^{(1)}\frac{\partial\mu_1}{\partial y}+Q_1^{(1)}\frac{\partial\mu_2}{\partial x}+Q_2^{(1)}\frac{\partial\mu_2}{\partial y}+\underline{R_1^{(1)}\frac{\partial\mu_3}{\partial x}}+R_2^{(1)}\frac{\partial\mu_3}{\partial y}\right)dx\,dy$$
$$+\iint\left\{\left(C_2^{(1)}-\frac{\partial z}{\partial x}B_1^{(1)}-\frac{\partial z}{\partial y}B_2^{(1)}\right)\mu_1+\left(\frac{\partial z}{\partial x}A_1^{(1)}+\frac{\partial z}{\partial y}A_2^{(1)}-C_1^{(1)}\right)\mu_2+(B_1^{(1)}-A_2^{(1)})\mu_3\right\}dx\,dy=0,$$

where the integrals are taken over the curve C of the surface (M) and over the region limited by it, and where ds designates the arc element of C.

Let us again apply Green's equation to the terms which contain the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect to x, y. We then obtain, due to the fact that λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary,

$$\frac{\partial A_1^{(1)}}{\partial x}+\frac{\partial A_2^{(1)}}{\partial y}=\Delta_1 X, \qquad F=A_1^{(1)}\frac{dy}{ds}-A_2^{(1)}\frac{dx}{ds},$$
$$\frac{\partial B_1^{(1)}}{\partial x}+\frac{\partial B_2^{(1)}}{\partial y}=\Delta_1 Y, \qquad G=B_1^{(1)}\frac{dy}{ds}-B_2^{(1)}\frac{dx}{ds},$$
$$\frac{\partial C_1^{(1)}}{\partial x}+\frac{\partial C_2^{(1)}}{\partial y}=\Delta_1 Z, \qquad H=C_1^{(1)}\frac{dy}{ds}-C^{(1)}\frac{dx}{ds},$$
$$\frac{\partial P_1^{(1)}}{\partial x}+\frac{\partial P_2^{(1)}}{\partial y}+C_2^{(1)}-\frac{\partial z}{\partial x}B_1^{(1)}-\frac{\partial z}{\partial y}B_2^{(1)}=\Delta_1 L, \qquad I=P_1^{(1)}\frac{dy}{ds}-P_2^{(1)}\frac{dx}{ds},$$
$$\frac{\partial Q_1^{(1)}}{\partial x}+\frac{\partial Q_2^{(1)}}{\partial y}+\frac{\partial z}{\partial x}A_1^{(1)}+\frac{\partial z}{\partial y}A_2^{(1)}-C_1^{(1)}=\Delta_1 M, \qquad J=Q_1^{(1)}\frac{dy}{ds}-Q_2^{(1)}\frac{dx}{ds},$$
$$\frac{\partial R_1^{(1)}}{\partial x}+\frac{\partial R_2^{(1)}}{\partial y}+B_1^{(1)}-A_2^{(1)}=\Delta_1 N, \qquad K=R_1^{(1)}\frac{dy}{ds}-R_2^{(1)}\frac{dx}{ds}.$$

These formulas can be derived *a posteriori* from those given previously. For example, let us consider those on the right. We saw previously (Section 35, 1) that F, G, H can be obtained by replacing, in the expressions /88

$$A_1\frac{d\rho_2}{ds}-A_2\frac{d\rho_1}{ds},\quad B_1\frac{d\rho_2}{ds}-B_2\frac{d\rho_1}{ds},\quad C_1\frac{d\rho_2}{ds}-C_2\frac{d\rho_1}{ds},$$

the terms $\frac{d\rho_1}{ds}$, $\frac{d\rho_2}{ds}$ by

$$-\frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial\rho_2}+\mu\frac{\partial y}{\partial\rho_2}+\nu\frac{\partial z}{\partial\rho_2}\right),\quad \frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial\rho_1}+\mu\frac{\partial y}{\partial\rho_1}+\nu\frac{\partial z}{\partial\rho_1}\right),$$

respectively, where λ, μ, ν are the direction cosines of the external normal to C. This results in

$$F=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)A_1^{(1)}+\left(\mu+\nu\frac{\partial z}{\partial y}\right)A_2^{(1)}\right],$$
$$G=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)B_1^{(1)}+\left(\mu+\nu\frac{\partial z}{\partial y}\right)B_2^{(1)}\right],$$
$$H=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)C_1^{(1)}+\left(\mu+\nu\frac{\partial z}{\partial y}\right)C_2^{(1)}\right],$$

and in the same way

$$I=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)P_1^{(1)}+\left(\mu+\nu\frac{\partial z}{\partial y}\right)P_2^{(1)}\right],$$

$$J = \frac{1}{\Delta_1}\left[\left(\lambda + \nu\frac{\partial z}{\partial x}\right)Q_1^{(1)} + \left(\mu + \nu\frac{\partial z}{\partial y}\right)Q_2^{(1)}\right],$$

$$H = \frac{1}{\Delta_1}\left[\left(\lambda + \nu\frac{\partial z}{\partial x}\right)R_1^{(1)} + \left(\mu + \nu\frac{\partial z}{\partial y}\right)R_2^{(1)}\right],$$

which is the same as saying that we have:

$$\frac{dy}{ds} = \frac{\lambda + \nu\frac{\partial z}{\partial x}}{\Delta_1}, \quad \frac{dx}{ds} = -\frac{\mu + \nu\frac{\partial z}{\partial y}}{\Delta_1}.$$

Thus these latter relationships mean that the equations:

$$\lambda\frac{dx}{ds} + \mu\frac{dy}{ds} + \nu\frac{dz}{ds} = 0, \quad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

have as a consequence

$$\frac{\frac{dx}{ds}}{-\left(\mu + \nu\frac{\partial z}{\partial y}\right)} = \frac{\frac{dy}{ds}}{\lambda + \nu\frac{\partial z}{\partial x}} = \frac{\frac{dz}{ds}}{\lambda\frac{\partial z}{\partial y} - \mu\frac{\partial z}{\partial x}} = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}.$$

The sign of the latter ratio corresponds to the direction in which C is traversed, which is contained in the rule for applying Green's equation.

38. Introduction of New Auxiliary Quantities Obtained from the Consideration of Non-Orthogonal Triads Formed from Mz_1' and the Tangents to the Curves (ρ_1) and (ρ_2). /89 In Section 35, 3, we considered an orthogonal triad $Mx_1'y_1'z_1'$ whose axis Mz_1' is normal to (M). The equations for F_0'', G_0'', H_0'', I_0'', J_0'', K_0'' led us to introduce new auxiliary quantities. However, we may also consider the indefinite equations and refer them to the triad, which is no longer orthogonal in general, and which is formed by the axis Mz_1' and by the tangents to the curves (ρ_1) and (ρ_2). This can be readily done by using the calculations already carried out with the intermediary of $Mx_1'y_1'z_1'$. It is sufficient to start with the equations obtained with the latter and to obtain expressions for the combinations:

$$X_0''\xi_1^{(1)} + Y_0''\eta_1^{(1)}, \quad X_0''\xi_2^{(1)} + Y_0''\eta_2^{(1)}, \quad L_0''\xi_1^{(1)} + M_0''\eta_1^{(1)}, \quad L_0''\xi_2^{(1)} + M_0''\eta_2^{(1)}.$$

Let us set

$$\mathcal{A}_1'' = \xi_2^{(1)}A_1'' + \eta_2^{(1)}B_1'', \quad \mathcal{A}_2'' = \xi_2^{(1)}A_2'' + \eta_2^{(1)}B_2'',$$
$$\mathcal{B}_1'' = \xi_1^{(1)}A_1'' + \eta_1^{(1)}B_1'', \quad \mathcal{B}_2'' = \xi_1^{(1)}A_2'' + \eta_1^{(1)}B_2'',$$

and four analogous equations for $\mathcal{L}_1''$, $\mathcal{Q}_1''$, $\mathcal{L}_2''$, $\mathcal{Q}_2''$. We derive from them:

$$A_1'' = \frac{\eta_2^{(1)}\mathcal{B}_1'' - \eta_1^{(1)}\mathcal{A}_1''}{\Delta}, \quad A_2'' = \frac{\eta_2^{(1)}\mathcal{B}_2'' - \eta_1^{(1)}\mathcal{A}_2''}{\Delta},$$
$$B_1'' = \frac{\xi_1^{(1)}\mathcal{A}_1'' - \xi_2^{(1)}\mathcal{B}_1''}{\Delta}, \quad B_2'' = \frac{\xi_1^{(1)}\mathcal{A}_2'' - \xi_2^{(1)}\mathcal{B}_2''}{\Delta},$$

and analogous equations for P_1'', Q_1'', P_2'', Q_2''. The equations may be written as:

$$\frac{\partial\mathcal{A}_1''}{\partial\rho_1} + \frac{\partial\mathcal{A}_2''}{\partial\rho_2} - \Sigma_2\mathcal{A}_1'' - \Sigma_1\mathcal{A}_2'' - \Theta_2\mathcal{B}_1'' - \Theta_1\mathcal{B}_2'' - \Delta\mathfrak{D}'C_1'' - \Delta\mathfrak{D}'C_2'' = \Delta_0(\xi_2^{(1)}X_0'' + \eta_2^{(1)}Y_0''),$$

$$\frac{\partial \mathfrak{B}_1''}{\partial \rho_1}+\frac{\partial \mathfrak{B}_2''}{\partial \rho_2}-\Sigma_1\mathfrak{A}_1''-\Sigma_2\mathfrak{A}_2''-\theta_1\mathfrak{B}_1''-\theta_2\mathfrak{B}_2''-\Delta\mathfrak{D}C_1''-\Delta\mathfrak{D}'C_2''=\Delta_0(\xi_1^{(1)}X_0''+\eta_1^{(1)}Y_0''),$$

$$\frac{\partial C_1''}{\partial \rho_1}+\frac{\partial C_2''}{\partial \rho_2}+\frac{\mathcal{E}\mathfrak{D}'-\mathcal{F}\mathfrak{D}}{\Delta}\mathfrak{A}_1''+\frac{\mathcal{G}\mathfrak{D}-\mathcal{F}\mathfrak{D}'}{\Delta}\mathfrak{B}_1''+\frac{\mathcal{E}\mathfrak{D}'-\mathcal{F}\mathfrak{D}'}{\Delta}\mathfrak{A}_2''+\frac{\mathcal{G}\mathfrak{D}-\mathcal{F}\mathfrak{D}'}{\Delta}\mathfrak{B}_2''=\Delta_0 Z_0'',$$

$$\frac{\partial \mathfrak{L}_1''}{\partial \rho_1}+\frac{\partial \mathfrak{L}_2''}{\partial \rho_2}-\Sigma_2\mathfrak{L}_1''-\Sigma_3\mathfrak{L}_2''-\theta_2\mathfrak{Q}_1''-\theta_3\mathfrak{Q}_2''-\Delta\mathfrak{D}'R_1''-\Delta\mathfrak{D}'R_2''-\Delta C_1''=\Delta_0(\xi_2^{(1)}L_0''+\eta_2^{(1)}M_0''),$$

$$\frac{\partial \mathfrak{Q}_1''}{\partial \rho_1}+\frac{\partial \mathfrak{Q}_2''}{\partial \rho_2}-\Sigma_1\mathfrak{L}_1''-\Sigma_2\mathfrak{L}_2''-\theta_1\mathfrak{Q}_1''-\theta_2\mathfrak{Q}_2''-\Delta\mathfrak{D}R_1''-\Delta\mathfrak{D}'R_2''+\Delta C_2''=\Delta_0(\xi_1^{(1)}L_0''+\eta_1^{(1)}M_0''),$$

$$\frac{\partial R_1''}{\partial \rho_1}+\frac{\partial R_2''}{\partial \rho_2}+\frac{\mathcal{E}\mathfrak{D}'-\mathcal{F}\mathfrak{D}}{\Delta}\mathfrak{L}_1''+\frac{\mathcal{G}\mathfrak{D}-\mathcal{F}\mathfrak{D}'}{\Delta}\mathfrak{Q}_1''+\frac{\mathcal{E}\mathfrak{D}'-\mathcal{F}\mathfrak{D}'}{\Delta}\mathfrak{L}_2''+\frac{\mathcal{G}\mathfrak{D}'-\mathcal{F}\mathfrak{D}'}{\Delta}\mathfrak{Q}_2''$$
$$+\frac{\mathcal{E}\mathfrak{A}_1''+\mathcal{F}(\mathfrak{A}_2''-\mathfrak{B}_1'')-\mathcal{G}\mathfrak{B}_2''}{\Delta}=\Delta_0 N_0'',$$

In these equations, Σ_1, Σ_2, Σ_3, θ_1, θ_2, θ_3 designate the six Christoffel symbols:

$$\Sigma_1=\frac{-\mathcal{E}\frac{\partial \mathcal{E}}{\partial \rho_2}+2\mathcal{E}\frac{\partial \mathcal{F}}{\partial \rho_1}-\mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_1}}{2\Delta^2},\qquad \theta_1=\frac{\mathcal{G}\frac{\partial \mathcal{E}}{\partial \rho_1}+\mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_1}-2\mathcal{F}\frac{\partial \mathcal{F}}{\partial \rho_1}}{2\Delta^2},$$

$$\Sigma_2=\frac{\mathcal{E}\frac{\partial \mathcal{G}}{\partial \rho_1}-\mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_2}}{2\Delta^2},\qquad \theta_2=\frac{\mathcal{G}\frac{\partial \mathcal{E}}{\partial \rho_2}-\mathcal{F}\frac{\partial \mathcal{G}}{\partial \rho_1}}{2\Delta^2},$$

$$\Sigma_3=\frac{\mathcal{E}\frac{\partial \mathcal{G}}{\partial \rho_2}+\mathcal{F}\frac{\partial \mathcal{G}}{\partial \rho_1}-2\mathcal{F}\frac{\partial \mathcal{F}}{\partial \rho_2}}{2\Delta^2},\qquad \theta_3=\frac{2\mathcal{G}\frac{\partial \mathcal{F}}{\partial \rho_2}-\mathcal{G}\frac{\partial \mathcal{G}}{\partial \rho_1}-\mathcal{F}\frac{\partial \mathcal{G}}{\partial \rho_2}}{2\Delta^2},$$

and $\Delta^2\mathfrak{D}$, $\Delta^2\mathfrak{D}'$, $\Delta^2\mathfrak{D}''$ designate the three determinants defined by the identity(9): /90

$$\Delta^2(\mathfrak{D}d\rho_1^2+2\mathfrak{D}'d\rho_1d\rho_2+\mathfrak{D}''d\rho_2^2)=\begin{vmatrix}\frac{\partial x}{\partial \rho_1}\frac{\partial x}{\partial \rho_2}\frac{\partial^2 x}{\partial \rho_1^2}d\rho_1^2+2\frac{\partial^2 x}{\partial \rho_1\partial \rho_2}d\rho_1d\rho_2+\frac{\partial^2 x}{\partial \rho_2^2}d\rho_2^2\\ \frac{\partial y}{\partial \rho_1}\frac{\partial y}{\partial \rho_2}\frac{\partial^2 y}{\partial \rho_1^2}d\rho_1^2+2\frac{\partial^2 y}{\partial \rho_1\partial \rho_2}d\rho_1d\rho_2+\frac{\partial^2 y}{\partial \rho_2^2}d\rho_2^2\\ \frac{\partial z}{\partial \rho_1}\frac{\partial z}{\partial \rho_2}\frac{\partial^2 z}{\partial \rho_1^2}d\rho_1^2+2\frac{\partial^2 z}{\partial \rho_1\partial \rho_2}d\rho_1d\rho_2+\frac{\partial^2 z}{\partial \rho_2^2}d\rho_2^2\end{vmatrix}.$$

In the preceding calculations we have employed the relationships:

$$p_1^{(1)}=\xi_1^{(1)}\mathfrak{D}'-\xi_2^{(1)}\mathfrak{D},\qquad q^{(1)}=\eta_1^{(1)}\mathfrak{D}'-\eta_2^{(1)}\mathfrak{D},$$
$$p_2^{(1)}=\xi_1^{(1)}\mathfrak{D}''-\xi_2^{(1)}\mathfrak{D}',\qquad q_2^{(1)}=\eta_1^{(1)}\mathfrak{D}''-\eta_2^{(1)}\mathfrak{D}',$$

$$\frac{\partial \xi_1}{\partial \rho_1}-\eta_1r_1=\theta_1\xi_1+\Sigma_1\xi_2,\quad \frac{\partial \xi_2}{\partial \rho_1}-\eta_2r_1=\frac{\partial \xi_1}{\partial \rho_2}-\eta_1r_2=\theta_2\xi_1+\Sigma_2\xi_2,$$
$$\frac{\partial \xi_2}{\partial \rho_2}-\eta_2r_2=\theta_3\xi_1+\Sigma_3\xi_2,$$

$$\frac{\partial \eta_1}{\partial \rho_1}+\xi_1r_1=\theta_1\eta_1+\Sigma_1\eta_2,\quad \frac{\partial \eta_2}{\partial \rho_1}+\xi_2r_1=\frac{\partial \eta_1}{\partial \rho_2}+\xi_1r_2=\theta_2\eta_1+\Sigma_2\eta_2,$$
$$\frac{\partial \eta_2}{\partial \rho_2}+\xi_2r_2=\theta_3\eta_1+\Sigma_3\eta_2.$$

39. External Virtual Work. Theorem Analogous to the One of Varignon and Saint-Guilhem. Remarks on the Auxiliary Quantities Introduced in the Preceding Sections. We will employ *external virtual work* over the deformed surface (M) for an arbitrary virtual deformation to designate the expression:

(9) As we will state again later on, by $\Delta^2\mathfrak{D}$, $\Delta^2\mathfrak{D}'$, $\Delta^2\mathfrak{D}''$ we designate the quantities which M. Darboux designated by D, D', D".

$$\delta\mathfrak{T}_e = -\int_{C_0} (F_0'\delta' x + G_0'\delta' y + H_0'\delta' z + I_0'\delta\imath' + J_0'\delta\jmath' + K_0'\delta\kappa')\, ds_0$$
$$+ \iint_{C_0} (X_0'\delta' x + Y_0'\delta' y + Z_0'\delta' z + L_0'\delta\imath' + M_0'\delta\jmath' + N_0'\delta\kappa')\, \Delta_0\, d\rho_1\, d\rho_2.$$

This equation may be given in other forms by introducing different elements. For example, let us assume that we have introduced the expressions X_0, Y_0, Z_0, L_0, M_0, N_0; F_0, G_0, H_0, I_0, J_0, K_0. For this purpose, let us designate by δI, δJ, δK the projections on fixed axes of the segment whose projections along Mx', My', Mz' are $\delta I'$, $\delta J'$, $\delta K'$, in such a way that, for example, we have

$$-\delta\imath = \alpha''\delta\alpha' + \beta''\delta\beta' + \gamma''\delta\gamma' = -(\alpha'\delta\alpha'' + \beta'\delta\beta'' + \gamma'\delta\gamma''),$$

where we have always assumed axes with the same configuration. We thus obtain:

$$\delta\mathfrak{T}_e = -\int_{C_0} (F_0\delta x + G_0\delta y + H_0\delta z + I_0\delta\imath + J_0\delta\jmath + K_0\delta\kappa)\, ds_0$$
$$+ \iint_{C_0} (X_0\delta x + Y_0\delta y + Z_0\delta z + L_0\delta\imath + M_0\delta\jmath + N_0\delta\kappa)\, \Delta_0 d\rho_1 d\rho_2.$$

The force (X_0', Y_0', Z_0') or (X_0, Y_0, Z_0), the moment (L_0', M_0', N_0') or (L_0, M_0, N_0) are referred to the unit of area of the *non-deformed* surface. The stress (F_0', G_0', H_0') or (F_0, G_0, H_0) and the moment of deformation (I_0', J_0' K_0') or (I_0, J_0, K_0) are referred to the unit of length of the *non-deformed* contour C_0. Let us begin with the equation: /91

$$\iint_{C_0} \delta(W\Delta_0)\, d\rho_1 d\rho_2 = -\delta\mathfrak{T}_e,$$

applied to an arbitrary part of the deformable surface limited by a contour C_0.

$\delta(W\Delta_0)$ must be identically zero due to the invariance of W and of Δ_0 for the group of Euclidian displacements, and when the variations δx, δy, δz are given by equations (9), page 960, i.e.:

$$\delta x = (a_1 + \omega_2 z - \omega_3 y)\,\delta t,$$
$$\delta y = (a_2 + \omega_3 x - \omega_1 z)\,\delta t,$$
$$\delta z = (a_3 + \omega_1 y - \omega_2 x)\,\delta t,$$

and when δI, δJ, δK are given by

$$\delta\imath = \omega_1\delta t, \qquad \delta\jmath = \omega_2\delta t, \qquad \delta\kappa = \omega_3\delta t,$$

no matter what the value of the constants a_1, a_2, a_3, ω_1, ω_2, ω_3 may be. From the preceding expression for $\delta\mathfrak{T}_e$ we may conclude that we have:

$$\int_{C_0} F_0 ds_0 - \iint_{C_0} X_0\Delta_0 d\rho_1 d\rho_2 = 0, \qquad \int_{C_0} G_0 ds_0 - \iint_{C_0} Y_0\Delta_0 d\rho_1 d\rho_2 = 0,$$
$$\int_{C_0} H_0 ds_0 - \iint_{C_0} Z_0\Delta_0 d\rho_1 d\rho_2 = 0,$$

$$\int_{C_0} (I_0 + yH_0 - zG_0)\, ds_0 - \iint_{C_0} (L_0 + yZ_0 - zY_0)\, \Delta_0 d\rho_1 d\rho_2 = 0,$$

and two analogous equations.

These six equations may be easily derived from those which are usually written down employing the principle of solidification[10]. *We may therefore state that the contour* C_0 *is variable in the preceding equations.*

The auxiliary quantities introduced in the preceding sections are not the only ones we may consider. If we limit ourselves to the consideration of these, we would like to make some simple remarks.

In effect we have introduced two systems of stresses and moments of deformation for a point M of the deformed surface. The first are those which are applied to the curves (ρ_1) and (ρ_2). The others are those which are orthogonal /92 curves which have an arbitrary definition and must be specified more precisely, and whose tangents Mx_1', My_1' have arbitrary, rectangular directions in the plane tangent to (M) at M, and which must be defined more precisely.

The equations we have indicated give the latter elements in terms of the former, and, conversely, the former elements in terms of the latter ones.

Let us now assume that we have introduced the function W. The first stresses and moments of deformation correspond to the expressions already indicated, and we may immediately obtain expressions for the second ones. However, in the calculations it is possible to make the functions explicit which must be isolated depending on the nature of the problem, for example, which could be x, y, z and three parameters[11] λ_1, λ_2, λ_3 which are used to define α, α', ... γ''.

If we introduce x, y, z, λ_1, λ_2, λ_3 and if we continue to let W designate the function containing ρ_1, ρ_2, the first derivatives of x, y, z with respect to ρ_1, ρ_2, and λ_1, λ_2, λ_3 and their first derivatives with respect to ρ_1, ρ_2, obtained by replacing the various quantities ξ_i, η_i, ζ_i, p_i, q_i, r_i in the function $W(\rho_i, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i)$ by their values given by equations (30) and (31), we will then have:

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(10) The transition from the elements referred to a unit of area of (M_0) and a unit of length of C_0 to elements referred to a unit of area of (M) and unit of length of C is immediately possible. Therefore, it is sufficient to limit ourselves to the first case, as we shall do.

(11) For example, the three auxiliary quantities λ_1, λ_2, λ_3 may be chosen to be the components of rotation which make the axes Ox, Oy, Oz become parallel to Mx', My', Mz', respectively.

$$A_i = \Delta_0 \frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}},\ B_i = \Delta_0 \frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}},\ C_i = \Delta_0 \frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}},\ \Pi_i = \Delta_0 \frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial \rho_i}},\ X_i = \Delta_0 \frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial \rho_i}},\ \Sigma_i = \Delta_0 \frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial \rho_i}}.$$

40. Concept of the Energy of Deformation. Natural State of the Deformable Surface. Let us consider two states (M_0) and (M) of the deformable surface limited by the contours C_0 and C. Let us consider an arbitrary sequence of states beginning with (M_0) and ending in (M). For this purpose it is sufficient to consider functions $x, y, z, \alpha, \alpha', \ldots \gamma''$ of ρ_1, ρ_2 and of a variable h such that, for the value 0 of the variable h they become $x_0, y_0, z_0, \alpha_0, \alpha_0', \ldots \gamma_0''$, respectively, and for the value h of h they become the values $x, y, z, \alpha, \alpha', \ldots \gamma''$ for (M) respectively.

If we let the parameter h vary continuously from 0 to h, we will obtain a continuous deformation passing from the state (M_0) to the state (M). For this continuous deformation, let us consider the *total work* performed by the external forces and moments applied to the various surface elements and by the stresses and moments of deformation applied to the various elements of the contour. In order to obtain this total work, it is sufficient to integrate the total differential obtained, starting with one of the expressions for $\delta\mathcal{T}_e$ of the preceding section, from 0 to h. It is necessary to substitute for the variations of $x, y, z, \alpha, \alpha', \ldots \gamma''$ the partial derivatives which correspond to an increase dh of h. The equation

$$\delta\mathcal{T}_e = -\iint_{C_0} \delta(W\Delta_0)\, d\rho_1 d\rho_2$$

results in the expression $-\iint_{C_0} \frac{\partial(W\Delta_0)}{\partial h}\, dh d\rho_1 d\rho_2$ for the present value of /93

$\delta\mathcal{T}_e$. We thus obtain the following for the total work

$$-\int_0^h \left(\iint_{C_0} \frac{\partial(W\Delta_0)}{\partial h} d\rho_1 d\rho_2\right) dh = -\iint_{C_0} \left[(W\Delta_0)_M - (W\Delta_0)_{M_0}\right] d\rho_1 d\rho_2.$$

The work considered is independent of the intermediary states and only depends on the extreme states considered (M_0) and (M).

This leads to the introduction of the concept of the *energy of deformation* which should be distinguished from that of the action considered previously. We will say that -W is the *density of the energy of deformation* referred to the unit of area of the nondeformed surface.

These considerations are only the repetition of those we presented previously in Section 12. This is also true for the observations we made regarding the *natural state* of the deformable line. which can be repeated for the deformable surface and which were the subject of Section 13.

41. Concept of the Concealed Triad and the Concealed W. In the study of the deformable surface, as in the case of the deformable line, it is natural to focus one's attention on the geometrical surface described by the deformable surface in a special way. This amounts to considering x, y, z and $\alpha, \alpha', \ldots \gamma''$

as simple auxiliary quantities. We may also express this by stating that we will ignore the existence of triads which determine the deformable surface and for which we know only the apexes of these triads. If we take this point of view, in order to discuss the partial differential equations to which one is led depending on the case at hand, we may introduce the concept of the *concealed triad*, and we are led to a classification according to the different circumstances which can occur in the elimination of α, α', ... γ''.

Let us first study the reductions which can occur in the elimination of α, α', ... γ''. In the corresponding special cases where one almost exclusively considers the point surface traced out by the deformed surface (M), it is even sometimes possible to make an abstraction of (M_0), and, consequently, of the deformation which allows the transition from (M_0) to (M). This latter point of view is assumed in the discussion of the surface which is called flexible and inextensible by the geometricians.

The triad can be used in another way. We may make special hypotheses regarding it, which is also the case for (M). This all amounts to considering special deformations of the deformable, free surface. If the relationships which we assume are simple relationships between ξ_i, η_i, ζ_i, p_i, q_i, r_i, as will be the case in the applications which we will study, we will take these relationships into account in the calculation of W and will derive more special functions from W. The interesting question which poses itself will be the simple introduction of special forms and the assumption that the general function W is *concealed*, which will be used as the point of departure. We will thus obtain a /94 *special theory for the particular deformations specified by relationships between* ξ_i, η_i, ζ_i, p_i, q_i, r_i.

We therefore see that it is possible, using the theory of the free deformable surface, to obtain by means of special cases and to derive from a common origin the equations resulting from special problems which could previously only be attempted up to the present. In these latter cases, one sometimes attempts to avoid the consideration of the deformations. In reality, these should be treated. This may carried out in practice by considering the infinitesimally small deformation.

Let us consider the case where the external force and moment at the most contain the first derivatives of the unknowns x, y, z and λ_1, λ_2, λ_3. The second derivatives of these unknowns will only be introduced in the partial differential equations by means of W. Thus, the derivatives of x, y, z only appear in ξ_i, η_i, ζ_i, and those of λ_1, λ_2, λ_3 only occur in p_i, q_i, r_i. We therefore see that if W only depends on ξ_i, η_i, ζ_i, or only depends on p_i, q_i, r_i, the orders of the derivatives entering into the system of partial differential equations will be reduced. We will first consider the first of these two cases.

42. Case in which W only Depends on ρ_1, ρ_2, ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2. The Surface which, in the Case of Infinitesimally Small Deformations, Leads to the

Membrane Studied by Poisson and Lamé. The Fluid Surface, which as a Special Case Includes the Surface Considered by Lagrange, Poisson and M. Duhem. Let us assume that W only depends on the quantities ρ_i, ξ_i, η_i, ζ_i and not on p_i, q_i, r_i. The equations can be reduced to the following:

$$\frac{\partial}{\partial\rho_1}\frac{\partial(W\Delta_0)}{\partial\frac{\partial x}{\partial\rho_1}}+\frac{\partial}{\partial\rho_2}\frac{\partial(W\Delta_0)}{\partial\frac{\partial x}{\partial\rho_2}}=\Delta_0X_0,\quad \frac{\partial W}{\partial\lambda_1}+\Delta_0\mathfrak{X}_0=0,$$

$$\frac{\partial}{\partial\rho_1}\frac{\partial(W\Delta_0)}{\partial\frac{\partial y}{\partial\rho_1}}+\frac{\partial}{\partial\rho_2}\frac{\partial(W\Delta_0)}{\partial\frac{\partial y}{\partial\rho_2}}=\Delta_0Y_0,\quad \frac{\partial W}{\partial\lambda_2}+\Delta_0\mathfrak{M}_0=0,$$

$$\frac{\partial}{\partial\rho_1}\frac{\partial(W\Delta_0)}{\partial\frac{\partial z}{\partial\rho_1}}+\frac{\partial}{\partial\rho_2}\frac{\partial(W\Delta_0)}{\partial\frac{\partial z}{\partial\rho_2}}=\Delta_0Z_0,\quad \frac{\partial W}{\partial\lambda_3}+\Delta_0\mathfrak{N}_0=0,$$

in which W only depends on ρ_1, ρ_2, $\frac{\partial x}{\partial\rho_1}$, ... $\frac{\partial z}{\partial\rho_2}$, λ_1, λ_2, λ_3. They indicate that, if we consider the simple case where X_0, Y_0, Z_0, $\mathfrak{X}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are given functions(12) of ρ_1, ρ_2, x, y, z, $\frac{\partial x}{\partial\rho_1}$, ... $\frac{\partial z}{\partial\rho_2}$, λ_1, λ_2, λ_3, the three equations on the right can be solved with respect to λ_1, λ_2, λ_3. One finally obtains three partial differential equations which, using our hypotheses, only contain ρ_1, ρ_2, x, y, z and their first and second derivatives.

Let us limit ourselves to the special case in which the given functions /95
$\mathfrak{X}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are zero. This would also be true for the corresponding values of the functions of any one of the systems (L_0, M_0, N_0), (L_0', M_0', N_0'), (L_0'', M_0'', N_0''). As a result we find that the equations

$$\frac{\partial W}{\partial\lambda_1}=0,\quad \frac{\partial W}{\partial\lambda_2}=0,\quad \frac{\partial W}{\partial\lambda_3}=0$$

imply either

$$\frac{\partial y}{\partial\rho_1}C_1-\frac{\partial z}{\partial\rho_1}B_1+\frac{\partial y}{\partial\rho_2}C_2-\frac{\partial z}{\partial\rho_2}B_2=0,$$
$$\frac{\partial z}{\partial\rho_1}A_1-\frac{\partial x}{\partial\rho_1}C_1+\frac{\partial z}{\partial\rho_2}A_2-\frac{\partial x}{\partial\rho_2}C_2=0,$$
$$\frac{\partial x}{\partial\rho_1}B_1-\frac{\partial y}{\partial\rho_1}A_1+\frac{\partial x}{\partial\rho_2}B_2-\frac{\partial y}{\partial\rho_2}A_2=0,$$

or

$$S_1=0,\quad S_2=0,\quad S_3=0,$$

in such a way that the stress at a point of an aribtrary curve is in the plane tangent to the deformed surface and the shear stresses applied along two rectangular directions are equal.

With these assumptions, let us note that if we begin with two positions

(12) We assume that X_0, Y_0, Z_0, $\mathfrak{X}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ do not contain the derivatives of λ_1, λ_2, λ_3 in order to simplify the derivation and in order to make our point more easily.

(M_0) and (M) assumed to be *given* and if we derive the functions $\mathcal{X}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ from them, as was done in Sections 34 and 35, it may happen that in the case where the three functions are zero the result is obtained by accident, i.e., for a certain ensemble of special deformations. However, it may happen that this result is obtained no matter what the deformation (M) may be, whether it is a consequence of the nature of (M), i.e., of the form of W.

Let us consider the latter case which is of special interest. W is then a simple function(13) of ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, where the three latter quantities are defined by equations (32) of Section 31. The equations derived in Sections 34 and 35 are reduced to either:

$$\sum_i \left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i'\right) = \Delta_0 X_0', \qquad F_0' = A_1' \frac{d\rho_2}{ds_0} - A_2' \frac{d\rho_1}{ds_0},$$
$$\sum_i \left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i'\right) = \Delta_0 Y_0', \qquad G_0' = B_1' \frac{d\rho_2}{ds_0} - B_2' \frac{d\rho_1}{ds_0},$$
$$\sum_i \left(\frac{\partial C_i'}{\partial \rho_i} + p_i' B_i' - q_i' A_i'\right) = \Delta_0 Z_0', \qquad H_0' = C_1' \frac{d\rho_2}{ds_0} - C_2' \frac{d\rho_1}{ds_0},$$

where we have:

$$A_1' = \Delta_0 \left(2\xi_1 \frac{\partial W}{\partial \mathcal{E}} + \xi_2 \frac{\partial W}{\partial \mathcal{F}}\right), \qquad B_1' = \Delta_0 \left(2\eta_1 \frac{\partial W}{\partial \mathcal{E}} + \eta_2 \frac{\partial W}{\partial \mathcal{F}}\right),$$
$$C_1' = \Delta_0 \left(2\zeta_1 \frac{\partial W}{\partial \mathcal{E}} + \zeta_2 \frac{\partial W}{\partial \mathcal{F}}\right),$$
$$A_2' = \Delta_0 \left(\xi_1 \frac{\partial W}{\partial \mathcal{F}} + 2\xi_2 \frac{\partial W}{\partial \mathcal{G}}\right), \qquad B_2' = \Delta_0 \left(\eta_1 \frac{\partial W}{\partial \mathcal{F}} + 2\eta_2 \frac{\partial W}{\partial \mathcal{G}}\right),$$
$$C_2' = \Delta_0 \left(\zeta_1 \frac{\partial W}{\partial \mathcal{F}} + 2\zeta_2 \frac{\partial W}{\partial \mathcal{G}}\right),$$

or /96

$$\frac{\partial A_1}{\partial \rho_1} + \frac{\partial A_2}{\partial \rho_2} = \Delta_0 X_0, \qquad F_0 = A_1 \frac{d\rho_2}{ds_0} - A_2 \frac{d\rho_1}{ds_0},$$
$$\frac{\partial B_1}{\partial \rho_1} + \frac{\partial B_2}{\partial \rho_2} = \Delta_0 Y_0, \qquad G_0 = B_1 \frac{d\rho_2}{ds_0} - B_2 \frac{d\rho_1}{ds_0},$$
$$\frac{\partial C_1}{\partial \rho_1} + \frac{\partial C_2}{\partial \rho_2} = \Delta_0 Z_0, \qquad H_0 = C_1 \frac{d\rho_2}{ds_0} - C_2 \frac{d\rho_1}{ds_0},$$

where we have

$$A_1 = \Delta_0 \left(2 \frac{\partial x}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{E}} + \frac{\partial x}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{F}}\right), \qquad B_1 = \Delta_0 \left(2 \frac{\partial y}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{E}} + \frac{\partial y}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{F}}\right),$$
$$C_1 = \Delta_0 \left(2 \frac{\partial z}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{E}} + \frac{\partial z}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{F}}\right),$$
$$A_2 = \Delta_0 \left(\frac{\partial x}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{F}} + 2 \frac{\partial x}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{G}}\right), \qquad B_2 = \Delta_0 \left(\frac{\partial y}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{F}} + 2 \frac{\partial y}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{G}}\right),$$
$$C_2 = \Delta_0 \left(\frac{\partial z}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{F}} + 2 \frac{\partial z}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{G}}\right),$$

(13) The triad is completely concealed. We may also assume that we have a simple point surface.

or:

$$\sum_i \left(\frac{\partial A_i''}{\partial \rho_i} + q_i^{(i)} C_i'' - r_i^{(i)} B_i''\right) = \Delta_0 X_0'', \quad E_0'' = A_1'' \frac{d\rho_2}{ds_0} - A_2'' \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial B_i''}{\partial \rho_i} + r_i^{(i)} A_i'' - p_i^{(i)} C_i''\right) = \Delta_0 Y_0'', \quad G_0'' = B_1'' \frac{d\rho_2}{ds_0} - B_2'' \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial C_i''}{\partial \rho_i} + p_i^{(i)} B_i'' - q_i^{(i)} A_i''\right) = \Delta_0 Z_0'', \quad H_0'' = C_1'' \frac{d\rho_2}{ds_0} - C_2'' \frac{d\rho_1}{ds_0},$$

where we have:

$$A_1'' = \Delta_0 \left(2\xi_1^{(1)} \frac{\partial W}{\partial \mathcal{E}} + \xi_2^{(1)} \frac{\partial W}{\partial \mathcal{F}}\right), \quad B_1'' = \Delta_0 \left(2\eta_1^{(1)} \frac{\partial W}{\partial \mathcal{E}} + \eta_2^{(1)} \frac{\partial W}{\partial \mathcal{F}}\right),$$

$$C_1'' = \Delta_0 \left(2\zeta_1^{(1)} \frac{\partial W}{\partial \mathcal{E}} + \zeta_2^{(1)} \frac{\partial W}{\partial \mathcal{F}}\right),$$

$$A_2'' = \Delta_0 \left(\xi_1^{(1)} \frac{\partial W}{\partial \mathcal{F}} + 2\xi_2^{(1)} \frac{\partial W}{\partial \mathcal{G}}\right), \quad B_2'' = \Delta_0 \left(\eta_1^{(1)} \frac{\partial W}{\partial \mathcal{F}} + 2\eta_2^{(1)} \frac{\partial W}{\partial \mathcal{G}}\right),$$

$$C_2'' = \Delta_0 \left(\zeta_1^{(1)} \frac{\partial W}{\partial \mathcal{F}} + 2\zeta_2^{(1)} \frac{\partial W}{\partial \mathcal{G}}\right),$$

or, finally:

$$\frac{\partial}{\partial \rho_1} \begin{vmatrix} N_1 & \xi_2^{(1)} \\ T & \eta_2^{(1)} \end{vmatrix} - r_1^{(1)} \begin{vmatrix} T & \xi_2^{(1)} \\ N_2 & \eta_2^{(1)} \end{vmatrix} + \frac{\partial}{\partial \rho_2} \begin{vmatrix} \xi_1^{(1)} & N_1 \\ \eta_1^{(1)} & T \end{vmatrix} - r_2^{(1)} \begin{vmatrix} \xi_1^{(1)} & T \\ \eta_1^{(1)} & N_2 \end{vmatrix} = \Delta_0 X_0'',$$

$$\frac{\partial}{\partial \rho_1} \begin{vmatrix} T & \xi_2^{(1)} \\ N_2 & \eta_2^{(1)} \end{vmatrix} + r_1^{(1)} \begin{vmatrix} N_1 & \xi_2^{(1)} \\ T & \eta_2^{(1)} \end{vmatrix} + \frac{\partial}{\partial \rho_2} \begin{vmatrix} \xi_1^{(1)} & T \\ \eta_1^{(1)} & N_2 \end{vmatrix} + r_2^{(1)} \begin{vmatrix} \xi_1^{(1)} & N_1 \\ \eta_1^{(1)} & T \end{vmatrix} = \Delta_0 Y_0'',$$

$$p_1^{(1)} \begin{vmatrix} T & \xi_2^{(1)} \\ N_2 & \eta_2^{(1)} \end{vmatrix} - q_1^{(1)} \begin{vmatrix} N_1 & \xi_2^{(1)} \\ T & \eta_2^{(1)} \end{vmatrix} + p_2^{(1)} \begin{vmatrix} \xi^{(1)} & T \\ \eta^{(1)} & N_2 \end{vmatrix} - q_2^{(1)} \begin{vmatrix} \xi_1^{(1)} & N_1 \\ \eta^{(1)} & T \end{vmatrix} = \Delta_0 Z_0'',$$

$$E_0'' \frac{ds_0}{ds} = \lambda'' N_1 + \mu'' T, \quad G_0'' \frac{ds_0}{ds} = \lambda'' T + \mu'' N_2, \quad H_0'' = 0,$$

where we have: /97

$$N_1 = 2\frac{\Delta_0}{\Delta} \left\{ (\xi_1^{(1)})^2 \frac{\partial W}{\partial \mathcal{E}} + \xi_1^{(1)} \xi_2^{(1)} \frac{\partial W}{\partial \mathcal{F}} + (\xi_2^{(1)})^2 \frac{\partial W}{\partial \mathcal{G}} \right\},$$

$$T = \frac{\Delta_0}{\Delta} \left\{ 2\xi_1^{(1)} \eta_1^{(1)} \frac{\partial W}{\partial \mathcal{E}} + (\xi_2^{(1)} \eta_1^{(1)} + \xi_1^{(1)} \eta_2^{(1)}) \frac{\partial W}{\partial \mathcal{F}} + 2\xi_2^{(1)} \eta_2^{(1)} \frac{\partial W}{\partial \mathcal{G}} \right\},$$

$$N_2 = 2\frac{\Delta_0}{\Delta} \left\{ (\eta_1^{(1)})^2 \frac{\partial W}{\partial \mathcal{E}} + \eta_1^{(1)} \eta_2^{(1)} \frac{\partial W}{\partial \mathcal{F}} + (\eta_2^{(1)}) \frac{\partial W}{\partial \mathcal{G}} \right\}.$$

In this way we have stated that the stress is in the plane tangent to the deformed surface. N_1 and N_2 are normal stresses, i.e., tensile or compression stresses. T is a stress tangent to the linear element along which it acts, i.e., a shearing stress.

By considering an infinitesimally small deformation applied to the preceding surface, we may find again the surface or membrane studied by Poisson and Lamé[14].

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(14) Poisson. *Mémoire sur le Mouvement des Corps Élastiques (Memoir on the Motion of Elastic Bodies).* Page 488 ff., *Mém. de l'Inst. (Memoirs of the Institute)*, Volume VIII, 1829; G. Lamé, *Lecons sur la Théorie Mathématique de l'Élasticité des Corps Solides (Lessons on the Mathematical Theory of Elasticity of Solid Bodies)*, 2nd Edition, 1866, 9th and 10th Lessons.

Let us note that in addition to the formula specifying Δ which has already been used, we have the following:

$$\mathcal{E} = (\xi_1^{(1)})^2 + (\eta_1^{(1)})^2, \quad \mathcal{F} = \xi_1^{(1)}\xi_2^{(1)} + \eta_1^{(1)}\eta_2^{(1)}, \quad \mathcal{G} = (\xi_2^{(1)})^2 + (\eta_2^{(1)})^2,$$

from which it follows that N_1, T, N_2 may be considered as limiting functions of ρ_1, ρ_2 and of $\xi_1^{(1)}$, $\xi_2^{(1)}$, $\eta_1^{(1)}$, $\eta_2^{(1)}$.

An interesting special case which we will call the case of the *fluid surface*, is obtained by assuming that we have

$$T = 0, \quad N_1 = N_2.$$

where we take into account the three functions defined in this way.

If we note that we have the identities(15):

$$(\xi_1^{(1)})^2\,\mathcal{G} - 2\,\xi_1^{(1)}\xi_2^{(1)}\,\mathcal{F} + (\xi_2^{(1)})^2\,\mathcal{E} = \Delta^2,$$
$$\xi_1^{(1)}\eta_1^{(1)}\mathcal{G} - (\xi_1^{(1)}\eta_2^{(1)} + \xi_2^{(1)}\eta_1^{(1)})\,\mathcal{F} + \xi_2^{(1)}\eta_2^{(1)}\,\mathcal{E} = 0,$$
$$(\eta_1^{(1)})^2\,\mathcal{G} - 2\,\eta_1^{(1)}\eta_2^{(1)}\,\mathcal{F} + (\eta_2^{(1)})^2\,\mathcal{E} = \Delta^2,$$

which result from the values:

$$\mathcal{E} = (\xi_1^{(1)})^2 + (\eta_1^{(1)})^2, \quad \mathcal{F} = \xi_1^{(1)}\xi_2^{(1)} + \eta_1^{(1)}\eta_2^{(1)}, \quad \mathcal{G} = (\xi_2^{(1)})^2 + (\eta_2^{(1)})^2$$

of the expressions $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ defined by equation (32), it can be seen that the two conditions which we have imposed amount to the following:

$$\frac{\frac{\partial W}{\partial \mathcal{E}}}{\mathcal{G}} = -\frac{\frac{\partial W}{\partial \mathcal{F}}}{2\mathcal{F}} = \frac{\frac{\partial W}{\partial \mathcal{G}}}{\mathcal{E}},$$

which means that W only depends on $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ through the intermediary of the /98
quantity $\Delta = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}$, and consequently it is a function of ρ_1, ρ_2 and of $\mu = \frac{\Delta}{\Delta_0} - I$. If we continue to let W designate the expression for W in terms of ρ_1, ρ_2, μ, we will have:

$$N_1 = N_2 = \frac{\partial W}{\partial \mu}, \quad T = 0.$$

It is easy to obtain the special form which the various systems of equations take on and which are only combinations or simple consequences of one of them. In particular, the system on page 1048 takes on the following form, due to the equations which the $\xi_i^{(1)}$, ... $r_i^{(1)}$ satisfy and where N designates the expression $\frac{\partial W}{\partial \mu}$:

$$\eta_2^{(1)}\frac{\partial N}{\partial \rho_1} - \eta_1^{(1)}\frac{\partial N}{\partial \rho_2} = \Delta_0 X_0'$$
$$-\xi_2^{(1)}\frac{\partial N}{\partial \rho_1} + \xi_1^{(1)}\frac{\partial N}{\partial \rho_2} = \Delta_0 Y_0',$$
$$N\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right) = \frac{\Delta_0}{\Delta} Z_0'$$

(15) Due to the second of these identities, if T = 0, *no matter what the linear element may be*, we are led to the conditions which follow and consequently to $N_1 = N_2$. We may be satisfied in setting T = 0.

where we have used the equation:

$$\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}=\frac{p_1^{(1)}\xi_1^{(1)}-p_2^{(1)}\xi_2^{(1)}+q_1^{(1)}\eta_1^{(1)}-q_2^{(1)}\eta_2^{(1)}}{\xi_1^{(1)}\eta_2^{(1)}-\xi_2^{(1)}\eta_1^{(1)}},$$

in which the principal radii of curvature $\mathcal{R}_1$ and $\mathcal{R}_2$ of the deformed surface (M) appear.

If we consider the special case in which W only depends on μ, in which (M_0) does not occur explicitly, we find the surface considered by Lagrange[16] and which was later studied by M. Duhem[17]. We should make completely analogous remarks to those which we presented for the flexible and extensible thread of Lagrange. If, as Lagrange and M. Duhem assumed, the surface (M_0) does not appear explicitly, the surface (M_0) only appears through the quantity μ. Its existence is only manifested by this quantitiy. If we assume that the function W is *given*, for the term μ, which we may introduce as an auxiliary unknown in the usual problems, we may substitute the unknown N. If the function W is concealed, N becomes an auxiliary unknown in addition, but the knowledge of it will say nothing *with respect to* (M_0).

In the case where the surface (M_0) only appears through the quantity μ, we may take two other arbitrary variables in the place of ρ_1, ρ_2, for example, x, y, and if W is given, we have two unknowns and three equations. If W were concealed, μ would not be present in W and we arrive at the same case. In the first case, the remark made by Poisson was repudiated by M. Duhem[18]. Let us /99 carry out an explicit development of this remark by writing the equations in the form given by Lagrange and given more explicitly by Poisson and M. Duhem[19].

Let us solve the preceding equations with respect to $\frac{\partial N}{\partial \rho_1}$ and with respect to $\frac{\partial N}{\partial \rho_2}$. We obtain:

$$\frac{\partial N}{\partial \rho_1}=+\frac{\Delta_0}{\Delta}(X_0''\xi_1^{(1)}+Y_0''\eta_1^{(1)}),\qquad \frac{\partial N}{\partial \rho_2}=+\frac{\Delta_0}{\Delta}(X_0''\xi_2^{(1)}+Y_0''\eta_2^{(1)});$$

However, by introducing for a moment the direction cosines l, l', l'' of Mx_1', m, m', m'' of My_1', n, n', n'' of Mz_1' with respect to fixed axes, we have:

$$\xi_i^{(1)}=l\frac{\partial x}{\partial \rho_i}+l'\frac{\partial y}{\partial \rho_i}+l''\frac{\partial z}{\partial \rho_i},$$
$$\eta_i^{(1)}=m\frac{\partial x}{\partial \rho_i}+m'\frac{\partial y}{\partial \rho_i}+m''\frac{\partial z}{\partial \rho_i},$$

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(16) Lagrange. *Mécanique Analytique (Analytical Mechanics)*, 1st Part, Section V, Chapter III, Section II, Nos. 44-45, p. 158-162 of the 4th Edition.

(17) P. Duhem. *Hydrodynamique, Elasticité, Acoustique (Hydrodynamics, Elasticity, Acoustics)*, Volume II, p. 78 ff.

(18) P. Duhem. Ibid, Volume II, p. 92, at the top of the page.

(19) P. Duhem. Ibid, Volume II, p. 86 and 91.

$$\zeta_i^{(1)} = n\frac{\partial x}{\partial \rho_i} + n'\frac{\partial y}{\partial \rho_i} + n''\frac{\partial z}{\partial \rho_i},$$

and

$$X_0''\xi_i^{(1)} + Y_0''\eta_i^{(1)} = X_0\frac{\partial x}{\partial \rho_i} + Y_0\frac{\partial y}{\partial \rho_i} + Z_0\frac{\partial z}{\partial \rho_i}.$$

The preceding system can be written as:

$$\frac{\partial N}{\partial \rho_1} = \frac{\Delta_0}{\Delta}\left(X_0\frac{\partial x}{\partial \rho_1} + Y_0\frac{\partial y}{\partial \rho_1} + Z_0\frac{\partial z}{\partial \rho_1}\right),$$
$$\frac{\partial N}{\partial \rho_2} = \frac{\Delta_0}{\Delta}\left(X_0\frac{\partial x}{\partial \rho_2} + Y_0\frac{\partial y}{\partial \rho_2} + Z_0\frac{\partial z}{\partial \rho_2}\right),$$
$$N\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right) = \frac{\Delta_0}{\Delta}(X_0 n + Y_0 n' + Z_0 n'');$$

which is the one to be found on page 86 of Volume II of the already cited work of M. Duhem, except for the notation (only the direction of the normal to (M) is changed).

Let us introduce the variables x, y in place of ρ_1, ρ_2. Let us note that these two relationships, which contain the derivatives of N can be summarized as follows:

$$dN = \frac{\Delta_0}{\Delta}(X_0 dx + Y_0 dy + Z_0 dz),$$

which corresponds to the remark made by M. Duhem on the top of page 90 of Volume II of his work for the special case where only μ appears.

If x, y are taken as variables we have the system /100

$$\frac{\partial N}{\partial x} = \frac{\Delta_0}{\Delta}\left(X_0 + Z_0\frac{\partial z}{\partial x}\right),$$
$$\frac{\partial N}{\partial y} = \frac{\Delta_0}{\Delta}\left(Y_0 + Z_0\frac{\partial z}{\partial y}\right),$$
$$N\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right) = \frac{\Delta_0}{\Delta}(X_0 n + Y_0 n' + Z_0 n'').$$

which is nothing else than the system consisting of equations (31) and (32) of M. Duhem, except for the notation and a convenient convention for the direction of the normal.

Along with Poisson and M. Duhem, let us consider the case where $\frac{\Delta_0}{\Delta}X_0$, $\frac{\Delta_0}{\Delta}Y_0$, $\frac{\Delta_0}{\Delta}Z_0$ are given functions of x, y, z (we can even assume derivatives of z). We then obtain three equations containing two unknowns N, z.

In the special case where the given functions of x, y, z which we have just discussed are such that $\frac{\Delta_0}{\Delta}(X_0 dx + Y_0 dy + Z_0 dz)$ is the total differential of a function V, the system of three equations, which can be written as follows according to what we have said,

$$dN = \frac{\Delta_0}{\Delta}(X_0 dx + Y_0 dy + Z_0 dz),$$

$$N\left(\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}\right)=\frac{\Delta_0}{\Delta}(X_0 n+Y_0 n'+Z_0 n''),$$

becomes the following:

$$N-V=const.=C,$$

$$N\left(\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}\right)=n\frac{\partial V}{\partial x}+n'\frac{\partial V}{\partial y}+n''\frac{\partial V}{\partial z};$$

N is calculated from the equation

$$N=V+C,$$

and the surface (M) satisfies the equation(20):

$$(V+C)\left(\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}\right)=n\frac{\partial V}{\partial x}+n'\frac{\partial V}{\partial y}+n''\frac{\partial V}{\partial z}.$$

43. The Flexible and Inextensible Surface of the Geometricians. The Incompressible Fluid Surface. The Surface of M. Daniele. We have considered the /101 special case where W does not depend on p_i, q_i, r_i as well as various special variants of this case. We will now show how, by studying special deformations, we may obtain surfaces already considered by other authors, at least in part.

Let us first consider the simple case in which the triad is concealed, i.e., more precisely, we are considering a simple point surface and we assume the general case in which W is an arbitrary function of ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$.

1. We may consider the case where we focus our attention only on deformations of the surface for which we have:

$$\mathcal{E}=\mathcal{E}_0,\quad \mathcal{F}=\mathcal{F}_0,\quad \mathcal{G}=\mathcal{G}_0.$$

In the definitions for the forces, etc., it is sufficient to introduce these hypotheses. If the forces, etc. are given, it is sufficient to introduce these three conditions. In the latter case, the usual problems in which the function W is given or in the general case where $\mathcal{E}-\mathcal{E}_0$, $\mathcal{F}-\mathcal{F}_0$, $\mathcal{G}-\mathcal{G}_0$ are not zero cannot occur unless the given quantities are of a special type.

Let us now assume that only the function W_0, obtained by setting $\mathcal{E}=\mathcal{E}_0$, $\mathcal{F}=\mathcal{F}_0$, $\mathcal{G}=\mathcal{G}_0$ in W (ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$) is given. Also, we assume that we do not know the values of the derivatives of W with respect to $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ for $\mathcal{E}=\mathcal{E}_0$, $\mathcal{F}=\mathcal{F}_0$, $\mathcal{G}=\mathcal{G}_0$. Thus, we assume that W is *concealed*. We see that N_1, T, N_2 become three auxiliary quantities which must be added to x, y, z

(20) See Duhem, *Elasticité (Elasticity)*, etc., Volume II, p. 92, which was inspired by pages 179-181 of Poisson, *Mémoire sur les Surfaces Élastiques (Memoir on Elastic Surfaces)*, read August 1, 1814, published as an excerpt in the May 1815 issue of Volume III of the *Correspondance sur l'Ecole Polytechnique (Correspondence of the Polytechnic School)*, p. 154-159, and then in *Mémoires de l'Institut de France (Memoirs of the Institute of France)*, year 1812, Second Part, which appeared in 1816.

in such a way that, in the case where the forces acting on the surface elements are given, we have six partial differential equations with six unknowns. We only obtain a deterministic problem if we add additional conditions. If the deformed figure is specified *a priori*, we have three equations for the unknown functions N_1, T, N_2.

The equations which we have just discussed are those which define the flexible and inextensible surface of the geometricians.

2. We may assume that we are attempting to define a deformable surface *sui generis*, for which *the definition already takes into account* the conditions

$$\mathcal{E} = \mathcal{E}_0, \quad \mathcal{T} = \mathcal{T}_0, \quad \mathcal{G} = \mathcal{G}_0.$$

For the definition of the new surface, we will also define F_0', G_0', ... N_0' by the following identity,

$$\iint_{c_0} \delta(W\Delta_0)\,d\rho_1 d\rho_2 = \int_{c_0} (F_0'\delta' x + G_0'\delta' y + \ldots + K_0'\delta\kappa')\,ds_0$$
$$- \iint_{c_0} (X_0'\delta' x + Y_0'\delta' y + \ldots + N_0'\delta\kappa')\,\Delta_0 d\rho_1 d\rho_2;$$

where we are using the same ideas as before. However, this identity need only be satisfied for the equations:

$$\mathcal{E} = \mathcal{E}_0, \quad \mathcal{T} = \mathcal{T}_0, \quad \mathcal{G} = \mathcal{G}_0$$

and other terms. We will consider a surface for which the theory will result by adding *a posteriori* conditions $\mathcal{E} = \mathcal{E}_0$, $\mathcal{T} = \mathcal{T}_0$, $\mathcal{G} = \mathcal{G}_0$ to the knowledge of a function $W(\rho_1, \rho_2, \mathcal{E}, \mathcal{T}, \mathcal{G})$ of the auxiliary functions μ_1, μ_2, μ_3 of /102
ρ_1, ρ_2. For this purpose we will use the identity:

$$\iint_{c_0} [\delta W + \mu_1\delta(\mathcal{E} - \mathcal{E}_0) + \mu_2\delta(\mathcal{T} - \mathcal{T}_0) + \mu_3\delta(\mathcal{G} - \mathcal{G}_0)]\,\Delta d\rho_1 d\rho_2$$
$$= \int_{c_0} (F_0'\delta' x + G_0'\delta' y + \ldots + K_0'\delta\kappa')\,ds_0 - \iint_{c_0} (X_0'\delta' x + Y_0'\delta' y + \ldots + N_0'\delta\kappa')\,\Delta_0 d\rho_1 d\rho_2.$$

This is the same as replacing W by $W_1 = W + \mu_1(\mathcal{E} - \mathcal{E}_0) + \mu_2(\mathcal{T} - \mathcal{T}_0) + \mu_3(\mathcal{G} - \mathcal{G}_0)$ in the preceding general theory, and then letting $\mathcal{E} = \mathcal{E}_0$, $\mathcal{T} = \mathcal{T}_0$, $\mathcal{G} = \mathcal{G}_0$.

It can be seen that we again obtain the theory of flexible surface corresponding to the function W_1 of ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{T}$, $\mathcal{G}$, in which we limit ourselves to studying deformations corresponding to $\mathcal{E} = \mathcal{E}_0$, $\mathcal{T} = \mathcal{T}_0$, $\mathcal{G} = \mathcal{G}_0$.

If we consider the case of the *concealed* W_1, assuming that we only know the value $W_0(\rho_1, \rho_2)$, which W and W_1 simultaneously have for $\mathcal{E} = \mathcal{E}_0$, $\mathcal{T} = \mathcal{T}_0$, $\mathcal{G} = \mathcal{G}_0$, we again arrive at the classical theory of the flexible and inextensible surface.

Let us note that if, in order to establish the flexible and inextensible surface, we take the conditions $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$ in W into account *a priori* by a change in the auxiliary quantities, we are led to replace W by $\mu_1(\mathcal{E} - \mathcal{E}_0) + \mu_2(\mathcal{F} - \mathcal{F}_0) + \mu_3(\mathcal{G} - \mathcal{G}_0)$ in the calculations for the general deformable surface, and we obtain the equations which lead to the study of a flexible surface restricted to deformations which correspond to $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$. Assuming that μ_1, μ_2, μ_3 are not known, these equations again lead us to the flexible and inextensible surface of the geometricians. From this point of view we identically obtain the derivation given by Beltrami in Section 2 of his well known memoir. We should note that, in the case where X_0, Y_0, Z_0, expressed in terms of equations, are the partial derivatives of a function ϕ of ρ_1, ρ_2, x, y, z with respect to x, y, z, then the equations in which X_0, Y_0, Z_0 appear are nothing else than the equations for the extremals of the problem of the calculus of variations which consists of determining an extremum of the integral

$$\iint \Delta_0 \varphi \, d\rho_1 d\rho_2,$$

for the conditions

$$\mathcal{E} = \mathcal{E}_0, \quad \mathcal{F} = \mathcal{F}_0, \quad \mathcal{G} = \mathcal{G}_0.$$

Let us consider the case where the surface (M_0) is not one of the knowns and does not enter the problem. The variables ρ_1, ρ_2 appear as a system of coordinates to which the surface is referred. If these variables are not among the knowns, it is possible to arbitrarily introduce two other variables in their place. Taking this approach, which is the one that is generally taken, the equations written down above result in the various known equations studied by the authors by virtue of special cases. We will limit ourselves to giving a few bibliographical entries in the following section. /103

Let us assume that we start with a surface formed by means of a function W of ρ_1, ρ_2, Δ, or, if we wish, of ρ_1, ρ_2 and of $\mu = \frac{\Delta}{\Delta_0} - 1$. Let us assume that we focus our attention only(21) on the deformations of the surface, for which we have:

$$\mu = 0.$$

We are therefore dealing with the case of the *incompressible fluid surface*. It is sufficient to introduce this hypothesis in the definitions of the forces, etc., and, if these forces are given, to impose this condition. In the latter case, the usual problems corresponding to specifying the function W and corresponding to the general case where μ is not zero require that the knowns be of a special type.

(21) It seems that this is the point of view that M. Duhem took in his work: *Hydrodynamique (Hydrodynamics)*, etc.; see p. 91 of Volume II, the four last lines, and p. 92 at the end of Section 5.

If we assume that *only* the function W_0, obtained by letting $\mu = 0$ in $W(\rho_1, \rho_2, \mu)$, is given, and that we do not know the value of $\frac{\partial W}{\partial \mu}$ for $\mu = 0$, and that therefore W is concealed, we can see that the expression N becomes an auxiliary quantity which we must add to x, y, z in such a way that in the case of given forces we have four equations and four unknowns.

We may also start with a function W which may contain the ξ_i, η_i, ζ_i as well as the p_i, q_i, r_i and may attempt to find the form which it must have such that the stress acting on an arbitrary linear element is normal to it and, in addition, that it is in the plane tangent to (M). It is necessary and sufficient that W does not depend on ξ_i, η_i, ζ_i except through the intermediary of the expression $\Delta = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}$.

We would also like to mention the case where the surface is derived from a function $(\rho_1, \rho_2, \mathcal{E}, \mathcal{G}, \mathcal{F})$ by adding conditions $\mathcal{E} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$. In the case where W does not depend on $\mathcal{F}$, we obtain a surface whose study was started by Daniele(22). The case where W depends on $\mathcal{F}$ may be approximated in an interesting way depending on the way this is done for the flexible and inextensible surface. It seems as though this latter one corresponds better to what one calls the reinforced surfaces, or to coverings such as those for balloons made of an elastic material, which has inextensible threads imbedded in it.

44. Some Bibliographical Entries for the Flexible and Inextensible Surface of the Geometricians. The flexible and inextensible surface of the geometricians has given rise to a large number of works in the area of mechanics. It seems useful to give the following bibliographical entries which deal with this surface.

Lagrange. *Mécanique Analytique (Analytical Mechanics)*, 3rd Edition, 1st Part, Section V, Chapter III, Section 2, p. 138-143; Note of J. Bertrand, p. 140; 4th Edition, Volume XI of the *OEuvres de Lagrange (Works of Lagrange)*, 1st Part, Section V, Chapter III, Section 2, p. 156-162; Note of M. Darboux, p. 160. /104

Poisson. *Report on Elastic Surfaces;* read August 14, 1814; included in the *Mémoires de la Classe des Sciences Mathématiques et Physiques de l'Institut de France (Memoirs of the Mathematics and Physics Science Class of the French Institute)*, year 1812, 2nd Part, p. 167-225.

Cisa de Grésy. *Considerations of the Equilibrium of Flexible and Inextensible Surfaces (Memorie della R. Accademia delle Scienze di Torino)*, Volume XXIII, Part 1, p. 259-294, 1818).

Bordoni. *Abstract Equilibrium of Rotation (Memorie di Matématica e di Fisica della Società Italiana delle Scienze*, resident in Modena, 19, p. 155-186, 1821); *Memorie dell' I. R. Istituto Lombardo di Scienze, Lettere ed Arti*,

(22) E. Daniele. *Equilibrium of Threads, Rend. del Circolo Matematico di Palermo*, 13, p. 28-85, 1899.

9, p. 126-142, 1863; *Stability and Equilibrium of an Embankment (Memorie di Matematica et di Fisica della Società Italiana delle Scienze*, resident in Modena, 24, p. 75-112, 1850); *Problems of Curves in Roads (Memorie dell' I. R. Istituto Lombardo di Scienze. Lettere ed Arti*, 9, p. 143-154, 1863).

Mossotti. *Lezioni di Meccanica Razionale (Lessons in Rational Mechanics)*, Firenze, 1851.

Brioschi. *Regarding Certain Points of the Theory of Surfaces (Annali di Tortolini*, 3, p. 293-321, 1852).

Jellet. *On the Properties of Inextensible Surfaces (Transactions of the Royal Irish Academy*, 22, p. 343-378, 1853).

Mainardi. *Note on Several Arguments of Rational and Applied Mechanics (Giornale dell' I. R. Istituto Lombardo di Scienze, Lettere ed Arti*, 8, p. 304-308, 1856).

Lecornu. *The Equilibrium of Flexible and Inextensible Surfaces.* C. R., 91, p. 809-812, 1880; *Journal de l'Ecole Polytechnique*, 48th Chapter, p. 1-109, 1880.

Beltrami. *The Equilibrium of Flexible and Inextensible Surfaces (Memorie della Academia delle Scienze dell' Instituto di Bologna)*, Series 4, 3, p. 217-265, 1882.

Kötter. *Equilibrium of Flexible, Inextensible Surfaces, Inaugural Dissertation*, Halle, February 6, 1883. Application of Abel Functions to a Problem of the Statics of Flexible, Inextensible Surfaces (*Journal für die Reine und Angewandte Mathematik*, 103, p. 44-74, 1888).

Morera. *The Equilibrium of Flexible and Inextensible Surfaces (Atti della Reale Accademia dei Lincei, Transunti*, Series 3, 7, p. 268-270, 1883).

Volterra. *The Equilibrium of Flexible and Inextensible Surfaces.* Note I and Note II *(Atti della R. Acc. dei Lincei. Transunti*, Series 3, 8, p. 214-217, 244-246, 1884); *The Deformation of Flexible and Inextensible Surfaces (Atti della R. Accad. dei Lincei, Rendiconti*, Series 4, 1, p. 274-278, 1885).

Maggi. *The EQuilibrium of Flexible and Inextensible Surfaces (Rendiconti del R. Istituto Lombardo di Scienze ed Lettere*, Series 2, 17, p. 683-694, 1884).

Padova. *Researches on the Equilibrium of Flexible and Inextensible Surfaces*, Note I and Note II *(Atti della R. Acc. dei Lincei, Rendiconti*, Series 4, 1, p. 269-274, 306-309, 1885).

Pennacchietti. *The Equilibrium of Flexible and Inextensible Surfaces (Palermo Rend.*, 9, p. 87-95, 1895). *The Equations of Equilibrium of Flexible and Inextensible Surfaces (Atti Acc. Gioenia* (4), 8, 1895), *The Integration of the Equations of Equilibrium of Flexible and Inextensible Surfaces (Atti Acc. Gioenia*, (4), 8, 1895).

Rakhmaninov. *Equilibrium of a Flexible Inextensible Surface* (In Russian). *(Recueil de la Soc. Math. de Moscou*, 19, p. 110-181, 1895).

Lecornu. *The Equilibrium of an Ellipsoidal Envelope (Comptes Rendus*, 122, p. 218-220, 1896; *Annales de l'Ecole Normale Supérieure* (3), 17, p. 501-539, 1900.

De Francesco. *The Motion of a Thread and the Equilibrium of a Flexible and Inextensible Surface. Napoli Rend.*, (3), 9, p. 227, 1903; *Napoli Atti* (2), 12, 1905. /105

45. Deformable Surface Obtained Assuming that Mz' is Normal to the Surface (M). We will attempt to introduce the condition that Mz' is normal to the surface (M). We may do this by either departing from the deformable surface described above and by studying only deformations of this surface which satisfy

the conditions

$$\zeta_1 = 0, \quad \zeta_2 = 0, \tag{42}$$

or by defining a new deformable surface for which we will develop a theory analogous to that of the first one but taking conditions (42) into account.

Let us assume the former case and let us study the deformations of (M) which satisfy (42). Also, let us assume that(23) we have $\zeta_1^{(0)} = \zeta_2^{(0)} = 0$, which comes from the study of infinitesimally small deformations, in order to have a continuous sequence of surfaces beginning with (M_0).

It is sufficient to introduce hypotheses (42) in the equations of Sections 34 and following in order to obtain expressions for the various elements which occur in this theory. Conversely, if, by way of an exampl , we specify the external forces and moments we must add the two equations (42) to the six equations which result from this specification, which shows that *if the function* W *which serves as the point of departure is given*, it is not possible to arbitrarily specify the external forces and moments.

However, let us note that by limiting ourselves to the study of deformations which satisfy (42), we are primarily interested in establishing a special surface. We are thus led to distinguishing three cases, by following this idea: 1. The function W is concealed, and we only know the function W_0 for the special deformations which are specified by and constitute the essential elements of these deformations; 2. The function W remains *concealed*, (i.e., it is not given); we know the relationships (differential relationships, for example), which relate W_0 and the outlines of the function W (three functions in this case); 3. The function W always remains concealed; we partially or completely know the functions which indicate the existence of W to us.

Let us develop this feature by carrying out the calculations in detail. Due to conditions (42), the triad, instead of depending on the six parameters x, y, z, λ_1, λ_2, λ_3 only depends on four parameters, for example on x, y, z, m, where m designates the angle defined by the equation:

$$\operatorname{tg} m = \frac{\eta_1}{\xi_1},$$

which is one of the angles the axis Mx' makes with the curve (ρ_2) of (M).

Let us designate by $\Delta^2 \mathcal{D}$, $\Delta^2 \mathcal{D}'$, $\Delta^2 \mathcal{D}''$ the determinants defined by the identity which we have written in Section 38, page 90, which only depends on the derivatives of x, y, z and are independent of m and its derivatives. Let us also recall the equations of the same paragraph (Christoffel symbols): /106

$$\Sigma_1 = \frac{-\mathcal{E}\frac{\partial \mathcal{S}}{\partial \rho_2} + 2\mathcal{E}\frac{\partial \mathcal{F}}{\partial \rho_1} - \mathcal{F}\frac{\partial \mathcal{S}}{\partial \rho_1}}{2\Delta^2},$$

(23) The conditions $\zeta_1^{(0)} = \zeta_2^{(0)} = 0$ can be omitted in our present derivation and do not appear in the study of the infinitesimally small deformation.

$$\Sigma_2 = \frac{\mathcal{E}\frac{\partial \mathcal{G}}{\partial \rho_1} - \mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_2}}{2\Delta^2},$$

and, from the usual conventions,

$$\Delta = \xi_1\eta_2 - \xi_2\eta_1.$$

We obtain the following equations[24] for determining the rotations p_1, q_1, r_1, p_2, q_2, r_2

$$p_1 = \xi_1\mathfrak{D}' - \xi_2\mathfrak{D}, \qquad q_1 = \eta_1\mathfrak{D}' - \eta_2\mathfrak{D},$$
$$p_2 = \xi_1\mathfrak{D}'' - \xi_2\mathfrak{D}', \qquad q_2 = \eta_1\mathfrak{D}'' - \eta_2\mathfrak{D}',$$
$$r_1 = -\frac{\partial m}{\partial \rho_1} + \frac{\Sigma_1\Delta}{\mathcal{E}}, \qquad r_2 = -\frac{\partial m}{\partial \rho_2} + \frac{\Sigma_2\Delta}{\mathcal{E}}.$$

The translations can be calculated beforehand by the system

$$\frac{\eta_1}{\xi_1} = \operatorname{tg} m, \qquad \xi_1^2 + \eta_1^2 = \mathcal{E}, \qquad \xi_1\xi_2 + \eta_1\eta_2 = \mathcal{F}, \qquad \xi_2^2 + \eta_2^2 = \mathcal{G}.$$

It can be seen that the translations are expressed in terms of m and first derivatives with respect to x, y, z. The rotations p_1, q_1, p_2, q_2 are expressed in terms of m and the first and second derivatives of x, y, z. Finally, the rotations r_1, r_2 are expressed in terms of the derivatives of m and the first and second derivatives of x, y, z.

If we introduce these values into the function obtained by setting $\zeta_1 = \zeta_2 = 0$ in W, a function which we will call W^0 in order to avoid confusion, we obtain the function W_0 of ρ_1, ρ_2, m, $\frac{\partial m}{\partial \rho_1}$, $\frac{\partial m}{\partial \rho_2}$ of x, y, z and of their first and second derivatives, which consequently depends on the quantities m, $\frac{\partial m}{\partial \rho_1}$, ... through the intermediary of the nine independent expressions:

$$m, \mathcal{E}, \mathcal{F}, \mathcal{G}, r_1, r_2, \mathfrak{D}, \mathfrak{D}', \mathfrak{D}'',$$

or, which amounts to the same thing, of the nine independent expressions ξ_1, η_1, ξ_2, η_2, r_1, r_2, $\mathfrak{D}$, $\mathfrak{D}'$, $\mathfrak{D}''$.

Let us designate by W_0' the function of these nine latter quantities which results in W_0 when their values are substituted. W_0' results from W^0 by the substitution of p_1, q_1, p_2, q_2.

We have a function W_0' which in addition to ρ_1, ρ_2 contains the *nine arguments* which have just been enumerated, whereas W, in addition to ρ_1, ρ_2, ζ_1, ζ_2

(24) M. Darboux. *Lecons (Lessons)*, Volume II, p. 363, p. 378-379, Nos. 495 and 503 give identical or equivalant equations. By $\Delta^2\mathfrak{D}$, $\Delta^2\mathfrak{D}'$, $\Delta^2\mathfrak{D}''$ we will designate the quantities which M. Darboux designated by D, D', D''.

contains the *ten arguments* ξ_1, η_1, ξ_2, η_2, p_1, q_1, r_1, p_2, q_2, r_2.

Let us pause here and consider this important point which results from the /107 treatment of one of the equations which M. Darboux designated as equations of Codazzi, i.e., $p_1\eta_2 - q_1\xi_2 - p_2\eta_1 + q_2\xi_1 = 0$. Let us study the equations of statics of the deformable surface in the present case.

The function W_0' is derived from W^0 by the substitution of the values p_1, q_1, p_2, q_2

$$p_1 = \xi_1\mathfrak{D}' - \xi_2\mathfrak{D}, \qquad q_1 = \eta_1\mathfrak{D}' - \eta_2\mathfrak{D},$$
$$p_2 = \xi_1\mathfrak{D}' - \xi_2\mathfrak{D}', \qquad q_2 = \eta_1\mathfrak{D}' - \eta_2\mathfrak{D}',$$

and we find that

$$\frac{\partial W_0'}{\partial \xi_1} = \frac{\partial W^0}{\partial \xi_1} + \frac{\partial W^0}{\partial p_1}\mathfrak{D}' + \frac{\partial W^0}{\partial p_2}\mathfrak{D}', \qquad \frac{\partial W_0'}{\partial \xi_2} = \frac{\partial W^0}{\partial \xi_2} - \frac{\partial W^0}{\partial p_1}\mathfrak{D} - \frac{\partial W^0}{\partial p_2}\mathfrak{D}',$$
$$\frac{\partial W_0'}{\partial \eta_1} = \frac{\partial W^0}{\partial \eta_1} + \frac{\partial W^0}{\partial q_1}\mathfrak{D}' + \frac{\partial W^0}{\partial q_2}\mathfrak{D}', \qquad \frac{\partial W_0'}{\partial \eta_2} = \frac{\partial W^0}{\partial \eta_2} - \frac{\partial W^0}{\partial q_1}\mathfrak{D} - \frac{\partial W^0}{\partial q_2}\mathfrak{D}',$$
$$\frac{\partial W_0'}{\partial r_1} = \frac{\partial W^0}{\partial r_1}, \qquad \frac{\partial W_0'}{\partial r_2} = \frac{\partial W^0}{\partial r_2},$$
$$\frac{\partial W_0'}{\partial \mathfrak{D}} = -\frac{\partial W^0}{\partial p_1}\xi_2 - \frac{\partial W^0}{\partial q_1}\eta_2, \qquad \frac{\partial W_0'}{\partial \mathfrak{D}'} = \xi_1\frac{\partial W^0}{\partial p_2} + \eta_1\frac{\partial W^0}{\partial q_2},$$
$$\frac{\partial W_0'}{\partial \mathfrak{D}'} = \xi_1\frac{\partial W^0}{\partial p_1} - \xi_2\frac{\partial W^0}{\partial p_2} + \eta_1\frac{\partial W^0}{\partial q_1} - \eta_2\frac{\partial W^0}{\partial q_2}.$$

We continue to designate by W^0 the result of the substitution of p_1, p_2, q_1, q_2.

Let us assume that we have introduced the expressions for the quantities into these equations by means of m, $\frac{\partial m}{\partial \rho_1}$, ... , and that we have taken (42) into account. We should note that the equations

$$C_1' = \Delta_0\frac{\partial W}{\partial \zeta_1}, \qquad C_2' = \Delta_0\frac{\partial W}{\partial \zeta_2}$$

do not allow us to calculate C_1', C_2' if W is concealed, because it is necessary to take (42) into account. However, the other equations will furnish the other expressions A_1', ..., in terms of the values of the derivatives of W^0. We have, for example:

$$\left(\frac{\partial W}{\partial p_1}\right)_{\zeta_1=0,\ \zeta_2=0} = \frac{\partial W^0}{\partial p_1}.$$

The nine equations which we have just written down now result in:

$$\Delta_0\frac{\partial W_0}{\partial \xi_1} = A_1' + \mathfrak{D}'P_1' + \mathfrak{D}'P_2', \qquad \Delta_0\frac{\partial W_0}{\partial \xi_2} = A_2' - \mathfrak{D}P_1' - \mathfrak{D}'P_2',$$
$$\Delta_0\frac{\partial W_0}{\partial \eta_1} = B_1' + \mathfrak{D}'Q_1' + \mathfrak{D}'Q_2', \qquad \Delta_0\frac{\partial W_0}{\partial \eta_2} = B_2' - \mathfrak{D}Q_1' - \mathfrak{D}'Q_2',$$
$$\Delta_0\frac{\partial W_0}{\partial r_1} = R_1', \qquad \Delta_0\frac{\partial W_0}{\partial r_2} = R_2',$$
$$\Delta_0\frac{\partial W_0}{\partial \mathfrak{D}} = -\xi_2P_1' - \eta_2Q_1', \qquad \Delta_0\frac{\partial W_0}{\partial \mathfrak{D}'} = \xi_1P_2' + \eta_1Q_2',$$
$$\Delta_0\frac{\partial W_0}{\partial \mathfrak{D}'} = \xi_1P_1' + \eta_1Q_1' - \xi_2P_2' - \eta_2Q_2',$$

where we write W_0 instead of W_0' to indicate that the arguments ξ_1, ... $\mathcal{D}''$ /108
must be replaced by their values as a function of m, $\frac{\partial m}{\partial \rho_1}$, ...

For the calculation of the ten auxiliary quantities A_1', ... in addition to C_1', C_2', we only have nine equations if the function W_0 is only known. Thus, *when* W_0 *alone is known*, there remain three arbitrary quantities.

In order to study the system of equations for the statics of the deformable surface, let us apply the equations for the triad $Mx_1'y_1'z_1'$ to the triad $Mx'y'z'$, in which the following auxiliary quantities appear and are defined by the equations:

$$\mathcal{A}_1' = \xi_2 A_1' + \eta_2 B_1', \qquad \mathcal{A}_2' = \xi_2 A_2' + \eta_2 B_2',$$
$$\mathcal{B}_1' = \xi_1 A_1' + \eta_1 B_1', \qquad \mathcal{B}_2' = \xi_1 A_2' + \eta_1 B_2',$$

and by four analogous ones for $\mathcal{P}_1'$, $\mathcal{Q}_1'$, $\mathcal{P}_2'$, $\mathcal{Q}_2'$. The nine preceding equations can be written as:

$$\Delta_0\left(\xi_2\frac{\partial W_0}{\partial \xi_1}+\eta_2\frac{\partial W_0}{\partial \eta_1}\right)=\mathcal{A}_1'+\mathcal{D}'\mathcal{P}_1'+\mathcal{D}''\mathcal{P}_2',\ \Delta_0\left(\xi_2\frac{\partial W_0}{\partial \xi_2}+\eta_2\frac{\partial W_0}{\partial \eta_2}\right)=\mathcal{A}_2'-\mathcal{D}\mathcal{P}_1'-\mathcal{D}'\mathcal{P}_2',$$
$$\Delta_0\left(\xi_1\frac{\partial W_0}{\partial \xi_1}+\eta_1\frac{\partial W_0}{\partial \eta_1}\right)=\mathcal{B}_1'+\mathcal{D}'\mathcal{Q}_1'+\mathcal{D}''\mathcal{Q}_2',\ \Delta_0\left(\xi_1\frac{\partial W_0}{\partial \xi_2}+\eta_1\frac{\partial W_0}{\partial \eta_2}\right)=\mathcal{B}_2'-\mathcal{D}\mathcal{Q}_1'-\mathcal{D}'\mathcal{Q}_2',$$
$$\Delta_0\frac{\partial W_0}{\partial r_1}=R_1', \qquad \Delta_0\frac{\partial W_0}{\partial r_2}=R_2',$$
$$\Delta_0\frac{\partial W_0}{\partial \mathcal{D}}=-\mathcal{P}_1', \qquad \Delta_0\frac{\partial W_0}{\partial \mathcal{D}'}=\mathcal{Q}_1'-\mathcal{P}_2', \qquad \Delta_0\frac{\partial W_0}{\partial \mathcal{D}''}=\mathcal{Q}_2'.$$

Let us consider the six equilibrium equations given in Section 38. The two first ones of the second group result in $\Delta C_1'$ and $\Delta C_2'$

$$\Delta C_1'=\frac{\partial \mathcal{P}_1'}{\partial \rho_1}+\frac{\partial \mathcal{P}_2'}{\partial \rho_2}-\Sigma_1\mathcal{P}_1'-\Sigma_2\mathcal{P}_2'-\theta_1\mathcal{Q}_1'-\theta_2\mathcal{Q}_2'-\Delta\mathcal{D}'R_1'-\Delta\mathcal{D}''R_2'-\Delta_0(\xi_2L_0'+\eta_2M_0')$$
$$-\Delta C_2'=\frac{\partial \mathcal{Q}_1'}{\partial \rho_1}+\frac{\partial \mathcal{Q}_2'}{\partial \rho_2}-\Sigma_1\mathcal{P}_1'-\Sigma_2\mathcal{P}_2'-\theta_1\mathcal{Q}_1'-\theta_2\mathcal{Q}_2'-\Delta\mathcal{D}R_1'-\Delta\mathcal{D}'R_2'-\Delta_0(\xi_1L_0'+\eta_1M_0')$$

Let us introduce these values in the three equations of the first group. Let us write the third equation of the second group, and we obtain the system:

$$\begin{aligned}
U_2=&\frac{\partial}{\partial \rho_1}(\mathcal{A}_1'+\mathcal{D}'\mathcal{P}_1'+\mathcal{D}''\mathcal{P}_2')+\frac{\partial}{\partial \rho_2}(\mathcal{A}_2'-\mathcal{D}\mathcal{P}_1'-\mathcal{D}'\mathcal{P}_2')+\mathcal{D}\frac{\partial \mathcal{P}_1'}{\partial \rho_2}-2\mathcal{D}'\frac{\partial \mathcal{P}_1'}{\partial \rho_2}\\
&+\mathcal{D}''\frac{\partial \mathcal{Q}_2'}{\partial \rho_2}+\mathcal{D}''\frac{\partial}{\partial \rho_1}(\mathcal{Q}_1'-\mathcal{P}_2')-\Sigma_1(\mathcal{A}_1'+\mathcal{D}'\mathcal{P}_1'+\mathcal{D}''\mathcal{P}_2')-\Sigma_2(\mathcal{A}_2'-\mathcal{D}\mathcal{P}_1'-\mathcal{D}'\mathcal{P}_2')\\
&-\theta_1(\mathcal{B}_1'+\mathcal{D}'\mathcal{Q}_1'+\mathcal{D}''\mathcal{Q}_2')-\theta_2(\mathcal{B}_2'-\mathcal{D}\mathcal{Q}_1'-\mathcal{D}'\mathcal{Q}_2')+2(2\Sigma_2\mathcal{D}'-\Sigma_1\mathcal{D}''-\Sigma_2\mathcal{D})\mathcal{P}_1'\\
&-(\theta_2\mathcal{D}-2\theta_2\mathcal{D}'+\theta_1\mathcal{D}'')(\mathcal{Q}_1'-\mathcal{P}_2')-\Delta(\mathcal{D}\mathcal{D}''-\mathcal{D}'^2)R_1'-\Delta_0(\xi_2X_0'+\eta_2Y_0')\\
&+\Delta_0\mathcal{D}'(\xi_2L_0'+\eta_2M_0')-\Delta_0\mathcal{D}''(\xi_1L_0'+\eta_1M_0')=0,
\end{aligned}$$

$$\begin{aligned}
U_1=&\frac{\partial}{\partial \rho_1}(\mathcal{B}_1'+\mathcal{D}'\mathcal{Q}_1'+\mathcal{D}''\mathcal{Q}_2')+\frac{\partial}{\partial \rho_2}(\mathcal{B}_2'-\mathcal{D}\mathcal{Q}_1'-\mathcal{D}'\mathcal{Q}_2')+\mathcal{D}\frac{\partial}{\partial \rho_2}(\mathcal{Q}_1'-\mathcal{P}_2')\\
&+2\mathcal{D}'\frac{\partial \mathcal{Q}_2'}{\partial \rho_2}-\mathcal{D}''\frac{\partial \mathcal{Q}_2'}{\partial \rho_1}-\mathcal{D}\frac{\partial \mathcal{P}_1'}{\partial \rho_1}-\Sigma_1(\mathcal{A}_1'+\mathcal{D}'\mathcal{P}_1'+\mathcal{D}''\mathcal{P}_2')-\Sigma_2(\mathcal{A}_2'-\mathcal{D}\mathcal{P}_1'-\mathcal{D}'\mathcal{P}_2')\\
&-\theta_1(\mathcal{B}_1'+\mathcal{D}'\mathcal{Q}_1'+\mathcal{D}''\mathcal{Q}_2')-\theta_2(\mathcal{B}_2'-\mathcal{D}\mathcal{Q}_1'-\mathcal{D}'\mathcal{Q}_2')+2(-2\theta_2\mathcal{D}'+\theta_1\mathcal{D}''+\theta_2\mathcal{D})\mathcal{Q}_2\\
&+(2\Sigma_2\mathcal{D}'-\Sigma_1\mathcal{D}''-\Sigma_2\mathcal{D})(\mathcal{Q}_1'-\mathcal{P}_2')+\Delta(\mathcal{D}\mathcal{D}''-\mathcal{D}'^2)R_2'-\Delta_0(\xi_1X_0'+\eta_1Y_0')\\
&+\Delta_0\mathcal{D}(\xi_2L_0'+\eta_2M_0')-\Delta_0\mathcal{D}'(\xi_1L_0'+\eta_1M'_0)=0.
\end{aligned}$$

$$
\begin{aligned}
V = {} & \frac{1}{\Delta}\left(\frac{\partial \mathfrak{L}_1'}{\partial \rho_1^2} - \frac{\partial^2(\mathfrak{Q}_1' - \mathfrak{L}_2')}{\partial \rho_1 \partial \rho_2} - \frac{\partial^2 \mathfrak{Q}_2'}{\partial \rho_2^2}\right) - \frac{\theta_1 + 2\Sigma_2}{\Delta}\frac{\partial \mathfrak{L}_1'}{\partial \rho_1} + \frac{\Sigma_1}{\Delta}\frac{\partial \mathfrak{L}_1'}{\partial \rho_2} + \frac{\Sigma_3}{\Delta}\frac{\partial}{\partial \rho_1}(\mathfrak{Q}_1' - \mathfrak{L}_2') \\
& + \frac{\theta_1}{\Delta}\frac{\partial}{\partial \rho_2}(\mathfrak{Q}_1' - \mathfrak{L}_2') - \frac{\theta_2}{\Delta}\frac{\partial \mathfrak{Q}_2'}{\partial \rho_1} + \frac{2\theta_2 + \Sigma_1}{\Delta}\frac{\partial \mathfrak{Q}_2'}{\partial \rho_2} + \left[\frac{\partial}{\partial \rho_2}\frac{\Sigma_1}{\Delta} - \frac{\partial}{\partial \rho_1}\frac{\Sigma_2}{\Delta} + \frac{\mathcal{G}}{\Delta}(\mathfrak{D}\mathfrak{D}'' - \mathfrak{D}'^2)\right]\mathfrak{L}_1' \\
& + \left[\frac{\partial}{\partial \rho_2}\frac{\theta_1}{\Delta} - \frac{\partial}{\partial \rho_1}\frac{\theta_2}{\Delta} - \frac{\mathcal{F}}{\Delta}(\mathfrak{D}\mathfrak{D}'' - \mathfrak{D}'^2)\right](\mathfrak{Q}_1' - \mathfrak{L}_2') + \left[\frac{\partial}{\partial \rho_2}\frac{\theta_2}{\Delta} - \frac{\partial}{\partial \rho_1}\frac{\theta_2}{\Delta} - \frac{\mathcal{G}}{\Delta}(\mathfrak{D}\mathfrak{D}'' - \mathfrak{D}'^2)\right]\mathfrak{Q}_2' \\
& + \frac{\mathcal{E}\mathfrak{D}'' - \mathcal{F}\mathfrak{D}}{\Delta}(\mathfrak{A}_1' + \mathfrak{D}'\mathfrak{L}_1' + \mathfrak{D}'\mathfrak{L}_2') + \frac{\mathcal{G}\mathfrak{D} - \mathcal{F}\mathfrak{D}'}{\Delta}(\mathfrak{B}_1' + \mathfrak{D}_1\mathfrak{Q}_1' + \mathfrak{D}'\mathfrak{Q}_2') \\
& + \frac{\mathcal{E}\mathfrak{D}'' - \mathcal{F}\mathfrak{D}'}{\Delta}(\mathfrak{A}_2' - \mathfrak{D}\mathfrak{L}_1' - \mathfrak{D}'\mathfrak{L}_2') + \frac{\mathcal{G}\mathfrak{D}'' - \mathcal{F}\mathfrak{D}''}{\Delta}(\mathfrak{B}_2' - \mathfrak{D}\mathfrak{Q}_1' - \mathfrak{D}'\mathfrak{Q}_2') \\
& - \frac{\partial}{\partial \rho_1}(\mathfrak{D}'R_1' + \mathfrak{D}'R_2') + \frac{\partial}{\partial \rho_2}(\mathfrak{D}R_1' + \mathfrak{D}'R_2') - \frac{\partial}{\partial \rho_1}\left[\frac{\Delta_0}{\Delta}(\xi_2 L_0' + \eta_2 M_0')\right] \\
& + \frac{\partial}{\partial \rho_2}\left[\frac{\Delta_0}{\Delta}(\xi_1 L_0' + \eta_1 M_0')\right] - \Delta_0 Z_0' = 0,
\end{aligned}
$$

$$
\begin{aligned}
W = {} & \frac{\partial R_1'}{\partial \rho_1} + \frac{\partial R_2'}{\partial \rho_2} + \frac{\mathcal{E}}{\Delta}(\mathfrak{A}_1' + \mathfrak{D}'\mathfrak{L}_1' + \mathfrak{D}'\mathfrak{L}_2') \\
& + \frac{\mathcal{F}}{\Delta}[\mathfrak{A}_2' - \mathfrak{D}\mathfrak{L}_1' - \mathfrak{D}'\mathfrak{L}_2' - (\mathfrak{B}_1' + \mathfrak{D}'\mathfrak{Q}_1' + \mathfrak{D}'\mathfrak{Q}_2')] - \frac{\mathcal{G}}{\Delta}(\mathfrak{B}_2' - \mathfrak{D}\mathfrak{Q}_1' - \mathfrak{D}'\mathfrak{Q}_2') - \Delta_0 N_0' = 0,
\end{aligned}
$$

noting that the equations of Codazzi are as follows, for the formation of the three first equations and using the present notation(25)

$$
\begin{aligned}
& \mathfrak{D}\mathfrak{D}'' - \mathfrak{D}'^2 = \mathfrak{K}, \\
& \frac{\partial \mathfrak{D}}{\partial \rho_2} - \frac{\partial \mathfrak{D}'}{\partial \rho_1} + \Sigma_3 \mathfrak{D} - 2\Sigma_2 \mathfrak{D}' + \Sigma_1 \mathfrak{D}'' = 0, \\
& \frac{\partial \mathfrak{D}''}{\partial \rho_1} - \frac{\partial \mathfrak{D}'}{\partial \rho_2} + \theta_3 \mathfrak{D} - 2\theta_2 \mathfrak{D}' + \theta_1 \mathfrak{D}'' = 0,
\end{aligned}
$$

$\mathfrak{K}$ designates the expression formed only with $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ and their first and second derivatives, and represents the total curvature of the surface. We should also note that

$$
\frac{\partial \log \Delta}{\partial \rho_1} = \theta_1 + \Sigma_2, \qquad \frac{\partial \log \Delta}{\partial \rho_2} = \theta_2 + \Sigma_3,
$$

and, consequently, setting the two values of $\frac{\partial^2 \log \Delta}{\partial \rho_1 \partial \rho_2}$ equal to each other,

$$
\frac{\partial \theta_2}{\partial \rho_1} - \frac{\partial \theta_1}{\partial \rho_2} = \frac{\partial \Sigma_2}{\partial \rho_2} - \frac{\partial \Sigma_3}{\partial \rho_1},
$$

where

$$
\begin{aligned}
\frac{\partial}{\partial \rho_2}\frac{\Sigma_2}{\Delta} - \frac{\partial}{\partial \rho_1}\frac{\Sigma_3}{\Delta} &= -\left(\frac{\partial}{\partial \rho_2}\frac{\theta_1}{\Delta} - \frac{\partial}{\partial \rho_1}\frac{\theta_2}{\Delta}\right) = \frac{1}{\Delta}\left(\frac{\partial \theta_2}{\partial \rho_1} - \frac{\partial \theta_1}{\partial \rho_2} - \theta_2\Sigma_2 + \theta_1\Sigma_3\right) \\
&= \frac{1}{\Delta}\left(\frac{\partial \Sigma_3}{\partial \rho_2} - \frac{\partial \Sigma_2}{\partial \rho_1} - \theta_2\Sigma_2 + \theta_1\Sigma_3\right).
\end{aligned}
$$

---

(25) These equations are immediately derived from those given in Volume III, p. 246, 248 of the *Leçons (Lessons)* of M. Darboux, by carrying out the change in notation and noting that

$$
\frac{\partial \log \Delta}{\partial \rho_1} = \theta_1 + \Sigma_2, \qquad \frac{\partial \log \Delta}{\partial \rho_2} = \theta_2 + \Sigma_3.
$$

46. Reduction of the System in the Preceding Section to a Form Analogous to the One which is Obtained in the Calculus of Variations. From the preceding calculation it follows that, when the expressions for the external forces and moments have been given specific forms, the auxiliary quantities A_1', ..., or, which is the same thing, the auxiliary quantities $\mathcal{A}_1'$, ... are all eliminated even though their number is higher than the number of equations by one. This is also a *a priori* consequence of the usual assumptions made in the calculus of variations. /110

We will now write the equations resulting from this elimination in the form which may be derived from the calculus of variations, in the case where the expressions for the external forces and moments are given in a specific way.

Let us begin by replacing in W_0 the arguments ξ_1, η_1, ξ_2, η_2 as a function of the arguments m, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ by their expressions derived from the equations:

$$\frac{\eta_1}{\xi_1} = \operatorname{tg} m, \qquad \xi_1^2 + \eta_1^2 = \mathcal{E}, \qquad \xi_1\xi_2 + \eta_1\eta_2 = \mathcal{F}, \qquad \xi_2^2 + \eta_2^2 = \mathcal{G},$$

to which we will add the equation already used:

$$\xi_1\eta_2 - \xi_2\eta_1 = \Delta,$$

which only determines the sign of ξ_2, η_2.

From this we derive:

$$\xi_1 = \sqrt{\mathcal{E}}\cos m, \qquad \xi_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\cos m - \frac{\Delta}{\sqrt{\mathcal{E}}}\sin m,$$

$$\eta_1 = \sqrt{\mathcal{E}}\sin m, \qquad \eta_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\sin m + \frac{\Delta}{\sqrt{\mathcal{E}}}\cos m,$$

where $\sqrt{\mathcal{E}}$ designates a determination of the root.

If, for a moment, we let $[W_0]$ designate the function of ρ_1, ρ_2 and of m, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, r_1, r_2, $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ obtained in this way, we obtain the relationships:

$$\frac{\partial[W_0]}{\partial\mathcal{E}} = \frac{1}{2\mathcal{E}}\left(\xi_1\frac{\partial W_0}{\partial\xi_1} + \eta_1\frac{\partial W_0}{\partial\eta_1}\right) - \frac{1}{2\mathcal{E}}\left(\xi_2\frac{\partial W_0}{\partial\xi_2} + \eta_2\frac{\partial W_0}{\partial\eta_2}\right) + \frac{\mathcal{G}}{2\mathcal{E}\Delta}\left(\xi_1\frac{\partial W_0}{\partial\eta_2} - \eta_1\frac{\partial W_0}{\partial\xi_2}\right),$$

$$\frac{\partial[W_0]}{\partial\mathcal{F}} = \frac{1}{\Delta}\left(\eta_2\frac{\partial W_0}{\partial\xi_2} - \xi_2\frac{\partial W_0}{\partial\eta_2}\right),$$

$$\frac{\partial[W_0]}{\partial\mathcal{G}} = \frac{1}{2\Delta}\left(\xi_1\frac{\partial W_0}{\partial\eta_2} - \eta_1\frac{\partial W_0}{\partial\xi_2}\right),$$

$$\frac{\partial[W_0]}{\partial m} = \xi_1\frac{\partial W_0}{\partial\eta_1} - \eta_1\frac{\partial W_0}{\partial\xi_1} + \xi_2\frac{\partial W_0}{\partial\eta_2} - \eta_2\frac{\partial W_0}{\partial\xi_2}.$$

For abbreviation, let us set

$$a_1' = \mathcal{A}_1' + \mathcal{D}'\mathcal{X}_1' + \mathcal{D}''\mathcal{X}_2', \qquad a_2' = \mathcal{A}_2' - \mathcal{D}\mathcal{X}_1' - \mathcal{D}'\mathcal{X}_2',$$

$$b_1' = \mathcal{B}_1' + \mathcal{D}'\Omega_1' + \mathcal{D}''\Omega_2', \qquad b_2' = \mathcal{B}_2' - \mathcal{D}\Omega_1' - \mathcal{D}'\Omega_2';$$

We obtain the relationships:

$$\xi_1\frac{\partial(W_0\Delta_0)}{\partial\xi_1} + \eta_1\frac{\partial(W_0\Delta_0)}{\partial\eta_1} = b_1', \qquad \xi_1\frac{\partial(W_0\Delta_0)}{\partial\xi_2} + \eta_1\frac{\partial(W_0\Delta_0)}{\partial\eta_2} = b_2',$$

$$\xi_1 \frac{\partial(W_0\Delta_0)}{\partial\xi_1} + \eta_1 \frac{\partial(W_0\Delta_0)}{\partial\eta_1} = a_1', \qquad \xi_2 \frac{\partial(W_0\Delta_0)}{\partial\xi_2} + \eta_2 \frac{\partial(W_0\Delta_0)}{\partial\eta_2} = a_2',$$

from which we derive the following expressions for the derivatives of $(W_0\Delta_0)$: /111

$$\frac{\partial(W_0\Delta_0)}{\partial\xi_1} = \frac{\eta_2 b_1' - \eta_1 a_1'}{\Delta}, \qquad \frac{\partial(W_0\Delta_0)}{\partial\xi_2} = \frac{\eta_2 b_2' - \eta_1 a_2'}{\Delta},$$

$$\frac{\partial(W_0\Delta_0)}{\partial\eta_1} = \frac{\xi_1 a_1' - \xi_2 b_1'}{\Delta}, \qquad \frac{\partial(W_0\Delta_0)}{\partial\eta_2} = \frac{\xi_1 a_2' - \xi_2 b_2'}{\Delta},$$

which allows us to calculate the various combinations formed with the derivatives of $(W_0\Delta_0)$, in terms of a_1', b_1', a_2', b_2'. We thus obtain:

$$\Delta_0 \frac{\partial[W_0]}{\partial\mathcal{E}} = \frac{1}{2\mathcal{E}} b_1' - \frac{1}{2\mathcal{E}} a_2' + \frac{\mathcal{G}}{2\mathcal{E}\Delta}\left(-\frac{\mathcal{F}}{\Delta} b_2' + \frac{\mathcal{E}}{\Delta} a_2'\right) = \frac{1}{2\mathcal{E}} b_1' + \frac{\mathcal{F}^2}{2\mathcal{E}\Delta^2} a_2' - \frac{\mathcal{F}\mathcal{G}}{2\mathcal{E}\Delta^2} b_2',$$

$$\Delta_0 \frac{\partial[W_0]}{\partial\mathcal{F}} = -\frac{\mathcal{F}}{\Delta^2} a_2' + \frac{\mathcal{G}}{\Delta^2} b_2',$$

$$\Delta_0 \frac{\partial[W_0]}{\partial\mathcal{E}} = \frac{\mathcal{E}}{2\Delta^2} a_2' - \frac{\mathcal{F}}{2\Delta^2} b_2',$$

$$\Delta_0 \frac{\partial[W_0]}{\partial m} = \frac{\mathcal{E}}{\Delta} a_1' + \frac{\mathcal{F}}{\Delta}(a_2' - b_1') - \frac{\mathcal{G}}{\Delta} b_2',$$

from which we may derive:

$$a_1' = \frac{\Delta}{\mathcal{E}} \Delta_0 \frac{\partial[W_0]}{\partial m} + 2\mathcal{F}\Delta_0 \frac{\partial[W_0]}{\partial\mathcal{E}} + \mathcal{G}\Delta_0 \frac{\partial[W_0]}{\partial\mathcal{F}},$$

$$b_1' = 2\mathcal{E}\Delta_0 \frac{\partial[W_0]}{\partial\mathcal{E}} + \mathcal{F}\Delta_0 \frac{\partial[W_0]}{\partial\mathcal{F}},$$

$$a_2' = \Delta\Delta_0 \left\{ \mathcal{F} \frac{\partial[W_0]}{\partial\mathcal{F}} + 2\mathcal{G} \frac{\partial[W_0]}{\partial\mathcal{G}} \right\},$$

$$b_2' = \Delta\Delta_0 \left\{ \mathcal{E} \frac{\partial[W_0]}{\partial\mathcal{F}} + 2\mathcal{F} \frac{\partial[W_0]}{\partial\mathcal{G}} \right\},$$

in such a way that if we designate by W_0 the function $[W_0]$, the *ten* auxiliary quantities other than C_1', C_2' are defined by the *nine* equations:

$$\mathcal{A}_1' + \mathcal{D}'\mathcal{P}_1' + \mathcal{D}''\mathcal{P}_2' = \frac{\Delta}{\mathcal{E}} \frac{\partial(W_0\Delta_0)}{\partial m} + 2\mathcal{F} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{E}} + \mathcal{G} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}},$$

$$\mathcal{R}_1' + \mathcal{D}'\mathcal{Q}_1' + \mathcal{D}''\mathcal{Q}_2' = 2\mathcal{E} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{E}} + \mathcal{F} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}},$$

$$\mathcal{A}_2' - \mathcal{D}\mathcal{P}_1' - \mathcal{D}'\mathcal{P}_2' = \Delta \left\{ \mathcal{F} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}} + 2\mathcal{G} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{G}} \right\},$$

$$\mathcal{R}_2' - \mathcal{D}\mathcal{Q}_1' - \mathcal{D}'\mathcal{Q}_2' = \Delta \left\{ \mathcal{E} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}} + 2\mathcal{F} \frac{\partial(W_0\Delta_0)}{\partial\mathcal{G}} \right\},$$

$$C_1' = \Delta_0 \frac{\partial W_0}{\partial r_1}, \qquad C_2' = \Delta_0 \frac{\partial W_0}{\partial r_2}.$$

Let us define the direction cosines γ, γ', γ'' of the normal Mz' to (M) by the equations:

$$\gamma = \frac{1}{\Delta} \frac{\partial(y, z)}{\partial(\rho_1, \rho_2)}, \qquad \gamma' = \frac{1}{\Delta} \frac{\partial(z, x)}{\partial(\rho_1, \rho_2)}, \qquad \gamma'' = \frac{1}{\Delta} \frac{\partial(x, y)}{\partial(\rho_1, \rho_2)}.$$

We thus obtain the following identity in which we introduce the notation we have just indicated in the place of the derivatives of W_0 /112

$$
\begin{aligned}
&\frac{\partial^2}{\partial\rho_1^2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial^2 x}{\partial\rho_1^2}}+\frac{\partial^2}{\partial\rho_1\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial^2 x}{\partial\rho_1\partial\rho_2}}+\frac{\partial^2}{\partial\rho_2^2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial^2 x}{\partial\rho_2^2}}-\frac{\partial}{\partial\rho_1}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial x}{\partial\rho_1}}-\frac{\partial}{\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial x}{\partial\rho_2}}\\
&\qquad+\frac{\partial(W_0\Delta_0)}{\partial x}=-\left\{\frac{\partial^2\left(\frac{\gamma}{\Delta}\mathfrak{P}_1'\right)}{\partial\rho_1^2}-\frac{\partial^2\left[\frac{\gamma}{\Delta}(\mathfrak{Q}_1'-\mathfrak{P}_2')\right]}{\partial\rho_1\partial\rho_2}-\frac{\partial^2\left(\frac{\gamma}{\Delta}\mathfrak{Q}_2'\right)}{\partial\rho_2^2}\right\}\\
&\qquad+\frac{\partial}{\partial\rho_1}\left\{\frac{\partial}{\partial\rho_1}\left(\frac{\mathcal{E}\frac{\partial x}{\partial\rho_2}-\mathcal{F}\frac{\partial x}{\partial\rho_1}}{\mathcal{E}\Delta}\mathrm{C}_1'\right)+\frac{\partial}{\partial\rho_2}\left(\frac{\mathcal{E}\frac{\partial x}{\partial\rho_2}-\mathcal{F}\frac{\partial x}{\partial\rho_1}}{\mathcal{E}\Delta}\mathrm{C}_2'\right)\right\}\\
&\qquad-\frac{\partial}{\partial\rho_1}\left\{\left(\frac{b_1'}{\mathcal{E}}+\frac{\mathcal{F}^2a_2'}{\mathcal{E}\Delta^2}-\frac{\mathcal{F}\mathcal{G}b_2'}{\mathcal{E}\Delta^2}\right)\frac{\partial x}{\partial\rho_1}+\left(-\frac{\mathcal{F}a_2'}{\Delta^2}+\frac{\mathcal{G}b_2'}{\Delta^2}\right)\frac{\partial x}{\partial\rho_2}\right.\\
&\qquad+\left[\frac{\partial}{\partial\rho_1}\left(\frac{\mathcal{E}\frac{\partial x}{\partial\rho_2}-\mathcal{F}\frac{\partial x}{\partial\rho_1}}{\mathcal{E}\Delta}\right)-\mathfrak{D}'\gamma\right]\mathrm{C}_1'+\left[\frac{\partial}{\partial\rho_2}\left(\frac{\mathcal{E}\frac{\partial x}{\partial\rho_2}-\mathcal{F}\frac{\partial x}{\partial\rho_1}}{\mathcal{E}\Delta}\right)-\mathfrak{D}'\gamma\right]\mathrm{C}_2'\\
&\qquad-\frac{\partial\mathfrak{D}}{\partial\frac{\partial x}{\partial\rho_1}}\mathfrak{P}_1'+\frac{\partial\mathfrak{D}'}{\partial\frac{\partial x}{\partial\rho_1}}(\mathfrak{Q}_1'-\mathfrak{P}_2')+\left.\frac{\partial\mathfrak{D}''}{\partial\frac{\partial x}{\partial\rho_1}}\mathfrak{Q}_2'\right\}-\frac{\partial}{\partial\rho_2}\left\{\left(-\frac{\mathcal{F}a_2'}{\Delta^2}+\frac{\mathcal{G}b_2'}{\Delta^2}\right)\frac{\partial x}{\partial\rho_1}\right.\\
&\qquad+\left(\frac{\mathcal{E}a_2'}{\Delta^2}-\frac{\mathcal{F}b_2'}{\Delta^2}\right)\frac{\partial x}{\partial\rho_2}+\mathfrak{D}\gamma\mathrm{C}_1'+\mathfrak{D}'\gamma\mathrm{C}_2'-\frac{\partial\mathfrak{D}}{\partial\frac{\partial x}{\partial\rho_2}}\mathfrak{P}_1'\\
&\qquad+\left.\frac{\partial\mathfrak{D}'}{\partial\frac{\partial x}{\partial\rho_2}}(\mathfrak{Q}_1'-\mathfrak{P}_2')+\frac{\partial\mathfrak{D}''}{\partial\frac{\partial x}{\partial\rho_2}}\mathfrak{Q}_2'\right\}.
\end{aligned}
$$

In order to obtain this identity in the above form, we have used the relationships(26):

$$
\begin{aligned}
&\frac{\partial\mathcal{E}}{\partial\rho_1}=2(\mathcal{E}\theta_1+\mathcal{F}\Sigma_1), && \frac{\partial\mathcal{E}}{\partial\rho_2}=2(\mathcal{E}\theta_2+\mathcal{F}\Sigma_2).\\
&2\frac{\partial\mathcal{F}}{\partial\rho_2}-\frac{\partial\mathcal{G}}{\partial\rho_1}=2(\mathcal{E}\theta_2+\mathcal{F}\Sigma_2), && 2\frac{\partial\mathcal{F}}{\partial\rho_1}-\frac{\partial\mathcal{E}}{\partial\rho_2}=2(\mathcal{F}\theta_1+\mathcal{G}\Sigma_1).\\
&\frac{\partial\mathcal{G}}{\partial\rho_1}=2(\mathcal{F}\theta_2+\mathcal{G}\Sigma_2). && \frac{\partial\mathcal{G}}{\partial\rho_2}=2(\mathcal{F}\theta_2+\mathcal{G}\Sigma_2).
\end{aligned}
$$

which, when solved, result in the values for the Christoffel symbols Σ_1, Σ_2, Σ_3, θ_1, θ_2, θ_3, or, conversely, the values of the six derivatives of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$. These allow us to eliminate these derivatives of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$. We also have used /113 the relationships(27):

$$
\begin{aligned}
\frac{\partial^2 x}{\partial\rho_1^2}&=\theta_1\frac{\partial x}{\partial\rho_1}+\Sigma_1\frac{\partial x}{\partial\rho_2}+\mathfrak{D}\Delta\gamma,\\
\frac{\partial^2 x}{\partial\rho_1\partial\rho_2}&=\theta_2\frac{\partial x}{\partial\rho_1}+\Sigma_2\frac{\partial x}{\partial\rho_2}+\mathfrak{D}'\Delta\gamma,\\
\frac{\partial^2 x}{\partial\rho_2^2}&=\theta_3\frac{\partial x}{\partial\rho_1}+\Sigma_3\frac{\partial x}{\partial\rho_2}+\mathfrak{D}''\Delta\gamma.
\end{aligned}
$$

(26) We have also continued to use $\frac{\partial\log\Delta}{\partial\rho_1}=\theta_1+\Sigma_2$, $\frac{\partial\log\Delta}{\partial\rho_2}=\theta_2+\Sigma_3$.

(27) Darboux. *Lecons (Lessons)*, Volume III, No. 702, p. 251.

which allows us to eliminate the second derivatives of x, y, z and which give rise to two series of analogous equations obtained by replacing x, γ by y, γ' and z, γ''. The direction cosines γ', γ'' are defined by equations which can be derived by cyclic permutation of those for γ.

Let us consider the various expressions which occurred in the preceding calculation.

Let us set

$$\frac{\partial r_1}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial \frac{\Sigma_1 \Delta}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial}{\partial \frac{\partial x}{\partial \rho_1}} \left(\frac{-\mathcal{E}\frac{\partial \mathcal{E}}{\partial \rho_2} + 2\mathcal{E}\frac{\partial \mathcal{F}}{\partial \rho_1} - \mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_1}}{2\Delta\mathcal{E}} \right)$$

$$= -\frac{2\Delta\Sigma_1}{\mathcal{E}^2}\frac{\partial x}{\partial \rho_1} - \frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}\right) + \frac{-\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{E}}{\partial \rho_2} + 2\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{F}}{\partial \rho_1} - \frac{1}{2}\frac{\partial x}{\partial \rho_2}\frac{\partial \mathcal{E}}{\partial \rho_1} - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1^2}}{\Delta\mathcal{E}}.$$

$$= -\frac{2\Delta\Sigma_1}{\mathcal{E}^2}\frac{\partial x}{\partial \rho_1} - \frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}\right) + \frac{2\frac{\partial x}{\partial \rho_1}(\mathcal{F}\theta_1 + \mathcal{G}\Sigma_1) - \frac{\partial x}{\partial \rho_2}(\mathcal{E}\theta_1 + \mathcal{F}\Sigma_1) - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1^2}}{\Delta\mathcal{E}};$$

On the other hand, we have

$$\frac{\partial}{\partial \rho_1}\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}} = \frac{\frac{\partial \mathcal{E}}{\partial \rho_1}\frac{\partial x}{\partial \rho_2} + \mathcal{E}\frac{\partial^2 x}{\partial \rho_1 \partial \rho_2} - \frac{\partial \mathcal{F}}{\partial \rho_1}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1^2}}{\Delta\mathcal{E}} - \frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}^2}\left[\mathcal{E}(\theta_1 + \Sigma_2) + \frac{\partial \mathcal{E}}{\partial \rho_1}\right]$$

$$= \frac{2\frac{\partial x}{\partial \rho_2}(\mathcal{E}\theta_1 + \mathcal{F}\Sigma_1) + \mathcal{E}\left(\theta_2\frac{\partial x}{\partial \rho_1} + \Sigma_2\frac{\partial x}{\partial \rho_2} + \mathcal{D}'\Delta\gamma\right) - \frac{\partial x}{\partial \rho_1}(\mathcal{F}\theta_1 + \mathcal{G}\Sigma_1 + \mathcal{E}\theta_2 + \mathcal{F}\Sigma_2) - \mathcal{F}\frac{\partial^2 x}{\partial \rho_2^2}}{\Delta\mathcal{E}}$$

$$- \frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}^2}[\mathcal{E}(\theta_1 + \Sigma_2) + 2(\mathcal{E}\theta_1 + \mathcal{F}\Sigma_1)],$$

from which it results that:

$$\frac{\partial \frac{\Sigma_1 \Delta}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_1}} - \frac{\partial}{\partial \rho_1}\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}} = -\mathcal{D}'\gamma.$$

In the same way we have: /114

$$\frac{\partial r_2}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial \frac{\Delta\Sigma_2}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial}{\partial \frac{\partial x}{\partial \rho_1}}\left(\frac{\mathcal{E}\frac{\partial \mathcal{G}}{\partial \rho_1} - \mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_2}}{2\Delta\mathcal{E}}\right)$$

$$= -\frac{2\Sigma_2\Delta}{\mathcal{E}^2}\frac{\partial x}{\partial \rho_1} - \frac{\Sigma_2}{\Delta\mathcal{E}}\left(\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}\right) + \frac{\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{G}}{\partial \rho_1} - \frac{1}{2}\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{E}}{\partial \rho_2} - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1 \partial \rho_2}}{\Delta\mathcal{E}}$$

$$= -\frac{2\Sigma_2\Delta}{\mathcal{E}^2}\frac{\partial x}{\partial \rho_1} - \frac{\Sigma_2}{\Delta\mathcal{E}}\left(\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}\right) + \frac{2\frac{\partial x}{\partial \rho_1}(\mathcal{F}\theta_2 + \mathcal{G}\Sigma_2) - \frac{\partial x}{\partial \rho_1}(\mathcal{E}\theta_2 + \mathcal{F}\Sigma_2) - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1 \partial \rho_2}}{\Delta\mathcal{E}};$$

but on the other hand

$$\frac{\partial}{\partial \rho_2}\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}=\frac{\frac{\partial \mathcal{E}}{\partial \rho_2}\frac{\partial x}{\partial \rho_2}+\mathcal{E}\frac{\partial^2 x}{\partial \rho_2^2}-\frac{\partial \mathcal{F}}{\partial \rho_2}\frac{\partial x}{\partial \rho_1}-\mathcal{F}\frac{\partial^2 x}{\partial \rho_1\partial \rho_2}}{\Delta\mathcal{E}}-\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}^2}\left[\mathcal{E}(\theta_2+\Sigma_2)+\frac{\partial \mathcal{E}}{\partial \rho_2}\right]$$

$$=\frac{2\frac{\partial x}{\partial \rho_2}(\mathcal{E}\theta_2+\mathcal{F}\Sigma_2)+\mathcal{E}\left(\theta_2\frac{\partial x}{\partial \rho_1}+\Sigma_2\frac{\partial x}{\partial \rho_2}+\mathcal{D}''\Delta\gamma\right)-\frac{\partial x}{\partial \rho_1}(\mathcal{E}\theta_2+\mathcal{F}\Sigma_2+\mathcal{F}\theta_2+\mathcal{G}\Sigma_2)-\mathcal{F}\frac{\partial^2 x}{\partial \rho_1\partial \rho_2}}{\Delta\mathcal{E}}$$

$$-\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}^2}[\mathcal{E}(\theta_2+\Sigma_2)+2(\mathcal{E}\theta_2+\mathcal{F}\Sigma_2)],$$

from which it follows that

$$\frac{\partial\frac{\Sigma_2\Delta}{\mathcal{E}}}{\partial\frac{\partial x}{\partial \rho_1}}-\frac{\partial}{\partial \rho_2}\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}=-\mathcal{D}'\gamma.$$

We also have:

$$\frac{\partial r_1}{\partial\frac{\partial x}{\partial \rho_2}}=\frac{\partial\frac{\Delta\Sigma_1}{\mathcal{E}}}{\partial\frac{\partial x}{\partial \rho_2}}=\frac{\partial}{\partial\frac{\partial x}{\partial \rho_2}}\left(\frac{-\mathcal{E}\frac{\partial \mathcal{E}}{\partial \rho_2}+2\mathcal{E}\frac{\partial \mathcal{F}}{\partial \rho_1}-\mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_1}}{2\Delta\mathcal{E}^2}\right)$$

$$=-\frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}\right)+\frac{\mathcal{E}\frac{\partial^2 x}{\partial \rho_2^2}-\frac{1}{2}\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{E}}{\partial \rho_1}}{\mathcal{E}\Delta}$$

$$=-\frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}\right)+\frac{\mathcal{E}\left(\theta_1\frac{\partial x}{\partial \rho_1}+\Sigma_1\frac{\partial x}{\partial \rho_2}+\mathcal{D}\Delta\gamma\right)-\frac{\partial x}{\partial \rho_1}(\mathcal{E}\theta_1+\mathcal{F}\Sigma_1)}{\Delta\mathcal{E}}=\mathcal{D}\gamma.$$

$$\frac{\partial r_2}{\partial\frac{\partial x}{\partial \rho_2}}=\frac{\partial\frac{\Delta\Sigma_2}{\mathcal{E}}}{\partial\frac{\partial x}{\partial \rho_2}}=\frac{\partial}{\partial\frac{\partial x}{\partial \rho_2}}\left(\frac{\mathcal{E}\frac{\partial \mathcal{G}}{\partial \rho_1}-\mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_2}}{2\Delta\mathcal{E}}\right)$$

$$=-\frac{\Sigma_2}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}\right)+\frac{\mathcal{E}\frac{\partial^2 x}{\partial \rho_1\partial \rho_2}-\frac{1}{2}\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{E}}{\partial \rho_2}}{\Delta\mathcal{E}}$$

$$=-\frac{\Sigma_2}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial \rho_2}-\mathcal{F}\frac{\partial x}{\partial \rho_1}\right)+\frac{\mathcal{E}\left(\theta_2\frac{\partial x}{\partial \rho_1}+\Sigma_2\frac{\partial x}{\partial \rho_2}+\mathcal{D}'\Delta\gamma\right)-\frac{\partial x}{\partial \rho_1}(\mathcal{E}\theta_2+\mathcal{F}\Sigma_2)}{\Delta\mathcal{E}}=\mathcal{D}'\gamma.$$

Let us modify the identity which we have obtained, and we should note that we also obtain two analogous ones by replacing x, γ by y, γ', and then by z, γ''. /115

We will now develop the parentheses in such a way as to isolate the first members of the equations of statics of the deformable surface. The abstraction for the forces has been carried out. For this purpose, we will employ the relationships(28):

$$\frac{\partial\gamma}{\partial \rho_1}=\frac{\mathcal{F}\mathcal{D}'-\mathcal{G}\mathcal{D}}{\Delta}\frac{\partial x}{\partial \rho_1}+\frac{\mathcal{F}\mathcal{D}-\mathcal{E}\mathcal{D}'}{\Delta}\frac{\partial x}{\partial \rho_2},$$

$$\frac{\partial\gamma}{\partial \rho_2}=\frac{\mathcal{F}\mathcal{D}'-\mathcal{G}\mathcal{D}'}{\Delta}\frac{\partial x}{\partial \rho_1}+\frac{\mathcal{F}\mathcal{D}'-\mathcal{E}\mathcal{D}'}{\Delta}\frac{\partial x}{\partial \rho_2},$$

(28) Darboux. *Lecons (Lessons)*, Volume III, No. 698, p. 244-245.

which give rise to two analogous systems obtained by replacing x, γ by y, γ', and then by z, γ''. This means that:

$$\frac{\partial \frac{\gamma}{\Delta}}{\partial \rho_1} = \frac{\mathcal{F}\mathfrak{D}' - \mathcal{G}\mathfrak{D}}{\Delta^2}\frac{\partial x}{\partial \rho_1} + \frac{\mathcal{F}\mathfrak{D} - \mathcal{E}\mathfrak{D}'}{\Delta^2}\frac{\partial x}{\partial \rho_2} - \frac{\Theta_1 + \Sigma_2}{\Delta}\gamma,$$

$$\frac{\partial \frac{\gamma}{\Delta}}{\partial \rho_2} = \frac{\mathcal{F}\mathfrak{D}' - \mathcal{G}\mathfrak{D}'}{\Delta^2}\frac{\partial x}{\partial \rho_1} + \frac{\mathcal{F}\mathfrak{D}' - \mathcal{E}\mathfrak{D}'}{\Delta^2}\frac{\partial x}{\partial \rho_2} - \frac{\Theta_2 + \Sigma_2}{\Delta}\gamma.$$

i.e.,

$$\frac{\partial \frac{\gamma}{\Delta}}{\partial \rho_2} = -\mathfrak{D}'\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta^2} - \mathfrak{D}\frac{\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}}{\Delta^2} - \frac{\Theta_1 + \Sigma_2}{\Delta}\gamma,$$

$$\frac{\partial \frac{\gamma}{\Delta}}{\partial \rho_2} = -\mathfrak{D}'\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta^2} - \mathfrak{D}'\frac{\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}}{\Delta^2} - \frac{\Theta_2 + \Sigma_2}{\Delta}\gamma.$$

We therefore arrive at the result that, if U_2, U_1, V, $\mathfrak{W}$ designate the first terms of the equations of statics of the deformable surface, the expression we have just treated contains all the terms independent of the external forces and moments which appear in:

$$-\gamma V - \frac{\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}}{\Delta^2}U_1 - \frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta^2}U_2 + \frac{\partial}{\partial \rho_1}\left\{\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}\mathfrak{W}\right\}.$$

Two analogous results are obtained by changing x, γ into y, γ', and then into z, γ''.

On the other hand, the equation $\mathfrak{W} = 0$ can be written as:

$$\frac{\partial}{\partial \rho_1}\frac{(W_0\Delta_0)}{\partial \frac{\partial m}{\partial \rho_1}} + \frac{\partial}{\partial \rho_2}\frac{\partial(W_0\Delta_0)}{\partial \frac{\partial m}{\partial \rho_2}} - \frac{\partial(W_0\Delta_0)}{\partial m} + \Delta_0 N_0' = 0.$$

It is therefore seen that if we set: /116

$$\Delta_0\mathfrak{X}_0 = \Delta_0 X_0 + \frac{\partial}{\partial \rho_1}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial \rho_2} + M_0\frac{\partial y}{\partial \rho_2} + N_0\frac{\partial z}{\partial \rho_2}\right) - \frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}\Delta_0 N_0'\right]$$
$$- \frac{\partial}{\partial \rho_2}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial \rho_1} + M_0\frac{\partial y}{\partial \rho_1} + N_0\frac{\partial z}{\partial \rho_1}\right)\right],$$

and two analogous equations obtained by replacing $\mathfrak{X}_0$, X_0, x, γ by $\mathfrak{Y}_0$, Y_0, y, γ', respectively, and then by $\mathfrak{Z}_0$, Z_0, z, γ'' (outside of $L_0\frac{\partial x}{\partial \rho_2} + \ldots$ and $L_0\frac{\partial x}{\partial \rho_1} + \ldots$), the equations of statics of the deformable surface can be summarized as follows(29):

(29) This relationship is analogous to the equation $\int_{t_0}^{t_1}(\delta T + U)dt = 0$ which Tisserand gives on p. 4 of Volume I of his *Traité de Mécanique Céleste (Treatise on Celestial Mechanics)*, for Hamilton's principle.

$$\iint \delta(W_0\Delta_0)d\rho_1 d\rho_2 + \iint \Delta_0(\mathfrak{X}_0\delta x + \mathfrak{Y}_0\delta y + \mathfrak{Z}_0\delta z - N_0'\delta m)d\rho_1 d\rho_2 = 0,$$

where we have only taken terms into account which occur under the double integral sign.

The preceding result can be generalized. Let us assume that ξ_1, η_1, ξ_2, η_2 is expressed as a function of m, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ by the equations:

$$\xi_1 = \sqrt{\mathcal{E}}\cos(m+u), \qquad \xi_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\cos(m+u) - \frac{\Delta}{\sqrt{\mathcal{E}}}\sin(m+u),$$

$$\eta_1 = \sqrt{\mathcal{E}}\sin(m+u), \qquad \eta_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\sin(m+u) + \frac{\Delta}{\sqrt{\mathcal{E}}}\cos(m+u),$$

where u designates an arbitrary function of only the quantities $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$. The equation $\mathcal{W} = 0$ can then be written as:

$$-\frac{\partial}{\partial\rho_1}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial m}{\partial\rho_1}} + \frac{\partial}{\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial m}{\partial\rho_2}} - \frac{\partial(W_0\Delta_0)}{\partial m} + \Delta_0 N_0' = 0$$

By forming the combination

$$-\gamma V - \frac{\mathcal{G}\frac{\partial x}{\partial\rho_1} - \mathcal{F}\frac{\partial x}{\partial\rho_2}}{\Delta^2}U_1 - \frac{\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}}{\Delta^2}U_2$$

$$+\frac{\partial}{\partial\rho_1}\left\{\left[\frac{\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}}{\Delta\mathcal{E}} - \frac{\partial u}{\partial\frac{\partial x}{\partial\rho_1}}\right]\mathfrak{W}\right\} + \frac{\partial}{\partial\rho_2}\left\{-\frac{\partial u}{\partial\frac{\partial x}{\partial\rho_2}}\mathfrak{W}\right\},$$

and two analogous equations obtained by replacing x, γ' by y, γ', then by z, γ'', /117
we obtain three equations the first of which is:

$$\frac{\partial^2}{\partial\rho_1^2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial^2 x}{\partial\rho_1^2}} + \frac{\partial^2}{\partial\rho_1\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial^2 x}{\partial\rho_1\partial\rho_2}} + \frac{\partial^2}{\partial\rho_2^2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial^2 x}{\partial\rho_2^2}} - \frac{\partial}{\partial\rho_1}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial x}{\partial\rho_1}} - \frac{\partial}{\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\frac{\partial x}{\partial\rho_2}} + \Delta_0\mathfrak{X}_0 = 0,$$

Setting:

$$\Delta_0\mathfrak{X}_0 = \Delta_0 X_0 + \frac{\partial}{\partial\rho_1}\left\{\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial\rho_2} + M_0\frac{\partial y}{\partial\rho_2} + N_0\frac{\partial z}{\partial\rho_2}\right) + \left(\frac{\partial u}{\partial\frac{\partial x}{\partial\rho_1}} - \frac{\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}}{\Delta\mathcal{E}}\right)\Delta_0 N_0'\right\}$$

$$-\frac{\partial}{\partial\rho_2}\left\{\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial\rho_1} + M_0\frac{\partial y}{\partial\rho_1} + N_0\frac{\partial z}{\partial\rho_1}\right) - \frac{\partial u}{\partial\frac{\partial x}{\partial\rho_2}}\Delta_0 N_0'\right\},$$

we can summarize the four equations as follows

$$\iint_{c_0}\left\{\delta(W_0\Delta_0) + \Delta_0(\mathfrak{X}_0\delta x + \mathfrak{Y}_0\delta y + \mathfrak{Z}_0\delta z - N_0'\delta m)\right\}d\rho_1 d\rho_2 = 0,$$

where we only take into consideration terms which are present under the double integral sign.

The summarized form we have just obtained and which must be dealt with according to the rules of the calculus of variations is particularly convenient

for performing changes in variables.

Assuming that the expressions $\mathcal{X}_0$, $\mathcal{Y}_0$, $\mathcal{Z}_0$, N_0' have a specific form, we will obtain the equations for the extremals of a problem of the calculus of variations.

Let us consider the particular case(30) where $W_0\Delta_0$ does not depend on r_1, r_2 and only depends on ξ_1, ξ_2, η_1, η_2 through the intermediary of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$. This is the same as stating that the final expression for $W_0\Delta_0$ does not depend on m, $\frac{\partial m}{\partial \rho_1}$, $\frac{\partial m}{\partial \rho_2}$ and is a function of ρ_1, ρ_2 and the six functions

$$\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{L}, \mathcal{M}, \mathcal{N}$$

of the six first and second derivatives of x, y, z.

Let us also assume that $N_0' = 0$. If $\mathcal{X}_0$, $\mathcal{Y}_0$, $\mathcal{Z}_0$ did not depend on m, we would finally obtain three equations in terms of x, y, z which can be summarized by the equation:

$$\iint_{c_0} \delta(W_0\Delta_0)d\rho_1 d\rho_2 + \iint_{c_0} \Delta_0(\mathcal{X}_0\delta x + \mathcal{Y}_0\delta y + \mathcal{Z}_0\delta z)d\rho_1 d\rho_2 = 0.$$

If we let U designate a function of ρ_1, ρ_2 and of x, y, z, $\gamma = \frac{I}{\Delta_0}\frac{\partial(y, z)}{\partial(\rho_1, \rho_2)}$, /118

γ', γ'', in the particular case where

$$X_0 = \frac{\partial U}{\partial x}, \quad Y_0 = \frac{\partial U}{\partial y}, \quad Z_0 = \frac{\partial U}{\partial z},$$

$$L_0\frac{\partial x}{\partial \rho_1} + M_0\frac{\partial y}{\partial \rho_1} + N_0\frac{\partial z}{\partial \rho_1} = -\left(\frac{\partial U}{\partial \gamma}\frac{\partial \Delta}{\partial \frac{\partial x}{\partial \rho_2}} + \frac{\partial U}{\partial \gamma'}\frac{\partial \Delta}{\partial \frac{\partial y}{\partial \rho_2}} + \frac{\partial U}{\partial \gamma''}\frac{\partial \Delta}{\partial \frac{\partial z}{\partial \rho_2}}\right),$$

$$L_0\frac{\partial x}{\partial \rho_2} + M_0\frac{\partial y}{\partial \rho_2} + N_0\frac{\partial z}{\partial \rho_2} = \left(\frac{\partial U}{\partial \gamma}\frac{\partial \Delta}{\partial \frac{\partial x}{\partial \rho_1}} + \frac{\partial U}{\partial \gamma'}\frac{\partial \Delta}{\partial \frac{\partial y}{\partial \rho_1}} + \frac{\partial U}{\partial \gamma''}\frac{\partial \Delta}{\partial \frac{\partial z}{\partial \rho_1}}\right),$$

$$L_0\gamma + M_0\gamma' + N_0\gamma'' = 0,$$

we then have:

$$\Delta_0\mathcal{X}_0 = \frac{\partial(U\Delta_0)}{\partial x} - \frac{\partial}{\partial \rho_1}\frac{\partial(U\Delta_0)}{\partial \frac{\partial x}{\partial \rho_1}} - \frac{\partial}{\partial \rho_2}\frac{\partial(U\Delta_0)}{\partial \frac{\partial x}{\partial \rho_2}},$$

and two analogous equations, and we obtain the three equations of the extremals of the integral:

$$\iint \Delta_0(W_0 + U)d\rho_1 d\rho_2;$$

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(30) The following can also be applied to the case where $W_0\Delta_0$ is arbitrary. The essential hypothesis is the one made for L_0, M_0, N_0. We may consider the case where $W_0\Delta_0$ is of the first degree with respect to r_1, r_2 and the coefficients of the latter quantities are constant or, more generally, independent of ρ_2 and ρ_1, respectively.

and the preceding equations amount to setting:

$$L_0 = \gamma' \frac{\partial U}{\partial \gamma''} - \gamma'' \frac{\partial U}{\partial \gamma'},$$

$$M_0 = \gamma'' \frac{\partial U}{\partial \gamma} - \gamma \frac{\partial U}{\partial \gamma''},$$

$$N_0 = \gamma \frac{\partial U}{\partial \gamma'} - \gamma' \frac{\partial U}{\partial \gamma};$$

All this results from the fact that γ, γ', γ'' satisfy the following systems which define a function F of $\frac{\partial x}{\partial \rho_1}$, $\frac{\partial y}{\partial \rho_1}$, ... $\frac{\partial z}{\partial \rho_2}$:

$$\frac{\frac{\partial F}{\partial \frac{\partial x}{\partial \rho_1}}}{\gamma} = \frac{\frac{\partial F}{\partial \frac{\partial y}{\partial \rho_1}}}{\gamma'} = \frac{\frac{\partial F}{\partial \frac{\partial z}{\partial \rho_1}}}{\gamma''}, \qquad \frac{\frac{\partial F}{\partial \frac{\partial x}{\partial \rho_2}}}{\gamma} = \frac{\frac{\partial F}{\partial \frac{\partial y}{\partial \rho_2}}}{\gamma'} = \frac{\frac{\partial F}{\partial \frac{\partial z}{\partial \rho_2}}}{\gamma''}.$$

An interesting special case of the preceding one is that in which the term $\frac{\Delta_0 W_0}{\Delta}$, taking x and y as variables, only depends, in addition to x, y, on all the derivatives of z with respect to x, y. It is easy to find the form of W_0.

Let us note that the two expressions

$$dx^2 + dy^2 + dz^2, \qquad -(d\gamma dx + d\gamma' dy + d\gamma'' dz)$$

can be written as /119

$$\mathcal{E} d\rho_1^2 + 2\mathcal{F} d\rho_1 d\rho_2 + \mathcal{G} d\rho_2^2, \qquad \Delta(\mathfrak{D} d\rho_1^2 + 2\mathfrak{D}' d\rho_1 d\rho_2 + \mathfrak{D}'' d\rho_2^2),$$

from which it results that, due to the equations:

$$dx = \frac{\partial x}{\partial \rho_1} d\rho_1 + \frac{\partial x}{\partial \rho_2} d\rho_2,$$

$$dy = \frac{\partial y}{\partial \rho_1} d\rho_1 + \frac{\partial y}{\partial \rho_2} d\rho_2,$$

we find the identities

$$\mathcal{E} d\rho_1^2 + 2\mathcal{F} d\rho_1 d\rho_2 + \mathcal{G} d\rho_2^2 = (1 + p^2) dx^2 + 2pq dx dy + (1 + q^2) dy^2$$

$$\Delta(\mathfrak{D} d\rho_1^2 + 2\mathfrak{D}' d\rho_1 d\rho_2 + \mathfrak{D}'' d\rho_2^2) = \frac{1}{\sqrt{1 + p^2 + q^2}} (r dx^2 + 2s dx dy + t dy^2).$$

From the theory of invariants of quadratic forms, we have

$$\mathcal{E}\mathcal{G} - \mathcal{F}^2 = (1 + p^2 + q^2) \left[\frac{\partial(x, y)}{\partial(\rho_1, \rho_2)}\right]^2,$$

$$\Delta^2(\mathfrak{D}\mathfrak{D}'' - \mathfrak{D}'^2) = \frac{rt - s^2}{1 + p^2 + q^2} \left[\frac{\partial(x, y)}{\partial(\rho_1, \rho_2)}\right]^2,$$

$$\Delta(\mathcal{G}\mathfrak{D} + \mathcal{E}\mathfrak{D}'' - 2\mathcal{F}\mathfrak{D}') = \frac{(1 + q^2) r + (1 + p^2) t - 2pqs}{\sqrt{1 + p^2 + q^2}} \left[\frac{\partial(x, y)}{\partial(\rho_1, \rho_2)}\right]^2,$$

and consequently, making use of the absolute invariants,

$$\mathfrak{D}\mathfrak{D}'' - \mathfrak{D}'^2 = \frac{rt - s^2}{(1 + p^2 + q^2)^2},$$

$$\frac{\mathfrak{G}\mathfrak{D}+\mathfrak{E}\mathfrak{D}'-2\mathfrak{F}\mathfrak{D}'}{\Delta}=\frac{(1+q^2)r-2pqs+(1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}}.$$

We find the two well known expressions of the total curvature and mean curvature. The case which we wish to consider is the one where $\frac{\Delta_0 W_0}{\Delta}$ is a function ϕ of ρ_1, ρ_2 and the two expressions:

$$\frac{1}{\mathfrak{R}_1\mathfrak{R}_2}=\mathfrak{D}\mathfrak{D}'-\mathfrak{D}'^2,\qquad \frac{1}{\mathfrak{R}_1}+\frac{1}{\mathfrak{R}_2}=\frac{\mathfrak{G}\mathfrak{D}+\mathfrak{E}\mathfrak{D}'-2\mathfrak{F}\mathfrak{D}'}{\Delta},$$

where $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are the principal radii of curvature.

The equations which summarize the equations of statics of the deformable surface can be written as follows, taking x, y as variables:

$$\delta\iint\varphi\sqrt{1+p^2+q^2}\,dxdy+\iint\frac{\Delta_0}{\Delta}(\mathfrak{X}_0\delta x+\mathfrak{Y}_0\delta y+\mathfrak{Z}_0\delta z)\sqrt{1+p^2+q^2}\,dxdy=0.$$

The function under the $\iint$ sign in the second integral is:

$$\frac{\Delta_0}{\Delta}\left\{\left(\mathfrak{X}_0\frac{\partial x}{\partial\rho_1}+\mathfrak{Y}_0\frac{\partial y}{\partial\rho_1}\right)\delta\rho_1+\left(\mathfrak{X}_0\frac{\partial x}{\partial\rho_2}+\mathfrak{Y}_0\frac{\partial y}{\partial\rho_2}\right)\delta\rho_2+\mathfrak{Z}_0\delta z\right\}\sqrt{1+p^2+q^2},$$

and consequently the equations of the problem will be as follows, because ϕ does not contain the derivatives of ρ_1, ρ_2, /120

$$\frac{\partial^2}{\partial x^2}\frac{\partial(\varphi\sqrt{1+p^2+q^2})}{\partial r}+\ldots-\frac{\partial}{\partial y}\frac{\partial(\varphi\sqrt{1+p^2+q^2})}{\partial q}+\frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}\,\mathfrak{Z}_0=0,$$

$$\frac{\partial}{\partial\rho_1}(\varphi\sqrt{1+p^2+q^2})+\left(\mathfrak{X}_0\frac{\partial x}{\partial\rho_1}+\mathfrak{Y}_0\frac{\partial y}{\partial\rho_1}\right)\frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}=0,$$

$$\frac{\partial}{\partial\rho_2}(\varphi\sqrt{1+p^2+q^2})+\left(\mathfrak{X}_0\frac{\partial x}{\partial\rho_2}+\mathfrak{Y}_0\frac{\partial y}{\partial\rho_2}\right)\frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}=0.$$

As a special case, let us assume that ϕ does not depend on ρ_1, ρ_2 and only depends on $\frac{1}{\mathfrak{R}_1+\mathfrak{R}_2}$ and $\frac{1}{\mathfrak{R}_1\mathfrak{R}_2}$. We obtain the equations:

$$\frac{\partial^2}{\partial x^2}\frac{\partial(\varphi\sqrt{1+p^2+q^2})}{\partial r}+\ldots+\frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}\,\mathfrak{Z}_0=0.$$

$$\mathfrak{X}_0=0,\quad \mathfrak{Y}_0=0.$$

We may write:

$$\Delta_0\mathfrak{X}_0=\left\{\frac{\Delta_0}{\Delta}X_0\sqrt{1+p^2+q^2}-\frac{\partial}{\partial y}\left[\gamma\frac{\Delta_0}{\Delta}(L_0+N_0p)\right]+\frac{\partial}{\partial x}\left[\gamma\frac{\Delta_0}{\Delta}(M_0+N_0q)\right]\right\}\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)},$$

$$\Delta_0\mathfrak{Y}_0=\left\{\frac{\Delta_0}{\Delta\gamma'}Y_0-\frac{\partial}{\partial y}\left[\gamma'\frac{\Delta_0}{\Delta}(L_0+N_0p)\right]+\frac{\partial}{\partial x}\left[\gamma'\frac{\Delta_0}{\Delta}(M_0+N_0q)\right]\right\}\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)},$$

$$\Delta_0\mathfrak{Z}_0=\left\{\frac{\Delta_0}{\Delta\gamma''}Z_0-\frac{\partial}{\partial y}\left[\gamma''\frac{\Delta_0}{\Delta}(L_0+N_0p)\right]+\frac{\partial}{\partial x}\left[\gamma''\frac{\Delta_0}{\Delta}(M_0+N_0q)\right]\right\}\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)}.$$

We may combine the two equations $\mathfrak{X}_0 = 0$, $\mathfrak{Y}_0 = 0$ with those of the preceding case. For example, we can rearrange in such a way that we introduce the combination $\gamma x_0 + \gamma'\mathfrak{Y}_0 + \gamma''z_0$, setting:

$$\frac{\partial^2}{\partial x^2}\frac{\partial(\varphi\sqrt{1+p^2+q^2})}{\partial r}+\ldots+\frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}\,\frac{1}{\gamma''}(\gamma\mathfrak{X}_0+\gamma'\mathfrak{Y}_0+\gamma''\mathfrak{Z}_0)=0.$$

If the knowns are suitable in the equation which we have just written or in such a combination that ρ_1, ρ_2 can no longer appear, we obtain, making use of the preceding equation, an equation for the surface. The equations

$$x_0 = 0, \quad y_0 = 0$$

define ρ_1, ρ_2 as a function of x, y (or vice versa) and can be set aside if we disregard the natural state.

Let us consider the special case in which the function ϕ is a linear function with constant coefficients with respect to $(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2})^2$ and $\frac{1}{\mathcal{R}_1 \mathcal{R}_2}$, i.e., a function of the form

$$A\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right)^2 + B\frac{1}{\mathcal{R}_1\mathcal{R}_2} + C,$$

where A, B, C are constants. The constant B disappears following a remark first made by Poisson in his memoir on elastic surfaces(31) and which was subsequently /121 generalized by Olinde Rodrigues(32). In the case where all the external forces are zero, we may summarize the equation as follows

$$\delta \iint \left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right)^2 \sqrt{1 + p^2 + q^2}\, dxdy + C\delta \iint \sqrt{1 + p^2 + q^2}\, dxdy = 0,$$

which is the conclusion which Poisson arrived at in his investigations.

Finally, we should observe that the general developments of this system easily lead to the theories of Thomson and Tait(33) and Lord Rayleigh(34) by considering infinitesimally small deformations. We will leave the work of this comparison to the reader, and let him study the case they treated(35) in detail.

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(31) Poisson. *Memoir on Elastic Surfaces*, read August 1, 1814. Mémoires de la Classe des Sciences mathématiques et physiques de l'Institut de France (Memoirs of the Mathematical and Physical Science Class of the French Institute), year 1812, 2nd Part, p. 167-225). An excerpt from this memoir first appeared in the Bulletin of the Philomatic Society and then in the Correspondance sur l'Ecole Polytechnique, Volume III, p. 154-159, 1815.

(32) Rodrigues. *Recherches sur la Théorie Analytique des Lignes et des Rayons de Courbure des Surfaces et sur la Transformation d'une Classe d'Intégrales Doubles, qui ont un Rapport Direct avec les Formules de cette Theorie (Researches on the Analytical Theory of Lines and the Radii of Curvature of Surfaces and on the Transformation of a Class of Double Integrals, which are Directly related with the Formulas of this Theory).* Correspondance sur l'Ecole Polytechnique (Correspondance of the Polytechnic School, Volume III, p. 162-182, 1815. In particular, see p. 173 and following.

(33) Thomson and Tait. *Treatise*, Part II, No. 644.

(34) Lord Rayleigh. *Theory of Sound*, Volume I, 2nd Edition, p. 352, 1894.

(35) We are dealing with an infinitesimally small deformation of an originally plane surface.

47. Dynamics of the Deformable Line. The dynamics of the deformable line can be related to the preceding discussion, where it is sufficient to look upon one of the parameters, ρ_2 for example, as the time t. We thus obtain a simultaneous action of deformation and of motion. Due to the triad, the speed of a point of the deformable line enters into W by means of the three arguments ξ_2, η_2, ζ_2 and we obtain the concept of *kinetic anisotropy* already considered by Rankine and which has been introduced in several theories of physics, in the theory of double refraction and that of rotary polarization, for example. Even if W is independent of the rotations and results in zero external moments, the argument of pure deformation $\xi_1^2 + \eta_1^2 + \zeta_1^2$ and the argument $\xi_2^2 + \eta_2^2 + \zeta_2^2$ are in general accompanied by the argument $\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2$. This type of argument is nothing new in mechanics, and occurs especially in the theory of distant forces, as we will show later on.

When W does not contain the mixed argument $\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2$, it is in general necessary to consider the state of deformation and of motion infinitesimally adjacent to the natural state in order to deal with the case of classical mechanics where *the action of deformation is completely separated from the kinetic action*. One then obtains the D'Alembert principle assuming that the external force and the external moment are zero, i.e., expressing the fact that the deformable line is not subjected to any action from the outside world, and introducing the fundamental concept of *an isolated system*, which /122 we discussed at the beginning of this article.

The dynamics of the deformable surface can be established in the same manner using the theory of the three-dimensional, deformable medium, which we will now describe.

IV Statics and Dynamics of the Deformable Medium

48. Deformable Medium. Natural State and Deformed State. The theories of the deformable line and the deformable surface which we have just discussed lead us, in a very general way, to the consideration of a deformable medium which is more general than is usually considered in the theory of elasticity. It seems to us that this was the goal that Lord Kelvin and Helmholtz had in mind in the theories of light and magnetism.

Let us consider a space (M_0) described by point M_0 whose coordinates are x_0, y_0, z_0 with respect to three rectangular fixed axes Ox, Oy, Oz. We may regard these coordinates as functions of three parameters ρ_1, ρ_2, ρ_3 chosen in an arbitrary way. However, for simplification purposes we will assume that these coordinates are taken to be independent variables. Let attach an orthogonal triad to each point M_0 of the space (M_0) whose axes $M_0x'_0$, $M_0y'_0$, $M_0z'_0$ have the direction cosines α_0, α'_0, α''_0; β_0, β'_0, β''_0; γ_0, γ'_0, γ''_0 respectively, with respect to the axes Ox, Oy, Oz, and which are functions of

the independent variables x_0, y_0, z_0.

The continuous, three-dimensional ensemble of these triads $M_0x_0'y_0'z_0'$ is what we shall call a *deformable medium*.

Let us impart a displacement M_0M to the point M_0. Let x, y, z be the coordinates of the point M with respect to the fixed triad Oxyz. Also, let us impart a rotation to the triad $M_0x_0'y_0'z_0'$, which brings its axes into coincidence with those of a triad Mx'y'z' which we will attach to the point M. We will define this rotation by specifying the direction cosines α, α', α''; β, β', β''; γ, γ', γ'' of the axes Mx', My', Mz' with respect to fixed axes Ox, Oy, Oz.

The continuous, three-dimensional ensemble of the triads Mx'y'z' is what we shall call *the deformed state* of the deformable medium under consideration, which, in its initial state, will be called the *natural state*.

49. Kinematic Elements for the States of the Deformable Medium. For convenience in writing and notation, we will sometimes introduce the letters ρ_1, ρ_2, ρ_3 in place of x_0, y_0, z_0 in the following, which amounts to a convenient change in the notation, expressed by the equations:

$$x_0 = \rho_1, \quad y_0 = \rho_2, \quad z_0 = \rho_3,$$

which should be borne in mind.

Let us designate by $\xi^{(0)}_i$, $\eta^{(0)}_i$, $\zeta^{(0)}_i$ the components of the velocity of the origin M_0 of the axes $M_0x'_0$, M_0y_0', M_0z_0' along these axes, when only ρ_i is /123 varied and plays the role of time. Also, let $p^{(0)}_i$, $q^{(0)}_i$, $r^{(0)}_i$ be the projections along these same axes of the instantaneous rotation of the triad $M_0x_0'y'_0z'_0$ with respect to the parameter ρ_i. We will use ξ_i, η_i, ζ_i and p_i, q_i, r_i to designate the analogous quantities for the triad Mx'y'z' when it is referred to the fixed triad Oxyz, as was done for the triad $M_0x_0'y_0'z_0'$.

The elements which we have just introduced are calculated in the ordinary way; in particular we have:

$$(43)\quad \begin{cases} \xi_i = \alpha \dfrac{\partial x}{\partial \rho_i} + \alpha' \dfrac{\partial y}{\partial \rho_i} + \alpha'' \dfrac{\partial z}{\partial \rho_i}, \\ \eta_i = \beta \dfrac{\partial x}{\partial \rho_i} + \beta' \dfrac{\partial y}{\partial \rho_i} + \beta'' \dfrac{\partial z}{\partial \rho_i}, \\ \zeta_i = \gamma \dfrac{\partial x}{\partial \rho_i} + \gamma' \dfrac{\partial y}{\partial \rho_i} + \gamma'' \dfrac{\partial z}{\partial \rho_i}, \end{cases} \qquad \begin{cases} p_i = \sum \gamma \dfrac{\partial \beta}{\partial \rho_i} = -\sum \beta \dfrac{\partial \gamma}{\partial \rho_i}, \\ q_i = \sum \alpha \dfrac{\partial \gamma}{\partial \rho_i} = -\sum \gamma \dfrac{\partial \alpha}{\partial \rho_i}, \\ r_i = \sum \beta \dfrac{\partial \alpha}{\partial \rho_i} = -\sum \alpha \dfrac{\partial \beta}{\partial \rho_i}. \end{cases} \quad (44)$$

The linear element of the deformed medium (M) referred to the independent variables x_0, y_0, z_0 is defined by the equation:

$$ds^2 = (1 + 2\varepsilon_1)dx_0^2 + (1 + 2\varepsilon_2)dy_0^2 + (1 + 2\varepsilon_3)dz_0^2 + 2\gamma_1 dy_0 dz_0 + 2\gamma_2 dz_0 dx_0 + 2\gamma_3 dx_0 dy_0,$$

where ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 are calculated by the following double equations:

$$\begin{cases} \varepsilon_1 = \frac{1}{2}\left[\left(\frac{\partial x}{\partial x_0}\right)^2 + \left(\frac{\partial y}{\partial x_0}\right)^2 + \left(\frac{\partial z}{\partial x_0}\right)^2 - 1\right] = \frac{1}{2}(\xi_1^2 + \eta_1^2 + \zeta_1^2 - 1), \\ \varepsilon_2 = \frac{1}{2}\left[\left(\frac{\partial x}{\partial y_0}\right)^2 + \left(\frac{\partial y}{\partial y_0}\right)^2 + \left(\frac{\partial z}{\partial y_0}\right)^2 - 1\right] = \frac{1}{2}(\xi_2^2 + \eta_2^2 + \zeta_2^2 - 1), \\ \varepsilon_3 = \frac{1}{2}\left[\left(\frac{\partial x}{\partial z_0}\right)^2 + \left(\frac{\partial y}{\partial z_0}\right)^2 + \left(\frac{\partial z}{\partial z_0}\right)^2 - 1\right] = \frac{1}{2}(\xi_3^2 + \eta_3^2 + \zeta_3^2 - 1), \\ \gamma_1 = \frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0} + \frac{\partial y}{\partial y_0}\frac{\partial y}{\partial z_0} + \frac{\partial z}{\partial y_0}\frac{\partial z}{\partial z_0} = \xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3, \\ \gamma_2 = \frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0} + \frac{\partial y}{\partial z_0}\frac{\partial y}{\partial x_0} + \frac{\partial z}{\partial z_0}\frac{\partial z}{\partial x_0} = \xi_3\xi_1 + \eta_3\eta_1 + \zeta_3\zeta_1, \\ \gamma_3 = \frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0} + \frac{\partial y}{\partial x_0}\frac{\partial y}{\partial y_0} + \frac{\partial z}{\partial x_0}\frac{\partial z}{\partial y_0} = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2. \end{cases} \quad (45)$$

Let x', y', z' designate the projections of the segment OM on the axes Mx', My', Mz', such that the coordinates of the fixed point O will be -x', -y', -z' with respect to these axes. We will have the following known equations:

$$\xi_i - \frac{\partial x'}{\partial p_i} - qz' + ry' = 0, \quad \eta_i - \frac{\partial y'}{\partial p_i} - rx' + pz' = 0, \quad \zeta_i - \frac{\partial z'}{\partial p_i} - py' + qx' = 0, \quad (46)$$

which give new expressions for ξ_i, η_i, ζ_i.

50. Expressions for the Variations of the Rates of Translation and Rotation of the Triad for the Deformed State. Let us assume that we impart an infinitesimally small displacement to each of the triads of the deformed state in a continuous way with respect to these triads. Let δx, δy, δz; $\delta x'$, $\delta y'$, $\delta z'$; $\delta\alpha$, $\delta\alpha'$, ..., $\delta\gamma''$ designate the respective variations of x, y, z; x', y', z'; α, α', ..., γ''. The variations $\delta\alpha$, $\delta\alpha'$, ..., $\delta\gamma''$ are expressed by means of equations such as the following: /124

$$\delta\alpha = \beta\delta\kappa' - \gamma\delta\iota', \quad (47)$$

in terms of three auxiliary quantities $\delta I'$, $\delta J'$, $\delta K'$ which are the well known components of the instantaneous rotation along Mx', My', Mz' connected with the infinitely small displacement under consideration. The variations δx, δy, δz are the projections of the infinitely small displacement experienced by the point M along Ox, Oy, Oz. The projections $\delta' x$, $\delta' y$, $\delta' z$ of this displacement along Mx', My', Mz' are immediately derived from them and have the following values:

$$\delta' x = \delta x' + z'\delta\iota' - y'\delta\kappa', \quad \delta' y = \delta y' + x'\delta\kappa' - z'\delta\iota', \quad \delta' z = \delta z' + y'\delta\iota' - x'\delta\iota'. \quad (48)$$

Let us attmept to determine the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i experienced by ξ_i, η_i, ζ_i, p_i, q_i, r_i, respectively. According to equations (44), we have:

$$\delta p_i = \sum\left(\frac{\partial\beta}{\partial p_i}\delta\gamma + \gamma\frac{\partial\delta\beta}{\partial p_i}\right),$$
$$\delta q_i = \sum\left(\frac{\partial\gamma}{\partial p_i}\delta\alpha + \alpha\frac{\partial\delta\gamma}{\partial p_i}\right),$$
$$\delta r_i = \sum\left(\frac{\partial\alpha}{\partial p_i}\delta\beta + \beta\frac{\partial\delta\alpha}{\partial p_i}\right),$$

Let us replace $\delta\alpha$ by its value $\beta\delta K' - \gamma\beta J'$, and $\delta\alpha'$, ..., $\delta\gamma''$ by their similar values. We have

$$\delta p_i = \frac{\partial \delta\alpha'}{\partial p_i} + q_i\delta\kappa' - r_i\delta\iota', \quad \delta q_i = \frac{\partial \delta\iota'}{\partial p_i} + r_i\delta\alpha' - p_i\delta\kappa', \quad \delta r_i = \frac{\partial \delta\kappa'}{\partial p_i} + p_i\delta\iota' - q_i\delta\alpha'. \tag{49}$$

Equations (46) also give us three equations, the first of which is:

$$\delta\xi_i = \frac{\partial \delta x'}{\partial p_i} + q_i\delta z' - r_i\delta y' + z'\delta q_i - y'\delta r_i;$$

let us replace δp_i, δq_i, δr_i by their values given by equations (49). We have:

$$\begin{cases} \delta\xi_i = \eta_i\delta\kappa' - \zeta_i\delta\iota' + \dfrac{\partial \delta' x}{\partial p_i} + q_i\delta' z - r_i\delta' y, \\ \delta\eta_i = \zeta_i\delta\alpha' - \xi_i\delta\kappa' + \dfrac{\partial \delta' y}{\partial p_i} + r_i\delta' x - p_i\delta' z, \\ \delta\zeta_i = \xi_i\delta\iota' - \eta_i\delta\alpha' + \dfrac{\partial \delta' z}{\partial p_i} + p_i\delta' y - q_i\delta' x, \end{cases} \tag{50}$$

in which we have introduced the three symbols $\delta' x$, $\delta' y$, $\delta' z$ defined by equations (48) to shorten the notation.

51. Euclidian Action of Deformation for a Deformable Medium. Let us retain the notations of Section 49 and introduce the known quantity Δ, defined by the equation: /125

$$\Delta = \frac{D(x, y, z)}{D(x_0, y_0, z_0)} = \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial y}{\partial x_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial z_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix}.$$

The square of it, which is formed by the rule of multiplication of determinants, can be expressed as follows as a function of ε_1, ε_2, ε_3, γ_1, γ_2, γ_3:

$$\Delta^2 = \begin{vmatrix} 1 + 2\varepsilon_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & 1 + 2\varepsilon_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & 1 + 2\varepsilon_3 \end{vmatrix}.$$

Let us consider a function W of *two infinitely close positions* of the triad $Mx'y'z'$, i.e., a function of x_0, y_0, z_0, of x, y, z, α, β, γ, α', β', γ', α'', β'', γ'' and their first derivatives with respect to x_0, y_0, z_0. Let us attempt to determine what must be the form of W such that the integral

$$\iiint W dx_0 dy_0 dz_0.$$

extended over an arbitrary portion of the space (M_0) has a zero variation, when the ensemble of all the triads of the deformable medium, taken in its deformed state, is subjected *to the same arbitrary, ininfinitesimal transformation of the group of Euclidian displacements.*

More precisely, it is necessary to determine W in such a way that we have:

$$\delta W = 0$$

when, on the one hand, the origin M of the triad Mx'y'z' carries out an infinitely small displacement, whose projections δx, δy, δz on the axes Ox, Oy, Oz are

$$\begin{cases} \delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t, \\ \delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t, \\ \delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t. \end{cases} \tag{51}$$

where a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constants and δt is an infinitesimally small quantity independent of x_0, y_0, z_0, and when, on the other hand, this triad Mx'y'z' carries out an infinitesimally small rotation whose components along the axes Ox, Oy, Oz are

$$\omega_1\delta t, \quad \omega_2\delta t, \quad \omega_3\delta t.$$

Let us note that in the present case the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$; δp_i, δq_i, δr_i of the eighteen expressions ξ_i, η_i, ζ_i; p_i, q_i, r_i are zero, which results from the well known theorem of the moving triad. We may also immediately verify it by means of equations (49) and (50), replacing $\delta'x$, $\delta'y$, $\delta'z$; $\delta I'$, /126
$\delta J'$, $\delta K'$ in them by their present values. It follows that we obtain a solution of the problem by taking W to be an arbitrary function of x_0, y_0, z_0 and of the eighteen expressions ξ_i, η_i, ζ_i; p_i, q_i, r_i. We will now show that we obtain the general solution(1) of the problem which we have posed ourselves in this way.

For this purpose, let us note that relationships (44) alow us to express the first derivatives of the nine cosines α, α', ..., γ'' with respect to x_0, y_0, z_0 in terms of these cosines and of p_i, q_i, r_i, using well known equations. On the other hand, equations (43) alow us to express the nine cosines α, α', ..., γ'' in terms of ξ_i, η_i, ζ_i and the first derivatives of x, y, z with respect to x_0, in terms of ξ_2, η_2, ζ_2 and the first derivatives of x, y, z with respect to y_0, or, finally, in terms of ξ_3, η_3, ζ_3 and first derivatives of x, y, z with respect to z_0. It is therefore useless, in the latter case, to make a hypothesis regarding the method of solution. It is clear that we will not obtain a more general form than the one we have started with, assuming finally that the function W which we are seeking is an arbitrary function of x_0, y_0, z_0 and x, y, z and of their first derivatives with respect to x_0, y_0, z_0, and finally of ξ_i, η_i, ζ_i, p_i, q_i, r_i. We indicate this fact, using the notation $\rho_1 = x_0$, $\rho_2 = y_0$, $\rho_3 = z_0$, by writing:

$$W = W\left(\rho_i, x, y, z, \frac{\partial x}{\partial \rho_i}, \frac{\partial y}{\partial \rho_i}, \frac{\partial z}{\partial \rho_i}, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i\right).$$

(1) In the following we assume that *the medium can carry out all possible deformations*, and that, consequently, *the deformed state can be assumed completely arbitrarily*.

Since the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i are zero in the present case according to what we have said above, we may finally write the following for the new form of W, which is true due to equations (51) and no matter what a_1, a_2, a_3, ω_1, ω_2, ω_3 may be,

$$\frac{\partial W}{\partial x}\delta x+\frac{\partial W}{\partial y}\delta y+\frac{\partial W}{\partial z}\delta z+\sum\left(\frac{\partial W}{\partial\frac{\partial x}{\partial\rho_i}}\delta\frac{\partial x}{\partial\rho_i}+\frac{\partial W}{\partial\frac{\partial y}{\partial\rho_i}}\delta\frac{\partial y}{\partial\rho_i}+\frac{\partial W}{\partial\frac{\partial z}{\partial\rho_i}}\delta\frac{\partial z}{\partial\rho_i}\right)=0.$$

Let us replace δx, δy, δz by their values (51) and $\delta\frac{\partial x}{\partial\rho_1}$, $\delta\frac{\partial y}{\partial\rho_1}$, $\delta\frac{\partial z}{\partial\rho_1}$ by the values which are derived by differentiation. Let us set the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3 equal to zero. We obtain the following six conditions:

$$\frac{\partial W}{\partial x}=0,\quad\frac{\partial W}{\partial y}=0,\quad\frac{\partial W}{\partial z}=0,$$

$$\sum\left(\frac{\partial W}{\partial\frac{\partial y}{\partial\rho_i}}\frac{\partial z}{\partial\rho_i}-\frac{\partial W}{\partial\frac{\partial z}{\partial\rho_i}}\frac{\partial y}{\partial\rho_i}\right)=0,\quad\sum\left(\frac{\partial W}{\partial\frac{\partial z}{\partial\rho_i}}\frac{\partial x}{\partial\rho_i}-\frac{\partial W}{\partial\frac{\partial x}{\partial\rho_i}}\frac{\partial z}{\partial\rho_i}\right)=0.$$

$$\sum\left(\frac{\partial W}{\partial\frac{\partial x}{\partial\rho_i}}\frac{\partial y}{\partial\rho_i}-\frac{\partial W}{\partial\frac{\partial y}{\partial\rho_i}}\frac{\partial x}{\partial\rho_i}\right)=0,$$

which are identities if we assume that the expressions appearing in W have been /127 reduced to their minimum number.

As could have easily have been predicted, the first three show us that W is independent of x, y, z. The three latter ones indicate that W does not depend on the first derivatives of x, y, z with respect to x_0, y_0, z_0 except through the intermediary of the quantities ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 defined by equations (45). We finally see that *the unknown function* W *has the following remarkable form:*

$$W(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i),$$

which is analogous to the one already encountered for the deformable line and the deformable surface.

If we multiply W by the volume element $dx_0dy_0dz_0$ of the space (M_0), the product $Wdx_0dy_0dz_0$ thus obtained is an invariant analogous to the volume element of the medium (M), for the group of Euclidian displacements.

In the same way, the common value of the integrals:

$$\iiint_{S_0}|\Delta|\,dx_0dy_0dz_0,\qquad\iiint_S dxdydz,$$

taken over the interior of a surface S_0 of the medium (M_0) and over the interior of the corresponding surface S of the medium (M), respectively, determines the volume of the region limited by the surface S. In the same way, by associating the concept of action in passing from the natural state W to the deformed state

(M) in our minds, we may connect the function W to the elements used to define the deformable medium. We will say that the integral

$$\iiint_{S_0} W dx_0 dy_0 dz_0$$

is the *action of deformation* in the interior of the surface S on the deformed surface.

On the other hand, we will say that W is the *density* of the action of deformation at a point of the deformed medium referred to a unit of volume of the non-deformed medium, and that $\frac{W}{|\Delta|}$ is the density of this action at a point referred to the unit of volume of the deformed medium.

52. External Force and Moment. External Stress and Moment of Deformation. Stress and Moment of Deformation at a Point of the Deformed Medium. Let us consider an arbitrary variation of the action of deformation at the interior of a surface S of the medium (M), i.e.:

$$\delta \iiint_{S_0} W dx_0 dy_0 dz_0$$
$$= \iiint_{S_0} \sum \left(\frac{\partial W}{\partial \xi_i} \delta \xi_i + \frac{\partial W}{\partial \eta_i} \delta \eta_i + \frac{\partial W}{\partial \zeta_i} \delta \zeta_i + \frac{\partial W}{\partial p_i} \delta p_i + \frac{\partial W}{\partial q_i} \delta q_i + \frac{\partial W}{\partial r_i} \delta r_i \right) dx_0 dy_0 dz_0.$$

We may write it as follows, due to equations (49) and (50) of Section 50: /128

$$\delta \iiint_{S_0} W dx_0 dy_0 dz_0 = \iiint_{S_0} \sum \left\{ \frac{\partial W}{\partial \xi_i} \left(\eta_i \delta \kappa' - \zeta_i \delta \iota' + \frac{\partial \delta' x}{\partial \rho_i} + q_i \delta' z - r_i \delta' y \right) \right.$$
$$+ \frac{\partial W}{\partial \eta_i} \left(\zeta_i \delta \iota' - \xi_i \delta \kappa' + \frac{\partial \delta' y}{\partial \rho_i} + r_i \delta' x - p_i \delta' z \right)$$
$$+ \frac{\partial W}{\partial \zeta_i} \left(\xi_i \delta \iota' - \eta_i \delta \iota' + \frac{\partial \delta' z}{\partial \rho_i} + p_i \delta' y - q_i \delta' x \right)$$
$$+ \frac{\partial W}{\partial p_i} \left(\frac{\partial \delta \iota'}{\partial \rho_i} + q_i \delta \kappa' - r_i \delta \iota' \right) + \frac{\partial W}{\partial q_i} \left(\frac{\partial \delta \iota'}{\partial \rho_i} + r_i \delta \iota' - p_i \delta \kappa' \right)$$
$$\left. + \frac{\partial W}{\partial r_i} \left(\frac{\partial \delta \kappa'}{\partial \rho_i} + p_i \delta \iota' - q_i \delta \iota' \right) \right\} dx_0 dy_0 dz_0.$$

Let us apply Green's formula to the terms explicitly containing a derivative with respect to one of the variables ρ_1, ρ_2, ρ_3. We obtain the following, where l_0, m_0, n_0 designate the direction cosines with respect to fixed axes Ox, Oy, Oz of the external normal to the surface S_0, which limits the medium before the deformation, and where $d\sigma_0$ designates the area element of this surface:

$$\delta \iiint_{S_0} W dx_0 dy_0 dz_0 = \iint_{S_0} \left\{ \left(l_0 \frac{\partial W}{\partial \xi_1} + m_0 \frac{\partial W}{\partial \xi_2} + n_0 \frac{\partial W}{\partial \xi_3} \right) \delta' x \right.$$
$$+ \left(l_0 \frac{\partial W}{\partial \eta_1} + m_0 \frac{\partial W}{\partial \eta_2} + n_0 \frac{\partial W}{\partial \eta_3} \right) \delta' y + \left(l_0 \frac{\partial W}{\partial \zeta_1} + m_0 \frac{\partial W}{\partial \zeta_2} + n_0 \frac{\partial W}{\partial \zeta_3} \right) \delta' z$$

$$+\left(l_0\frac{\partial W}{\partial p_1}+m_0\frac{\partial W}{\partial p_2}+n_0\frac{\partial W}{\partial p_3}\right)\delta\iota'+\left(l_0\frac{\partial W}{\partial q_1}+m_0\frac{\partial W}{\partial q_2}+n_0\frac{\partial W}{\partial q_3}\right)\delta\lambda'$$

$$+\left(l_0\frac{\partial W}{\partial r_1}+m_0\frac{\partial W}{\partial r_2}+n_0\frac{\partial W}{\partial r_3}\right)\delta\kappa'\Bigg\}\,d\sigma_0$$

$$-\iiint_{S_0}\Bigg\{\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\xi_i}+q_i\frac{\partial W}{\partial\zeta_i}-r_i\frac{\partial W}{\partial\eta_i}\right)\right]\delta'x$$

$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\eta_i}+r_i\frac{\partial W}{\partial\xi_i}-p_i\frac{\partial W}{\partial\zeta_i}\right)\right]\delta'y$$

$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\zeta_i}+p_i\frac{\partial W}{\partial\eta_i}-q_i\frac{\partial W}{\partial\xi_i}\right)\right]\delta'z$$

$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial p_i}+q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial\zeta_i}-\zeta_i\frac{\partial W}{\partial\eta_i}\right)\right]\delta\iota'$$

$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial q_i}+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\zeta_i\frac{\partial W}{\partial\xi_i}-\xi_i\frac{\partial W}{\partial\zeta_i}\right)\right]\delta\lambda'$$

$$+\left[\sum\left(\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial r_i}+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial\eta_i}-\eta_i\frac{\partial W}{\partial\xi_i}\right)\right]\kappa'\Bigg\}\,dx_0dy_0dz_0.$$

Let us set: /129

$$F_0'=l_0\frac{\partial W}{\partial\xi_1}+m_0\frac{\partial W}{\partial\xi_2}+n_0\frac{\partial W}{\partial\xi_3},\quad I_0'=l_0\frac{\partial W}{\partial p_1}+m_0\frac{\partial W}{\partial p_2}+n_0\frac{\partial W}{\partial p_3}.$$

$$G_0'=l_0\frac{\partial W}{\partial\eta_1}+m_0\frac{\partial W}{\partial\eta_2}+n_0\frac{\partial W}{\partial\eta_3},\quad J_0'=l_0\frac{\partial W}{\partial q_1}+m_0\frac{\partial W}{\partial q_2}+n_0\frac{\partial W}{\partial q_3}.$$

$$H_0'=l_0\frac{\partial W}{\partial\zeta_1}+m_0\frac{\partial W}{\partial\zeta_2}+n_0\frac{\partial W}{\partial\zeta_3},\quad K_0'=l_0\frac{\partial W}{\partial r_1}+m_0\frac{\partial W}{\partial r_2}+n_0\frac{\partial W}{\partial r_3}.$$

$$X_0'=\sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\xi_i}+q_i\frac{\partial W}{\partial\zeta_i}-r_i\frac{\partial W}{\partial\eta_i}\right],$$

$$Y_0'=\sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\eta_i}+r_i\frac{\partial W}{\partial\xi_i}-p_i\frac{\partial W}{\partial\zeta_i}\right],$$

$$Z_0'=\sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial\zeta_i}+p_i\frac{\partial W}{\partial\eta_i}-q_i\frac{\partial W}{\partial\xi_i}\right],$$

$$L_0'=\sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial p_i}+q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial\zeta_i}-\zeta_i\frac{\partial W}{\partial\eta_i}\right],$$

$$M_0'=\sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial q_i}+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\zeta_i\frac{\partial W}{\partial\xi_i}-\xi_i\frac{\partial W}{\partial\zeta_i}\right],$$

$$N_0'=\sum\left[\frac{\partial}{\partial\rho_i}\frac{\partial W}{\partial r_i}+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial\eta_i}-\eta_i\frac{\partial W}{\partial\xi_i}\right],$$

We will have:

$$\delta\iiint_{S_0}W\,dx_0dy_0dz_0=\iint_{S_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta\iota'+J_0'\delta\lambda'+K_0'\delta\kappa')\,d\sigma_0$$

$$-\iiint_{S_0}(X_0'\delta'x+Y_0'\delta'y+Z_0'\delta'z+L_0'\delta\iota'+M_0'\delta\lambda'+N_0'\delta\kappa')\,dx_0dy_0dz_0.$$

Let us first consider the triple integral which appears in the expression for $\delta\iiint_{S_0}W\,dx_0dy_0dz_0$. We will call *external force and external moment at the*

point M *referred to the unit of volume of the non-deformed medium*, the segments having their origin at M and whose projections on the axes Mx', My', Mz' are X_0', Y_0', Z_0' and L_0', M_0', N_0' respectively.

If we now consider the surface integral appearing in

$$\delta \iiint_{\Sigma_0} W dx_0 dy_0 dz_0,$$

we will designate by *external stress and external moment of deformation at the point* M *of the surface* S, which limits the deformed medium, *referred to the unit of area of the surface* S_0, the segments emanating from this point M and whose projections on the axes Mx', My', Mz' are $-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$, respectively. At a determined point M of (S), these six latter /130 quantities only depend on the direction of the external normal to the surface (S). They remain invariable if the region of (M_0) under consideration is varied, if the region of the external normal does not change. They change their sign if this direction is replaced by the opposite one. Let us assume that at the interior of the deformed medium limited by the surface (S) we draw a surface (Σ) which circumscribes a part of (A) of the medium, either with or without a part of the surface (S). By (B) let us designate the remainder outside of the part (A). Let (Σ_0) be the surface of (M_0) which corresponds to the surface (Σ) of (M), and let (A_0) and (B_0) be the regions of (M_0) which correspond to regions (A) and (B) of (M). In our minds let us separate the two parts (A) and (B). We may regard the two segments $(-F_0', -G_0', -H_0')$ and $(-I_0', -J_0', -K_0')$ determined for the point M and for the direction of the normal directed from (Σ_0) through the exterior of (A_0) as the external stress and moment of deformation at the point M of the boundary (Σ) of the region (A). In the same way, we may regard the two segments (F_0', G_0', H_0') and (I_0', J_0', K_0') as the external stress and moment of deformation at the point M of the boundary (Σ) of the region (B). Due to this fact we will say that $-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$ $-J_0'$, $-K_0'$ are the components along the axes Mx', My', Mz' *of the stress and moment of deformation exerted at* M *on the portion* (A) *of the medium* (M), and that F_0', G_0', H_0' and I_0', J_0', K_0' are the components along the axes Mx', My', Mz' *of the stress and moment of deformation exerted at* M *on the portion* (B) *of the medium* (M).

We may repeat the observation made at the end of Sections 9 and 34 regarding the replacement of the triad Mx'y'z' by a triad which is invariably connected to it, without any modifications.

53. Various Conditions for the Stress and Moment of Deformation. Let us set:

$$A_i' = \frac{\partial W}{\partial \xi_i}, \quad B_i' = \frac{\partial W}{\partial \eta_i}, \quad C_i' = \frac{\partial W}{\partial \zeta_i},$$

$$P' = \frac{\partial W}{\partial p_i}, \quad Q_i' = \frac{\partial W}{\partial q_i}, \quad R_i' = \frac{\partial W}{\partial r_i};$$

A_i', B_i', C_i' and P_i', Q_i', R_i' represent the projections on Mx', My', Mz' of the stress and moment of deformation which are applied to a surface at point M which, before deformation, had an interior normal at the point M_0 parallel to the coordinate axes Ox, Oy, Oz which correspond to the index i. It is sufficient to recall that we have already found it convenient to replace x_0, y_0, z_0 by ρ_1, ρ_2, ρ_3, which correspond to the indices 1, 2, 3, respectively, according to this convention. This stress and moment of deformation are referred to the unit of area of the non-deformed surface, it should be recalled.

The new stresses and new moments of deformation which we have just defined are related to the elements introduced in the preceding section according to the following relationships: /131

$$F_0' = l_0A_1' + m_0A_2' + n_0A_3', \quad L_0' = l_0P_1' + m_0P_2' + n_0P_3',$$
$$G_0' = l_0B_1' + m_0B_2' + n_0B_3', \quad J_0' = l_0Q_1' + m_0Q_2' + n_0Q_3',$$
$$H_0' = l_0C_1' + m_0C_2' + n_0C_3', \quad K_0' = l_0R_1' + m_0R_2' + n_0R_3'.$$

$$\sum\left(\frac{\partial A_i'}{\partial \rho_i} + q_iC_i' - r_iB_i'\right) - X_0' = 0,$$

$$\sum\left(\frac{\partial B_i'}{\partial \rho_i} + r_iA_i' - p_iC_i'\right) - Y_0' = 0,$$

$$\sum\left(\frac{\partial C_i'}{\partial \rho_i} + p_iB_i' - q_iA_i'\right) - Z_0' = 0,$$

$$\sum\left(\frac{\partial P_i'}{\partial \rho_i} + q_iR_i' - r_iQ_i' + \eta_iC_i' - \zeta_iB_i'\right) - L_0' = 0,$$

$$\sum\left(\frac{\partial Q_i'}{\partial \rho_i} + r_iP_i' - p_iR_i' + \zeta_iA_i' - \xi_iC_i'\right) - M_0' = 0,$$

$$\sum\left(\frac{\partial R_i'}{\partial \rho_i} + p_iQ_i' - q_iP_i' + \xi_iB_i' - \eta_iA_i'\right) - N_0' = 0,$$

Let us attempt to transform these relationships which we have written down in a way which is independent of the values of the quantities which occur in them and which are calculated in terms of W. These relationships hold for the segments associated with the point M and which we have named. Instead of defining these segments by their projections on Mx', My', Mz', we may just as well define them by their projections on other axes. These latter projections will be related by relationships that are the transforms of the preceding ones.

The transformed relationships are immediately obtained by noting that the initial equations allow simple and immediate interpretation(2) by making an association between the moving axes and axes parallel to them and displaced through the point O.

1. Let us limit ourselves to the consideration of fixed axes Ox, Oy, Oz. Let X_0, Y_0, Z_0 and L_0, M_0, N_0, respectively, designate the projections on these

(2) An interesting interpretation to be noted is analogous to the one given by P. Saint-Guilhem for the dynamics of a triad.

axes of the external force and external moment at an arbitrary point M of the deformed medium. Let F_0, G_0, H_0 and I_0, J_0, K_0 designate the projections of the stress and moment of deformation on a surface whose internal normal has the direction cosines l_0, m_0, n_0 before deformation. Let A_i, B_i, C_i and P_i, Q_i, R_i be the projections of the stress (A_i', B_i', C_i') and the moment of deformation (P_i', Q_i', R_i'). The transforms of the preceding relationships are obviously: /132

$$
\begin{aligned}
F_0 &= l_0A_1 + m_0A_2 + n_0A_3, & I_0 &= l_0P_1 + m_0P_2 + n_0P_3,\\
G_0 &= l_0B_1 + m_0B_2 + n_0B_3, & J_0 &= l_0Q_1 + m_0Q_2 + n_0Q_3,\\
H_0 &= l_0C_1 + m_0C_2 + n_0C_3, & K_0 &= l_0R_1 + m_0R_2 + n_0R_3.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0} - X_0 = 0,\\
&\frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0} - Y_0 = 0,\\
&\frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0} - Z_0 = 0,\\
&\frac{\partial P_1}{\partial x_0} + \frac{\partial P_2}{\partial y_0} + \frac{\partial P}{\partial z_0} + C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} - B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial y_0} - B_3\frac{\partial z}{\partial z_0} - L_0 = 0,\\
&\frac{\partial Q_1}{\partial x_0} + \frac{\partial Q_2}{\partial y_0} + \frac{\partial Q_3}{\partial z_0} + A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} - C_1\frac{\partial x}{\partial x_0} - C_2\frac{\partial x}{\partial y_0} - C_3\frac{\partial x}{\partial z_0} - M_0 = 0,\\
&\frac{\partial R_1}{\partial x_0} + \frac{\partial R_2}{\partial y_0} + \frac{\partial R_3}{\partial z_0} + B_1\frac{\partial x}{\partial x_0} + B_2\frac{\partial x}{\partial y_0} + C_3\frac{\partial x}{\partial z_0} - A_1\frac{\partial y}{\partial x_0} - A_2\frac{\partial y}{\partial y_0} - A_3\frac{\partial y}{\partial z_0} - N_0 = 0.
\end{aligned}
$$

These are three-dimensional generalizations of the equations of Lord Kelvin and Tait for one and two dimensions.

2. Let us now note that we may express the nine cosines α, α', ... γ'' in terms of three auxiliary quantities. Let λ_1, λ_2, λ_3 be three such auxiliary quantities. Let us set:

$$
\begin{aligned}
\sum \gamma d\beta &= -\sum \beta d\gamma = \varpi_1' d\lambda_1 + \varpi_2' d\lambda_2 + \varpi_3' d\lambda_3,\\
\sum \alpha d\gamma &= -\sum \gamma d\alpha = \chi_1' d\lambda_1 + \chi_2' d\lambda_2 + \chi_3' d\lambda_3,\\
\sum \beta d\alpha &= -\sum \alpha d\beta = \sigma_1' d\lambda_1 + \sigma_2' d\lambda_2 + \sigma_3' d\lambda_3.
\end{aligned}
$$

The functions ϖ_i', χ_i', σ_i' of λ_1, λ_2, λ_3 defined in this way satisfy the relationships:

$$
\begin{aligned}
&\frac{\partial \varpi_j'}{\partial \lambda_i} - \frac{\partial \varpi_i'}{\partial \lambda_j} + \chi_i'\sigma_j' - \chi_j'\sigma_i' = 0,\\
&\frac{\partial \chi_j'}{\partial \lambda_i} - \frac{\partial \chi_i'}{\partial \lambda_j} + \sigma_i'\varpi_j' - \sigma_j'\varpi_i' = 0, \qquad (i, j) = 1, 2, 3.\\
&\frac{\partial \sigma_j'}{\partial \lambda_i} - \frac{\partial \sigma_i'}{\partial \lambda_j} + \varpi_i'\chi_j' - \varpi_j'\chi_i' = 0,
\end{aligned}
$$

and we have:

$$
\begin{aligned}
p_i &= \varpi_1' \frac{\partial \lambda_1}{\partial \rho_i} + \varpi_2' \frac{\partial \lambda_2}{\partial \rho_i} + \varpi_3' \frac{\partial \lambda_3}{\partial \rho_i},\\
q_i &= \chi_1' \frac{\partial \lambda_1}{\partial \rho_i} + \chi_2' \frac{\partial \lambda_2}{\partial \rho_i} + \chi_3' \frac{\partial \lambda_3}{\partial \rho_i},\\
r_i &= \sigma_1' \frac{\partial \lambda_1}{\partial \rho_i} + \sigma_2' \frac{\partial \lambda_2}{\partial \rho_i} + \sigma_3' \frac{\partial \lambda_3}{\partial \rho_i}.
\end{aligned}
\qquad \text{(where } x_0 = \rho_1,\ y_0 = \rho_2,\ z_0 = \rho_3\text{)}
$$

Let ϖ_i, χ_i, σ_i designate the projections on fixed axes Ox, Oy, Oz of the segment whose projections on the axes Mx', My', Mz' are ϖ_i', χ_i', σ_i'. We will /133 have:

$$\sum \alpha' dx' = -\sum \alpha'' dx' = \varpi_1 d\lambda_1 + \varpi_2 d\lambda_2 + \varpi_3 d\lambda_3,$$
$$\sum \alpha'' dx = -\sum \alpha dx'' = \chi_1 d\lambda_1 + \chi_2 d\lambda_2 + \chi_3 d\lambda_3,$$
$$\sum \alpha dx' = -\sum \alpha' dx = \sigma_1 d\lambda_1 + \sigma_2 d\lambda_2 + \sigma_3 d\lambda_3,$$

from which it follows that the new functions ϖ_i, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy the relationships(3):

$$\frac{\partial \varpi_j}{\partial \lambda_i} - \frac{\partial \varpi_i}{\partial \lambda_j} = \chi_i \sigma_j - \chi_j \sigma_i,$$
$$\frac{\partial \chi_j}{\partial \lambda_i} - \frac{\partial \chi_i}{\partial \lambda_j} = \sigma_i \varpi_j - \sigma_j \varpi_i, \qquad (i, j) = 1, 2, 3.$$
$$\frac{\partial \sigma_j}{\partial \lambda_i} - \frac{\partial \sigma_i}{\partial \lambda_j} = \varpi_i \chi_j - \varpi_j \chi_i.$$

We should also like to note a fact which will be of use to us later, namely, that if $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ designate the variations of λ_1, λ_2, λ_3 which correspond to the variations $\delta\alpha$, $\delta\alpha'$, ... $\delta\gamma''$ of α, α', ... γ'', we will have:

$$\delta I' = \varpi_1' \delta\lambda_1 + \varpi_2' \delta\lambda_2 + \varpi_3' \delta\lambda_3,$$
$$\delta J' = \chi_1' \delta\lambda_1 + \chi_2' \delta\lambda_2 + \chi_3' \delta\lambda_3,$$
$$\delta K' = \sigma_1' \delta\lambda_1 + \sigma_2' \delta\lambda_2 + \sigma_3' \delta\lambda_3,$$
$$\delta I = \alpha \delta I' + \beta \delta J' + \gamma \delta K' = \varpi_1 \delta\lambda_1 + \varpi_2 \delta\lambda_2 + \varpi_3 \delta\lambda_3,$$
$$\delta J = \alpha' \delta I' + \beta' \delta J' + \gamma' \delta K' = \chi_1 \delta\lambda_1 + \chi_2 \delta\lambda_2 + \chi_3 \delta\lambda_3,$$
$$\delta K = \alpha'' \delta I' + \beta'' \delta J' + \gamma'' \delta K' = \sigma_1 \delta\lambda_1 + \sigma_2 \delta\lambda_2 + \sigma_3 \delta\lambda_3,$$

where δI, δJ, δK are the projections on fixed axes of the segment whose projections on Mx', My', Mz' are $\delta I'$, $\delta J'$, $\delta K'$.

Let us now set

$$\mathfrak{J}_0 = \varpi_1' I_0' + \chi_1' J_0' + \sigma_1' K_0' = \varpi_1 I_0 + \chi_1 J_0 + \sigma_1 K_0,$$
$$\mathfrak{J}_0 = \varpi_2' I_0' + \chi_2' J_0' + \sigma_2' K_0' = \varpi_2 I_0 + \chi_2 J_0 + \sigma_2 K_0,$$
$$\mathfrak{K}_0 = \varpi_3' I_0' + \chi_3' J_0' + \sigma_3' K_0' = \varpi_3 I_0 + \chi_3 J_0 + \sigma_3 K_0,$$
$$\mathfrak{L}_0 = \varpi_1' L_0' + \chi_1' M_0' + \sigma_1' N_0' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0,$$
$$\mathfrak{M}_0 = \varpi_2' L_0' + \chi_2' M_0' + \sigma_2' N_0' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0,$$
$$\mathfrak{N}_0 = \varpi_3' L_0' + \chi_3' M_0' + \sigma_3' N_0' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0.$$

Let us also introduce the following notation:

$$\Pi_i = \varpi_1' P_i' + \chi_1' Q_i' + \sigma_1' R_i' = \varpi_1 P_i + \chi_1 Q_i + \sigma_1 R_i,$$
$$X_i = \varpi_2' P_i' + \chi_2' Q_i' + \sigma_2' R_i' = \varpi_2 P_i + \chi_2 Q_i + \sigma_2 R_i,$$
$$\Sigma_i = \varpi_3' P_i' + \chi_3' Q_i' + \sigma_3' R_i' = \varpi_3 P_i + \chi_3 Q_i + \sigma_3 R_i,$$

---

(3) These equations may be used to directly define the functions ϖ_i, χ_i, σ_i and can be substituted in

$$\varpi_i = \alpha \varpi_i' + \beta \chi_i' + \gamma \sigma_i',$$
$$\chi_i = \alpha' \varpi_i' + \beta' \chi_i' + \gamma' \sigma_i', \qquad (i = 1, 2, 3)$$
$$\sigma_i = \alpha'' \varpi_i' + \beta'' \chi_i' + \gamma'' \sigma_i'.$$

Instead of the latter system where P_i', Q_i', R_i' or P_i, Q_i, R_i appear, we have /134 the following:

$$\mathfrak{L}_0=\sum_i\Big[\frac{\partial\Pi_i}{\partial\rho_i}-P_i'\Big(\frac{\partial\varpi_i'}{\partial\rho_i}+q_i\sigma_i'-r_i\chi_i'\Big)-Q_i'\Big(\frac{\partial\chi_i'}{\partial\rho_i}+r_i\varpi_i'-p_i\sigma_i'\Big)-R_i'\Big(\frac{\partial\sigma_i'}{\partial\rho_i}+p_i\chi_i'-q_i\varpi_i'\Big)$$
$$+A_i'(\chi_i'\zeta_i-\sigma_i'\eta_i)+B_i'(\sigma_i'\xi_i-\varpi_i'\zeta_i)+C_i'(\varpi_i'\eta_i-\chi_i'\xi_i)\Big].$$

and two analogous equations. If we note that the functions ξ_i, η_i, ζ_i, p_i, q_i, r_i, of λ_1, λ_2, λ_3, $\frac{\partial\lambda_1}{\partial\rho_i}$, $\frac{\partial\lambda_2}{\partial\rho_i}$, $\frac{\partial\lambda_3}{\partial\rho_i}$ result in the equations:

$$\frac{\partial\xi_i}{\partial_j}+\chi_j'\zeta_i-\sigma_j'\eta_i=0,\qquad \frac{\partial p_i}{\partial\lambda_j}=\frac{\partial\varpi_j'}{\partial\rho_i}+q_i\sigma_j'-r_i\chi_j',$$
$$\frac{\partial\eta_i}{\partial\lambda_j}+\sigma_j'\xi_i-\varpi_j'\zeta_i=0,\qquad \frac{\partial q_i}{\partial\lambda_j}=\frac{\partial\chi_j'}{\partial\rho_i}+r_i\varpi_j'-p_i\sigma_j'.$$
$$\frac{\partial\zeta_i}{\partial\lambda_j}+\varpi_j'\eta_i-\chi_j'\xi_i=0,\qquad \frac{\partial r_i}{\partial\lambda_j}=\frac{\partial\sigma_j'}{\partial\rho_i}+p_i\chi_j'-q_i\varpi_j',$$

which are a consequence of the defining relationships for the functions ϖ_i', χ_i', σ_i' and the nine identities which they satisfy, we may change the preceding system into the new form:

$$\mathfrak{L}_0=\sum_i\Big[\frac{\partial\Pi_i}{\partial\rho_i}-A_i'\frac{\partial\xi_i}{\partial\lambda_1}-B_i'\frac{\partial\eta_i}{\partial\lambda_1}-C_i'\frac{\partial\zeta_i}{\partial\lambda_1}-P_i'\frac{\partial p_i}{\partial\lambda_1}-Q_i'\frac{\partial q_i}{\partial\lambda_1}-R_i'\frac{\partial r_i}{\partial\lambda_1}\Big].$$

and two similar equations.

3. The equations which we have previously introduced constitute the generalization of those which we developed in an earlier work[4]. We may transform them in such a way that we can obtain the generalization of the well known equations of the theory of elasticity expressed in terms of the stress. For this purpose it is sufficient to use the method already employed in the work to which we have just referred.

In order to abbreviate the writing, let us for a moment designate by $\mathfrak{X}'_0$, $\mathfrak{Y}'_0$, $\mathfrak{Z}'_0$, $\mathfrak{L}'_0$, $\mathfrak{M}'_0$, $\mathfrak{N}'_0$ the first terms of the relationships to be transformed, which respectively contain X_0, Y_0, Z_0, L_0, M_0, N_0. Let us note that we may summarize the twelve relationships we have established in the following one:

$$\iiint(\mathfrak{X}_0'\lambda_1+\mathfrak{Y}_0'\lambda_2+\mathfrak{Z}_0'\lambda_3+\mathfrak{L}_0'\mu_1+\mathfrak{M}_0'\mu_2+\mathfrak{N}_0'\mu_3)dx_0dy_0dz_0$$
$$-\iint\{(F_0-l_0A_1-m_0A_2-n_0A_3)\lambda_1+(G_0-l_0B_1-m_0B_2-n_0B_3)\lambda_2$$
$$+(H_0-l_0C_1-m_0C_2-n_0C_3)\lambda_3+(I_0-l_0P_1-m_0P_2-n_0P_3)\mu_1$$
$$+(J_0-l_0Q_1-m_0Q_2-n_0Q_3)\mu_2+(K_0-l_0R_1-m_0R_2-n_0R_3)\mu_3\}\,d\sigma_0=0,$$

---

(4) E. and F. Cosserat. *First Report on the Theory of Elasticity; Annales de la Faculté des Sciences de Toulouse* (1), 10, p. I_1 - I_{116}, 1896.

where λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary functions and where the integrals are taken over the surface S_0 of the medium (M_0) and over the region limited by it. If we apply Green's equation, the relationship we obtain is as follows: /135

$$
\begin{aligned}
&\iiint (X_0\lambda_1 + Y_0\lambda_2 + Z_0\lambda_3 + L_0\mu_1 + M_0\mu_2 + N_0\mu_3)dx_0dy_0dz_0 \\
&- \iint (F_0\lambda_1 + G_0\lambda_2 + H_0\lambda_3 + I_0\mu_1 + J_0\mu_2 + K_0\mu_3)dz_0 \\
&+ \iiint \Big(A_1\frac{\partial\lambda_1}{\partial x_0} + A_2\frac{\partial\lambda_1}{\partial y_0} + A_3\frac{\partial\lambda_1}{\partial z_0} + B_1\frac{\partial\lambda_2}{\partial x_0} + B_2\frac{\partial\lambda_2}{\partial y_0} + B_3\frac{\partial\lambda_2}{\partial z_0} \\
&\qquad + C_1\frac{\partial\lambda_3}{\partial x_0} + C_2\frac{\partial\lambda_3}{\partial y_0} + C_3\frac{\partial\lambda_3}{\partial z_0}\Big)dx_0dy_0dz_0 \\
&+ \iiint \Big(P_1\frac{\partial\mu_1}{\partial x_0} + P_2\frac{\partial\mu_1}{\partial y_0} + P_3\frac{\partial\mu_1}{\partial z_0} + Q_1\frac{\partial\mu_2}{\partial x_0} + Q_2\frac{\partial\mu_2}{\partial y_0} + Q_3\frac{\partial\mu_2}{\partial z_0} \\
&\qquad + R_1\frac{\partial\mu_3}{\partial x_0} + R_2\frac{\partial\mu_3}{\partial y_0} + R_3\frac{\partial\mu_3}{\partial z_0}\Big)dx_0dy_0dz_0 \\
&- \iiint \Big(C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} + B_1\frac{\partial z}{\partial x_0} + B_2\frac{\partial z}{\partial y_0} + B_3\frac{\partial z}{\partial z_0}\Big)\mu_1dx_0dy_0dz_0 \\
&- \iiint \Big(A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} + C_1\frac{\partial x}{\partial x_0} + C_2\frac{\partial x}{\partial y_0} + C_3\frac{\partial x}{\partial z_0}\Big)\mu_2dx_0dy_0dz_0 \\
&- \iiint \Big(B_1\frac{\partial x}{\partial x_0} + B_2\frac{\partial x}{\partial y_0} + B_3\frac{\partial x}{\partial z_0} + A_1\frac{\partial y}{\partial x_0} + A_2\frac{\partial y}{\partial y_0} + A_3\frac{\partial y}{\partial z_0}\Big)\mu_3dx_0dy_0dz_0 = 0.
\end{aligned}
$$

Let us attempt to transform this latter relationship and let us take the new variables to be the functions x, y, z of x_0, y_0, z_0. If ϕ designates an arbitrary function of x_0, y_0, z_0 and which thus becomes a function of x, y, z, the elementary equations for the change of variables are given by:

$$
\begin{aligned}
\frac{\partial\phi}{\partial x_0} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x_0} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x_0}, \\
\frac{\partial\phi}{\partial y_0} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial y_0} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial y_0} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y_0}, \\
\frac{\partial\phi}{\partial z_0} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial z_0}.
\end{aligned}
$$

Let us apply these equations to the functions λ_1, λ_2, λ_3, μ_1, μ_2, μ_3. S is still used to designate the surface of the medium (M) which corresponds to the surface S_0 of (M_0). In addition, let us designate by X, Y, Z, L, M, N the projections on Ox, Oy, Oz of the external force and external moment applied at the point M and referred to the unit of volume of the deformed medium (M). Let F, G, H, I, J, K designate the projections on Ox, Oy, Oz of the stress and moment of deformation which are applied at the point M on S and are referred to the unit of area of S. Finally, let us introduce eighteen new auxiliary quantities p_{xx}, p_{yx}, p_{zx}, p_{xy}, p_{zy}, p_{yy}, p_{xz}, p_{yz}, p_{zz}, q_{xx}, q_{yx}, q_{zx}, q_{xy}, q_{yy}, q_{zy}, q_{xz}, q_{yz}, q_{zz}, by means of the equations: /136

$$\Delta p_{xx} = A_1 \frac{\partial x}{\partial x_0} + A_2 \frac{\partial x}{\partial y_0} + A_3 \frac{\partial x}{\partial z_0}, \quad \Delta q_{xx} = P_1 \frac{\partial x}{\partial x_0} + P_2 \frac{\partial x}{\partial y_0} + P_3 \frac{\partial x}{\partial z_0},$$

$$\Delta p_{yx} = A_1 \frac{\partial y}{\partial x_0} + A_2 \frac{\partial y}{\partial y_0} + A_3 \frac{\partial y}{\partial z_0}, \quad \Delta q_{yx} = P_1 \frac{\partial y}{\partial x_0} + P_2 \frac{\partial y}{\partial y_0} + P_3 \frac{\partial y}{\partial z_0},$$

$$\Delta p_{zx} = A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0}, \quad \Delta q_{zx} = P_1 \frac{\partial z}{\partial x_0} + P_2 \frac{\partial z}{\partial y_0} + P_3 \frac{\partial z}{\partial z_0},$$

and by analogous ones obtained by replacing:

$$A_1, A_2, A_3, p_{xx}, p_{yx}, p_{zx}, P_1, P_2, P_3, q_{xx}, q_{yx}, q_{zx},$$

by:

$$B_1, B_2, B_3, p_{xy}, p_{yy}, p_{zy}, Q_1, Q_2, Q_3, q_{xy}, q_{yy}, q_{zy},$$

respectively, and then by:

$$C_1, C_2, C_3, p_{xz}, p_{yz}, p_{zz}, R_1, R_2, R_3, q_{xz}, q_{yz}, q_{zz}.$$

We obtain the transformed relationship as follows:

$$\begin{aligned}
&\iiint (X\lambda_1 + Y\lambda_2 + Z\lambda_3 + L\mu_1 + M\mu_2 + N\mu_3)\,dxdydz \\
&- \iint (F\lambda_1 + G\lambda_2 + H\lambda_3 + I\mu_1 + J\mu_2 + K\mu_3)\,d\sigma \\
&+ \iiint \Big(p_{xx}\frac{\partial\lambda_1}{\partial x} + p_{yx}\frac{\partial\lambda_1}{\partial y} + p_{zx}\frac{\partial\lambda_1}{\partial z} + p_{xy}\frac{\partial\lambda_2}{\partial x} + p_{yy}\frac{\partial\lambda_2}{\partial y} + p_{zy}\frac{\partial\lambda_2}{\partial z} \\
&\qquad + p_{xz}\frac{\partial\lambda_3}{\partial x} + p_{yz}\frac{\partial\lambda_3}{\partial y} + p_{zz}\frac{\partial\lambda_3}{\partial z}\Big)\,dxdydz \\
&+ \iiint \Big(q_{xx}\frac{\partial\mu_1}{\partial x} + q_{yx}\frac{\partial\mu_1}{\partial y} + q_{zx}\frac{\partial\mu_1}{\partial z} + q_{xy}\frac{\partial\mu_2}{\partial x} + q_{yy}\frac{\partial\mu_2}{\partial y} + q_{zy}\frac{\partial\mu_2}{\partial z} \\
&\qquad + q_{xz}\frac{\partial\mu_3}{\partial x} + q_{yz}\frac{\partial\mu_3}{\partial y} + q_{zz}\frac{\partial\mu_3}{\partial z}\Big)\,dxdydz \\
&- \iiint \{(p_{yz} - p_{zy})\mu_1 + (p_{zx} - p_{xz})\mu_2 + (p_{xy} - p_{yx})\mu_3\}\,dxdydz = 0,
\end{aligned}$$

where the integrals are taken over the surface S of the medium (M) and over the region limited by it. $d\sigma$ designates the area element of S.

Let us again apply Green's formula to the terms containing the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect to x, y, z. Let us designate by l, m, n the direction cosines of the exterior normal to the surface S with respect to fixed axes. Since λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary, we have /137

$$\begin{aligned}
F &= lp_{xx} + mp_{yx} + np_{zx}, & I &= lq_{xx} + mq_{yx} + nq_{zx}, \\
G &= lp_{xy} + mp_{yy} + np_{zy}, & J &= lq_{xy} + mq_{yy} + nq_{zy}, \\
H &= lp_{xz} + mp_{yz} + np_{zz}, & K &= lq_{xz} + mq_{yz} + nq_{zz},
\end{aligned}$$

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} - X = 0,$$

$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} - Y = 0,$$

$$\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} - Z = 0,$$

$$\frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + p_{yz} - p_{zy} - L = 0,$$
$$\frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z} + p_{zx} - p_{xz} - M = 0,$$
$$\frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z} + p_{xy} - p_{yx} - N = 0.$$

The meaning of the eighteen new auxiliary quantities p_{xx}, ..., q_{xx}, ... immediately results from the relationships we have just found. It is clear that the coefficients p_{xx}, p_{xy}, p_{xz} of l in the expressions for F, G, H represent the projections on Ox, Oy, Oz of the stress applied at the point M on a surface whose interior normal is parallel to Ox, and that the coefficients q_{xx}, q_{xy}, q_{xz} of l in the expressions for I, J, K are the projections on Ox, Oy, Oz of the moment of deformation at M with respect to this same surface. The coefficients of m and of n give rise to a similar interpretation regarding surfaces whose interior normals are parallel to Oy and Oz. The auxiliary quantities which we have just introduced and the equations relating them do not seem to have been treated up to the present in such a general form. As far as we know, they have only been treated in the particular case where the nine quantities q_{xx}, ..., q_{zz} are zero, and the first work which treats this question seems to be that of M. Voigt[5].

Finally, let us note that if we carry out a change of variables in the /138 equations which contain X, Y, Z, F, G, H in such a way as to introduce the original variables x_0, y_0, z_0, we immediately find equations, the first three of which represent the generalization of the equations established by M. Boussinesq.

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(5) Waldemar Voigt. *Theoretical Studies of the Elasticity Behavior of Crystals*, I, II, Abhandlungen der königlichen Gesellschaft der Wissenschaften zu Göttingen (Transactions of the Royal Association of Scientists in Gottingen), Vol. 34, 1887. The first section entitled: Derivation of the Fundamental Equations for the Assumption of Molecules with Polarity, has 49 pages (3-52); the second one entitled: Investigation of the Elastic Behavior of a Cylinder Made of a Crystalline Substance Whose Outer Surface is Subjected to No Forces and When the Stresses Acting Along the Cylinder Axis are Constant in Its Interior, contains 48 pages (53-100). One could also consult the work of M. Voigt: The Present State of Our Knowledge of the Elasticity of Crystals (Report presented at the International Congress of Physics held in Paris in 1900, Volume I, pp. 277-347) where the paper by Poisson, *Memoires de l'Acadamie (Memoirs of the Academy)* Volume XVIII, p. 3, 1842 (see page 289). Also see Larmor, *On the Propagation of a Disturbance in a Gyrostatically Loaded Medium* (Proc. Lond. Math. Soc., Nov., 1891); Love, *Treatise on the Mathematical Theory of Elasticity* (Camb. University Press, First Edition, 1892, Second Edition, 1906); Combebiac, The General Equations of Elasticity, Bull. de la Soc. Math. de France, Volume XXX, pp. 108-110 and pp. 242-247, 1902.

54. Virtual External Work; Theorem Analogous to That of Varignon and Saint-Guilhem. Remarks Regarding the Auxiliary Quantities Introduced in the Preceding Sections. For an arbitrary virtual deformation, we will give the name of *external virtual work* for the deformed medium (M) to the expression

$$\delta\mathfrak{T}_e = -\iint_{\sigma_0} (F_0'\delta'x + G_0'\delta'y + H_0'\delta'z + I_0'\delta_I' + J_0'\delta_J' + K_0'\delta_K')d\sigma_0$$
$$+ \iiint_{\tau_0} (X_0'\delta'x + Y_0'\delta'y + Z_0'\delta'z + L_0'\delta_I' + M_0'\delta_J' + N_0'\delta_K')dx_0dy_0dz_0.$$

Let us refer to the notation of Section 50 and let us designate by δ_I, δ_J, δ_K the projections on fixed axes of the segment whose projections on Mx', My', Mz' are $\delta_{I'}$, $\delta_{J'}$, $\delta_{K'}$. Thus, we have, for example,

$$-\delta_I = \alpha''\delta\alpha' + \beta''\delta\beta' + \gamma''\delta\gamma' = -(\alpha'\delta\alpha'' + \beta'\delta\beta'' + \gamma'\delta\gamma''),$$

assuming that the axes under consideration have the same configuration.

With this, let us assume that in Section 53 we attribute the meaning defined by the following equations to the arbitrary quantities λ_1, λ_2, λ_3, μ_1, μ_2, μ_3:

$$\lambda_1 = \delta x,\ \lambda_2 = \delta y,\ \lambda_3 = \delta z,\ \mu_1 = \delta_I,\ \mu_2 = \delta_J,\ \mu_3 = \delta_K.$$

We thus see that the relationships obtained previously between the new auxiliary quantities introduced only express the following condition:

No matter what the virtual displacements of Section 50 imposed on the deformed medium may be, the external virtual work $\delta\mathfrak{T}_e$ is given either by the relationship:

$$\delta\mathfrak{T}_e = -\iiint \left(p_{xx}\frac{\partial\delta x}{\partial x} + p_{yx}\frac{\partial\delta x}{\partial y} + p_{zx}\frac{\partial\delta x}{\partial z} + p_{xy}\frac{\partial\delta y}{\partial x} + p_{yy}\frac{\partial\delta y}{\partial y} + p_{zy}\frac{\partial\delta y}{\partial z}\right.$$
$$\left. + p_{xz}\frac{\partial\delta z}{\partial x} + p_{yz}\frac{\partial\delta z}{\partial y} + p_{zz}\frac{\partial\delta z}{\partial z}\right) dxdydz$$
$$-\iiint \left(q_{xx}\frac{\partial\delta_I}{\partial x} + q_{yx}\frac{\partial\delta_I}{\partial y} + q_{zx}\frac{\partial\delta_I}{\partial z} + q_{xy}\frac{\partial\delta_J}{\partial x} + q_{yy}\frac{\partial\delta_J}{\partial y} + q_{zy}\frac{\partial\delta_J}{\partial z}\right.$$
$$\left. + q_{xz}\frac{\partial\delta_K}{\partial x} + q_{yz}\frac{\partial\delta_K}{\partial y} + q_{zz}\frac{\partial\delta_K}{\partial z}\right) dxdydz$$
$$+ \iiint \left\{(p_{yz} - p_{zy})\delta_I + (p_{zx} - p_{xy})\delta_J + (p_{xy} - p_{yx})\delta_K\right\} dxdydz,$$

where the integrals are extended over the deformed medium, or by the relationship: /139

$$\delta\mathfrak{T}_e = -\iiint \left(A_1\frac{\partial\delta x}{\partial x_0} + A_2\frac{\partial\delta x}{\partial y_0} + A_3\frac{\partial\delta x}{\partial z_0} + B_1\frac{\partial\delta y}{\partial x_0} + B_2\frac{\partial\delta y}{\partial y_0} + B_3\frac{\partial\delta y}{\partial z_0}\right.$$
$$\left. + C_1\frac{\partial\delta z}{\partial x_0} + C_2\frac{\partial\delta z}{\partial y_0} + C_3\frac{\partial\delta z}{\partial z_0}\right) dx_0dy_0dz_0$$
$$-\iiint \left(P_1\frac{\partial\delta_I}{\partial x_0} + P_2\frac{\partial\delta_I}{\partial y_0} + P_3\frac{\partial\delta_I}{\partial z_0} + Q_1\frac{\partial\delta_J}{\partial x_0} + Q_2\frac{\partial\delta_J}{\partial y_0} + Q_3\frac{\partial\delta_J}{\partial z_0}\right.$$
$$\left. + R_1\frac{\partial\delta_K}{\partial x_0} + R_2\frac{\partial\delta_K}{\partial y_0} + R_3\frac{\partial\delta_K}{\partial z_0}\right) dx_0dy_0dz_0$$

$$+\iiint\left(C_1\frac{\partial y}{\partial x_0}+C_2\frac{\partial y}{\partial y_0}+C_3\frac{\partial y}{\partial z_0}-B_1\frac{\partial z}{\partial x_0}-B_2\frac{\partial z}{\partial y_0}-B_3\frac{\partial z}{\partial z_0}\right)\delta\iota\, dx_0dy_0dz_0$$

$$+\iiint\left(A_1\frac{\partial z}{\partial x_0}+A_2\frac{\partial z}{\partial y_0}+A_3\frac{\partial z}{\partial z_0}-C_1\frac{\partial x}{\partial x_0}-C_2\frac{\partial x}{\partial y_0}-C_3\frac{\partial x}{\partial z_0}\right)\delta\iota\, dx_0dy_0dz_0$$

$$+\iiint\left(B_1\frac{\partial x}{\partial x_0}+B_2\frac{\partial x}{\partial y_0}+B_3\frac{\partial x}{\partial z_0}-A_1\frac{\partial y}{\partial x_0}-A_2\frac{\partial y}{\partial y_0}-A_3\frac{\partial y}{\partial z_0}\right)\delta\kappa\, dx_0dy_0dz_0,$$

where the integrals are extended over the non-deformed medium, because the equation

$$\delta\mathfrak{T}_e=-\iint_{s_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'y+I_0'\delta\iota'+J_0'\delta\iota'+K_0'\delta\kappa')\,d\sigma_0$$
$$+\iiint_{s_0}(X_0'\delta'x+Y_0'\delta'y+Z_0'\delta'z+L_0'\delta\iota'+M_0'\delta\iota'+N_0'\delta\kappa')\,dx_0dy_0dz_0$$

given above for the definition of the external virtual work can also be written as:

$$\delta\mathfrak{T}_e=-\iint_{s_0}(F_0\delta x+G_0\delta y+H_0\delta z+I_0\delta\iota+J_0\delta\iota+K_0\delta\kappa)\,d\sigma_0$$
$$+\iiint_{s_0}(X_0\delta x+Y_0\delta y+Z_0\delta z+L_0\delta\iota+M_0\delta\iota+N_0\delta\kappa)\,dx_0dy_0dz_0,$$

due to the meaning of X_0, Y_0, ... N_0, F_0, G_0, ... K_0. Also we have

$$\delta\mathfrak{T}_e=-\iint_{s}(F\delta x+G\delta y+H\delta z+I\delta\iota+J\delta\iota+K\delta\kappa)\,d\sigma_0$$
$$+\iiint_{s}(X\delta x+Y\delta y+Z\delta z+L\delta\iota+M\delta\iota+N\delta\kappa)\,dxdydz,$$

due to the meaning of X, Y, ... N, F, G, ... K.

Let us start from the equation:

$$\iiint_{s_0}\delta W\,dx_0dy_0dz_0+\delta\mathfrak{T}_e=0,$$

applied to an arbitrary part of the medium limited by a surface S_0. /140

Since δW must be identically zero, due to the invariance of W for the group of Euclidian displacements, where the variations δx, δy, δz are given by equation (51), i.e.,

$$\delta x=(a_1+\omega_3z-\omega_2y)\delta t,$$
$$\delta y=(a_2+\omega_1x-\omega_3z)\delta t,$$
$$\delta z=(a_3+\omega_2y-\omega_1x)\delta t,$$

and δI, δJ, δK by

$$\delta\iota=\omega_1\delta t,\qquad \delta\iota=\omega_2\delta t,\qquad \delta\kappa=\omega_3\delta t,$$

which holds no matter what the values of the constants a_1, a_2, a_3, ω_1, ω_2, ω_3 are, we conclude that -- using the expressions for $\delta\mathcal{T}_e$ we are imposing here(6) -- we have:

$$\iint_{S_0} \Gamma_0 dx_0 - \iiint_{S_0} X_0 dx_0 dy_0 dz_0 = 0, \quad \iint_{S_0} (L_0 + H_0 y - G_0 z) dx_0 - \iiint_{S_0} (L_0 + Z_0 y - Y_0 z) dx_0 dy_0 dz_0 = 0,$$

and four analogous equations. These six equations are easily derived from those which are usually written down using the principle of solidification.

In these equations it is conceivable that the boundary S_0 *is variable.*

The auxiliary quantities introduced in the preceding paragraphs are not the only possible ones. Since we are limiting ourselves to a consideration of them, we would like to add some simple remarks.

Specifically we have introduced two systems of stresses and moments of deformation for a point M of the deformed medium. The first are those which are applied to surfaces which, before deformation, have normals parallel to one of the fixed axes Ox, Oy, Oz. The second ones are those which are applied to surfaces whose normals are parallel to one of the same fixed axes Ox, Oy, Oz.

The equations we have indicated give the latter elements in terms of the former. However, by carrying out a straightforward solution which we will not dwell upon, we conversely obtain the first elements in terms of the second ones.

Let us now assume that we have introduced the function W. The first stresses and moments of deformation have the expressions already given, and from them we may immediately derive expressions for the second ones. However, in these calculations, we may explicitely write the functions in terms of which we desire to have the solution. Depending on the problem, these will be, for /141 example, x, y, z or x', y', z' and three parameters(7) λ_1, λ_2, λ_3 in terms of which α, α' ... γ'' are expressed.

Let us introduce x, y, z, λ_1, λ_2, λ_3 and continue to designate by W the function containing x_0, y_0, z_0, the first derivatives of x, y, z with respect to x_0, y_0, z_0, λ_1, λ_2, λ_3 and their first derivatives with respect to x_0, y_0, z_0. It is obtained by replacing the various quantities ξ_i, η_i, ζ_i, p_i, q_i, r_i

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(6) The transition from elements referred to the unit of volume of the non-deformed medium and to units of area of the boundary S_0 to elements referred to units of volume of the deformed medium and units of area of the boundary S is so straightforward that it is sufficient to limit ourselves to the first case, for example, as we are doing.

(7) As these auxiliary quantities λ_1, λ_2, λ_3 we may for example take the components of rotation which make the axes Ox, Oy, Oz become parallel to Mx', My', Mz', respectively.

in the function $W(x_0, y_0, z_0, \xi_1, \eta_1, \zeta_1, p_1, q_1, r_1)$ by their values given by equations (43) and (44). We will have:

$$A_1 = \frac{\partial W}{\partial \frac{\partial x}{\partial x_0}}, \quad A_2 = \frac{\partial W}{\partial \frac{\partial x}{\partial y_0}}, \quad A_3 = \frac{\partial W}{\partial \frac{\partial x}{\partial z_0}},$$

$$B_1 = \frac{\partial W}{\partial \frac{\partial y}{\partial x_0}}, \quad B_2 = \frac{\partial W}{\partial \frac{\partial y}{\partial y_0}}, \quad B_3 = \frac{\partial W}{\partial \frac{\partial y}{\partial z_0}},$$

$$C_1 = \frac{\partial W}{\partial \frac{\partial z}{\partial x_0}}, \quad C_2 = \frac{\partial W}{\partial \frac{\partial z}{\partial y_0}}, \quad C_3 = \frac{\partial W}{\partial \frac{\partial z}{\partial z_0}},$$

$$\Pi_i = \frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial \xi_i}}, \quad X_i = \frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial \xi_i}}, \quad \Sigma_i = \frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial \xi_i}},$$

55. Concept of the Energy of Deformation. Theorem which Leads to that of Clapeyron as a Special Case. Let us consider the two states (M_0) and (M) of the deformable medium limited by the surfaces (S_0) and (S), and let us consider an arbitrary sequence of states beginning with (M_0) and ending with (M). For this purpose, it is sufficient to consider functions $x, y, z, \alpha, \alpha', \ldots \gamma''$ of x_0, y_0, z_0 and of a variable h which, for the value zero of the variable h, are reduced to $x_0, y_0, z_0, \alpha_0, \alpha_0', \ldots, \gamma_0''$, respectively, and for the value h of h they are reduced to the values $x, y, z, \alpha, \alpha', \ldots \gamma''$, respectively, at (M).

By letting the parameter h vary from 0 to h in a continuous manner, we obtain a continuous deformation which allows us to pass from the state (M_0) to the state (M). In this continuous deformation, let us consider the *total work* performed by the external forces and moments applied to the various volume elements of the medium and by the stresses and moments of deformation applied to the various surface elements of the boundary. In order to obtain this total work, it is sufficient to integrate from 0 to h the total differential obtained by beginning with one of the expressions for $\delta\mathcal{T}_e$ of the preceding paragraph and by substituting the partial differentials corrresponding to the increase dh of h for the variations of $x, y, z, \alpha, \alpha', \ldots, \gamma''$. The equation

$$\delta\mathcal{T}_e = -\iiint_{s_0} \delta W dx_0 dy_0 dz_0,$$

specifies that $-\iiint_{s_0} \frac{\partial W}{\partial h} dh dx_0 dy_0 dz_0$ gives the present value of $\delta\mathcal{T}_e$. We /142
obtain the following for the total work

$$-\int_0^h \left(\iiint_{s_0} \frac{\partial W}{\partial h} dx_0 dy_0 dz_0\right) dh = -\iiint_{s_0} (W_M - W_{M_0}) dx_0 dy_0 dz_0.$$

The work under consideration is independent of the intermediate states and only depends on the extreme states (M_0) and (M) under consideration.

This leads to the introduction of the concept of *energy of deformation*,

which must be distinguished from the concept of action considered previously. We will say that -W is the density of the *energy of deformation* referred to the unit of volume of the non-deformed medium.

The problem which we have just posed ourselves and which consists of determining the *total work* performed by the external forces and moments, and by the stresses and moments of deformation applied at the boundary, results in the *theorem of* Clapeyron(8) by considering an infinitesimally small deformation and a special medium. Let us simply introduce the hypothesis, which we will take up again in Section 58 in a more general form, that W is a simple function of ε_1, ε_2, ε_3, γ_1, γ_2, γ_3. We may therefore consider the equations:

$$\Omega_1 = \frac{\partial W}{\partial \varepsilon_1}, \quad \Omega_2 = \frac{\partial W}{\partial \varepsilon_2}, \quad \Omega_3 = \frac{\partial W}{\partial \varepsilon_3}, \quad \Xi_1 = \frac{\partial W}{\partial \gamma_1}, \quad \Xi_2 = \frac{\partial W}{\partial \gamma_2}, \quad \Xi_3 = \frac{\partial W}{\partial \gamma_3},$$

as equations which define a change of variables from ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 to Ω_1, Ω_2, Ω_3, Ξ_1, Ξ_2, Ξ_3. Due to this change in variables, W becomes a function W' of Ω_1, Ω_2, Ω_3 Ξ_1, Ξ_2, Ξ_3.

Assuming this, let us pass to the infinitesimally small deformation, and let us consider the case considered in Section 31, p. 74-76 of the First Memoir on the Theory of Elasticity. W and W' become quadratic forms W_2 of e_1, e_2, e_3, g_1, g_2, g_3 and W_2' of $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$. This latter expression is what one calls the form associated with W_2, up to the factor $\frac{1}{4}$. We obtain the following for the total work discussed above, in the case of the infinitesimally small deformation

$$\iiint W_2 dx_0 dy_0 dz_0.$$

Let us be more specific and assume that we have(9):

$$W_2(e_i, g_i) = -\left(\frac{\lambda}{2} + \mu\right)(e_1 + e_2 + e_3)^2 - \frac{\mu}{2}(g_1^2 + g_2^2 + g_3^2 - 4e_2e_3 - 4e_3e_1 - 4e_1e_2).$$

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(8) Lamé must be credited with making the theorem of Clapeyron known, in his Note of *Comptes Rendus*, V. XXXV, p. 459-464, 1852, and the in his *Lecons sur la Théorie Mathématique de l'Élasticité des Corps Solides (Lessons on the Mathematical Theory of Elasitcity of Solid Bodies)* (1st Edition, 1852, 2nd Edition, 1866). The following note did not appear until February 1, 1858: Clapeyron, *Mémoire sur le Travail des Forces Élastiques, dans un Corps Solide Déformé par l'Action de Forces Extérieures (Memoir on the Work of Elastic Forces in a Solid Deformed Body by the Effect of External Forces)*, *Comptes Rendus*, Volume XLVI, p. 208, 1858. Also see Todhunter and Pearson, *A History of the Theory of Elasticity, etc.*, Sections 1041 and 1067-1070.

(9) E. and F. Cosserat. *Premier Mémoire sur la Théorie de l'Élasticité (First Memoir on the Theory of Elasticity)*, p. 77.

we will have /143

$$W_2'(\mathfrak{N}_i, \mathfrak{T}_i) = -\frac{1}{2}\left\{\frac{\mathfrak{N}_1^2+\mathfrak{N}_2^2+\mathfrak{N}_3^2}{2\mu} - \frac{\lambda}{2\mu}\frac{(\mathfrak{N}_1+\mathfrak{N}_2+\mathfrak{N}_3)^2}{3\lambda+2\mu} + \frac{\mathfrak{T}_1^2+\mathfrak{T}_2^2+\mathfrak{T}_3^2}{\mu}\right\},$$

or

$$W_2'(\mathfrak{N}_i, \mathfrak{T}_i) = -\frac{1}{2}\left\{\frac{1+\frac{\lambda}{\mu}}{3\lambda+2\mu}(\mathfrak{N}_1+\mathfrak{N}_2+\mathfrak{N}_3)^2 - \frac{1}{\mu}(\mathfrak{N}_2\mathfrak{N}_3+\mathfrak{N}_3\mathfrak{N}_1+\mathfrak{N}_1\mathfrak{N}_2-\mathfrak{T}_1^2-\mathfrak{T}_2^2-\mathfrak{T}_3^2)\right\}.$$

It is seen that Lamé's result is again found exactly if it is noted that, in the case of the infintely small deformation, the total work performed by the external forces and by the boundary stresses is obviously reduced to the expression which he indicates.

56. Natural State of the Deformable Medium. In the preceding we started from a so-called natural state of the deformable medium, and we have produced a so-called deformed state. We have pointed out formulas which make it possible to calculate the external force and similar elements as soon as the W function, representing the action of deformation at one point, is associated with the deformable medium.

Let us, as before, discuss for a short period of time this concept of the *natural state*.

This natural state has hitherto been a state which has undergone no deformation. Let us imagine that functions x, y, z, α, α', ... γ'', which specify the deformed state, depend on a parameter and that for a particular value of this parameter we are dealing with the natural state. The latter then seems to us a particular case of the deformed state, and we are led to apply the concepts relative to the deformed state to it.

Without changing the values of the elements defined by the formulas in Section 52, we may replace the W function by the same function increased by any *determinate* function of x_0, y_0, z_0. If we take our inspiration from the idea of *action* which we associate with the passage from the natural (M_0) to the deformed state (M), we may, if we wish, assume that *the function of* x_0, y_0, z_0 defined by expression

$$W(x_0, y_0, z_0, \xi_i^{(0)}, \eta_i^{(0)}, \zeta_i^{(0)}, p_i^{(0)}, q_i^{(0)}, r_i^{(0)})$$

is identically zero. The values obtained for the natural state for the external force and the similar elements will not necessarily be zero. We will say that they define the external force and the similar elements relative to the natural state(10).

(10) We may then speak of force, stress, etc., because we regard the natural state as the limit of a series of states for which we know the force, stress, etc. Force, stress, etc. have not yet been defined by us, except when there is deformation capable of manifesting and measuring them.

In the above, the natural state is presented as the initial state of a series of deformed states, as a state from which we start in order to study the deformation. Consequently, we are led to ask ourselves whether it is not possible to have one of the deformed states play the role of the initial state, so that the elements defined in Section 52 (external force and moment, external deformation stress and moment), calculated for the other deformed states have the same value, if we relate the first ones to the unit of volume of the deformed medium, and the second ones to the unit of area of a deformed surface. This question cannot be answered unless we introduce and state precisely the concept of the action corresponding to the transition from the deformed state to another deformed state. /144

The simplest hypothesis consists of assuming that this last action is obtained by subtracting the action, corresponding to the transition from the natural state (M_0) to the second deformed state (M), from the action corresponding to the transition from the natural state (M_0) to the first deformed state (M'). If we use ξ_i', η_i', ζ_i', p_i', q_i', r_i', with respect to (M') to designate the quantities analogous(11) to ξ_i, η_i, ζ_i, p_i, q_i, r_i with respect to (M), we are then led to adopt the following expression, for the action of deformation relative to the transition from state (M') to state (M).

$$\iiint_{S_0} \left\{ W(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i) - W(x_0, y_0, z_0, \xi_i', \eta_i', \zeta_i', p_i', q_i', r_i') \right\} dx_0 dy_0 dz_0, \tag{52}$$

which we can write, if Δ' is the value of Δ for (M'):

$$\iiint_{S_0} W_0'(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i) \, |\Delta'| \, dx_0 dy_0 dz_0, \tag{53}$$

designating by S' the surface of (M') that corresponds to S_0 of (M_0), and by $W_0'(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i)$ the expression

$$\left\{ W(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i) - W(x_0, y_0, z_0, \xi_i', \eta_i', \zeta_i', p_i', q_i', r_i') \right\} \frac{1}{|\Delta'|}.$$

If desired, we can substitute the following in expression (53), according to the remarks given at the beginning of this section:

$$\iiint_{S_0} W'(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i) \, |\Delta'| \, dx_0 dy_0 dz_0 \tag{53'}$$

where $W'(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i)$ indicates the expression

$$W(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i) \frac{1}{|\Delta'|}.$$

We may note that we have, for example,

$$|\Delta'| \frac{\partial W'(x_0, y_0, z_0, \xi_i, \dots r_i)}{\partial \xi_i} = \frac{\partial W(x_0, y_0, z_0, \xi_i, \dots r_i)}{\partial \xi_i},$$

(11) We have to remark that ξ_i', η_i', ζ_i', p_i', q_i', r_i' are not analogous to $\xi_i^{(0)}$, $\eta_i^{(0)}$, $\zeta_i^{(0)}$, $p_i^{(0)}$, $q_i^{(0)}$, $r_i^{(0)}$, since they are not formed by the coordinates x', y', z' of (M') in the same way that $\xi_i^{(0)}$, $\eta_i^{(0)}$, $\zeta_i^{(0)}$, $p_i^{(0)}$, $q_i^{(0)}$, $r_i^{(0)}$ are in terms of x_0, y_0, z_0.

Let us apply the formulas analogous to those of Section 52 to expression (53) or to expression (53'), starting from (M') as the natural state, *but supposing that* (M') *is related to the system of coordinates* x_0, y_0, z_0 *and assuming that the formulas of* § 52 *are consequently modified.* It is clear that this gives the same values for the external force and moment, for the state (M), referred to the unit of volume of (M), as well as the same values for the stress and moment of deformation referred to the unit of area of (S). /145

Thus we can consider (M) as a deformed state, the natural state of which would be (M'), provided that the function W associated with the state (M) is actually(12) W_0' or W'.

By way of example, let us assume that, in accordance with these indications, the external force and moment are given in terms of simple functions of x_0, y_0, z_0 and of the elements that determine the position of the triad Mx'y'z'. Let us also assume that the natural state is given. We can consider the equations of Section 52 relative to the external force and moment as partial differential equations with respect to the unknowns x, y, z and depending on three parameters λ_1, λ_2, λ_3 in terms of which we express α, α', ... γ''. The expressions ξ_i, η_i, ζ_i, p_i, q_i, r_i are then functions of $\frac{\partial x}{\partial \rho_i}$, $\frac{\partial y}{\partial \rho_i}$, $\frac{\partial z}{\partial \rho_i}$, λ_1, λ_2, λ_3, $\frac{\partial \lambda_1}{\partial \rho_i}$, $\frac{\partial \lambda_2}{\partial \rho_i}$, $\frac{\partial \lambda_3}{\partial \rho_i}$ [always letting $\rho_1 = x_0$, $\rho_2 = y_0$, $\rho_3 = z_0$], that we will calculate by means of the formulas (43) and (44).

Let us assume that X_0', Y_0', Z_0', L_0', M_0', N_0', or, which is the same, X_0, Y_0, Z_0, L_0, M_0, N_0, are given functions of x_0, y_0, z_0, x, y, z, λ_1, λ_2, λ_3. After substituting the values of ξ_i, η_i, ζ_i, p_i, q_i, r_i derived from the formulas (43) and (44), expression W is then a determined function of x_0, y_0, z_0, $\frac{\partial x}{\partial x_0}$, ..., $\frac{\partial z}{\partial z_0}$, λ_1, λ_2, λ_3, $\frac{\partial \lambda_1}{\partial x_0}$, ..., $\frac{\partial \lambda_3}{\partial z_0}$, which we will continue to designate by W. The equations of the problem can then be written:

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(12) This allows us, as we said at the beginning of the paragraph, to generalize the notion of the natural state which we first introduced. Instead of having this term correspond simply to a particular state, we can, in a more general way, make it correspond to the idea of any state from which /146 we start studying the deformation. The fact that x_0, y_0, z_0, was introduced at the beginning of the theory seems to make (M_0) play a particular role. However, we must not consider x_0, y_0, z_0 as anything but coordinates serving to define the *different media* and not (M_0); these coordinates were particularly chosen, and in connection with a particular medium (M_0), in order that attention may then be drawn to (M_0) in regard to the infinitesimally small deformation.

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial x}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial x}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial x}{\partial z_0}}=X_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial y}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial y}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial y}{\partial z_0}}=Y_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}}=Z_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial z_0}}-\frac{\partial W}{\partial \lambda_1}=\mathcal{L}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial z_0}}-\frac{\partial W}{\partial \lambda_2}=\mathcal{M}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial x_0}}+\frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial y_0}}+\frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial z_0}}-\frac{\partial W}{\partial \lambda_3}=\mathcal{N}_0,$$

where $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are the functions of x_0, y_0, z_0, x, y, z, λ_1, λ_2, λ_3 that result from the definitions of Section 53. /146

This results directly from the formulas in the preceding sections, or more immediately from the fact that the definition formulas of X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ may be summarized in the relationship

$$\delta\iiint W dx_0 dy_0 dz_0 + \delta\mathfrak{T}_e = 0,$$

i.e., in

$$\delta\iiint W dx_0 dy_0 dz_0 = \iint (F_0\delta x + G_0\delta y + H_0\delta z + \mathfrak{I}_0\delta\lambda_1 + \mathfrak{J}_0\delta\lambda_2 + \mathfrak{K}_0\delta\lambda_3)d\sigma$$
$$-\iiint (X_0\delta x + Y_0\delta y + Z_0\delta z + \mathfrak{L}_0\delta\lambda_1 + \mathfrak{M}_0\delta\lambda_2 + \mathfrak{N}_0\delta\lambda_3)dx_0 dy_0 dz_0.$$

57. Concepts of the Concealed Triad and the Concealed W. In studies of the deformable medium it is natural, as in those of the deformable line and surface, to direct attention particularly toward the *point medium* outlined by the deformable medium. This amounts to regarding x, y, z separately and to considering α, α', ..., γ" as simple auxiliary quantities. We may likewise express this by imagining that the existence of the triads that determine the deformable medium is ignored, and that only the vertices of these triads are known. From this viewpoint, we can introduce the concept of the *concealed triad* to consider the partial differential equations to which we may be led depending on the case, and we are led to a classification resulting from the different situations which may be produced when α, α', ..., γ" are eliminated.

One of the first studies to appear is therefore that of the reductions which may take place by eliminating α, α', ..., γ". In the particular corresponding cases where attention is drawn almost exclusively to the point medium outlined by the deformed medium (M), one may sometimes even disregard (M_0), and

consequently the deformation which permits passage from (M_0) to (M).

The triad may be used in another way, as we have already stated for the deformable line and surface. We may make special assumptions about it and likewise about medium (M); all of this amounts to considering particular deformed states of the free deformable medium. If the relationships which we impose are simple ones between ξ_i, η_i, ζ_i, p_i, q_i, r_i, as will be the case in /147 the applications which we will have to study, we will be able to take these relationships into account in calculating W and to derive more special functions from W. The interesting question which will be posed is whether to introduce simply these particular forms and to assume in some manner that the general W function, which has been the starting point, is concealed. We will thus have *a theory which is specific to the particular deformed states evidenced by the given relations between* ξ_i, η_i, ζ_i, p_i, q_i, r_i.

We will see that by means of the theory of the free deformable medium the equations which are the result of the special theories which are encountered in physics(13) can thus be gathered together under the heading of particular cases, stemming from the same origin.

In the latter theories, we are sometimes placed in circumstances which are apt to keep deformations from being considered, but in reality these need to be considered. This can be done in practical applications by considering the infinitesimally small deformation.

Let us take the case where external force and moment comprise at the most only the first derivatives of the unknowns x, y, z and λ_1, λ_2, λ_3. The second derivatives of these unknowns will be introduced into the partial differential equations only by W; now the derivatives of x, y, z appear only in ξ_i, η_i, ζ_i, and those of λ_1, λ_2, λ_3 show up only in p_i, q_i, r_i. We therefore see that, if W depends only on ξ_i, η_i, ζ_i or only on p_i, q_i, r_i, there will be a reduction in the order of the derivatives entering into the system of partial differential equations. We will examine the first of these cases, which corresponds to the ordinary theory of elasticity of material media and to the theory of different ether media considered in the doctrine of light waves.

58. Case where W Depends only on x_0, y_0, z_0, ξ_i, η_i, ζ_i and is Independent of p_i, q_i, r_i. How to Find the Equations Relative to the Deformable Body in the Classic Theory and to the Hydrostatic Medium. Let us assume that W depends only on the quantities x_0, y_0, z_0, ξ_i, η_i, ζ_i and not on the quantities p_i, q_i,

(13) All our considerations apply up to this point equally well to material media as to different ether media. We have nowhere used the word *matter*, and what we are setting forth is, as we said at the beginning, a *theory of action on space and motion*. To gain a more complete idea of the concept of matter, we will later on explain how matter is to be connected with the concept of *entropy* in accord with the profound insight which Lippmann has introduced into electricity.

r_1; the equations of Section 56 which reduce to

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial x}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial x}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial x}{\partial z_0}} = X_0, \qquad \frac{\partial W}{\partial \lambda_1} + \mathfrak{L}_0 = 0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial y}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial y}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial y}{\partial z_0}} = Y_0, \qquad \frac{\partial W}{\partial \lambda_2} + \mathfrak{M}_0 = 0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}} = Z_0, \qquad \frac{\partial W}{\partial \lambda_3} + \mathfrak{N}_0 = 0,$$

in which W depends only on x_0, y_0, z_0, $\frac{\partial x}{\partial x_0}$, ..., $\frac{\partial z}{\partial z_0}$, λ_1, λ_2, λ_3 show us /148
that if we take the simple case where X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$, are given functions(14) of x_0, y_0, z_0, x, y, z, $\frac{\partial x}{\partial x_0}$, ..., $\frac{\partial z}{\partial z_0}$, λ_1, λ_2, λ_3 the three equations on the right may be solved for λ_1, λ_2, λ_3. Finally, we obtain three partial differential equations which under our assumptions contain only x_0, y_0, z_0 and x, y, z and their first and second derivatives.

Let us first consider the particular case in which the given functions $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are zero. The corresponding values of the functions of any one of the systems (L_0', M_0', N_0'), (L_0, M_0, N_0) (L, M, N) will be zero. It results from this that the equations

$$\frac{\partial W}{\partial \lambda_1} = 0, \qquad \frac{\partial W}{\partial \lambda_2} = 0, \qquad \frac{\partial W}{\partial \lambda_3} = 0$$

become

$$C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} - B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial y_0} - B_3\frac{\partial z}{\partial z_0} = 0,$$

$$A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} - C_1\frac{\partial x}{\partial x_0} - C_2\frac{\partial x}{\partial y_0} - C_3\frac{\partial x}{\partial z_0} = 0,$$

$$B_1\frac{\partial x}{\partial x_0} + B_2\frac{\partial x}{\partial y_0} + B_3\frac{\partial x}{\partial z_0} - A_1\frac{\partial y}{\partial x_0} - A_2\frac{\partial y}{\partial y_0} - A_3\frac{\partial y}{\partial z_0} = 0,$$

i.e., become

$$p_{yz} = p_{zy}, \qquad p_{zx} = p_{xz}, \qquad p_{xy} = p_{yx},$$

which can be directly interpreted.

Under this assumption, let us remark that if we start from the two positions (M_0) and (M) which are assumed as *given* and if from these -- as in Section 53 -- we deduce the functions $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$, it may happen that in the case where

(14) We assume that X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ contain no derivatives of λ_1, λ_2, λ_3 in order to simplify the presentation and to indicate more easily that to which we wish to refer.

these three functions are zero this appears as an accidental result, i.e., for a certain set of particular deformed states. However, it may happen that this result occurs no matter what the deformed (M) state and that this is a consequence of the nature of the (M) media, i.e., of the form of W.

Let us consider the latter case, which is of particular interest. Here W is a simple function(15) of ρ_1, ρ_2, ρ_3 and of the six expressions ε_1, ε_2, ε_3, γ_1, γ_2, γ_3 defined by formulas (45).

The equations deduced from Sections 52 and 53 reduce either to /149

$$\sum_i \left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i'\right) = X_0' \qquad F_0' = l_0 A_1' + m_0 A_2' + n_0 A_3',$$
$$\sum_i \left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i'\right) = Y_0' \qquad G_0' = l_0 B_1' + m_0 B_2' + n_0 B_3',$$
$$\sum_i \left(\frac{\partial C_i'}{\partial \rho_i} + p_i B_i' - q_i A_i'\right) = Z_0' \qquad H_0' = l_0 C_1' + m_0 C_2' + n_0 C_3',$$

where we have

$$\left.\begin{aligned} A_i' &= \xi_i \frac{\partial W}{\partial \varepsilon_i} + \xi_k \frac{\partial W}{\partial \gamma_j} + \xi_j \frac{\partial W}{\partial \gamma_k} \\ B_i' &= \eta_i \frac{\partial W}{\partial \varepsilon_i} + \eta_k \frac{\partial W}{\partial \gamma_j} + \eta_j \frac{\partial W}{\partial \gamma_k} \\ C_i' &= \zeta_i \frac{\partial W}{\partial \varepsilon_i} + \zeta_k \frac{\partial W}{\partial \gamma_j} + \zeta_j \frac{\partial W}{\partial \gamma_k} \end{aligned}\right\} (i, j, k = 1, 2, 3).$$

or else to(16)

$$\frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0} = X_0, \qquad F_0 = l_0 A_1 + m_0 A_2 + n_0 A_3,$$
$$\frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0} = Y_0, \qquad G_0 = l_0 B_1 + m_0 B_2 + n_0 B_3,$$
$$\frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0} = Z_0, \qquad H_0 = l_0 C_1 + m_0 C_2 + n_0 C_3,$$

where we have

$$A_1 = \Omega_1 \frac{\partial x}{\partial x_0} + \Xi_3 \frac{\partial x}{\partial y_0} + \Xi_2 \frac{\partial x}{\partial z_0},$$
$$A_2 = \Xi_3 \frac{\partial x}{\partial x_0} + \Omega_2 \frac{\partial x}{\partial y_0} + \Xi_1 \frac{\partial x}{\partial z_0},$$
$$A_3 = \Xi_2 \frac{\partial x}{\partial x_0} + \Xi_1 \frac{\partial x}{\partial y_0} + \Omega_3 \frac{\partial x}{\partial z_0},$$
$$B_1 = \Omega_1 \frac{\partial y}{\partial x_0} + \Xi_3 \frac{\partial y}{\partial y_0} + \Xi_2 \frac{\partial y}{\partial z_0},$$
$$B_2 = \Xi_3 \frac{\partial y}{\partial x_0} + \Omega_2 \frac{\partial y}{\partial y_0} + \Xi_1 \frac{\partial y}{\partial z_0},$$

(15) The triad is completely concealed; we may also imagine that we have a simple point medium.

(16) Compare E. and F. Cosserat, *Premier Mémoire sur la Théorie de l'Élasticité (First Note on Elasticity Theory)*, p. 45, 46, 65.

$$B_1 = \Xi_3 \frac{\partial y}{\partial x_0} + \Xi_1 \frac{\partial y}{\partial y_0} + \Omega_2 \frac{\partial y}{\partial z_0},$$
$$C_1 = \Omega_1 \frac{\partial z}{\partial x_0} + \Xi_2 \frac{\partial z}{\partial y_0} + \Xi_3 \frac{\partial z}{\partial z_0},$$
$$C_2 = \Xi_2 \frac{\partial z}{\partial x_0} + \Omega_2 \frac{\partial z}{\partial y_0} + \Xi_1 \frac{\partial z}{\partial z_0},$$
$$C_3 = \Xi_3 \frac{\partial z}{\partial x_0} + \Xi_1 \frac{\partial z}{\partial y_0} + \Omega_3 \frac{\partial z}{\partial z_0},$$

by setting $\Omega_i = \frac{\partial W}{\partial \tau_i}$, $\Xi_i = \frac{\partial W}{\partial \gamma_i}$ for purposes of abbreviation or else it reduces to(17) /150

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} = X, \qquad F = lp_{xx} + mp_{yx} + np_{zx},$$
$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} = Y, \qquad G = lp_{xy} + mp_{yy} + np_{zy},$$
$$\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} = Z, \qquad H = lp_{xz} + mp_{yz} + np_{zz},$$

where we have

$$p_{xx} = \frac{1}{\Delta}\left[\Omega_1\left(\frac{\partial x}{\partial x_0}\right)^2 + \Omega_2\left(\frac{\partial x}{\partial y_0}\right)^2 + \Omega_3\left(\frac{\partial x}{\partial z_0}\right)^2 + 2\Xi_1\frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0} + 2\Xi_2\frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0} + 2\Xi_3\frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0}\right],$$

and similar formulas for p_{yx}, ... (Δ has the meaning attributed to it in Section 54 which we are going to recall in a moment).

It can be seen that we will again find the deformable continuous medium which is treated in the ordinary theory of elasticity.

A particular case of interest is obtained when searching for the form of W in which we have the identities

$$p_{yz} = 0, \qquad p_{yx} = 0, \qquad p_{zy} = 0,$$

whatever $\frac{\partial x}{\partial x_0}$, ... may be. We find that W must be a simple function of x_0, y_0, z_0 and of expression Δ defined by the formulas(18)

$$\Delta = \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}, \qquad \Delta^2 = \begin{vmatrix} 1 + 2\varepsilon_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & 1 + 2\varepsilon_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & 1 + 2\varepsilon_3 \end{vmatrix},$$

which can be seen, for example, by noting that if we go back to the formulas given before(19) which provide p_{xx}, p_{yx}, p_{zx}, ... as a function of A_1, ... we

(17) Compare E. and F. Cosserat, *Premier Mémoire sur la Théorie de l'Élasticité (First Note on Elasticity Theory)*, p. 40, 44, 65.

(18) Compare E. and F. Cosserat. *Premier Mémoire sur la Théorie de l'Élasticité (First Note on Elasticity Theory)*, p. 23, 24.

(19) These formulas are in fact those on page 47 of our *Premier Mémoire sur la Théorie de l'Élasticité (First Note on Elasticity Theory)*.

have

$$\frac{\dfrac{\partial W}{\partial \dfrac{\partial x}{\partial x_0}}}{\dfrac{\partial \Delta}{\partial \dfrac{\partial x}{\partial x_0}}} = \frac{\dfrac{\partial W}{\partial \dfrac{\partial x}{\partial y_0}}}{\dfrac{\partial \Delta}{\partial \dfrac{\partial x}{\partial y_0}}} = \frac{\dfrac{\partial W}{\partial \dfrac{\partial x}{\partial z_0}}}{\dfrac{\partial \Delta}{\partial \dfrac{\partial x}{\partial z_0}}},$$

and two similar systems. Since W is assumed to be a simple function of x_0, y_0, /151
z_0 and of Δ, we then have

$$p_{xx} = p_{yy} = p_{zz} = \frac{\partial W}{\partial \Delta}.$$

If we examine the particular case in which W depends only on Δ and if we grant that X, Y, Z are given as a function of x, y, z, the equations in question, which are

$$\frac{\partial p}{\partial x} = X, \frac{\partial p}{\partial y} = Y, \frac{\partial p}{\partial z} = Z, F = lp, G = mp, H = np,$$

become those which serve as the basis for hydrostatics(20) when $p = \frac{dW}{d\Delta}$. The initial medium (M_0) appears only through Δ, and the unknown Δ may be replaced by the unknown p which is linked to it by the relationship $p = \frac{dW}{d\Delta}$. If the W function is not given, i.e., if it is *concealed*, we have the preceding equations where p is an auxiliary quantity whose meaning is well known.

It is sufficient for us to indicate that the case where the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are not zero encompasses the theory of all the ether media which from MacCullagh to Lord Kelvin have been considered for the study of light waves, but here the theory of these media is entirely mechanical. We would also like to mention that the more general case, which in the expressions of the external moment contains a trace of the derivatives of the action W with respect to rotations p_i, q_i, r_i, leads in the most natural fashion to the concept of *magnetic induction* introduced by Maxwell.

59. The Invariable Body. We have considered the particular case where W does not depend on p_i, q_i, r_i and different particularizations of this case. By studying either particular deformations or new media defined by a bond theory which benefits from results already acquired, we can arrive at other media which, at least in part, are considered by the authors.

Let us, for example, start from the simple case in which the triad is *concealed*, i.e., definitively from a *point* medium in which W is a function of x_0, y_0, z_0, ε_1, ε_2, ε_3, γ_1, γ_2, γ_3.

1. We may imagine that attention is directed solely to deformations of

(20) Compare Duhem. *Hydrodynamique, Elasticité, Acoustique (Hydrodynamics, Elasticity, Acoustics).*

the medium for which we have

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0.$$

In the definitions of forces, etc. it is sufficient to introduce these assumptions and, if the forces are given, to introduce these six conditions. In the latter case, the *usual* problems in which the W function is given and the general case where ε_i, γ_i are not zero may be posited only if the data are particular.

If we assume that *only* the W_0 function, obtained by setting $\varepsilon_2 = \varepsilon_2 = \varepsilon_3 =$ /152
$= \gamma_1 = \gamma_2 = \gamma_3 = 0$ in $W(\rho_1, \rho_2, \varepsilon_1 \ldots)$, is given, that we do not know the values of the derivatives of W with respect to ε_1, ε_2, ... γ_3 for $\varepsilon_1 = \varepsilon_2 = \ldots = \gamma_3 = 0$, and that W is thus *concealed*, we see that p_{xx}, ... p_{zz}, for example, become six auxiliary quantities which must be added to x, y, z, so that in the case where the forces acting on the volume elements are given we have nine partial differential equations with nine unknowns, to which we must add secondary conditions.

Let us now note that the system

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0;$$

can be integrated. Since the deformation is assumed to be continuous, the integration corresponds to a total displacement of the medium; it therefore remains to determine the six integration constants and the auxiliary quantities p_{xx}, ...

If the forces and stresses acting on the medium are given, the six equations of Section 54, with the simplifications contributed by the form of W and applied to the whole body, will -- assuming that X, ... are known as functions of x, y, z -- determine the six integration constants. Therefore, to finish we will have only to determine p_{xx}, ... later. It may be seen that, omitting this subsequent determination, we will again find the usual problems of the mechanics of invariable bodies, a mechanics in which it is perhaps more usually assumed that the concealed W function depends only on Δ.

2. We may imagine that the attempt is being made to define a medium *sui generis*, the definition of which already takes into account the conditions

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0.$$

To define the new medium, retaining the same sequence of ideas as in the preceding, we will further define F_0', ... N_0' by the identity

$$\iiint_{s_0} \delta W dx_0 dy_0 dz_0 = \iint_{s_0} (F_0'\delta' x + \ldots + K_0'\delta\varepsilon') \, dz_0$$

$$- \iiint_{s_0} (X_0'\delta' x + \ldots + N_0'\delta\varepsilon') \, dx_0 dy_0 dz_0;$$

This can no longer hold except if $\varepsilon_1 = \ldots = \gamma_3 = 0$. In other words, we are

faced with a medium whose theory is going to result in the *a posteriori* addition of the conditions $\varepsilon_1 = \ldots = \gamma_3 = 0$ to the knowledge of a function $W(x_0, y_0, z_0, \varepsilon_1, \varepsilon_2, \ldots \gamma_3)$ and of the six auxiliary functions $\mu_1, \ldots \mu_6$ of x_0, y_0, z_0 by means of the identity

$$\iiint_{s_0} (\delta W + \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \ldots + \mu_6\gamma_3)\,dx_0dy_0dz_0 = \iint_{s_0} (F_0'\delta' x + \ldots)\,ds_0$$
$$- \iiint_{s_0} (X_0'\delta' x + \ldots)\,dx_0dy_0dz_0$$

This amounts to setting $\varepsilon_1 = \ldots = \gamma_3 = 0$ in the general theory which precedes, /153 where we began to replace W by $W_1 = W + \mu_1\varepsilon_1 + \ldots + \mu_6\gamma_3$.

It is seen that we fall back on *the theory of the elastic medium corresponding to the* W_1 *function of* $x_0, y_0, z_0, \varepsilon_1, \ldots \gamma_3$ *whose deformations corresponding to* $\varepsilon_1 = \ldots, = \gamma_3 = 0$ *are the ones to which we restrict our study.* If, therefore, we are concerned with the case where W_1 is *concealed*, and if we assume that only the value of $W_0(x_0, y_0, z_0)$ -- taken simultaneously by W and W_1 for $\varepsilon_1 = \ldots = \gamma_3 = 0$ -- is known, we will again find the theory which encompases the usual theory of the invariable body.

Let us note that if, in order to formulate the invariable body, we account *a priori* in W for the conditions $\varepsilon_1 = \ldots = \gamma_3 = 0$ by changing the auxiliary quantities, we are led to replace W by $\mu_1\varepsilon_1 + \ldots + \mu_6\gamma_3$ in the calculations for the general medium. We likewise find formulas which also lead back to the study of an elastic medium where we restrict ourselves to deformations which correspond to $\varepsilon_1 = \ldots = \gamma_3 = 0$. These formulas -- assuming that $\mu_1, \ldots, \mu_6$ are *not known* -- take us back again to the theory which includes the usual theory of the invariable body. From this latter point of view, we come back to presenting what can be done according to the ideas of Lagrange. We may observe, in particular, that in the cases where X_0, Y_0, Z_0 are given as partial derivatives with respect to x, y, z of a function ϕ of x_0, y_0, z_0, x, y, z, the equations which contain X_0, Y_0, Z_0 are none other than the equations to which one is led when trying to determine an extremum of the integral

$$\iiint \varphi\,dx_0dy_0dz_0,$$

under the conditions

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0,$$

3. Let us present a third procedure(21) to formulate a medium whose theory

(21) Compare Thomson and Tait. *Treatise*, Volume I, Part I, p. 271, beginning with the 11th line from the bottom.

always leads to the same equations and which will be a limiting case of the original medium. This procedure is to be compared with the first, and also applies to different cases of deformable line and surface.

Let us imagine that the W which serves to define the original medium is variable, and more specifically, let us assume that, for the values of $\varepsilon_1, \ldots \gamma_3$ near zero, it may be expanded into a Maclaurin series by the formula

$$W = W_1 + W_2 + \ldots + W_i + \ldots,$$

where W_i represents the aggregate of the $i\underline{th}$ degree terms. Let us assume that the coefficients of W_2 (which may depend on x_0, y_0, z_0) increase indefinitely in their variation. *If we want* W *to maintain a finite value* we must assume that $\varepsilon_1, \ldots, \gamma_3$ tend toward zero -- in other words, we may then consider only the deformations which satisfy $\varepsilon_1 = \ldots = \gamma_3 = 0$. Stated otherwise, the body which /154 we obtain in the limit is susceptible only to the displacements of the ensemble. We may assume that derivatives $\frac{\partial W}{\partial \varepsilon_1}$, ... tend toward limits, while W, varying as we have just stated, is also made to vary likewise because of a studied deformation of this medium.

To explain the preceding more exactly, let us imagine that the coefficients of W_1, W_2, ... depend on a parameter h, so that when h tends toward zero the coefficients of W_2 increase indefinitely. More specifically, let us assume that these latter coefficients are linear with respect to $\frac{1}{h}$. Let us likewise imagine that x, y, z defining a studied deformation vary with h and in such a way that ε_1, ... tend toward zero. We will also assume that ε_1, ... are infinitely small of the first order with respect to h, for example, that ε_1, ... can be expanded in powers of h, the first terms of the expansion being those of h. Under these conditions, when W tends toward zero $\frac{\partial W}{\partial \varepsilon_1}$, ..., $\frac{\partial W}{\partial \gamma_3}$ tend toward certain limits (which may be functions of x_0, y_0, z_0). If, therefore, we consider the equations in Section 52 which serve to define the external force and moment, they finally lead us to formulas which permit us to define them and which are none other than the equations with which we started, *in which the concept of the* W *function has disappeared* and in which the six auxiliary quantities F_0', G_0', H_0', I_0', J_0', K_0' appear.

60. Deformable Medium in Motion. The theory of motion of the deformable line and that of motion of the deformable surface have naturally occurred as special cases of the theory of the deformable surface and that of the deformable medium. For this, it is sufficient to give one of the surface or medium parameters ρ_i the meaning of time. Since we are not considering the statics of a medium of more than three dimensions, we will discuss the theory of motion of a deformable medium directly below, but nevertheless we will not give it a form

entirely analogous to that which we have indicated for the dynamics of the deformable line and the deformable surface.

Let us again consider a space (M_0) described by a point M_0 whose coordinates are x_0, y_0, z_0 with respect to three fixed orthogonal axes Ox, Oy, Oz, and let us connect to each point M_0 of the space (M_0) an orthogonal triad whose axes M_0x_0*, $M_0y'_0$, $M_0'z_0$ have direction cosines which are respectively α_0, α_0', α_0''; β_0, β_0', β_0''; γ_0, γ_0', γ_0'' with respect to the axes Ox, Oy, Oz and which are functions of the independent variables x_0, y_0, z_0.

The continuous three-dimensional set $M_0x_0'y_0'z_0'$ of such triads may be considered as the position of a deformable medium defined at a definite moment t_0 as follows.

Let us give to point M_0 displacement M_0M which is a function of t and of the position of the point M_0, and becomes zero when $t = t_0$; let x, y, z be the coordinates of point M, which we will consider as functions of x_0, y_0, z_0, t. Let us moreover impart a rotation to the triad $M_0x_0'y_0'z'_0$ which will finally /155 cause its axes to be those of a triad Mx'y'z' which we will attach to point M. We will define this rotation by prescribing the direction cosines α, α', α''; β, β', β''; γ, γ', γ'' of the axes Mx', My', Mz' with respect to the fixed axes Ox, Oy, Oz. All these cosines will, like x, y, z, be functions of x_0, y_0, z_0, t.

The continuous three-dimensional set of triads Mx'y'z' for a given value of time t will be what we shall call the *deformed state* of the deformable medium considered at this instant t. The continuous four-dimensional set of triads Mx'y'z' obtained by causing t to vary will be the *trajectory of the deformed state* of the deformable medium.

For convenience in writing and notation we will, as we have already done, sometimes introduce the terms ρ_1, ρ_2, ρ_3 instead of x_0, y_0, z_0 in what follows. We will continue to designate by $\xi_i^{(0)}$, $\eta_i^{(0)}$, $\zeta_i^{(0)}$ the velocity components of origin M_0 of axes M_0x_0', $M_0y'_0$, M_0z_0' along these axes *when* ρ_i *alone varies* and by $p_i^{(0)}$, $q_i^{(0)}$, $r_i^{(0)}$ the projections on these same axes of the instantaneous rotation of triad $M_0x_0'y_0'z_0'$ with respect to the parameter ρ_i. We will designate by ξ_i, η_i, ζ_i and by p_i, q_i, r_i the analogous quantities for the triad Mx'y'z' when it, like triad $M_0x_0'y_0'z_0'$, is referred to the fixed triad Oxyz.

When time t varies and the motion of triad Mx'y'z' is referred to the fixed triad Oxyz, the origin M has a velocity whose components along the Mx', My', Mz' axes will be designated by ξ, η, ζ,** and the instantaneous rotation of triad Mx'y'z' will be defined by the components p, q, r.

* Note: The original is probably incorrect and should read $M_0x'_0$.
** Note: The original is probably incorrect and should read ξ, η, ζ.

The elements which we have just introduced are calculated just as in Section 49. First, we have the formulas

$$(54)\quad \begin{cases} \xi_i = \alpha \dfrac{\partial x}{\partial \rho_i} + \alpha' \dfrac{\partial y}{\partial \rho_i} + \alpha'' \dfrac{\partial z}{\partial \rho_i}, \\ \eta_i = \beta \dfrac{\partial x}{\partial \rho_i} + \beta' \dfrac{\partial y}{\partial \rho_i} + \beta'' \dfrac{\partial z}{\partial \rho_i}, \\ \zeta_i = \gamma \dfrac{\partial x}{\partial \rho_i} + \gamma' \dfrac{\partial y}{\partial \rho_i} + \gamma'' \dfrac{\partial z}{\partial \rho_i}, \end{cases} \qquad \begin{cases} p_i = \sum \gamma \dfrac{\partial \beta}{\partial \rho_i} = -\sum \beta \dfrac{\partial \gamma}{\partial \rho_i}, \\ q_i = \sum \alpha \dfrac{\partial \gamma}{\partial \rho_i} = -\sum \gamma \dfrac{\partial \alpha}{\partial \rho_i}, \\ r_i = \sum \beta \dfrac{\partial \alpha}{\partial \rho_i} = -\sum \alpha \dfrac{\partial \beta}{\partial \rho_i}. \end{cases} \quad (55)$$

to which are added the following

$$(54')\quad \begin{cases} \xi = \alpha \dfrac{dx}{dt} + \alpha' \dfrac{dy}{dt} + \alpha'' \dfrac{dz}{dt}, \\ \eta = \beta \dfrac{dx}{dt} + \beta' \dfrac{dy}{dt} + \beta'' \dfrac{dz}{dt}, \\ \zeta = \gamma \dfrac{dx}{dt} + \gamma' \dfrac{dy}{dt} + \gamma'' \dfrac{dz}{dt}, \end{cases} \qquad \begin{cases} p = \sum \gamma \dfrac{d\beta}{dt} = -\sum \beta \dfrac{d\gamma}{dt}, \\ q = \sum \alpha \dfrac{d\gamma}{dt} = -\sum \gamma \dfrac{d\alpha}{dt}, \\ r = \sum \beta \dfrac{d\alpha}{dt} = -\sum \alpha \dfrac{d\beta}{dt}, \end{cases} \quad (55')$$

by introducing from this point on a distinction in the notation for the derivatives with respect to time, depending on whether x_0, y_0, z_0, t or x, y, z, t are taken as independent variables.

Let us assume that each of the triads of the trajectory of the deformed state is subjected to an infinitely small displacement which may vary continously with these triads. With the same notation as in Section 50, we will have /156

$$\delta x = \beta \delta \kappa' - \gamma \delta \lambda', \quad (56)$$

$$\delta' x = \delta x' + z' \delta \lambda' - y' \delta \kappa', \quad \delta' y = \delta y' + x' \delta \kappa' - z' \delta \iota', \quad \delta' z = \delta z' + y' \delta \iota' - x' \delta \lambda', \quad (57)$$

$$(58)\quad \begin{cases} \delta \xi_i = \eta_i \delta \kappa' - \zeta_i \delta \lambda' + \dfrac{\partial \delta' x}{\partial \rho_i} + q_i \delta' z - r_i \delta' y, \\ \delta \eta_i = \zeta_i \delta \iota' - \xi_i \delta \kappa' + \dfrac{\partial \delta' y}{\partial \rho_i} + r_i \delta' x - p_i \delta' z, \\ \delta \zeta_i = \xi_i \delta \lambda' - \eta_i \delta \iota' + \dfrac{\partial \delta' z}{\partial \rho_i} + p_i \delta' y - q_i \delta' x, \end{cases} \qquad \begin{cases} \delta p_i = \dfrac{\partial \delta \iota'}{\partial \rho_i} + q_i \delta \kappa' - r_i \delta \lambda', \\ \delta q_i = \dfrac{\partial \delta \lambda'}{\partial \rho_i} + r_i \delta \iota' - p_i \delta \kappa', \\ \delta r_i = \dfrac{\partial \delta \kappa'}{\partial \rho_i} + p_i \delta \lambda' - q_i \delta \iota', \end{cases} \quad (59)$$

$$(58')\quad \begin{cases} \delta \xi = \eta \delta \kappa' - \zeta \delta \lambda' + \dfrac{d \delta' x}{dt} + q \delta' z - r \delta' y, \\ \delta \eta = \zeta \delta \iota' - \xi \delta \kappa' + \dfrac{d \delta' y}{dt} + r \delta' x - p \delta' z, \\ \delta \zeta = \xi \delta \lambda' - \eta \delta \iota' + \dfrac{d \delta' z}{dt} + p \delta' y - q \delta' x, \end{cases} \qquad \begin{cases} \delta p = \dfrac{d \delta \iota'}{dt} + q \delta \kappa' - r \delta \lambda', \\ \delta q = \dfrac{d \delta \lambda'}{dt} + r \delta \iota' - p \delta \kappa', \\ \delta r = \dfrac{d \delta \kappa'}{dt} + p \delta \lambda' - q \delta \iota'. \end{cases} \quad (59')$$

61. Euclidean Action of Deformation and Motion for a Deformable Medium in Motion. Let us consider a W function of *two infinitely close positions of* triad Mx'y'z', i.e., a function of x_0, y_0, z_0, t, of x, y, z, α, α', ..., γ'' and of their first derivatives with respect to x_0, y_0, z_0, t. Let us set ourselves the task of determining what the form of W must be so that the quadruple integral

$$\iiiint W dx_0 dy_0 dz_0 dt,$$

extended over any portion of space (M_0) and over the time interval between t_1

and t_2 may have a zero variation when the ensemble of all the triads comprising what we have called the trajectory of the deformable medium taken in its deformed state is subjected to the *same arbitrary infinitesimal transformation from the group of Euclidean displacements.*

It is a question, finally, of determining W in such a way that we will have

$$\delta W = 0,$$

when, on the one hand, origin M of triad Mx'y'z' undergoes an infinitesimally small displacement whose projections δx, δy, δz on the Ox, Oy, Oz axes are

$$\begin{cases} \delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t, \\ \delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t, \\ \delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t, \end{cases} \tag{60}$$

where a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constants and δt is an infinitesimally small quantity independent of x_0, y_0, z_0, t, and when, on the other hand, this triad Mx'y'z' undergoes an infinitesimally small rotation whose components along the Ox, Oy, Oz axes are

$$\omega_1 \delta t, \quad \omega_2 \delta t, \quad \omega_3 \delta t.$$

It is sufficient to repeat the line of reasoning which we have already used/157 several times to see that *the desired* W *function has the remarkable form*

$$W(x_0, y_0, z_0, t, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i, \xi, \eta, \zeta, p, q, r),$$

similar to that encountered for the deformable line, surface, and medium at rest.

We will say that the integral

$$\int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt$$

is the *action of deformation and of motion* inside surface S for the deformed medium in motion and in the time interval between moments t_1 and t_2. We will say, on the other hand, that W is the *density* of action of deformation and motion *at a point* in the deformed medium taken *at a given instant* and referred to the volume unit of the non-deformed medium and the time unit. With the same meaning as given to Δ in Section 51, the ratio $\frac{W}{|\Delta|}$ is the density of this action taken at a given point and a given instant referred to the volume unit of the deformed medium and the time unit.

62. External Force and Moment; External Deformation Stress and Moment; Stress, Deformation Moment, Momentum, and Moment of Momentum of the Deformed Medium in Motion at a Given Point and at a Given Instant. Let us consider *any* variation of the action of deformation and motion inside a surface (S) of the medium (M) and in the time interval between moments t_1 and t_2, i.e.,

$$\delta\int_{t_1}^{t_2}\iiint_{s_0} W dx_0 dy_0 dz_0 dt = \int_{t_1}^{t_2}\iiint_{s_0}\Big\{\sum\Big(\frac{\partial W}{\partial \xi_i}\delta\xi_i + \frac{\partial W}{\partial \eta_i}\delta\eta_i + \frac{\partial W}{\partial \zeta_i}\delta\zeta_i + \frac{\partial W}{\partial p_i}\delta p_i + \frac{\partial W}{\partial q_i}\delta q_i + \frac{\partial W}{\partial r_i}\delta r_i\Big) + \frac{\partial W}{\partial \xi}\delta\xi + \frac{\partial W}{\partial \eta}\delta\eta + \frac{\partial W}{\partial \zeta}\delta\zeta + \frac{\partial W}{\partial p}\delta p + \frac{\partial W}{\partial q}\delta q + \frac{\partial W}{\partial r}\delta r\Big\} dx_0 dy_0 dz_0 dt.$$

Because of formulas (58), (58'), (59), and (59') we may write it as

$$\begin{aligned}\delta\int_{t_1}^{t_2}\iiint_{s_0} W dx_0 dy_0 dz_0 dt = \int_{t_1}^{t_2}\iiint_{s_0}\Big\{\sum\Big[&\frac{\partial W}{\partial \xi_i}\Big(\eta_i\delta\kappa' - \zeta_i\delta\lambda' + \frac{\partial\delta' x}{\partial p_i} + q_i\delta' z - r_i\delta' y\Big)\\ &+\frac{\partial W}{\partial \eta_i}\Big(\zeta_i\delta\lambda' - \xi_i\delta\kappa' + \frac{\partial\delta' y}{\partial p_i} + r_i\delta' x - p_i\delta' z\Big) + \frac{\partial W}{\partial \zeta_i}\Big(\xi_i\delta\lambda' - \eta_i\delta\lambda' + \frac{\partial\delta' z}{\partial p_i} + p_i\delta' y - q_i\delta' x\Big)\\ &+\frac{\partial W}{\partial p_i}\Big(\frac{\partial\delta\lambda'}{\partial p_i} + q_i\delta\kappa' - r_i\delta\lambda'\Big) + \frac{\partial W}{\partial q_i}\Big(\frac{\partial\delta\lambda'}{\partial p_i} + r_i\delta\lambda' - p_i\delta\kappa'\Big) + \frac{\partial W}{\partial r_i}\Big(\frac{\partial\delta\kappa'}{\partial p_i} + p_i\delta\lambda' - q_i\delta\lambda'\Big)\Big]\\ &+\frac{\partial W}{\partial \xi}\Big(\eta\delta\kappa' - \zeta\delta\lambda' + \frac{d\delta' x}{dt} + q\delta' z - r\delta' y\Big) + \frac{\partial W}{\partial \eta}\Big(\zeta\delta\lambda' - \xi\delta\kappa' + \frac{d\delta' y}{dt} + r\delta' x - p\delta' z\Big)\\ &+\frac{\partial W}{\partial \zeta}\Big(\xi\delta\lambda' - \eta\delta\lambda' + \frac{d\delta' z}{dt} + p\delta' y - q\delta' x\Big) + \frac{\partial W}{\partial p}\Big(\frac{d\delta\lambda'}{dt} + q\delta\kappa' - r\delta\lambda'\Big)\\ &+\frac{\partial W}{\partial q}\Big(\frac{d\delta\lambda'}{dt} + r\delta\lambda' - p\delta\kappa'\Big) + \frac{\partial W}{\partial r}\Big(\frac{d\delta\kappa'}{dt} + p\delta\lambda' - q\delta\lambda'\Big)\Big\} dx_0 dy_0 dz_0 dt.\end{aligned}$$

Let us apply Green's formula to the terms which explicitly contain a derivative with respect to one of the variables ρ_1, ρ_2, ρ_3, and let us integrate by parts over the terms which explicitly contain a derivative with respect to time t. Designating by l_0, m_0, n_0 the direction cosines with respect to the fixed axes Ox, Oy, Oz of the external normal to surface S_0 which delimits the medium before the deformation at time t, and by $d\sigma_0$ the area element of this surface, we have /158

$$\begin{aligned}&\delta\int_{t_1}^{t_2}\iiint_{s_0} W dx_0 dy_0 dz_0 dt =\\ &\int_{t_1}^{t_2}\iint_{s_0}\Big\{\Big(l_0\frac{\partial W}{\partial \xi_1} + m_0\frac{\partial W}{\partial \xi_2} + n_0\frac{\partial W}{\partial \xi_3}\Big)\delta' x + \Big(l_0\frac{\partial W}{\partial \eta_1} + m_0\frac{\partial W}{\partial \eta_2} + n_0\frac{\partial W}{\partial \eta_3}\Big)\delta' y\\ &+\Big(l_0\frac{\partial W}{\partial \zeta_1} + m_0\frac{\partial W}{\partial \zeta_2} + n_0\frac{\partial W}{\partial \zeta_3}\Big)\delta' z + \Big(l_0\frac{\partial W}{\partial p_1} + m_0\frac{\partial W}{\partial p_2} + n_0\frac{\partial W}{\partial p_3}\Big)\delta\lambda'\\ &+\Big(l_0\frac{\partial W}{\partial q_1} + m_0\frac{\partial W}{\partial q_2} + n_0\frac{\partial W}{\partial q_3}\Big)\delta\mu' + \Big(l_0\frac{\partial W}{\partial r_1} + m_0\frac{\partial W}{\partial r_2} + n_0\frac{\partial W}{\partial r_3}\Big)\delta\kappa'\Big\} d\sigma_0 dt\\ &+\Big\{\iiint_{s_0}\Big(\frac{\partial W}{\partial \xi}\delta' x + \frac{\partial W}{\partial \eta}\delta' y + \frac{\partial W}{\partial \zeta}\delta' z + \frac{\partial W}{\partial p}\delta\lambda' + \frac{\partial W}{\partial q}\delta\mu' + \frac{\partial W}{\partial r}\delta\kappa'\Big) dx_0 dy_0 dz_0\Big\}_{t_1}^{t_2}\\ &-\int_{t_1}^{t_2}\iiint_{s_0}\Big\{\Big[\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \xi_i} + q_i\frac{\partial W}{\partial \zeta_i} - r_i\frac{\partial W}{\partial \eta_i}\Big) + \frac{d}{dt}\frac{\partial W}{\partial \xi} + q\frac{\partial W}{\partial \zeta} - r\frac{\partial W}{\partial \eta}\Big]\delta' x\\ &+\Big[\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \eta_i} + r_i\frac{\partial W}{\partial \xi_i} - p_i\frac{\partial W}{\partial \zeta_i}\Big) + \frac{d}{dt}\frac{\partial W}{\partial \eta} + r\frac{\partial W}{\partial \xi} - p\frac{\partial W}{\partial \zeta}\Big]\delta' y\\ &+\Big[\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \zeta_i} + p_i\frac{\partial W}{\partial \eta_i} - q_i\frac{\partial W}{\partial \xi_i}\Big) + \frac{d}{dt}\frac{\partial W}{\partial \zeta} + p\frac{\partial W}{\partial \eta} - q\frac{\partial W}{\partial \xi}\Big]\delta' z\\ &+\Big[\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial p_i} + q_i\frac{\partial W}{\partial r_i} - r_i\frac{\partial W}{\partial q_i} + \eta_i\frac{\partial W}{\partial \zeta_i} - \zeta_i\frac{\partial W}{\partial \eta_i}\Big)\end{aligned}$$

$$+\frac{d}{dt}\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}+\eta\frac{\partial W}{\partial \zeta}-\zeta\frac{\partial W}{\partial \eta}\Big]\delta\iota'$$

$$+\Big[\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial q_i}+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\zeta_i\frac{\partial W}{\partial \xi_i}-\xi_i\frac{\partial W}{\partial \zeta_i}\Big)$$

$$+\frac{d}{dt}\frac{\partial W}{\partial q}+r\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial r}+\zeta\frac{\partial W}{\partial \xi}-\xi\frac{\partial W}{\partial \zeta}\Big]\delta\jmath'$$

$$+\Big[\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial r_i}+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial \eta_i}-\eta_i\frac{\partial W}{\partial \xi_i}\Big)$$

$$+\frac{d}{dt}\frac{\partial W}{\partial r}+p\frac{\partial W}{\partial q}-q\frac{\partial W}{\partial p}+\xi\frac{\partial W}{\partial \eta}-\eta\frac{\partial W}{\partial \xi}\Big]\delta\kappa'\Big\}dx_0dy_0dz_0dt.$$

As in Section 52, let us set

$$F_0'=l_0\frac{\partial W}{\partial \xi_1}+m_0\frac{\partial W}{\partial \xi_2}+n_0\frac{\partial W}{\partial \xi_3},\qquad I_0'=l_0\frac{\partial W}{\partial p_1}+m_0\frac{\partial W}{\partial p_2}+n_0\frac{\partial W}{\partial p_3},$$

$$G_0'=l_0\frac{\partial W}{\partial \eta_1}+m_0\frac{\partial W}{\partial \eta_2}+n_0\frac{\partial W}{\partial \eta_3},\qquad J_0'=l_0\frac{\partial W}{\partial q_1}+m_0\frac{\partial W}{\partial q_2}+n_0\frac{\partial W}{\partial q_3},$$

$$H_0'=l_0\frac{\partial W}{\partial \zeta_1}+m_0\frac{\partial W}{\partial \zeta_2}+n_0\frac{\partial W}{\partial \zeta_3},\qquad K_0'=l_0\frac{\partial W}{\partial r_1}+m_0\frac{\partial W}{\partial r_2}+n_0\frac{\partial W}{\partial r_3},$$

and also /159

$$A'=\frac{\partial W}{\partial \xi},\qquad B'=\frac{\partial W}{\partial \eta},\qquad C'=\frac{\partial W}{\partial \zeta},$$

$$P'=\frac{\partial W}{\partial p},\qquad Q'=\frac{\partial W}{\partial q},\qquad R'=\frac{\partial W}{\partial r};$$

On the other hand, let us set

$$X_0'=\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \xi_i}+q_i\frac{\partial W}{\partial \zeta_i}-r_i\frac{\partial W}{\partial \eta_i}\Big)+\frac{d}{dt}\frac{\partial W}{\partial \xi}+q\frac{\partial W}{\partial \zeta}-r\frac{\partial W}{\partial \eta},$$

$$Y_0'=\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \eta_i}+r_i\frac{\partial W}{\partial \xi_i}-p_i\frac{\partial W}{\partial \zeta_i}\Big)+\frac{d}{dt}\frac{\partial W}{\partial \eta}+r\frac{\partial W}{\partial \xi}-p\frac{\partial W}{\partial \zeta},$$

$$Z_0'=\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \zeta_i}+p_i\frac{\partial W}{\partial \eta_i}-q_i\frac{\partial W}{\partial \xi_i}\Big)+\frac{d}{dt}\frac{\partial W}{\partial \zeta}+p\frac{\partial W}{\partial \eta}-q\frac{\partial W}{\partial \xi},$$

$$L_0'=\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial p_i}+q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial \zeta_i}-\zeta_i\frac{\partial W}{\partial \eta_i}\Big)$$

$$+\frac{d}{dt}\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}+\eta\frac{\partial W}{\partial \zeta}-\zeta\frac{\partial W}{\partial \eta},$$

$$M_0'=\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial q_i}+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\zeta_i\frac{\partial W}{\partial \xi_i}-\xi_i\frac{\partial W}{\partial \zeta_i}\Big)$$

$$+\frac{d}{dt}\frac{\partial W}{\partial q}+r\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial r}+\zeta\frac{\partial W}{\partial \xi}-\xi\frac{\partial W}{\partial \zeta},$$

$$N_0'=\sum\Big(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial r_i}+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial \eta_i}-\eta_i\frac{\partial W}{\partial \xi_i}\Big)$$

$$+\frac{d}{dt}\frac{\partial W}{\partial r}+p\frac{\partial W}{\partial q}-q\frac{\partial W}{\partial p}+\xi\frac{\partial W}{\partial \eta}-\eta\frac{\partial W}{\partial \xi};$$

We will have

$$\delta\int_{t_1}^{t_2}\iiint_{s_0}W\,dx_0dy_0dz_0dt$$

$$=\int_{t_1}^{t_2}\iint_{s_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta\iota'+J_0'\delta\jmath'+K_0'\delta\kappa')ds_0dt$$

$$+\Big\{\iiint_{s_0}(A'\delta'x+B'\delta'y+C'\delta'z+P'\delta\iota'+Q'\delta\jmath'+R'\delta\kappa')dx_0dy_0dz_0\Big\}_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\iiint_{s_0}(X_0'\delta'x+Y_0'\delta'y+Z_0'\delta'z+L_0'\delta\iota'+M_0'\delta\jmath'+N_0'\delta\kappa')dx_0dy_0dz_0dt.$$

Considering first the quadruple integral which appears in the expression of $\delta\int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt$, we will call the segments originating in M and having the respective projections X_0', Y_0', Z_0' and L_0', M_0', N_0' on axes Mx', My', Mz' *the external force and external moment at point* M *at instant* t *referred to the volume unit for the position of the medium at instant* t_0.

Considering the triple integral taken with respect to time and extended over surface S_0, we will give the name of *external stress and external moment of deformation at point* M *of surface* S *delimiting the deformed medium at instant* t *and referred to the area unit of surface* S_0 to the segments which emanate from this point M and whose projections on the Mx', My', Mz' axes are, respectively, /160
$-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$. These latter six quantities at a certain point M of (S) depend only on the direction of the external normal to surface S. They remain unchanged if, when the considered region of (M_0) changes, this direction of the normal does not change, and they change sign if this direction is replaced by the opposite direciton.

Let us assume that inside the deformed medium delimited by surface S a surface Σ is drawn which, either alone or with a portion of surface S, circumscribes a part (A) of the medium, and let us designate by (B) what remains of the medium outside part (A). Let Σ_0 be the surface of (M_0) which corresponds to surface Σ of (M), and let (A_0) and (B_0) be the regions of (M_0) which correspond to regions (A) and (B) of (M). Let us keep the two parts (A) and (B) separate in our minds. The two segments $(-F_0', -G_0', -H_0')$ and $(-I_0', -J_0', -K_0')$, determined by point M and for the direction of the normal drawn with respect to Σ_0 and exterior to (A_0), may be regarded as external stress and moment of deformation at point M of boundary Σ of region (A). Likewise, we may regard the two segments (F_0', G_0', H_0') and (I_0', J_0', K_0') as external stress and moment of deformation at point M of boundary Σ of region B. Due to this statement, we will say that $-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$ are the components of the *deformation stress and moment exerted in* M *on part* (A) *of medium* (M) along axes Mx', My', Mz', and that F_0', G_0', H_0' and I_0', J_0', K_0' are the components of *deformation stress and moment exerted in* M *on part* (B) *of medium* (M) along axes Mx', My', Mz'.

Finally, considering the triple integral extended over the volume of (M) at time t, with its values taken at the limiting times t_1 and t_2, we will give the name of *momentum and moment of momentum at point* M *of deformed medium* (M) *at instant* t to the segments which have their origin in M and whose projections on the Mx', My', Mz' axes are A', B', C' and P', Q', R', respectively.

63. Various Specifications for the Stress and Moment of Deformation, for Momentum and for Moment of Momentum. As in Section 53 let us set

$$A_i' = \frac{\partial W}{\partial \xi_i}, \quad B_i' = \frac{\partial W}{\partial \eta_i}, \quad C_i' = \frac{\partial W}{\partial \zeta_i},$$
$$P_i' = \frac{\partial W}{\partial p_i}, \quad Q_i' = \frac{\partial W}{\partial q_i}, \quad R_i' = \frac{\partial W}{\partial r_i};$$

A_i', B_i', C_i' and P_i', Q_i', R_i', respectively, represent the projections on Mx', My', Mz' of the stress and moment of deformation which are exerted at point M on a surface whose normal before deformation at point M_0 was the line parallel to that of the coordinate axes Ox, Oy, Oz, corresponding to subscript i. It is indeed sufficient to remember that we have already agreed to replace the terms x_0, y_0, z_0 by ρ_1, ρ_3, ρ_3 which by this convention correspond, respectively, to subscripts 1, 2, 3. This stress and this moment of deformation are, let us recall, referred to the area unit of the non-deformed surface at instant t. /161

The new stresses and the new moments of deformation which we have just defined are linked to the elements introduced in the preceding section by the following relationships:

$$F_0' = l_0A_1' + m_0A_2' + n_0A_3', \qquad I_0' = l_0P_1' + m_0P_2' + n_0P_3'$$
$$G_0' = l_0B_1' + m_0B_2' + n_0B_3', \qquad J_0' = l_0Q_1' + m_0Q_2' + n_0Q_3'$$
$$H_0' = l_0C_1' + m_0C_2' + n_0C_3', \qquad K_0' = l_0R_1' + m_0R_2' + n_0R_3'$$

$$\sum\left(\frac{\partial A_i'}{\partial \rho_i} + q_iC_i' - r_iB_i'\right) + \frac{dA'}{dt} + qC' - rB' - X_0' = 0$$
$$\sum\left(\frac{\partial B_i'}{\partial \rho_i} + r_iA_i' - p_iC_i'\right) + \frac{dB'}{dt} + rA' - pC' - Y_0' = 0$$
$$\sum\left(\frac{\partial C_i'}{\partial \rho_i} + p_iB_i' - q_iA_i'\right) + \frac{dC'}{dt} + pB' - qA' - Z_0' = 0$$

$$\sum\left(\frac{\partial P_i'}{\partial \rho_i} + q_iR_i' - r_iQ_i' + \eta_iC_i' - \zeta_iB_i'\right) + \frac{dP'}{dt} + qR' - rQ' + \eta C' - \zeta B' - L_0' = 0,$$
$$\sum\left(\frac{\partial Q_i'}{\partial \rho_i} + r_iP_i' - p_iR_i' + \zeta_iA_i' - \xi_iC_i'\right) + \frac{dQ'}{dt} + rP' - pR' + \zeta A' - \xi C' - M_0' = 0,$$
$$\sum\left(\frac{\partial R_i'}{\partial \rho_i} + p_iQ_i' - q_iP_i' + \xi_iB_i' - \eta_iA_i'\right) + \frac{dR'}{dt} + pQ' - qP' + \xi B' - \eta A' - N_0' = 0.$$

Let us attempt to transform the relationships which we have just written independently of the values calculated in terms of W and of the quantities which are contained in them. These relationships hold true, in fact, between the segments connected to point M to which we have given names. Instead of defining these segments by their projections on Mx', My', Mz', we can just as well define them by their projections on other axes. These latter projections will be connected by relationships which are transformations of the preceding ones. The transformed relationships will moreover be immediately obtained, if it is noted that the original formulas have simple interpretations(22) by adding to the mobile axes those axes which are drawn parallel to them through point 0.

(22) An interpretation which is interesting to note is the analogue of the one given by P. Saint-Guilhem in reference to triad dynamics.

1. Just as for statics, let us restrict ourselves to consideration of the fixed axes Ox, Oy, Oz. Let us designate respectively by X_0, Y_0, Z_0 and L_0, M_0, N_0 the projections upon these axes of the external force and external moment at any point M of the deformed medium at instant t; by F_0, G_0, H_0 and I_0, J_0, K_0 the projections of deformation stress and moment on a surface whose external normal before deformation and at instant t has the direction cosines l_0, m_0, n_0; by A_i, B_i, C_i and P_i, Q_i, R_i the projections of stress (A_i', B_i', C_i') and of moment of deformation (P_i', Q_i', R_i'); and by A, B, C and P, Q, R the projections of momentum (A, B, C) and of moment of momentum (P, Q, R). The transforms of the foregoing relationships are obviously /162

$$F_0 = l_0A_1 + m_0A_2 + n_0A_3, \qquad I_0 = l_0P_1 + m_0P_2 + n_0P_3,$$
$$G_0 = l_0B_1 + m_0B_2 + n_0B_3, \qquad J_0 = l_0Q_1 + m_0Q_2 + n_0Q_3,$$
$$H_0 = l_0C_1 + m_0C_2 + n_0C_3, \qquad K_0 = l_0R_1 + m_0R_2 + n_0R_3,$$
$$\frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0} + \frac{dA}{dt} - X_0 = 0,$$
$$\frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0} + \frac{dB}{dt} - Y_0 = 0,$$
$$\frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0} + \frac{dC}{dt} - Z_0 = 0,$$
$$\frac{\partial P_1}{\partial x_0} + \frac{\partial P_2}{\partial y_0} + \frac{\partial P_3}{\partial z_0} + \frac{dP}{dt} + C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} + C\frac{dy}{dt}$$
$$- B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial y_0} - B_3\frac{\partial z}{\partial z_0} - B\frac{dz}{dt} - L_0 = 0,$$
$$\frac{\partial Q_1}{\partial x_0} + \frac{\partial Q_2}{\partial y_0} + \frac{\partial Q_3}{\partial z_0} + \frac{dQ}{dt} + A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} + A\frac{dz}{dt}$$
$$- C_1\frac{\partial x}{\partial x_0} - C_2\frac{\partial x}{\partial y_0} - C_3\frac{\partial x}{\partial z_0} - C\frac{dx}{dt} - M_0 = 0,$$
$$\frac{\partial R_1}{\partial x_0} + \frac{\partial R_2}{\partial y_0} + \frac{\partial R_3}{\partial z_0} + \frac{dR}{dt} + B_1\frac{\partial x}{\partial x_0} + B_2\frac{\partial x}{\partial y_0} + B_3\frac{\partial x}{\partial z_0} + B\frac{dx}{dt}$$
$$- A_1\frac{\partial y}{\partial x_0} - A_2\frac{\partial y}{\partial y_0} - A_3\frac{\partial y}{\partial z_0} - A\frac{dy}{dt} - N_0 = 0.$$

2. Let us now note that we may express the nine cosines α, α', ... γ'' by means of the three auxiliary quantities λ_1, λ_2, λ_3. Let us set

$$\Sigma\gamma d\beta = -\Sigma\beta d\gamma = \varpi_1' d\lambda_1 + \varpi_2' d\lambda_2 + \varpi_3' d\lambda_3,$$
$$\Sigma\alpha d\gamma = -\Sigma\gamma d\alpha = \chi_1' d\lambda_1 + \chi_2' d\lambda_2 + \chi_3' d\lambda_3,$$
$$\Sigma\beta d\alpha = -\Sigma\alpha d\beta = \sigma_1' d\lambda_1 + \sigma_2' d\lambda_2 + \sigma_3' d\lambda_3.$$

Functions $\bar{\omega}_i$, χ_i, σ_i of λ_1, λ_2, λ_3, thus defined, satisfy the relationships which we have already written several times

$$\frac{\partial \varpi_j'}{\partial \lambda_i} - \frac{\partial \varpi_i'}{\partial \lambda_j} + \chi_i'\sigma_j' - \chi_j'\sigma_i' = 0,$$
$$\frac{\partial \chi_j'}{\partial \lambda_i} - \frac{\partial \chi_i'}{\partial \lambda_j} + \sigma_i'\varpi_j' - \sigma_j'\varpi_i' = 0, \qquad (i, j = 1, 2, 3),$$
$$\frac{\partial \sigma_j'}{\partial \lambda_i} - \frac{\partial \sigma_i'}{\partial \lambda_j} + \varpi_i'\chi_j' - \varpi_j'\chi_i' = 0,$$

and we have

$$p_i = \varpi_1' \frac{\partial\lambda_1}{\partial\rho_i} + \varpi_2' \frac{\partial\lambda_2}{\partial\rho_i} + \varpi_3' \frac{\partial\lambda_3}{\partial\rho_i}, \qquad p = \varpi_1' \frac{d\lambda_1}{dt} + \varpi_2' \frac{d\lambda_2}{dt} + \varpi_3' \frac{d\lambda_3}{dt},$$
$$q_i = \chi_1' \frac{\partial\lambda_1}{\partial\rho_i} + \chi_2' \frac{\partial\lambda_2}{\partial\rho_i} + \chi_3' \frac{\partial\lambda_3}{\partial\rho_i}, \qquad q = \chi_1' \frac{d\lambda_1}{dt} + \chi_2' \frac{d\lambda_2}{dt} + \chi_3' \frac{d\lambda_3}{dt},$$
$$r_i = \sigma_1' \frac{\partial\lambda_1}{\partial\rho_i} + \sigma_2' \frac{\partial\lambda_2}{\partial\rho_i} + \sigma_3' \frac{\partial\lambda_3}{\partial\rho_i}, \qquad r = \sigma_1' \frac{d\lambda_1}{dt} + \sigma_2' \frac{d\lambda_2}{dt} + \sigma_3' \frac{d\lambda_3}{dt},$$

where $x_0 = \rho_1$, $y_0 = \rho_2$, $z_0 = \rho_3$. Let us designate by $\bar{\omega}_i$, χ_i, σ_i the projections/163 on the fixed axes Ox, Oy, Oz of the segment whose projections on the Mx', My', Mz' axes are $\bar{\omega}_i'$, χ_i', σ_i', and we will have

$$\Sigma x' dx'' = -\Sigma x'' dx' = \varpi_1 d\lambda_1 + \varpi_2 d\lambda_2 + \varpi_3 d\lambda_3,$$
$$\Sigma x'' dx = -\Sigma x dx'' = \chi_1 d\lambda_1 + \chi_2 d\lambda_2 + \chi_3 d\lambda_3,$$
$$\Sigma x dx' = -\Sigma x' dx = \sigma_1 d\lambda_1 + \sigma_2 d\lambda_2 + \sigma_3 d\lambda_3,$$

by virtue of which(23) the new functions $\bar{\omega}_i$, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy relationships

$$\frac{\partial\varpi_j}{\partial\lambda_i} - \frac{\partial\varpi_i}{\partial\lambda_j} = \chi_i\sigma_j - \chi_j\sigma_i,$$
$$\frac{\partial\chi_j}{\partial\lambda_i} - \frac{\partial\chi_i}{\partial\lambda_j} = \sigma_i\varpi_j - \sigma_j\varpi_i, \qquad (i, j = 1, 2, 3).$$
$$\frac{\partial\sigma_j}{\partial\lambda_i} - \frac{\partial\sigma_i}{\partial\lambda_j} = \varpi_i\chi_j - \varpi_j\chi_i;$$

We would like to make the additional remark, which will be of service to us further on, that if $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ denote the variations in λ_1, λ_2, λ_3 which correspond to the variations $\delta\alpha$, $\delta\alpha'$, ... $\delta\gamma''$ of α, α', ... γ'', we will have

$$\delta i' = \varpi_1'\delta\lambda_1 + \varpi_2'\delta\lambda_2 + \varpi_3'\delta\lambda_3$$
$$\delta j' = \chi_1'\delta\lambda_1 + \chi_2'\delta\lambda_2 + \chi_3'\delta\lambda_3,$$
$$\delta k' = \sigma_1'\delta\lambda_1 + \sigma_2'\delta\lambda_2 + \sigma_3'\delta\lambda_3,$$
$$\delta i = \alpha\delta i' + \beta\delta j' + \gamma\delta k' = \varpi_1\delta\lambda_1 + \varpi_2\delta\lambda_2 + \varpi_3\delta\lambda_3,$$
$$\delta j = \alpha'\delta i' + \beta'\delta j' + \gamma'\delta k' = \chi_1\delta\lambda_1 + \chi_2\delta\lambda_2 + \chi_3\delta\lambda_3,$$
$$\delta k = \alpha''\delta i' + \beta''\delta j' + \gamma''\delta k' = \sigma_1\delta\lambda_1 + \sigma_2\delta\lambda_2 + \sigma_3\delta\lambda_3,$$

where δI, δJ, δK are the projections upon the fixed axes of the segment whose projections on Mx', My', Mz' are $\delta I'$, $\delta J'$, $\delta K'$. Let us now set

$$\mathfrak{I}_0 = \varpi_1' I_0' + \chi_1' J_0' + \sigma_1' K_0' = \varpi_1 I_0 + \chi_1 J_0 + \sigma_1 K_0,$$
$$\mathfrak{J}_0 = \varpi_2' I_0' + \chi_2' J_0' + \sigma_2' K_0' = \varpi_2 I_0 + \chi_2 J_0 + \sigma_2 K_0,$$
$$\mathfrak{K}_0 = \varpi_3' J_0' + \chi_3' J_0' + \sigma_3' K_0' = \varpi_3 I_0 + \chi_3 J_0 + \sigma_3 K_0,$$
$$\mathfrak{L}_0 = \varpi_1' L_0' + \chi_1' M_0' + \sigma_1' N_0' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0,$$
$$\mathfrak{M}_0 = \varpi_2' L_0' + \chi_2' M_0' + \sigma_2' N_0' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0,$$
$$\mathfrak{N}_0 = \varpi_3' L_0' + \chi_3' M_0' + \sigma_3' N_0' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0.$$

Let us also introduce the following notation:

$$\Pi_i = \varpi_1 P_i' + \chi_1' Q_i' + \sigma_1' R_i' = \varpi_1 P_i + \chi_1 Q_i + \sigma_1 R_i,$$
$$X_i = \varpi_2' P_i' + \chi_2' Q_i' + \sigma_2' R_i' = \varpi_2 P_i + \chi_2 Q_i + \sigma_2 R_i,$$

(23) These formulas may serve to define functions $\bar{\omega}_i$, χ_i, σ_i directly and to be substituted for

$$\varpi_i = \alpha\varpi_i' + \beta\chi_i' + \gamma\sigma_i',$$
$$\chi_i = \alpha'\varpi_i' + \beta'\chi_i' + \gamma'\sigma_i', \qquad (i = 1, 2, 3).$$
$$\sigma_i = \alpha''\varpi_i' + \beta''\chi_i' + \gamma''\sigma_i'.$$

$$\Sigma_i = \varpi_3' P_i' + \chi_3' Q_i' + \sigma_3' R_i' = \varpi_3 P_i + \chi_3 Q_i + \sigma_3 R_i,$$
$$\Pi = \varpi_1' P' + \chi_1' Q' + \sigma_1' R' = \varpi_1 P + \chi_1 Q + \sigma_1 R,$$
$$X = \varpi_2' P' + \chi_2' Q' + \sigma_2' R' = \varpi_2 P + \chi_2 Q + \sigma_2 R,$$
$$\Sigma = \varpi_3' P' + \chi_3' Q' + \sigma_3' R' = \varpi_3 P + \chi_3 Q + \sigma_3 R,$$

In place of the last system, where either P_i', Q_i', R_i', P', Q', R' or P_i, Q_i, /164
R_i, P, Q, R appear, we have the following

$$-\mathfrak{L}_0 + \sum_i \Big[\frac{\partial \Pi_i}{\partial \rho_i} - P_i'\Big(\frac{\partial \varpi_1'}{\partial \rho_i} + q_i\sigma_1' - r_i\chi_1'\Big) - Q_i'\Big(\frac{\partial \chi_1'}{\partial \rho_i} + r_i\varpi_1' - p_i\sigma_1'\Big) - R_i'\Big(\frac{\partial \sigma_1'}{\partial \rho_i} + p_i\chi_1' - q_i\varpi_1'\Big)$$
$$+ A_i'(\chi_1'\zeta_i - \sigma_1'\eta_i) + B_i'(\sigma_1'\xi_i - \varpi_1'\zeta_i) + C_i'(\varpi_1'\eta_i - \chi_1'\xi_i)\Big]$$
$$+ \frac{d\Pi}{dt} - P'\Big(\frac{d\varpi_1'}{dt} + q\sigma_1' - r\chi_1'\Big) - Q'\Big(\frac{d\chi_1'}{dt} + r\varpi_1' - p\sigma_1'\Big) - R'\Big(\frac{d\sigma_1'}{dt} + p\chi_1' - q\varpi_1'\Big)$$
$$+ A'(\chi_1'\zeta - \sigma_1'\eta) + B'(\sigma_1'\xi - \varpi_1'\zeta) + C'(\varpi_1'\eta - \chi_1'\xi) = 0,$$

with two similar equations. If it is noted that functions ξ_i, η_i, ζ_i, p_i, q_i, r_i and ξ, η, ζ, p, q, r of λ_1, λ_2, λ_3, $\frac{\partial\lambda_1}{\partial\rho_i}$, $\frac{\partial\lambda_2}{\partial\rho_i}$, $\frac{\partial\lambda_3}{\partial\rho_i}$, $\frac{d\lambda_1}{dt}$, $\frac{d\lambda_2}{dt}$, $\frac{d\lambda_3}{dt}$ give rise to formulas

$$\frac{\partial \xi_i}{\partial \lambda_j} + \chi_j'\zeta_i - \sigma_j'\eta_i = 0, \qquad \frac{\partial p_i}{\partial \lambda_j} = \frac{\partial \varpi_j'}{\partial \rho_i} + q_i\sigma_j' - r_i\chi_j',$$
$$\frac{\partial \eta_i}{\partial \lambda_j} + \sigma_j'\xi_i - \varpi_j'\zeta_i = 0, \qquad \frac{\partial q_i}{\partial \lambda_j} = \frac{\partial \chi_j'}{\partial \rho_i} + r_i\varpi_j' - p_i\sigma_j',$$
$$\frac{\partial \zeta_i}{\partial \lambda_j} + \varpi_j'\eta_i - \chi_j'\xi_i = 0, \qquad \frac{\partial r_i}{\partial \lambda_j} = \frac{\partial \sigma_j'}{\partial \rho_i} + p_i\chi_j' - q_i\varpi_j',$$
$$\frac{\partial \xi}{\partial \lambda_j} + \chi_j'\zeta - \sigma_j'\eta = 0, \qquad \frac{\partial p}{\partial \lambda_j} = \frac{d\varpi_j'}{dt} + q\sigma_j' - r\chi_j',$$
$$\frac{\partial \eta}{\partial \lambda_j} + \sigma_j'\xi - \varpi_j'\zeta = 0, \qquad \frac{\partial q}{\partial \lambda_j} = \frac{d\chi_j'}{dt} + r\varpi_j' - p\sigma_j',$$
$$\frac{\partial \zeta}{\partial \lambda_j} + \varpi_j'\eta - \chi_j'\xi = 0, \qquad \frac{\partial r}{\partial \lambda_j} = \frac{d\sigma_j'}{dt} + p\chi_j' - q\varpi_j',$$

which result from the relationships defining functions $\bar{\omega}_i'$, χ_i', σ_i', and the nine identities which they satisfy, the new form

$$-\mathfrak{L}_0 + \sum_i \Big[\frac{\partial \Pi_i}{\partial \rho_i} - A_i'\frac{\partial \xi_i}{\partial \lambda_1} - B_i'\frac{\partial \eta_i}{\partial \lambda_1} - C_i'\frac{\partial \zeta_i}{\partial \lambda_1} - P_i'\frac{\partial p_i}{\partial \lambda_1} - Q_i'\frac{\partial q_i}{\partial \lambda_1} - R_i'\frac{\partial r_i}{\partial \lambda_1}\Big]$$
$$+ \frac{d\Pi}{dt} - A'\frac{\partial \xi}{\partial \lambda_1} - B'\frac{\partial \eta}{\partial \lambda_1} - C'\frac{\partial \zeta}{\partial \lambda_1} - P'\frac{\partial p}{\partial \lambda_1} - Q'\frac{\partial q}{\partial \lambda_1} - R'\frac{\partial r}{\partial \lambda_1} = 0,$$

with two analogous equations may be given to the preceding system.

3. We are finally going to subject the equations which we introduced in the preceding to a transformation similar to the one which led us in Section 53 to a generalization of the elasticity theory equations in terms of stress.

For purposes of abbreviation, let us for a moment use $\mathfrak{X}_0'$, $\mathfrak{Y}_0'$, $\mathfrak{Z}_0'$, $\mathfrak{L}_0'$, $\mathfrak{M}_0'$, $\mathfrak{N}_0'$ to designate the first terms of the relationships to be

transformed, which respectively contain X_0, Y_0, Z_0, L_0, M_0, N_0, and let us observe that the twelve relationships which we have established may be summarized in the following: /165

$$\int_{t_1}^{t_2}\iiint_{s_0}(\mathfrak{X}_0'\lambda_1+\mathfrak{Y}_0'\lambda_2+\mathfrak{Z}_0'\lambda_3+\mathfrak{L}_0'\mu_1+\mathfrak{M}_0'\mu_2+\mathfrak{N}_0'\mu_3)\,dx_0dy_0dz_0dt$$
$$+\int_{t_1}^{t_2}\iint_{s_0}\{(F_0-l_0A_1-m_0A_2-n_0A_3)\lambda_1+(G_0-l_0B_1-m_0B_2-n_0B_3)\lambda_2$$
$$+(H_0-l_0C_1-m_0C_2-n_0C_3)\lambda_3+(I_0-l_0P_1-m_0P_2-n_0P_3)\mu_1$$
$$+(J_0-l_0Q_1-m_0Q_2-n_0Q_3)\mu_2+(K_0-l_0R_1-m_0R_2-n_0R_3)\mu_3\}\,dz_0dt=0,$$

where λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary functions and where the integrals are extended, on the one hand, over the time interval between times t_1 and t_2 and, on the other hand, are taken over the surface S_0 of medium (M_0) and over the domain limited by this surface. With the application of Green's formula and integration by parts, the relationship which we have just described becomes

$$-\int_{t_1}^{t_2}\iiint_{s_0}(X_0\lambda_1+Y_0\lambda_2+Z_0\lambda_3+L_0\mu_1+M_0\mu_2+N_0\mu_3)\,dx_0dy_0dz_0dt$$
$$+\int_{t_1}^{t_2}\iint_{s_0}(F_0\lambda_1+G_0\lambda_2+H_0\lambda_3+I_0\mu_1+J_0\mu_2+K_0\mu_3)\,dz_0dt$$
$$+\left\{\iiint_{s_0}(A\lambda_1+B\lambda_2+C\lambda_3+P\mu_1+Q\mu_2+R\mu_3)\,dx_0dy_0dz_0\right\}_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\iiint_{s_0}\left(A_1\frac{\partial\lambda_1}{\partial x_0}+A_2\frac{\partial\lambda_1}{\partial y_0}+A_3\frac{\partial\lambda_1}{\partial z_0}+A\frac{d\lambda_1}{dt}+B_1\frac{\partial\lambda_2}{\partial x_0}+B_2\frac{\partial\lambda_2}{\partial y_0}+B_3\frac{\partial\lambda_2}{\partial z_0}+B\frac{d\lambda_2}{dt}\right.$$
$$\left.+C_1\frac{\partial\lambda_3}{\partial x_0}+C_2\frac{\partial\lambda_3}{\partial y_0}+C_3\frac{\partial\lambda_3}{\partial z_0}+C\frac{d\lambda_3}{dt}\right)dx_0dy_0dz_0dt$$
$$-\int_{t_1}^{t_2}\iiint_{s_0}\left(P_1\frac{\partial\mu_1}{\partial x_0}+P_2\frac{\partial\mu_1}{\partial y_0}+P_3\frac{\partial\mu_1}{\partial z_0}+P\frac{d\mu_1}{dt}+Q_1\frac{\partial\mu_2}{\partial x_0}+Q_2\frac{\partial\mu_2}{\partial y_0}+Q_3\frac{\partial\mu_2}{\partial z_0}+Q\frac{d\mu_2}{dt}\right.$$
$$\left.+R_1\frac{\partial\mu_3}{\partial x_0}+R_2\frac{\partial\mu_3}{\partial y_0}+R_3\frac{\partial\mu_3}{\partial z_0}+R\frac{d\mu_3}{dt}\right)dx_0dy_0dz_0dt$$
$$+\int_{t_1}^{t_2}\iiint_{s_0}\left(C_1\frac{\partial y}{\partial x_0}+C_2\frac{\partial y}{\partial y_0}+C_3\frac{\partial y}{\partial z_0}+C\frac{dy}{dt}\right.$$
$$\left.-B_1\frac{\partial z}{\partial x_0}-B_2\frac{\partial z}{\partial y_0}-B_3\frac{\partial z}{\partial z_0}-B\frac{dz}{dt}\right)\mu_1dx_0dy_0dz_0dt$$
$$+\int_{t_1}^{t_2}\iiint_{s_0}\left(A_1\frac{\partial z}{\partial x_0}+A_2\frac{\partial z}{\partial y_0}+A_3\frac{\partial z}{\partial z_0}+A\frac{dz}{dt}\right.$$
$$\left.-C_1\frac{\partial x}{\partial x_0}-C_2\frac{\partial x}{\partial y_0}-C_3\frac{\partial x}{\partial z_0}-C\frac{dx}{dt}\right)\mu_2dx_0dy_0dz_0dt$$
$$+\int_{t_1}^{t_2}\iiint_{s_0}\left(B_1\frac{\partial x}{\partial x_0}+B_2\frac{\partial x}{\partial y_0}+B_3\frac{\partial x}{\partial z_0}+B\frac{dx}{dt}\right.$$
$$\left.-A_1\frac{\partial y}{\partial x_0}-A_2\frac{\partial y}{\partial y_0}-A_3\frac{\partial y}{\partial z_0}-A\frac{dy}{dt}\right)\mu_3dx_0dy_0dz_0dt=0.$$

Let us determine the transform of this last relationship by taking the functions /166
x, y, z for the other new variables, while retaining t. Let us apply the

elementary formulas for the change of variables (see Section 53) to the functions $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$. With S still indicating the surface of medium (M) at instant t corresponding to surface S_0 of (M_0), let us additionally designate by X, Y, Z, L, M, N the projections upon Ox, Oy, Oz of the external force and external moment applied at point M at the time t referred to the unit volume of the deformed medium (M), and by F, G, H, I, J, K the projections on Ox, Oy, Oz of the stress and moment of deformation which are applied at the point M on S, referred to the unit of area of S. Finally, let us, just as in Section 53, introduce eighteen new auxiliary quantities $p_{xx}, \ldots, q_{xx}, \ldots$ by the formulas

$$\Delta p_{xx} = A_1 \frac{\partial x}{\partial x_0} + A_2 \frac{\partial x}{\partial y_0} + A_3 \frac{\partial x}{\partial z_0}, \quad \Delta q_{xx} = P_1 \frac{\partial x}{\partial x_0} + P_2 \frac{\partial x}{\partial y_0} + P_3 \frac{\partial x}{\partial z_0},$$

$$\Delta p_{yx} = A_1 \frac{\partial y}{\partial x_0} + A_2 \frac{\partial y}{\partial y_0} + A_3 \frac{\partial y}{\partial z_0}, \quad \Delta q_{yx} = P_1 \frac{\partial y}{\partial x_0} + P_2 \frac{\partial y}{\partial y_0} + P_3 \frac{\partial y}{\partial z_0},$$

$$\Delta p_{zx} = A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0}, \quad \Delta q_{zx} = P_1 \frac{\partial z}{\partial x_0} + P_2 \frac{\partial z}{\partial y_0} + P_3 \frac{\partial z}{\partial z_0},$$

and by similar ones obtained by replacing

$$A_1, A_2, A_3, p_{xx}, p_{yx}, p_{zx}, P_1, P_2, P_3, q_{xx}, q_{yx}, q_{zx},$$

respectively by

$$B_1, B_2, B_3, p_{xy}, p_{yy}, p_{zy}, Q_1, Q_2, Q_3, q_{xy}, q_{yy}, q_{zy},$$

then by

$$C_1, C_2, C_3, p_{xz}, p_{yz}, p_{zz}, R_1, R_2, R_3, q_{xz}, q_{yz}, q_{zz},$$

while the quantity Δ has the same expression as in Section 53. We obtain the transformed relationship

$$\begin{aligned}
&-\int_{t_1}^{t_2} \iiint_s (X\lambda_1 + Y\lambda_2 + Z\lambda_3 + L\mu_1 + M\mu_2 + N\mu_3)\, dxdydzdt \\
&+\int_{t_1}^{t_2} \iiint_s (F\lambda_1 + G\lambda_2 + H\lambda_3 + I\mu_1 + J\mu_2 + K\mu_3)\, dsdt \\
&+\left\{\iiint_s \left(\frac{A}{\Delta}\lambda_1 + \frac{B}{\Delta}\lambda_2 + \frac{C}{\Delta}\lambda_3 + \frac{P}{\Delta}\mu_1 + \frac{Q}{\Delta}\mu_2 + \frac{R}{\Delta}\mu_3\right) dxdydz\right\}_{t_1}^{t_2} \\
&-\int_{t_1}^{t_2} \iiint_s \left(p_{xx}\frac{\partial\lambda_1}{\partial x} + p_{yx}\frac{\partial\lambda_1}{\partial y} + p_{zx}\frac{\partial\lambda_1}{\partial z} + p_{xy}\frac{\partial\lambda_2}{\partial x} + \ldots + p_{zz}\frac{\partial\lambda_3}{\partial z}\right. \\
&\qquad\qquad \left. + \frac{A}{\Delta}\frac{d\lambda_1}{dt} + \frac{B}{\Delta}\frac{d\lambda_2}{dt} + \frac{C}{\Delta}\frac{d\lambda_3}{dt}\right) dxdydzdt \\
&-\int_{t_1}^{t_2} \iiint_s \left(q_{xx}\frac{\partial\mu_1}{\partial x} + q_{yx}\frac{\partial\mu_1}{\partial y} + q_{zx}\frac{\partial\mu_1}{\partial z} + q_{xy}\frac{\partial\mu_2}{\partial x} + \ldots + q_{zz}\frac{\partial\mu_3}{\partial z}\right. \\
&\qquad\qquad \left. + \frac{P}{\Delta}\frac{d\mu_1}{dt} + \frac{Q}{\Delta}\frac{d\mu_2}{dt} + \frac{R}{\Delta}\frac{d\mu_3}{dt}\right) dxdydzdt \\
&+\int_{t_1}^{t_2} \iiint_s \left\{\left(p_{yz} - p_{zy} + \frac{C}{\Delta}\frac{dy}{dt} - \frac{B}{\Delta}\frac{dz}{dt}\right)\mu_1 + \left(p_{zx} - p_{xz} + \frac{A}{\Delta}\frac{dz}{dt} - \frac{C}{\Delta}\frac{dx}{dt}\right)\mu_2\right. \\
&\qquad\qquad \left. + \left(p_{xy} - p_{yx} + \frac{B}{\Delta}\frac{dx}{dt} - \frac{A}{\Delta}\frac{dy}{dt}\right)\mu_3\right\} dxdydzdt = 0,
\end{aligned}$$

where the integrals are taken over the time interval between t_1 and t_2, and /167
over the surface S of the medium (M) at time t and over the region limited by it. $d\sigma$ designates the area element of S.

Let us apply Green's formula again to the terms that contain the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect to x, y, z, and let us integrate by parts the terms that contain the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect to t(24). Let us denote by l, m, n the direction cosines, with respect to the fixed axes, of the external normal to the surface S at the instant t. Since λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary, we obtain:

$$\begin{aligned}
F &= lp_{xx} + mp_{yx} + np_{zx}, & I &= lq_{xx} + mq_{yx} + nq_{zx},\\
G &= lp_{xy} + mp_{yy} + np_{zy}, & J &= lq_{xy} + mq_{yy} + nq_{zy},\\
H &= lp_{xz} + mp_{yz} + np_{zz}, & K &= lq_{xz} + mq_{yz} + nq_{zz},
\end{aligned}$$

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} + \frac{1}{\Delta}\frac{dA}{dt} - X = 0,$$

$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} + \frac{1}{\Delta}\frac{dB}{dt} - Y = 0,$$

$$\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} + \frac{1}{\Delta}\frac{dC}{dt} - Z = 0,$$

$$\frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + p_{yz} - p_{zy} + \frac{1}{\Delta}\frac{dP}{dt} + \frac{C}{\Delta}\frac{dy}{dt} - \frac{B}{\Delta}\frac{dz}{dt} - L = 0,$$

$$\frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z} + p_{zx} - p_{xz} + \frac{1}{\Delta}\frac{dQ}{dt} + \frac{A}{\Delta}\frac{dz}{dt} - \frac{C}{\Delta}\frac{dx}{dt} - M = 0,$$

$$\frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z} + p_{xy} - p_{yx} + \frac{1}{\Delta}\frac{dR}{dt} + \frac{B}{\Delta}\frac{dx}{dt} - \frac{A}{\Delta}\frac{dy}{dt} - N = 0.$$

The meaning of the eighteen new auxiliary quantities is p_{xx}, ..., q_{xx}, ... immediately obtained from the relationships that we just found. It is clear, in fact, that the coefficients p_{xx}, p_{xy}, p_{xz} of l in the expressions of F, G, H, represent the projections on Ox, Oy, Oz of the stress exerted at the point M onto a surface whose internal normal is parallel to Ox. It is also clear that the coefficients q_{xx}, q_{xy}, q_{xz} of l in the expressions of I, J, K, are the projections on Ox, Oy, Oz of the moment of deformation at M on the same surface.

64. <u>External Virtual Work; Theorem Analogous to those of Varignon and of Saint-Guilhem. Remarks on the Auxiliary Quantities Introduced in the Preceding Paragraphs</u>. We will call the following expression *the external virtual work* for the deformed medium (M) between the instants t_1 and t_2 during an arbitrary virtual deformation:

$$\begin{aligned}
\delta\mathcal{T}_e = &-\left\{\iiint_{\Delta_0}(A'\delta'x + B'\delta'y + C'\delta'z + P'\delta\iota' + Q'\delta\jmath' + R'\delta\kappa')dx_0dy_0dz_0\right\}_{t_1}^{t_2}\\
&-\int_{t_1}^{t_2}\iint_{s_0}(F_0'\delta'x + G_0'\delta'y + H_0'\delta'z + I_0'\delta\iota' + J_0'\delta\jmath' + K_0'\delta\kappa')d\sigma_0dt\\
&+\int_{t_1}^{t_2}\iiint_{\Delta_0}(X_0'\delta'x + Y_0'\delta'y + Z_0'\delta'z + L_0'\delta\iota' + M_0'\delta\jmath' + N_0'\delta\kappa')dx_0dy_0dz_0dt.
\end{aligned}$$

(24) Since the region of integration actually varies with t, we carry out the integration by parts by using the intermediate variables x_0, y_0, z_0. We assume that Δ is positive and equal to $|\Delta|$.

Let us refer back to the notation of Section 60, and moreover let us designate /168
by δI, δJ, δK the projections on fixed axes of the segment whose projections on Mx', My', Mz' are $\delta I'$, $\delta J'$, $\delta K'$, so that we have, for example:

$$-\delta I = \alpha'\delta\alpha' + \beta'\delta\beta' + \gamma'\delta\gamma' = -(\alpha'\delta\alpha'' + \beta'\delta\beta'' + \gamma'\delta\gamma''),$$

always assuming that the axes we are considering have the same configuration.

If that is the case, let us assume that in Section 63.3 we attribute to the arbitrary quantities λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 the meaning defined by the following formulas:

$$\lambda_1 = \delta x,\ \lambda_2 = \delta y,\ \lambda_3 = \delta z,\ \mu_1 = \delta I,\ \mu_2 = \delta J,\ \mu_3 = \delta K.$$

We then see that the relationships obtained previously for the new auxiliary quantities only express the following condition:

Whatever the virtual displacements of Section 60 associated with the trajectory of the deformed medium may be, the external virtual work $\delta\mathcal{T}_e$ is given either by the relationship

$$\begin{aligned}
-\delta\mathcal{T}_e = &\int_{t_1}^{t_2}\iiint_s \Big(p_{xx}\frac{\partial\delta x}{\partial x} + p_{yx}\frac{\partial\delta x}{\partial y} + p_{zx}\frac{\partial\delta x}{\partial z} + p_{xy}\frac{\partial\delta y}{\partial x} + \dots + p_{zz}\frac{\partial\delta z}{\partial z} \\
&\qquad + \frac{A}{\Delta}\frac{d\delta x}{dt} + \frac{B}{\Delta}\frac{d\delta y}{dt} + \frac{C}{\Delta}\frac{d\delta z}{dt}\Big)\,dx\,dy\,dz\,dt \\
&+ \int_{t_1}^{t_2}\iiint_s \Big(q_{xx}\frac{\partial\delta I}{\partial x} + q_{yx}\frac{\partial\delta I}{\partial y} + q_{zx}\frac{\partial\delta I}{\partial z} + q_{xy}\frac{\partial\delta J}{\partial x} + \dots + q_{zz}\frac{\partial\delta K}{\partial z} \\
&\qquad + \frac{P}{\Delta}\frac{d\delta I}{dt} + \frac{Q}{\Delta}\frac{d\delta J}{dt} + \frac{R}{\Delta}\frac{d\delta K}{dt}\Big)\,dx\,dy\,dz\,dt \\
&- \int_{t_1}^{t_2}\iiint_s \Big\{\Big(p_{yz} - p_{zy} + \frac{C}{\Delta}\frac{dy}{dt} - \frac{B}{\Delta}\frac{dz}{dt}\Big)\delta I + \Big(p_{zx} - p_{xz} + \frac{A}{\Delta}\frac{dz}{dt} - \frac{C}{\Delta}\frac{dx}{dt}\Big)\delta J \\
&\qquad + \Big(p_{xy} - p_{yx} + \frac{B}{\Delta}\frac{dx}{dt} - \frac{A}{\Delta}\frac{dy}{dt}\Big)\delta K\Big\}\,dx\,dy\,dz\,dt,
\end{aligned}$$

where the integrals are extended over the time interval between the times t_1 and t_2 and over the deformed medium, or by the relationship

$$\begin{aligned}
-\delta\mathcal{T}_e = &\int_{t_1}^{t_2}\iiint_{s_0} \Big(A_1\frac{\partial\delta x}{\partial x_0} + A_2\frac{\partial\delta x}{\partial y_0} + A_3\frac{\partial\delta x}{\partial z_0} + B_1\frac{\partial\delta y}{\partial x_0} + \dots + C_3\frac{\partial\delta z}{\partial z_0} \\
&\qquad + A\frac{d\delta x}{dt} + B\frac{d\delta y}{dt} + C\frac{d\delta z}{dt}\Big)\,dx_0\,dy_0\,dz_0\,dt \\
&+ \int_{t_1}^{t_2}\iiint_{s_0} \Big(P_1\frac{\partial\delta I}{\partial x_0} + P_2\frac{\partial\delta I}{\partial y_0} + P_3\frac{\partial\delta I}{\partial z_0} + Q_1\frac{\partial\delta J}{\partial x_0} + \dots + R_3\frac{\partial\delta K}{\partial z_0} \\
&\qquad + P\frac{d\delta I}{dt} + Q\frac{d\delta J}{dt} + R\frac{d\delta K}{dt}\Big)\,dx_0\,dy_0\,dz_0\,dt \\
&- \int_{t_1}^{t_2}\iiint_{s_0} \Big(C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} + C\frac{dy}{dt} \\
&\qquad - B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial y_0} - B_3\frac{\partial z}{\partial z_0} - B\frac{dz}{dt}\Big)\,\delta I\,dx_0\,dy_0\,dz_0\,dt \\
&- \int_{t_1}^{t_2}\iiint_{s_0} \Big(A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} + A\frac{dz}{dt} \\
&\qquad - C_1\frac{\partial x}{\partial x_0} - C_2\frac{\partial x}{\partial y_0} - C_3\frac{\partial x}{\partial z_0} - C\frac{dx}{dt}\Big)\,\delta J\,dx_0\,dy_0\,dz_0\,dt \\
&- \int_{t_1}^{t_2}\iiint_{s_0} \Big(B_1\frac{\partial x}{\partial x_0} + B_2\frac{\partial x}{\partial y_0} + B_3\frac{\partial x}{\partial z_0} + B\frac{dx}{dt}
\end{aligned}$$

$$-A_1\frac{\partial y}{\partial x_0}-A_2\frac{\partial y}{\partial y_0}-A_3\frac{\partial y}{\partial z_0}-A\frac{dy}{dt}\Big)\delta\kappa\, dx_0 dy_0 dz_0 dt$$

where the integrals are extended over the time interval between t_1 *and* t_2 *and* /169
over the non-deformed medium at the instant t, *because the formula:*

$$\delta\mathfrak{T}_e=-\left\{\iiint_{s_0}(A'\delta'x+B'\delta'y+C'\delta'z+P'\delta\iota'+Q'\delta\jmath'+R'\delta\kappa')dx_0dy_0dz_0\right\}_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\iint_{s_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta\iota'+J_0'\delta\jmath'+K_0'\delta\kappa')d\sigma_0 dt$$
$$+\int_{t_1}^{t_2}\iiint_{s_0}(X_0'\delta'x+Y_0'\delta'y+Z_0'\delta'z+L_0'\delta\iota'+M_0'\delta\jmath'+N_0'\delta\kappa')dx_0dy_0dz_0dt,$$

given eariler as the definition of external virtual work, can also be written as:

$$\delta\mathfrak{T}_e=-\left\{\iiint_{s_0}(A\delta x+B\delta y+C\delta z+P\delta\iota+Q\delta\jmath+R\delta\kappa)\,dx_0dy_0dz_0\right\}_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\iint_{s_0}(F_0\delta x+G_0\delta y+H_0\delta z+I_0\delta\iota+J_0\delta\jmath+K_0\delta\kappa)d\sigma_0 dt$$
$$+\int_{t_1}^{t_2}\iiint_{s_0}(X_0\delta x+Y_0\delta y+Z_0\delta z+L_0\delta\iota+M_0\delta\jmath+N_0\delta\kappa)dx_0dy_0dz_0dt,$$

by virtue of the meaning of X_0, Y_0, Z_0, L_0, M_0, N_0, F_0, G_0, H_0, I_0, J_0, K_0, A, B, C, P, Q, R and similarly as:

$$\delta\mathfrak{T}_e=-\left\{\iiint_s\left(\frac{A}{\Delta}\delta x+\frac{B}{\Delta}\delta y+\frac{C}{\Delta}\delta z+\frac{P}{\Delta}\delta\iota+\frac{Q}{\Delta}\delta\jmath+\frac{R}{\Delta}\delta\kappa\right)dxdydz\right\}_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\iint_s(F\delta x+G\delta y+H\delta z+I\delta\iota+J\delta\jmath+K\delta\kappa)d\sigma dt$$
$$+\int_{t_1}^{t_2}\iiint_s(X\delta x+Y\delta y+Z\delta z+L\delta\iota+M\delta\jmath+N\delta\kappa)dxdydzdt,$$

by virtue of the meaning of X, Y, .., N, F, G, ..., K.

Let us start with the formula

$$\int_{t_1}^{t_2}\iiint_{s_0}\delta W dx_0dy_0dz_0dt+\delta\mathfrak{T}_e=0,$$

applied to a certain portion of the medium bounded by a surface S_0 and to the time interval between times t_1 and t_2. Since δW has to be identically zero, by virtue of the invariance of W for the group of Euclidean displacements, when the variations δx, δy, δz are given by the formulas (60) of Section 61, i.e.:

$$\delta x=(a_1+\omega_2 z-\omega_3 y)\delta t,$$
$$\delta y=(a_2+\omega_3 x-\omega_1 z)\delta t,$$
$$\delta z=(a_3+\omega_1 y-\omega_2 x)\delta t,$$

and δI, δJ, δK are given by /170

$$\delta I = \omega_1 \delta t, \qquad \delta J = \omega_2 \delta t, \qquad \delta K = \omega_3 \delta t,$$

whatever the values of the constants a_1, a_2, a_3, ω_1, ω_2, ω_3 may be, we reach the conclusion that we have the following, based on the expressions for $\delta \mathcal{C}_e$ that we just derived(25):

$$\left\{\iiint_{s_0} A\,dx_0 dy_0 dz_0\right\}_{t_1}^{t_2} + \int_{t_1}^{t_2} \iint_{s_0} F_0 dz_0 dt - \int_{t_1}^{t_2} \iiint_{s_0} X_0 dx_0 dy_0 dz_0 dt = 0,$$

$$\left\{\iiint_{s_0} (P + Cy - Bz) dx_0 dy_0 dz_0\right\}_{t_1}^{t_2} + \int_{t_1}^{t_2} \iint_{s_0} (I_0 + H_0 y - G_0 z) d\sigma_0 dt$$

$$- \int_{t_1}^{t_2} \iiint_{s_0} (L_0 + Z_0 y - Y_0 z) dx_0 dy_0 dz_0 dt = 0,$$

and four analogous equations. *It is conceivable, in these formulas, that the boundary* S_0 *is variable.*

The auxiliary quantities that we introduced in the previous paragraphs are not the only ones we could consider. In discussing these auxiliary quantities, we will add the same remarks as in Section 54.

We have introduced in a definite way two systems of stresses and of moments of deformation relative to a point M of the deformed medium at the time t. The first are those that are exerted on surfaces which, before deformation, have their normal parallel to one of the fixed axes Ox, Oy, Oz; the second are those which are exerted on surfaces that have their normal parallel to one of the same fixed axes Ox, Oy, Oz. The formulas that we have indicated give the last in terms of the first. However, by a very straightforward solution, on which we are not going to dwell, we obtain conversely the first elements in terms of the second ones.

Let us now assume that we introduce the function W; the first stresses and deformation moments correspond to the expressions already indicated, and we immediately derive the expressions for the second ones. In these calculations, however, we can state explicitely the functions we are interested in, according to the nature of the problem, which will be, *for example*, x, y, z and three parameters(26) λ_1, λ_2, λ_3 in terms of which we express α, α', ... γ''.

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(25) The transition from the elements referred to the unit volume of the non-deformed medium and to the unit area of the boundary S_0, to the elements referred to the unit volume of the deformed medium and to the unit area of the boundary S at time t, is so immediate that it is sufficient to restrict ourselves to the former, for example, as we are doing.

(26) As such auxiliary quantities λ_1, λ_2, λ_3 we can take, for example, the components of the rotation that makes the axes Ox, Oy, Oz parallel to Mx', My', Mz', respectively.

If we introduce x, y, z, λ_1, λ_2, λ_3, and if we continue to denote by W the function depending on x_0, y_0, z_0, on the first derivatives of x, y, z with respect to x_0, y_0, z_0, t, on λ_1, λ_2, λ_3 and on their first derivatives with respect to x_0, y_0, z_0, t -- with this function obtained by replacing the different quantities ξ_i, η_i, ζ_i, p_i, q_i, r_i, ξ, η, ζ, p, q, r by their values given /171 by the formulas (54), (55) and (54'), (55') into the function $W(x_0, y_0, z_0, t, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i, \xi, \eta, \zeta, p, q, r)$-- we will have:

$$A_1 = \frac{\partial W}{\partial \frac{\partial x}{\partial x_0}}, \quad A_2 = \frac{\partial W}{\partial \frac{\partial x}{\partial y_0}}, \quad A_3 = \frac{\partial W}{\partial \frac{\partial x}{\partial z_0}}, \quad A = \frac{\partial W}{\partial \frac{dx}{dt}},$$

$$B_1 = \frac{\partial W}{\partial \frac{\partial y}{\partial x_0}}, \quad B_2 = \frac{\partial W}{\partial \frac{\partial y}{\partial y_0}}, \quad B_3 = \frac{\partial W}{\partial \frac{\partial y}{\partial z_0}}, \quad B = \frac{\partial W}{\partial \frac{dy}{dt}},$$

$$C_1 = \frac{\partial W}{\partial \frac{\partial z}{\partial x_0}}, \quad C_2 = \frac{\partial W}{\partial \frac{\partial z}{\partial y_0}}, \quad C_3 = \frac{\partial W}{\partial \frac{\partial z}{\partial z_0}}, \quad C = \frac{\partial W}{\partial \frac{dz}{dt}}.$$

$$\Pi_i = \frac{\partial W}{\partial \frac{\partial \lambda_i}{\partial p_i}}, \quad X_i = \frac{\partial W}{\partial \frac{\partial \lambda_i}{\partial q_i}}, \quad \Sigma_i = \frac{\partial W}{\partial \frac{\partial \lambda_i}{\partial r_i}},$$

$$\Pi = \frac{\partial W}{\partial \frac{d\lambda_1}{dt}}, \quad X = \frac{\partial W}{\partial \frac{d\lambda_2}{dt}}, \quad \Sigma = \frac{\partial W}{\partial \frac{d\lambda_3}{dt}}.$$

65. Concept of the Energy of Deformation and of Motion. We would like to note that the present discussion contains the statics of deformable bodies as a special case. It is sufficient, in fact, to consider a *virtual reversible modification of the action*, in the sense of M. Duhem, instead of regarding, as we just did, a *feasible virtual modification*.

This observation will lead us to the concept of the energy of deformation and motion. We intend to determine, for a *real modification*, the work of external forces and moments, and of stresses and moments of deformation during a certain time interval. For this purpose, it is sufficient to calculate the elementary work corresponding to the time dt. This is equal to

$$\left\{ \iiint_{s_0} (\xi X_0' + \eta Y_0' + \ldots)\, dx_0 dy_0 dz_0 - \iint_{s_0} (\xi F_0' + \eta G_0' + \ldots)\, ds \right\} dt;$$

If we replace X_0', Y_0', ..., F_0', G_0', ... by their expressions as functions of the action, and if we perform a calculation which is the inverse of the one that led to their definition, we immediately obtain, by virtue of Codazzi's equations:

$$\left\{ \iiint_{s_0} \left(\frac{dE}{dt} + \frac{\partial W}{\partial t} \right) dx_0 dy_0 dz_0 \right\} dt,$$

by letting

$$E = \xi \frac{\partial W}{\partial \xi} + \eta \frac{\partial W}{\partial \eta} + \zeta \frac{\partial W}{\partial \zeta} + p \frac{\partial W}{\partial p} + q \frac{\partial W}{\partial q} + r \frac{\partial W}{\partial r} - W.$$

If we consider in particular the case in which W does not contain t explicitly, so that $\frac{\partial W}{\partial t}$ is zero, the previous value becomes the time differential /172
with respect to time of the expression:

$$\iiint_{0} E dx_0 dy_0 dz_0,$$

that can be called the *energy of deformation and of motion at the time* t.

At the point where we have just arrived, we can make some important general remarks which will again be applied in the theory of Euclidean action.

The concept of the Euclidean action of deformation and of motion itself is sufficient to provide us, in a very extended case, with a *constructive* definition of momentum and of the moment of momentum, of the stress and deformation moment, of the external forces and moments. We can distinguish between a dynamic part and a static part in the external force and moment, by grouping the terms that contain the dynamic acceleration in one part, and the terms that contain what we could call the *kinematic acceleration* in the other part. This distinction expresses an extension of d'Alembert's principle. Likewise, let us assume that the external work is zero, and the energy of deformation and of motion remains invariable in time. We thus obtain the concept of *conservation of energy*, which may be simply translated into the hypothesis that the medium is *isolated* from the external world. Consequently, we will find again all the fundamental ideas of classical mechanics, and it is plain that the particular form they take on in it arises from the fact that we consider only the state of motion and of deformation *infinitely close to the natural state*, when we assume that W and all its derivatives are zero.

66. Initial State and Natural States. General Ideas on the Problems to which we are Led by Considering the Deformable Medium. So far we have considered the trajectory of the deformed state, and, after considering the *initial position* (M_0) at a determined instant t_0 of the deformed state, we referred to its position (M) at some instant t. Considerations analogous to those developed in Section 56, in which the parameter introduced is actually replaced by the time t, can be resumed here in order to have one of the deformed states play the role that we assigned to the initial state (M_0).

We can also conceive, however, that the functions x, y, z, ..., determining the trajectory of the deformed state depend on a parameter, and that we can distinguish a particular value of this parameter. Likewise, we will define a series of states that we can call *natural states*, and their trajectory can be called the *trajectory of the natural states*. We will be able to use our new parameter as we have done in our *Note on the Dynamics of the Point and of the Invariable Body* and can study in particular the trajectory of deformed states which is infinitely close to the trajectory of natural states.

Based on the preceding considerations, let us assume that the external /173
force and moments are, by way of an example, given in terms of simple functions of x_0, y_0, z_0, t and of the elements that define the position of the triad Mx'y'z.

We can consider the equations of Section 62 for the external force and moment as partial differential equations with respect to x, y, z with three parameters λ_1, λ_2, λ_3, by means of which we express α, α', ... γ''. This point of view is one of those that follows most naturally. The expressions ξ_i, η_i, ζ_i, p_i, q_i, r_i, ξ, η, ζ, p, q, r will be functions of $\frac{\partial x}{\partial \rho_i}$, $\frac{\partial y}{\partial \rho_i}$, $\frac{\partial z}{\partial \rho_i}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, λ_1, ... , $\frac{\partial \lambda_1}{\partial \rho_i}$, ... , $\frac{d\lambda_1}{dt}$, ... (by setting always $\rho_1 = x_0$, $\rho_2 = y_0$, $\rho_3 = z_0$) which we will calculate by means of the formulas (54), (55) and (54'), (55').

Let us assume that X_0', Y_0', Z_0', L_0', M_0', N_0', or, which is the same, X_0, Y_0, Z_0, L_0, M_0, N_0, are given functions of x_0, y_0, z_0, t, x, y, z, λ_1, λ_2, λ_3. The expression W is, after substitution of the values of ξ_i, ..., r_i, ξ, ..., r taken from the formulas (54), (55) and (54'), (55'), a determined function of

$$x_0, y_0, z_0, t, \frac{\partial x}{\partial x_0}, \ldots, \frac{\partial z}{\partial z_0}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \lambda_1, \lambda_2, \lambda_3, \frac{\partial \lambda_1}{\partial z_0}, \ldots, \frac{\partial \lambda_3}{\partial z_0}, \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, -$$

which we will continue to designate by W, and the equations of the problem can be written:

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial x}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial x}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial x}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{dx}{dt}} = X_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial y}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial y}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial y}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{dy}{dt}} = Y_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{dz}{dt}} = Z_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_1}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{d\lambda_1}{dt}} - \frac{\partial W}{\partial \lambda_1} = \mathfrak{L}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_2}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{d\lambda_2}{dt}} - \frac{\partial W}{\partial \lambda_2} = \mathfrak{M}_0,$$

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial \lambda_3}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial \frac{d\lambda_3}{dt}} - \frac{\partial W}{\partial \lambda_3} = \mathfrak{N}_0,$$

$\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ being the functions of x_0, y_0, z_0, t, x, y, z, λ_1, λ_2, λ_3 that result from the definitions of Section 63. This follows directly from the formulas of the preceding paragraphs, in any case in a more immediate fashion than the defining equations of X_0, Y_0, Z_0, $\mathfrak{L}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ can be summarized in /174 the relationship

$$\delta \int_{t_1}^{t_2} \iiint_{\tau_0} W dx_0 dy_0 dz_0 dt + \delta \mathfrak{T}_\varepsilon = 0,$$

that is, in

$$\delta\int_{t_1}^{t_2}\iiint_{v_0} W dx_0 dy_0 dz_0 dt$$
$$=\left\{\iiint_{v_0}(A\delta x + B\delta y + C\delta z + P\delta\lambda_1 + Q\delta\lambda_2 + R\delta\lambda_3)dx_0 dy_0 dz_0\right\}_{t_1}^{t_2}$$
$$+\int_{t_1}^{t_2}\iint_{s_0}(F_0\delta x + G_0\delta y + H_0\delta z + \mathfrak{I}_0\delta\lambda_1 + \mathfrak{J}_0\delta\lambda_2 + \mathfrak{K}_0\delta\lambda_3)d\sigma_0 dt$$
$$-\int_{t_1}^{t_2}\iiint_{v_0}(X_0\delta x + Y_0\delta y + Z_0\delta z + \mathfrak{L}_0\delta\lambda_1 + \mathfrak{M}_0\delta\lambda_2 + \mathfrak{N}_0\delta\lambda_3)dx_0 dy_0 dz_0 dt.$$

67. Concept of the Concealed Triad and of Concealed W. Case in which W Depends only on x_0, y_0, z_0, t, ξ_i, η_i, ζ_i, ξ, η, ζ and in Independent of p_i, q_i, r_i, p, q, r. Extension of Classical Dynamics of the Deformable Body. The Gyrostatic Medium and Kinetic Anisotropy. The considerations that we presented earlier in regard to the concealed triad and concealed W can still be applied to the deformable medium in motion. It only suffices to add that the concealed W can correspond to a concealed motion.

We will examine in particular the case in which W depends only on the quantities x_0, y_0, z_0, t, ξ_i, η_i, ζ_i, ξ, η, ζ and not on p_i, q_i, r_i, p, q, r. Then the equations of Section 66 may be reduced to the following:

$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial\frac{\partial x}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial\frac{\partial x}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial\frac{\partial x}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial\frac{dx}{dt}} = X_0, \qquad \frac{\partial W}{\partial\lambda_1} + \mathfrak{L}_0 = 0,$$
$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial\frac{\partial y}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial\frac{\partial y}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial\frac{\partial y}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial\frac{dy}{dt}} = Y_0, \qquad \frac{\partial W}{\partial\lambda_2} + \mathfrak{M}_0 = 0,$$
$$\frac{\partial}{\partial x_0}\frac{\partial W}{\partial\frac{\partial z}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial\frac{\partial z}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial\frac{\partial z}{\partial z_0}} + \frac{d}{dt}\frac{\partial W}{\partial\frac{dz}{dt}} = Z_0, \qquad \frac{\partial W}{\partial\lambda_3} + \mathfrak{N}_0 = 0,$$

in which W depends only on x_0, y_0, z_0, t, $\frac{\partial x}{\partial x_0}$, ..., $\frac{\partial z}{\partial z_0}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, λ_1, λ_2, λ_3. They show that, if we take the simple case in which X_0, Y_0, Z_0, $\mathfrak{K}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are given functions[27] of x_0, y_0, z_0, t, x, y, z, $\frac{\partial x}{\partial x_0}$, ..., $\frac{\partial z}{\partial z_0}$, /175
$\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, λ_1, λ_2, λ_3, the three equations on the right can be solved with respect to λ_1, λ_2, λ_3. Thus, we obtain finally three partial differential equations containing, following our assumptions, only x_0, y_0, z_0, t and also x, y, z

(27) We assume that X_0, Y_0, Z_0, $\mathfrak{K}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ do not contain the derivatives of λ_1, λ_1, λ_3, in order to simplify the discussion, and in order to show more easily that to which we refer.

and their derivatives of first and second order.

Let us regard the particular case in which the given functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are zero. Then the same will be true of the corresponding values of the functions of a certain one of the systems (L_0', M_0', N_0'), (L_0, M_0, N_0), (L, M, N). As a result, the equations

$$\frac{\partial W}{\partial \lambda_1} = 0, \qquad \frac{\partial W}{\partial \lambda_2} = 0, \qquad \frac{\partial W}{\partial \lambda_3} = 0$$

become

$$C_1 \frac{\partial y}{\partial x_0} + C_2 \frac{\partial y}{\partial y_0} + C_3 \frac{\partial y}{\partial z_0} - B_1 \frac{\partial z}{\partial x_0} - B_2 \frac{\partial z}{\partial y_0} - B_3 \frac{\partial z}{\partial z_0} = B \frac{dz}{dt} - C \frac{dy}{dt},$$
$$A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0} - C_1 \frac{\partial x}{\partial x_0} - C_2 \frac{\partial x}{\partial y_0} - C_3 \frac{\partial x}{\partial z_0} = C \frac{dx}{dt} - A \frac{dz}{dt},$$
$$B_1 \frac{\partial x}{\partial x_0} + B_2 \frac{\partial x}{\partial y_0} + B_3 \frac{\partial x}{\partial z_0} - A_1 \frac{\partial y}{\partial x_0} - A_2 \frac{\partial y}{\partial y_0} - A_3 \frac{\partial y}{\partial z_0} = A \frac{dy}{dt} - B \frac{dx}{dt},$$

that is,

$$p_{yz} - p_{zy} = \frac{1}{\Delta}\left(B \frac{dz}{dt} - C \frac{dy}{dt}\right), \qquad p_{zx} - p_{xz} = \frac{1}{\Delta}\left(C \frac{dx}{dt} - A \frac{dz}{dt}\right),$$
$$p_{xy} - p_{yx} = \frac{1}{\Delta}\left(A \frac{dy}{dt} - B \frac{dx}{dt}\right),$$

which we can interpret by saying that the motion, in the deformable body under consideration which encompasses the classical theory of elasticity as a special case, creates a *moment* the three components of which are:

$$\frac{1}{\Delta}\left(B \frac{dz}{dt} - C \frac{dy}{dt}\right), \quad \frac{1}{\Delta}\left(C \frac{dx}{dt} - A \frac{dz}{dt}\right), \quad \frac{1}{\Delta}\left(A \frac{dy}{dt} - B \frac{dx}{dt}\right),$$

and which has the effect of destroying the equalities

$$p_{yz} = p_{zy}, \quad p_{zx} = p_{xz}, \quad p_{xy} = p_{yx}.$$

Under this assumption, we observe that if we start with a trajectory assumed to be given and if we deduce, as in Section 63, the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, it can happen, in the case where the three functions are zero, that this result is obtained accidentally -- that is, for a certain set of particular trajectories. It can also happen, however, that this is produced for any trajectory (M), as a consequence of the nature of the media (M) and of their motions -- that is, as a consequence of the form of W.

Let us consider this last case, which is particularly interesting. W is then a simple function[28] of x_0, y_0, z_0, t and of the ten expressions ε_1, ε_2, ε_3, γ_1, γ_2, γ_3, ϕ_1, ϕ_2, ϕ_3, v^2 defined by the following formulas: /176

(28) The triad is completely concealed; we can also conceive of the case in which the medium is simply a point.

$$\varepsilon_1 = \frac{1}{2}\left\{\left(\frac{\partial x}{\partial x_0}\right)^2 + \left(\frac{\partial y}{\partial x_0}\right)^2 + \left(\frac{\partial z}{\partial x_0}\right)^2 - 1\right\} = \frac{1}{2}(\xi_1^2 + \eta_1^2 + \zeta_1^2 - 1),$$

$$\varepsilon_2 = \frac{1}{2}\left\{\left(\frac{\partial x}{\partial y_0}\right)^2 + \left(\frac{\partial y}{\partial y_0}\right)^2 + \left(\frac{\partial z}{\partial y_0}\right)^2 - 1\right\} = \frac{1}{2}(\xi_2^2 + \eta_2^2 + \zeta_2^2 - 1),$$

$$\varepsilon_3 = \frac{1}{2}\left\{\left(\frac{\partial x}{\partial z_0}\right)^2 + \left(\frac{\partial y}{\partial z_0}\right)^2 + \left(\frac{\partial z}{\partial z_0}\right)^2 - 1\right\} = \frac{1}{2}(\xi_3^2 + \eta_3^2 + \zeta_3^2 - 1),$$

$$\gamma_1 = \frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0} + \frac{\partial y}{\partial y_0}\frac{\partial y}{\partial z_0} + \frac{\partial z}{\partial y_0}\frac{\partial z}{\partial z_0} = \xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3,$$

$$\gamma_2 = \frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0} + \frac{\partial y}{\partial z_0}\frac{\partial y}{\partial x_0} + \frac{\partial z}{\partial z_0}\frac{\partial z}{\partial x_0} = \xi_3\xi_1 + \eta_3\eta_1 + \zeta_3\zeta_1,$$

$$\gamma_3 = \frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0} + \frac{\partial y}{\partial x_0}\frac{\partial y}{\partial y_0} + \frac{\partial z}{\partial x_0}\frac{\partial z}{\partial y_0} = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2,$$

$$\varphi_1 = \frac{dx}{dt}\frac{\partial x}{\partial x_0} + \frac{dy}{dt}\frac{\partial y}{\partial x_0} + \frac{dz}{dt}\frac{\partial z}{\partial x_0} = \xi\,\xi_1 + \eta\,\eta_1 + \zeta\,\zeta_1,$$

$$\varphi_2 = \frac{dx}{dt}\frac{\partial x}{\partial y_0} + \frac{dy}{dt}\frac{\partial y}{\partial y_0} + \frac{dz}{dt}\frac{\partial z}{\partial y_0} = \xi\,\xi_2 + \eta\,\eta_2 + \zeta\,\zeta_2,$$

$$\varphi_3 = \frac{dx}{dt}\frac{\partial x}{\partial z_0} + \frac{dy}{dt}\frac{\partial y}{\partial z_0} + \frac{dz}{dt}\frac{\partial z}{\partial z_0} = \xi\,\xi_3 + \eta\,\eta_3 + \zeta\,\zeta_3,$$

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \xi^2 + \eta^2 + \zeta^2.$$

The equations deduced from Sections 62 and 63 reduce either to

$$\sum\left(\frac{\partial A_i'}{\partial p_i} + q_iC_i' - r_iB_i'\right) + \frac{dA'}{dt} + qC' - rB' = X_0', \qquad F_0' = l_0A_1' + m_0A_2' + n_0A_3',$$

$$\sum\left(\frac{\partial B_i'}{\partial p_i} + r_iA_i' - p_iC_i'\right) + \frac{dB'}{dt} + rA' - pC' = Y_0', \qquad G_0' = l_0B_1' + m_0B_2' + n_0B_3',$$

$$\sum\left(\frac{\partial C_i'}{\partial p_i} + p_iB_i' - q_iA_i'\right) + \frac{dC'}{dt} + pB' - qA' = Z_0', \qquad H_0' = l_0C_1' + m_0C_2' + n_0C_3',$$

where we have

$$A_i' = \xi_i\frac{\partial W}{\partial \varepsilon_i} + \xi_k\frac{\partial W}{\partial \gamma_j} + \xi_j\frac{\partial W}{\partial \gamma_k} + \xi\frac{\partial W}{\partial \varphi_i},$$

$$B_i' = \eta_i\frac{\partial W}{\partial \varepsilon_i} + \eta_k\frac{\partial W}{\partial \gamma_j} + \eta_j\frac{\partial W}{\partial \gamma_k} + \eta\frac{\partial W}{\partial \varphi_i}, \qquad (i, j, k = 1, 2, 3),$$

$$C_i' = \zeta_i\frac{\partial W}{\partial \varepsilon_i} + \zeta_k\frac{\partial W}{\partial \gamma_j} + \zeta_j\frac{\partial W}{\partial \gamma_k} + \zeta\frac{\partial W}{\partial \varphi_i},$$

$$A' = \frac{1}{v}\frac{\partial W}{\partial v}\xi + \sum \xi_i\frac{\partial W}{\partial \varphi_i},$$

$$B' = \frac{1}{v}\frac{\partial W}{\partial v}\eta + \sum \eta_i\frac{\partial W}{\partial \varphi_i},$$

$$C' = \frac{1}{v}\frac{\partial W}{\partial v}\zeta + \sum \zeta_i\frac{\partial W}{\partial \varphi_i},$$

or to /177/

$$\frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0} + \frac{dA}{dt} = X_0, \qquad F_0 = l_0A_1 + m_0A_2 + n_0A_3,$$

$$\frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0} + \frac{dB}{dt} = Y_0, \qquad G_0 = l_0B_1 + m_0B_2 + n_0B_3,$$

$$\frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0} + \frac{dC}{dt} = Z_0, \qquad H_0 = l_0C_1 + m_0C_2 + n_0C_3,$$

where we have:

$$A_1 = \Omega_1 \frac{\partial x}{\partial x_0} + \Xi_1 \frac{\partial x}{\partial y_0} + \Xi_2 \frac{\partial x}{\partial z_0} + \Phi_1 \frac{dx}{dt},$$
$$B_1 = \Omega_1 \frac{\partial y}{\partial x_0} + \Xi_1 \frac{\partial y}{\partial y_0} + \Xi_2 \frac{\partial y}{\partial z_0} + \Phi_1 \frac{dy}{dt},$$
$$C_1 = \Omega_1 \frac{\partial z}{\partial x_0} + \Xi_1 \frac{\partial z}{\partial y_0} + \Xi_2 \frac{\partial z}{\partial z_0} + \Phi_1 \frac{dz}{dt},$$

with analogous expressions for A_2, B_2, C_2, A_3, B_3, C_3, and

$$A = \Phi_1 \frac{\partial x}{\partial x_0} + \Phi_2 \frac{\partial x}{\partial y_0} + \Phi_3 \frac{\partial x}{\partial z_0} + \frac{1}{v} \frac{\partial W}{\partial v} \frac{dx}{dt},$$
$$B = \Phi_1 \frac{\partial y}{\partial x_0} + \Phi_2 \frac{\partial y}{\partial y_0} + \Phi_3 \frac{\partial y}{\partial z_0} + \frac{1}{v} \frac{\partial W}{\partial v} \frac{dy}{dt},$$
$$C = \Phi_1 \frac{\partial z}{\partial x_0} + \Phi_2 \frac{\partial z}{\partial y_0} + \Phi_3 \frac{\partial z}{\partial z_0} + \frac{1}{v} \frac{\partial W}{\partial v} \frac{dz}{dt},$$

by letting

$$\Omega_i = \frac{\partial W}{\partial \varepsilon_i}, \qquad \Xi_i = \frac{\partial W}{\partial \gamma_i}, \qquad \Phi_i = \frac{\partial W}{\partial \varphi_i},$$

or even to

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} + \frac{1}{\Delta} \frac{dA}{dt} = X, \qquad F = lp_{xx} + mp_{yx} + np_{zx},$$
$$\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} + \frac{1}{\Delta} \frac{dB}{dt} = Y, \qquad G = lp_{xy} + mp_{yy} + np_{zy},$$
$$\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} + \frac{1}{\Delta} \frac{dC}{dt} = Z, \qquad H = lp_{xz} + mp_{yz} + np_{zz},$$

where we have:

$$p_{xx} = \frac{1}{\Delta}\left\{\Omega_1\left(\frac{\partial x}{\partial x_0}\right)^2 + \Omega_2\left(\frac{\partial x}{\partial y_0}\right)^2 + \Omega_3\left(\frac{\partial x}{\partial z_0}\right)^2 + 2\Xi_1\frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0} + 2\Xi_2\frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0} + 2\Xi_3\frac{\partial x}{\partial x_0}\frac{\partial x}{\partial y_0} + \left(\Phi_1\frac{\partial x}{\partial x_0} + \Phi_2\frac{\partial x}{\partial y_0} + \Phi_3\frac{\partial x}{\partial z_0}\right)\frac{dx}{dt}\right\},$$

$$p_{yx} = \frac{1}{\Delta}\left\{\Omega_1\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial x_0} + \Omega_2\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial y_0} + \Omega_3\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial z_0} + \Xi_1\left(\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial z_0} + \frac{\partial x}{\partial z_0}\frac{\partial y}{\partial y_0}\right) + \Xi_2\left(\frac{\partial x}{\partial z_0}\frac{\partial y}{\partial x_0} + \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial z_0}\right) + \Xi_3\left(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0}\right) + \left(\Phi_1\frac{\partial y}{\partial x_0} + \Phi_2\frac{\partial y}{\partial y_0} + \Phi_3\frac{\partial y}{\partial z_0}\right)\frac{dx}{dt}\right\},$$

$$p_{zx} = \frac{1}{\Delta}\left\{\Omega_1\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial x_0} + \Omega_2\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial y_0} + \Omega_3\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial z_0} + \Xi_1\left(\frac{\partial z}{\partial y_0}\frac{\partial x}{\partial z_0} + \frac{\partial z}{\partial z_0}\frac{\partial x}{\partial y_0}\right) + \Xi_2\left(\frac{\partial z}{\partial z_0}\frac{\partial x}{\partial x_0} + \frac{\partial z}{\partial x_0}\frac{\partial x}{\partial z_0}\right) + \Xi_3\left(\frac{\partial z}{\partial x_0}\frac{\partial x}{\partial y_0} + \frac{\partial z}{\partial y_0}\frac{\partial x}{\partial x_0}\right) + \left(\Phi_1\frac{\partial z}{\partial x_0} + \Phi_2\frac{\partial z}{\partial y_0} + \Phi_3\frac{\partial z}{\partial z_0}\right)\frac{dx}{dt}\right\},$$

with similar expressions for p_{xy}, p_{yy}, p_{zy}, p_{xz}, p_{yz}, p_{zz}. We thus obtain the /178 most general equations for the motion of the classic deformable body.

For the stress to satisfy relationships

$$p_{yz} = p_{zy}, \qquad p_{zx} = p_{xz}, \qquad p_{xy} = p_{yx},$$

it is sufficient that we have

$$\varphi_1 = 0, \qquad \varphi_2 = 0, \qquad \varphi_3 = 0,$$

i.e., that W be independent of the arguments ϕ_1, ϕ_2, ϕ_3. If, more particularly, we should have

$$p_{yz} = p_{zy} = 0, \qquad p_{zx} = p_{xz} = 0, \qquad p_{xy} = p_{yx} = 0,$$

W becomes a simple function of Δ and of v, and we find that

$$p_{xx} = p_{yy} = p_{zz} = \frac{\partial W}{\partial \Delta};$$

We are then dealing with motion of a *perfect* liquid.

Since functions $\mathfrak{K}_0$, $\mathfrak{M}_0$, $\mathfrak{N}_0$ are not zero, W contains as arguments the twelve translations ξ_i, η_i, ζ_i, ξ, η, ζ. On one hand, the medium may be regarded as *gyrostatic* -- if we extend the meaning of this word created by Lord Kelvin -- and, on the other hand, the medium is endowed with *kinetic anisotropy* in the sense attributed to it by Rankine and later by Lord Rayleigh. Thus, for example, we base the theory of double refraction of light, as set forth by Lord Kelvin and Glazebrook, on a purely mechanical foundation.

V. Euclidean Distance Action. Action of Constraint, and Dissipative Action

68. Euclidean Action of Deformation and of Motion on a Discontinuous Medium. Let us consider a discrete system of n triads with each triad distinguished by a subscript i which consequently assumes the values 1, 2, ..., n. Let $M_i\, x_i'\, y_i'\, z_i'$ be the triad of subscript i whose vertex M_i has the coordinates x_i, y_i, z_i and the axes $M_i x_i'$, $M_i y_i'$, $M_i z'_i$ with the direction cosines α_i, α_i', α_i''; β_i, β_i', β_i''; γ_i, γ_i', γ_i'' with respect to three fixed rectangular axes Ox, Oy, Oz. We will assume that the quantities x_i, y_i, z_i, α_i, α_i', ..., γ_i'' are functions of time t, and we will introduce the six arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i which have subscript i and are defined by formulas (54') and (55') of Section 60.

Let us consider a function W of two infinitely close positions of the triad system $M_i x_i' y_i' z_i'$, i.e., a function of t, of x_i, y_i, z_i, α_i, α_i', ..., γ_i'', and of their first derivatives with respect to t (i assumes the values 1, 2, ..., n). Let us set ourselves the task of determining what the form of W should be in order that this function may remain invariant for any infinitesimal transformation of the group of Euclidean displacements like equation (60). Let us note that the relationships (54') and (55') of Section 60 with the subscript i permit us, by means of well known formulas, to express the first derivatives of the nine cosines α_i, α_i', ..., γ_i'' with respect to t by these cosines and p_i, /179
q_i, r_i. On the other hand, we can express the nine cosines α_i, α_i', ..., γ_i'' by means of ξ_i, η_i, ζ_i and the first derivatives of x_i, y_i, z_i with respect to t. The function W which we seek may therefore finally be expressed as a function of t, of x_i, y_i, z_i, and of their first derivatives, and finally as a function of ξ_i, η_i, ζ_i, p_i, q_i, r_i. We will indicate this by writing

$$W = W\left(t, x_i, y_i, z_i, \frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i\right).$$

Since the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i are zero in the present case, as results from the well known theory of the mobile triad, we must write

$$\sum_i \left(\frac{\partial W}{\partial x_i}\delta x_i + \frac{\partial W}{\partial y_i}\delta y_i + \frac{\partial W}{\partial z_i}\delta z_i + \frac{\partial W}{\partial \frac{dx_i}{dt}}\delta\frac{dx_i}{dt} + \frac{\partial W}{\partial \frac{dy_i}{dt}}\delta\frac{dy_i}{dt} + \frac{\partial W}{\partial \frac{dz_i}{dt}}\delta\frac{dz_i}{dt}\right) = 0.$$

for the new form of W which is obtained from formulas (60) taken with subscript i, whatever the variables a_1, a_2, a_3, ω_1, ω_2, ω_3.

Let us replace δx_i, δy_i, δz_i by their values in expression (60) and $\delta\frac{dx_i}{dt}$, $\delta\frac{dy_i}{dt}$, $\delta\frac{dz_i}{dt}$ by the values derived from them by differentiation. Let us set the coefficients a_1, a_2, a_3, ω_1, ω_2, ω_3 equal to zero. We obtain the following six conditions

$$\sum_i \frac{\partial W}{\partial x_i} = 0, \quad \sum_i \frac{\partial W}{\partial y_i} = 0, \quad \sum_i \frac{\partial W}{\partial z_i} = 0, \tag{63}$$

and

$$\sum_i \left(y_i \frac{\partial W}{\partial z_i} - z_i \frac{\partial W}{\partial y_i} + \frac{dy_i}{dt}\frac{\partial W}{\partial \frac{dz_i}{dt}} - \frac{dz_i}{dt}\frac{\partial W}{\partial \frac{dy_i}{dt}}\right) = 0, \tag{64}$$

with similar relationships.

If we assume that *the points* (x_i, y_i, z_i) *can describe all the possible trajectories*, we thus arrive at identities satisfied by function W of the 6n arguments x_i, y_i, z_i, $\frac{dx_i}{dt}$, $\frac{dy_i}{dt}$, $\frac{dz_i}{dt}$ and the previous arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i, which we may leave aside for a moment. Let us attempt to determine the form resulting for W.

Let us begin by treating the case of the three-equation system

$$\left\{\begin{aligned} &\sum_{i=1}^{i=p}\left(y_i \frac{\partial W}{\partial z_i} - z_i \frac{\partial W}{\partial y_i}\right) = 0, \\ &\sum_{i=1}^{i=p}\left(z_i \frac{\partial W}{\partial x_i} - x_i \frac{\partial W}{\partial z_i}\right) = 0, \\ &\sum_{i=1}^{i=p}\left(x_i \frac{\partial W}{\partial y_i} - y_i \frac{\partial W}{\partial x_i}\right) = 0, \end{aligned}\right. \tag{65}$$

which determine a W function of the 3n arguments x_i, y_i, z_i. We have already /180 come across this system in connection with the statics of the line, surface,

and continuous three-dimensional medium in the case where p = 1, p = 2, p = 3. Let us lay aside the p = 1 case where the equations are reduced to two. For p = 2 and p = 3, we have three equations which form a complete system. For p = 2 we have three equations, six variables, and the three independent solutions

$$x_i^2 + y_i^2 + z_i^2 \quad (i = 1,2), \quad x_1x_2 + y_1y_2 + z_1z_2;$$

for p = 3 we have three equations, nine variables, and the six independent solutions

$$x_i^2 + y_i^2 + z_i^2 \quad (i = 1,2,3), \quad x_ix_j + y_iy_j + z_iz_j \quad (i,j = 1,2,3).$$

When p > 3 the system is still complete. To prove it, we need only show that the system has 3p-3 independent solutions, since the number of equations is 3 and the number of variables is 3p. We actually have first the p solutions

$$x_i^2 + y_i^2 + z_i^2 \quad (i = 1,2,\ldots,p,)$$

then the solution

$$x_1x_2 + y_1y_2 + z_1z_2,$$

and finally the 2(p-2) solutions

$$x_1x_i + y_1y_i + z_1z_i, \quad x_2x_i + y_2y_i + z_2z_i \quad (i = 3,4,5,\ldots,p).$$

all of which are independent. W is therefore a function of the 3(p-1) independent arguments which we have just enumerated.

Let us now return to the proposed system formed by conditions (63) and (64). Conditions (63) prove that W depends on $x_1 \ldots x_n$, $y_1, \ldots y_n$, $z_1, \ldots z_n$ only through the expressions

$$\begin{array}{llll} X_2 = x_2 - x_1, & X_3 = x_3 - x_1, & \ldots, & X_n = x_n - x_1, \\ Y_2 = y_2 - y_1, & Y_3 = y_3 - y_1, & \ldots, & Y_n = y_n - y_1, \\ Z_2 = z_2 - z_1, & Z_3 = z_3 - z_1, & \ldots, & Z_n = z_n - z_1. \end{array}$$

Moreover, let us assume that

$$\frac{dx_i}{dt} = X_{n+i}, \quad \frac{dy_i}{dt} = Y_{n+i}, \quad \frac{dz_i}{dt} = Z_{n+i}.$$

and let us indicate that equations (64) are satisfied by the function W of the arguments $X_2, X_3, \ldots, X_{2n}$; $Y_2, Y_3, \ldots, Y_{2n}$; $Z_2, Z_3, \ldots, Z_{2n}$. Let us, for example, take a look at the first of equations (64). We have

$$-y_1\left(\frac{\partial W}{\partial Z_2} + \frac{\partial W}{\partial Z_3} + \ldots + \frac{\partial W}{\partial Z_n}\right) + z_1\left(\frac{\partial W}{\partial Y_2} + \frac{\partial W}{\partial Y_3} + \ldots + \frac{\partial W}{\partial Y_n}\right)$$
$$+ (y_1 + Y_2)\frac{\partial W}{\partial Z_2} - (z_1 + Z_2)\frac{\partial W}{\partial Y_2} + \ldots = 0;$$

y_1 and z_1 disappear, and the first of the following equations remains: /181

$$\sum_{i=2}^{i=2n}\left(Y_i\frac{\partial W}{\partial Z_i} - Z_i\frac{\partial W}{\partial Y_i}\right) = 0.$$

$$\sum_{i=2}^{i=2n}\left(Z_i\frac{\partial W}{\partial X_i}-X_i\frac{\partial W}{\partial Z_i}\right)=0,$$

$$\sum_{i=2}^{i=2n}\left(X_i\frac{\partial W}{\partial Y_i}-Y_i\frac{\partial W}{\partial X_i}\right)=0.$$

We therefore arrive at system (65) where x_i, y_i, z_i are replaced by X_{i+1}, Y_{i+1}, Z_{i+1} and p by 2n-1.

If we first assume that n = 2, we see that, disregarding the arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i, W is a function of the independent expressions

$$X_2^2+Y_2^2+Z_2^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2,$$

$$X_3^2+Y_3^2+Z_3^2=\left(\frac{dx_1}{dt}\right)^2+\left(\frac{dy_1}{dt}\right)^2+\left(\frac{dz_1}{dt}\right)^2=\xi_1^2+\eta_1^2+\zeta_1^2,$$

$$X_4^2+Y_4^2+Z_4^2=\left(\frac{dx_2}{dt}\right)^2+\left(\frac{dy_2}{dt}\right)^2+\left(\frac{dz_2}{dt}\right)^2=\xi_2^2+\eta_2^2+\zeta_2^2,$$

$$X_2X_3+Y_2Y_3+Z_2Z_3=(x_2-x_1)\frac{dx_1}{dt}+(y_2-y_1)\frac{dy_1}{dt}+(z_2-z_1)\frac{dz_1}{dt},$$

$$X_2X_4+Y_2Y_4+Z_2Z_4=(x_2-x_1)\frac{dx_2}{dt}+(y_2-y_1)\frac{dy_2}{dt}+(z_2-z_1)\frac{dz_2}{dt},$$

$$X_3X_4+Y_3Y_4+Z_3Z_4=\frac{dx_1}{dt}\frac{dx_2}{dt}+\frac{dy_1}{dt}\frac{dy_2}{dt}+\frac{dz_1}{dt}\frac{dz_2}{dt},$$

Therefore, finally, W is a function of t, ξ_i, η_i, ζ_i, p_i, q_i, r_i and of the *four arguments*

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2,$$

$$(x_2-x_1)\frac{dx_1}{dt}+(y_2-y_1)\frac{dy_1}{dt}+(z_2-z_1)\frac{dz_1}{dt},$$

$$(x_2-x_1)\frac{dx_2}{dt}+(y_2-y_1)\frac{dy_2}{dt}+(z_2-z_1)\frac{dz_2}{dt},$$

$$\frac{dx_1}{dt}\frac{dx_2}{dt}+\frac{dy_1}{dt}\frac{dy_2}{dt}+\frac{dz_1}{dt}\frac{dz_2}{dt}.$$

If we assume that n > 2, we see that, disregarding arguments ξ_i, η_i, ζ_i, /182
p_i, q_i, r_i, W is a function of the 6(n-1) independent expressions

$$X_i^2+Y_i^2+Z_i^2=\begin{cases}\text{either } (x_i-x_1)^2+(y_i-y_1)^2+(z_i-z_1)^2 & (i=1,2,\dots,n),\\ \text{either } \left(\frac{dx_k}{dt}\right)^2+\left(\frac{dy_k}{dt}\right)^2+\left(\frac{dz_k}{dt}\right)^2=\xi_k^2+\eta_k^2+\zeta_k^2,\end{cases}$$

$$X_2X_3+Y_2Y_3+Z_2Z_3=(x_2-x_1)(x_3-x_1)+(y_2-y_1)(y_3-y_1)+(z_2-z_1)(z_3-z_1),$$

$$X_2X_i+Y_2Y_i+Z_2Z_i=\begin{cases}\text{either } (x_2-x_1)(x_i-x_1)+(y_2-y_1)(y_i-y_1)+(z_2-z_1)(z_i-z_1),\\ \text{either } (x_2-x_1)\frac{dx_k}{dt}+(y_2-y_1)\frac{dy_k}{dt}+(z_2-z_1)\frac{dz_k}{dt},\end{cases}$$

$$X_3X_i+Y_3Y_i+Z_3Z_i=\begin{cases}\text{either } (x_3-x_1)(x_i-x_1)+(y_3-y_1)(y_i-y_1)+(z_3-z_1)(z_i-z_1)\\ \text{either } (x_3-x_1)\frac{dx_k}{dt}+(y_3-y_1)\frac{dy_k}{dt}+(z_3-z_1)\frac{dz_k}{dt}.\end{cases}$$

Let us note that we have

$$(x_i-x_j)(x_i-x_k)+(y_i-y_j)(y_i-y_k)+(z_i-z_j)(z_i-z_k)=\frac{1}{2}(r_{ij}^2+r_{ik}^2-r_{kj}^2),$$

where r is the distance between two points in the system. For reasons of symmetry, we may need to let *non-independent* arguments be present in W. Independently of ξ_i, η_i, ζ_i, p_i, q_i, r_i, the following arguments may be considered:

$$r_{ij}^2=(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2,$$
$$\psi_{ij}=\frac{dx_i}{dt}\frac{dx_j}{dt}+\frac{dy_i}{dt}\frac{dy_j}{dt}+\frac{dz_i}{dt}\frac{dz_j}{dt},$$
$$\lambda_{ijk}=(x_i-x_j)\frac{dx_k}{dt}+(y_i-y_j)\frac{dy_k}{dt}+(z_i-z_j)\frac{dz_k}{dt};$$

The latter include two-subscript arguments λ_{iji} and three-subscript arguments λ_{ijk}. The latter play a part only when there are more than two points, and in this case we see that the action at the two points is affected by all the other points. It is easy to establish the relations of rather complex form which exist between these non-independent arguments. They are similar to the known relationships between the distances r_{ij} when the number of points is $\geqq 5$.

Knowing the expression of Euclidean action W on the triad system under consideration, we may easily find the expression for the external force and moment on any triad by a computation which repeats those which we have performed before. Since the action W is a function of x_i, y_i, z_i, $\frac{dx_i}{dt}$, $\frac{dy_i}{dt}$, $\frac{dz_i}{dt}$ by means of r_{ij}, ψ_{ij}, λ_{ijk}, it is convenient first to regard W as a function of x_i, y_i, z_i, $\frac{dx_i}{dt}$, $\frac{dy_i}{dt}$, $\frac{dz_i}{dt}$ and of ξ_i, η_i, ζ_i, p_i, q_i, r_i. We will have

$$\delta\int_{t_1}^{t_2} W dt$$
$$=\left[\sum_i(A_i\delta x_i+B_i\delta y_i+C_i\delta z_i+P_i\delta\iota_i+Q_i\delta\jmath_i+R_i\delta\kappa_i)\right]_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\sum_i(X_i\delta x_i+Y_i\delta y_i+Z_i\delta z_i+L_i\delta\iota_i+M_i\delta\jmath_i+N_i\delta\kappa_i)dt,$$

by assuming that

$$A_i=\alpha_i\frac{\partial W}{\partial\xi_i}+\beta_i\frac{\partial W}{\partial\eta_i}+\gamma_i\frac{\partial W}{\partial\zeta_i},\quad P_i=\alpha_i\frac{\partial W}{\partial p_i}+\beta_i\frac{\partial W}{\partial q_i}+\gamma_i\frac{\partial W}{\partial r_i},$$
$$B_i=\alpha_i'\frac{\partial W}{\partial\xi_i}+\beta_i'\frac{\partial W}{\partial\eta_i}+\gamma_i'\frac{\partial W}{\partial\zeta_i},\quad Q_i=\alpha_i'\frac{\partial W}{\partial p_i}+\beta_i'\frac{\partial W}{\partial q_i}+\gamma_i'\frac{\partial W}{\partial r_i},$$
$$C_i=\alpha_i''\frac{\partial W}{\partial\xi_i}+\beta_i''\frac{\partial W}{\partial\eta_i}+\gamma_i''\frac{\partial W}{\partial\zeta_i},\quad R_i=\alpha_i''\frac{\partial W}{\partial p_i}+\beta_i''\frac{\partial W}{\partial q_i}+\gamma_i''\frac{\partial W}{\partial r_i},$$

where (A_i, B_i, C_i) and (P_i, Q_i, R_i) are, respectively, the momentum and the mo- /183
ment of momentum of the triad with subscript i, and

$$X_i=\frac{dA_i}{dt}+\frac{d}{dt}\left(\frac{\partial W}{\partial\frac{dx_i}{dt}}\right)-\frac{\partial W}{\partial x_i},\quad L_i=\frac{dP_i}{dt}+C_i\frac{dy_i}{dt}-B_i\frac{dz_i}{dt},$$

$$Y_i = \frac{dB_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dy_i}{dt}}\right) - \frac{\partial W}{\partial y_i}, \quad M_i = \frac{dQ_i}{dt} + A_i \frac{dz_i}{dt} - C_i \frac{dy_i}{dt},$$

$$Z_i = \frac{dC_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dz_i}{dt}}\right) - \frac{\partial W}{\partial z_i}, \quad N_i = \frac{dR_i}{dt} + B_i \frac{dx_i}{dt} - A_i \frac{dy_i}{dt},$$

where (X_i, Y_i, Z_i) and (L_i, M_i, N_i) are, respectively, the external force and the external moment on the triad with subscript i. In this computation, it is easy to isolate the arguments r_{ij}, ψ_{ij}, γ_{ijk}.

Let us note that the expression for the external force is broken up into two parts; the first, which depends on segments (A_i, B_i, C_i), (P_i, Q_i, R_i) and their derivatives, is the strictly dynamic part. The second, which results from the presence of arguments r_{ij}, ψ_{ij}, λ_{ijh} in W, corresponds to the force which the triad with subscript i experiences due to all the other triads of the system. Let us consider the expression

$$\sum_i \Big[X_i \frac{dx_i}{dt} + Y_i \frac{dy_i}{dt} + Z_i \frac{dz_i}{dt} + L_i(\alpha_i p_i + \beta_i q_i + \gamma_i r_i)$$
$$+ M_i(\alpha_i' p_i + \beta_i' q_i + \gamma_i' r_i) + N_i(\alpha_i'' p_i + \beta_i'' q_i + \gamma_i'' r_i) \Big] dt,$$

which represents the sum of the elementary work performed by the forces applied to the different triads. If we calculate them by replacing X_i, Y_i, Z_i, L_i, M_i, N_i by the preceding values, we find the following expression:

$$\sum_i \Big[\frac{d}{dt}\left(\xi_i \frac{\partial W}{\partial \xi_i} + \eta_i \frac{\partial W}{\partial \eta_i} + \zeta_i \frac{\partial W}{\partial \zeta_i} + p_i \frac{\partial W}{\partial p_i} + q_i \frac{\partial W}{\partial q_i} + r_i \frac{\partial W}{\partial r_i}\right)$$
$$- \left(\frac{\partial W}{\partial \xi_i}\frac{d\xi_i}{dt} + \frac{\partial W}{\partial \eta_i}\frac{d\eta_i}{dt} + \dots + \frac{\partial W}{\partial r_i}\frac{dr_i}{dt}\right)\Big] dt,$$

for the elementary work of the external force and the external moment corresponding to the dynamic part, and the expression /184

$$\sum_i \Bigg[\frac{d}{dt}\left(\frac{dx_i}{dt}\frac{\partial W}{\partial \frac{dx_i}{dt}} + \frac{dy_i}{dt}\frac{\partial W}{\partial \frac{dy_i}{dt}} + \frac{dz_i}{dt}\frac{\partial W}{\partial \frac{dz_i}{dt}}\right)$$
$$- \left(\frac{\partial W}{\partial \frac{dx_i}{dt}}\frac{d^2x_i}{dt^2} + \frac{\partial W}{\partial \frac{dy_i}{dt}}\frac{d^2y_i}{dt^2} + \frac{\partial W}{\partial \frac{dz_i}{dt}}\frac{d^2z_i}{dt^2} + \frac{\partial W}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial W}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial W}{\partial z_i}\frac{dz_i}{dt}\right)\Bigg] dt.$$

for the elementary work corresponding to the forces applied to the system triads.

If we add these two expressions, and if we set

$$E = \sum_i \Bigg(\xi_i \frac{\partial W}{\partial \xi_i} + \eta_i \frac{\partial W}{\partial \eta_i} + \zeta_i \frac{\partial W}{\partial \zeta_i} + p_i \frac{\partial W}{\partial p_i} + q_i \frac{\partial W}{\partial q_i} + r_i \frac{\partial W}{\partial r_i}$$

$$+\frac{dx_i}{dt}\frac{\partial W}{\partial \frac{dx_i}{dt}}+\frac{dy_i}{dt}\frac{\partial W}{\partial \frac{dy_i}{dt}}+\frac{dz_i}{dt}\frac{\partial W}{\partial \frac{dz_i}{dt}}-W\Bigg).$$

we see that the sum of the elementary work performed is

$$dE+\frac{\partial W}{\partial t}\,dt;$$

By assuming W to be independent of t and by giving E the name of *energy of motion and position* of the triad system under consideration, we obtain a theorem quite similar to that of Section 65.

From the preceding, it is easy to deduce a theory of dynamics of a system which is established according to the same plan as the classical theory, without limiting ourselves to consideration of the central forces, as is done in classical theory. The present discussion moreover has the advantage of connecting with their true origin the various laws of distant force studied by Gauss, Riemann, Weber, and Clausius(1), whose laws all merely introduce the arguments r_{ij}, ψ_{ij}, γ_{ijk}.

69. Euclidean Action of Constraint and Dissipative Euclidean Action. The ideas which we have just developed in regard to distant Euclidean action lead in the most natural way to the idea of *constraint*, which we owe to Gauss and which, as we know, Hertz applied to studying the foundations of mechanics by following a path already traveled by Beltrami, R. Lipschitz, and M. G. Darboux(2).

For purposes of simplification, let there be a point which, when its motion/185 is *free*, describes a path defined by the three functions x_0, y_0, z_0 of time t. Let us moreover designate by x, y, z the functions of time t which define its trajectory when the point is subjected to constraints. We can imagine the two points (x, Y, Z), (X_0, Y_0, Z_0) whose coordinates are obtained, for example, by the formulas

$$X=x+\frac{dx}{dt}dt+\frac{1}{2}\frac{d^2x}{dt^2}dt^2,\quad X_0=x_0+\frac{dx_0}{dt}dt+\frac{1}{2}\frac{d^2x_0}{dt^2}dt^2,$$
$$Y=y+\frac{dy}{dt}dt+\frac{1}{2}\frac{d^2y}{dt^2}dt^2,\quad Y_0=y_0+\frac{dy_0}{dt}dt+\frac{1}{2}\frac{d^2y_0}{dt^2}dt^2,$$
$$Z=z+\frac{dz}{dt}dt+\frac{1}{2}\frac{d^2z}{dt^2}dt^2,\quad Z_0=z_0+\frac{dz_0}{dt}dt+\frac{1}{2}\frac{d^2z_0}{dt^2}dt^2,$$

which come from the Taylor expansion limited to the first three terms. By

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(1) See R. Reiff and A. Sommerfeld, *Encyklopädie der Mathematischen Wissenschaften*, 52, pp. 3-62.

(2) Beltrami. *On the General Theory of Differential Parameters, Memoire della Reale Accademia di Bologna*, February 25, 1869.
R. Lipschitz. *Studies of a Calculus of Variations Problem Containing the Problem of Mechanics, Journal für die Reine und Angewandte Mathematik*, 74, 1872, pp. 116-149; *Notes on the Principle of Least Constraint*, ibid., 82, 1877, pp. 311-342.
G. Darboux. *Leçons sur la Théorie Générale des Surfaces (Lessons on the General Theory of Surfaces)*, Part 2, Book 5, Chapter 6-8. Paris, 1889.

assuming that the constraints are *frictionless*, we may state that at the time t under consideration we have

$$x = x_0, \quad y = y_0, \quad z = z_0, \quad \frac{dx}{dt} = \frac{dx_0}{dt}, \quad \frac{dy}{dt} = \frac{dy_0}{dt}, \quad \frac{dz}{dt} = \frac{dz_0}{dt}.$$

When this has been assumed and after the *Euclidean distant action* $U_1(r)$ has been considered for the two points (X, Y, Z) and (X_0, Y_0, Z_0) whose distance we designate by r, the introduction of the Gaussian idea of constraint amounts to replacing r by its value, so that we are led to function U of argument γ defined by the formula

$$\gamma^2 = \left(\frac{d^2x}{dt^2} - \frac{d^2x_0}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2} - \frac{d^2y_0}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2} - \frac{d^2z_0}{dt^2}\right)^2.$$

If we then apply the method of variable action, we will have

$$\delta U = \mathfrak{X}\left(\delta\frac{d^2x}{dt^2} - \delta\frac{d^2x_0}{dt^2}\right) + \mathfrak{Y}\left(\delta\frac{d^2y}{dt^2} - \delta\frac{d^2y_0}{dt^2}\right) + \mathfrak{Z}\left(\delta\frac{d^2z}{dt^2} - \delta\frac{d^2z_0}{dt^2}\right).$$

by setting

$$\mathfrak{X} = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2x}{dt^2} - \frac{d^2x_0}{dt^2}\right), \quad \mathfrak{Y} = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2y}{dt^2} - \frac{d^2y_0}{dt^2}\right), \quad \mathfrak{Z} = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2z}{dt^2} - \frac{d^2z_0}{dt^2}\right);$$

If with Gauss we call the argument γ *constraint*, the force ($\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$) may be called the *force of constraint* applied to point (x, y, z) and may be regarded having the effect of preventing the point from carrying out its *free* motion. The force $-\mathfrak{X}$, $-\mathfrak{Y}$, $-\mathfrak{Z}$, - applied to point (x_0, y_0, z_0) has, on the contrary, the effect of changing free motion into constrained motion.

The essential difference between the present concept of force and that which ensues from Newton's laws of motion is the following: In the latter, we consider the action relative to two infinitely close points, one present and one future, *on the same path*. In the concept of Gauss and Hertz, the action relates to two future positions, one on the path called *free* and the other on the path called *constrained*. In both cases, we apparently have a theory which /186 permits *prediction* of future motion -- this is the aim of point dynamics. Besides that, and this is the point which we particularly wanted to make, the action is *Euclidean*.

It is interesting to note in this regard that Gauss explicitly compared constraining action and the *law of errors*. These two effects have in fact the same form. We thus see that the basic nature of the law of errors is the *Euclidean invariance* of that law, and that the new branch of mechanics created by Maxwell, Boltzmann, and W. Gibbs under the name of statistical mechanics can also be given the deductive form which we are here trying to give to ordinary mechanics.

We will again point out that both in the mechanics which may be derived from Newton's ideas and in the mechanics derivable from Gauss' concept of concept of constraint, the binding forces convey an *indeterminacy* which comes from the definition of force and leads to introduction of Lagrange multipliers.

Gauss' idea may also be applied to friction by considering an Euclidean action at the two points

$$X = x + \frac{dx}{dt}\,dt, \qquad X_0 = x_0 + \frac{dx_0}{dt}\,dt,$$
$$Y = y + \frac{dy}{dt}\,dt, \qquad Y_0 = y_0 + \frac{dy_0}{dt}\,dt,$$
$$Z = z + \frac{dz}{dt}\,dt, \qquad Z_0 = z_0 + \frac{dz_0}{dt}\,dt,$$

where point x_0, y_0, z_0 refers to a free path and point x, y, z refers to a path traveled with friction. As we are here dealing with sliding friction, we may set $x = x_0$, $y = y_0$, $x = z_0$, $\frac{dx}{dt} = \mu \frac{dx_0}{dt}$, $\frac{dy}{dt} = \mu \frac{dy_0}{dt}$, $\frac{dz}{dt} = \mu \frac{dz_0}{dt}$. For the action we are led to a function of velocity $v_0 = \sqrt{\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{dy_0}{dt}\right)^2 + \left(\frac{dz_0}{dt}\right)^2}$ containing the factor $I - \mu$. This corresponds exactly to the concept of *dissipation of free action at the point* x_0, y_0, z_0.

The arguments r_{ij}, ψ_{ij}, λ_{ijk} which we have considered in Section 68 definitively convey a similar idea in regard to a triad considered alone in the contemplated system of n triads. We may, if we wish, make a distinction between these arguments and say that r_{ij} is a *potential* argument, and that ψ_{ij}, λ_{ijk} are *dissipative* arguments. The hypothesis of central forces thus reduces to considering in mechanics only the dynamics of systems without *distant friction*. From the arguments r_{ij}, ψ_{ij}, λ_{ijk} we may moreover derive the particular Weber argument $\frac{dr_{ij}}{dt}$, and if we pass from the discontinuous medium to the *continuous* medium -- the concept of which is based on consideration of the ds^2 of space -- we are led to introduce into action W the *viscosity arguments* $\frac{d\varepsilon_1}{dt}$, $\frac{d\varepsilon_2}{dt}$, $\frac{d\varepsilon_3}{dt}$, $\frac{d\gamma_1}{dt}$, $\frac{d\gamma_2}{dt}$, $\frac{d\gamma_3}{dt}$. Beside such arguments which were first conceived /187 by Navier and Poisson, we must apparently also place arguments like $\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2$, which was discussed in Section 47, and like ϕ_1, ϕ_2, ϕ_3 from Section 67. We will limit ourselves to these brief remarks on viscosity, which has not hitherto been studied in a sufficiently systematic way.

VI. Euclidean Action from the Eulerian Point of View

70. Independent Variables of Lagrange and Euler. The Auxiliary Quantities Considered from the Point of View of Hydrodynamics. In the statics and dynamics of the deformable medium, we have taken as independent variables x_0, y_0, z_0 and x_0, y_0, z_0, t, respectively. In the first case, (static) we designate by x_0, y_0, z_0 the coordinates of point M_0 of the natural state (M_0) by imagining that the natural state is deformed in an infinitely slow manner without its points acquiring velocity, and that it passes continuously from position (M_0) to position

(M)[1]. In the second case (dynamic) we designate by x_0, y_0, z_0 the coordinates of position M_0 at a determined instant t_0 of the point which at instant t is at M. Position (M_0) of the medium *plays a particular role*.

The deformable medium (M) has been assumed to be generated by a triad Mx'y'z' whose vertex M has the coordinates x, y, z and whose edges have α, α', α''; β, β', β''; γ, γ', γ'' for direction cosines relative to fixed axes Ox, Oy, Oz. In the case of statics, x, y, z, α, α', ..., γ'' were considered as functions of the three independent variables x_0, y_0, z_0, and in the case of dynamics, they were considered as functions of the four independent variables x_0, y_0, z_0, t. In either case, we will say that the independent variables under consideration are *Lagrange's variables*. If we wish to specify that concept, we will write

$$x = x(x_0, y_0, z_0), \quad y = y(x_0, y_0, z_0), \quad z = z(x_0, y_0, z_0), \tag{66}$$

or

$$x = x(x_0, y_0, z_0, t), \quad y = y(x_0, y_0, z_0, t), \quad z = z(x_0, y_0, z_0, t), \tag{66'}$$

and likewise let

$$\alpha = \alpha(x_0, y_0, z_0), \quad \alpha' = \alpha'(x_0, y_0, z_0), \quad \alpha'' = \alpha''(x_0, y_0, z_0), \tag{67}$$

or /188

$$\alpha = \alpha(x_0, y_0, z_0, t), \quad \alpha' = \alpha'(x_0, y_0, z_0, t), \quad \alpha'' = \alpha''(x_0, y_0, z_0, t), \tag{67'}$$

with similar formulas for β, β', β'', γ, γ', γ''.

But we may now imagine that a change in variables is effected on the independent variables. In particular, by analogy with the procedure in hydrodynamics, we may imagine that x, y, z, or x, y, z, t are taken as independent variables. We will then say that we are considering *Eulerian variables*.

How does the matter appear at first glance? In the theory formerly developed, whether it is considered as defining the force elements, etc., or whether, inversely, the attempt is made to determine position (M), we clearly have functions x, y, z, α, α', ..., γ'' of x_0, y_0, z_0 or of x_0, y_0, z_0, t defined by expressions (66), (67) or by expressions (66'), (67'). Let us now imagine that equations (66) or (66') are solved with respect to x_0, y_0, z_0. We have

$$x_0 = x_0(x, y, z), \quad y_0 = y_0(x, y, z), \quad z_0 = z_0(x, y, z), \tag{68}$$

or

$$x_0 = x_0(x, y, z, t), \quad y_0 = y_0(x, y, z, t), \quad z_0 = z_0(x, y, z, t), \tag{68'}$$

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[1] In this concept of infinitely slow deformation of a medium, which resembles the reversible transformation of thermodynamics, we have defined the external force and moment, the stress and moment of deformation which may both be called *static*, then the work performed in passing from (M_0) to (M). Consequently, we have obtained the concept of *deformation energy*, which has taken its place beside the concept of *action* from which it started.

and by substitution into equation (67) or (67') we have

$$\alpha = \alpha(x, y, z), \quad \alpha' = \alpha'(x, y, z), \quad \alpha'' = \alpha''(x, y, z), \tag{69}$$

or

$$\alpha = \alpha(x, y, z, t), \quad \alpha' = \alpha'(x, y, z, t), \quad \alpha'' = \alpha''(x, y, z, t). \tag{69'}$$

We now clearly have the functions x_0, y_0, z_0, α, α', ..., γ'' of x, y, z or of x, y, z, t, and conversely by solving expressions (68), (69) or (68'), (69') we will return again to equations (66), (67) or to (66'), (67').

But what has just been stated must be supplemented by noting that in either case it may be convenient to introduce auxiliary quantities.

If we consider the case of Lagrange variables, it may happen that functions x, y, z are not explicitly included in the matter[2]. It may be suitable to introduce the first derivatives of x, y, z with respect to x_0, y_0, z_0 or x_0, y_0, z_0, t as auxiliary quantities[3]. By considering x, y, z, α, α', ..., γ'', we may in this case also introduce the translations and rotations ξ_i, ..., r_i, ξ, .. /189 .., r as auxiliary quantities if only x_0, y_0, z_0 or x_0, y_0, z_0 t, for example, appear in the given quantities.

If we now consider the case of Eulerian variables, we may point out similar situations where the use of auxiliary quantities can offer advantages. Let us first assume that the hypothesis which we have just considered is realized with Lagrange variables. We will be able to keep the indicated auxiliary quantities. The only essential difference from the preceding case will be in the *subsequent* determination -- if it is made -- of formulas (66), (67) or the analogous ones. Let us assume moreover that x_0, y_0, z_0 are not included in the question; then the derivatives of x_0, y_0, z_0 with respect to x, y, z or to x, y, z, t may then be introduced as auxiliary quantities.

From these indications, it may be seen that it can be useful to present the equations serving as the point of departure in a form which is convenient for the introduction of auxiliary quantities. It will be noted that this aim has already been achieved by the previously derived equations in which the auxiliary quantities ξ_i, ..., r_i, ξ, ..., r already appear.

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(2) This is what will normally happen if we start with results like those given by our discussion, and if we do not modify the expressions of force, etc. by virtue of formulas (66), (67) or (66'), (67'). The terms x, y, z do not in fact explicitly appear in W.

(3) These auxiliary quantities will now be related by relationships which are easy to formulate. The same remark will apply in general. It is not introduced in hydrodynamics where the auxiliary quantities are the derivatives with respect to the sole variable t (and where the use of these auxiliary quantities is often limited to the case where Eulerian variables are introduced).

71. Expressions of $\xi_1, \ldots, r_1$ (or of $\xi_1, \ldots, r_1, \xi, \ldots, r$) by Means of Functions $x_0, y_0, z_0, \alpha, \alpha', \ldots, \gamma''$ of x, y, z (or of $x, y, z\ t$) and of their Derivatives; Introduction of Eulerian Arguments. It follows from the explanations which have just been given that it may be useful to have the expressions of $\xi_1, \ldots, r_1$ or $\xi_1, \ldots, r_1, \xi, \ldots, r$ no longer evaluated by formulas (66), (67) or (66'), (67'), which assume that x_0, y_0, z_0 or x_0, y_0, z_0, t are independent variables, but by formulas (68), (69) or (68'), (69'), which introduce the functions $x_0, y_0, z_0, \alpha, \alpha', \ldots, \gamma''$ of x, y, z or of x, y, z, t.

Let us discuss more generally the case where t appears. The formulas derived will, in particular, give the case where $x, y, z\ \alpha, \alpha', \ldots, \gamma''$ are independent of t. By virtue of formulas (66'), (67'), the quantities $\xi_1, \ldots$ are calculated by the formulas[4]

$$\begin{cases} \xi_i = \alpha \dfrac{\partial x}{\partial \rho_i} + \alpha' \dfrac{\partial y}{\partial \rho_i} + \alpha'' \dfrac{\partial z}{\partial \rho_i}, & \xi = \alpha \dfrac{dx}{dt} + \alpha' \dfrac{dy}{dt} + \alpha'' \dfrac{dz}{dt}, \\ \eta_i = \beta \dfrac{\partial x}{\partial \rho_i} + \beta' \dfrac{\partial y}{\partial \rho_i} + \beta'' \dfrac{\partial z}{\partial \rho_i}, & \eta = \beta \dfrac{dx}{dt} + \beta' \dfrac{dy}{dt} + \beta'' \dfrac{dz}{dt}, \\ \zeta_i = \gamma \dfrac{\partial x}{\partial \rho_i} + \gamma' \dfrac{\partial y}{\partial \rho_i} + \gamma'' \dfrac{\partial z}{\partial \rho_i}, & \zeta = \gamma \dfrac{dx}{dt} + \gamma' \dfrac{dy}{dt} + \gamma'' \dfrac{dz}{dt}, \end{cases} \tag{70}$$

$$\begin{cases} p_i = \sum \gamma \dfrac{\partial \beta}{\partial \rho_i} = -\sum \beta \dfrac{\partial \gamma}{\partial \rho_i}, & p = \sum \gamma \dfrac{d\beta}{dt} = -\sum \beta \dfrac{d\gamma}{dt}, \\ q_i = \sum \alpha \dfrac{\partial \gamma}{\partial \rho_i} = -\sum \gamma \dfrac{\partial \alpha}{\partial \rho_i}, & q = \sum \alpha \dfrac{d\gamma}{dt} = -\sum \gamma \dfrac{d\alpha}{dt}, \\ r_i = \sum \beta \dfrac{\partial \alpha}{\partial \rho_i} = -\sum \alpha \dfrac{\partial \beta}{\partial \rho_i}, & r = \sum \beta \dfrac{d\alpha}{dt} = -\sum \alpha \dfrac{d\beta}{dt}, \end{cases} \tag{71}$$

(where $\rho_1 = x_0$, $\rho_2 = y_0$, $\rho_3 = z_0$) and by means of $x_0, y_0, z_0, \alpha, \alpha', \ldots, \gamma''$ and their derivatives with respect to x, y, z calculated from formulas (68'), (69'). /190

For this purpose we are going to show that the quantities $\xi_1, \ldots, r_1, \xi, \ldots, r$, which from now on will be called *Lagrangian arguments*, are expressed simply by means of the following auxiliary quantities, which we will call Eulerian arguments:

$$\begin{cases} (\xi_i) = \alpha[\xi_i] + \alpha'[\eta_i] + \alpha''[\zeta_i], & (\xi) = \dfrac{\partial \rho_1}{\partial t}, \\ (\eta_i) = \beta[\xi_i] + \beta'[\eta_i] + \beta''[\zeta_i], & (\eta) = \dfrac{\partial \rho_2}{\partial t}, \\ (\zeta_i) = \gamma[\xi_i] + \gamma'[\eta_i] + \gamma''[\zeta_i], & (\zeta) = \dfrac{\partial \rho_3}{\partial t}, \end{cases} \tag{72}$$

$$\begin{cases} (p_i) = \alpha[p_i] + \alpha'[q_i] + \alpha''[r_i], & (p) = \sum \gamma \dfrac{\partial \beta}{\partial t} = -\sum \beta \dfrac{\partial \gamma}{\partial t}, \\ (q_i) = \beta[p_i] + \beta'[q_i] + \beta''[r_i], & (q) = \sum \alpha \dfrac{\partial \gamma}{\partial t} = -\sum \gamma \dfrac{\partial \alpha}{\partial t}, \end{cases} \tag{73}$$

(4) We shall employ the usual notation for derivatives with respect to t. (See, for example, Appel, *Traité de Mécanique (Treatise on Mechanics)*, Volume 3, First Edition, p. 277).

$$\left\{ (r_i) = \gamma[p_i] + \gamma'[q_i] + \gamma''[r_i], \quad (r) = \sum \beta \frac{\partial x}{\partial t} = -\sum \alpha \frac{\partial \beta}{\partial t}, \right. \tag{73}$$

where we assume that

$$\begin{cases} [\xi_i] = \frac{\partial p_i}{\partial x}, & [\eta_i] = \frac{\partial p_i}{\partial y}, & [\zeta_i] = \frac{\partial p_i}{\partial z}, \\ [p_1] = \sum \gamma \frac{\partial \beta}{\partial x} = -\sum \beta \frac{\partial \gamma}{\partial x}, & [q_1] = \sum \gamma \frac{\partial \beta}{\partial y} = -\sum \beta \frac{\partial \gamma}{\partial y}, & [r_1] = \sum \gamma \frac{\partial \beta}{\partial z} = -\sum \beta \frac{\partial \gamma}{\partial z}, \end{cases} \tag{74}$$

with similar formulas for $[p_2]$, $[q_2]$, $[r_2]$ and for $[p_3]$, $[q_3]$, $[r_3]$ derived by first changing γ, β into α, γ, and then into β, α, where we employ the familiar notation(5) $\frac{\partial \alpha}{\partial t}$, $\frac{\partial \beta}{\partial t}$, $\frac{\partial \gamma}{\partial t}$, ...

Let us successively differentiate the relationships (68') with respect to the Lagrangian variables. The result is four systems of three equations which, because of the notation (70) and (72), may be written

$$\xi_i(\xi_i) + \eta_i(\eta_i) + \zeta_i(\zeta_i) = 1, \quad \xi_j(\xi_k) + \eta_j(\eta_k) + \zeta_j(\zeta_k) = 0, \quad (j \neq k), \tag{75}$$

$$\begin{cases} (\xi) + \xi(\xi_1) + \eta(\eta_1) + \zeta(\zeta_1) = 0, \\ (\eta) + \xi(\xi_2) + \eta(\eta_2) + \zeta(\zeta_2) = 0, \\ (\zeta) + \xi(\xi_3) + \eta(\eta_3) + \zeta(\zeta_3) = 0, \end{cases} \tag{76}$$

These last three relationships (76) may be written as follows due to the preceding expressions (75) (just as will result from formulas (78) given further on):

$$\begin{cases} \xi + \xi_1(\xi) + \xi_2(\eta) + \xi_3(\zeta) = 0, \\ \eta + \eta_1(\xi) + \eta_2(\eta) + \eta_3(\zeta) = 0, \\ \zeta + \zeta_1(\xi) + \zeta_2(\eta) + \zeta_3(\zeta) = 0. \end{cases} \tag{76'}$$

Before we solve equations (75) and (76), let us note that these systems may be replaced by equivalent ones, which are obtained by differentiating relationships (66') successively with respect to the Eulerian variables x, y, z, t and which, by virtue of the notation (72) (multiplying by α, α', α'' and adding, etc.), may be written /191

$$\begin{cases} \alpha = \sum (\xi_i) \frac{\partial x}{\partial p_i}, & \beta = \sum (\eta_i) \frac{\partial x}{\partial p_i}, & \gamma = \sum (\zeta_i) \frac{\partial x}{\partial p_i}, \\ \alpha' = \sum (\xi_i) \frac{\partial y}{\partial p_i}, & \beta' = \sum (\eta_i) \frac{\partial y}{\partial p_i}, & \gamma' = \sum (\zeta_i) \frac{\partial y}{\partial p_i}, \\ \alpha'' = \sum (\xi_i) \frac{\partial z}{\partial p_i}, & \beta'' = \sum (\eta_i) \frac{\partial z}{\partial p_i}, & \gamma'' = \sum (\zeta_i) \frac{\partial z}{\partial p_i}, \end{cases} \tag{75''}$$

to which we add (76'). System (75") may then be written as follows, multiplying by α, α', α'' and adding, etc.,

$$\begin{cases} \sum \xi_i(\xi_i) = 1, & \sum \xi_i(\eta_i) = 0, & \sum \xi_i(\zeta_i) = 0, \\ \sum \eta_i(\xi_i) = 0, & \sum \eta_i(\eta_i) = 1, & \sum \eta_i(\zeta_i) = 0, \\ \sum \zeta_i(\xi_i) = 0, & \sum \zeta_i(\eta_i) = 0, & \sum \zeta_i(\zeta_i) = 1. \end{cases} \tag{75'}$$

Let us additionally note that the following form, which involves expression (75), is intermediate between expressions (75") and (75) and finally results

(5) Ibid.

from formulas (70) combined with expression (75) and from formulas (74):

$$\begin{cases} \alpha = \sum \xi_i[\xi_i], & \beta = \sum \eta_i[\xi_i], & \gamma = \sum \zeta_i[\xi_i], \\ \alpha' = \sum \xi_i[\eta_i], & \beta' = \sum \eta_i[\eta_i], & \gamma' = \sum \zeta_i[\eta_i], \\ \alpha'' = \sum \xi_i[\zeta_i], & \beta'' = \sum \eta_i[\zeta_i], & \gamma'' = \sum \zeta_i[\zeta_i]. \end{cases} \qquad (75''')$$

It can be seen that the Lagrangian arguments are functions solely of the Eulerian arguments, and vice versa (at least insofar as the translations are concerned).

Let us first determine the Lagrangian arguments by means of the Eulerian arguments. Let us designate by Δ the determinant

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{vmatrix}, \text{ which is } \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}, \quad \text{if} \quad \begin{vmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{vmatrix} = 1;$$

Let ξ_1', η_1', ζ_1', ξ_2', η_2', ζ_2', ξ_3', η_3', ζ_3' be the coefficients of the elements of determinant Δ, i.e., the minors provided with the suitable sign. This amounts to setting

$$\xi_1' = \eta_2\zeta_3 - \eta_3\zeta_2, \quad \eta_1' = \zeta_2\xi_3 - \zeta_3\xi_2, \quad \zeta_1' = \xi_2\eta_3 - \xi_3\eta_2, \ldots.$$

We obtain

$$\begin{cases} (\xi_i) = \dfrac{\xi_i'}{\Delta}, & (\xi) = -\dfrac{\xi\xi_1' + \eta\eta_1' + \zeta\zeta_1'}{\Delta}, \\ (\eta_i) = \dfrac{\eta_i'}{\Delta}, & (\eta) = -\dfrac{\xi\xi_2' + \eta\eta_2' + \zeta\zeta_2'}{\Delta}, \\ (\zeta_i) = \dfrac{\zeta_i'}{\Delta}, & (\zeta) = -\dfrac{\xi\xi_3' + \eta\eta_3' + \zeta\zeta_3'}{\Delta}, \end{cases} \qquad (77)$$ /192

by solving equations (75) for (ξ_i), (η_i), (ζ_i), (ξ), (η), (ζ) and then substituting into expression (76).

Inversely, we may determine ξ_i, η_i, ζ_i, ξ, η, ζ as functions of (ξ_i), (η_i), (ζ_i), (ξ), (η), (ζ). We note that the determinant whose elements are $\Delta(\xi_i)$, $\Delta(\eta_i)$, $\Delta(\zeta_i)$ is the adjunct determinant(6) of Δ, so that we must use $\frac{1}{\Delta}$ to designate the determinant

$$\frac{1}{\Delta} = \begin{vmatrix} (\xi_1) & (\eta_1) & (\zeta_1) \\ (\xi_2) & (\eta_2) & (\zeta_2) \\ (\xi_3) & (\eta_3) & (\zeta_3) \end{vmatrix}. \qquad (78)$$

Let us solve formulas (75) and (76) for ξ_i, η_i, ζ_i, ξ, η, ζ. When the coefficients of the elements of determinant (78) are designated by (ξ_i'), (η_i'), (ζ_i')(7),

(6) This adjunct determinant is the square of Δ.

(7) The first nine formulas of expression (79) ($i = 1, 2, 3$) are, if desired, known consequences of the theory of adjunct determinants. It is clear that all the present calculations could be linked to the theory of forms and to that of linear substitutions.

we have

$$\begin{cases} \xi_i = \Delta(\xi_i'), & \xi = -\Delta\{(\xi)(\xi_1') + (\eta)(\xi_2') + (\zeta)(\xi_3')\}, \\ \eta_i = \Delta(\eta_i'), & \eta = -\Delta\{(\xi)(\eta_1') + (\eta)(\eta_2') + (\zeta)(\eta_3')\}, \\ \zeta_i = \Delta(\zeta_i'), & \zeta = -\Delta\{(\xi)(\zeta_1') + (\eta)(\zeta_2') + (\zeta)(\zeta_3')\}. \end{cases} \tag{79}$$

Let us now set ourselves the task of determining the rotations.

Let us differentiate relationships (67') successively with respect to x, y, z, t. Still using the familiar notation for derivatives with respect to time[8], we will have

$$\begin{aligned} \frac{\partial\alpha}{\partial x} &= \frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial x} + \frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial x} + \frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial x}, \\ \frac{\partial\alpha}{\partial y} &= \frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial y} + \frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial y} + \frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial y}, \\ \frac{\partial\alpha}{\partial z} &= \frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial z} + \frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial z} + \frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial z}, \\ \frac{\partial\alpha}{\partial t} &= \frac{\partial\alpha}{\partial x_0}\frac{\partial x_0}{\partial t} + \frac{\partial\alpha}{\partial y_0}\frac{\partial y_0}{\partial t} + \frac{\partial\alpha}{\partial z_0}\frac{\partial z_0}{\partial t} + \frac{d\alpha}{dt}, \end{aligned}$$

with similar formulas for the cosines β, γ, ..., γ''. /193

Formulas (74) then give

$$\begin{aligned} [p_1] &= \sum p_i[\xi_i], & [p_2] &= \sum q_i[\xi_i], & [p_3] &= \sum r_i[\xi_i], \\ [q_1] &= \sum p_i[\eta_i], & [q_2] &= \sum q_i[\eta_i], & [q_3] &= \sum r_i[\eta_i], \\ [r_1] &= \sum p_i[\zeta_i], & [r_2] &= \sum q_i[\zeta_i], & [r_3] &= \sum r_i[\zeta_i], \end{aligned}$$

and, taking formulas (72) into account, formulas (73) give

$$\begin{cases} (p_1) = \sum p_i(\xi_i), & (p_2) = \sum q_i(\xi_i), & (p_3) = \sum r_i(\xi_i), \\ (q_1) = \sum p_i(\eta_i), & (q_2) = \sum q_i(\eta_i), & (q_3) = \sum r_i(\eta_i), \\ (r_1) = \sum p_i(\zeta_i), & (r_2) = \sum q_i(\zeta_i), & (r_3) = \sum r_i(\zeta_i), \\ & (p) = p_1(\xi) + p_2(\eta) + p_3(\zeta) + p, \\ & (q) = q_1(\xi) + q_2(\eta) + q_3(\zeta) + q, \\ & (r) = r_1(\xi) + r_2(\eta) + r_3(\zeta) + r, \end{cases} \tag{80}$$

which give us the last Eulerian arguments (p_i), (q_i), (r_i), (p), (q), (r) by means of the Lagrange arguments (it is sufficient to replace (ξ_1), ... by their

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(8) We distinguish $\frac{d\alpha}{dt}$ and $\frac{\partial z}{\partial t}$, ..., according to the notation employed by M. Appell, *Traité de Mécanique*, (*Treatise on Mechanics*) Volume III, p. 277. For x_0, y_0, z_0 we have not had to introduce $\frac{dx_0}{dt}$, $\frac{dy_0}{dt}$, $\frac{dz_0}{dt}$, which are zero. It will be noted that the present x_0, y_0, z_0 are functions of x, y, z, t, which, when set equal to the former x_0, y_0, z_0, define functions x, y, z, which are thus implicit functions. We will subsequently return to this point.

values).

Inversely, to obtain the last Lagrangian arguments p_1, ... we could solve system (80), but relationships (69') may also be directly differentiated with respect to x_0, y_0, z_0, t, and we have

$$\frac{\partial x}{\partial x_0} = \frac{\partial x}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial x_0} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial x_0},$$
$$\frac{\partial x}{\partial y_0} = \frac{\partial x}{\partial x}\frac{\partial x}{\partial y_0} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial y_0} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial y_0},$$
$$\frac{\partial x}{\partial z_0} = \frac{\partial x}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial z_0},$$
$$\frac{dx}{dt} = \frac{\partial x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial x}{\partial t}.$$

Relationships (71), with due account taken of expression (70), then give us

$$\begin{cases} p_1 = (p_1)\xi_1 + (q_1)\eta_1 + (r_1)\zeta_1, \\ q_1 = (p_2)\xi_1 + (q_2)\eta_1 + (r_2)\zeta_1, \\ r_1 = (p_3)\xi_1 + (q_3)\eta_1 + (r_3)\zeta_1, \end{cases} \tag{81}$$

which may be written in the intermediate form

$$p_1 = [p_1]\frac{\partial x}{\partial x_0} + [q_1]\frac{\partial y}{\partial x_0} + [r_1]\frac{\partial z}{\partial x_0},$$
$$q_1 = [p_2]\frac{\partial x}{\partial x_0} + [q_2]\frac{\partial y}{\partial x_0} + [r_2]\frac{\partial z}{\partial x_0},$$
$$r_1 = [p_3]\frac{\partial x}{\partial x_0} + [q_3]\frac{\partial y}{\partial x_0} + [r_3]\frac{\partial z}{\partial x_0},$$

with similar formulas for p_2, q_2, r_2; p_3, q_3, r_3 which are obtained by changing /194 ξ_1, η_1, ζ_1 into ξ_2, η_2, ζ_2 and then into ξ_3, η_3, ζ_3, or by changing x_0 into y_0 and then into z_0. Furthermore, we have

$$\begin{cases} p = (p_1)\xi + (q_1)\eta + (r_1)\zeta + (p), \\ q = (p_2)\xi + (q_2)\eta + (r_2)\zeta + (q), \\ r = (p_3)\xi + (q_3)\eta + (r_3)\zeta + (r). \end{cases} \tag{81'}$$

72. Equations of the Statics of the Deformable Medium Relative to the Eulerian Variables which are First Deduced from Equations Derived by Lagrangian Variables. In regard to the statics of the deformable medium, we have already passed from Lagrangian to Eulerian variables. It is first sufficient for us to complete the results obtained(9).

In Section 53, we found formulas like the following:

$$\Delta p_{xx} = \frac{\partial x}{\partial x_0}A_1 + \frac{\partial x}{\partial y_0}A_2 + \frac{\partial x}{\partial z_0}A_3, \qquad \Delta q_{xx} = \frac{\partial x}{\partial x_0}P_1 + \frac{\partial x}{\partial y_0}P_2 + \frac{\partial x}{\partial z_0}P_3,$$
$$\Delta p_{yx} = \frac{\partial y}{\partial x_0}A_1 + \frac{\partial y}{\partial y_0}A_2 + \frac{\partial y}{\partial z_0}A_3, \qquad \Delta q_{yx} = \frac{\partial y}{\partial x_0}P_1 + \frac{\partial y}{\partial y_0}P_2 + \frac{\partial y}{\partial z_0}P_3,$$
$$\Delta p_{zx} = \frac{\partial z}{\partial x_0}A_1 + \frac{\partial z}{\partial y_0}A_2 + \frac{\partial z}{\partial z_0}A_3, \qquad \Delta q_{zx} = \frac{\partial z}{\partial x_0}P_1 + \frac{\partial z}{\partial y_0}P_2 + \frac{\partial z}{\partial z_0}P_3,$$

(9) We will try to obtain the definitive results directly.

where we have

$$A_i = \alpha \frac{\partial W}{\partial \xi_i} + \beta \frac{\partial W}{\partial \eta_i} + \gamma \frac{\partial W}{\partial \zeta_i}, \qquad P_i = \alpha \frac{\partial W}{\partial p_i} + \beta \frac{\partial W}{\partial q_i} + \gamma \frac{\partial W}{\partial r_i}.$$

Let us assume that W is expressed by the arguments (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i), and let us set

$$W = \Delta\Omega.$$

Because of formulas (77) in the preceding section, we will have

$$\frac{\partial W}{\partial \xi_i} = \Delta \frac{\partial \Omega}{\partial \xi_i} + \Omega \xi_i' = \Delta \left\{ \frac{\partial \Omega}{\partial \xi_i} + \Omega(\xi_i) \right\},$$
$$\frac{\partial W}{\partial \eta_i} = \Delta \frac{\partial \Omega}{\partial \eta_i} + \Omega \eta_i' = \Delta \left\{ \frac{\partial \Omega}{\partial \eta_i} + \Omega(\eta_i) \right\},$$
$$\frac{\partial W}{\partial \zeta_i} = \Delta \frac{\partial \Omega}{\partial \zeta_i} + \Omega \zeta_i' = \Delta \left\{ \frac{\partial \Omega}{\partial \zeta_i} + \Omega(\zeta_i) \right\},$$

and consequently, since Δ does not depend on the p_i, q_i, r_i, we have /195

$$A_i = \Delta \left\{ \alpha \frac{\partial \Omega}{\partial \xi_i} + \beta \frac{\partial \Omega}{\partial \eta_i} + \gamma \frac{\partial \Omega}{\partial \zeta_i} + \Omega[\xi_i] \right\},$$
$$P_i = \Delta \left\{ \alpha \frac{\partial \Omega}{\partial p_i} + \beta \frac{\partial \Omega}{\partial q_i} + \gamma \frac{\partial \Omega}{\partial r_i} \right\}.$$

By differentiating relationships (75) with respect to ξ_i, we have

$$\xi_i \frac{\partial(\xi_j)}{\partial \xi_i} + \eta_i \frac{\partial(\eta_j)}{\partial \xi_i} + \zeta_i \frac{\partial(\zeta_j)}{\partial \xi_i} = -(\xi_i), \quad \xi_j \frac{\partial(\xi_k)}{\partial \xi_i} + \eta_j \frac{\partial(\eta_k)}{\partial \xi_i} + \zeta_j \frac{\partial(\zeta_k)}{\partial \xi_i} = 0 \quad (i \neq j),$$

from which we deduce

$$\frac{\partial(\xi_j)}{\partial \xi_i} = -(\xi_j) \frac{\xi_i'}{\Delta} = -(\xi_i)(\xi_j),$$
$$\frac{\partial(\eta_j)}{\partial \xi_i} = -(\xi_j) \frac{\eta_i'}{\Delta} = -(\eta_i)(\xi_j),$$
$$\frac{\partial(\zeta_j)}{\partial \xi_i} = -(\xi_j) \frac{\zeta_i'}{\Delta} = -(\zeta_i)(\xi_j);$$

By means of relationships (80), we have

$$\frac{\partial(p_j)}{\partial \xi_i} = -(p_j)(\xi_i),$$
$$\frac{\partial(q_j)}{\partial \xi_i} = -(p_j)(\eta_i),$$
$$\frac{\partial(r_j)}{\partial \xi_i} = -(p_j)(\zeta_i),$$

with similar formulas for the derivatives with respect to η_i, ζ_i. If we set

$$(A_i') = \frac{\partial \Omega}{\partial(\xi_i)}, \quad (B_i') = \frac{\partial \Omega}{\partial(\eta_i)}, \quad (C_i') = \frac{\partial \Omega}{\partial(\zeta_i)},$$
$$(P_i') = \frac{\partial \Omega}{\partial(p_i)}, \quad (Q_i') = \frac{\partial \Omega}{\partial(q_i)}, \quad (R_i') = \frac{\partial \Omega}{\partial(r_i)},$$

we thus have

$$\frac{1}{\Delta}A_i=\Omega[\xi_i]-\Big[\{(\xi_i)(A_1')+(\eta_i)(B_1')+(\zeta_i)(C_1')\}[\xi_1]+\{(\xi_i)(P_1')+(\eta_i)(Q_1')+(\zeta_i)(R_1')\}[p_1]$$
$$+\{(\xi_i)(A_2')+(\eta_i)(B_2')+(\zeta_i)(C_2')\}[\xi_2]+\{(\xi_i)(P_2')+(\eta_i)(Q_2')+(\zeta_i)(R_2')\}[p_2]$$
$$+\{(\xi_i)(A_3')+(\eta_i)(B_3')+(\zeta_i)(C_3')\}[\xi_3]+\{(\xi_i)(P_3')+(\eta_i)(Q_3')+(\zeta_i)(R_3')\}[p_3]\Big].$$

Because of formulas (72), (73), (74), (75") and using $[A_i]$, $[B_i]$, $[C_i]$; $[P_i]$, $[Q_i]$, $[R_i]$ to designate the components relative to axes Ox, Oy, Oz of the two vectors whose components with respect to the Mx', My', Mz' axes are, respectively, (A_i'), (B_i'), (C_i'); (P_i'), (Q_i'), (R_i'), we obtain the three following formulas:

$$p_{xx}=\Omega-\sum[A_i][\xi_i]-\sum[P_i][p_i],$$
$$p_{yx}=\quad-\sum[B_i][\xi_i]-\sum[Q_i][p_i],$$
$$p_{zx}=\quad-\sum[C_i][\xi_i]-\sum[R_i][p_i],$$

with similar formulas for B_i, C_i, and p_{xy}, p_{yy}, p_{zy}, p_{xz}, p_{yz}, p_{zz}. We then /196 have

$$\frac{1}{\Delta}P_i=\alpha\left\{(\xi_i)\frac{\partial\Omega}{\partial(p_1)}+(\eta_i)\frac{\partial\Omega}{\partial(q_1)}+(\zeta_i)\frac{\partial\Omega}{\partial(r_1)}\right\}$$
$$+\beta\left\{(\xi_i)\frac{\partial\Omega}{\partial(p_2)}+(\eta_i)\frac{\partial\Omega}{\partial(q_2)}+(\zeta_i)\frac{\partial\Omega}{\partial(r_2)}\right\}$$
$$+\gamma\left\{(\xi_i)\frac{\partial\Omega}{\partial(p_3)}+(\eta_i)\frac{\partial\Omega}{\partial(q_3)}+(\zeta_i)\frac{\partial\Omega}{\partial(r_3)}\right\},$$

and, again taking expression (75") into account, we obtain the following three formulas:

$$q_{xx}=\alpha[P_1]+\beta[P_2]+\gamma[P_3],$$
$$q_{yx}=\alpha[Q_1]+\beta[Q_2]+\gamma[Q_3],$$
$$q_{zx}=\alpha[R_1]+\beta[R_2]+\gamma[R_3],$$

with similar formulas for Q_i, R_i, and q_{xy}, q_{yy}, q_{zy}, q_{xz}, q_{yz}, q_{zz}.

73. Equations of the Dynamics of the Deformable Medium Relative to the Eulerian Variables and First Deduced from Equations Derived by Lagrangian Variables. With respect to the dynamics of the deformable medium, we have also passed from the Lagrangian to the Eulerian variables. We are first going to complete the results obtained.

A_i is increased by

$$\Delta\Big[\Big\{\alpha\frac{\partial(\xi)}{\partial\xi_i}+\beta\frac{\partial(\xi)}{\partial\eta_i}+\gamma\frac{\partial(\xi)}{\partial\zeta_i}\Big\}\frac{\partial\Omega}{\partial(\xi)}+\Big\{\alpha\frac{\partial(p)}{\partial\xi_i}+\beta\frac{\partial(p)}{\partial\eta_i}+\gamma\frac{\partial'p)}{\partial\zeta_i}\Big\}\frac{\partial\Omega}{\partial(p)}$$
$$+\Big\{\alpha\frac{\partial(\eta)}{\partial\xi_i}+\beta\frac{\partial(\eta)}{\partial\eta_i}+\gamma\frac{\partial(\eta)}{\partial\zeta_i}\Big\}\frac{\partial\Omega}{\partial(\eta)}+\Big\{\alpha\frac{\partial(q)}{\partial\xi_i}+\beta\frac{\partial(q)}{\partial\eta_i}+\gamma\frac{\partial'q)}{\partial\zeta_i}\Big\}\frac{\partial\Omega}{\partial(q)}$$
$$+\Big\{\alpha\frac{\partial(\zeta)}{\partial\xi_i}+\beta\frac{\partial(\zeta)}{\partial\eta_i}+\gamma\frac{\partial(\zeta)}{\partial\zeta_i}\Big\}\frac{\partial\Omega}{\partial(\zeta)}+\Big\{\alpha\frac{\partial(r)}{\partial\xi_i}+\beta\frac{\partial(r)}{\partial\eta_i}+\gamma\frac{\partial(r)}{\partial\zeta_i}\Big\}\frac{\partial\Omega}{\partial(r)}\Big];$$

According to expressions (76) and (80),

$$\frac{\partial(\xi)}{\partial\xi_1} = -(\xi_1)(\xi), \quad \frac{\partial(\xi)}{\partial\xi_2} = -(\xi_1)(\eta), \quad \frac{\partial(\xi)}{\partial\xi_3} = -(\xi_1)(\zeta),$$
$$\frac{\partial(p)}{\partial\xi_1} = -(p_1)(\xi), \quad \frac{\partial(p)}{\partial\xi_2} = -(p_1)(\eta), \quad \frac{\partial(p)}{\partial\xi_3} = -(p_1)(\zeta),$$

with similar formulas, so that by setting

$$(A') = \frac{\partial\Omega}{\partial(\xi)}, \quad (B') = \frac{\partial\Omega}{\partial(\eta)}, \quad (C') = \frac{\partial\Omega}{\partial(\zeta)},$$
$$(P') = \frac{\partial\Omega}{\partial(p)}, \quad (Q') = \frac{\partial\Omega}{\partial(q)}, \quad (R') = \frac{\partial\Omega}{\partial(r)},$$

we have to add to the value given for A_i in the preceding section /197

$$A(\xi), \quad A(\eta), \quad A(\zeta),$$

respectively, for i = 1, 2, 3 by setting

$$-\frac{A}{\Delta} = (A')[\xi_1] + (B')[\xi_2] + (C')[\xi_3] + (P')[p_1] + (Q')[p_2] + (R')[p_3].$$

The expressions to be added to the values of p_{xx}, p_{xy}, p_{xz} in the preceding section are therefore

$$\frac{A}{\Delta}\left\{(\xi)\frac{\partial x}{\partial x_0} + (\eta)\frac{\partial x}{\partial y_0} + (\zeta)\frac{\partial x}{\partial z_0}\right\}, \quad \frac{A}{\Delta}\left\{(\xi)\frac{\partial y}{\partial x_0} + (\eta)\frac{\partial y}{\partial y_0} + (\zeta)\frac{\partial y}{\partial z_0}\right\},$$
$$\frac{A}{\Delta}\left\{(\xi)\frac{\partial z}{\partial x_0} + (\eta)\frac{\partial z}{\partial y_0} + (\zeta)\frac{\partial z}{\partial z_0}\right\};$$

According to the values in expression (76) for (ξ), (η), (ζ), we have

$$(\xi)\frac{\partial x}{\partial x_0} + (\eta)\frac{\partial x}{\partial y_0} + (\zeta)\frac{\partial x}{\partial z_0} = -\xi\sum(\xi_i)\frac{\partial x}{\partial p_i} - \eta\sum(\eta_i)\frac{\partial x}{\partial p_i} - \zeta\sum(\zeta_i)\frac{\partial x}{\partial p_i},$$
$$(\xi)\frac{\partial y}{\partial x_0} + (\eta)\frac{\partial y}{\partial y_0} + (\zeta)\frac{\partial y}{\partial z_0} = -\xi\sum(\xi_i)\frac{\partial y}{\partial p_i} - \eta\sum(\eta_i)\frac{\partial y}{\partial p_i} - \zeta\sum(\zeta_i)\frac{\partial y}{\partial p_i},$$
$$(\xi)\frac{\partial z}{\partial x_0} + (\eta)\frac{\partial z}{\partial y_0} + (\zeta)\frac{\partial z}{\partial z_0} = -\xi\sum(\xi_i)\frac{\partial z}{\partial p_i} - \eta\sum(\eta_i)\frac{\partial z}{\partial p_i} - \zeta\sum(\zeta_i)\frac{\partial z}{\partial p_i},$$

i.e., by virtue of formulas (75"),

$$(\xi)\frac{\partial x}{\partial x_0} + (\eta)\frac{\partial x}{\partial y_0} + (\zeta)\frac{\partial x}{\partial z_0} = -(\alpha\xi + \beta\eta + \gamma\zeta),$$
$$(\xi)\frac{\partial y}{\partial x_0} + (\eta)\frac{\partial y}{\partial y_0} + (\zeta)\frac{\partial y}{\partial z_0} = -(\alpha'\xi + \beta'\eta + \gamma'\zeta),$$
$$(\xi)\frac{\partial z}{\partial x_0} + (\eta)\frac{\partial z}{\partial y_0} + (\zeta)\frac{\partial z}{\partial z_0} = -(\alpha''\xi + \beta''\eta + \gamma''\zeta),$$

so that the expressions to be added to the values of p_{xx}, p_{xy}, p_{xz} in the preceding section are

$$-\frac{A}{\Delta}\frac{dx}{dt}, \quad -\frac{A}{\Delta}\frac{dy}{dt}, \quad -\frac{A}{\Delta}\frac{dz}{dt}.$$

There will be similar expressions for p_{yx}, ..., p_{zx}, ..., by the obvious change in A in two analogous expressions B and C, which are deduced from it by replacing the $[\xi_i]$, $[p_i]$ by the corresponding quantities $[\eta_i]$, $[q_i]$ and $[\zeta_i]$,

$[r_i]$.

We have just introduced the notation A, B, C. Let us show that it is identical to the notation introduced in the Lagrangian theory:

$$A = \alpha \frac{\partial W}{\partial \xi} + \beta \frac{\partial W}{\partial \eta} + \gamma \frac{\partial W}{\partial \zeta}, \ldots;$$

We have, in fact,

$$\frac{A}{\Delta} = \alpha \left[(A') \frac{\partial(\xi)}{\partial \xi} + (B') \frac{\partial(\eta)}{\partial \xi} + \ldots + (R') \frac{\partial(r)}{\partial \xi} \right] + \beta \left[(A') \frac{\partial(\xi)}{\partial \eta} + \ldots \right] + \gamma \left[(A') \frac{\partial(\xi)}{\partial \zeta} + \ldots \right];$$

According to formulas (76) and (80), we have /198

$$\frac{\partial(\xi)}{\partial \xi} = -(\xi_1), \quad \frac{\partial(\eta)}{\partial \xi} = -(\xi_2), \quad \frac{\partial(\zeta)}{\partial \xi} = -(\xi_3),$$
$$\frac{\partial(p)}{\partial \xi} = -(p_1), \quad \frac{\partial(q)}{\partial \xi} = -(p_2), \quad \frac{\partial(r)}{\partial \xi} = -(p_3),$$

and similar relationships for η, ζ. Because of relationships (72), we therefore obtain

$$-\frac{A}{\Delta} = (A')[\xi_1] + (B')[\xi_2] + (C')[\xi_3] + (P')[p_1] + (Q')[p_2] + (R')[p_3].$$

Likewise, for the P, Q, R of the Lagrangian theory, i.e.,

$$P = \alpha \frac{\partial W}{\partial p} + \beta \frac{\partial W}{\partial q} + \gamma \frac{\partial W}{\partial r}, \ldots,$$

because of relationships (80) we have

$$\frac{P}{\Delta} = \alpha(P') + \beta(Q') + \gamma(R'), \ldots.$$

Let us consider, finally, the modification which must be made in the formulas of the preceding section in order to have the q_{xx}, ... relative to the present case in dynamics.

The quantities which we have called p_i are increased by

$$\Delta\left[(P')\left\{\alpha\frac{\partial(p)}{\partial p_1}+\beta\frac{\partial(p)}{\partial q_1}+\gamma\frac{\partial(p)}{\partial r_1}\right\}+(Q')\left\{\alpha\frac{\partial(q)}{\partial p_1}+\ldots\right\}+(R')\left\{\alpha\frac{\partial(r)}{\partial p_1}+\ldots\right\}\right]$$
$$\Delta\left[(P')\left\{\alpha\frac{\partial(p)}{\partial p_2}+\beta\frac{\partial(p)}{\partial q_2}+\gamma\frac{\partial(p)}{\partial r_2}\right\}+(Q')\left\{\alpha\frac{\partial(q)}{\partial p_2}+\ldots\right\}+(R')\left\{\alpha\frac{\partial(r)}{\partial p_2}+\ldots\right\}\right]$$
$$\Delta\left[(P')\left\{\alpha\frac{\partial(p)}{\partial p_3}+\beta\frac{\partial(p)}{\partial q_3}+\gamma\frac{\partial(p)}{\partial r_3}\right\}+(Q')\left\{\alpha\frac{\partial(q)}{\partial p_3}+\ldots\right\}+(R')\left\{\alpha\frac{\partial(r)}{\partial p_3}+\ldots\right\}\right]$$

respectively, or by

$$\Delta(\xi)\left\{\alpha(P')+\beta(Q')+\gamma(R')\right\}$$
$$\Delta(\eta)\left\{\alpha(P')+\beta(Q')+\gamma(R')\right\}$$
$$\Delta(\zeta)\left\{\alpha(P')+\beta(Q')+\gamma(R')\right\}.$$

for i = 1, 2, 3 because of formulas (80). It is seen that these increases are

$$P(\xi),\quad P(\eta),\quad P(\zeta);$$

The expressions to be added to the values of q_{xx}, q_{xy}, q_{xz} in the preceding section are therefore /199

$$\frac{P}{\Delta}\left\{(\xi)\frac{\partial x}{\partial x_0}+(\eta)\frac{\partial x}{\partial y_0}+(\zeta)\frac{\partial x}{\partial z_0}\right\},$$

$$\frac{P}{\Delta}\left\{(\xi)\frac{\partial y}{\partial x_0}+(\eta)\frac{\partial y}{\partial y_0}+(\zeta)\frac{\partial y}{\partial z_0}\right\},\quad \frac{P}{\Delta}\left\{(\xi)\frac{\partial z}{\partial x_0}+(\eta)\frac{\partial z}{\partial y_0}+(\zeta)\frac{\partial z}{\partial z_0}\right\},$$

i.e.,

$$-\frac{P}{\Delta}(\alpha\xi+\beta\eta+\gamma\zeta),\quad -\frac{P}{\Delta}(\alpha'\xi+\beta'\eta+\gamma'\zeta),\quad -\frac{P}{\Delta}(\alpha''\xi+\beta''\eta+\gamma''\zeta),$$

or, finally,

$$-\frac{P}{\Delta}\frac{dx}{dt},\quad -\frac{P}{\Delta}\frac{dy}{dt},\quad -\frac{P}{\Delta}\frac{dz}{dt}.$$

There will be similar expressions for q_{yx}, ...; q_{zx}, ... by the change of P into Q and then into R.

74. Variations in Eulerian Arguments Deduced from those of Lagrangian Arguments. In order to formulate directly the Eulerian equations relative to the deformable medium, we are going to calculate the variations in the Eulerian arguments. We will begin by deducing them from the variations of the Lagrangian arguments in order to have a check; then we will calculate them directly.

When the sign δ is applied to equations (75), we obtain three systems like

$$\begin{aligned}
\xi_1\delta(\xi_1)+\eta_1\delta(\eta_1)+\zeta_1\delta(\zeta_1)&=-(\xi_1)\delta\xi_1-(\eta_1)\delta\eta_1-(\zeta_1)\delta\zeta_1,\\
\xi_2\delta(\xi_1)+\eta_2\delta(\eta_1)+\zeta_2\delta(\zeta_1)&=-(\xi_1)\delta\xi_2-(\eta_1)\delta\eta_2-(\zeta_1)\delta\zeta_2,\\
\xi_3\delta(\xi_1)+\eta_3\delta(\eta_1)+\zeta_3\delta(\zeta_1)&=-(\xi_1)\delta\xi_3-(\eta_1)\delta\eta_3-(\zeta_1)\delta\zeta_3,
\end{aligned}$$

from which, when relationships (77) are taken into consideration, we obtain

$$\begin{aligned}
-\delta(\xi_1)&=(\xi_1)\{(\xi_1)\delta\xi_1+(\eta_1)\delta\eta_1+(\zeta_1)\delta\zeta_1\}+(\xi_2)\{(\xi_1)\delta\xi_2+\ldots\}+(\xi_3)\{(\xi_1)\delta\xi_3+\ldots\}\\
&=(\xi_1)\sum(\xi_i)\delta\xi_i+(\eta_1)\sum(\xi_i)\delta\eta_i+(\zeta_1)\sum(\xi_i)\delta\zeta_i,
\end{aligned}$$

or replacing $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$ by their values and taking relationships (75') and (80) into account, we have

$$\begin{aligned}
\delta(\xi_1)=(\eta_1)\delta z'-(\zeta_1)\delta y'&-(\xi_1)\left\{(\xi_1)\frac{\partial\delta' x}{\partial x_0}+(\xi_2)\frac{\partial\delta' x}{\partial y_0}+(\xi_3)\frac{\partial\delta' x}{\partial z_0}+(p_2)\delta' z-(p_3)\delta' y\right\}\\
&-(\eta_1)\left\{(\xi_1)\frac{\partial\delta' y}{\partial x_0}+(\xi_2)\frac{\partial\delta' y}{\partial y_0}+(\xi_3)\frac{\partial\delta' y}{\partial z_0}+(p_3)\delta' x-(p_1)\delta' z\right\}\\
&-(\zeta_1)\left\{(\xi_1)\frac{\partial\delta' z}{\partial x_0}+(\xi_2)\frac{\partial\delta' z}{\partial y_0}+(\xi_3)\frac{\partial\delta' z}{\partial z_0}+(p_1)\delta' y-(p_2)\delta' x\right\};
\end{aligned}$$

However, for example, due to equations (75") we have /200

$$\sum(\xi_i)\frac{\partial\delta' x}{\partial p_i}=\frac{\partial\delta' x}{\partial x}\sum(\xi_i)\frac{\partial x}{\partial p_i}+\frac{\partial\delta' x}{\partial y}\sum(\xi_i)\frac{\partial y}{\partial p_i}+\frac{\partial\delta' x}{\partial z}\sum(\xi_i)\frac{\partial z}{\partial p_i}$$
$$=\alpha\frac{\partial\delta' x}{\partial x}+\alpha'\frac{\partial\delta' x}{\partial y}+\alpha''\frac{\partial\delta' x}{\partial z},$$

We thus obtain the following relationship:

$$\delta(\xi_1)=(\eta_1)\delta z'-(\zeta_1)\delta y'-(\xi_1)\left\{\alpha\frac{\partial\delta' x}{\partial x}+\alpha'\frac{\partial\delta' x}{\partial y}+\alpha''\frac{\partial\delta' x}{\partial z}+(p_2)\delta' z-(p_3)\delta' y\right\}$$
$$-(\eta_1)\left\{\alpha\frac{\partial\delta' y}{\partial x}+\alpha'\frac{\partial\delta' y}{\partial y}+\alpha''\frac{\partial\delta' y}{\partial z}+(p_3)\delta' x-(p_1)\delta' z\right\}$$
$$-(\zeta_1)\left\{\alpha\frac{\partial\delta' z}{\partial x}+\alpha'\frac{\partial\delta' z}{\partial y}+\alpha''\frac{\partial\delta' z}{\partial z}+(p_1)\delta' y-(p_2)\delta' x\right\};$$

To have $\delta(\eta_1)$, $\delta(\zeta_1)$, it is sufficient to subject (ξ_1), (η_1), (ζ_1) to circular permutation and to replace α, α', α'' by β, β', β'', and then by γ, γ', γ'', as well as to replace the p's by the q_i's, and then by the r_i's. There are systems of similar formulas for $\delta(\xi_2)$, $\delta(\eta_2)$, $\delta(\zeta_2)$; $\delta(\xi_3)$, $\delta(\eta_3)$, $\delta(\zeta_3)$.

Next, by means of expression (76) and the values of $\delta\xi$, $\delta\eta$, $\delta\zeta$ we have

$$\delta(\xi)=-\left\{\xi\delta(\xi_1)+\eta\delta(\eta_1)+\zeta\delta(\zeta_1)\right\}-\left\{(\xi_1)\delta\xi+(\eta_1)\delta\eta+(\zeta_1)\delta\zeta\right\}$$
$$=-(\xi_1)\left[\frac{d\delta' x}{dt}-(\alpha\xi+\beta\eta+\gamma\zeta)\frac{\partial\delta' x}{\partial x}-(\alpha'\xi+\beta'\eta+\gamma'\zeta)\frac{\partial\delta' x}{\partial y}-(\alpha''\xi+\beta''\eta+\gamma''\zeta)\frac{\partial\delta' x}{\partial z}\right.$$
$$\left.+\left\{q-(p_2)\xi-(q_2)\eta-(r_2)\zeta\right\}\delta' z-\left\{r-(p_3)\xi-(q_3)\eta-(r_3)\zeta\right\}\delta' y\right]$$
$$-(\eta_1)\left[\frac{d\delta' y}{dt}-(\alpha\xi+\beta\eta+\gamma\zeta)\frac{\partial\delta' y}{\partial x}-(\alpha'\xi+\beta'\eta+\gamma'\zeta)\frac{\partial\delta' y}{\partial y}-(\alpha''\xi+\beta''\eta+\gamma''\zeta)\frac{\partial\delta' y}{\partial z}\right.$$
$$\left.+\left\{r-(p_3)\xi-(q_3)\eta-(r_3)\zeta\right\}\delta' x-\left\{p-(p_1)\xi-(q_1)\eta-(r_1)\zeta\right\}\delta' z\right]$$
$$-(\zeta_1)\left[\frac{d\delta' z}{dt}-(\alpha\xi+\beta\eta+\gamma\zeta)\frac{\partial\delta' z}{\partial x}-(\alpha'\xi+\beta'\eta+\gamma'\zeta)\frac{\partial\delta' z}{\partial y}-(\alpha''\xi+\beta''\eta+\gamma''\zeta)\frac{\partial\delta' z}{\partial z}\right.$$
$$\left.+\left\{p-(p_1)\xi-(q_1)\eta-(r_1)\zeta\right\}\delta' y-\left\{q-(p_2)\xi-(q_2)\eta-(r_2)\zeta\right\}\delta' x\right];$$

Relations (80) yield the following due to (76)

$$(p_1)\xi+(q_1)\eta+(r_1)\zeta=-\left\{p_1(\xi)+p_2(\eta)+p_3(\zeta)\right\},$$
$$(p_2)\xi+(q_2)\eta+(r_2)\zeta=-\left\{q_1(\xi)+q_2(\eta)+q_3(\zeta)\right\},$$
$$(p_3)\xi+(q_3)\eta+(r_3)\zeta=-\left\{r_1(\xi)+r_2(\eta)+r_3(\zeta)\right\},$$

from which we finally obtain

$$\delta(\xi)=-(\xi_1)\left\{\frac{d\delta' x}{dt}-\frac{dx}{dt}\frac{\partial\delta' x}{\partial x}-\frac{dy}{dt}\frac{\partial\delta' x}{\partial y}-\frac{dz}{dt}\frac{\partial\delta' x}{\partial z}+(q)\delta' z-(r)\delta' y\right\}$$
$$-(\eta_1)\left\{\frac{d\delta' y}{dt}-\frac{dx}{dt}\frac{\partial\delta' y}{\partial x}-\frac{dy}{dt}\frac{\partial\delta' y}{dy}-\frac{dz}{dt}\frac{\partial\delta' y}{\partial z}+(r)\delta' x-(p)\delta' z\right\}$$
$$-(\zeta_1)\left\{\frac{d\delta' z}{dt}-\frac{dx}{dt}\frac{\partial\delta' z}{\partial x}-\frac{dy}{dt}\frac{\partial\delta' z}{\partial y}-\frac{dz}{dt}\frac{\partial\delta' z}{\partial z}+(p)\delta' y-(q)\delta' x\right\}.$$

There will be similar values for $\delta(\eta)$, $\delta(\zeta)$ by changing (ξ_1), (η_1), (ζ_1) /201 into (ξ_2), (η_2), (ζ_2) and then into (ξ_3), (η_3), (ζ_3).

According to expression (80), we now have

$$\delta(p_1) = (\xi_1)\delta p_1 + (\xi_2)\delta p_2 + (\xi_3)\delta p_3 + p_1\delta(\xi_1) + p_2\delta(\xi_2) + p_3\delta(\xi_3),$$

i.e., by means of formulas (75") we have

$$\begin{aligned}\delta(p_1) = {} & (q_1)\delta\kappa' - (r_1)\delta\tau' \\ & + \alpha\frac{\partial\delta\tau'}{\partial x} + \alpha'\frac{\partial\delta\tau'}{\partial y} + \alpha''\frac{\partial\delta\tau'}{\partial z} + (p_2)\delta\kappa' - (p_3)\delta\tau' \\ & - (p_1)\left\{\alpha\frac{\partial\delta' x}{\partial x} + \alpha'\frac{\partial\delta' x}{\partial y} + \alpha''\frac{\partial\delta' x}{\partial z} + (p_2)\delta' z - (p_3)\delta' y\right\} \\ & - (q_1)\left\{\alpha\frac{\partial\delta' y}{\partial x} + \alpha'\frac{\partial\delta' y}{\partial y} + \alpha''\frac{\partial\delta' y}{\partial z} + (p_3)\delta' x - (p_1)\delta' z\right\} \\ & - (r_1)\left\{\alpha\frac{\partial\delta' z}{\partial x} + \alpha'\frac{\partial\delta' z}{\partial y} + \alpha''\frac{\partial\delta' z}{\partial z} + (p_1)\delta' y - (p_2)\delta' x\right\}\end{aligned}$$

with similar formulas for $\delta(q_1)$, $\delta(r_1)$ and for $\delta(p_2)$, $\delta(q_2)$, $\delta(q_2)$, $\delta(r_2)$; $\delta(p_3)$, $\delta(q_3)$, $\delta(r_3)$.

We then have

$$\delta(p) = \delta p + (\xi)\delta p_1 + (\eta)\delta p_2 + (\zeta)\delta p_3 + p_1\delta(\xi) + p_2\delta(\eta) + p_3\delta(\zeta),$$

i.e., because of formulas (75"), (76), and (80):

$$\begin{aligned}\delta(p) = {} & \frac{d\delta\tau'}{dl} - \frac{\partial\delta\tau'}{\partial x}\frac{dx}{dl} - \frac{\partial\delta\tau'}{\partial y}\frac{dy}{dl} - \frac{\partial\delta\tau'}{\partial z}\frac{dz}{dl} + (q)\delta\kappa' - (r)\delta\tau' \\ & - (p_1)\left\{\frac{d\delta' x}{dl} - \frac{\partial\delta' x}{\partial x}\frac{dx}{dl} - \frac{\partial\delta' x}{\partial y}\frac{dy}{dl} - \frac{\partial\delta' x}{\partial z}\frac{dz}{dl} + (q)\delta' z - (r)\delta' y\right\} \\ & - (q_1)\left\{\frac{d\delta' y}{dl} - \frac{\partial\delta' y}{\partial x}\frac{dx}{dl} - \frac{\partial\delta' y}{\partial y}\frac{dy}{dl} - \frac{\partial\delta' y}{\partial z}\frac{dz}{dl} + (r)\delta' x - (p)\delta' z\right\} \\ & - (r_1)\left\{\frac{d\delta' z}{dl} - \frac{\partial\delta' z}{\partial x}\frac{dx}{dl} - \frac{\partial\delta' z}{\partial y}\frac{dy}{dl} - \frac{\partial\delta' z}{\partial z}\frac{dz}{dl} + (p)\delta' y - (q)\delta' x\right\}\end{aligned}$$

with similar formulas for $\delta(q)$, $\delta(r)$.

Let us now try to determine the form of the formulas just established when the auxiliary quantities δx, δy, δz, δI, δJ, δK, defined as we have previously done, are introduced. We have, for example,

$$\frac{\partial\delta x}{\partial x} = \alpha\frac{\partial\delta' x}{\partial x} + \beta\frac{\partial\delta' y}{\partial x} + \gamma\frac{\partial\delta' z}{\partial x} + \frac{\partial\alpha}{\partial x}\delta' x + \frac{\partial\beta}{\partial x}\delta' y + \frac{\partial\gamma}{\partial x}\delta' y$$

and similar expressions for $\frac{\partial\delta y}{\partial x}$, $\frac{\partial\delta z}{\partial x}$, from which we obtain the system

$$\begin{aligned}\alpha\frac{\partial\delta x}{\partial x} + \alpha'\frac{\partial\delta y}{\partial x} + \alpha''\frac{\partial\delta z}{\partial x} &= \frac{\partial\delta' x}{\partial x} + [p_2]\delta' z - [p_3]\delta' y, \\ \beta\frac{\partial\delta x}{\partial x} + \beta'\frac{\partial\delta y}{\partial x} + \beta''\frac{\partial\delta z}{\partial x} &= \frac{\partial\delta' y}{\partial x} + [p_3]\delta' x - [p_1]\delta' z, \\ \gamma\frac{\partial\delta x}{\partial x} + \gamma'\frac{\partial\delta y}{\partial x} + \gamma''\frac{\partial\delta z}{\partial x} &= \frac{\partial\delta' z}{\partial x} + [p_1]\delta' y - [p_2]\delta' x,\end{aligned}$$

and similar systems for the derivatives with respect to y and z. There are /202
similar formulas connecting $\delta I'$, $\delta J'$, $\delta K'$ and δI, δJ, δK. Because of formulas (72) and under the assumption that the determinant $|\alpha'\beta'\gamma''| = 1$, we then have

$$\delta(\xi_1) = -[\xi_1]\left(\alpha\frac{\partial\delta x}{\partial x} + \alpha'\frac{\partial\delta x}{\partial y} + \alpha''\frac{\partial\delta x}{\partial z}\right) + (\alpha'[\zeta_1] - \alpha''[\eta_1])\delta I \qquad (82)$$

$$-[\eta_1]\left(\alpha\frac{\partial\delta y}{\partial x}+\alpha'\frac{\partial\delta y}{\partial y}+\alpha''\frac{\partial\delta y}{\partial z}\right)+(\alpha''[\xi_1]-\alpha[\zeta_1])\delta J$$
$$-[\zeta_1]\left(\alpha\frac{\partial\delta z}{\partial x}+\alpha'\frac{\partial\delta z}{\partial y}+\alpha''\frac{\partial\delta z}{\partial z}\right)+(\alpha[\eta_1]-\alpha'[\xi_1])\delta K, \tag{82}$$

with similar formulas.

The value of $\delta(\xi)$ on page 200 may be given in the form

$$\delta(\xi)=-(\xi_1)\left\{\frac{\partial\delta' x}{\partial t}+(q)\delta' z-(r)\delta' y\right\}$$
$$-(\eta_1)\left\{\frac{\partial\delta' y}{\partial t}+(r)\delta' x-(p)\delta' z\right\}$$
$$-(\zeta_1)\left\{\frac{\partial\delta' z}{\partial t}+(p)\delta' y-(q)\delta' x\right\};$$

but by virtue of formulas (73) which define (p), (q), (r) we have formulas like the following:

$$\frac{\partial\delta' x}{\partial t}+(q)\delta' z-(r)\delta' y=\alpha\frac{\partial\delta x}{\partial t}+\alpha'\frac{\partial\delta y}{\partial t}+\alpha''\frac{\partial\delta z}{\partial t},\ \ldots$$

and consequently due to formulas (72):

$$\delta(\xi)=-\left([\xi_1]\frac{\partial\delta x}{\partial t}+[\eta_1]\frac{\partial\delta y}{\partial t}+[\zeta_1]\frac{\partial\delta z}{\partial t}\right), \tag{83}$$

a formula in which we may return to the derivatives $\frac{d}{dt}$, as we will demonstrate in detail farther on.

By means of the formulas which define δx, δy, δz, δI, δJ, δK we have

$$\delta(p_1)=\alpha\left(\alpha\frac{\partial\delta I}{\partial x}+\alpha'\frac{\partial\delta J}{\partial x}+\alpha''\frac{\partial\delta K}{\partial x}\right)+[\gamma(q_1)-\beta(r_1)]\delta I$$
$$+\alpha'\left(\alpha\frac{\partial\delta I}{\partial y}+\alpha'\frac{\partial\delta J}{\partial y}+\alpha''\frac{\partial\delta K}{\partial y}\right)+[\gamma'(q_1)-\beta'(r_1)]\delta J$$
$$+\alpha''\left(\alpha\frac{\partial\delta I}{\partial z}+\alpha'\frac{\partial\delta J}{\partial z}+\alpha''\frac{\partial\delta K}{\partial z}\right)+[\gamma''(q_1)-\beta''(r_1)]\delta K$$
$$-(p_1)\left[\alpha\left(\alpha\frac{\partial\delta x}{\partial x}+\alpha'\frac{\partial\delta y}{\partial x}+\alpha''\frac{\partial\delta z}{\partial x}\right)+\alpha'\left(\alpha\frac{\partial\delta x}{\partial y}+\alpha'\frac{\partial\delta y}{\partial y}+\alpha''\frac{\partial\delta z}{\partial y}\right)+\alpha''\left(\alpha\frac{\partial\delta x}{\partial z}+\ldots\right)\right]$$
$$-(q_1)\left[\alpha\left(\beta\frac{\partial\delta x}{\partial x}+\beta'\frac{\partial\delta y}{\partial x}+\beta''\frac{\partial\delta z}{\partial x}\right)+\alpha'\left(\beta\frac{\partial\delta x}{\partial y}+\beta'\frac{\partial\delta y}{\partial y}+\beta''\frac{\partial\delta z}{\partial y}\right)+\alpha''\left(\beta\frac{\partial\delta x}{\partial z}+\ldots\right)\right]$$
$$-(r_1)\left[\alpha\left(\gamma\frac{\partial\delta x}{\partial x}+\gamma'\frac{\partial\delta y}{\partial x}+\gamma''\frac{\partial\delta z}{\partial x}\right)+\alpha'\left(\gamma\frac{\partial\delta x}{\partial y}+\gamma'\frac{\partial\delta y}{\partial y}+\gamma''\frac{\partial\delta z}{\partial y}\right)+\alpha''\left(\gamma\frac{\partial\delta x}{\partial z}+\ldots\right)\right].$$

which because of formulas (73) may be written /203

$$\delta(p_1)=\alpha\left(\alpha\frac{\partial\delta I}{\partial x}+\alpha'\frac{\partial\delta I}{\partial y}+\alpha''\frac{\partial\delta I}{\partial z}\right)+(\alpha'[r_1]-\alpha''[q_1])\delta I$$
$$+\alpha'\left(\alpha\frac{\partial\delta J}{\partial x}+\alpha'\frac{\partial\delta J}{\partial y}+\alpha''\frac{\partial\delta J}{\partial z}\right)+\alpha''[p_1]-\alpha[r_1])\delta J \tag{84}$$
$$+\alpha''\left(\alpha\frac{\partial\delta K}{\partial x}+\alpha'\frac{\partial\delta K}{\partial y}+\alpha''\frac{\partial\delta K}{\partial z}\right)+(\alpha[q_1]-\alpha'[p_1])\delta K$$
$$-[p_1]\left(\alpha\frac{\partial\delta x}{\partial x}+\alpha'\frac{\partial\delta x}{\partial y}+\alpha''\frac{\partial\delta x}{\partial z}\right)$$

$$
\begin{aligned}
&- [q_1]\left(\alpha \frac{\partial \delta y}{\partial x} + \alpha' \frac{\partial \delta y}{\partial y} + \alpha'' \frac{\partial \delta y}{\partial z}\right) \\
&- [r_1]\left(\alpha \frac{\partial \delta z}{\partial x} + \alpha' \frac{\partial \delta z}{\partial y} + \alpha'' \frac{\partial \delta z}{\partial z}\right),
\end{aligned} \tag{84}
$$

and we have similar results for $\delta(q_1)$, ...

Finally, let us note that we can write

$$
\begin{aligned}
\delta(p) = &\frac{\partial \delta x}{\partial t} + (q)\delta z' - (r)\delta y' \\
&- (p_1)\left[\frac{\partial \delta' x}{\partial t} + (q)\delta' z - (r)\delta' y\right] \\
&- (q_1)\left[\frac{\partial \delta' y}{\partial t} + (r)\delta' x - (p)\delta' z\right] \\
&- (r_1)\left[\frac{\partial \delta' z}{\partial t} + (p)\delta' y - (q)\delta' x\right],
\end{aligned}
$$

or

$$
\begin{aligned}
\delta(p) = &\alpha \frac{\partial \delta x}{\partial t} + \alpha' \frac{\partial \delta y}{\partial t} + \alpha'' \frac{\partial \delta z}{\partial t} \\
&- (p_1)\left(\alpha \frac{\partial \delta x}{\partial t} + \alpha' \frac{\partial \delta y}{\partial t} + \alpha'' \frac{\partial \delta z}{\partial t}\right) \\
&- (q_1)\left(\beta \frac{\partial \delta x}{\partial t} + \beta' \frac{\partial \delta y}{\partial t} + \beta'' \frac{\partial \delta z}{\partial t}\right) \\
&- (r_1)\left(\gamma \frac{\partial \delta x}{\partial t} + \gamma' \frac{\partial \delta y}{\partial t} + \gamma'' \frac{\partial \delta z}{\partial t}\right),
\end{aligned}
$$

or at last

$$
\delta(p) = \alpha \frac{\partial \delta x}{\partial t} + \alpha' \frac{\partial \delta y}{\partial t} + \alpha'' \frac{\partial \delta z}{\partial t} - [p_1]\frac{\partial \delta x}{\partial t} - [q_1]\frac{\partial \delta y}{\partial t} - [r_1]\frac{\partial \delta z}{\partial t}, \tag{85}
$$

a formula in which we may also return to the derivatives $\frac{d}{dt}$. There are two similar formulas for $\delta(q)$, $\delta(r)$.

75. Variations in Directly Determined Eulerian Arguments. We assume that *the functions* x, y, z of x_0, y_0, z_0, t are subjected to the variations δx, δy, δz. Let us consider the relationships which are obtained by successively /204 differentiating relationships (68') with respect to the Lagrangian variables. From them, we may deduce

$$
\frac{\partial x}{\partial p_i}\delta[\xi_i] + \frac{\partial y}{\partial p_i}\delta[\eta_i] + \frac{\partial z}{\partial p_i}\delta[\zeta_i] + [\xi_i]\frac{\partial \delta x}{\partial p_i} + [\eta_i]\frac{\partial \delta y}{\partial p_i} + [\zeta_i]\frac{\partial \delta z}{\partial p_i} = 0;
$$

but we have

$$
\begin{aligned}
\frac{\partial \delta x}{\partial p_i} &= \frac{\partial \delta x}{\partial x}\frac{\partial x}{\partial p_i} + \frac{\partial \delta x}{\partial y}\frac{\partial y}{\partial p_i} + \frac{\partial \delta x}{\partial z}\frac{\partial z}{\partial p_i}, \\
\frac{\partial \delta y}{\partial p_i} &= \frac{\partial \delta y}{\partial x}\frac{\partial x}{\partial p_i} + \frac{\partial \delta y}{\partial y}\frac{\partial y}{\partial p_i} + \frac{\partial \delta y}{\partial z}\frac{\partial z}{\partial p_i}, \\
\frac{\partial \delta z}{\partial p_i} &= \frac{\partial \delta z}{\partial x}\frac{\partial x}{\partial p_i} + \frac{\partial \delta z}{\partial y}\frac{\partial y}{\partial p_i} + \frac{\partial \delta z}{\partial z}\frac{\partial z}{\partial p_i};
\end{aligned}
$$

If the values of these derivatives are introduced into the preceding relationship, we have

$$\frac{\partial x}{\partial p_i}\left\{\delta[\xi_i]+[\xi_i]\frac{\partial x}{\partial x}+[\eta_i]\frac{\partial \delta y}{\partial x}+[\zeta_i]\frac{\partial \delta z}{\partial x}\right\}$$
$$+\frac{\partial y}{\partial p_i}\left\{\delta[\eta_i]+[\xi_i]\frac{\partial \delta x}{\partial y}+[\eta_i]\frac{\partial \delta y}{\partial y}+[\zeta_i]\frac{\partial \delta z}{\partial y}\right\}$$
$$+\frac{\partial z}{\partial p_i}\left\{\delta[\zeta_i]+[\xi_i]\frac{\partial \delta x}{\partial z}+[\eta_i]\frac{\partial \delta y}{\partial z}+[\zeta_i]\frac{\partial \delta z}{\partial z}\right\}=0;$$

The parentheses in this last equation are therefore zero, and we have

$$\delta[\xi_i]=-\left\{[\xi_i]\frac{\partial \delta x}{\partial x}+[\eta_i]\frac{\partial \delta y}{\partial x}+[\zeta_i]\frac{\partial \delta z}{\partial x}\right\},$$
$$\delta[\eta_i]=-\left\{[\xi_i]\frac{\partial \delta x}{\partial y}+[\eta_i]\frac{\partial \delta y}{\partial y}+[\zeta_i]\frac{\partial \delta z}{\partial y}\right\},$$
$$\delta[\zeta_i]=-\left\{[\xi_i]\frac{\partial \delta x}{\partial z}+[\eta_i]\frac{\partial \delta y}{\partial z}+[\zeta_i]\frac{\partial \delta z}{\partial z}\right\}.$$

In the same fashion, we have

$$\delta(\xi)=-\frac{dx}{dt}\delta[\xi_1]-\frac{dy}{dt}\delta[\eta_1]-\frac{dz}{dt}\delta[\zeta_1]-[\xi_1]\frac{d\delta x}{dt}-[\eta_1]\frac{d\delta y}{dt}-[\zeta_1]\frac{d\delta z}{dt};$$

By replacing $\delta[\xi_1]$, $\delta[\eta_1]$, $\delta[\zeta_1]$ by the values which we have just obtained, we derive /205

$$\delta(\xi)=\frac{dx}{dt}\left\{[\xi_1]\frac{\partial \delta x}{\partial x}+[\eta_1]\frac{\partial \delta y}{\partial x}+[\zeta_1]\frac{\partial \delta z}{\partial x}\right\}$$
$$+\frac{dy}{dt}\left\{[\xi_1]\frac{\partial \delta x}{\partial y}+[\eta_1]\frac{\partial \delta y}{\partial y}+[\zeta_1]\frac{\partial \delta z}{\partial y}\right\}$$
$$+\frac{dz}{dt}\left\{[\xi_1]\frac{\partial \delta x}{\partial z}+[\eta_1]\frac{\partial \delta y}{\partial z}+[\zeta_1]\frac{\partial \delta z}{\partial z}\right\},$$
$$-[\xi_1]\frac{d\delta x}{dt}-[\eta_1]\frac{d\delta y}{dt}-[\zeta_1]\frac{d\delta z}{dt},$$

with similar formulas for $\delta(\eta)$, $\delta(\zeta)$. To return to the formula which we derived in Section 74, it is sufficient to note that we have

$$\frac{d\delta x}{dt}=\frac{\partial \delta x}{\partial x}\frac{dx}{dt}+\frac{\partial \delta x}{\partial y}\frac{dy}{dt}+\frac{\partial \delta x}{\partial z}\frac{dz}{dt}+\frac{\partial \delta x}{\partial t},$$
$$\frac{d\delta y}{dt}=\frac{\partial \delta y}{\partial x}\frac{dx}{dt}+\frac{\partial \delta y}{\partial y}\frac{dy}{dt}+\frac{\partial \delta y}{\partial z}\frac{dz}{dt}+\frac{\partial \delta y}{\partial t},$$
$$\frac{d\delta z}{dt}=\frac{\partial \delta z}{\partial x}\frac{dx}{dt}+\frac{\partial \delta z}{\partial y}\frac{dy}{dt}+\frac{\partial \delta z}{\partial z}\frac{dz}{dt}+\frac{\partial \delta z}{\partial t};$$

but we will not subsequently use the formula on page 1154* and its analogues. It is valid to note that, in the case where x, y, z, t are the *independent variables*, the integration field of the integrals with respect to x, y, z which we will consider *will depend on* t, and the integration with respect to x, y, z and with respect to t cannot be transposed in the usual manner (as with x_0, y_0, z_0, t).

If it is desired to integrate by parts with respect to t, it will be necessary to introduce the auxiliary variables x_0, y_0, z_0 and to use only derivatives

*Translators Note: Reference not given.

with respect to t with the form $\frac{d}{dt}$, which will necessitate the use of formulas like the one we described above for $\delta(\xi)$.

The calculation which must be made to obtain $\delta(p_i)$, $\delta(q_i)$, $\delta(r_i)$, $\delta(p)$, $\delta(q)$, $\delta(r)$, like the one which leads to the expressions of $\delta(\xi_i)$, $\delta(\eta_i)$, $\delta(\zeta_i)$, $\delta(\xi)$, $\delta(\eta)$, $\delta(\zeta)$, is based at present on the formulas which we have just obtained for $\delta[\xi_i]$, $\delta[\eta_i]$, $\delta[\zeta_i]$. The transformation to which the expressions $\delta(p)$, $\delta(q)$, $\delta(r)$ given in Section 74 are subjected in order to have only derivatives with respect to t, of the form $\frac{d}{dt}$, is the same as we have indicated for $\delta(\xi)$, $\delta(\eta)$, $\delta(\zeta)$.

76. The Action of Deformation and Motion with Eulerian Variables. Invariance of Eulerian Arguments. Application of Variable Action Method. The action of deformation and of motion with the Lagrangian variables was

$$\int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

where W is a function of x_0, y_0, z_0, t; ξ_i, η_i, ζ_i, p_i, q_i, r_i; ξ, η, ζ, p, q, r.

According to formulas (79) and (81), (81') it may also be said that W is a function of x_0, y_0, z_0, t; (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i); (ξ), (η), (ζ), (p), (q), (r). If we set[10]

$$\Omega = \frac{W}{\Delta},$$

the preceding action may also be written

$$\int_{t_1}^{t_2} \iiint_{S} \Omega dx dy dz dt;$$

The integration relative to x, y, z is extended to the medium S, *that is, to a domain which is variable with time.* /206

It may also be imagined that we arrive at this last action independently of the first. In fact, the Lagrangian arguments are, as was seen before, *Euclidean invariants*. However, since the Eulerian arguments are uniquely functions of the Lagrangian arguments according to formulas (77) and (80), this results in the fact that they are also Euclidean invariants. This may be easily established *directly* by means of formulas (82), (83) and (84), (85) by setting the following:

$$\delta x = (a_1 + \omega_2 z - \omega_3 y)\, dt,$$
$$\delta y = (b_1 + \omega_3 x - \omega_1 z)\, dt,$$
$$\delta z = (c_1 + \omega_1 y - \omega_2 x)\, dt,$$
$$\delta \imath = \omega_1 \delta t, \qquad \delta \jmath = \omega_2 \delta t, \qquad \delta \kappa = \omega_3 \delta t.$$

[10] We assume Δ to be positive and therefore equal to $|\Delta|$.

The result of this is that we are thus directly led to take the following integral:

$$\int_{t_1}^{t_2} \iiint_S \Omega dx dy dz dt,$$

where *function* Ω *has the unique form*

$$\Omega\left(x_0, y_0, z_0, t, (\xi_i), (\eta_i), (\zeta_i), (p_i), (q_i), (r_i), (\xi), (\eta), (\zeta), (p), (q), (r)\right).$$

as the *form of the action of deformation and motion with the Eulerian variables* inside surface S of the deformed medium and in the time interval between moments t_1 and t_2.

Let us consider *any* variation of the action of deformation and motion inside surface (S) of medium (M) and in the time interval between moments t_1 and t_2. For this purpose, let us give to x, ... the variations δx, ...

Let us for a moment write the integral in the form

$$\int_{t_1}^{t_2} \iiint_{S_0} \Omega \Delta dx_0 dy_0 dz_0 dt;$$

Its variation is

$$\int_{t_1}^{t_2} \iiint_{S_0} (\Delta \delta\Omega + \Omega \delta\Delta) dx_0 dy_0 dz_0 dt,$$

or

$$\int_{t_1}^{t_2} \iiint_S \left(\delta\Omega + \Omega \frac{\delta\Delta}{\Delta}\right) dx dy dz dt.$$

Now, /207

$$\Delta = \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)},$$

$$\begin{aligned}\delta\Delta &= \frac{\partial(y, z)}{\partial(y_0, z_0)} \frac{\partial\delta x}{\partial x_0} + \frac{\partial(y, z)}{\partial(z_0, x_0)} \frac{\partial\delta x}{\partial y_0} + \frac{\partial(y, z)}{\partial(x_0, y_0)} \frac{\partial\delta x}{\partial z_0} + \ldots\ldots \\ &= \left\{\frac{\partial(y, z)}{\partial(y_0, z_0)} \frac{\partial x}{\partial x_0} + \frac{\partial(y, z)}{\partial(z_0, x_0)} \frac{\partial x}{\partial y_0} + \frac{\partial(y, z)}{\partial(x_0, y_0)} \frac{\partial x}{\partial z_0}\right\} \frac{\partial\delta x}{\partial x} + \ldots\ldots \\ &= \frac{\partial\delta x}{\partial x} \Delta + \frac{\partial\delta y}{\partial y} \Delta + \frac{\partial\delta z}{\partial z} \Delta,\end{aligned}$$

i.e.,

$$\frac{\delta\Delta}{\Delta} = \frac{\partial\delta x}{\partial y} + \frac{\partial\delta y}{\partial y} + \frac{\partial\delta z}{\partial z},$$

and consequently the variation of the integral is

$$\int_{t_1}^{t_2} \iiint_S \left\{\Omega\left(\frac{\partial\delta x}{\partial x} + \frac{\partial\delta y}{\partial y} + \frac{\partial\delta z}{\partial z}\right) + \delta\Omega\right\} dx dy dz dt.$$

Variation $\delta\Omega$ of Ω is

$$\delta\Omega = \sum\left\{\frac{\partial\Omega}{\partial(\xi_1)}\delta(\xi_1) + \frac{\partial\Omega}{\partial(\eta_1)}\delta(\eta_1) + \dots\right\} + \frac{\partial\Omega}{\partial(p)}\delta(p) + \dots + \frac{\partial\Omega}{\partial(r)}\delta(r),$$

where $\delta(\xi_1)$, $\delta(\eta_1)$, ..., $\delta(r)$ are determined by the formulas in Sections 74 and 75, written so that only derivatives having the form of $\frac{d}{dt}$ with respect to t appear therein. We may apply Green's formula to the terms which explicitly contain a derivative with respect to one of the variables x, y, z. With respect to the terms explicitly containing a derivative with respect to time, following is the procedure (the integration field with respect to x, y, z *varies with time):* Let

$$\int_{t_1}^{t_2}\iiint_{\sigma} g\frac{dh}{dt}\,dxdydzdt,$$

be such a term. If we go over to the variables x_0, y_0, z_0, we obtain

$$\int_{t_1}^{t_2}\iiint_{\sigma_0} g\Delta\frac{dh}{dt}\,dx_0dy_0dz_0dt,$$

where, integrating by parts, we have

$$\begin{aligned}
&\iiint_{\sigma_0}[g\Delta h]_{t_1}^{t_2}\,dx_0dy_0dz_0 - \int_{t_1}^{t_2}\iiint_{\sigma_0} h\frac{d(g\Delta)}{dt}\,dx_0dy_0dz_0dt\\
&= \left[\iiint_{\sigma_0} g\Delta h\,dx_0dy_0dz_0\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\iiint_{\sigma_0} h\frac{d(g\Delta)}{dt}\,dx_0dy_0dz_0dt\\
&= \left[\iiint_{\sigma} ghdxdydz\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\iiint_{\sigma}\frac{h}{\Delta}\frac{d(g\Delta)}{dt}\,dxdydzdt,
\end{aligned}$$

by returning(11) to the variables x, y, z. /208

Thus, using l, m, n to designate the direction cosines with respect to the fixed axes Ox, Oy, Oz of the external normal to the surface S, which delimits the medium after deformation at instant t, and using dσ to designate the element of area of this surface, we obtain

$$\begin{aligned}
&\delta\int_{t_1}^{t_2}\iiint\omega\,dxdydzdt\\
&= \int_{t_1}^{t_2}\iint_{S}\{(lp_{xx}+mp_{yx}+np_{zx})\delta x + (lp_{xy}+mp_{yy}+np_{zy})\delta y + (lp_{xz}+mp_{yz}+np_{zz})\delta z\\
&+ (lq_{xx}+mq_{yx}+nq_{zx})\delta\imath + (lq_{xy}+mq_{yy}+nq_{zy})\delta\jmath + (lq_{zx}+mq_{yz}+nq_{zz})\delta\kappa\}\,d\sigma dt\\
&+ \left\{\iiint_{\sigma}\left(\frac{A}{\Delta}\delta x + \frac{B}{\Delta}\delta y + \frac{C}{\Delta}\delta z + \frac{P}{\Delta}\delta\imath + \frac{Q}{\Delta}\delta\jmath + \frac{R}{\Delta}\delta\kappa\right)dxdydz\right\}_{t_1}^{t_2}\\
&- \int_{t_1}^{t_2}\iiint_{\sigma}\left\{\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} + \frac{1}{\Delta}\frac{dA}{dt}\right)\delta x\right.
\end{aligned}$$

(11) Here $\frac{d\Delta}{dt}$ may be replaced by its value taken from

$$\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{dy}{dt}\right) + \frac{\partial}{\partial z}\left(\frac{dz}{dt}\right).$$

$$
\begin{aligned}
&+\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}+\frac{1}{\Delta}\frac{dB}{dt}\right)\delta y\\
&+\left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}+\frac{1}{\Delta}\frac{dC}{dt}\right)\delta z\\
&+\left(\frac{\partial q_{xx}}{\partial x}+\frac{\partial q_{yx}}{\partial y}+\frac{\partial q_{zx}}{\partial z}+\frac{1}{\Delta}\frac{dP}{dt}+p_{yz}-p_{zy}+\frac{C}{\Delta}\frac{dy}{dt}-\frac{B}{\Delta}\frac{dz}{dt}\right)\delta_1\\
&+\left(\frac{\partial q_{xy}}{\partial x}+\frac{\partial q_{yy}}{\partial y}+\frac{\partial q_{zy}}{\partial z}+\frac{1}{\Delta}\frac{dQ}{dt}+p_{zx}-p_{xz}+\frac{A}{\Delta}\frac{dz}{dt}-\frac{C}{\Delta}\frac{dx}{dt}\right)\delta_2\\
&+\left(\frac{\partial q_{xz}}{\partial y}+\frac{\partial q_{yz}}{\partial y}+\frac{\partial q_{zz}}{\partial z}+\frac{1}{\Delta}\frac{dR}{dt}+p_{xy}-p_{yx}+\frac{B}{\Delta}\frac{dx}{dt}-\frac{A}{\Delta}\frac{dy}{dt}\right)\delta_3\Big\}\,dx\,dy\,dz\,dt,
\end{aligned}
$$

where with the notation of Section 73 we have set

$$
\begin{aligned}
\frac{A}{\Delta}&=-(A')[\xi_1]-(B')[\xi_2]-(C')[\xi_3]-(P')[p_1]-(Q')[p_2]-(R')[p_3],\\
\frac{B}{\Delta}&=-(A')[\eta_1]-(B')[\eta_2]-(C')[\eta_3]-(P')[q_1]-(Q')[q_2]-(R')[q_3],\\
\frac{C}{\Delta}&=-(A')[\zeta_1]-(B')[\zeta_2]-(C')[\zeta_3]-(P')[r_1]-(Q')[r_2]-(R')[r_3],\\
\frac{P}{\Delta}&=[P]=\alpha\,(P')+\beta\,(Q')+\gamma\,(R'),\\
\frac{Q}{\Delta}&=[Q]=\alpha'\,(P')+\beta'\,(Q')+\gamma'\,(R'),\\
\frac{R}{\Delta}&=[R]=\alpha''(P')+\beta''(Q')+\gamma''(R'),
\end{aligned}
$$

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$$
\begin{aligned}
p_{xx}&=\Omega-\sum[A_i][\xi_i]-\sum[P_i][p_i]-\frac{A}{\Delta}\frac{dx}{dt},\\
p_{yx}&=-\sum[B_i][\xi_i]-\sum[Q_i][p_i]-\frac{A}{\Delta}\frac{dy}{dt},\\
p_{zx}&=-\sum[C_i][\xi_i]-\sum[R_i][p_i]-\frac{A}{\Delta}\frac{dz}{dt},\\
p_{xy}&=-\sum[A_i][\eta_i]-\sum[P_i][q_i]-\frac{B}{\Delta}\frac{dx}{dt},\\
p_{yy}&=\Omega-\sum[B_i][\eta_i]-\sum[Q_i][q_i]-\frac{B}{\Delta}\frac{dy}{dt},\\
p_{zy}&=-\sum[C_i][\eta_i]-\sum[R_i][q_i]-\frac{B}{\Delta}\frac{dz}{dt},\\
p_{xz}&=-\sum[A_i][\zeta_i]-\sum[P_i][r_i]-\frac{C}{\Delta}\frac{dx}{dt},\\
p_{yz}&=-\sum[B_i][\zeta_i]-\sum[Q_i][r_i]-\frac{C}{\Delta}\frac{dy}{dt},\\
p_{zz}&=\Omega-\sum[C_i][\zeta_i]-\sum[R_i][r_i]-\frac{C}{\Delta}\frac{dz}{dt},
\end{aligned}
$$

and also

$$
\begin{aligned}
q_{xx}&=\alpha\,[P_1]+\beta\,[P_2]+\gamma\,[P_3]-\frac{P}{\Delta}\frac{dx}{dt},\\
q_{yx}&=\alpha\,[Q_1]+\beta\,[Q_2]+\gamma\,[Q_3]-\frac{P}{\Delta}\frac{dy}{dt},\\
q_{zx}&=\alpha\,[Q_1]+\beta\,[R_2]+\gamma\,]R_3]-\frac{P}{\Delta}\frac{dz}{dt},
\end{aligned}
$$

with similar formulas for q_{xy}, q_{yy}, q_{zy}, q_{xz}, q_{yz}, q_{zz}.

77. Remarks on Variations Introduced into the Preceding Sections. Application of the Method of Variable Action in Accord with the Usual Method in the Calculus of Variations. In the preceding section, we have utilized the previously calculated variations. It is advantageous to define their meaning more exactly and to connect them with the variations calculated by Jordan[12].

In completing this, let us recall the Jordan discussion. He looks for the variation of

$$\int \phi\, dx\, dy\, dz,$$

by assuming that, on the one hand, x, y, z undergo variations and, on the other, the functions appearing in ϕ also undergo variations. Due to this fact, ϕ undergoes two variations, whose effects are additive. Jordan successively considers the variation due to the variation of the functions appearing in ϕ, and then that due to the variation of x, y, z which he compares to the preceding.

We may also investigate the total effect of the juxtaposition of the two /210
variations on the quantities u, ..., $u_{\alpha\beta\gamma}$, ... which appear in ϕ, where we call δu, ... the complete variations. We have

$$\delta\phi = \frac{\partial\phi}{\partial u}\delta u + \dots$$

for the *total* variation $\delta\phi$ of ϕ.

With this we may note that the variations calculated previously are what we have called the *total* variations and that the calculation of the previous section was carried out with this in mind.

If we wish to present this in a form identical to that of M. Jordan, we may proceed as follows. In the following, we will introduce the functions x_0, y_0, z_0, α, α', ... γ'' of x, y, z which either entirely, or in part, appear explicitly as well as their derivatives. The functions x_0, y_0, z_0 of x, y, z, t are the ones which must be substituted in the second terms of (68') in order to obtain x, y, z as a function of x_0, y_0, z_0. Due to the fact that x, y, z experience variations δx, δy, δz, the result is that *these functions* x_0, y_0, z_0 of x, y, z also are subjected to variations which we will designate[13] by (δx_0), ..., such that we obtain the equations:

---

(12) Jordan, *Cours d'Analyse de l'Ecole Polytechnique (The Polytechnic School Course in Analysis)*, First Edition, Volume 3, No. 339, pp. 533-535; Second Edition, Volume 3, No. 396, pp. 528-530.

(13) In a general way we will designate by (δ) the variations obtained by leaving x, y, z fixed in order to avoid any confusion.

$$\begin{cases} 0 = (\delta x_0) + \frac{\partial x_0}{\partial x}\delta x + \frac{\partial x_0}{\partial y}\delta y + \frac{\partial x_0}{\partial z}\delta z, \\ 0 = (\delta y_0) + \frac{\partial y_0}{\partial x}\delta x + \frac{\partial y_0}{\partial y}\delta y + \frac{\partial y_0}{\partial z}\delta z, \\ 0 = (\delta z_0) + \frac{\partial z_0}{\partial x}\delta x + \frac{\partial z_0}{\partial y}\delta y + \frac{\partial z_0}{\partial z}\delta z, \end{cases} \tag{86}$$

The variations (δx_0), (δy_0), (δz_0) which appear in these last three equations are the variations which appear in the derivation copied from that of M. Jordan and which we have given. This remark seems to be close to the one developed by M. C. Neumann in his researches(14) on the equations of Maxwell and Hertz. On the other hand, he adhered to the rules of calculation adapted by M. H. Poincaré in his memoir *On the Dynamics of the Electron*(15), which we will discuss later on.

For α, α', ..., γ'', we will have the variations $(\delta\alpha)$, ... in the sense of M. Jordan. However, the variations introduced in the preceding sections and which we will continue to designate by $\delta\alpha$, ..., will be total variations, in such a way that we will have /211

$$\delta\alpha = (\delta\alpha) + \frac{\partial\alpha}{\partial x}\delta x + \frac{\partial\alpha}{\partial y}\delta y + \frac{\partial\alpha}{\partial z}\delta z.$$

This is the same as saying that by introducing the variations $(\delta\alpha)$, ... in the sense of M. Jordan, we introduce the auxiliary quantities $\delta I'$, $\delta J'$, $\delta K'$ among other things, which we will *define* in terms of $(\delta\alpha)$, δx, ..., as follows

$$\begin{cases} \delta I' = \sum\gamma\delta\beta = \sum\gamma(\delta\beta) + [p_1]\delta x + [q_1]\delta y + [r_1]\delta z, \\ \delta J' = \sum\alpha\delta\gamma = \sum\alpha(\delta\gamma) + [p_2]\delta x + [q_2]\delta y + [r_2]\delta z, \\ \delta K' = \sum\beta\delta\alpha = \sum\beta(\delta\alpha) + [p_3]\delta x + [q_3]\delta y + [r_3]\delta z. \end{cases} \tag{87}$$

As can be seen the fundamental convention is the one expressed by the relationships (86). It can be found in a later work on the *theory of temperature*, for example, for functions whose differentials parameters appear in the case of a point medium, if we disregard the equations which contain the coefficients of total variation of these functions.

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(14) G. Neumann. *Die Elektrischen Kräfte (Electrical Forces)*, Volume II, Leipzig, 1898; *On the Maxwell-Hertz Theory, (Abhandl. der k. Sächs Gesells. der Wiss. zu Leipzig; Math.-Phys. Klasse*, Volume XXVII, Nos. 2 and 8, 1901-1902).

(15) H. Poincaré. *Rend. di Palermo*, Volume XXI, p. 129 ff. (1905), 1906. M. H. Poincaré uses notation different from ours, especially for the derivatives with respect to t, where our signs d, ∂, which are those of M. Appell (*Treatise on Mechanics*, Volume III, 1st Edition, p. 277), are interchanged. He distinguishes the ordinary variation $(\delta\phi)$ of a function ϕ in the sense of Jordan, which he designates by $\frac{d\phi}{\partial\varepsilon}d\varepsilon$, from its variation $\delta\phi$ (which we call *total*), which he designates by $\frac{\partial\phi}{\partial\varepsilon}\delta\varepsilon$ (in particular, see equation (11 bis), page 140).

It should be observed that in the present case the simplest way of carrying out the calculations is not to follow the derivation of M. Jordan, but instead it consists of determining the total variation of the function under the integration sign, as we have done previously. Nevertheless, it will be useful to also follow the path of M. Jordan, in order to make comparisons later on during the development of the two points of view which the concept of *temperature* suggests.

We have

$$\begin{aligned}\delta\int_{t_1}^{t_2}\iiint_s \Omega\,dx\,dy\,dz\,dt = \int_{t_1}^{t_2}\iiint_s\Big[&\frac{\partial\Omega}{\partial x_0}(\delta x_0)+\frac{\partial\Omega}{\partial y_0}(\delta y_0)+\frac{\partial\Omega}{\partial z_0}(\delta z_0)\\ &+\sum\Big\{\frac{\partial\Omega}{\partial(\xi_i)}\big(\delta(\xi_i)\big)+\dots+\frac{\partial\Omega}{\partial(r_i)}\big(\delta(r_i)\big)\Big\}+\frac{\partial\Omega}{\partial(\xi)}\big(\delta(\xi)\big)+\dots+\frac{\partial\Omega}{\partial(r)}\big(\delta(r)\big)\\ &+\frac{d}{dx}(\Omega\delta x)+\frac{d}{dy}(\Omega\delta y)+\frac{d}{dz}(\Omega\delta z)\Big]dx\,dy\,dz\,dt,\end{aligned} \tag{88}$$

where the sign (δ) corresponds to the variation obtained by keeping x, y, z fixed, so that in a general way we have

$$(\delta\mathfrak{F}) = \delta\mathfrak{F} - \frac{d\mathfrak{F}}{dx}\delta x - \frac{d\mathfrak{F}}{dy}\delta y - \frac{d\mathfrak{F}}{dz}\delta z. \tag{89}$$

We will substitute the auxiliary quantities δx, δy, δz, $\delta I'$, $\delta J'$, $\delta K'$, defined by the equations (86), (87) for the variations (δx_0), .. . We should remember, *regarding the integration with respect to* t, that the region of integration with respect to x, y, z varies with t and that we cannot interchange the integration with respect to t and the system of integrations with respect to x, y, z *in the usual way used for the variables* x_0, y_0, z_0. /212

If we replace (δx_0), (δy_0), (δz_0), $(\delta(\xi_i))$, ... by their values defined by (89), which contain (86), we obtain:

$$\begin{aligned}\delta\int_{t_1}^{t_2}\iiint_s \Omega\,dx\,dy\,dz\,dt = \int_{t_1}^{t_2}\iiint_s\Big[&-\frac{d\Omega}{dx}\delta x-\frac{d\Omega}{dy}\delta y-\frac{d\Omega}{dz}\delta z\\ &+\sum\Big\{\frac{\partial\Omega}{\partial(\xi_i)}\delta(\xi_i)+\dots+\frac{\partial\Omega}{\partial(r_i)}\delta(r_i)\Big\}+\frac{\partial\Omega}{\partial(\xi)}\delta(\xi)+\dots+\frac{\partial\Omega}{\partial(r)}\delta(r)\\ &+\frac{d}{dx}(\Omega\delta x)+\frac{d}{dy}(\Omega\delta y)+\frac{d}{dz}(\Omega\delta z)\Big]dx\,dy\,dz\,dt.\end{aligned} \tag{90}$$

If we consider

$$\begin{aligned}\int_{t_1}^{t_2}\iiint_s\Big[&-\frac{d\Omega}{dx}\delta x-\frac{d\Omega}{dy}\delta y-\frac{d\Omega}{dz}\delta z\\ &+\frac{d}{dx}(\Omega\delta x)+\frac{d}{dy}(\Omega\delta y)+\frac{d}{dz}(\Omega\delta z)\Big]dx\,dy\,dz\,dt,\end{aligned} \tag{91}$$

separately as well as

$$\int_{t_1}^{t_2}\iiint_s\Big[\sum\Big\{\frac{\partial\Omega}{\partial(\xi_i)}\delta(\xi_i)+\dots\Big\}+\frac{\partial\Omega}{\partial(\xi)}\delta(\xi)+\dots+\frac{\partial\Omega}{\partial(r)}\delta(r)\Big]dx\,dy\,dz\,dt, \tag{92}$$

as well as the following, as was done in the preceding section

$$\int_{t_1}^{t_2}\iiint_S \Omega\left(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\right)dxdydzdt, \qquad (91')$$

and (92), it can be seen that the calculation is identical to the one which we have carried previously.

78. The Concepts of Action of Lagrange and Euler. The Method of the Action Variable Applied to the Concept of Action of Euler Expressed in Terms of Euler Variables. In his work *on the dynamics of the electron for an infinite field* presented July 23, 1905 at the meeting of the Palermo Circle, M. H. Poincaré seems to have introduced a concept of action different from that which we have considered up to the present. If we formulate the idea of M. H. Poincaré more precisely by considering a *finite field*, one is led to distinguish between the two concepts of action, one due to Lagrange and the other due to Euler.

We may subject the general function W or Ω to integrations with respect to independent variables[16] x_0, y_0, z_0 or with respect to the independent variables[17] x, y, z *in fixed fields* and may then integrate with respect to t.

1. Beginning with the space (M_0), let us consider an observer attached to /213 reference axes who focuses his attention on a portion (S_0) of this space and to the various positions which it takes on subsequently, i.e.: (S) at the arbitrary time t, (S_1) and (S_2) at the times t_1 and t_2. We consider the integral

$$\int_{t_1}^{t_2}\iiint_S \Omega dxdydzdt,$$

in which the region of integration (S) with respect to x, y, z *varies with* t and which takes on the following form by carrying out a change of variables defined by (66') or by (68')

$$\int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

where W designates the expression obtained by replacing the quantities x, y, z in $\Omega\Delta$ by their expressions (66'), and in which the region of integration (S_0) with respect to x_0, y_0, z_0 is *independent of* t. We thus have the concept of action of Lagrange.

2. Let us continue to consider an observer *fixed with respect to reference axes*, and let us assume that he constantly fixes his attention on a *determined and fixed* part of the space (M). Let us designate by x_0, y_0, z_0 the coordinates of the point M_0 of (M_0), which were calculated by means of equation (68') and which become the coordinates x, y, z at the point M of (M) at the time t. Let (S_0) be the region described by M_0 and which becomes (S) at the time t. We may

(16) In this case, W designates the function.

(17) In this case, Ω designates the function.

designate by (S_{01}), (S_{02}) the region corresponding to (S_0), which varies with t, for the values t_1 and t_2 of t.

Ω contains x, y, z and the functions expressed by equations (66') at the same time. Let us consider

$$\int_{t_1}^{t_2} \iiint_S \Omega dx dy dz dt,$$

where this time the region of integration (S) with respect to x, y, z is *independent of* t, and which takes on the following form by carrying out the change of variables defined by (66') or by (68')

$$\int_{t_1}^{t_2} \iiint_{S_0} W dx_0 dy_0 dz_0 dt,$$

where the region of integration (S_0) with respect to x_0, y_0, z_0 *varies with* t. We thus obtain the concept of action attributed to Euler.

We have considered the first case in the preceding paragraphs. We will now treat the second one. Equation (88) is thus replaced by the following[18]:

$$\begin{aligned}\left(\delta \int_{t_1}^{t_2} \iiint_S \Omega dx dy dz dt\right) = \int_{t_1}^{t_2} \iiint_S \Big[& \frac{\partial \Omega}{\partial x_0}(\delta x_0) + \frac{\partial \Omega}{\partial y_0}(\delta y_0) + \frac{\partial \Omega}{\partial z_0}(\delta z_0) \\ & + \sum \left\{ \frac{\partial \Omega}{\partial(\xi_i)}(\delta(\xi_i)) + \dots + \frac{\partial \Omega}{\partial(r_i)}(\delta(r_i)) \right\} + \frac{\partial \Omega}{\partial(\xi)}(\delta(\xi)) + \dots + \frac{\partial \Omega}{\partial(r)}(\delta(r)) \Big] dx dy dz dt;\end{aligned} \tag{88'}$$

due to equation (89), equation (90) is replaced by the following /214

$$\begin{aligned}\left(\delta \int_{t_1}^{t_2} \iiint_S \Omega dx dy dz dt\right) = \int_{t_1}^{t_2} \iiint_S \Big[& -\frac{d\Omega}{dx}\delta x - \frac{d\Omega}{dy}\delta y - \frac{d\Omega}{dz}\delta z \\ & + \sum \left\{ \frac{\partial \Omega}{\partial(\xi_i)}\delta(\xi_i) + \dots \right\} + \frac{\partial \Omega}{\partial(\xi)}\delta(\xi) + \dots + \frac{\partial \Omega}{\partial(r)}\delta(r) \Big] dx dy dz dt.\end{aligned} \tag{90'}$$

The sequence of calculations resembles those of Section 77. In any case, a difference occurs regarding the derivatives with respect to time. In the present case, it is possible to interchange the integration with respect to t and the integration with respect to the set of variables x, y, z. If this interchange is carried out, the integration with respect to time must be carried out by considering x, y, z as constants. The integrations by parts with respect to time must be carried out by making them applicable with respect to the derivatives $\frac{\partial}{\partial t}$ and not with respect to the derivatives $\frac{d}{dt}$, as was done in Sections 76 and 77, and conforming with the remarks made in Sections 75 and 76.

In the present case the integration by parts results in:

---

[18] Referring to the derivation of M. Jordan, we may observe that the part $\frac{d}{dx}(\Omega\delta x) + \frac{d}{dy}(\Omega\delta y) + \frac{d}{dz}(\Omega\delta z)$ comes from the fact that the field is in motion and corresponds to the variation of the quantities x, y, z as well as independent variables.

$$
\begin{aligned}
&\left(\delta\int_{t_1}^{t_2}\iiint \Omega\, dx\,dy\,dz\,dt\right)\\
&=\int_{t_1}^{t_2}\iint_{\Sigma}\{(l'p'_{xx}+m'p'_{yx}+n'p'_{zx})\delta x+(l'p'_{xy}+m'p'_{yy}+n'p'_{zy})\delta y+(l'p'_{xz}+m'p'_{yz}+n'p'_{zz})\delta z\\
&\quad+(l'q'_{xx}+m'q'_{yx}+n'q'_{zx})\delta\lambda+(l'q'_{xy}+m'q'_{yy}+n'q'_{zy})\delta\mu+(l'q'_{xz}+m'q'_{yz}+n'q'_{zz})\delta\kappa\}\,d\sigma\,dt\\
&\quad+\left\{\iiint_{\Sigma}\left(\frac{A'}{\Delta}\delta x+\frac{B'}{\Delta}\delta y+\frac{C'}{\Delta}\delta z+\frac{P'}{\Delta}\delta'\lambda+\frac{Q'}{\Delta}\delta'\mu+\frac{R'}{\Delta}\delta'\kappa\right)dx\,dy\,dz\right\}_{t_1}^{t_2}\\
&\quad-\int_{t_1}^{t_2}\iiint_{\Sigma}\left\{\left(\frac{\partial p'_{xx}}{\partial x}+\frac{\partial p'_{yx}}{\partial y}+\frac{\partial p'_{zx}}{\partial z}+\frac{\partial}{\partial t}\frac{A'}{\Delta}+\frac{d\Omega}{dx}\right)\delta x\right.\\
&\qquad+\left(\frac{\partial p'_{xy}}{\partial x}+\frac{\partial p'_{yy}}{\partial y}+\frac{\partial p'_{zy}}{\partial z}+\frac{\partial}{\partial t}\frac{B'}{\Delta}+\frac{d\Omega}{dy}\right)\delta y\\
&\qquad+\left(\frac{\partial p'_{xz}}{\partial x}+\frac{\partial p'_{yz}}{\partial y}+\frac{\partial p'_{zz}}{\partial z}+\frac{\partial}{\partial t}\frac{C'}{\Delta}+\frac{d\Omega}{dz}\right)\delta z\\
&\qquad+\left(\frac{\partial q'_{xx}}{\partial x}+\frac{\partial q'_{yx}}{\partial y}+\frac{\partial q'_{zx}}{\partial z}+\frac{\partial}{\partial t}\frac{P'}{\Delta}+p'_{yz}-p'_{zy}\right)\delta\lambda\\
&\qquad+\left(\frac{\partial q'_{xy}}{\partial x}+\frac{\partial q'_{yy}}{\partial y}+\frac{\partial q'_{zy}}{\partial z}+\frac{\partial}{\partial t}\frac{Q'}{\Delta}+p'_{zx}-p'_{xz}\right)\delta\mu\\
&\qquad\left.+\left(\frac{\partial q'_{xz}}{\partial x}+\frac{\partial q'_{yz}}{\partial y}+\frac{\partial q'_{zz}}{\partial z}+\frac{\partial}{\partial t}\frac{R'}{\Delta}+p'_{xy}-p'_{yx}\right)\delta\kappa\right\}dx\,dy\,dz\,dt,
\end{aligned}
$$

where we have set the following using the notation of Sections 72 and 73, /215

$$
\begin{aligned}
\frac{A'}{\Delta}&=\frac{A}{\Delta}=-(A')[\xi_1]-(B')[\xi_2]-(C')[\xi_3]-(P')[p_1]-(Q')[p_2]-(R')[p_3],\\
\frac{B'}{\Delta}&=\frac{B}{\Delta}=-(A')[\eta_1]-(B')[\eta_2]-(C')[\eta_3]-(P')[q_1]-(Q')[q_2]-(R')[q_3],\\
\frac{C'}{\Delta}&=\frac{C}{\Delta}=-(A')[\zeta_1]-(B')[\zeta_2]-(C')[\zeta_3]-(P')[r_1]-(Q')[r_2]-(R')[r_3],\\
\frac{P'}{\Delta}&=\frac{P}{\Delta}=[P]=\alpha(P')+\beta(Q')+\gamma(R'),\\
\frac{Q'}{\Delta}&=\frac{Q}{\Delta}=[Q]=\alpha'(P')+\beta'(Q')+\gamma'(R'),\\
\frac{R'}{\Delta}&=\frac{R}{\Delta}=[R]=\alpha''(P')+\beta''(Q')+\gamma''(R'),\\
p'_{xx}&=-\sum\{[A_i][\xi_i]+[P_i][p_i]\}\\
p'_{yx}&=-\sum\{[B_i][\xi_i]+[Q_i][p_i]\}\\
p'_{zx}&=-\sum\{[C_i][\xi_i]+[R_i][p_i]\}
\end{aligned}
$$

and similar equations for p'_{xy}, p'_{yy}, p'_{zy}; p'_{xz}, p'_{yz}, p'_{zz}, which one obtains by changing $[\xi_i]$, $[p_i]$ into $[\eta_i]$, $[q_i]$, respectively, and then into $[\zeta_i]$, $[r_i]$. Among other things we have

$$
\begin{aligned}
q'_{zx}&=\alpha[P_1]+\beta[P_2]+\gamma[P_3],\\
q'_{yx}&=\alpha[Q_1]+\beta[Q_2]+\gamma[Q_3],\\
q'_{zx}&=\alpha[R_1]+\beta[R_2]+\gamma[R_3],
\end{aligned}
$$

and similar equations for q'_{xy}, q'_{yy}, q'_{zy}; q'_{xz}, q'_{yz}, q'_{zz} which we obtain by changing α, β, γ into α', β', γ', then into α'', β'', γ'', respectively.

Let us note that

$$\frac{\partial}{\partial t}\frac{A'}{\Delta} = \frac{d}{dt}\frac{A'}{\Delta} - \frac{dx}{dt}\frac{\partial}{\partial x}\frac{A'}{\Delta} - \frac{dy}{dt}\frac{\partial}{\partial y}\frac{A'}{\Delta} - \frac{dz}{dt}\frac{\partial}{\partial z}\frac{A'}{\Delta}$$

can, due to the relationship

$$\frac{1}{\Delta}\frac{d\Delta}{dt} = \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial z}\frac{dz}{dt},$$

be written as

$$\frac{\partial}{\partial t}\frac{A'}{\Delta} = \frac{1}{\Delta}\frac{dA'}{dt} - \frac{\partial}{\partial x}\left(\frac{A'}{\Delta}\frac{dx}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{A'}{\Delta}\frac{dy}{dt}\right) - \frac{\partial}{\partial z}\left(\frac{A'}{\Delta}\frac{dz}{dt}\right);$$

as well as

$$\frac{\partial}{\partial t}\frac{P'}{\Delta} = \frac{1}{\Delta}\frac{dP'}{dt} - \frac{\partial}{\partial x}\left(\frac{P'}{\Delta}\frac{dx}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{P'}{\Delta}\frac{dy}{dt}\right) - \frac{\partial}{\partial z}\left(\frac{P'}{\Delta}\frac{dz}{dt}\right).$$

On the other hand, $A' = A$, $P' = P$. The result is: /216

$$\frac{\partial p'_{xx}}{\partial x} + \frac{\partial p'_{yx}}{\partial y} + \frac{\partial p'_{zx}}{\partial z} + \frac{\partial}{\partial t}\frac{A'}{\Delta} + \frac{d\Omega}{dx} = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} + \frac{1}{\Delta}\frac{d\Delta}{dt},$$

and

$$\frac{\partial q'_{xx}}{\partial x} + \frac{\partial q'_{yx}}{\partial y} + \frac{\partial q'_{zx}}{\partial z} + \frac{\partial}{\partial t}\frac{P'}{\Delta} + p'_{yz} - p'_{zy}$$
$$= \frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + \frac{1}{\Delta}\frac{dP}{dt} + p_{yz} - p_{zy} + \frac{C}{\Delta}\frac{dy}{dt} - \frac{B}{\Delta}\frac{dz}{dt},$$

with similar relationships.

The external force and moment have therefore the same definition as in Sections 62 and 63. However, this is not true for the stress and moment of deformation. According to Sections 72, 76 we have

$$\begin{cases} p_{xx} - p'_{xx} = \pi_{xx} = \Omega - \dfrac{A}{\Delta}\dfrac{dx}{dt}, \\ p_{yx} - p'_{yx} = \pi_{yx} = \quad - \dfrac{A}{\Delta}\dfrac{dy}{dt}, \\ p_{zx} - p'_{zx} = \pi_{zx} = \quad - \dfrac{A}{\Delta}\dfrac{dz}{dt}, \end{cases} \tag{93}$$

and similar expressions for π_{xy}, π_{yy}, π_{zy}; π_{xz}, π_{yz}, π_{zz} which we obtain by cyclic permutation of A, B, C and of x, y, z. Among other things

$$\begin{cases} q_{xx} - q'_{xx} = \chi_{xx} = - \dfrac{P}{\Delta}\dfrac{dx}{dt}, \\ q_{yx} - q'_{yx} = \chi_{yx} = - \dfrac{P}{\Delta}\dfrac{dy}{dt}, \\ q_{zx} - q'_{zx} = \chi_{zx} = - \dfrac{P}{\Delta}\dfrac{dz}{dt}, \end{cases} \tag{93'}$$

and similar expressions for χ_{xy}, χ_{yy}, χ_{zy}; χ_{xz}, χ_{yz}, χ_{zz} which we obtain by cyclic permutation of P, Q, R and of x, y, z.

79. <u>The Method of the Action Variable Applied to the Euler Concept of Action Expressed in Terms of Lagrange Variables.</u> We will now develop the Euler concept of the action using Lagrange variables. We will start with the integral

$$\int_{t_1}^{t_2} \iiint_{s_0} W dx_0 dy_0 dz_0 dt,$$

where at present the region of integration with respect to x_0, y_0, z_0 varies with time t and corresponds to the fixed region of integration described by the point (x, y, z).

Following the derivation of M. Jordan, we have: /217

$$\delta \int_{t_1}^{t_2} \iiint_{s_0} W dx_0 dy_0 dz_0 dt$$
$$= \int_{t_1}^{t_2} \iiint_{s_0} \Big[\sum \left(\frac{\partial W}{\partial \xi_i} \delta \xi_i + \ldots + \frac{\partial W}{\partial r_i} \delta r_i \right) + \frac{\partial W}{\partial \xi} \delta \xi + \ldots + \frac{\partial W}{\partial r} \delta r$$
$$+ \frac{d}{dx_0} \left(W(\delta x_0) \right) + \frac{d}{dy_0} \left(W(\delta y_0) \right) + \frac{d}{dz_0} \left(W(\delta z_0) \right) \Big] dx_0 dy_0 dz_0 dt,$$

where (δx_0), (δy_0), (δz_0) are defined by equations (86) using the auxiliary quantities δx, δy, δz.

The sequence of calculations resembles that which we encountered in the dynamics of the deformable medium. However, a difference regarding the derivatives with respect to time is introduced. In the present case, we cannot interchange the integration with respect to time and the integration with respect to the variables x_0, y_0, z_0. We may therefore apply a reasoning similar to that of Section 76. We will therefore only introduce derivatives with respect to time of the form $\frac{\partial}{\partial t}$, using the equation

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{d\mathcal{F}}{dt} + \frac{\partial \mathcal{F}}{\partial x_0} \frac{\partial x_0}{\partial t} + \frac{\partial \mathcal{F}}{\partial y_0} \frac{\partial y_0}{\partial t} + \frac{\partial \mathcal{F}}{\partial z_0} \frac{\partial z_0}{\partial t},$$

where $\frac{\partial x_0}{\partial t}$, $\frac{\partial y_0}{\partial t}$, $\frac{\partial z_0}{\partial t}$ designate the derivatives of functions x_0, y_0, z_0 of x, y, z, t with respect to t, which are obtained from equations (66'). Using the notation already introduced, the preceding formula can be written as:

$$\frac{d\mathcal{F}}{dt} = \frac{\partial \mathcal{F}}{\partial t} - (\xi) \frac{\partial \mathcal{F}}{\partial x_0} - (\eta) \frac{\partial \mathcal{F}}{\partial y_0} - (\zeta) \frac{\partial \mathcal{F}}{\partial z_0}. \tag{94}$$

If there is a term of the form

$$\int_{t_1}^{t_2} \iiint_{s_0} g \frac{\partial h}{\partial t} dx_0 dy_0 dz_0 dt,$$

it will be written as

$$\int_{t_1}^{t_2} \iiint_{s} \frac{g}{\Delta} \frac{\partial h}{\partial t} dx dy dz dt,$$

and, by integrating by parts,

$$\iiint_{\Delta}\left\{\frac{g}{\Delta}h\right\}_{t_1}^{t_2}dxdydz-\int_{t_1}^{t_2}\iiint_{\Delta}h\frac{\partial}{\partial t}\left(\frac{g}{\Delta}\right)dxdydzdt$$
$$=\left\{\iiint_{\Delta}\frac{g}{\Delta}hdxdydz\right\}_{t_1}^{t_2}-\int_{t_1}^{t_2}\iiint_{\Delta}h\frac{\partial}{\partial t}\left(\frac{g}{\Delta}\right)dxdydzdt,$$

i.e., returning to the variables x_0, y_0, z_0, we obtain /218

$$\left\{\iiint_{\Delta_0}ghdx_0dy_0dz_0\right\}_{t_1}^{t_2}-\int_{t_1}^{t_2}\iiint_{\Delta_0}h\Delta\frac{\partial}{\partial t}\left(\frac{g}{\Delta}\right)dx_0dy_0dz_0dt.$$

With this, it follows from the preceding equations of dynamics of the deformable medium and from (94) that we obtain the following by integration by parts,

$$\begin{aligned}
&\delta\int_{t_1}^{t_2}\iiint_{\Delta_0}W\,dx_0dy_0dz_0dt\\
&=\int_{t_1}^{t_2}\iint_{\Delta_0}(F_0'\delta'x+G_0'\delta'y+H_0'\delta'z+I_0'\delta\iota'+J_0'\delta\lambda'+K_0'\delta\kappa')\,dx_0dt\\
&+\left\{\iiint_{\Delta_0}(A'\delta'x+B'\delta'y+C'\delta'z+P'\delta\iota'+Q'\delta\lambda'+R'\delta\kappa')\,dx_0dy_0dz_0\right\}_{t_1}^{t_2}\\
&-\int_{t_1}^{t_2}\iiint_{\Delta_0}(X_0'\delta'x+Y_0'\delta'y+Z_0'\delta'z+L_0'\delta\iota'+M_0'\delta\lambda'+N_0'\delta\kappa')\,dx_0dy_0dz_0dt.
\end{aligned}$$

by setting

$$F_0'=l_0\left\{\frac{\partial W}{\partial\xi_1}-(\xi_1)W-(\xi)\frac{\partial W}{\partial\xi}\right\}+m_0\left\{\frac{\partial W}{\partial\xi_2}-(\xi_2)W-(\eta)\frac{\partial W}{\partial\xi}\right\}+n_0\left\{\frac{\partial W}{\partial\xi_3}-(\xi_3)W-(\zeta)\frac{\partial W}{\partial\xi}\right\},$$
$$G_0'=l_0\left\{\frac{\partial W}{\partial\eta_1}-(\eta_1)W-(\xi)\frac{\partial W}{\partial\eta}\right\}+m_0\left\{\frac{\partial W}{\partial\eta_2}-(\eta_2)W-(\eta)\frac{\partial W}{\partial\eta}\right\}+n_0\left\{\frac{\partial W}{\partial\eta_3}-(\eta_3)W-(\zeta)\frac{\partial W}{\partial\eta}\right\},$$
$$H_0'=l_0\left\{\frac{\partial W}{\partial\zeta_1}-(\zeta_1)W-(\xi)\frac{\partial W}{\partial\zeta}\right\}+m_0\left\{\frac{\partial W}{\partial\zeta_2}-(\zeta_2)W-(\eta)\frac{\partial W}{\partial\zeta}\right\}+n_0\left\{\frac{\partial W}{\partial\zeta_3}-(\zeta_3)W-(\zeta)\frac{\partial W}{\partial\zeta}\right\},$$
$$I_0'=l_0\left\{\frac{\partial W}{\partial p_1}-(\xi)\frac{\partial W}{\partial p}\right\}+m_0\left\{\frac{\partial W}{\partial p_2}-(\eta)\frac{\partial W}{\partial p}\right\}+n_0\left\{\frac{\partial W}{\partial p_3}-(\zeta)\frac{\partial W}{\partial p}\right\},$$
$$J_0'=l_0\left\{\frac{\partial W}{\partial q_1}-(\xi)\frac{\partial W}{\partial q}\right\}+m_0\left\{\frac{\partial W}{\partial q_2}-(\eta)\frac{\partial W}{\partial q}\right\}+n_0\left\{\frac{\partial W}{\partial q_3}-(\zeta)\frac{\partial W}{\partial q}\right\},$$
$$K_0'=l_0\left\{\frac{\partial W}{\partial r_1}-(\xi)\frac{\partial W}{\partial r}\right\}+m_0\left\{\frac{\partial W}{\partial r_2}-(\eta)\frac{\partial W}{\partial r}\right\}+n_0\left\{\frac{\partial W}{\partial r_3}-(\zeta)\frac{\partial W}{\partial r}\right\},$$
$$\begin{aligned}
X_0'=&\frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial\xi_1}-(\xi)\frac{\partial W}{\partial\xi}\right)+\frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial\xi_2}-(\eta)\frac{\partial W}{\partial\xi}\right)+\frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial\xi_3}-(\zeta)\frac{\partial W}{\partial\xi}\right)\\
&+\sum\left(q_i\frac{\partial W}{\partial\zeta_i}-r_i\frac{\partial W}{\partial\eta_i}\right)+\Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial\xi}\right)+q\frac{\partial W}{\partial\zeta}-r\frac{\partial W}{\partial\eta},
\end{aligned}$$
$$\begin{aligned}
Y_0'=&\frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial\eta_1}-(\xi)\frac{\partial W}{\partial\eta}\right)+\frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial\eta_2}-(\eta)\frac{\partial W}{\partial\eta}\right)+\frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial\eta_3}-(\zeta)\frac{\partial W}{\partial\eta}\right)\\
&+\sum\left(r_i\frac{\partial W}{\partial\xi_i}-p_i\frac{\partial W}{\partial\zeta_i}\right)+\Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial\eta}\right)+r\frac{\partial W}{\partial\xi}-p\frac{\partial W}{\partial\zeta},
\end{aligned}$$
$$\begin{aligned}
Z_0'=&\frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial\zeta_1}-(\xi)\frac{\partial W}{\partial\zeta}\right)+\frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial\zeta_2}-(\eta)\frac{\partial W}{\partial\zeta}\right)+\frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial\zeta_3}-(\zeta)\frac{\partial W}{\partial\zeta}\right)\\
&+\sum\left(p_i\frac{\partial W}{\partial\eta_i}-q_i\frac{\partial W}{\partial\xi_i}\right)+\Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial\zeta}\right)+p\frac{\partial W}{\partial\eta}-q\frac{\partial W}{\partial\xi},
\end{aligned}$$
$$L_0'=\frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial p_1}-(\xi)\frac{\partial W}{\partial p}\right)+\frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial p_2}-(\eta)\frac{\partial W}{\partial p}\right)+\frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial p_3}-(\zeta)\frac{\partial W}{\partial p}\right)$$

$$+\sum\left(q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial \zeta_i}-\zeta_i\frac{\partial W}{\partial \eta_i}\right)+\Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial p}\right)+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}+\eta\frac{\partial W}{\partial \zeta}-\zeta\frac{\partial W}{\partial \eta}.$$ /219

$$M'_0=\frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial q_1}-(\xi)\frac{\partial W}{\partial q}\right)+\frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial q_2}-(\eta)\frac{\partial W}{\partial q}\right)+\frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial q_3}-(\zeta)\frac{\partial W}{\partial q}\right)$$
$$+\sum\left(r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\zeta_i\frac{\partial W}{\partial \xi_i}-\xi_i\frac{\partial W}{\partial \zeta_i}\right)+\Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial q}\right)+r\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial r}+\zeta\frac{\partial W}{\partial \xi}-\xi\frac{\partial W}{\partial \zeta}.$$

$$N'_0=\frac{\partial}{\partial x_0}\left(\frac{\partial W}{\partial r_1}-(\xi)\frac{\partial W}{\partial r}\right)+\frac{\partial}{\partial y_0}\left(\frac{\partial W}{\partial r_2}-(\eta)\frac{\partial W}{\partial r}\right)+\frac{\partial}{\partial z_0}\left(\frac{\partial W}{\partial r_3}-(\zeta)\frac{\partial W}{\partial r}\right)$$
$$+\sum\left(p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial \eta_i}-\eta_i\frac{\partial W}{\partial \xi_i}\right)+\Delta\frac{\partial}{\partial t}\left(\frac{1}{\Delta}\frac{\partial W}{\partial r}\right)+p\frac{\partial W}{\partial q}-q\frac{\partial W}{\partial p}+\xi\frac{\partial W}{\partial \eta}-\eta\frac{\partial W}{\partial \xi}.$$

We should observe that N_0', for example, can be written as follows, due to (94),

$$X_0'=\sum\left(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \xi_i}+q_i\frac{\partial W}{\partial \zeta_i}-r_i\frac{\partial W}{\partial \eta_i}\right)+\frac{d}{dt}\frac{\partial W}{\partial \xi}+q\frac{\partial W}{\partial \zeta}-r\frac{\partial W}{\partial \eta}$$
$$-\left(\frac{1}{\Delta}\frac{\partial \Delta}{\partial t}+\frac{\partial(\xi)}{\partial x_0}+\frac{\partial(\eta)}{\partial y_0}+\frac{\partial(\zeta)}{\partial z_0}\right)\frac{\partial W}{\partial \xi};$$

and thus we have

$$\frac{1}{\Delta}\frac{\partial \Delta}{\partial t}=-\left(\frac{\partial(\xi)}{\partial x_0}+\frac{\partial(\eta)}{\partial y_0}+\frac{\partial(\zeta)}{\partial z_0}\right),$$

and consequently X_0' *has the same value*

$$X_0'=\sum\left(\frac{\partial}{\partial p_i}\frac{\partial W}{\partial \xi_i}+q_i\frac{\partial W}{\partial \zeta_i}-r_i\frac{\partial W}{\partial \eta_i}\right)+\frac{d}{dt}\frac{\partial W}{\partial \xi}+q\frac{\partial W}{\partial \zeta}-r\frac{\partial W}{\partial \eta}$$

as in Section 62. The same remark is applied to Y_0', Z_0', L_0', M_0', N_0'. However, *this is not true for the stress and moment of deformation.* Using simple transformations, we again find the relationships (93) and (93') of Section 78.

80. The Concept of Radiation Energy of Deformation and of Motion. We have seen that the density of the energy of deformation and of motion, expressed in terms of Lagrangian arguments and referred to the space (x_0, y_0, z_0) is

$$E=\xi\frac{\partial W}{\partial \xi}+\eta\frac{\partial W}{\partial \eta}+\zeta\frac{\partial W}{\partial \zeta}+p\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial q}+r\frac{\partial W}{\partial r}-W; \tag{95}$$

this same density, *referred to the space* (x, y, z) and expressed in terms of the function Ω of Euler arguments (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i); (ξ), (η), (ζ), (p), (q), (r) is

$$\frac{E}{\Delta}=(\xi)\frac{\partial \Omega}{\partial(\xi)}+(\eta)\frac{\partial \Omega}{\partial(\eta)}+(\zeta)\frac{\partial \Omega}{\partial(\zeta)}+(p)\frac{\partial \Omega}{\partial(p)}+(q)\frac{\partial \Omega}{\partial(q)}+(r)\frac{\partial \Omega}{\partial(r)}-\Omega. \tag{96}$$

This result is obtained either by transforming the expression (95) using relationships which we indicated previously between the Lagrangian arguments and the Euler arguments, or by directly repeating the reasoning of Section 65 for the elementary work /220

$$dt\left\{\iiint_{\omega_0}(\xi X_0'+\eta Y_0'+\zeta Z_0'+pL_0'+qM_0'+rN_0')dx_0dy_0dz_0\right.$$

$$-\iint_{\sigma_0}(\xi F_0' + \eta G_0' + \zeta H_0' + pI_0' + qJ_0' + rK_0')d\sigma_0\Big\}$$

of the external forces and moments and the external stresses and moments of deformation which are applied to the part (M) of the medium which at the time t occupies the part (M_0) of the natural state. Using this latter method, we again obtain the following expression for the elementary work

$$dt\left\{\iiint_{\Omega_0}\frac{dE}{dt}dx_0dy_0dz_0\right\},$$

Ω is assumed to be independent of t.

If we note that we have the following identity:

$$\frac{1}{\Delta}\frac{dE}{dt} = \frac{\partial}{\partial t}\left(\frac{E}{\Delta}\right) + \frac{\partial}{\partial x}\left(\frac{E}{\Delta}\frac{dx}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{E}{\Delta}\frac{dy}{dt}\right) + \frac{\partial}{\partial z}\left(\frac{E}{\Delta}\frac{dz}{dt}\right),$$

used by M. Poincaré in the memoir which we cited in Section 77, and which may be applied to an arbitrary function, we obtain the following new expression for the elementary work under consideration:

$$dt\left\{\frac{\partial}{\partial t}\iiint_{\Omega}\frac{E}{\Delta}dxdydz\right.$$
$$\left.+\iiint_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{E}{\Delta}\frac{dx}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{E}{\Delta}\frac{dy}{dt}\right) + \frac{\partial}{\partial z}\left(\frac{E}{\Delta}\frac{dz}{dt}\right)\right]dxdydz\right\},$$

or

$$dt\left\{\frac{\partial}{\partial t}\iiint_{\Omega}\frac{E}{\Delta}dxdydz + \iint_{S}\frac{E}{\Delta}\left(l\frac{dx}{dt} + m\frac{dy}{dt} + n\frac{dz}{dt}\right)dS\right\}. \tag{97}$$

The second integral in (97) expresses the *energy flux of deformation and of motion* through a *fixed surface* S in the deformed body.

Let us now consider the Euler concept of the action. In the preceding paragraphs, we found that the values of the external forces and moments remain the same, but the following terms vanish in the expressions for the stresses p_{xx}, p_{xy}, p_{xz}

$$\pi_{xx} = \Omega - \frac{A}{\Delta}\frac{dx}{dt},$$
$$\pi_{xy} = -\frac{B}{\Delta}\frac{dx}{dt},$$
$$\pi_{xz} = -\frac{C}{\Delta}\frac{dx}{dt},$$

and the following terms vanish in the expressions for the moments of deformation /221
q_{xx}, q_{xy}, q_{xz},

$$\chi_{xx} = -\frac{P}{\Delta}\frac{dx}{dt},$$
$$\chi_{xy} = -\frac{Q}{\Delta}\frac{dx}{dt},$$
$$\chi_{xz} = -\frac{R}{\Delta}\frac{dx}{dt},$$

with similar expressions for the quantities π_{yx}, π_{yy}, π_{yz}, π_{zx}, π_{zy}, π_{zz} and X_{yx}, X_{yy}, X_{yz}, X_{zx}, X_{zy}, X_{zz}. From this it results that to the elementary work obtained in the preceding case we must add a new surface integral, which is as follows

$$dt \left\{ \iint_s \left(l \frac{dx}{dt} + m \frac{dy}{dt} + n \frac{dz}{dt} \right) \left[\Omega - \frac{1}{\Delta} \left(A \frac{dx}{dt} + B \frac{dy}{dt} + C \frac{dz}{dt} \right) - \left\{ p(P') + q(Q') + r(R') \right\} \right] d\sigma \right\}.$$

We may call this new integral the *flux of energy radiation through the surface* S *of the deformed body*.

The reasoning used in Section 64 and based on the *Euclidean invariance* of the action density does not lead to the same conclusions regarding the external forces and moments, as well as regarding the *new* external stresses and moments. We may interpret this by saying that the new stresses and moments of deformation no longer satisfy what M. Poincaré called the *reaction principle*. As is well known, this latter conclusion is also found in the electricity theories of M. Lorentz. However, the existence of the radiation which we have derived allows us to associate the stresses and moments of deformation π_{xx}, π_{yx}, ..., X_{xx}, X_{yx}, ... with what Maxwell (using ideas from the electromagnetic theory of light) and Bartoli (using thermodynamics) called the *energy radiation pressure*, and in this way we may again observe the *reaction principle*.

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