NASA TECHNICAL TRANSLATION NASA TT F-11,576

NASA TT F-11,576

EXPERIMENTAL STUDIES ON THE THEORY OF W. RITZ ON TRANSVERSE VIBRATIONS OF SQUARE PLATES

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Translation of "Experimentelle Untersuchungen zur W. Ritzschen Theorie der Transversalschwingungen Quadratischer Platten". Annalen der Physik, Vol.86, pp 717-750, 1928.

FACILITY FORM 602
N68-31982
(ACCESSION NUMBER)
(THRU)
35
(PAGES)
1
(CODE)
(NASA CR OR TMX OR AD NUMBER)
32
(CATEGORY)

AUG 1968
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 APRIL 1967.

NASA TT F-11,576

EXPERIMENTAL STUDIES ON THE THEORY OF W. RITZ ON TRANSVERSE VIBRATIONS OF SQUARE PLATES

A. Lemke

ABSTRACT. Transverse vibrations of plane elastic plates have been studied. Experimental data are correlated to theoretical requirements. The accuracy of most of the measurements is not very great. Extensive tables have been compiled and are acclaimed to be of general significance over and above the immediate concern of the paper.

I. Introduction /717*

The problem of transverse vibrations of plane elastic plates with free edges, which - ever since the experiments of Chladni were conducted - has been treated by mathematicians and physicists, requires the integration of the differential equation:

$$\frac{\partial^2 w}{\partial t^2} + \frac{E \cdot D^2}{3\varrho \cdot (1 - \mu^2)} \cdot \Delta \Delta w = 0 \quad [16] \tag{1}$$

under certain boundary conditions. In this equation E represents Young's modulus of elasticity, μ Poisson's elasticity number, ϱ the density of the plate material, 2 D is the thickness of the plates, w the displacement in the direction perpendicular to the plane of the plate which progresses with the velocity $\partial w/\partial t$. The equation is valid for square plates in the case of which the length of the edge was introduced as unity.

According to W. Ritz [17, 18] the problem may be reduced to a problem of variation which in the form of an expansion in a power series results in a solution of the differential equation with theoretically any degree of approximations. Thus the deflection w may be represented by a convergent serial development of arbitrary functions of an orthogonal system. The convergence of the progression and thus the practical value of this method depend in essence upon the appropriate choice of functions whereby the choice of functions is facilitated by the result of the experiment. Ritz applies as

*Numbers in the margin indicate pagination in the foreign text.

suitable function - in analogy of the plate to the elastic rod - the characteristic function of oscillating rods.

In order to experimentally prove his method, W. Ritz /718 resorted to the relative pitches of tone as they had been observed by Chladni and to the acoustic figures as measured by Strehlke [18b]. Statements made by Chladni are in several respects uncertain. In an effort evidently to establish a mathematical interrelationship of the oscillation frequencies, Chladni overlooked the dependence of the relative pitches of tone from the material of the plates and thus did not make any statement thereon. It remains also uncertain whether the specialized intervals refer to beats at constant temperature. Ritz made the assumption that Chladni used glass plates - a fact to which he, by the way, refers as being more feasible than metal plates - that is, he chose a mean value of different μ-values found for the individual types of glass for which he carried through the calculation with 3-6 terms in a progressive development and found for this arbitrarily chosen μ- value his theory confirmed. In doing so he does, however, not specify frequencies of oscillation but rather pitches of tone in the terminology of music. A +- sign indicates that the tone is somewhat higher, a -- sign, that the tone is somewhat lower.

The more exact investigation requires in addition to the determination of the absolute pitches of tone knowledge of the coefficient of elasticity E and the value μ for the plates used. Ritz has already pointed out the possibility of determining the coefficients of elasticity by exact observation of the frequencies of oscillation of several suitably chosen tones of the plates [18c]. The frequency of oscillation n is by

$$n = \frac{D}{l \cdot \pi} \cdot \sqrt{\frac{E}{3\varrho \cdot (1-\mu^2)}} \cdot \sqrt{\lambda(\mu)}, \qquad (2)$$

where, except for λ, all terms are constants of the plate while $\lambda(\mu)$ is a characteristic function of μ for the corresponding tone which can be calculated numerically according to the Ritz approach with any desirable degree of approximation.

The ratio of two given frequencies of oscillation of a plate is only dependent upon λ:

$$\frac{n_a}{n_b} = \frac{\sqrt{\lambda_a(\mu)}}{\sqrt{\lambda_b(\mu)}} = F(\mu) \qquad (3)$$

and it is thus possible to calculate the Poisson elasticity /719

number μ. Equation 2 allows the calculation of the modulus of elasticity E for absolute determinations of the frequency of oscillation and with the help of the known relationship

$$\mu = \frac{E - 2F}{2F} \tag{4}$$

it will finally also be possible to determine the modulus of torsional shear F.

The coefficients of elasticity of circular plates have been determined by Professor Kalähne [7] with the help of the formulas calculated by Kirchhoff. The coefficient of elasticity μ was obtained with 10% accuracy.

In the present paper we aimed at:

1. a test of the Ritz procedure by measuring the pitches of tone and the shape of figures for as many characteristic tones of a plate as possible and the comparison with the results of the theory,
2. an attempt to determine the coefficients of elasticity of given plates from the observed pitches of tone.

2. Numerical Application of the Ritz Calculation Approach

Let us assume u_m and u_n are the characteristic functions of vibrating rods [8a]:

$$\begin{cases} u_m(x) = \dfrac{\mathfrak{Cof}\, k_m \cdot \cos k_m x + \cos k_m \cdot \mathfrak{Cof}\, k_m x}{\sqrt{\mathfrak{Cof}^2 k_m + \cos^2 k_m}} & \text{for even } m \\ u_n(x) = \dfrac{\mathfrak{Sin}\, k_n \cdot \sin k_n x + \sin k_n \cdot \mathfrak{Sin}\, k_n x}{\sqrt{\mathfrak{Sin}^2 k_n - \sin^2 k_n}} & \text{for odd } n \end{cases} \tag{5}$$

where the length of the rod equals 1. The terms k_m and k_n are the roots of the equations

$$\begin{cases} \operatorname{tg} x + \mathfrak{Tg}\, x = 0 & (\text{roots } k_m) \\ \operatorname{tg} x - \mathfrak{Tg}\, x = 0 & (\text{roots } k_n) \end{cases} \tag{6}$$

which have the following values:

m	k_m	n	k_n
0	0	1	0
2	2,36502	3	3,92660
4	5,49780	5	7,06858
6	8,63938	7	10,21017
8	11,78096	9	13,35175
10	14,92255		

*

The functions of the expansion in a power series are set equal to the products of two each characteristic functions of oscillating rods in which for the purpose of better distinction these functions are designated for the argument x with u, for the argument y with v:

$$\psi_{mn} = u_m(x) \cdot v_n(y). \tag{7}$$

The expansion in a power series thus assumes for each individual tone the following form:

$$\begin{cases} w = A_{00} \cdot u_0(x) \cdot v_0(y) + A_{10} \cdot u_1(x) \cdot v_0(y) + A_{01} \cdot u_0(x) \cdot v_1(y) \\ \qquad + A_{11} \cdot u_1(x) \cdot v_1(y) + A_{02} \cdot u_0(x) \cdot v_2(y) \\ \qquad + \dots \end{cases} \tag{8}$$

in which the coefficients A_{00} etc. can be determined with the help of the Ritz approach by a minimum condition. It requires the solution of the equations:

$$\begin{cases} 0 = (\alpha_{00}^{00} - \lambda) \cdot A_{00} + \alpha_{01}^{00} \cdot A_{01} \\ \qquad + \alpha_{10}^{00} \cdot A_{10} + \dots \quad + \alpha_{ss}^{00} \cdot A_{ss} + \dots \\ 0 = \alpha_{00}^{01} \cdot A_{00} + (\alpha_{01}^{01} - \lambda) \cdot A_{01} + \dots \quad + \alpha_{ss}^{01} \cdot A_{ss} + \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 0 = \alpha_{00}^{ss} \cdot A_{00} + \alpha_{01}^{ss} \cdot A_{01} \\ \qquad + \alpha_{10}^{ss} \cdot A_{10} + \dots + (\alpha_{ss}^{ss} - \lambda) \cdot A_{ss} + \dots \\ 0 = \dots\dots\dots\dots\dots\dots \end{cases} \tag{9}$$

The α-terms of these equations signify the following:

$$\alpha_{mn}^{mn} = k_m^4 + k_n^4 + 2\mu \cdot \omega_{mm} \cdot \omega_{nn} + 2(1-\mu) \cdot \gamma_{mm} \cdot \gamma_{nn}, \tag{10}$$

*Translator's note: Commas represent decimal points in reproduced material.

$$\alpha_{mn}^{pq} = \mu \cdot (\omega_{mp} \cdot \omega_{qn} + \omega_{pm} \cdot \omega_{nq}) + 2(1-\mu) \cdot \gamma_{nq} \cdot \gamma_{mp}, \tag{11}$$

where ω and γ are given by the following formula:

$$\omega_{mn} = \frac{2 k_m^4 [u_m' \cdot u_n - u_m \cdot u_n'](x=1)}{k_m^4 - k_n^4}, \quad \omega_{0m} = \omega_{1m} = 0, \quad (m \neq n) \tag{12}$$

$$\gamma_{mn} = \frac{2 [k_m^4 \cdot u_m \cdot u_n' - k_n^4 \cdot u_m' \cdot u_n](x=1)}{k_m^4 - k_n^4}, \quad \gamma_{0n} = \gamma_{0n} = 0. \quad (m \neq n) \tag{13}$$

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By contrast, for m = n the following is valid:

a. for even m:

$$\omega_{mm} = \frac{-k_m^2 (\mathfrak{Cof}^2 k_m - \cos^2 k_m)}{\mathfrak{Cof}^2 k_m + \cos^2 k_m} + \frac{2 k_m \cdot \cos^2 k_m \cdot \mathfrak{Cof}^2 k_m \cdot \mathfrak{Tg}\, k_m}{\mathfrak{Cof}^2 k_m + \cos^2 k_m}, \tag{14}$$

$$\gamma_{mm} = \frac{k_m^2 \cdot (\mathfrak{Cof}^2 k_m - \cos^2 k_m)}{\mathfrak{Cof}^2 k_m + \cos^2 k_m} + \frac{6 k_m \cdot \cos^2 k_m \cdot \mathfrak{Cof}^2 k_m \cdot \mathfrak{Tg}\, k_m}{\mathfrak{Cof}^2 k_m + \cos^2 k_m}; \tag{15}$$

b. for odd n:

$$\omega_{nn} = \frac{-k_n^2 (\mathfrak{Sin}^2 k_n + \sin^2 k_n)}{\mathfrak{Sin}^2 k_n - \sin^2 k_n} + \frac{2 k_n \mathfrak{Sin}^2 k_n \cdot \sin^2 k_n \cdot \mathfrak{Cotg}\, k_n}{\mathfrak{Sin}^2 k_n - \sin^2 k_n}, \tag{16}$$

$$\gamma_{nn} = \frac{k_n^2 \cdot (\mathfrak{Sin}^2 k_n + \sin^2 k_n)}{\mathfrak{Sin}^2 k_n - \sin^2 k_n} + \frac{6 k_n \mathfrak{Sin}^2 k_n \cdot \sin^2 k_n \cdot \mathfrak{Cotg}\, k_n}{\mathfrak{Sin}^2 k_n - \sin^2 k_n}. \tag{17}$$

The numerically calculated ω- and γ- values up to index 10 are compiled in Tables 1 and 2. The α-terms are linearly dependent upon the Poisson ratio μ. Tables 3 and 4 contain the α-terms calculated from $\mu = 0.36$; Tables 5 and 6 contain their derivations according to μ so that the α-terms may be calculated for each μ-value from equations (10) and (11) which are linear in μ.

Thus, one is in a position to determine the α-values for given μ-values from Tables 3 through 6 by suitable interpolation and thus the frequency parameters; this makes these numerical tables valuable over and above the specific investigation.

The calculation of the system of equations is facilitated by the fact that due to symmetry properties of the sound patterns the number of terms is reduced. One distinguishes acoustic singlets and acoustic doublets. If a tone is associated by two acoustic patterns which do not coincide, this tone is referred to as an acoustic doublet. For an acoustic

singlet, that is for one to which only a single acoustic pattern belongs, w(-x,+y) by its absolute value is identical with w(+x,+y); if this were not the case, one would be able to construct by exchange of -x with +x another figure which cannot be brought to coincide with the first one, a fact which is in contradiction to the assumption. To acoustic singlets, therefore, only the following two conditions are applicable, either w(-x,+y) = -w(+x,y) or w(-x,+y) = +w(x,y); the solution thus is either an even or an odd function. /723
Since x and y may be exchanged arbitrarily for square plates, w(x,y) and w(y,x) can also differ by the sign; thus the function corresponding to acoustic singlets can only be either symmetrical or antisymmetrical.

Table 1

	γ_{mn}										
m	γ_{m0}	γ_{m1}	γ_{m2}	γ_{m3}	γ_{m4}	γ_{m5}	γ_{m6}	γ_{m7}	γ_{m8}	γ_{m9}	$\gamma_{m\,10}$
0	0	0	0	0	0	0	0	0	0	0	0
1	0	+3	0	−3,4641	0	+3,4641	0	−3,4641	0	+3,4641	0
2	0	0	+12,370	0	−8,8442	0	+9,1521	0	−9,2332	0	+9,2628
3	0	−3,4641	0	+27,2316	0	−14,3966	0	+15,1565	0	−15,4348	0
4	0	0	−8,8442	0	+46,717	0	−19,5263	0	+20,740	0	−21,284
5	0	+3,4641	0	−14,3966	0	+71,1705	0	−24,526	0	+26,131	0
6	0	0	+9,1521	0	−19,5263	0	+100,556	0	−29,444	0	+31,377
7	0	−3,4641	0	+15,1565	0	−24,526	0	+134,88	0	−34,311	0
8	0	0	−9,2332	0	+20,740	0	−29,444	0	+174,133	0	−39,142
9	0	+3,4641	0	−15,4348	0	+26,131	0	−34,311	0	+218,32	0
10	0	0	+9,2628	0	−21,284	0	+31,377	0	−39,142	0	+267,45

Table 2

	ω_{mn}										
m	ω_{m0}	ω_{m1}	ω_{m2}	ω_{m3}	ω_{m4}	ω_{m5}	ω_{m6}	ω_{m7}	ω_{m8}	ω_{m9}	$\omega_{m\,10}$
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0		0	0	0	0	0	0	0
2	−4,6473	0	−3,0756	0	−0,4502	0	+0,14268	0	−0,061542	0	+0,031816
3	0	−10,1485	0	−11,5124	0	−1,32145	0	+0,56186	0	−0,28405	0
4	+10,9953	0	−13,1468	0	−24,726	0	−2,4651	0	+1,25135	0	−0,70765
5	0	+21,022	0	−13,8775	0	−42,896	0	−3,7477	0	+2,1426	0
6	−17,2787	0	+25,406	0	−15,0318	0	−66,00	0	−5,1129	0	+3,1810
7	0	−31,905	0	+25,686	0	−16,314	0	−94,037	0	−6,5303	0
8	+23,562	0	−37,891	0	+26,384	0	−17,6791	0	−127,01	0	−7,9829
9	0	+42,788	0	−37,9735	0	+27,275	0	−19,0968	0	−164,92	0
0	−29,845	0	+50,428	0	−38,407	0	+28,314	0	−20,5495	0	−

Solution of the system of equations according to the approach described in detail by W. Ritz[18d] by successive correction results in approximate values λ for the frequency parameter λ and approximate values for the coefficients A of the progressive development. Since the coefficients of elasticity of the plate had to be determined, the parameters λ to which the frequency of oscillation is linked by (2) had to be calculated as functions of μ with an accuracy as great as possible.

This requires for each λ-value the solution of a system of equations (9) of s equations with s unknowns. The present study initially used the value $\mu = 0.225$ as chosen by Ritz. The calculation revealed several differences as opposed to the values calculated by Ritz - evidently due to errors in calculation encountered there. In the following compilation of the results of the calculation the values calculated by Ritz are therefore given for the purpose of comparison. It is not immediately evident with what accuracy the λ-values are obtained in the case of a development with s terms; likewise the progression developed with s terms cannot simply be extended by additional terms. Rather, each preceeding term of the progression changes with the addition of extra terms. Therefore, the progressive development was carried out for 6 low tones with 6 terms, then with 15 terms for different μ-values initially for $\mu = 0.225$. Since, however, not glass plates but only metal plates were available for which μ is much greater, other μ-values had to be chosen for the calculation as is evident from the following compilation of the results of the calculation[1].

[1] I am greatly indebted to the Director of the Observatory of Königsberg, Prof. Przybyllok, for his permission to use the Brunswiga calculator.

Table 3

α-values (odd)

α_{mn}^{pq}	α_{11}^{pq}	α_{13}^{pq}	α_{31}^{pq}	α_{33}^{pq}	α_{15}^{pq}	α_{51}^{pq}
α_{mn}^{11}	+11,52	−13,302	−13,302	+15,36	+13,302	+13,302
α_{mn}^{13}	−13,302	+342,29	+52,437	−78,689	−55,283	−92,163
α_{mn}^{31}	−13,302	+52,437	+342,29	−78,689	−92,163	−55,283
α_{mn}^{33}	+15,36	−78,689	−78,689	+1520,1	+114,54	+114,5
α_{mn}^{15}	+13,302	−55,28	−92,16	+114,54	+2769,8	+174,46
α_{mn}^{51}	+13,302	−92,16	−55,28	+114,54	+174,46	+2769,8
α_{mn}^{35}	−15,36	+68,66	+33,624	−438,83	−158,85	−168,86
α_{mn}^{53}	−15,36	+33,624	+68,66	−438,83	−168,86	−158,85
α_{mn}^{55}	+15,36	−73,836	−73,836	+278,50	−9,07	−9,07
α_{mn}^{17}	−13,302	+58,201	+131,92	−161,04	−94,18	−256,82
α_{mn}^{71}	−13,302	+131,92	+58,201	−161,04	−256,82	−94.18
α_{mn}^{37}	+15,36	−69,258	+11,479	+419,53	+122,44	+223,23
α_{mn}^{73}	+15,36	+11,479	−69,258	+419,53	+223,23	+122,44
α_{mn}^{57}	−15,36	+71,457	+79,013	−294,33	−137,11	+177,13
α_{mn}^{75}	−15,36	+79,013	+71,457	−294,33	+177,13	−137,11
α_{mn}^{77}	+15,36	−73,658	−73,658	+304,43	+151,8	+151,8
α_{mn}^{19}	+13,302	−59,269	−171,68	+207,17	+100,34	+339,18
α_{mn}^{91}	+13,302	−171,68	−59,269	+207,17	+339,18	+100,34
α_{mn}^{39}	−15,36	+69,476	−56,581	−379,46	−123,70	−277,60
α_{mn}^{93}	−15,36	−56,581	+69,476	−379,46	−277,60	−123,70
α_{mn}^{59}	+15,36	−70,588	−84,190	+303,91	+132,08	−345,18
α_{mn}^{95}	+15,36	−84,190	−70,588	+303,91	−345,18	+132,08
α_{mn}^{79}	−15,36	+71,70	+75,86	−309,75	−140,48	−166,48
α_{mn}^{97}	−15,36	+75,86	+71,70	−309,75	−166,48	−140,48
α_{mn}^{99}	+15,36	−72,813	−72,813	+312,71	+148,87	+148,87

Table 3 (Continued)

α-values (odd)

α_{mn}^{pq}	α_{35}^{pq}	α_{53}^{pq}	α_{55}^{pq}	α_{17}^{pq}	α_{71}^{pq}	α_{37}^{pq}
α_{mn}^{11}	−15,36	−15,36	+15,36	−13,302	−13,302	+15,36
α_{mn}^{13}	+68,663	+33,624	−73,836	+58,201	+131,92	−69,258
α_{mn}^{31}	+33,624	+68,663	−73,836	+131,92	+58,201	+11,479
α_{mn}^{33}	−438,83	−438,83	+278,5	−161,04	−161,04	+419,5
α_{mn}^{15}	−158,85	−168,86	−9,07	−94,180	−256,82	+122,44
α_{mn}^{51}	−168,86	−158,85	−9,070	−256,82	−94,180	+223,23
α_{mn}^{35}	+5570,6	+335,26	−1076,8	+168,35	+261,59	−771,77
α_{mn}^{53}	+335,26	+5570,6	−1076,8	+261,59	+168,35	−407,89
α_{mn}^{55}	−1076,8	−1076,8	+12801	−232,21	−232,21	+535,25
α_{mn}^{17}	+168,35	+261,59	−232,21	+11385,5	+381,82	−254,49
α_{mn}^{71}	+261,59	+168,35	−232,21	+381,82	+11385,5	−362,26
α_{mn}^{37}	−771,77	−407,89	+535,25	−254,49	−362,23	+16586
α_{mn}^{73}	−407,89	−771,77	+535,25	−362,23	−254,49	+531,66
α_{mn}^{57}	+478,44	+975,40	−1924,5	−113,62	+296,13	−1971,0
α_{mn}^{75}	+975,40	+478,44	−1924,5	+296,13	−113,62	−627,43
α_{mn}^{77}	−513,77	−513,77	+813,98	+482,04	+482,04	+1728,1
α_{mn}^{19}	−215,5	−355,82	+322,29	−131,75	−506,81	+221,90
α_{mn}^{91}	−355,82	−215,5	+322,29	−506,81	−131,75	+504,59
α_{mn}^{39}	+788,94	+474,28	−618,82	+175,99	+462,86	−1089,8
α_{mn}^{93}	+474,28	+788,94	−618,82	+462,86	+175,99	−650,63
α_{mn}^{59}	−505,22	−815,28	+1926,2	−201,56	−360,04	+673,98
α_{mn}^{95}	−815,28	−505,22	+1926,2	−360,04	−201,56	+707,95
α_{mn}^{79}	+532,29	+537,45	−869,73	+227,14	−850,46	−729,89
α_{mn}^{97}	+537,45	+532,29	−869,73	−850,46	+227,14	−1369,6
α_{mn}^{99}	−548,34	−548,34	+916,12	−252,73	−252,73	+769,08

Table 3 (Continued) /726

α-values (odd)

α_{mn}^{pq}	α_{73}^{pq}	α_{57}^{pq}	α_{75}^{pq}	α_{77}^{pq}	α_{19}^{pq}	α_{91}^{pq}
α_{mn}^{11}	+15,36	−15,36	−15,36	+15,36	+13,302	+13,302
α_{mn}^{13}	+11,479	+71,457	+79,013	−73,658	−59,269	−171,68
α_{mn}^{31}	−69,258	+79,013	+71,457	−73,658	−171,68	−59,269
α_{mn}^{33}	+419,5	−294,33	−294,33	+304,43	+207,17	+207,17
α_{mn}^{15}	+223,23	−137,11	+177,13	+151,8	+100,34	+339,18
α_{mn}^{51}	+122,44	+177,13	−137,11	+151,8	+339,18	+100,34
α_{mn}^{35}	−407,89	+478,44	+975,41	−513,77	−215,52	−355,82
α_{mn}^{53}	−771,77	+975,41	+478,44	−513,77	−355,82	−215,52
α_{mn}^{55}	+535,25	−1924,5	−1924,5	+813,98	+322,29	+322,29
α_{mn}^{17}	−362,26	−113,62	+296,13	+482,04	−131,75	−506,81
α_{mn}^{71}	−254,49	+296,13	−113,62	+482,04	−506,81	−131,75
α_{mn}^{37}	+531,66	−1971,0	−627,43	+1728,2	+221,90	+504,59
α_{mn}^{73}	+16586	−627,43	−1971,0	+1728,2	+504,59	+221,90
α_{mn}^{57}	−627,43	+28555	+870,83	−3555	−296,66	−429,15
α_{mn}^{75}	−1971,0	+870,83	+28555	−3555	−429,15	−296,66
α_{mn}^{77}	+1728,1	−3555,2	−3555,2	+51388	+371,48	+371,48
α_{mn}^{19}	+504,59	−296,66	−429,15	+371,48	+32618	+674,44
α_{mn}^{91}	+221,90	−429,15	−296,66	+371,48	+674,44	+32618
α_{mn}^{39}	−650,63	+730,78	+759,60	−843,54	−365,54	−653,36
α_{mn}^{93}	−1089,8	+759,60	+730,78	−843,54	−653,36	−365,54
α_{mn}^{59}	+707,95	−2729,9	−983,43	+1198,09	−280,04	+536,01
α_{mn}^{95}	+673,98	−983,43	−2729,9	+1198,09	+536,01	−280,04
α_{mn}^{79}	−1369,6	+1141,3	+3515,5	−5055,9	+926,17	−446,30
α_{mn}^{97}	−729,89	+3515,5	+1141,3	−5055,9	−446,30	+926,17
α_{mn}^{99}	+769,08	−1226,5	−1226,5	+1596,6	−1572,3	−1572,3

values [8b]:

.0

Table 3 (Continued)

α-values (odd)

α_{mn}^{pq}	α_{39}^{pq}	α_{93}^{pq}	α_{59}^{pq}	α_{95}^{pq}	α_{79}^{pq}	α_{97}^{pq}	α_{99}^{pq}
α_{mn}^{11}	−15,36	−15,36	+15,36	+15,36	−15,36	−15,36	+15,36
α_{mn}^{13}	+69,48	−56,58	−70,588	−84,19	+71,70	+75,86	−72.813
α_{mn}^{31}	−56,58	+6948	−84,19	−70,588	+75,86	+71,70	−72,813
α_{mn}^{33}	−379,46	−379,46	+303,91	+303,91	−309,75	−309,75	+312,71
α_{mn}^{15}	−123,69	−277,6	+132,08	−345,18	−140,48	−166,48	+148,87
α_{mn}^{51}	−277,6	−123,69	−345,18	+132,08	−166,48	−140,48	+148.87
α_{mn}^{35}	+788,94	+474,28	−505,22	−815,28	+532,29	+537,45	−548,34
α_{mn}^{53}	+474,28	+788,94	−815,28	−505,22	+537,45	+532,29	−548,34
α_{mn}^{55}	−618,82	−618,82	+1926,2	+1926,2	−869,73	−869,73	+916,12
α_{mn}^{17}	+175,99	+462,86	−201,56	−360,04	+227,14	−850,46	−252,72
α_{mn}^{71}	+462,86	+175,99	−360,04	−201,56	−850,46	+227,14	−252,72
α_{mn}^{37}	−1089,8	−650,63	+673,98	+707,95	−729,89	−1369,6	+769,08
α_{mn}^{73}	−650,63	−1089,8	+707,95	+673,98	−1369,6	−729,89	+769.08
α_{mn}^{57}	+730,78	+759,60	−2729,9	−983,43	+1141,2	+3515,5	−1226.5
α_{mn}^{75}	+759,60	+730,78	−983,43	−2729,9	+3515,5	+1141,2	−1226,5
α_{mn}^{77}	−843,54	−843,54	+1198,1	+1198,1	−5055,9	−5055,9	+1596.6
α_{mn}^{19}	−365,54	−653,36	−280,04	+536,01	+926,17	−446,30	−1572.3
α_{mn}^{91}	−653,36	−365,54	+536,01	−280,04	−446,3	+926,17	−1572.3
α_{mn}^{39}	+40995	+824,08	−3120,84	−889,34	+2677,2	+939,59	−2041,9
α_{mn}^{93}	+824,08	+40995	−889,34	−3120,84	+939,59	+2677,2	−2041,9
α_{mn}^{59}	−3120,8	−889,34	+59259	+1143,5	−5662,8	−1340,2	+5555.9
α_{mn}^{95}	−889,34	−3120,8	+1143,5	+59259	−1340,2	−5662,8	+5555.9
α_{mn}^{79}	+2677,2	+939,59	−5662,8	−1340,2	+91506	+1653,5	−8066,8
α_{mn}^{97}	+939,59	+2677,2	−1340,2	−5662,8	+1653,5	+91506	−8066,8
α_{mn}^{99}	−2041,9	−2041,9	+5555,9	+5555,9	−8066,8	−8066,8	+144154

Table 4

α-values (even)

α_{mn}^{pq}	$mn = 02$	α_{20}^{pq}	α_{22}^{pq}	α_{04}^{pq}	α_{40}^{pq}	α_{24}^{pq}
$pq = 02$	+31,285	+7,775	+5,1455	0	−18,395	+0,7532
α_{mn}^{20}	+7,775	+31,285	+5,1455	−18,395	0	−12,174
α_{mn}^{22}	+5,1455	+5,1455	+265,24	+21,995	+21,995	−124,98
α_{mn}^{04}	0	−18,395	+21,995	+913,60	+43,523	+41,367
α_{mn}^{40}	−18,395	0	+21,995	+43,523	+913,60	−52,039
α_{mn}^{24}	+0,7532	−12,174	−124,98	+41,367	−52,039	+1739,3
α_{mn}^{42}	−12,174	+0,7532	−124,98	−52,039	+41,367	+162,42
α_{mn}^{44}	−1,7820	−1,782	+104,38	−97,872	−97,872	−407,83
α_{mn}^{06}	0	+28,907	−42,505	0	−68,394	+25,148
α_{mn}^{60}	+28,907	0	−42,505	−68,394	0	+100,57
α_{mn}^{26}	−0,2387	+19,131	+116,62	+4,1242	+81,777	−289,80
α_{mn}^{62}	+19,131	−0,2387	+116,62	+81,777	+4,1242	−223,87
α_{mn}^{46}	+0,5648	+2,8003	−108,40	−9,7578	+153,80	+235,15
α_{mn}^{64}	+2,8003	+0,5648	−108,40	+153,80	−9,7578	+319,85
α_{mn}^{66}	−0,8875	−0,8875	+109,82	+15,334	+15,334	−252,06
α_{mn}^{08}	0	−39,419	+63,393	0	+93,265	−44,141
α_{mn}^{80}	−39,419	0	+63,393	+93,265	0	−149,99
α_{mn}^{28}	+0,10296	−26,088	−104,17	−2,0935	−111,51	+297,79
α_{mn}^{82}	−26,088	+0,10296	−104,17	−111,51	−2,0935	+283,87
α_{mn}^{48}	−0,2436	−3,8186	+110,96	+4,9533	−209,73	−244,99
α_{mn}^{84}	−3,8186	−0,2436	+110,96	−209,73	+4,9533	−214,29
α_{mn}^{68}	+0,3828	+1,2102	−110,67	−7,7838	−20,910	+255,76
α_{mn}^{86}	+1,2102	+0,3828	−110,67	−20,910	−7,7838	+264,73
α_{mn}^{88}	−0,5220	−0,5220	+110,8	+10,614	+10,614	−262,77
$\alpha_{mn}^{0,10}$	0	+49,931	−84,366	0	−118,14	+64,256
$\alpha_{mn}^{10,0}$	+49,931	0	−84,366	−118,14	0	+199,61

Table 4 (Continued)

α-values (even)

α^{pq}_{42}	α^{pq}_{44}	α^{pq}_{06}	α^{pq}_{60}	α^{pq}_{26}	α^{pq}_{62}	α^{pq}_{46}
−12,174	−1,782	0	+28,908	−0,2387	+13,131	+0,5648
+0,7532	−1,782	+28,908	0	+13,131	−0,2387	+2,8003
−124,98	+104,38	−42,505	−42,505	+116,62	+116,62	−108,4
−52,039	−97,872	0	−68,394	+4,1242	+81,777	−9,7578
+41,367	−97,872	−68,394	0	+81,777	+4,1242	+153,80
+162,42	−407,83	+25,148	+100,57	−289,80	−223,87	+235,15
+1739,3	−407,83	+100,57	+25,148	−223,87	−289,80	+319,85
−407,83	+5060,9	−59,50	−59,50	+292,59	+292,59	−1011,9
+100,57	−59,50	+5570,9	+107,48	+110,42	−158,03	−261,25
+25,148	−59,50	+107,48	+5570,9	−158,03	+110,42	+93,502
−223,87	+292,59	+110,42	−158,03	+7340,5	+339,59	−815,30
−289,80	+292,59	−158,03	+110,42	+339,59	+7340,5	−366,36
+319,85	−1011,9	−261,25	+93,502	−815,3	−366,35	+13672,5
+235,15	−1011,9	+93,502	−261,25	−366,35	−815,3	+571,57
−252,06	+514,71	+410,54	+410,54	+570,96	+570,96	−2097,6
−149,99	+104,44	0	−146,56	+29,577	+235,70	−69,979
−44,141	+104,44	−146,56	0	+235,70	+29,577	−164,12
+283,87	−359,86	+8,554	+215,50	−440,97	−454,73	+417,83
+297,79	−359,86	+215,50	+8,554	−454,73	−440,97	+484,34
−214,29	+994,21	−20,239	−127,50	+360,39	+435,87	−1557,8
−244,99	+994,21	−127,50	−20,239	+435,87	+360,39	−662,25
+264,73	−548,56	+31,804	−559,82	−392,60	−286,67	+779,28
+255,76	−548,56	−559,82	+31,804	−286,67	−392,60	+2012,9
−262,77	+574,36	−43,369	−43,369	+418,12	+418,12	−838,19
+199,61	−152,03	0	+185,64	−47,370	−313,68	+112,08
+64,256	−152,03	+185,65	0	−313,68	−47,370	+238,90

Table 4 (Continued)

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α-values (even)

α_{mn}^{pq}	α_{64}^{pq}	α_{66}^{pq}	α_{08}^{pq}	α_{80}^{pq}	α_{28}^{pq}	α_{82}^{pq}
$pq = 02$	+2,8003	−0,8875	0	−39,419	+0,10296	−26,088
α_{mn}^{20}	+0,5648	−0,8875	−39,419	0	−26,088	+0,10296
α_{mn}^{22}	−108,4	+109,82	+63,393	+63,393	−104,17	−104,17
α_{mn}^{04}	+153,80	+15,334	0	+93,265	−2,0935	−111,51
α_{mn}^{40}	−9,7578	+15,334	+93,265	0	−111,51	−2,0935
α_{mn}^{24}	+319,85	−252,06	−44,141	−149,98	+297,79	+283,87
α_{mn}^{42}	+235,15	−252,06	−149,98	−44,141	+283,87	+297,79
α_{mn}^{44}	−1011,9	+514,71	+104,44	+104,44	−359,86	−359,86
α_{mn}^{06}	+93,502	+410,54	0	−146,56	+8,554	+215,50
α_{mn}^{60}	−261,25	+410,54	−146,56	0	+215,50	+8,554
α_{mn}^{26}	−366,36	+570,96	+29,577	+235,70	−440,97	−454,73
α_{mn}^{62}	−815,30	+570,96	+235,70	+29,577	−454,73	−440,97
α_{mn}^{46}	+571,57	−2097,6	−69,979	−164,12	+417,83	+484,34
α_{mn}^{64}	+13672,5	−2097,6	−164,12	−69,979	+484,34	+417,83
α_{mn}^{66}	−2097,6	+27221	+109,97	+109,97	−506,89	−506,89
α_{mn}^{08}	−164,12	+109,97	+19263	+199,86	+212,49	−321,40
α_{mn}^{80}	−69,979	+109,97	+199,86	+19263	−321,40	+212,49
α_{mn}^{28}	+484,34	−506,89	+212,49	−321,40	+22332,5	+625,99
α_{mn}^{82}	+417,83	−506,89	−321,40	+212,49	+625,99	+22332,5
α_{mn}^{48}	−662,25	+836,13	−502,75	+223,80	−1349,6	−605,04
α_{mn}^{84}	−1557,8	+836,13	+223,80	−502,75	−605,04	−1349,6
α_{mn}^{68}	+2012,9	−3248,3	+790,04	−149,96	+871,7	+589,25
α_{mn}^{86}	+779,28	−3248,3	−149,96	+790,04	+589,25	+871,7
α_{mn}^{88}	−838,19	+1174,8	−1077,3	−1077,3	−322,64	−322,64
$\alpha_{mn}^{0,10}$	+238,90	−176,12	0	−253,15	+34,38	+427,74
$\alpha_{mn}^{10,0}$	+112,08	−176,12	−253,15	0	+427,74	+34,38

Table 4 (Continued) /731

α-values (even)

α_{48}^{pq}	α_{84}^{pq}	α_{68}^{pq}	α_{86}^{pq}	α_{88}^{pq}	$\alpha_{0,10}^{pq}$	$\alpha_{10,0}^{pq}$
−0,2436	−3,8186	+0,3828	+1,2102	−0,522	0	+49,931
−3,8186	−0,2436	+1,2102	+0,3828	−0,522	+49,931	0
+110,96	+110,96	−110,67	−110,67	+110,80	−84,366	−84,366
+4,9533	−209,73	−7,7838	−20,910	+10,614	0	−118,14
−209,73	+4,9533	−20,910	−7,7838	+10,614	−118,14	0
−244,99	−214,29	+255,76	+264,73	−262,77	+64,256	+199,61
−214,29	−244,99	+264,73	+255,76	−262,77	+199,61	+64,256
+994,21	+994,21	−548,56	−548,56	+574,36	−152,03	−152,03
−20,239	−127,5	+31,804	−559,82	−43,369	0	+185,64
−127,5	−20,239	−559,82	+31,804	−43,369	+185,64	0
+360,39	+435,87	−392,60	−286,67	+418,12	−313,68	−47,370
+435,87	+360,39	−286,67	−392,60	+418,12	−47,370	−313,68
−1557,8	−662,25	+779,28	+2012,9	−838,19	+112,08	+238,90
−662,25	−1557,8	+2012,9	+779,28	−838,19	+238,90	+112,08
+836,13	+836,13	−3248,3	−3248,3	+1174,8	−176,12	−176,12
−502,75	+223,80	+790,04	−149,96	−1077,3	0	−253,15
+223,80	−502,75	−149,96	+790,04	−1077,3	−253,15	0
−1349,6	−605,04	+871,7	+589,25	−322,66	+34,38	+427,74
−605,04	−1349,6	+589,25	+871,7	−322,66	+427,74	+34,38
+32850	+801,75	−3552,2	−951,88	+3359,2	−81,341	−325,78
+801,75	+32850	−951,88	−3552,2	+3359,2	−325,78	−81,341
−3552,2	−951,88	+53282	+1231,6	−5520,7	+127,82	+240,16
−951,88	−3552,2	+1231,6	+53282	−5520,7	+240,16	+127,82
+3359,2	+3359,2	−5520,7	−5520,7	+88952,7	−174,31	−174,31
−81,341	−325,78	+127,82	+240,16	−174,31	+49586	+320,66
−325,78	−81,341	+240,16	+127,82	−174,31	+320,66	+49586

Table 5

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Derivations α' (odd)

α'^{pq}_{mn}	α'^{pq}_{11}	$mn = 13$	$mn = 31$	$mn = 33$	$mn = 15$	$mn = 51$
α^{11}_{mn}	−18	+20,8	+20,8	−24	−20,8	−20,8
$pq = 13$	+20,8	−163,3	+79,0	+305,6	+86,37	−189,3
$pq = 31$	+20,8	+79,0	−163,3	+305,6	−189,3	+86,37
$pq = 33$	−24	+305,6	+305,6	−1219,3	+41,0	+41,0
$pq = 15$	−20,8	+86,37	−189,3	+41,0	−427,3	+417,7
$pq = 51$	−20,8	−189,3	+86,37	+41,0	+417,7	−427,3
$pq = 35$	+24	−86,3	−430,8	+959,3	+928,3	−192
$pq = 53$	+24	−430,8	−86,3	+959,3	−192	+928,3
$pq = 55$	−24	+71,97	+71,97	+378,0	−1394,5	−1394,5
$pq = 17$	+20,8	−90,93	+300	+156,0	+147,2	−646,7
$pq = 71$	+20,8	+300	−90,93	+156,0	−646,7	+147,2
$pq = 37$	−24	+99,33	+556,2	−1127	−132	+343
$pq = 73$	−24	+556,2	+99,33	−1127	+343	−132
$pq = 57$	+24	−93,2	−57,57	+394,8	+91,0	+1862
$pq = 75$	+24	−57,57	−93,2	+394,8	+1862	+91,0
$pq = 77$	−24	+87,08	+87,08	−430,5	−50,7	−50,7
$pq = 19$	−20,8	+92,6	−410,3	−278,7	−156,7	+875,3
$pq = 91$	−20,8	−410,3	+92,6	−278,7	+875,3	−156,7
$pq = 39$	+24	−104,2	−681,5	+1282	+159	−493,3
$pq = 93$	+24	−681,5	−104,2	+1282	−493,3	+159
$pq = 59$	−24	+101	+431,8	−390,3	−136	−2329
$pq = 95$	−24	+431,8	+101	−390,3	−2329	−136
$pq = 79$	+24	−97,83	−81,0	+439,3	+1128	+9,67
$pq = 97$	+24	−81,0	−97,83	+439,3	+9,67	+1128
$pq = 99$	−24	+94,7	+94,7	−455	−89,3	−89,3

Table 5 (Continued)

α'^{pq}_{mn}	$mn = 73$	$mn = 57$	$mn = 75$	$mn = 77$	$mn = 19$	$mn = 91$
α^{11}_{mn}	−24	+24	+24	−24	−20,8	−20,8
$pq = 13$	+556,2	−93,2	−57,57	+87,08	+92,6	−410,3
$pq = 31$	+99,33	−57,57	−93,2	+87,08	−410,3	+92,6
$pq = 33$	−1127	+394,8	+394,8	−430,5	−278,7	−278,7
$pq = 15$	+343	+91,0	+1862	−50,7	−156,7	+875,3
$pq = 51$	−132	+1862	+91,0	−50,7	+875,3	−156,7
$pq = 35$	+79,3	−632,7	−3284	+638,1	−95,7	−691,3
$pq = 53$	+1568	−3284	−632,7	+638,1	−691,3	−95,7
$pq = 55$	−475	+4353	+4353	−1081	+392	+392
$pq = 17$	−713,3	−2911	+350,7	+3935	+205,7	−1341
$pq = 71$	+1888,7	+350,7	−2911	+3935	−1341	+205,7
$pq = 37$	+201	+5314,7	+322,3	−6560	−43,7	+1104,7
$pq = 73$	−5177	+322,3	+5314,7	−6560	+1104,7	−43,7
$pq = 57$	+322,3	−11110	−923,2	+8500	−163,8	−688,9
$pq = 75$	+5314,7	−923,2	−11110	+8500	−688,9	−163,8
$pq = 77$	−6560	+8500	+8500	−18697	+371,7	+371,7
$pq = 19$	+1104,7	−163,8	−688,9	+371,7	−1300	+1807,3
$pq = 91$	−43,7	−688,9	−163,8	+371,7	+1807,3	−1300
$pq = 39$	−508	−714	−90,7	+545,3	+3187	−1518,3
$pq = 93$	+2167	−90,7	−714	+545,3	−1518,3	+3187
$pq = 59$	−136,3	+5983	+829,3	−1347	−4980	+985,7
$pq = 95$	−872	+829,3	+5983	−1347	+958,7	−4980
$pq = 79$	+7762	−1503	−9817	+11660	+6775	−579,3
$pq = 97$	+861,5	−9817	−1503	+11660	−579,3	+6775
$pq = 99$	−805,3	+1575	+1575	−2103	−8570	−8570

Table 5 (Continued)

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Derivations α' (odd)

$mn = 35$	$mn = 53$	$mn = 55$	$mn = 17$	$mn = 71$	$mn = 37$
+24	+24	−24	+20,8	+20,8	−24
−86,3	−430,8	+71,97	−90,93	+300	+99,33
−430,8	−86,3	+71,97	+300	−90,93	+556,2
+959,3	+959,3	+378,0	+156,0	+156,0	−1127
+928,3	−192	−1394,5	+147,2	−646,7	−132
−192	+928,3	−1394,5	−646,7	+147,2	+343
−2890	−220,3	+2702	−4,30	+435	+1568
−220,3	−2890	+2702	+435	−4,30	+79,3
+2702	+2702	−6440	−173	−173	−475
−4,30	+435	−173	−808,3	+993,7	+1888,7
+435	−4,30	−173	+993,7	−808,3	−713,3
+1568	+79,3	−475	+1888,7	−713,3	−5177
+79,3	+1568	−475	−713,3	+1888,7	+201
−632,7	−3284	+4353	−2911	+350,7	+5314,7
−3284	−632,7	+4353	+350,7	−2911	+322,3
+638,1	+638,1	−1081	+3935	+3935	−6560
−95,7	−691,3	+392	+205,7	−1341	−43,7
−691,3	−95,7	+392	−1341	+205,7	+1104,7
−1763	+82,7	+371,3	−171,3	+994	+2167
+82,7	−1763	+371,3	+994	−171,3	−508
+687	+3835	−4980	+100,7	−528,3	−872
+3835	+687	−4980	−528,3	+100,7	−136,3
−722	−610,2	+1145	−29,3	−4958	+861,5
−610,2	−722	+1145	−4958	−29,3	+7762
+718	+718	−1249,7	−42,0	−42,0	−805,3

Table 5 (continued)

$mn = 39$	$mn = 93$	$mn = 59$	$mn = 95$	$mn = 79$	$mn = 97$	$mn = 99$
+24	+24	−24	−24	+24	+24	−24
−104,2	−681,5	+101	+431,8	−97,83	−81,0	+94,7
−681,5	−104,2	+431,8	+101	−81,0	−97,83	+94,7
+1282	+1282	−390,3	−390,3	+439,3	+439,3	−455
+159	−493,3	−136	−2329	+1128	+9,67	−89,3
−493,3	+159	−2329	−136	+9,67	+1128	−89,3
−1763	+82,7	+687	+3835	−722	−610,2	+718
+82,7	−1763	+3835	+687	−610,2	−722	+718
+371,3	+371,3	−4980	−4980	+1145	+1145	−1249,7
−171,3	+994	+100,7	−528,3	−29,3	−4958	−42,0
+994	−171,3	−528,3	+100,7	−4958	−29,3	−42,0
+2167	−508	−872	−136,3	+861,5	+7762	−805,3
−508	+2167	−136,3	−872	+7762	+861,5	−805,3
−714	−90,7	+5983	+829,3	−1503	−9817	+1575
−90,7	−714	+829,3	+5983	−9817	−1503	+1575
+545,3	+545,3	−1347	−1347	+11660	+11660	−2103
+3187	−1518,3	−4980	+985,7	+6775	−579,3	−8570
−1518,3	+3187	+985,7	−4980	−579,3	+6775	−8570
−8100	+965,7	+8795,7	−329,7	−10950	−332	+13050
+965,7	−8100	−329,7	+8795,7	−332	−10950	+13050
+8795,7	−329,7	−16937	−617,7	+14020	+1260	−16267
−329,7	+8795,7	−617,7	−16937	+1260	+14020	−16267
−10950	−332	+14020	+1260	−27900	−1946,7	+19211
−332	−10950	+1260	+14020	−1946,7	−27900	+19211
+13050	+13050	−16267	−16267	+19211	+19211	−40966,7

Table 6

α'-values (even)

α'^{pq}_{mn}	$mn = 02$	$mn = 20$	$mn = 22$	$mn = 04$	$mn = 40$	$mn = 24$
$pq=02$	0	+21,6	+14,295	0	−51,11	+2,092
$pq=20$	+21,6	0	+14,295	−51,11	0	−33,82
$pq=22$	+14.295	+14,295	−287	+61,0	+61,0	+260,7
$pq=04$	0	−51,11	+61,0	0	+120,9	+114,92
$pq=40$	−51,11	0	+61,0	+120,9	0	−144,57
$pq=24$	+2,092	−33,82	+260,7	+114,92	−144,57	−1002,3
$pq=42$	−33.82	+2,092	+260,7	−144,57	+114,92	+16,3
$pq=44$	−4,951	−4,951	−144,3	−271,9	−271,9	+1162,7
$pq=06$	0	+80,29	−118,1	0	−190,0	+69,87
$pq=60$	+80.29	0	−118,1	−190,0	0	+279,17
$pq=26$	−0,663	+253.15	−305	+11,457	+227,18	+537
$pq=62$	+253,15	−0,663	−305	+227,18	+11,457	−172,3
$pq=46$	+1,568	+7,783	+148,7	−27,10	+427,3	−306,17
$pq=64$	+7,783	+1,568	+148,7	+427,3	−27,10	−1486,7
$pq=66$	−2,466	−2,466	−160,17	+42,58	+42,58	+292,7
$pq=08$	0	−109,53	+176,08	0	+259,1	−122,62
$pq=80$	−109,53	0	+176,08	+259,1	0	−416,77
$pq=28$	+0.286	−72,47	+345,07	−5,817	−309,93	−598
$pq=82$	−72,47	+0,286	+345,07	−309,93	−5,817	+334,7
$pq=48$	−0,677	−10,612	−145,53	+13,757	−582,7	+305,3
$pq=84$	−10.612	−0,677	−145,53	−582,7	+13,757	+1801
$pq=68$	+1,063	+3,363	+162	−21,6	−58,07	−344
$pq=86$	+3,363	+1,063	+162	−58,07	−21,6	−266,3
$pq=88$	−1,45	−1,45	−165,7	+29,5	+29,5	+334
$pq=0,10$	0	+138,7	−234,4	0	−328,03	+178,5
$pq=10,0$	+138,7	0	−234,4	−328.03	0	+554,3

Table 6 (Continued)

α'^{pq}_{mn}	$mn = 64$	$mn = 66$	$mn = 08$	$mn = 80$	$mn = 28$	$mn = 82$
$pq=02$	+7,783	−2,466	0	−109,53	+0,286	−72,47
$pq=20$	+1,568	−2,466	−109,53	0	−72,47	+0,286
$pq=22$	+148,7	−160,17	+176,08	+176,08	+345,07	+345,07
$pq=04$	+427,3	+42,58	0	+259,1	−5,817	−309,93
$pq=40$	−27,10	+42,58	+259,1	0	−309,93	−5,817
$pq=24$	−1486,7	+292,7	−122,62	−416,77	−598	+334,7
$pq=42$	−306,17	+292,7	−416,77	−122,62	+334,7	−598
$pq=44$	+2257,3	−688,3	+290	+290	+19,3	+19,3
$pq=06$	+259,8	+1140,3	0	−407,3	+23,77	+598,7
$pq=60$	−725,7	+1140,3	−407,3	0	+598,7	+23,77
$pq=26$	−24,7	−3527,3	+821,7	+654,7	+798,7	−793,7
$pq=62$	+2676,7	−3527,3	+654,7	+821,7	−793,7	+798,7
$pq=46$	−530,7	+5083,3	−194,4	−455,7	−286,3	+291
$pq=64$	−6131,3	+5083,3	−455,7	−194,4	+291	−286,3
$pq=66$	+5083,3	−11513	+305,3	+305,3	+89,0	+89,0
$pq=08$	−455,7	+305,3	0	+555	+590,3	−893
$pq=80$	−194,4	+305,3	+555	0	−893	+590,3
$pq=28$	+291	+89,0	+590,3	−893	−3527	+1265
$pq=82$	−286,3	+89,0	−893	+590,3	+1265	−3527
$pq=48$	+410	−871,7	−1397	+621,7	+4807	−617
$pq=84$	+3313,3	−871,7	+621,7	−1397	−617	+4807
$pq=68$	−5997	+7427	+2194,7	−416,7	−6431,7	+126.7
$pq=86$	−1029,7	+7427	−416,7	+2194,7	+126,7	−6431,7
$pq=88$	+1064,7	−1553	−2993	−2993	+8037	+8037
$pq=0,10$	+664	−489,3	0	−703,3	+95,3	+1188
$pq=10,0$	+284,7	−489,3	−703,3	0	+1188	+95.3

Table 6 (Continued)

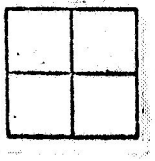
α'-values (even)

$mn = 42$	$mn = 44$	$mn = 06$	$mn = 60$	$mn = 26$	$mn = 62$	$mn = 46$
−33,82	−4,951	0	+80,29	−0,663	+253,15	+1,568
+2,092	−4,951	+80,29	0	+253,15	−0,663	+7,783
+260,7	−144,3	−118,1	−118,1	−305	−305	+148,7
−144,57	−271,9	0	−190,0	+11,457	+227,18	−27,10
+114,92	−271,9	−190,0	0	+227,18	+11,457	+427,3
+16,3	+1162,7	+69,87	+279,17	+537	−172,3	−306,17
−1002,3	+1162,7	+279,17	+69,87	−172,3	+537	−1486,7
+1162,7	−3142	−165,3	−165,3	−146,7	−146,7	+2257,3
+279,17	−165,3	0	+298,53	+306,7	−439	−725,7
+69,87	−165,3	+298,53	0	−439	+306,7	+259,8
−172,3	−146,7	+306,7	−439	−2081,7	+478	+2676,7
+537	−146,7	−439	+306,7	+478	−2081,7	−24,7
−1486,7	+2257,3	−725,7	+259,8	+2676,7	−24,7	−6131,3
−306,17	+2257,3	+259,8	−725,7	−24,7	+2676,7	−530,7
+292,7	−688,3	+1140,3	+1140,3	−3527,3	−3527,3	+5083,3
−416,77	+290	0	−407,3	+821,7	+654,7	−194,4
−122,62	+290	−407,3	0	+654,7	+821,7	−455,7
+334,7	+19,3	+23,77	+598,7	+798,7	−793,7	−286,3
−598	+19,3	+598,7	+23,77	−793,7	+798,7	+291
+1801	−2621	−56,2	−354,3	−445,7	+209,17	+3313,3
+305,3	−2621	−354,3	−56,2	+209,17	−445,7	+410
−266,3	+726,3	+88,3	−1555,5	+406,7	+4362	−1029,7
−344	+726,3	−1555,5	+88,3	+4362	+406,7	−5997
+334	−794,3	−120,47	−120,47	−349	−349	+1064,7
+554,3	−422,17	0	+315,83	−871,3	−131,6	+284,7
+178,5	−422,17	+315,83	0	−131,6	−871,3	+664

Table 6 (Continued)

$mn = 48$	$mn = 84$	$mn = 68$	$mn = 86$	$mn = 88$	$mn = 0,10$	$mn = 10,0$
−0,677	−10,612	+1,063	+3,363	−1,45	0	+138,7
−10,612	−0,677	+3,363	+1,063	−1,45	+138,7	0
−145,53	−145,53	+162	+162	−165,7	−234,4	−234,4
+13,757	−582,7	−21,6	−58,07	+29,5	0	−328,03
−582,7	+13,757	−58,07	−21,6	+29,5	−328,03	0
+305,3	+1801	−344	−266,3	+334	+178,5	+554,3
+1801	+305,3	−266,3	−344	+334	+554,3	+178,5
−2621	−2621	+726,3	+726,3	−794,3	−422,17	−422,17
−56,2	−354,3	+88,3	−1555,5	−120,47	0	+315,83
−354,3	−56,2	−1555,5	+88,3	−120,47	+315,83	0
−445,7	+209,17	+406,7	+4362	−349	−871,3	−131,6
+209,17	−445,7	+4362	+406,7	−349	−131,6	−871,3
+3313,3	+410	−1029,7	−5997	+1064,7	+284,7	+664
+410	+3313,3	−5997	−1029,7	+1064,7	+664	+284,7
−871,7	−871,7	+7427	+7427	−1553	−489,3	−489,3
−1397	+621,7	+2194,7	−416,7	−2993	0	−703,3
+621,7	−1397	−416,7	+2194,7	−2993	−703,3	0
+4807	−617	−6431,7	+126,7	+8037	+95,3	+1188
−617	+4807	+126,7	−6431,7	+8037	+1188	+95,3
−9983,3	−162,3	+9023	+748,3	−10735	−225,97	−905
−162,3	−9983,3	+748,3	+9023	−10735	−905	−225,97
+9023	+748,3	−18247	−1393	+13150	+355,3	+667,3
+748,3	+9023	−1393	−18247	+13150	+667,3	+355,3
−10735	−10735	+13150	+13150	−28380	−484	−484
−225,97	−905	+355,3	+667,3	−484	0	+890,7
−905	−225,97	+667,3	+355,3	−484	+890,7	0

Table 6a
I



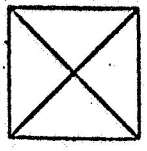
Frequency parameter	$\mu=0{,}225$	$\mu=0{,}343$	$\mu=0{,}360$	$\mu=0{,}390$
λ Calculated with 6 terms	12,49	10,73	10,47	—
λ Calculated with 15 terms	12,43 (n. Ritz) —	10,703	10,445	9,987

Standard function:

$$w = u_1 v_1 + A_{13}\cdot(u_1 v_3 + u_3 v_1) + A_{33}\cdot u_3\cdot v_3 + A_{15}\cdot(u_1 v_5 + u_5 v_1) + A_{35}(u_3 v_5 + u_5 v_3) + A_{55} u_5 v_5 + \ldots.$$

Coefficients of the progressive development

	$\mu=0{,}225$	$\mu=0{,}343$		$\mu=0{,}360$		$\mu=0{,}360$
A_{13}	+0,0378	+0,0328	+0,0325	+0,0320	+0,0318	+0,0306
A_{33}	−0,00435	−0,00541	−0,0050	−0,00555	−0,00514	−0,00537
A_{15}	−0,0034	−0,00265	−0,00257	−0,00255	−0,00246	−0,002285
A_{35}	+0,00118	+0,00139	+0,00121	+0,00141	+0,001235	+0,001276
A_{55}	−0,00045	−0,000474	−0,000365	−0,00048	−0,000366	−0,000370
A_{17}	—	—	+0,000413	—	+0,000382	+0,000328
A_{37}	—	—	−0,000431	—	−0,000440	−0,000456
A_{57}	—	—	+0,000148	—	+0,000149	+0,000150
A_{77}	—	—	−0,0000703	—	−0,0000701	−0,000070
A_{19}	—	—	−0,0000767	—	−0,0000638	−0,0000413
A_{39}	—	—	+0,000196	—	+0,000201	+0,0002086
A_{59}	—	—	−0,0000720	—	−0,0000727	−0,0000733
A_{79}	—	—	+0,0000382	—	+0,0000382	+0,00003805
A_{99}	—	—	−0,000023	—	−0,0000230	−0,0000228



Frequency parameter	$\mu=0{,}225$	$\mu=0{,}343$	$\mu=0{,}360$	$\mu=0{,}390$
λ Calculated with 4 terms	26,24	23,296	22,869	—
λ Calculated with 11 terms	26,40 (n. Ritz) —	23,115	22,669	21,872

Standard function:

$$w = (u_0 v_2 - u_2 v_0) + A_{04}(u_0 v_4 - u_4 v_0) + A_{24}(u_2 v_4 - u_4 v_2) + A_{06}\cdot(u_0 v_6 - u_6 v_0) + A_{26}(u_2 v_6 - u_6 v_2) + \ldots.$$

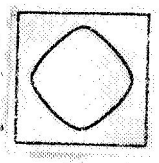
Coefficients of the progressive development

	$\mu=0{,}225$	$\mu=0{,}343$		$\mu=0{,}360$		$\mu=0{,}390$
A_{04}	−0,131	−0,0204	−0,02042	−0,02142	−0,02146	−0,023312
A_{24}	−0,0043	−0,00643	−0,006105	−0,00675	−0,00642	−0,006976
A_{06}	—	+0,00522	+0,00518	+0,00549	+0,00545	+0,005927
A_{26}	—	—	+0,00207	—	+0,00217	+0,00235
A_{46}	—	—	+0,000098	—	+0,0001006	+0,000105
A_{08}	—	—	−0,002042	—	−0,00215	−0,002337
A_{28}	—	—	−0,000929	—	−0,000975	−0,001054
A_{48}	—	—	−0,0000613	—	−0,0000631	−0,0000658
A_{68}	—	—	+0,0000080	—	+0,0000083	+0,0000087
A_{010}	—	—	+0,001008	—	+0,00106	+0,001154

Table 6a (Continued)

III

Frequency parameter	$\mu=0{,}225$	$\mu=0{,}343$	$\mu=0{,}360$	$\mu=0{,}390$
λ Calculated with 6 terms	35,90	37,93	38,22	—
λ Calculated with 15 terms	35,73 (n.Ritz) —	37,75	38,01	38,447



Standard function:

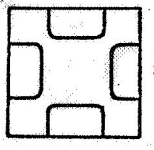
$$w = (u_0 v_2 + u_2 v_0) + A_{22} u_2 v_2 + A_{04} (u_0 v_4 + u_4 v_0) + A_{24} (u_2 v_4 + u_4 v_2) + A_{44} u_4 v_4 + \dots$$

Coefficients of the progressive development

	$\mu=0{,}225$	$\mu=0{,}343$		$\mu=0{,}360$		$\mu=0{,}390$
A_{22}	−0,0236	−0,0447	−0,0449	−0,0484	−0,0488	−0,0563
A_{04}	+0,00132	+0,02011	+0,0202	+0,02115	+0,0213	+0,02324
A_{24}	+0,0022	+0,00384	+0,00363	+0,00409	+0,00385	+0,00426
A_{44}	+0,00166	+0,00282	+0,00252	+0,00302	+0,00271	+0,00306
A_{06}	—	−0,00503	−0,00505	−0,00529	−0,00531	−0,00580
A_{26}	—	—	−0,00194	—	−0,00206	−0,00229
A_{08}	—	—	+0,00199	—	+0,00209	+0,00228
A_{46}	—	—	−0,000822	—	−0,000884	−0,000994
A_{28}	—	—	+0,000987	—	+0,00105	+0,001166
$A_{0\,10}$	—	—	−0,000976	—	−0,00103	−0,001121
A_{66}	—	—	+0,000293	—	+0,000316	+0,000353
A_{48}	—	—	+0,000355	—	+0,000382	+0,0004303
A_{68}	—	—	−0,000138	—	−0,000146	−0,000163
A_{88}	—	—	+0,000069	—	+0,000073	+0,000081

IV

Frequency parameter	$\mu=0{,}225$	$\mu=0{,}343$	$\mu=0{,}360$	$\mu=0{,}390$
Calculated with 6 terms	275,58	249,32	245,41	—
Calculated with 15 terms	266 (n.Ritz) —	245,51	241,78	235,08



Standard function:

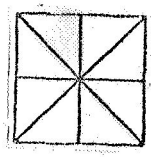
$$w = A_{02} (u_0 v_2 + u_2 v_0) + u_2 \cdot v_2 + A_{04} (u_0 v_4 + u_4 v_0) + A_{24} (u_2 v_4 + u_4 v_2) + A_{44} u_4 v_4 + \dots$$

Coefficients of the progressive development

	$\mu=0{,}225$	$\mu=0{,}343$		$\mu=0{,}360$		$\mu=0{,}390$
A_{02}	+0,0118	+0,02266	+0,0228	+0,0246	+0,0248	+0,02864
A_{04}	−0,020	−0,0288	−0,0275	−0,03005	−0,02875	−0,0310
A_{24}	+0,0876	+0,0730	+0,0690	+0,0709	+0,06704	+0,06350
A_{44}	−0,0047	−0,00951	−0,00674	−0,01015	−0,007355	−0,00841
A_{06}	—	+0,00529	+0,00540	+0,00556	+0,00568	+0,00619
A_{26}	—	—	−0,00971	—	−0,00921	−0,00830
A_{46}	—	—	+0,00314	—	+0,00330	+0,003574
A_{66}	—	—	−0,00148	—	−0,00151	−0,00158
A_{08}	—	—	−0,00211	—	−0,00222	−0,002425
A_{28}	—	—	+0,00204	—	+0,00183	+0,00147
A_{48}	—	—	−0,00153	—	−0,00160	−0,00172
A_{68}	—	—	+0,00076	—	+0,000778	+0,000808
A_{88}	—	—	−0,000435	—	−0,000441	−0,000452
$A_{0\,10}$	—	—	+0,001006	—	+0,00106	+0,00116

Table 6a (Continued)

V



Frequency parameter	$\mu=0{,}225$	$\mu=0{,}343$	$\mu=0{,}360$	$\mu=0{,}390$
λ Calculated with 3 terms	322,47	293,3	288,94	—
λ Calculated with 10 terms	316,1 (n.Ritz) —	291,95	287,34	279,06

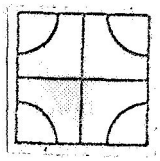
Standard function:

$$w = (u_1 v_3 - u_3 v_1) + A_{15}(u_1 v_5 - u_5 v_1) + A_{35}(u_3 v_5 - u_5 v_3) + A_{17}(u_1 v_7 - u_7 v_1) + \ldots$$

Coefficients of the progressive development

	$\mu=0{,}225$	$\mu=0{,}343$		$\mu=0{,}360$		$\mu=0{,}390$
A_{15}	+0,00024	−0,01394	−0,01423	−0,0160	−0,01634	−0,02008
A_{35}	+0,00216	−0,005895	−0,00511	−0,00707	−0,00623	−0,008235
A_{17}	—	—	+0,00643	—	+0,00709	+0,00826
A_{37}	—	—	+0,004255	—	+0,00470	+0,005495
A_{57}	—	—	+0,00031	—	+0,000315	+0,000322
A_{19}	—	—	−0,00338	—	−0,00367	−0,00419
A_{39}	—	—	−0,002713	—	−0,002934	−0,00333
A_{59}	—	—	−0,000232	—	−0,000236	−0,000241
A_{79}	—	—	+0,000415	—	+0,0000421	+0,000043

VI



Frequency parameter	$\mu=0{,}225$	$\mu=0{,}343$	$\mu=0{,}360$	$\mu=0{,}390$
λ Calculated with 6 terms	379,09	377,62	377,16	—
λ Calculated with 15 terms	378 (n.Ritz) —	374,23	373,54	372,12

Standard function:

$$w = A_{11}\cdot u_1\cdot v_1 + (u_1 v_3 + u_3 v_1) + A_{33}\cdot u_3\cdot v_3 + A_{15}\cdot(u_1 v_5 + u_5 v_1) + A_{35}\cdot(u_3 v_5 + u_5 v_3) + \ldots$$

Coefficients of the progressive development

	$\mu=0{,}225$	$\mu=0{,}343$		$\mu=0{,}360$		$\mu=0{,}390$
A_{11}	−0,0746	−0,06456	−0,0641	−0,0631	−0,0627	−0,06035
A_{33}	+0,171	+0,1295	+0,1252	+0,1227	+0,1184	+0,10562
A_{15}	+0,0431	+0,05066	+0,0489	+0,0518	+0,0500	+0,05194
A_{35}	−0,0084	−0,00480	−0,00347	−0,00419	−0,00285	−0,00172
A_{55}	+0,00546	+0,00814	+0,00645	+0,00856	+0,00684	+0,00755
A_{17}	—	—	−0,01286	—	−0,01321	−0,01384
A_{37}	—	—	−0,001936	—	−0,00229	−0,00295
A_{57}	—	—	−0,00290	—	−0,003052	−0,00334
A_{77}	—	—	+0,00139	—	+0,00146	+0,001575
A_{19}	—	—	+0,00515	—	+0,00531	+0,00560
A_{39}	—	—	+0,00184	—	+0,002046	+0,00242
A_{59}	—	—	+0,00150	—	+0,00158	+0,001724
A_{79}	—	—	−0,000766	—	−0,000798	−0,000858
A_{99}	—	—	+0,000448	—	+0,000466	+0,000497

Table 7. Table of the λ-functions and their derivations calculated numerically with 6 terms of the progressive development. /739

	Standard function	$\lambda(\mu_1 = 0{,}225)$	$\lambda(\mu_2 = 0{,}343)$	$\lambda(\mu_3 = 0{,}360)$	$\varepsilon = \frac{\partial \lambda}{\partial \mu}$ s between μ_2 and μ_3	s between μ_1 and μ_2
1	$u_1 v_1 + \ldots$	12,49	10,73	10,47	− 14,9	− 15,3
2	$(u_0 v_2 - u_2 v_0) + \ldots$	26,24	23,296	22,869	− 24,9	− 25,1
3	$(u_0 v_2 + u_2 v_0) + \ldots$	35,90	37,93	38,22	+ 17,2	+ 17,1
4	$u_2 v_2 + \ldots$	275,6	249,3	245,41	−223	−230
5	$u_1 v_3 - u_3 v_1 + \ldots$	322,5	293,3	288,94	−248	−256
6	$u_1 v_3 + u_3 v_1 + \ldots$	379,1	377,6	377,16	− 12,5	− 25,8

Table of the function and its derivation calculated numerically with 15 terms of the progressive development.

	Standard function	$\lambda(\mu_2 = 0{,}343)$	$\lambda(\mu_3 = 0{,}360)$	$\lambda(\mu_4 = 0{,}390)$	s between μ_2 and μ_3	s between μ_3 and μ_4
1	$u_1 v_1 + \ldots$	10,703	10,445	9,987		− 15,27
2	$(u_0 v_2 - u_2 v_0) + \ldots$	23,115	22,669	21,872		− 26,57
3	$(u_0 v_2 + u_2 v_0) + \ldots$	37,75	38,01	38,447		+ 14,57
4	$u_2 v_2 + \ldots$	245,51	241,78	235,08		−223,3
5	$u_1 v_3 - u_3 v_1 + \ldots$	291,95	287,34	279,06		−276
6	$u_1 v_3 + n_3 v_1 + \ldots$	374,23	373,54	372,12		−47,33

Since the graphic representation of the $\lambda(\mu)$-functions is an almost straight line one may without great error consider the slope ε of the curves between two adjacent μ-values as constant and by combination of two suitable tones determine a μ-value $\mu = \mu_0 + \delta\mu$ by linear interpolation. From all 6 tones for which the λ-curves have been calculated, an additional correction for μ may be applied by equalization according to the method of least squares.

The accuracy of the determination of the coefficient of elasticity, however, is very low. The influence of the number to be determined is so small that it has only a bearing in the last digits; these, however, cannot be determined accurately enough due to the insufficient convergence of the progressive development, so that for the calculation of the coefficient of elasticity only an accuracy of approximately 10% can be obtained.

3. Experiments /740

The experiments were conducted with two square-shaped metal plates which are described more closely in the following:

1. Aluminum Plate

Length of the square edge . . . . l = 198.4 mm (± 0.03 mm)
Thickness of the plate. . . . . $2D$ = 1.97 mm (± 0.01 mm)
Density . . . . . ϱ = 2.71

2. Brass Plate

Length of square edge . . . . . l = 240.0 mm (± 0.05 mm)
Thickness of plate. . . . . $2D$ = 3.89 mm (± 0.045 mm)
Density. . . . . ϱ = 8.48

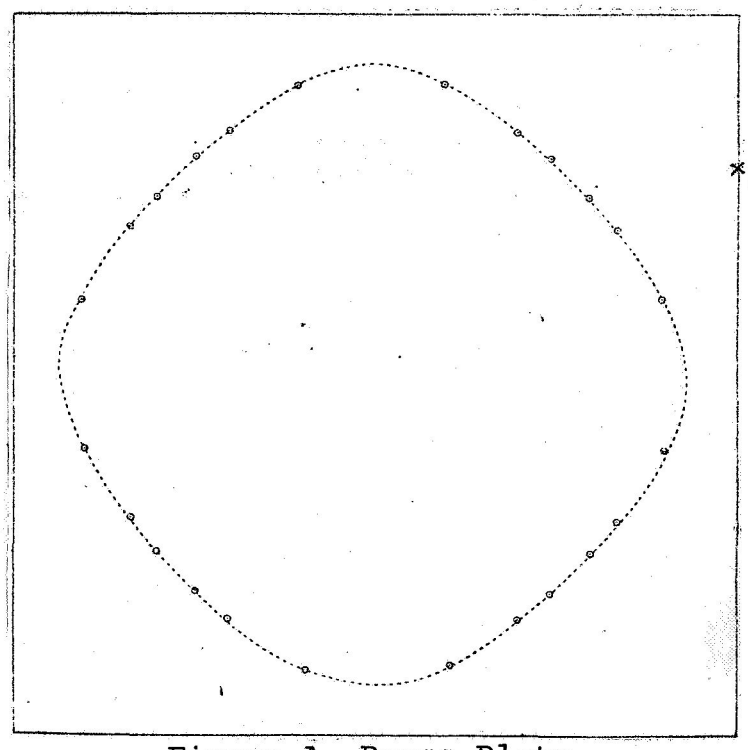


Figure A. Brass Plate

Standard function:

$$
\begin{aligned}
w = {} & u_0 v_2 + u_2 v_0 - 0{,}0563\, u_2 v_2 + 0{,}0232\,(u_0 v_4 + u_4 v_0) \\
& + 0{,}00426\,(u_2 v_4 + u_4 v_2) + 0{,}00306\, u_4 v_4 - 0{,}0058\,(u_0 v_6 + u_6 v_0) \\
& - 0{,}00229\,(u_2 v_6 + u_6 v_2) - 0{,}00099\,(u_4 v_6 + u_6 v_4) + 0{,}000353\, u_6 v_6 \\
& + 0{,}00228\,(u_0 v_8 + u_8 v_0) + 0{,}00117\,(u_2 v_8 + u_8 v_2) \\
& + 0{,}00043\,(u_4 v_8 + u_8 v_4) - 0{,}000163\,(u_6 v_8 + u_8 v_6) + \ldots
\end{aligned}
$$

Experiments were already conducted in 1924 at a time when the procurement of new experimental equipment was associated with great difficulties. /741

In order to produce acoustic figures, the plates were placed on conical corks at the point of nodal lines, being prevented from lateral displacement by conical corks which touched the edge at the intersections with nodal lines. The plates were stroked with the cello bow. The sand figures

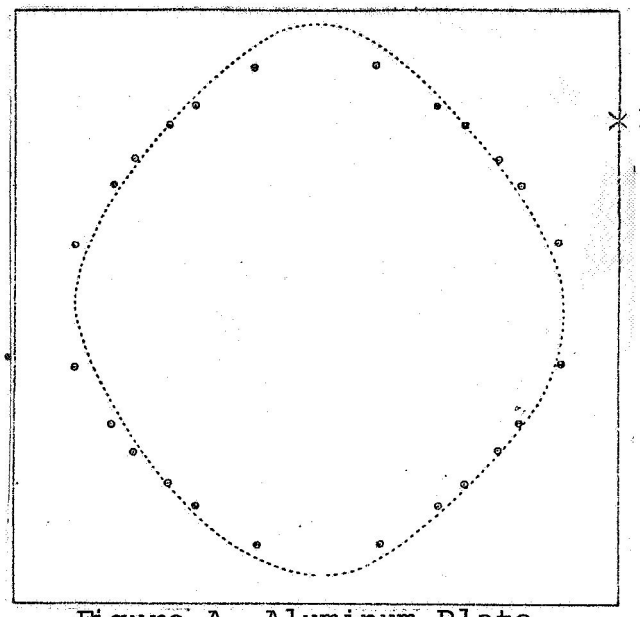


Figure A. Aluminum Plate

Standard function:

$$
\begin{aligned}
w = {} & u_0 v_2 + u_2 v_0 - 0{,}0488\, u_2 \cdot v_2 + 0{,}0213\,(u_0 v_4 + u_4 v_0) \\
& + 0{,}0039\,(u_2 v_4 + u_4 v_2) + 0{,}00271\, u_4 \cdot v_4 - 0{,}00531\,(u_0 v_6 + u_6 v_0) \\
& - 0{,}00206\,(u_2 v_6 + u_6 v_2) + 0{,}00209\,(u_0 v_8 + u_8 v_0) \\
& - 0{,}00088\,(u_4 v_6 + u_6 v_4) + 0{,}00105\,(u_2 v_8 + u_8 v_2) \\
& - 0{,}00103\,(u_0 v_{10} + u_{10} v_0) + 0{,}000316\, u_6 \cdot v_6 + 0{,}00038\,(u_4 v_8 + u_8 v_4) \\
& + \ldots
\end{aligned}
$$

obtained were transferred to rubberized black glossy paper and copied onto transparent paper with the help of carbon paper. The paper was then placed upon the theoretically calculated curves of nodal lines for the purpose of comparison (cf. the earlier Figs. A and B).

The frequencies of oscillation, for the better part, were determined with the help of a monochord and taken as mean value over several observations (accuracy 1%). /742

For the determination of the coefficient of elasticity μ, the accuracy of the frequencies of oscillation determined with the monochord was not sufficient. The 6 tones required for the calculation of each plate (No. I - VI of the tables on p. 743 which are identified in the final tables on pp. 744 and 746 with an *) were therefore determined with an accuracy of approximately 1 per mille with calibrated sound rods by the beat method[1].

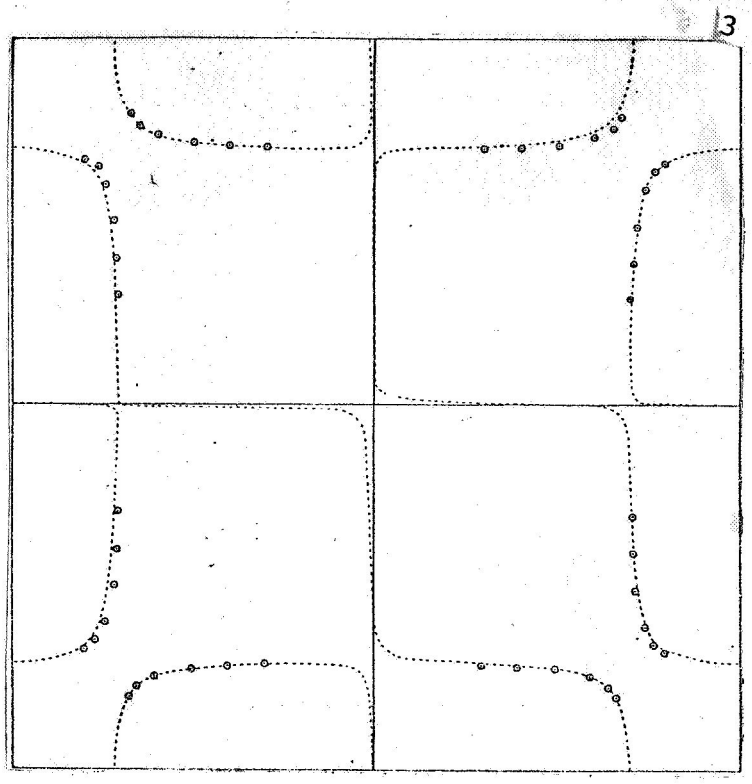


Figure B. Brass Plate

Standard function:

$$
\begin{aligned}
w = {} & + 0{,}008\, u_1 v_1 - 0{,}0493\,(u_1 v_3 + u_3 v_1) + u_3 v_3 - 0{,}0564\,(u_1 v_5 + u_5 v_1) \\
& + 0{,}0809\,(u_3 v_5 + u_5 v_3) - 0{,}0068\, u_5 v_5 + 0{,}0102\,(u_1 v_7 + u_7 v_1) \\
& - 0{,}016\,(u_3 v_7 + u_7 v_3) + 0{,}0045\,(u_5 v_7 + u_7 v_5) - 0{,}0027\, u_7 v_7 \\
& - 0{,}004\,(u_1 v_9 + u_9 v_1)
\end{aligned}
$$

The pitches of tone so determined and the data required for the interpolatory determination of μ are compiled in the following table. /743

[1] In the Technische Hochschule in Danzig where Prof. Kalähne was kind enough to make available his sonometer for the determination of the frequencies of oscillation.

Table 8

	λ ($\mu = 0{,}360$)	λ ($\mu = 0{,}390$)	ε	n brass	n aluminum
1	10,445	9,987	− 15,27	146,93	175,5
2	22,669	21,872	− 26,57	214,48 ± 0,05	235,8
3	38,01	38,447	+ 14,57	296,85 ± 0,01	315,13
4	241,78	235,08	−223,3	—	773,93 ± 0,04
5	287,34	279,06	−276	780,88 ± 0,6	841,44
6	373,54	372,12	− 47,33	906,7 ± 1	961,66 ± 0,5

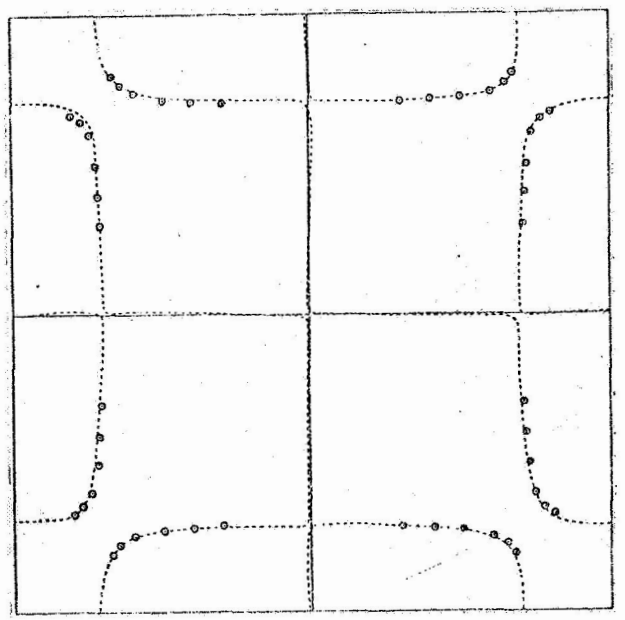


Figure B. Aluminum Plate

Standard function:

$$
\begin{aligned}
w = {} & + 0{,}008\, u_1 v_1 - 0{,}0493\,(u_1 v_3 + u_3 v_1) + u_3 v_3 - 0{,}0564\,(u_1 v_5 + u_5 v_1) \\
& + 0{,}0809\,(u_3 v_5 + u_5 v_3) - 0{,}0068\, u_5 v_5 + 0{,}0102\,(u_1 v_7 + u_7 v_1) \\
& - 0{,}016\,(u_3 v_7 + u_7 v_3) + 0{,}0045\,(u_5 v_7 + u_7 v_5) - 0{,}0027\, u_7 v_7 \\
& - 0{,}004\,(u_1 v_9 + u_9 v_1)
\end{aligned}
$$

Table 9. The frequency parameter λ and the frequencies of oscillation n of the aluminum plate. /744

Note to tables 9 and 10: for the purpose of identifying the tones only the main terms of the progressive development are given. A +- sign indicates that for the given tones the progressive development had been carried on further.

Frequencies of oscillation observed with the tonometer are identified by an *.

	Standard function	λ (μ=0,365)	λ (μ=0,390)	λ (μ=0,410)	(calculated) μ=0,365 $E=7147\frac{kg}{mm^2}$	μ=0,390 $E=7118\frac{kg}{mm^2}$	μ=0,410 $E=7090\frac{kg}{mm^2}$	obs.
1 (I)	$u_1 v_1$ + ...	10,369	9,987	9,682	160,5	158,9	157,7	157,5*
2 (II)	$u_0 v_2 - u_2 v_0$ + ...	22,536	21,872	21,341	236,6	235,24	234,1	235,8*
3 (III)	$u_0 v_2 + u_2 v_0$ + ...	38,083	38,447	38,738	307,6	311,9	315,4	315,1*
4	$u_1 v_2$ + ...	71,66	69,806	68,	421,9	420,2	418,9	420
5	$u_0 v_3 - u_3 v_0$	237,7	237,7	237,72	768,5	775,5	781,3	750
6 (IV)	$u_2 v_2$ + ...	240,66	235,08	230,61	773,8	771,2	769,6	773,9*
7 (V)	$u_1 v_3 - u_3 v_1$ + ...	285,96	279,06	273,54	842,9	840,3	838,13	841,4*
8 (VI)	$u_1 v_3 + u_3 v_1$ + ...	373,3	372,12	371,17	963,1	970,3	976,3	961,7*
9	$u_2 v_3$	722,7	707,6	695,53	1340	1338	1336	1260
10	$u_0 v_4 - u_4 v_0$	869,5	866,45	864,01	1470	1481	1489	1425
11	$u_0 v_4 + u_4 v_0$	957,7	960,75	963,2	1543	1559	1573	1490
12	$u_1 v_4$	1092	1084,6	1079	1647	1656	1664	1600
13	$u_3 v_3$ + ...	1418,7	1388	1364	1877	1874	1871	1820
14	$u_2 v_4 - u_4 v_2$	1572	1546	1526	1976	1978	1979	1935
15	$u_2 v_4 + u_4 v_2$	1897	1872	1852	2171	2170	2181	2080
16	$u_0 v_5$	2496	2496	2496	2490	2513	2532	2370
17	$u_1 v_5 - u_5 v_1$	2591	2570	2553	2537	2550	2560	2500
18	$u_1 v_5 + u_5 v_1$	2944	2944	2944	2705	2729	2749	2510
19	$u_3 v_4$	2975	2925	2886	2719	2720	2722	2570
20	$u_2 v_5$	3742	3705	3675	3049	3061	3072	2790
21	$u_4 v_4$	5045	4967	4904	3541	3545	3549	3200
22	$u_3 v_5 - u_5 v_3$	5222	5155	5102	3602	3611	3620	3400

/745

Table 9 (Continued)

	Standard function	$\lambda(\mu=0{,}365)$	$\lambda(\mu=0{,}390)$	$\lambda(\mu=0{,}410)$	(calculated) $\mu=0{,}365$ $E=7147\frac{kg}{mm^2}$	$\mu=0{,}390$ $E=7118\frac{kg}{mm^2}$	$\mu=0{,}410$ $E=7090\frac{kg}{mm^2}$	obs.
23	$u_0 v_6 - u_9 v_0$	5462	5454	5448	3684	3715	3741	3360
24	$u_0 v_6 + u_6 v_0$	5680	5687	5693	3757	3793	3824	3500
25	$u_3 v_5 + u_5 v_3$	5890	5812	5750	3826	3835	3843	3620
26	$u_1 v_6 - u_6 v_1$	5954	5939	5927	3846	3876	3901	3580
27	$u_2 v_6 - u_6 v_2$	6988	6924	6873	4167	4185	4201	3870
28	$u_2 v_6 + u_6 v_2$	7672	7632	7600	4366	4394	4418	4080

Exact agreement must only be expected for the frequencies of oscillation in heavy print, for which the progressive development was carried out with 10-16 terms. Others are only approximated with the main term, that is the frequencies of oscillation observed are too high as opposed to those observed.

Table 10. Frequency parameters and frequencies of oscillation of the brass plate. /746

	Standard function	$\lambda(\mu=0{,}404)$	$\lambda(\mu=0{,}405)$	$\lambda(\mu=0{,}420)$	(calculated) $\mu=0{,}404$ $E=7212\frac{kg}{mm^2}$	$\mu=0{,}405$ $E=7210\frac{kg}{mm^2}$	$\mu=0{,}420$ $E=7180\frac{kg}{mm^2}$	obs.
1 (I)	$u_1 v_1$	9,773	9,758	9,529	147	146,9	146,0	146,9*
	$+ \dots$							
2 (II)	$u_0 v_2 - u_2 v_0$	21,50	21,473	21,075	218,1	218	217,13	214,5*
	$+ \dots$							
3 (III)	$u_0 v_2 + u_2 v_0$	38,65	38,666	38,884	292,4	292,5	294,9	296,8*
	$+ \dots$							
4	$u_1 v_2$	68,77	68,693	67,579	390	389,9	388,8	384
	$+ \dots$							
5	$u_0 v_3$	237,72	237,72	237,72	725,1	725,3	729,2	714
6 (IV)	$u_2 v_2$	232	231,73	228,4	716,2	716,1	714,7	717
	$+ \dots$							
7 (V)	$u_1 v_3 - u_3 v_1$	275,2	274,9	270,8	780,1	780	778,3	780,9*
	$+ \dots$							
8 (VI)	$u_1 v_3 + u_3 v_1$	371,46	371,41	370,7	906,4	906,6	910,6	906,7*
	$+ \dots$							
9	$u_2 v_3$	699,1	698,5	689,5	1243	1243	1242	1180
10	$u_0 v_4 - u_4 v_0$	864,8	864,6	862,8	1383	1383	1389	1375
11	$u_0 v_4 + u_4 v_0$	962,4	962,56	964,4	1459	1459	1469	1418
12	$u_1 v_4$	1081	1080	1076	1546	1546	1551	1465
13	$u_3 v_3$	1371	1370	1352	1741	1741	1739	1740
	$+ \dots$							
14	$u_2 v_4 - u_4 v_2$	1532	1531	1516	1841	1840,7	1841	1840
15	$u_2 v_4 + u_4 v_2$	1858	1857	1842	2027	2027	2030	1890
16	$u_0 v_5$	2496,5	2496,5	2496,5	2350	2350	2363	2020
17	$u_1 v_5 - u_5 v_1$	2558	2557	2545	2378	2379	2386	2290
18	$u_1 v_5 + u_5 v_1$	2944	2944	2944	2551	2552	2566	2365
19	$u_3 v_4$	2899	2896	2866	2531	2531	2532	2390
20	$u_3 v_5 - u_5 v_3$	5118	5115	5075	3364	3365	3369	3020
21	$u_4 v_5$	8230	8226	8158	4266	4267	4272	4000

4. Results /747

The tables on pp. 744-746 contain the frequencies of oscillation calculated with different μ-values by comparison with the observed ones. The determination of the coefficient of elasticity μ from frequencies of oscillation is only possible within very wide limitations. It is obvious from Table 7 (p. 739) that in particular for the higher tones the convergence of the progressive development is not sufficient in order to determine the third digit after the decimal point of the λ-values with accuracy; this, naturally, made the effect of μ on the corresponding λ-value an illusion. The higher the tone the poorer the convergence of the series. Therefore, the μ-values calculated from the ratio of two frequencies of oscillation differ considerably. Since it is impossible to recognize which tones are best suited for the determination of μ, a μ-value was determined from all 6 tones (I - VI) by equalization according to the method of the least squares. The result was $\mu = 0.365$; $E = 7147$ kg/mm^2 for the aluminum plate and $\mu = 0.404$; $E = 7212$ kg/mm^2 for the brass plate. The calculation of the frequencies of oscillation showed for the tone III a distinct systematic deviation for the two plates. Exclusion of III results for the value $\mu = 0.365$ in a minimum of the relative square of error for the aluminum plate, $\mu = 0.404$ for the brass plate. For higher μ-values the deviation of the tone III (as evident from the tables) will become considerably smaller, however, greater differences are encountered with VI. In addition to tone I - VI, calculation of λ-values was also carried out for the tones (4) and (13) with the greater accuracy, with 16 terms (13) with 10 terms in a progressive development; (4) gave satisfactory agreement, (13) showed already greater deviation since here the convergence of the serial development becomes increasingly poorer. For even higher tones the convergence is poorer so that a satisfactory accuracy of the calculation is practically no longer possible.

The calculation of the nodal line was carried out in 2 cases. In one case - standard function $u_3v_3 + \ldots$ (13 on table 9 - shows for both plates very good agreement of the calculated with the observed nodal lines. In the second case - standard function $w = u_0v_2 + u_2v_0 \ldots$ (III of table 10) - the observed figure shows for the aluminum plate an unusual asymmetry, which varies in dependence on the point of stroke and /749
is possibly associated with the relatively higher load of the lighter aluminum plate by the bow. In the case of the brass plate the distortion is barely noticeable.

Table 11

/748

x	0,18	0,2	0,21	0,4	0,46	0,5
$u_0(x)$	+0,70711	+0,70711	+0,70711	+0,70711	+0,70711	+0,70711
$u_2(x)$	+0,75907	+0,73582	+0,72325	+0,38469	+0,24266	+0,14029
$u_4(x)$	+0,55789	+0,46360	+0,41444	−0,56131	−0,78152	−0,87840
$u_6(x)$	+0,01509	−0,157156	−0,24186	−0,95502	−0,67963	−0,39206
$u_8(x)$	−0,52247	−0,70707	−0,78518	+0,000602	+0,65061	+0,92581
$u_{10}(x)$	−0,89805	−0,98764	−1,0000	+0,95096	+0,83557	+0,38224

Table 11 (Continued)

x	0,6	0,65	0,68	0,8	0,82
$u_0(x)$	+0,70711	+0,70711	+0,70711	+0,70711	+0,70711
$u_2(x)$	−0,13823	−0,28725	−0,37911	−0,75968	−0,8245
$u_4(x)$	−0,90910	−0,80480	−0,70516	−0,07351	+0,00011
$u_6(x)$	+0,43163	+0,75086	+0,87318	+0,68346	+0,54666
$u_8(x)$	+0,71348	+0,20652	−0,14012	−0,93296	−0,88746
$u_{10}(x)$	−0,89286	−0,96632	−0,75603	+0,77334	+0,89782

Table 11 (Continued)

x	0,3	0,4	0,5	0,6	0,65	0,67
$u_1(x)$	+0,36743	+0,4899	+0,61238	+0,73486	+0,79609	+0,82058
$u_3(x)$	+0,88128	+0,93625	+0,82693	+0,56166	+0,37805	+0,296415
$u_5(x)$	+0,85245	+0,31917	−0,36204	−0,84941	−0,93343	−0,9312
$u_7(x)$	+0,07766	−0,81065	−0,92814	−0,16754	+0,32628	+0,50484
$u_9(x)$	−0,76043	−0,80886	+0,38354	+0,99104	+0,68541	+0,45231

Table 11 (Continued)

x	0,7	0,71	0,72	0,725	0,74	0,75
$u_1(x)$	+0,85733	+0,86958	+0,88182	+0,88795	+0,90632	+0,91857
$u_3(x)$	+0,16589	+0,120815	+0,074379	+0,051149	−0,020372	−0,06917
$u_5(x)$	−0,88859	−0,86242	−0,83190	−0,81501	−0,75582	−0,71079
$u_7(x)$	+0,72765	+0,7860	+0,83609	+0,85704	+0,90636	+0,93019
$u_9(x)$	+0,09063	−0,04022	−0,17089	−0,23451	−0,43836	−0,53106

Table 11 (Continued)

x	0,76	0,78	0,8	0,82	0,86	0,88
$u_1(x)$	+0,93081	+0,95530	+0,97981	+1,00430	+1,0533	+1,07778
$u_3(x)$	−0,118196	−0,21882	−0,3216	−0,42648	−0,64096	−0,74996
$u_5(x)$	−0,6605	−0,54668	−0,41577	−0,26984	+0,06008	+0,23998
$u_7(x)$	+0,93465	+0,91922	+0,85929	+0,75605	+0,43117	+0,21812
$u_9(x)$	−0,6326	−0,79831	−0,90209	−0,93491	−0,77461	−0,58652

The u-functions used for the calculation of the nodal lines are given in Table 11. The serial development for w is given under the measured figures. The zero-locations of w have been determined with the help of Table 11 by a trial and error approach and subsequent to sufficient limitation by interpolation. The points thus calculated are identified by small circles. The dotted lines are exact copies of the impressions on the transparent paper of the copied sound pattern.

If one places the origin of the coordinates into the center of the plate and specifies that one half of the length of the square edge is 1, one then obtains the following pairs of values:

Fig. A		Fig. B	
x	y	x	y
0.2	0.804	0.3	0.700
0.4	0.675	0.4	0.714
0.6	0.49	0.5	0.721
0.8	0.209	0.6	0.7425
		0.65	0.769
		0.67	0.798

In the case of Fig. B the axes of coordinates are nodal lines in themselves.

The agreement of the shape of the curve with the theory may be assumed to be satisfactory with the exception of the

1. distortion mentioned above,
2. the hyperbolic deviations in points where according to the theory two nodal lines are intersecting.

Ritz attempted to explain the hyperbolic deviation in the following way: "In order to overcome friction the amplitude of oscillation has to exceed a certain minimum, that is, the sand remains in a state of indifferent equilibrium within an area limited by hyperbolas" [18e]. The observation shows that the formation of hyperbolas occurs always in certain quadrants, the position of which - with regard to the point of stroke - is always the same, that is, it changes with the latter.

I am grateful to Prof. Kaufmann, Königsberg in Prussia, /750
who suggested this study and to Prof. Kalähne, Danzig, who kindly permitted the use of his tonometer.

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Translated for the National Aeronautics and Space Administration by the Translation and Interpretation Division of the Institute of Modern Languages, Inc., under contract NASw-1693.