NASA TECHNICAL TRANSLATION

NASA TT F-490

NASA TT F-490

TECH LIBRARY KAFB, NM
0069002

LOAN COPY: RETURN TO
AFWL (WLIL-2)
KIRTLAND AFB, N MEX

THEORY OF MOTION OF THE PLANET PLUTO

Part One: Plutonian Perturbations of the First Order With Respect To Perturbing Masses

by Sh. G. Sharaf

Transactions of the Institute of Theoretical Astronomy, Vol. 4
USSR Academy of Sciences Press, Moscow-Leningrad, 1955

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1969

TECH LIBRARY KAFB, NM

0069002

NASA TT F-490

THEORY OF MOTION OF THE PLANET PLUTO

Part One: Plutonian Perturbations of the First Order With Respect to Perturbing Masses

By Sh. G. Sharaf

Translation of "Teoriya Dvizheniya Plutona. Chast' Pervaya. Vozmushcheniya Plutona Pervogo Poryadka Otnositel'no Vozmushchayushchikh Mass."

Trudy Instituta Teoreticheskoy Astronomii. Vol. 4.
USSR Academy of Sciences Press, Moscow-Leningrad, 1955.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

TABLE OF CONTENTS

PART ONE. PLUTONIAN PERTURBATIONS OF THE FIRST ORDER WITH RESPECT TO PERTURBING MASSES

INTRODUCTION 1

CHAPTER I. EXPANSION OF THE PERTURBATION FUNCTION 7
1. The Method of Newcomb's Operators 7
2. Verification and Computation of the Coefficients of the Newcomb Operators 15
3. Computing the Derivatives of Laplace Coefficients 23
4. Use of Analytic Computers for Expanding the Perturbation Function and Its Derivatives 26

CHAPTER II. PERTURBATIONS OF COORDINATES 33
5. Perturbations of the Logarithms of the Radius Vector and Longitude in Orbit 33
6. Perturbations of Longitude of Node and Inclination 36
7. Determining Heliocentric Longitude and Latitude 38
8. Determining the Constants of Integration 38

CHAPTER III. PERTURBATIONS OF THE FIRST ORDER BY JUPITER 43
9. Elements 43
10. Expansion of the Perturbation Function 44
11. Computing Perturbations ρ' and w' 52
12. Determining the Perturbation of Ω' and i' 63
13. Perturbations of Pluto by Jupiter 63

CHAPTER IV. PERTURBATIONS OF THE FIRST ORDER BY SATURN 65
14. Elements 65
15. Expansion of the Perturbation Function 65
16. Determining the Perturbations of the Logarithm of the Radius, Vector, Longitude in Orbit, Node and Inclination 70

CHAPTER V. PERTURBATIONS OF THE FIRST ORDER BY URANUS 73
17. Elements 73
18. Expansion of the Perturbation Function 73
19. Determination of the Logarithm of the Radius Vector, Longitude in Orbit, Longitude of the Node, and Inclination 77

CHAPTER VI. PERTURBATIONS BY NEPTUNE 84
20. Perturbations of Pluto by Neptune 84

CHAPTER VII. IMPROVING THE ELEMENTS OF PLUTO 88
21. Observations of Pluto and Compilation of Normal Positions 88
22. Improvement of the Center of Pluto's Elements 100

APPENDIX..... 105
I. SYMBOLIC EXPANSION OF PERTURBATION FUNCTION..... 105
II. TABLES OF COEFFICIENTS OF NEWCOMB'S OPERATORS..... 110
III. PARTIAL VALUES OF OPERATORS FOR COMPUTING CORRECTIONS FOR SECOND TERM..... 138
IV. CERTAIN ERRORS IN THE EXPRESSION FOR NEWCOMB'S OPERATORS..... 139

REFERENCES..... 140

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL TRANSLATION EVALUATION

Budget Bureau No. 104-R037
Approval Expires: Sept. 30, 1969

TO:
THE USERS OF THIS TRANSLATION →

NASA TTF NO. 490

MAINTAINING THE QUALITY OF NASA TRANSLATIONS REQUIRES A CONTINUOUS EVALUATION PROGRAM. PLEASE COMPLETE AND MAIL THIS FORM TO AID IN THE EVALUATION OF THE USEFULNESS AND QUALITY OF THE TRANSLATING SERVICE.

THIS PUBLICATION *(Check one or more)*

- ☐ FURNISHED VALUABLE NEW DATA OR A NEW APPROACH TO RESEARCH.
- ☐ VERIFIED INFORMATION AVAILABLE FROM OTHER SOURCES.
- ☐ FURNISHED INTERESTING BACKGROUND INFORMATION.
- ☐ OTHER *(Explain)*: _____

FOLD LINE FOLD LINE

TRANSLATION TEXT *(Check one)*

- ☐ IS TECHNICALLY ACCURATE.
- ☐ IS SUFFICIENTLY ACCURATE FOR OUR PURPOSE.
- ☐ IS SATISFACTORY, BUT CONTAINS MINOR ERRORS.
- ☐ IS UNSATISFACTORY BECAUSE OF *(Check one or more)*:
  - ☐ POOR TERMINOLOGY.
  - ☐ NUMERICAL INACCURACIES.
  - ☐ INCOMPLETE TRANSLATION.
  - ☐ ILLEGIBLE SYMBOLS, TABULATIONS, OR CURVES.
  - ☐ OTHER *(Explain)*: _____

FOLD LINE FOLD LINE

REMARKS

FROM

DATE

NOTE: REMOVE THIS SHEET FROM THE PUBLICATION, FOLD AS INDICATED, STAPLE OR TAPE, AND MAIL. NO POSTAGE NECESSARY.

CUT ALONG THIS LINE

CUT ALONG THIS LINE

THEORY OF MOTION OF THE PLANET PLUTO.
PART I.
PLUTONIAN PERTURBATIONS OF THE FIRST ORDER WITH RESPECT TO PERTURBING MASSES.

Sh. G. Sharaf

ABSTRACT. After the Plutonian perturbations of the first order by the four large planets were determined, a preliminary improvement of Bauer's elements was performed, based on observations made between 1914 and 1951. In this report Newcomb's method of expanding the perturbation function, the methods of controlling the operator coefficients, and the technique for calculating the expansion coefficients of the perturbation function on analytic computers are outlined. Methods for determining the perturbations of the logarithm of the radius vector, length of the orbit, length of the node and inclination, and the constants of integration are given, as well as the perturbations by Neptune between 1912 and 1955. A summary of the Plutonian observations and of the normal positions is presented, as well as a method for correcting the elements applied to Pluto and the new Plutonian elements. The appendix contains the symbolic expansion of the perturbation function, a table of operator coefficients, the partial values of the operators for computing corrections for the second term, and some errors in the expressions for Newcomb's operators.

INTRODUCTION

The problem of the study of motion of the large planets is one of the basic problems in celestial mechanics. /5*

The classical methods of celestial mechanics developed in the eighteenth and nineteenth centuries were intended precisely for the solution of this problem. As the result of works by Laplace, Leverrier, Newcomb, and Hill, construction of tables of motion of the eight large planets had been fundamentally completed by the beginning of the nineteenth century. At present this problem is again becoming urgent. The discovery in 1930 of the ninth large planet of our system (Pluto) requires the construction of its analytical theory of motion in just as accurate a form as the other large planets.

In 1946 the Institute of Theoretical Astronomy included in its agenda the

*Numbers in the margin indicate pagination in the foreign text.

elaboration of the analytical theory of Plutonian motion. We present here the first part of this work, the determination of the perturbations of the first order with respect to the perturbing masses.

Before proceeding to the exposition of the theory, let us briefly consider certain works in regard to Pluto which appeared in print after its discovery.

The Lowell Observatory reported the discovery of Pluto on 13 March 1930, and as early as March and April of that year many astronomers had presented various systems of its elements. The number of systems exceeded forty by 1931, but not all works on the determination of Plutonian elements are of equal value. Particularly noteworthy among them are the elements determined by Bower, Nicolson and Mayall, and Zagar.

On 15 April 1930, Bower and Whipple (1930a) published Plutonian elements which were determined from three observations in 1930 and which well satisfied the twenty-four observations in the same year. As new observations were communicated, the authors corrected their elements and each time they gave an ephemeris for the preceding years based on the improved new elements. Their ephemerides led to the discovery on old photographic plates of observations in 1919, 1921, and 1927. On 17 June 1931, Bower and Whipple (1930b) published a system of elements (Table 1), the eighteenth, according to their numeration derived by improving the preceding system from the observations of 1919, 1927, and 1930. In order partially to take into account the perturbations, the masses of all the large planets were in this case added to the mass of the sun, in other words, the magnitude $k\sqrt{1+\Sigma m}=0.017213628$ was taken instead of the Gaussian constant. From these elements was given an ephemeris of Pluto from 1890 to 1930 and for the opposition of 1930-1931. On 28 April 1931, Bower gave his nineteenth barycentric system of the elements (Table 1) derived by correcting the eighteenth system from observations in the oppositions of 1914-1930. To account for the perturbations by the four large planets (Jupiter, Saturn, Uranus, and Neptune), tables of perturbations of the rectangular coordinates were computed, in which the origin of the coordinates was placed at the center of gravity of the system; the sun (to the mass of which were added the masses of the four inner planets: Jupiter, Saturn, Uranus, and Neptune).

Ephemerides of Pluto were computed on the basis of these elements, taking into consideration perturbations by the four planets (numerically) which were printed until 1935 in the Lick Observatory Bulletin, from 1935 in the Berliner Astronomisches Jahrbuch (BJ), and from 1950 in The Nautical Almanac (NA), The American Ephemeris, and in other astronomical annuals.

In 1951, in Astronomical Papers, Eckert, Brouwer, and Clemence published eleven-place heliocentric coordinates of the five outer planets at 40-day /6
intervals for the period from 1653 to 2060. The coordinates were derived by numerical integration on high-speed electronic computers under the direction of Eckert, Brouwer and Clemence. In the case of Pluto the original elements taken were those of Bower (nineteenth system).

Nicolson and Mayall in November of 1930 presented a system of elements derived from observations in 1919, 1921, 1927, and 1930. They numerically derived the perturbations by the four large planets in rectangular heliocentric coordinates. Nicolson and Mayall printed two identical systems -- barycentric and heliocentric (Table 1), the values of which were very close to Bower's elements.

These elements were listed in The American Ephemeris until 1950.

On 23 November 1930, Zagar published Plutonian elements obtained from eight observations during five oppositions (1919-1930). In order to account for perturbations by Uranus and Neptune he compiled special tables of perturbations in rectangular coordinates. As the origin of coordinates he took the center of gravity of the sun and of the six planets: Mercury, Venus, Earth, Mars, Jupiter, and Saturn. The published elements are in terms of the center of the sun weighted by the masses of the six planets.

Table 1 gives the above-mentioned systems of the elements of Pluto.

TABLE I. SYSTEMS OF ELEMENTS OF PLUTO GIVEN BY VARIOUS AUTHORS
(All elements in terms of the ecliptic and equinox of 1930.0)

Elements	Bower & Whipple (heliocentric elements XVIII	Bower (Barycentric elements XIX; moment of osculation 1930 September 20.0)	Nicolson & Mayall (heliocentric elements; moment of osculation 1930 Jan 7.0)	Nicolson & Mayall (barycentric elements; moment of osculation 1930 Jan 7.0)	Zagar (heliocentric elements; epoch and osculation 1930 April 1.0)
T	1989 Feb. 27.473	1989 Oct. 0.0344	1989 Oct. 2.03	1989 Nov. 6.98	$M_0 = 274°3'15''.4$
ω	113° 8′26″.1	113°32′ 0″.0	113° 1′41″.3	113°52′50″.6	113° 8′ 0″.5
Ω	109 21 36.9	109 21 44.2	109 21 39.4	109 21 43.7	109 21 38.9
i	17 8 57.1	17 8 41.6	17 6 58.4	17 8 38.1	17 6 50.8
a	39.59673	39.517738	39.60038	39.45743	39.579 436
e	0.253741	0.248644	0.246086	0.248520	0.247 196
n	14″.2498	14″.283	14″.23833	14″.325530	14″.258462
P	249.1661	248.43015	249.2097	247.6968	248.8579

In 1935 Roure began to print his work on the analytical theory of Plutonian motion (Roure, 1935, 1936, and 1937). He used Hill's method in the form given to it by Andwaye. In this case the role of the Moon is played by Pluto, that of the Sun by Neptune, and that of the Earth by the Sun. As the initial solution an intermediate orbit is taken, where eccentricity and inclination of Pluto are set equal to zero, while in the perturbation function, moreover, those terms are discarded which depend on eccentricity and inclination of Neptune and on the ratio of the major semiaxes. Roure derives his solution in rectangular coordinates presented as trigonometric series with a period of $\frac{2\pi}{n-n'}$, where the coefficients will be arranged in powers of $\mu' = \frac{1}{2}\frac{m}{1+m}\left(\frac{n}{n-n'}\right)^3$.

Subsequently Roure, proceeding from this basic solution, sought corrections for it depending on the eccentricities and inclinations of Pluto and Neptune. Here the ratio of the major semiaxes is given a numerical value from the very beginning. /7

The final solution is sought in the form of series in powers of eccentricities and inclinations.

To determine the perturbations by the other planets, Roure sets up equations for the perturbations of the initial coordinates, taking into consideration on the right side the terms of the perturbation function which depend on the action of the other planets. The equations are solved by successive approximations.

Roure did not carry his work to conclusion. He derived only four intermediate orbits of Pluto for the combinations of Pluto and the four outer planets, and determined the inequalities in Plutonian motion under the effect of Neptune which are dependent on the first degree of Plutonian eccentricity and inclination.

In addition, Roure (1940a, 1940b, and 1940c) published three works devoted to the determining long-period inequalities in Plutonian motion.

Thus, as is evident from the above-given survey of the literature, there is at the present time no analytical theory of Plutonian motion; but the compiling of such a work is now completely justified and realistic.

This theory will give not only good elements and ephemerides for Pluto, but also the opportunity to form a concept of the general nature and features of Plutonian motion. The observations of Pluto embrace a period from 1914 to the present time. The arc traveled by Pluto in this period amounts to about 50° in mean longitude. Although this arc is small, the elements derived on its basis will be satisfactory because Pluto moves slowly and experiences only slight perturbations by the large planets. Therefore, in the future, after more precisely rendering the values of the elements and mass of Pluto, it will be possible to introduce small corrections in the perturbation tables.

The Leplace-Newcomb method, which has completely justified itself in the construction of the theory for the large planets, has been used in the present work as the foundation for determining perturbations in Plutonian coordinates by Jupiter, Saturn, and Uranus.

In the expansion of the perturbation function we have retained the method of Newcomb's operators in a form corresponding to expansion in multiple mean anomalies (Newcomb, 1895a).

In constructing the theory of the four inner planets Newcomb used his method of expansion of the perturbation function in eccentric anomalies (Newcomb, 1891a and 1891b), believing that the series in eccentric anomalies converged more rapidly than did those in mean anomalies. The final expansion of the perturbation function, however, was given by Newcomb in the form of series in mean anomalies by passing from some arguments to others

by means of Bessel functions. Therefore the advantage of using expansions in eccentric anomalies is lost.

In determining the secular perturbations of the four inner planets, Newcomb utilized an expansion in mean anomalies (Newcomb, 1895b). This same method was successfully used by B. A. Orlov (1936) for computing certain terms of the perturbation function in his work, Application of the DeLaunay-Hill Method to the Case of the Commensurability 3:4 (Tule).

The slight dissemination of this method is chiefly explained by the lack of tables of numerical values of the coefficients in the operators. Compilation of such tables (given in Appendix II) is one of the important parts of the present work. When there are tables of operator coefficients, the work in expanding the perturbation function and its derivatives may be conducted very successfully on analytical computers.

The perturbations of Pluto by Neptune were derived by means of numerical integration. After determining the first-order perturbations of Pluto by the four planets it was decided to make a preliminary improvement of the elements (Bower's elements). The improvement was made on the basis of observations from 1914 to 1951 (twenty-four normal positions in all) by Eckert's modified method (Samoylova-Yakhontova, 1945).

The present work consists of seven chapters and appendices.

Chapter I sets forth Newcomb's method of expanding the perturbation function, methods of checking the operator coefficients, and methods of computing the coefficients of the expansion of the perturbation function on analytical computers. /8

Chapter II sets forth methods of determining perturbations of the logarithm of the radius vector, longitude in orbit, longitude of node, and inclination, and questions of determining the constants of integration.

Chapter III contains a detailed exposition of the utilization of the above-mentioned methods for determining the perturbations of Pluto by Jupiter.

Chapters IV and V briefly give the main stages in determining the perturbations of Pluto by Saturn and Uranus.

Chapter VI gives the perturbations of Pluto by Neptune from 1912 to 1955.

Chapter VII gives a summary of the observations of Pluto, a summary of the normal positions, the method applied to Pluto for correcting the elements, and new elements of Pluto.

The appendix contains:

I - a symbolic expansion of the perturbation function;
II - tables of operator coefficients;
III - partial operator values for computing corrections for the second term; and

IV - Certain errors in the expressions for Newcomb's operators.

The present work was accomplished under the general direction of Professor M. F. Subbotin. Substantial assistance at all stages of the computations was rendered by G. A. Petrenko and N. A. Budnikova. A. G. Mal'kova also took part in certain computations of the perturbations by Saturn.

Use of analytical computers for forming the laborious computations was possible due to the consultations of D. K. Kulikov and M. B. Zheleznyak.

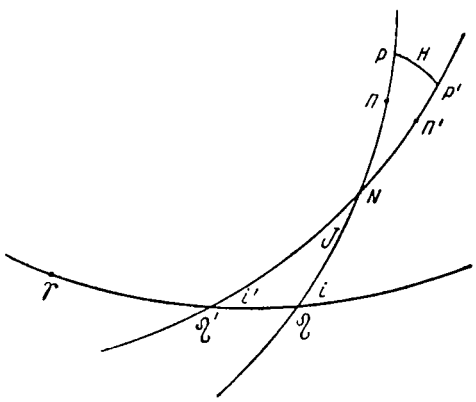
I consider it my pleasant duty to express my deep thanks to all these persons.

CHAPTER I

EXPANSION OF THE PERTURBATION FUNCTION

1. The Method of Newcomb's Operators

Expansion of the perturbation function in the case of circular motions of the perturbing and the perturbed planet is accomplished very simply. The general case of elliptical motion of both planets already demands special methods of expanding the main part of the perturbation function. One of these methods is the method of operators proposed by Newcomb in 1870. In his later work (Newcomb, 1891a and 1895a) this method was developed in more detail and reduced to tables giving the numerical values of the coefficients of the operators in the case where the perturbation function was expanded in series of multiples of the eccentric anomalies. The use of the method of operators in constructing the theory of the motion of the four inner planets proved the expediency of the operators and disclosed a number of advantages in this method over other known methods of expanding the perturbation function. /9



The convenience in using eccentric anomalies, however, is definitely less than that of using mean anomalies, at least in the final expressions of the perturbations. It is obvious that conversion to mean anomalies (by the use of Bessel functions) after the perturbations have already been obtained in the function of the eccentric anomalies to a considerable measure diminishes the advantages of using eccentric anomalies when expanding the perturbation function. Therefore, the final development of the method of operators with respect to the expansion in mean anomalies is an important part of the present work. This development is of absolute importance, since Newcomb, in his work, limits himself only to a general algebraic expression of the operator coefficients.

But before proceeding to the determination of the coefficients, let us briefly recall the basic ideas of the method of operators itself. In this chapter, as in the entire work, we will use the convention of providing magnitudes referring to an outer planet with primes, that of considering the outer planet the perturbed one, and that of keeping to the following notation:

M, M' - mean anomalies;
Ω, Ω' - longitudes of the ascending nodes of the planets;
i, i' - inclinations of the planets to the ecliptic;
a, a'.- major semiaxes;

e, e' - eccentricities;
n, n' - average daily motions;
v, v' - true anomalies of the planets;
E, E' - eccentric anomalies of the planets;
r, r' - radius vectors; /10
m, m' - masses of the planets;
N - distance of the common node of the planets from the ascending node of the inner planet;
N' - distance of the common node of the planets from the ascending node of the outer planet;
J - mutual inclination;
Π, Π' - distances of the perihelia from the common node;
W, W' - longitudes of planets calculated from the common node;
L, L' - mean longitudes of planets calculated from the common node;
H - angular distance between the planets;
α - ratio of major semiaxis of the inner planet to major semiaxis of the outer planet.

The perturbation function in the case of an effect by an outer planet on an inner planet has the form:

$$k^2 m R = k^2 m \left(\frac{1}{\Delta} - \frac{r' \cos H}{r^2}\right). \tag{1}$$

From triangle NPP' (see figure), we determine cos H

$$\cos H = \cos W \cos W' + \sin W \sin W' \cos J$$

or

$$\cos H = \cos (W' - W) - 2\sigma^2 \sin W' \sin W.$$

If in the perturbation equation we replace cos H by W, W', $\sigma = \sin \frac{J}{2}$ and if we introduce instead of r and r' their logarithms ρ, ρ', we will then derive R in the form of a function of ρ, ρ', W', W, and σ

$$R = R(\rho', \rho, W', W, \sigma).$$

In the case of circular motion when e' = e = 0 and, consequently:

$$^* \rho = \lg a, \ \rho' = \lg a', \ W = L, \ W' = L',$$

R will become R_0

$$R_0 = R(\lg a', \lg a, L', L, \sigma).$$

First, let us take a look at the expansion of the main part of perturbation function R_0. (The second term $\frac{r' \cos H}{r^2}$ may be subsequently introduced by comparatively simple transformations.) Since Newcomb's method may use any

*Translators note: lg=log.

expansion of $\frac{a'}{\Delta_0}$ in powers of the mutual inclination, the ratio of, the major semi-axes, and cosines of the arcs which are divisible by the distances of the planets from the common node, then for $\frac{a'}{\Delta_0}$ we will use the familiar expansion

$$
\begin{aligned}
a'\Delta_0^{-1} = \frac{1}{2}\sum_{-\infty}^{\infty} a'A_i \cos\left[iL'-iL\right] \\
+\sigma^2\sum_{-\infty}^{\infty} a'B_i \cos\left[(i+1)L'-(i-1)L\right] \\
+\sigma^4\sum_{-\infty}^{\infty} a'C_i \cos\left[(i+2)L'-(i-2)L\right] \\
+\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}
\tag{2}
$$

where $a'A_i$, $a'B_i$, $a'C_i$. . . are the functions of the major semiaxes in σ. If we take into consideration Laplace's coefficients b_n^i, then $a'A_i$, $a'B_i$ ... will be represented as linear functions of $c_n^i = \alpha^{\frac{n-1}{2}} b_n^i$:

$$
\begin{aligned}
a'A_i = c_1^i - \frac{1}{2}\sigma^2\left(c_3^{i+1} + c_3^{i-1}\right) + \frac{3}{8}\sigma^4\left(c_5^{i+2} + 4c_5^i + c_5^{i-2}\right) - \\
\frac{5}{16}\sigma^6\left(c_7^{i+3} + 9c_7^{i+1} + 9c_7^{i-1} + c_7^{i-3}\right) + \frac{35}{128}\sigma^8\left(c_9^{i+4} + 16c_9^{i+2} + 36c_9^i + 16c_9^{i-2} + c_9^{i-4}\right) - \\
-\frac{63}{256}\sigma^{10}\left(c_{11}^{i+5} + 25c_{11}^{i+3} + 100c_{11}^{i+1} + 100c_{11}^{i-1} + 25c_{11}^{i-3} + c_{11}^{i-5}\right) + \\
\frac{231}{1024}\sigma^{12}\left(c_{13}^{i+6} + 36c_{13}^{i+4} + 225c_{13}^{i+2} + 400c_{13}^i + 225c_{13}^{i-2} + 36c_{13}^{i-4} + c_{13}^{i-6}\right) - \ldots, \\
a'B_i = \frac{1}{2}c_3^i - \frac{3}{4}\sigma^2\left(c_5^{i-1} + c_5^{i+1}\right) + \frac{15}{16}\sigma^4\left(c_7^{i-2} + 3c_7^i + c_7^{i+2}\right) - \\
\frac{35}{32}\sigma^6\left(c_9^{i-3} + 6c_9^{i-1} + 6c_9^{i+1} + c_9^{i+3}\right) + \frac{315}{256}\sigma^8\left(c_{11}^{i-4} + 10c_{11}^{i-2} + 20c_{11}^i + 10c_{11}^{i+2} + c_{11}^{i+4}\right) \\
-\frac{693}{512}\sigma^{10}\left(c_{13}^{i-5} + 15c_{13}^{i-3} + 50c_{13}^{i-1} + 50c_{13}^{i+1} + 15c_{13}^{i+3} + c_{13}^{i+5}\right) + \ldots, \\
a'C_i = \frac{3}{8}c_5^i - \frac{15}{16}\sigma^2\left(c_7^{i-1} + c_7^{i+1}\right) + \frac{35}{64}\sigma^4\left(3c_9^{i-2} + 8c_9^i + 3c_9^{i+2}\right) - \\
\frac{315}{128}\sigma^6\left(c_{11}^{i-3} + 5c_{11}^{i-1} + 5c_{11}^{i+1} + c_{11}^{i+3}\right) + \frac{3465}{1024}\sigma^8\left(c_{13}^{i-4} + 8c_{13}^{i-2} + 15c_{13}^i + 8c_{13}^{i+2} + c_{13}^{i+4}\right) - \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{3}
$$

/11

The problem now consists of proceeding from expansion R_0 to expansion R, by representing the coefficients as functions of the major semiaxes and of

the mutual inclination, and as series in the eccentricities. For this purpose we expand R into a series in powers of $\Delta\rho' = \rho' - \lg a'$, $\Delta\rho = \rho - \lg a$, $f' = W' - L'$ and $f = W - L$. In turn these magnitudes are functions of the eccentricities and of the eccentric or mean anomalies. Consequently, after performing the appropriate operations, we will be able to represent R in the form that we desire. Depending on which anomalies we consider, we will derive two different methods of expanding the perturbation function.

Let us examine the expansions over the mean anomalies:

$$\left.\begin{aligned} \rho' &= \lg a' + \varphi(e', M') = \lg a' + \rho_1' e' + \rho_2' e'^2 + \rho_3' e'^3 + \ldots, \\ \rho &= \lg a + \varphi(e, M) = \lg a + \rho_1 e + \rho_2 e^2 + \rho_3 e^3 + \ldots, \\ W' &= L' + \psi(e', M') = L' + f_1' e' + f_2' e'^2 + f_3' e'^3 + \ldots, \\ W &= L + \psi(e, M) = L + f_1 e + f_2 e^2 + f_3 e^3 + \ldots \end{aligned}\right\} \tag{4}$$

From these qualities it follows that:

$$\frac{\partial R}{\partial \rho'} = \frac{\partial R}{\partial \lg a'}, \qquad \frac{\partial R}{\partial \rho} = \frac{\partial R}{\partial \lg a},$$

$$\frac{\partial R}{\partial W'} = \frac{\partial R}{\partial L'}, \qquad \frac{\partial R}{\partial W} = \frac{\partial R}{\partial L},$$

$$\frac{\partial^{m'+m+n'+n} R}{(\partial\rho')^{m'}(\partial\rho)^{m}(\partial W')^{n'}(\partial W)^{n}} = \frac{\partial^{m'+m+n'+n} R}{(\partial \lg a')^{m'}(\partial \lg a)^{m}(\partial L')^{n'}(\partial L)^{n}}.$$

In other words anv derivative to R with respect to ρ', ρ, W', W may be replaced by the corresponding derivative with respect to log a', log a, L', and L. /12

Let us expand R into the Taylor series. Let us consider the case e' = 0, then $\Delta\rho' = f' = 0$, , and the symbolic expansion of R with respect to $\Delta\rho$ and f will be written as:

$$R(\lg a', \lg a + \Delta\rho, L', L + f, \sigma) = \exp\left(\frac{\partial}{\partial \lg a}\Delta\rho + \frac{\partial}{\partial L} f\right) R(\lg a', \lg a, L', L, \sigma), \tag{5}$$

where

$$\exp(x) = 1 + x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \ldots,$$

and $\left(\frac{\partial}{\partial \lg a}\right)^m \left(\frac{\partial}{\partial L}\right)^n R$ must be substituted for $\frac{\partial^{m+n} R}{(\partial \lg a)^m (\partial L)^n}$.

We will introduce the differentiation symbols $D=\frac{\partial}{\partial \lg a}=\frac{\partial}{\partial \lg \alpha}$ and $D_1=\frac{\partial}{\partial L}$.

Then, taking into consideration that $R_0=\Sigma A(s, s')\exp i(s'L'+sL)$ and

$$\frac{\partial R_0}{\partial L}=\Sigma\, isA(s, s')\exp i(s'L'+sL)$$

(here $i=\sqrt{-1}$), we may rewrite Expression (5) as follows:

$$R(\lg a', \lg a+\Delta\rho, L', L+f, \sigma)=\sum \exp(D\Delta\rho+isf)A(s, s')\exp i(s'L'+sL)=$$
$$=\sum\left[1+(D\Delta\rho+isf)+\frac{1}{2!}(D\Delta\rho+isf)^2+\ .\ .\ .\right]A(s,s')\exp i(s'L'+sL).$$

For $\Delta\rho$ and f let us substitute their expansions in Equation (4), remove the parentheses and expand the obtained expression in powers of e; we will then obtain:

$$R=\sum\left\{1+e[\rho_1 D+isf_1]+e^2\left[\frac{1}{2}(\rho_1 D+isf_1)^2+(\rho_2 D+isf_2)\right]+e^3\left[\frac{1}{6}(\rho_1 D+isf_1)^3+\right.\right.$$
$$\left.\left.(\rho_1 D+isf_1)(\rho_2 D+isf_2)+(\rho_3 D+isf_3)\right]+\ .\ .\ .\right\}A(s,s')\exp i(s'L'+sL). \qquad (6)$$

From the expansions of the logarithm of the radius vector and the equation of the center in powers of eccentricity and the arcs divisible by the mean anomaly, we have for $\rho_1, \rho_2, \ldots$ and $f_1, f_2, \ldots$ the following expressions :

$$\rho_1=-\frac{1}{2}(\exp iM+\exp -iM),$$
$$\rho_2=\frac{1}{4}-\frac{3}{8}(\exp 2iM+\exp -2iM),$$
$$\rho_3=\frac{3}{16}(\exp iM+\exp -iM)-\frac{17}{48}(\exp 3iM+\exp -3iM),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_1=-i(\exp iM-\exp -iM),$$
$$f_2=-\frac{5}{8}i(\exp 2iM-\exp -2iM),$$
$$f_3=\frac{1}{8}i(\exp iM-\exp -iM)-\frac{13}{24}i(\exp 3iM-\exp -3iM)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Sustituting these expressions in Expression (6) we will obtain /13

$$R(\lg a', \lg a+\Delta\rho, L', L+f, \sigma)=\sum\left(1+e\left\{\left[-\frac{1}{2}D+s\right]\exp iM+\right.\right.$$
$$\left.\left[-\frac{1}{2}D-s\right]\exp -iM\right\}+e^2\left\{\left[\frac{1}{8}D^2+\left(-\frac{1}{2}s-\frac{3}{8}\right)D+\frac{1}{2}s^2+\frac{5}{8}s\right]\exp 2iM+\right.$$

$$\left[\frac{1}{4}D^2+\frac{1}{4}D-s^2\right]+\left[\frac{1}{8}D^2+\left(\frac{1}{2}s-\frac{3}{8}\right)D+\frac{1}{2}s^2-\frac{5}{8}s\right]\exp-2iM\Big\}+$$
$$+\ .\ .\ .\Big)A(s,s')\exp(s'L'+sL).$$

Newcomb called the coefficients of the different degrees of eccentricity and the expansion, which are polynomials of index s and differentiation symbol D, operators, and designated them by Π_m^n, where n is the degree of eccentricity and m is the number of times mean anomaly M is contained in the expansion. Then the expansion $R(\lg a', \lg a+\Delta\rho, L', L+f, \sigma)$ may be written thus:

$$R(\lg a', \rho, L', W, \sigma)=\sum e^n\Pi_m^n(s,D)A(s,s')\exp i(s'L'+sL+mM). \tag{7}$$

For the case $e'\neq 0$ we will reason as in the preceding and derive:

$$\begin{aligned}R(\lg a'+\Delta\rho', \rho, L'+f', W, \sigma)&=\sum e'^{n'}\Pi_{m'}^{n'}(s',D')\exp im'M'R(\lg a', \rho', L', W, \sigma)=\\&=\sum e'^{n'}\Pi_{m'}^{n'}(s',D')e^n\Pi_m^n(s,D)A(s,s')\exp i(s'L'+sL+m'M'+mM),\end{aligned} \tag{7'}$$

where $\Pi_{m'}^{n'}(s',D')$—are polynomials of the type $\Pi_m^n(s,D)$, where $D'=\frac{\partial}{\partial\lg a'}$.

Symbol D' is connected by a simple relationship with the symbol D. In fact, because the perturbation function is a homogeneous function with respect to a and a', the identity:

$$a'\frac{\partial R}{\partial a'}+a\frac{\partial R}{\partial a}=-R,$$

takes place; hence,

$$(D'+D)R=-R \text{ or } D'=-1-D.$$

Substituting the value $D'=-1-D$ in the operators $\Pi_{m'}^{n'}(s',D')$, we obtain polynomials of another sort distributed with respect to D,

$$\Pi_{om'}^{on'}(s',D).$$

The action of operators $\Pi_{om'}^{on'}(s',D)$ on $\Pi_m^n(s,D)A(s,s')$ may be represented as the action of the complex operators $\Pi_{mm'}^{nn'}(s,s',D)$ on $A(s,s')$. $\Pi_{mm'}^{nn'}(s,s',D)$ are formed by the term-by-term multiplication of simple operators $\Pi_{om'}^{on'}(s',D)$ and $\Pi_m^n(s,D)$.

From Expression (2) it is evident that s and s' are not independent. In fact,

for the first line we have s' = i, s = - i;
for the second line we have s' = i + 1, s = - i + 1;
for the third line we have s' = i + 2, s = - i + 2.
. . . . . .

Therefore, for each line, we form its operators dependent only on i and D, and we call the operators applied to the first line $(a'A_i)$, the operators of the zero class $\Pi^{nn'}_{mm'}$; the operators applied to the second line, or to $\sigma^2\alpha\beta_i$, we call the operators of the first class $\Pi'^{nn'}_{mm'}$, and so on.

Here $\Pi^{nn'}_{mm'}(i, D)$ is derived from $\Pi^{nn'}_{mm'}(s, s', D)$ by replacing s by - i and s' /14
by i; $\Pi'^{nn'}_{mm'}(i, D)$ is derived from $\Pi^{nn'}_{mm'}(s, s', D)$ by replacing s with - 1 + 1 and s' with i + 1; $\Pi''^{nn'}_{mm'}(i, D)$ is derived from $\Pi^{nn'}_{mm'}(s, s', D)$ by replacing s by - i + 2 and s' by i + 2.

Taking all these remarks into consideration the final expansion of the main part of the perturbation function may be written as:

$$\left.\begin{aligned}\frac{a'}{\Delta} = \frac{1}{2}\sum \Pi^{nn'}_{mm'}\, a'A_i e^n e'^{n'} \cos\left[iL' - iL + m'M' + mM\right] \\ + \sigma^2 \sum \Pi'^{nn'}_{mm'}\, a'B_i e^n e'^{n'} \cos\left[(i+1)L' - (i-1)L + m'M' + mM\right] \\ + \sigma^4 \sum \Pi''^{nn'}_{mm'}\, a'C_i e^n e'^{n'} \cos\left[(i+2)L' - (i-2)L + m'M' + mM\right] \\ + \cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}\right| \qquad (8)$$

In order to derive the expansion of the whole perturbation function we must take into account the effect of its second term R_1. In the case of the action of an inner planet on an outer planet,

$$R_1 = -\frac{r' \cos H}{r^2}$$

or

$$r'R_1 = -\left(\frac{r'}{r}\right)^2 \left[\cos(W' - W) - \sigma^2 \cos(W' - W) + \sigma^2 \cos(W' + W)\right] ;$$

and for a circular orbit

$$a'R_1 = -\alpha^{-2}(1-\sigma^2)\cos(L' - L) - \alpha^{-2}\sigma^2 \cos(L' + L).$$

Hence, in order to take into account the effect of the second term of the perturbation function, it suffices to introduce the appropriate corrections into $a'A_{-1}$, $a'A_{+1}$, $a'B_0$, and into their derivatives. It is easy to note that in the case of the action of an inner planet on an outer planet these corrections will be:

$$\left.\begin{array}{ll}\delta a'A_{\pm 1} = -\alpha^{-2}(1-\sigma^2), & \delta a'B_0 = -\alpha^{-2}, \\ \delta Da'A_{\pm 1} = 2\alpha^{-2}(1-\sigma^2), & \delta Da'B_0 = 2\alpha^{-2}, \\ \cdots\cdots\cdots & \cdots\cdots \\ \delta D^m a'A_{\pm 1} = -(-2)^m \alpha^{-2}(1-\sigma^2), & \delta D^m a'B_0 = -(-2)^m \alpha^{-2}, \\ \cdots\cdots\cdots & \cdots\cdots\end{array}\right| \qquad (9)$$

$$
\left.
\begin{array}{ll}
\delta D_\sigma a' A_{\pm 1} = 2\alpha^{-2}\sigma, & \delta D_\sigma \sigma^2 a' B_0 = -2\alpha^{-2}\sigma, \\
\delta D_\sigma D a' A_{\pm 1} = -4\alpha^{-2}\sigma, & \delta D_\sigma D \sigma^2 a' B_0 = 4\alpha^{-2}\sigma, \\
\dots\dots\dots & \dots\dots\dots \\
\delta D_\sigma D^m a' A_{\pm 1} = -(-2)^{m+1}\alpha^{-2}\sigma, & \delta D_\sigma D^m \sigma^2 a' B_0 = +(-2)^{m+1}\alpha^{-2}\sigma, \\
\dots\dots\dots & \dots\dots\dots
\end{array}
\right|
$$

From Expansion (8) may easily be derived perturbation function derivatives which ordinarily also enter into the right sides of the motion equations.

Let us rewrite Expression (8) in a somewhat different form:

$$
\left.
\begin{aligned}
a'R = \frac{1}{2} \sum P^{nn'}_{mm'} e^n e'^{n'} \cos(iL' - iL + m'M' + mM) \\
+ \sum P'^{nn'}_{mm'} e^n e'^{n'} \cos[(i+1)L' - (i-1)L + m'M' + mM] \\
+ \sum P''^{nn'}_{mm'} e^n e'^{n'} \cos[(i+2)L' - (i-2)L + m'M' + mM \\
+ \dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\right\} \qquad (8')
$$

where

$$
\begin{aligned}
P^{nn'}_{mm'} &= \Pi^{nn'}_{mm'} a' A_i, \\
P'^{nn'}_{mm'} &= \Pi'^{nn'}_{mm'} \sigma^2 a' B_i, \\
P''^{nn'}_{mm'} &= \Pi''^{nn'}_{mm'} \sigma^4 a' C_i, \\
&\dots\dots\dots,
\end{aligned}
\qquad /15
$$

then

$$
\left.
\begin{aligned}
Da'R &= \frac{1}{2} \sum DP^{nn'}_{mm'} e^n e'^{n'} \cos[iL' - iL + m'M' + mM] \\
&+ \sum DP'^{nn'}_{mm'} e^n e'^{n'} \cos[(i+1)L' - (i-1)L + m'M' + mM] \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots; \\
a'D_\sigma R &= \frac{1}{2} \sum D_\sigma P^{nn'}_{mm'} e^n e'^{n'} \cos[iL' - iL + m'M' + mM] \\
&+ \sum D_\sigma P'^{nn'}_{mm'} e^n e'^{n'} \cos[(i+1)L' - (i-1)L + m'M' + mM] \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots; \\
a'\frac{\partial R}{\partial L'} &= \frac{1}{2} \sum -iP^{nn'}_{mm'} e^n e'^{n'} \sin[iL' - iL + m'M' + mM] \\
&+ \sum -(i+1) P'^{nn'}_{mm'} e^n e'^{n'} \sin[(i+1)L' - (i-1)L + m'M' + mM] \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots; \\
a'\frac{\partial R}{\partial L} &= \frac{1}{2} \sum iP^{nn'}_{mm'} e^n e'^{n'} \sin[iL' - iL + m'M' + mM] \\
&+ \sum (i-1) P'^{nn'}_{mm'} e^n e'^{n'} \sin[(i+1)L' - (i-1)L + m'M' + mM] \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\right\} \qquad (10)
$$

And, finally, the partial derivative with respect to time $D'_t R = \frac{\partial R}{\partial \rho'} \frac{d\rho'}{dt} + \frac{\partial R}{\partial W'} \frac{dw'}{dt}$, when only the time entering through the coordinates of the perturbed planet is taken into consideration, may also be written:

$$\frac{1}{n'} a' D'_t R = \frac{1}{2} \sum - (i + m') P^{nn'}_{mm'} e^n e'^{n'} \sin [iL' - iL + m'M' + mM]$$

$$+ \sum - (i + 1 + m') P'^{nn'}_{mm'} e^n e'^{n'} \sin [(i + 1) L' - (i - 1) L + m'M' + mM]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The derivatives of the operators with respect to log a and to σ, which enter into these expressions, are, as is easily seen,

$$DP^{nn'}_{mm'} = \Pi^{nn'}_{mm'} Da' A_i,$$

$$DP'^{nn'}_{mm'} = \Pi'^{nn'}_{mm'} D\sigma^2 a' B_i$$

$$\ldots\ldots\ldots\ldots$$

$$D_\sigma P^{nn'}_{mm'} = \Pi^{nn'}_{mm'} D_\sigma a' A_i,$$

$$D_\sigma P'^{nn'}_{mm'} = \Pi^{nn'}_{mm'} D_\sigma \sigma^2 a' B_i,$$

$$\ldots\ldots\ldots\ldots$$

The derivatives with respect to log a and to σ, of the coefficients of expansion of the circular motion are easily derived from Expression (3) /16

$$\left.\begin{aligned}
D^k a' A_i &= D^k c^i_1 - \frac{1}{2}\sigma^2 \left(D^k c^{i+1}_3 + D^k c^{i-1}_3\right) + \\
&\quad + \frac{3}{8}\sigma^4 \left(D^k c^{i+2}_5 + 4D^k c^i_5 + D^k c^{i-2}_5\right) - \ldots \\
D^k \sigma^2 a' B_i &= \frac{1}{2}\sigma^2 D^k c^i_3 - \frac{3}{4}\sigma^4 \left(D^k c^{i-1}_5 + D^k c^{i+1}_5\right) + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
D^k D_\sigma a' A_i &= -\sigma \left(D^k c^{i+1}_3 + D^k c^{i-1}_3\right) + \frac{3}{2}\sigma^3 \left(D^k c^{i+2}_5 + 4D^k c^i_5 + D^k c^{i-2}_5\right) - \ldots \\
D^k D_\sigma \sigma^2 a' B_i &= \sigma D^k c^i_3 - 3\sigma^3 \left(D^k c^{i-1}_5 + D^k c^{i+1}_5\right) + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (11)$$

Appendix I gives the symbolic expansion of the perturbation function for the zero, first, and second classes.

2. Verification and Computation of the Coefficients of the Newcomb Operators

When determining the numerical values of the coefficients of the operators, verification is of tremendous significance. It is necessary that the verification reveal not only the computational errors, but also possible misprints in the literal expressions of the coefficients given by Newcomb. Let us consider more throughly the methods of verification.

First verification. If in the expansion $\frac{a'}{\Delta_1}$, by the Newcomb method, we set e' = 0 and M' = M = 0, then we will obtain:

$$\begin{aligned}\frac{a'}{\Delta_1} &= \frac{1}{2}\sum\left[\Pi_0^0 + e\left(\Pi_1' + \Pi_{-1}'\right) + e^2\left(\Pi_2^2 + \Pi_0^2 + \Pi_{-2}^2\right) + \ldots\right] a'A_i \cos i(\Pi' - \Pi) \\ &+ \sigma^2\sum\left[\Pi_0'^0 + e\left(\Pi_1'^1 + \Pi_{-1}'^1\right) + e^2\left(\Pi_2'^2 + \Pi_0'^2 + \Pi_{-2}'^2\right) + \ldots\right] a'B_i \cos\left[(i+1)\Pi' - (i-1)\Pi\right] \\ &+ \sigma^4\sum\left[\Pi_0''^0 + e\left(\Pi_1''^1 + \Pi_{-1}''^1\right) + e^2\left(\Pi_2''^2 + \Pi_0''^2 + \Pi_{-2}''^2\right) + \ldots\right] a'C_i \cos\left[(i+2)\Pi' - (i-2)\Pi\right] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{12}$$

On the other hand

$$\frac{a'}{\Delta_1} = \left[1 - 2\alpha(1-e)\cos H_0 + \alpha^2(1-e)^2\right]^{-\frac{1}{2}},$$

where

$$\cos H_0 = \cos\Pi' \cos\Pi + \sin\Pi' \sin\Pi \cos J.$$

If in the expansion $\frac{a'}{\Delta}$, for the case of circular motion (e = e' = 0), we place M = M' = 0, we will have:

$$\left.\begin{aligned}\frac{a'}{\Delta_0} &= (1 - 2\alpha\cos H + \alpha^2)^{-\frac{1}{2}} = \sum G\cos(i'\Pi' - i\Pi) = \\ &= \sum\frac{1}{2}a'A_i\cos(i\Pi' - i\Pi) \\ &+ \sum\sigma^2 a'B_i\cos\left[(i+1)\Pi' - (i-1)\Pi\right] \\ &+ \sum\sigma^4 a'C_i\cos\left[(i+2)\Pi' - (i-2)\Pi\right] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \tag{13}$$

Let us expand $\frac{a'}{\Delta_1}$ into a series of the same sort: /17

$$\frac{a'}{\Delta_1} = \left[1 - 2\alpha(1-e)\cos H_0 + \alpha^2(1-e)^2\right]^{-\frac{1}{2}} = \sum G'\cos(i'\Pi' - i\Pi). \tag{14}$$

Expansion (14) may be derived from Expansion (13) by substituting α for α(1 - e) or log α + log (1 - e).

Let us express coefficients G' by G. For convenience we assume ε = - log (1 - e). Then:

$$G = \varphi(\lg\alpha), \quad G' = \varphi(\lg\alpha - \varepsilon).$$

We expand G' into a Taylor series in powers of

$$G' = \left(1 - \varepsilon D + \frac{\varepsilon^2}{2!}D^2 - \frac{\varepsilon^3}{3!}D^3 + \ldots\right)G \tag{15}$$

and we expand ε with respect to powers of e

$$\varepsilon = -\lg(1-e) = e + \frac{1}{2}e^2 + \frac{1}{3}e^3 + \ldots$$

We substitute ε in Expression (6) and, designating the coefficients of the different degrees of e by K, we obtain

$$G' = (1 + K_1 e + K_2 e^2 + \ldots)\, G.$$

Hence

$$\left.\begin{aligned}\frac{a'}{\Delta_1} = &\sum (1 + K_1 e + K_2 e^2 + \ldots)\frac{1}{2}a' A_i \cos(i\Pi' - i\Pi)\\ &+ \sum (1 + K_1 e + K_2 e^2 + \ldots)\sigma^2 a' B_i \cos[(i+1)\Pi' - (i-1)\Pi]\\ &+ \sum (1 + K_1 e + K_2 e^2 + \ldots)\sigma^4 a' C_i \cos[(i+2)\Pi' - (i-2)\Pi]\\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \quad (16)$$

For $\frac{a'}{\Delta_1}$ we will obtain two expansions: (12) and (16). Equating the terms with the same powers of e, σ^2, and identical arguments of the cosines, we derive:

$$\begin{aligned}
K_1 &= \Pi_1^1 + \Pi_{-1}^1, & K_1 &= \Pi_1'^1 + \Pi_{-1}'^1,\\
K_2 &= \Pi_2^2 + \Pi_0^2 + \Pi_{-2}^2, & K_2 &= \Pi_2'^2 + \Pi_0'^2 + \Pi_{-2}'^2,\\
K_3 &= \Pi_3^3 + \Pi_1^3 + \Pi_{-1}^3 + \Pi_{-3}^3, & K_3 &= \Pi_3'^3 + \Pi_1'^3 + \Pi_{-1}^3 + \Pi_{-3}'^3,\\
&\ldots\ldots\ldots\ldots, & &\ldots\ldots\ldots\ldots.
\end{aligned}$$

The values of K may be determined by the recursion relationship proposed by Newcomb:

$$nK_n = -DK''_{n-1},$$

where K" is the result of substituting D - 1 instead of D in K; $K_1 = -D$,

whence are also determined the remaining values of K;

$$\left.\begin{aligned}
K_1 &= -D,\\
2K_2 &= -D + D^2,\\
6K_3 &= -2D + 3D^2 - D^3,\\
24K_4 &= -6D + 11D^2 - 6D^3 + D^4,\\
120K_5 &= -24D + 50D^2 - 36D^3 + 10D^4 - D^5,\\
720K_6 &= -120D + 274D^2 - 225D^3 + 85D^4 - 15D^5 + D^6,\\
5040K_7 &= -720D + 1764D^2 - 1624D^3 + 735D^4 - 175D^5 + 21D^6 - D^7,\\
40320K_8 &= -5040D + 13068D^2 - 13132D^3 + 6769D^4 - 1960D^5 + 322D^6 - 28D^7 + D^8,
\end{aligned}\right\} \quad (17)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The operators $\Pi_{om'}^{on'}$ are also verified in a similar fashion: /18

$$K_1' = \Pi_{01}^{01} + \Pi_{0-1}^{01}, \qquad K_1' = \Pi_{01}'^{01} + \Pi_{0-1}'^{01},$$

$$K_2' = \Pi_{02}^{02} + \Pi_{00}^{02} + \Pi_{0-2}^{02}, \qquad K_2' = \Pi_{02}'^{02} + \Pi_{00}'^{02} + \Pi_{0-2}'^{02},$$

$$K_3' = \Pi_{03}^{03} + \Pi_{01}^{03} + \Pi_{0-1}^{03} + \Pi_{0-3}^{03}, \qquad K_3' = \Pi_{03}'^{03} + \Pi_{01}'^{03} + \Pi_{0-1}'^{03} + \Pi_{0-3}'^{03},$$

. . . . . .

K' is derived from the expressions for K by substituting D for - 1 - D:

$$\left.\begin{aligned}
K_1' &= 1 + D,\\
2K_2' &= 2 + 3D + D^2,\\
6K_3' &= 6 + 11D + 6D^2 + D^3,\\
24K_4' &= 24 + 50D + 35D^2 + 10D^3 + D^4,\\
120K_5' &= 120 + 274D + 225D^2 + 85D^3 + 15D^4 + D^5,\\
720K_6' &= 720 + 1764D + 1624D^2 + 736D^3 + 175D^4 + 21D^5 + D^6,\\
5040K_7' &= 5040 + 13068D + 13132D^2 + 6769D^3 + 1960D^4 + 322D^5 + 28D^6 + D^7,\\
40320K_8' &= 40320 + 109584D + 118124D^2 + 67284D^3 + 22449D^4 +\\
&\quad + 4536D^5 + 546D^6 + 36D^7 + D^8,
\end{aligned}\right\} \qquad (18)$$

. . . . . .

This method verifies only the coefficients of the even powers with subscript i. To verify the coefficients of the odd powers of the subscripts, a second verification is made.

Second verification. If in expansion $\frac{a'}{\Delta}$, by the Newcomb method, we set e' = 0 and M' = 0, we obtain

$$\frac{a'}{\Delta_1} = \frac{1}{2}\sum\Big(\Pi_0^0 \cos i\,[\Pi' - \Pi - M] + e\,\{\Pi_1^1 \cos\,[i\,(\Pi' - \Pi) - (i-1)\,M] +$$

$$\Pi_{-1}^1 \cos\,[i\,(\Pi' - \Pi) - (i+1)\,M]\} + e^2\{\Pi_2^2 \cos\,[i\,(\Pi' - \Pi) - (i-2)\,M] +$$

$$\Pi_0^2 \cos\,[i\,(\Pi' - \Pi) - iM] + \Pi_{-2}^2 \cos\,[i\,(\Pi' - \Pi) - (i+2)\,M] + \ldots\Big)\,a'A_i +$$

$$+ \sum\Big(\Pi_0'^0 \cos\,[(i+1)\,\Pi' - (i-1)\,\Pi - (i-1)\,M] +$$

$$+ e\,\{\Pi_1'^1 \cos\,[(i+1)\,\Pi' - (i-1)\,\Pi - (i-2)\,M] +$$

$$\Pi_{-1}'^1 \cos\,[(i+1)\,\Pi' - (i-1)\,\Pi - (i+2)\,M]\} + \ldots\Big)\,e^2 a' B_i + \ldots$$

Let us take the derivative of $\frac{a'}{\Delta_1}$ with respect to M and set M equal to 0. Then

$$a' \frac{\partial\left(\frac{1}{\Delta_1}\right)}{\partial M} = \frac{1}{2} \sum \left[i\Pi_0^0 + e(i-1)\Pi_1^1 + e(i+1)\Pi_{-1}^1 + e^2(i-2)\Pi_2^2 + e^2 i\Pi_0^2 + e^2(i+2)\Pi_{-2}^2 + \ldots \right] a' A_i \sin i(\Pi' - \Pi) + \sum \left[(i-1)\Pi_0'^1 + e(i-2)\Pi_1'^1 + ei\Pi_{-1}'^1 + e^2(i-3)\Pi_2'^2 + e^2(i-1)\Pi_0'^2 + e^2(i+1)\Pi_{-2}'^2 + \ldots \right] \sigma^2 a' B_i \sin\left[(i+1)\Pi' - (i-1) + \ldots \right. \tag{19}$$

or, taking Expression (16) into consideration

$$\left.\begin{aligned} a' \frac{\partial\left(\frac{1}{\Delta_1}\right)}{\partial M} &= \frac{1}{2} \sum i(1 + K_1 e + K_2 e^2 + \ldots) a' A_i \sin i(\Pi' - \Pi) + \\ &\sum (i-1)(1 + K_1 e + K_2 e^2 + \ldots) \sigma^2 a' B_i \sin[(i+1)\Pi' - (i-1)\Pi] + \ldots \\ &\frac{1}{2} \sum \left[e\left(-\Pi_1^1 + \Pi_{-1}^1\right) + e^2\left(-2\Pi_2^2 + 2\Pi_{-2}^2\right) + e^3\left(-3\Pi_3^3 - \Pi_1^3 + \Pi_{-1}^3 + 3\Pi_{-3}^3\right) + \ldots \right] a' A_i \sin i(\Pi' - \Pi) + \\ &\sum \left[e\left(-\Pi_1'^1 + \Pi_{-1}'^1\right) + e^2\left(-2\Pi_2'^2 + 2\Pi_{-2}'^2\right) + e^3\left(-3\Pi_3'^3 - \Pi_1'^3 + \Pi_{-1}'^3 + 3\Pi_{-3}'^3\right) + \ldots \right] \sigma^2 a' B_i \sin[(i+1)\Pi' - (i-1)\Pi] + \ldots \end{aligned}\right\} \tag{20}$$

For the case e' = 0, M' = 0, and M ≠ 0 Expression (12) has the form:

$$\frac{a'}{\Delta_1} = \sum G' \cos[i'\Pi' - i(\Pi + v)].$$

Taking the derivative with respect to M and setting M equal to zero we obtain:

$$a' \frac{\partial\left(\frac{1}{\Delta_1}\right)}{\partial M} = \sum iG' \left(\frac{\partial v}{\partial M}\right)_0 \sin(i\Pi' - i\Pi).$$

Since

$$\left(\frac{\partial v}{\partial M}\right)_{M=0} = 1 + 2e + \frac{5}{2}e^2 + 3e^3 + \ldots$$

and

$$G' = [1 + K_1 e + K_2 e^2 + \ldots] G,$$

then

$$a' \frac{\partial\left(\frac{1}{\Delta_1}\right)}{\partial M} = \frac{1}{2} \sum i(1 + K_1 e + K_2 e^2 + \ldots) a' A_i \sin i(\Pi' - \Pi) +$$

$$
\begin{gathered}
\sum (i-1)(1+K_1e+K_2e^2+\ldots)\sigma^2 a' B_i \sin[(i+1)\Pi'-(i-1)\Pi]+ \\
\frac{1}{2}\sum i\left[2e+\left(\frac{5}{2}+2K_1\right)e^2+\left(3+\frac{5}{2}K_1+2K_2\right)e^3+\ldots\right]a'A_i \sin i(\Pi'-\Pi)+ \\
+\sum (i-1)\left[2e+\left(\frac{5}{2}+2K_1\right)e^2+\right. \\
\left.\left(3+\frac{5}{2}K_1+2K_2\right)e^3+\ldots\right]\sigma^2 a' B_i \sin[(i+1)\Pi'-(i-1)\Pi]+ \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}
\tag{21}
$$

Setting the coefficients of the identical powers of e, σ, and of the identical arguments of the sines in Expressions (20) and (21) we obtain /20

$$
\begin{aligned}
&iH_1=2i=-\Pi_1^1+\Pi_{-1}^1, && (i-1)H_1=-\Pi_1'^1+\Pi_{-1}'^1,\\
&iH_2=i\left(\frac{5}{2}+2K_1\right)=-2\Pi_2^2+2\Pi_{-2}^2, && (i-1)H_2=-2\Pi_2'^2+2\Pi_2'^2,\\
&iH_3=-3\Pi_3^3-\Pi_1^3+\Pi_{-1}^3+3\Pi_{-3}^3, && (i-1)H_3=-3\Pi_3'^3-\Pi_1'^3+\Pi_{-1}'^3+3\Pi_{-3}'^3,\\
&\ldots\ldots\ldots\ldots, && \ldots\ldots\ldots\ldots\\
&iH_k=-k\Pi_k^k-(k-2)\Pi_{k-2}^k-\ldots+(k-2)\Pi_{-k+2}^k+k\Pi_{-k}^k, && \\
&\ldots\ldots\ldots\ldots, && \ldots\ldots\ldots\ldots
\end{aligned}
$$

Substituting corresponding values of K in H we obtain

$$
\begin{aligned}
H_1&=2\\
2H_2&=5-4D,\\
2H_3&=6-7D+2D^2,\\
24H_4&=81-118D+54D^2-8D^3,\\
24H_5&=90-149D+88D^2-22D^3+2D^4,\\
240H_6&=575-1791D+1240D^2-410D^3+65D^4-4D^5,\\
240H_7&=3150-6226D+4853D^2-1920D^3+410D^4-45D^5+2D^6,\\
40320H_8&=187425-395232D+336854D^2-151984D^3+39410D^4-5908D^5+476D^6-16D^7,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

The operators pertaining to the outer planet are also verified in a similar fashion. The values of H' are derived from H by substituting D for - 1 - D. Here

$$
iH_n'=n\Pi_{0\,n}^{0\,n}+(n-1)\Pi_{0\,n-2}^{0\,n}+\ldots-(n-1)\Pi_{0-(n-2)}^{0\,n}-n\Pi_{0-n}^{0\,n}
$$

and

$$
\begin{aligned}
H_1'&=2,\\
2H_2'&=9+4D,
\end{aligned}
$$

$$2H_3' = 15 + 11D + 2D^2,$$

$$24H_4' = 261 + 250D + 78D^2 + 8D^3,$$

$$24H_5' = 351 + 399D + 166D^2 + 30D^3 + 2D^4,$$

$$240H_6' = 4485 + 5781D + 2900D^2 + 710D^3 + 85D^4 + 4D^5,$$

$$720H_7' = 16605 + 23568D + 13553D^2 + 4050D^3 + 665D^4 + 57D^5 + 2D^6,$$

$$40320H_8' = 1117305 + 1715040D + 1095822D^2 + 378784D^3 + 76650D^4 + 9100D^5 + \\ + 588D^6 + 16D^7,$$

. . . . . .

Verification of Complex Operators. We have

$$\left.\begin{aligned} K_1 &= \Pi_n^n + \Pi_{n-2}^n + \ldots \Pi_{-n}^n, \\ K_1' &= \Pi_{on'}^{on'} + \Pi_{o\,n'-2}^{o\,n'} + \ldots \Pi_{o-n'}^{on'}. \end{aligned}\right\} \tag{22}$$

If we multiply these two equalities we obtain /21

$$K_n K_{n'} \quad \sum_{mm'} \Pi_{mm'}^{nn'}, \quad \text{where } m = n,\ n-2,\ \ldots\ -n,\ m' = n',\ n'-2,\ \ldots\ -n'.$$

$$K_1K_1' = -D - D^2,$$

$$2K_1K_2' = -2D - 3D^2 - D^3,$$

$$6K_1K_3' = -6D - 11D^2 - 6D^3 - D^4,$$

$$24K_1K_4' = -24D - 50D^2 - 35D^3 - 10D^4 - D^5,$$

$$120K_1K_5' = -120D - 274D^2 - 225D^3 - 85D^4 - 15D^5 - D^6,$$

$$720K_1K_6' = -720D - 1764D^2 - 1624D^3 - 735D^4 - 175D^5 - 21D^6 - D^7,$$

. . . . . .,

$$2K_2K_1' = -D + D^3,$$

$$4K_2K_2' = -2D - D^2 + 2D^3 + D^4,$$

$$12K_2K_3' = -6D - 5D^2 + 5D^3 + 5D^4 + D^5,$$

$$48K_2K_4' = -24D - 26D^2 + 15D^3 + 25D^4 + 9D^5 + D^6,$$

$$240K_2K_5' = -120D - 154D^2 + 49D^3 + 140D^4 + 70D^5 + 14D^6 + D^7,$$

. . . . . .,

$$6K_3K_1' = -2D + D^2 + 2D^3 - D^4,$$

$$12K_3K_2' = -4D + 5D^3 - D^5,$$

$$36K_3K_3' = -12D - 4D^2 + 15D^3 + 5D^4 - 3D^5 - D^6,$$

$$144K_3K_4' = -48D - 28D^2 + 56D^3 + 35D^4 - 7D^5 - 7D^6 - D^7,$$

. . . . . .,

$$74K_4K_1' = -6D + 5D^2 + 5D^3 - 5D^4 + D^5,$$
$$48K_4K_2' = -12D + 4D^2 + 15D^3 - 5D^4 + D^6,$$
$$144K_4K_3' = -36D + 49D^2 - 6D^4 - 14D^5 + D^7,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$120K_5K_1' = -24D + 26D^2 + 15D^3 - 24D^4 + 9D^5 - D^6,$$
$$240K_5K_2' = -48D + 28D^2 + 91D^3 - 35D^4 - 7D^5 + 7D^6 - D^7,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$720K_6K_1' = -120D + 154D^2 + 49D^3 - 140D^4 + 70D^5 - 14D^6 + D^7,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Third Verification. If we replace D in the complex operators $\Pi_{mm'}^{nn'}(i)$ by $-1-D$, then we obtain $\Pi_{m'm}^{n'n}(-i)$, which in turn equals $\Pi_{-m'-m}^{n'n}(i)$.

These verifications completely guarantee the correctness of the coefficients of the operators.

In Newcomb's method the expansion of the perturbation function is reduced to computing the operators. When we must have individual terms of the expansion for any single particular case, we may take advantage of methods developed for numerical determination of the coefficients of the expansion, e.g., Innes's tables. In our case where it was necessary to compute operators for several expansions of the perturbation function (Pluto-Jupiter, Pluto-Saturn, etc), it was decided to compile a table of numerical values of the operator coefficients for the expansion in mean anomalies resembling the table given by Newcomb for expansion in eccentric anomalies. /22

The coefficients in the simple operators were derived from expressions given by Newcomb.*

The coefficients of the complex operators were determined by multiplying the corresponding coefficients of the simple operators. Verification of the simple operators was accomplished by the first and second method. Verification of the complex operators was accomplished by Formulas (22); moreover, in the complex operators one of the lines was computed directly from Newcomb's formulas. Thus we successfully avoided the cumbersome third verification. Tables of the operator coefficients were obtained as follows:

$\Pi_{mm'}^{nn'}$ for the values
$$i = \text{from } -7 \text{ to } +7,$$
$$2n + n' = \text{from } 0 \text{ to } +7,$$
$$2m + m' = \text{from } 0 \text{ to } +7;$$

$\Pi_{mm'}^{'nn'}$ for the values
$$i = \text{from } -5 \text{ to } +5,$$
$$2n + n' = \text{from } 0 \text{ to } +5,$$
$$2m + m' = \text{from } -5 \text{ to } +5;$$

*In Appendix IV the correct values of the algebraic expressions of several Newcomb operators are given.

$\Pi^{nn'}_{mm'}$ for the values

$$\begin{aligned} i &= \text{from } -3 \text{ to } +3, \\ 2n + n' &= \text{from } -3 \text{ to } +3, \\ 2m + m' &= \text{from } 0 \text{ to } +3. \end{aligned}$$

3. Computing the Derivatives of Laplace Coefficients

The values $a'A_i$, $a'B_i$, $a'C_i$, ... and their derivatives with respect to log α, which are linear functions of the Laplace coefficients b^i_n or, more accurately, of the values $c^i_n = \alpha^{\frac{n-1}{2}} b^i_n$, and, respectively, their derivatives with respect to log α, enter into the expansion of the perturbation function. The Laplace coefficients, and consequently c^i_n may easily be derived from the tables of Brown and Brouwer (1932). These tables with the argument $p = \frac{\alpha^2}{(1-\alpha^2)}$ give the values of $\log G^i_{\frac{n}{2}}$ connected to b^i_n by the relationship

$$b^i_n = \frac{\alpha^i}{(1-\alpha^2)^{\frac{n}{2}}} G^i_{\frac{n}{2}}$$

or, since we need c^i_n,

$$c^i_n = \frac{\alpha^{\frac{n-1}{2}} \alpha^i}{(1-\alpha^2)^{\frac{n}{2}}} G^i_{\frac{n}{2}}.$$

To determine the derivatives of the Laplace coefficients there are several methods, two of which, those of Newcomb (1880) and of Innes (1909), are the most interesting. Practical comparison of the two methods has shown the clear advantage of Innes' method, especially for large values of α. Here there is no need to occupy ourselves with the detailed deduction of the corresponding formulas. Therefore we shall limit ourselves to adducing a system of formulas which gives all the necessary derivatives and their verification.

The Brown and Brouwer tables will give the values of c^i_n. To determine the first derivative we have two formulas, one of which serves as the verification:

$$Dc^i_1 = ic^i_1 + (1+2i)p\left[c^i_1 - \frac{c^{i+1}_1}{\alpha}\right], \tag{23}$$

where /23

$$p = \frac{\alpha^2}{1-\alpha^2}$$

and

$$2Dc^i_1 = \left(\frac{1}{\alpha} - \alpha\right) c^i_3 - c^i_1. \tag{23}$$

For the second derivative we have formula:

$$\left(D^2-i^2\right) c_1^i = p\left[2Dc_1^i + c_1^i\right].$$

We successively take the derivatives with respect to log α, noting that $Dp = 2p(1 + p)$, and derive (for brevity we omit $c_{\frac{1}{n}}^{i}$):

$$\left.\begin{aligned}
&D(D^2-i^2)=p\,[4D^2+5D+2(1-i^2)],\\
&D^2(D^2-i^2)=p\,[6D^3+13D^2+(12-4i^2)D+4(1-i^2)],\\
&D^3(D^2-i^2)=p\,[8D^4+25D^3+(38-6i^2)D^2+(28-12i^2)D+8(1-i^2)],\\
&D^4(D^2-i^2)=p\,[10D^5+41D^4+(88-8i^2)D^3+(104-24i^2)D^2+(64-32i^2)D+\\
&\qquad 16(1-i^2)],\\
&D^5(D^2-i^2)=p\,[12D^6+61D^5+(170-10i^2)D^4+(280-40i^2)D^3+\\
&\qquad (272-80i^2)D^2+(144-80i^2)D+32(1-i^2)],\\
&D^6(D^2-i^2)=p\,[14D^7+85D^6+(292-12i^2)D^5+(620-60i^2)D^4+\\
&\qquad (832-160i^2)D^3+(688-240i^2)D^2+(320-192i^2)D+64(1-i^2)],\\
&D^7(D^2-i^2)=p\,[16D^8+113D^7+(462-14i^2)D^6+(1204-84i^2)D^5+\\
&\qquad (2072-280i^2)D^4+(2352-560i^2)D^3+(1696-672i^2)D^2+\\
&\qquad (704-448i^2)D+128(1-i^2)],\\
&D^8(D^2-i^2)=p\,[18D^9+145D^8+(688-16i^2)D^7+(2128-112i^2)D^6+\\
&\qquad (4480-448i^2)D^5+(6496-1120i^2)D^4+(6400-1792i^2)D^3+\\
&\qquad (4096-1792i^2)D^2+(1536-1024i^2)D+256(1-i^2)],\\
&D^9(D^2-i^2)=p\,[20D^{10}+181D^9+(978-18i^2)D^8+(3504-144i^2)D^7+\\
&\qquad (8736-672i^2)D^6+(15456-2016i^2)D^5+(19392-4032i^2)D^4+\\
&\qquad (16896-5376i^2)D^3+(9728-4608i^2)D^2+(3328-2304i^2)D+\\
&\qquad 512(1-i^2)],\\
&D^{10}(D^2-i^2)=p\,[22D^{11}+221D^{10}+(1340-20i^2)D^9+(5460-180i^2)D^8+\\
&\qquad (15744-960i^2)D^7+(32928-3360i^2)D^6+(50304-8064i^2)D^5+\\
&\qquad (55680-13440i^2)D^4+(43520-15360i^2)D^3+\\
&\qquad (22784-11520i^2)D^2+(7168-5120i^2)D+1024(1-i^2),\\
&D^{11}(D^2-i^2)=p\,[24D^{12}+265D^{11}+(1782-22i^2)D^{10}+(8140-220i^2)D^9+\\
&\qquad (26664-1320i^2)D^8+(64416-5280i^2)D^7+(116160-14784i^2)D^6+\\
&\qquad (156288-29568i^2)D^5+(154880-42240i^2)D^4+\\
&\qquad (109824-42240i^2)D^3+(52736-28160i^2)D^2+\\
&\qquad (15360-11264i^2)D+2048(1-i^2)],\\
&D^{12}(D^2-i^2)=p\,[26D^{13}+313D^{12}+(2312-24i^2)D^{11}+(11704-264i^2)D^{10}+\\
&\qquad (42944-1760i^2)D^9+(117744-7920i^2)D^8+\\
&\qquad (244992-25344i^2)D^7+(388608-59136i^2)D^6+\\
&\qquad (467456-101376i^2)D^5+(419584-126720i^2)D^4+\\
&\qquad (272384-112640i^2)D^3+(120832-67584i^2)D^2+
\end{aligned}\right\}\quad(24)$$

/24 & 25

$$(32768-24576i^2)D+4096(1-i^2)],$$

$$D^{13}(D^2-i^2)=p[28D^{14}+365D^{13}+(2938-26i^2)D^{12}+(16328-312i^2)D^{11}+$$
$$(66352-2288i^2)D^{10}+(203632-11440i^2)D^9+$$
$$(480480-41184i^2)D^8+(878592-109824i^2)D^7+$$
$$(1244672-219648i^2)L^6+(1354496-329472i^2)D^5+$$
$$(1111552-366080i^2)D^4+(665600-292864i^2)D^3+$$
$$(274432-159744i^2)D^2+(69632-53248i^2)D+8192(1-i^2)],$$

$$D^{14}(D^2-i^2)=p[30D^{15}+421D^{14}+(3668-28i^2)D^{13}+(22204-364i^2)D^{12}+$$
$$(99008-2912i^2)D^{11}+(336336-16016i^2)D^{10}+$$
$$(887744-64064i^2)D^9+(1839552-192192i^2)D^8+$$
$$(3001856-439296i^2)D^7+(3843840-768768i^2)D^6+$$
$$(3820544-1025024i^2)D^5+$$
$$(2888704-1025024i^2)D^4+(1605632-745472i^2)D^3+$$
$$(618496-372736i^2)D^2+(147456-114688i^2)D+16384(1-i^2)],$$

$$D^{15}(D^2-i^2)=p[32D^{16}+481D^{15}+(4510-30i^2)D^{14}+(29540-420i^2)D^{13}+$$
$$(143416-3640i^2)D^{12}+(534352-21840i^2)D^{11}+$$
$$(1560416-96096i^2)D^{10}+(3615040-320320i^2)D^9+$$
$$(6680960-823680i^2)D^8+(9847552-1647360i^2)D^7+$$
$$(11508224-2562560i^2)D^6+(10529792-3075072i^2)D^5+$$
$$(7383040-2795520i^2)D^4+(3829760-1863680i^2)D^3+$$
$$(1384448-860160i^2)D^2+(311296-245760i^2)D+32768(1-i^2)],$$

$$D^{16}(D^2-i^2)=p[34D^{17}+545D^{16}+(5472-32i^2)D^{15}+(38560-480i^2)D^{14}+$$
$$(202496-4480i^2)D^{13}+(821184-29120i^2)D^{12}+$$
$$(2629120-139776i^2)D^{11}+(6735872-512512i^2)D^{10}+$$
$$(13911040-1464320i^2)D^9+(23209472-3294720i^2)D^8+$$
$$(31203328-5857280i^2)D^7+(33546240-8200192i^2)D^6+$$
$$(28442624-8945664i^2)D^5+(18595840-7454720i^2)D^4+$$
$$(9043968-4587520i^2)D^3+(3080192-1966080i^2)D^2+$$
$$(655360-524288i^2)D+65536(1-i^2)],$$

$$D^{17}(D^2-i^2)=p[36D^{18}+613D^{17}+(6562-34i^2)D^{16}+(49504-544i^2)D^{15}+$$
$$(279616-5440i^2)D^{14}+(1226176-38080i^2)D^{13}+$$
$$(4271488-198016i^2)D^{12}+(11994112-792064i^2)D^{11}+$$
$$(27382784-2489344i^2)D^{10}+(51031552-6223360i^2)D^9+$$
$$(77622272-12446720i^2)D^8+(95952896-19914752i^2)D^7+$$
$$(95535104-25346048i^2)D^6+(75481088-25346048i^2)D^5+$$
$$(46235648-19496960i^2)D^4+(21168128-11141120i^2)D^3+$$
$$(6815744-4456448i^2)D^2+(1376256-1114112i^2)D+$$
$$131072(1-i^2)],$$

$$D^{18}(D^2-i^2)=p[38D^{19}+685D^{18}+(7788-36i^2)D^{17}+(62628-612i^2)D^{16}+$$
$$(378624-6528i^2)D^{15}+(1785408-48960i^2)D^{14}+$$

$$
\left.\begin{array}{l}
(6723840-274176i^2)\,D^{13}+(20537088-1188096i^2)\,D^{12}+\\
(51371008-4073472i^2)\,D^{11}+(105797120-11202048i^2)\,D^{10}+\\
(179685376-24893440i^2)\,D^{9}+(251197440-44808192i^2)\,D^{8}+\\
(287440896-65185552i^2)\,D^{7}+(266551296-76038144i^2)\,D^{6}+\\
(197197824-70189056i^2)\,D^{5}+(113639424-50135040i^2)D^{4}+\\
(49152000-26738688i^2)\,D^{3}+(15007744-10027008i^2)\,D^{2}+\\
(2883584-2359296i^2)\,D+262144\,(1-i^2)].
\end{array}\right\}
$$

We use these formulas only to determine $D^m c_1^0$ and $D^m c_1^1$, for the verification of which serve the relationships:

$$
\left.\begin{array}{c}
Dc_1^0=\alpha Dc_1^1,\\
(D-1)\,Dc_1^0=\alpha D^2 c_1^1,\\
(D-1)^2 Dc_1^0=\alpha D^3 c_1^1,\\
\cdots\cdots\cdots\cdots\\
(D-1)^m Dc_1^0=\alpha D^{m+1} c_1^1,\\
\cdots\cdots\cdots\cdots
\end{array}\right\} \tag{25}
$$

After all values Dc_1^i, $D^m c_1^0$, and $D^m c_1^1$, are determined, we may also find the remaining values $D^m c_1^i$, from the recursion formula:

$$
D^{m+2}c_1^{i+2}=D^m\left\{\left[D^2+D-i\,(i+1)\right]c_1^i-\left[D-(i+1)\,(i+2)\right]c_1^{i+2}\right\}. \tag{26}
$$

To determine $D^m c_n^i$ when $n > 1$ we have the following two recursion formulas, one of which may be used for verification:

$$
\left.\begin{array}{l}
n^2 D^m c_{n+2}^{i+1}=D^m\left[D^2+D-\left(i+\frac{n-1}{2}\right)\left(i+\frac{n+1}{2}\right)\right]c_n^i,\\
n^2 D^m c_{n+2}^{i-1}=D^m\left[D^2+D-\left(i-\frac{n-1}{2}\right)\left(i-\frac{n+1}{2}\right)\right]c_n^i.
\end{array}\right\} \tag{26'}
$$

Thus all the stages of the computations are verified and there is no need to conduct computations at second hand.

4. Use of Analytic Computers for Expanding the Perturbation Function and Its Derivatives

Expansion of the perturbation function is one of the most laborious parts of the work in constructing analytical theories of planetary motions. Analytical computers can render great aid in performing the computational part of this work. Of particularly great effect is the use of these computing means for

expanding the perturbation function by Newcomb's method. The formulas giving the coefficients of expansions in this case are such that they require no special conversion for the computer. The mass nature of the homogeneous operations make possible productive utilization of the analytical computer and comparatively rapid obtainment of the necessary results.

Let us recall the form of expansion of the perturbation function and its derivatives:

$$\left.\begin{aligned}
a'R &= \sum P^{(\kappa)nn'}_{mm'} e^n e'^{n'} \cos[(i+k)L' - (i-k)L + m'M' + mM],\\
a'DR &= \sum DP^{(\kappa)nn'}_{mm'} e^n e'^{n'} \cos[(i+k)L' - (i-k)L + m'M' + mM],\\
a'D_\sigma R &= \sum D_\sigma P^{(\kappa)nn'}_{mm'} e^n e'^{n'} \cos[(i+k)L' - (i-k)L + m'M' + mM],\\
a'\frac{\partial R}{\partial L'} &= \sum -(i+k) P^{(\kappa)nn'}_{mm'} e^n e'^{n'} \sin[(i+k)L' - (i-k)L + m'M' + mM],\\
a'\frac{\partial R}{\partial L} &= \sum -(-i+k) P^{(\kappa)nn'}_{mm'} e^n e'^{n'} \sin[(i+k)L' - (i-k)L + m'M' + mM],\\
\frac{1}{n'} a'D'_t R &= \sum -(i+k+m') P^{(\kappa)nn'}_{mm'} e^n e'^{n'} \sin[(i+k)L' - (i-k)L + m'M' + mM],
\end{aligned}\right\} \quad (*)$$

where k is the number of the operator class, and

$$a'D'R = -a'R - a'DR.$$

The work falls into several stages.

1. Computation of $P^{(\kappa)nn'}_{mm'}$, $DP^{(\kappa)nn'}_{mm'}$, $D_\sigma P^{(\kappa)nn'}_{mm'}$.

2. Computation of the expansion coefficients $a'R$, $a'DR$, $a'D_\sigma R$ $\cos[(i+k)L' - (i-k)L + m'M' + mM]$.

3. Computation of expansion coefficients $a'\frac{\partial R}{\partial L'}$, $a'\frac{\partial R}{\partial L}$, $\frac{1}{n'}a'D'_t R$ with $\sin[(i+k)L' - (i-k)L + m'M' + mM]$.

4. Transfer from expansions with respect to arguments $[(i+k)L' - (i-k)L + m'M' + mM]$ to expansions over the arguments $(j'M' + jM)$.

We will pause separately on each stage.

1. As was shown in Section 1 of this chapter:

$$\begin{aligned}
P^{nn'}_{mm'} &= \Pi^{nn'}_{mm'} a'A_i = p_0 a'A_i + p_1 Da'A_i + \ldots + p_{n+n'} D^{n+n'} a'A_i,\\
P'^{nn'}_{mm'} &= \Pi'^{nn'}_{mm'} \sigma^2 a'B_i = p'_0 \sigma^2 a'B_i + \ldots + p'_{n+n'} D^{n+n'} \sigma^2 a'B_i,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
DP^{nn'}_{mm'} &= \Pi^{nn'}_{mm'} Da'A_i = p_0 Da'A_i + \ldots + p_{n+n'} D^{n+n'+1} a'A_i,\\
DP'^{nn'}_{mm'} &= \Pi'^{nn'}_{mm'} D\sigma^2 a'B_i = p'_0 D\sigma^2 a'B_i + \ldots + p'_{n+n'} P^{n+n'+1} \sigma^2 a'B_i,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

$$D_\sigma P^{nn'}_{mm'} = \Pi^{nn'}_{mm'} D_\sigma a' A_i = p_0 D_\sigma a' A_i + \ldots + p_{n+n'} D^{n+n'} D_\sigma a' A_i,$$

$$D_\sigma P'^{nn'}_{mm'} = \Pi'^{nn'}_{mm'} D_\sigma \sigma^2 a' B_i + p'_0 D_\sigma \sigma^2 a' B_i + \ldots + p'_{n+n'} D^{n+n'} D_\sigma \sigma^2 a' B_i,$$

. . . . . .

To compute $P^{(\kappa)nn'}_{mm'}$, $DP^{(\kappa)nn'}_{mm'}$ and $D_\sigma P^{(\kappa)nn'}_{mm'}$ we need tables of operator coefficients P_0, P_1 . . . and tables of the derivatives of the expansion coefficients of circular motion $D^l a' A_i$, $D^l D_\sigma a' A_i$, $D^l \sigma^2 a' B_i$, $D^l D_\sigma \sigma^2 a' B_i$ . . .

Tables of coefficients of operators for various values of subscript i have been computed and are given in Appendix II of the present work. The tables of values of expansion coefficient derivatives for circular motion were previously computed by hand for each combination of planets individually from Formulas (11) with an accuracy for the given specific problem. With such initial tables we may proceed to creating the basic blocks. /27

The block of coefficients of operators 0_1 are punched according to the following pattern:

O_1

Columns	1	2—3	4	5	6	7—8	9—10	11—12	13—14	I	II	III	IV	V	VI	VII	VIII
Content	k	No. of Operator	n	n'	$n+n'$	m	m'	$m+m'$	i	First Column Free Term p_0	Second Column Coefficient with D p_1	Third Column Coefficient with D^2 p_2	Fourth Column Coefficient with D^3 p_3	Fifth Column Coefficient with D^4 p_4	Sixth Column Coefficient with D^5 p_5	Seventh Column Coefficient with D^6 p_6	Eighth Column Coefficient with D^7 p_7

We form the block of factors M_1 on the following model:

M_1

1	13	15	II	III	IV
k	$\lvert i \rvert$	$l+1$ (No. of Operator)	$D^l a' A_i$ $D^l \sigma^2 a' B_i$ . . .	$D^{l+1} a' A_i$ $D^{l+1} \sigma^2 a' B_i$ . .	$D^l D_\sigma a' A_i$ $D^l D_\sigma \sigma^2 a' B_i$ . .

On the basis of the block of coefficients of operators 0_1 we form the block of products 0_2. For this we take eight copies of block 0_1 on the reproducer, in

which the data of columns 1 through 14 remain unchanged, while in column 15 we punch the number of the copy and into field I we transfer fields I, II ..., and VIII, according to each copy.

O_2

1	2—3	4	5	6	7—8	9—10	11—12	13—14	15	I	II	III	IV	V	VI	VII
k	No. of Operator	n	n'	$n+n'$	m	m'	$m+m'$	i	$l+1$	p_l	$D^l a' A_i$	$D^{l+1} a' A_i$	$D^l D_\sigma a' A_i$	$p_l D^l a' A_i$	$p_l D^{l+1} a' A_i$	$p_l D^l D_\sigma a' A_i$
										p_l'	$D^l \sigma^2 a' B_i$	$D^{l+1} \sigma^2 a' B_i$	$D^l D_\sigma \sigma^2 a' B_i$	$p_l' D^l \sigma^2 a' B_i$	$p_l' D^{l+1} \sigma^2 a' B_i$	$p_l' D^l D_\sigma \sigma^2 a' B_i$
											.....	.....	.....	.....	.....	.....

We sort blocks 0_2 and M_1 together with respect to i (column 13), L + 1, k and transfer field II, III, and IV on the reproducer from block M_1 to block 0_2.

After this we pass 0_2 through the multiplier three times to get the multiplications

$$\begin{aligned} \text{I} \times \text{II} &= \text{V}, \\ \text{I} \times \text{III} &= \text{VI}, \\ \text{I} \times \text{IV} &= \text{VII} \end{aligned}$$

and we punch the products in fields V, VI, and VII.

We sort the block of products obtained with respect to subscripts i, operator number, k and pass them through the tabulator to sum fields V, VI, and VII, with respect to i. /28

The tabulogram will contain the corresponding values $P_{mm'}^{(k)\,nn'}$, $DP_{mm'}^{(k)\,nn'}$, $D_\sigma P_{mm'}^{(k)\,nn'}$.

2. We proceed to the second stage of the computations.

We punch the derived values of $P_{mm'}^{(k)\,nn'}$, $DP_{mm'}^{(k)\,nn'}$, $D_\sigma P_{mm'}^{(k)\,nn'}$ according to the pattern adjoined. Let us designate this block by 0_3.

O_3

1	2—3	4	5	6	7—8	9—10	11—12	13—14	I	II	III	IV	V	VI	VII
k	No. of Operator	n	n'	$n+n'$	m	m'	$m+m'$	i	$P_{mm'}^{(k)nn'}$	$DP_{mm'}^{(k)nn'}$	$D_\sigma P_{mm'}^{(k)nn'}$	$e^n e'^{n'}$	$e^n e'^{n'} P_{mm'}^{(k)nn'}$	$e^n e'^{n'} DP_{mm'}^{(k)nn'}$	$e^n e'^{n'} D_\sigma P_{mm'}^{(k)nn'}$

We will compile the block of eccentricities M_2 on the model:

M_2

4	5	6	IV
n	n'	$n+n'$	$e^n e'^{n'}$

We jointly sort blocks 0_3 and M_2 with respect to n and n' and transfer values 1 from block M_2 to 0_3.

We pass 0_3 through the multiplier three times for the multiplication:

I X IV = V,
II X IV = VI,
III X IV = VII,

and we punch the result in fields V, VI, and VII.

We sort the 0_3 with respect to i, m', m, and k and pass fields V, VI, and VII through the tabulator for summation with respect to i. Using the overthrow from one counter to the next, we simultaneously derive ΣV, Σ—(V + VI), ΣVI on the tabulogram.

Thus the tabulogram will give us expansion coefficient a'R, a'D'R, $A'D_\sigma R$, with cos [(i + k)L' - (i - k)L + m'M' + mM].

3. In order to derive $a'\frac{\partial R}{\partial L'}$, $a'\frac{\partial R}{\partial L}$, $\frac{1}{n'}\ a'D'_t R$, we must manually finish computing on the tabulogram the values i + k, - i + k, i + m' + k, - i + m + k (minor computations not significant enough to be conducted on the computers), and we must punch the data from the tabulogram according to the pattern:

O_4

1	2—3	4—5	6—8	9—11	12—13	I	II	III	IV	V	VI
k	$i+k$	$-i+k$	$i+m'+k$	$-i+m+k$	i	$a'R$	$a'D'R$	$a'D_\sigma R$	$a'\frac{\partial R}{\partial L'}$	$a'\frac{\partial R}{\partial L}$	$\frac{1}{n'}a'D'_t R$

We pass the derived block 0_4 through the multiplier for the multiplication: /29

I X (i + k) = IV,
I X (- i + k) = V,
I X (i + m' + k) = VI.

The signs of fields IV, V, and VI depend on the combinations of signs of the factors, and it is to be remembered that the sine must be reversed, as is evident from formula (*). Thus, fields IV, V, and VI will give the expansion coefficients $a' \frac{\partial R}{\partial L'}$, $a' \frac{\partial R}{\partial L}$, $\frac{1}{n'} a' D_t' R$ with respect to sin [(i + k)L' - (i - k)l + m'M' + mM].

4. Let us pass from expansions with respect to argument [(i + k)L' - (i - k)L + m'M' + mM] to expansion over the argument (j'M' + jM).

The common term of the expansions a'R, a'D'R, $A'D_\sigma R$ has the form:

$$H \cos [(i+k+m') M' - (i-k-m) M + (i+k) \Pi' - (i-k) \Pi],$$

and if we proceed to expand over the argument (j'M' = jM), then the common term will become:

$$H \cos [(i+k) \Pi' - (i-k) \Pi] \cos [(i+k+m') M' - (i-k-m) M] - H \sin [(i+k) \Pi' - (i-k) \Pi] \sin [(i+k+m') M' - (i-k-m) M].$$

Similarly, for $a' \frac{\partial R}{\partial L'}$, $a' \frac{\partial R}{\partial L}$, $\frac{1}{n'} a' D_t' R$ we have

$$H' \sin [(i+k+m') M' - (i-k-m) M + (i+k) \Pi' - (i-k) \Pi] = H' \sin [(i+k) \Pi' - (i-k) \Pi] \cos [(i+k+m') M' - (i-k-m) M] + H' \cos [(i+k) \Pi' - (i-k) \Pi] \sin [(i+k+m') M' - (i-k-m) M].$$

Taking these formulas into consideration let us create a tabular block M_3 consisting of $\pm \frac{\sin}{\cos} [(i+k) \Pi' - (i-k) \Pi]$ after the pattern:

M_3				
2—3	4—5	14	15	IV
$i+k$	$-i+k$	1 or 2	1 or 2	$\pm \frac{\sin}{\cos} [(i+k) \Pi' - (i-k) \Pi]$

In column 14 we punch 1 or 2 depending on whether it is a matter of expansion $a' \frac{\partial R}{\partial L'}$, $a' \frac{\partial R}{\partial L}$, $\frac{1}{n'} a' D_t' R$ or $a'R$, $a'D'R$, $a'D_\sigma R$.

In column 15 we punch 1 or 2 depending on whether the final expansion will be obtained with respect to sin (j'M' + jM) or cos (j'M' + jM).

Depending on the combinations of punches in column 14 and 15 there will be in field IV either $\pm \sin [(i+k) \Pi' - (i-k) \Pi]$ or $\pm \cos [(i+k) \Pi' - (i-k) \Pi]$.

We make four copies of block 0_4 on the reproducer and form block 0_5. Into the first and second copies we transfer

$$1-12 \rightarrow 1-12$$
$$\mathrm{I} \rightarrow \mathrm{I}$$
$$\mathrm{II} \rightarrow \mathrm{II}$$
$$\mathrm{III} \rightarrow \mathrm{III}.$$

We punch in column 14 from constant pulse 2, in column 15 in the first copy 1, in the second copy 2. /30

Into the third and fourth copies we transfer

$$1-12 \rightarrow 1-12$$
$$\mathrm{IV} \rightarrow \mathrm{I}$$
$$\mathrm{V} \rightarrow \mathrm{II}$$
$$\mathrm{VI} \rightarrow \mathrm{III}.$$

We punch in column 14 from constant pulse 1, in column 15 in the third copy 1, in the fourth copy 2.

We will designate the block obtained by 0_5.

We sort jointly 0_5 and M_3, with respect to $-1 + k$, $i + k$, and columns 14 and 15 from block M_3, we enter field IV → IV into block 0_5 and pass it three times through the multiplier for multiplication. As a result we obtain fields V, VI, and VII

$$\mathrm{I} \times \mathrm{IV} = \mathrm{V},$$
$$\mathrm{II} \times \mathrm{IV} = \mathrm{VI},$$
$$\mathrm{III} \times \mathrm{IV} = \mathrm{VII}.$$

We pass 0_5 through the tabulator and sum fields V, VI, and VII with respect to $i + m' + k$. The tabulogram will give the expansions: (a) if in column 14 there is a 2, and in column 15 there is a 1 and 2, $a'R$, $a'D'R$, $A'D_\sigma R$, respectively, with respect to $\sin(j'M' + jM)$ and $\cos(j'M' + jM)$; (b) if in column 14 there is a 1, while in 15 there is a 1 and 2 $a'\frac{\partial R}{\partial L}$, $a'\frac{\partial R}{\partial L}$, $\frac{1}{n'}\,a'D_t'R$, respectively, with respect to $\sin(j'M' + jM)$ and $\cos(j'M' + jM)$.

Thus the work on expanding the perturbation function will have been accomplished.

For verification all the operations are to be conducted twice, shifting the column but using different calculators, and, if possible, different combinations of computers.

CHAPTER II

PERTURBATIONS OF COORDINATES

5. Perturbations of the Logarithms of the Radius Vector and Longitude in Orbit

When determining perturbations of the logarithms of the radius vector and longitude in orbit we adhere to the method of Laplace with the appropriate changes introduced into this method by Newcomb (1873).

Let us consider briefly the derivation of the basic equations.

When the plane of the osculating orbit is taken as the basic plane, the equations of motion in polar coordinates have the form:

$$\left.\begin{aligned} &\frac{d^2r'}{dt^2} - r'\left(\frac{dw'}{dt}\right)^2 + \frac{k^2(1+m')}{r'^2} = k^2 m \frac{\partial R}{\partial r'}, \\ &\frac{d}{dt}\left(r'^2 \frac{dw'}{dt}\right) = k^2 m \frac{\partial R}{\partial w'}. \end{aligned}\right\} \tag{27}$$

Let us proceed as Newcomb suggests, from radius vector r' to its logarithm $\rho' = \log r'$, and determine perturbations ρ'. Noting that

$$\frac{\partial R}{\partial r'} = \frac{1}{r'}\frac{\partial R}{\partial \rho'}, \quad \frac{\partial R}{\partial w'} = \frac{\partial R}{\partial W'},$$

we represent Expression (27) in the form

$$\left.\begin{aligned} &r'\frac{d^2r'}{dt^2} - r'^2\left(\frac{dw'}{dt}\right)^2 + \frac{k^2(1+m')}{r'} = k^2 m \frac{\partial R}{\partial \rho'}, \\ &\frac{d}{dt}\left(r'^2 \frac{dw'}{dt}\right) = k^2 m \frac{\partial R}{\partial W'}. \end{aligned}\right\} \tag{28}$$

Multiplying the first of the equalities in Equation (28) by $2\frac{d\rho'}{dt'}$, the second by $2\frac{dw'}{dt}$, summing the products, and integrating, we obtain

$$\left(\frac{dr'}{dt}\right)^2 + r'^2\left(\frac{dw'}{dt}\right)^2 - \frac{2k^2(1+m')}{r'} = 2k^2 m \int D_t' R\, dt + C_0, \tag{29}$$

where C_0 is the constant of integration and

$$D_t' R = \frac{\partial R}{\partial \rho'}\frac{d\rho'}{dt} + \frac{\partial R}{\partial W'}\frac{dw'}{dt}.$$

Let us add this equation to the first of the equations of (28). /32
Then after carrying out a few indicated operations we will arrive at the equation:

$$\frac{1}{2}\frac{d^2r'^2}{dt^2}-\frac{k^2(1+m')}{r'}=2k^2m\int D_t' R\,dt+k^2m\frac{\partial R}{\partial\rho'}+C_0. \tag{30}$$

In the case of unperturbed motion, the Jacobian formula reduces to:

$$\frac{1}{2}\frac{d^2r_0'^2}{dt^2}-\frac{k^2(1+m')}{r_0'}=-\frac{k^2(1+m')}{a'},$$

where r_0' is the radius vector of unperturbed motion.

Term by term computation of this equality from Expression (30) gives

$$\frac{1}{2}\frac{d^2}{dt^2}\left(r'^2-r_0'^2\right)-k^2(1+m')\left(\frac{1}{r'}-\frac{1}{r_0'}\right)=k^2m\left(2\int D_t' R\,dt+\frac{\partial R}{\partial\rho'}\right)+C_0+\frac{k^2(1+m')}{a'}. \tag{31}$$

Let us expand $(r'^2 - r_0'^2)$ and $r'^{-1} - r_0'^{-1})$ in powers of $\partial\rho' = \rho'-\rho_0'$:

$$r'^2-r_0'^2=r_0'^2\left(2\delta\rho'+2\delta\rho'^2+\ldots\right),$$
$$r'^{-1}-r_0'^{-1}=r_0'^{-1}\left(-\delta\rho'+\frac{1}{2}\delta\rho'^2+\ldots\right).$$

Let us substitute the derived expressions in Expression (31). Then, restricting ourselves to terms merely of the first order $\delta\rho'$, we will arrive at the equation

$$\frac{d^2}{dt^2}\left(r_0'^2\delta\rho'\right)+\frac{k^2(1+m')}{r_0'^3}\left(r_0'^2\delta\rho'\right)=k^2m\,(Q+2C_1), \tag{32}$$

where

$$Q=2\int D_t' R\,dt+\frac{\partial R}{\partial\rho'},$$
$$2k^2mC_1=C_0+\frac{k^2(1+m')}{a'}.$$

Thus the determination of perturbations of the logarithm of the radius vector reduces to solving an equation of the type

$$\frac{d^2x}{dt^2}+\frac{k^2(1+m)}{r^3}x=k^2mX. \tag{33}$$

If two linearly independent partial solutions p and q of Equation (33) without a free term are known, then the general solution will be:

$$x=\frac{k^2m}{p\dot{q}-q\dot{p}}\left(q\int pX\,dt-p\int qX\,dt+K_1p+K_2q\right), \tag{34}$$

where K_1 and K_2 are constants of integration.

The coordinates of elliptic motion $\xi = a(\cos E - e)$ and $\eta = a\cos\varphi \sin E$ satisfy Equation (33) without a free term. We take as the partial solutions:

$$p = \cos E - e,$$

$$q = \sin E,$$

then, since

$$p\dot{q}-q\dot{p}=n,$$

$$x=\frac{k^2m}{n}\left(q\int pX\,dt-p\int qX\,dt+K_1p+K_2q\right). \tag{35}$$

Hence, assuming that $x = r_0'^2\,\delta\rho'$, $X = Q + 2C_1$ and noting that $k^2 =$ /33
$k^2=\frac{n'^2a'^3}{1+m'}$, we derive

$$\delta\rho'=\frac{m}{1+m'}\left(\frac{a'}{r_0'}\right)^2\left[n'q\int p\,(a'Q+2a'C_1)\,dt-n'p\int q\,(a'Q+2a'C_1)\,dt+a'K_1p+a'K_2q\right]. \tag{36}$$

To determine the perturbations of longitude in orbit we will turn to the second equality in Equation (28)

$$\frac{d}{dt}\left(r'^2\frac{dw'}{dt}\right)=k^2m\frac{\partial R}{\partial W'}.$$

After integration we have

$$\frac{dw'}{dt}=\frac{k^2m}{r'^2}\int\frac{\partial R}{\partial W'}\,dt+C. \tag{37}$$

In the case of unperturbed motion, the integral of the areas gives:

$$\frac{dw_0'}{dt}=\left(\frac{a'}{r_0'}\right)^2 n'\cos\varphi', \tag{38}$$

where w'_0 is the longitude in the unperturbed orbit.

Subtracting the area integral in Equation (38) term-by-term from Expression (37) and leaving the terms of the first order, we arrive at the equation

$$\frac{d\delta w'}{dt}=\left(\frac{a'}{r_0'}\right)^2 n'\left[\frac{m}{1+m'}\left(n'\int a'\frac{\partial R}{\partial W'}\,dt+C_2\right)-2\cos\varphi'\delta\rho'\right], \tag{39}$$

where: $a'^2 n' m\, C_2 = C - a'^2 n' \cos\varphi'$.

After integration we will obtain

$$\delta w' = n' \int \left(\frac{a'}{r_0'}\right)^2 \left[\frac{m}{1+m'}\left(n' \int a' \frac{\partial R}{\partial W'} dt + C_2\right) - 2\cos\varphi' \delta\varphi'\right] dt + C_3, \tag{40}$$

where C_3 is a constant of integration.

6. Perturbations of Longitude of Node and Inclination

Let us take Lagrange's equations for determining perturbations of inclination i' and longitude of ascending node Ω'

$$\left.\begin{aligned} \frac{di'}{dt} &= -\frac{m}{1+m'} a'n' \sec\varphi' \left[\csc i' \frac{\partial R}{\partial \Omega'} + \tan\frac{i'}{2}\left(\frac{\partial R}{\partial \pi'} + \frac{\partial R}{\partial \varepsilon'}\right)\right], \\ \frac{d\Omega'}{dt} &= \frac{m}{1+m'} a'n' \sec\varphi' \csc i' \frac{\partial R}{\partial i'}. \end{aligned}\right\} \tag{41}$$

As was pointed out in Chapter I, the perturbation function is expanded into a series of the type $R = \Sigma h \cos N$,

where: $N = s'L' + sL + m'M' + mM = (s'+m')L' + (s+m)L - m'\Pi' - m\Pi$;

h is a function of e, e', α, σ;

$$\begin{aligned} \Pi' &= \pi' - \tau'; & L' &= n't + \varepsilon' - \tau'; & \tau' &= N' + \Omega'; \\ \Pi &= \pi - \tau; & L &= nt + \varepsilon - \tau; & \tau &= N + \Omega. \end{aligned} \tag{34}$$

We must express the derivatives to R with respect to i', Ω, π', and ε', which enter into Expression (41) by derivatives from R with respect to L', L, Π', Π, and σ, over which we expand R.

$$\begin{aligned} \frac{\partial R}{\partial i'} &= \frac{\partial R}{\partial \Pi'}\frac{\partial \Pi'}{\partial \tau'}\frac{\partial \tau'}{\partial i'} + \frac{\partial R}{\partial \Pi}\frac{\partial \Pi}{\partial \tau}\frac{\partial \tau}{\partial i'} + \frac{\partial R}{\partial L'}\frac{\partial L'}{\partial \tau'}\frac{\partial \tau'}{\partial i'} + \frac{\partial R}{\partial L}\frac{\partial L}{\partial \tau}\frac{\partial \tau}{\partial i'} + \frac{\partial R}{\partial \sigma}\frac{\partial \sigma}{\partial J}\frac{\partial J}{\partial i'}, \\ \frac{\partial R}{\partial \Omega'} &= \frac{\partial R}{\partial \Pi'}\frac{\partial \Pi'}{\partial \tau'}\frac{\partial \tau'}{\partial \Omega'} + \frac{\partial R}{\partial \Pi}\frac{\partial \Pi}{\partial \tau}\frac{\partial \tau}{\partial \Omega'} + \frac{\partial R}{\partial L'}\frac{\partial L'}{\partial \tau'}\frac{\partial \tau'}{\partial \Omega'} + \frac{\partial R}{\partial L}\frac{\partial L}{\partial \tau}\frac{\partial \tau}{\partial \Omega'} + \frac{\partial R}{\partial \sigma}\frac{\partial \sigma}{\partial J}\frac{\partial J}{\partial \Omega'}, \\ \frac{\partial R}{\partial \pi'} &= \frac{\partial R}{\partial \Pi'}\frac{\partial \Pi'}{\partial \pi'}, \\ \frac{\partial R}{\partial \varepsilon'} &= \frac{\partial R}{\partial L'}\frac{\partial L'}{\partial \varepsilon'}. \end{aligned} \tag{42}$$

In order to obtain derivatives from τ, τ', J, with respect to i' and Ω', we will use differential formulas from spherical trigonometry:

$$\left.\begin{aligned} dJ &= \cos(\tau - \Omega)\, di - \cos(\tau' - \Omega')\, di' - \sin i' \sin(\tau' - \Omega')\, d(\Omega' - \Omega), \\ \sin J\, d(\tau - \Omega) &= -\cos J \sin(\tau - \Omega)\, di + \sin(\tau' - \Omega')\, di' - \\ &\quad - \sin i' \cos(\tau' - \Omega')\, d(\Omega' - \Omega), \\ \sin J\, d(\tau' - \Omega') &= -\sin(\tau - \Omega)\, di + \cos J \sin(\tau' - \Omega')\, di' - \\ &\quad \sin i \cos(\tau - \Omega)\, d(\Omega' - \Omega). \end{aligned}\right\} \tag{43}$$

From this we obtain

$$\left.\begin{aligned}
&\frac{\partial J}{\partial i'} = -\cos(\tau' - \Omega'), \qquad \frac{\partial J}{\partial \Omega'} = -\sin i' \sin(\tau' - \Omega'),\\
&\frac{\partial \tau}{\partial i'} = \frac{\sin(\tau' - \Omega')}{\sin J}, \quad \frac{\partial \tau}{\partial \Omega'} = -\frac{\sin i' \cos(\tau' - \Omega')}{\sin J},\\
&\frac{\partial \tau'}{\partial i'} = \frac{\cos J \sin(\tau' - \Omega')}{\sin J}, \quad \frac{\partial \tau'}{\partial \Omega'} = 1 - \frac{\sin i \cos(\tau - \Omega)}{\sin J},
\end{aligned}\right\} \tag{44}$$

and

$$\frac{\partial \Pi'}{\partial \tau'} = -1, \quad \frac{\partial \Pi}{\partial \tau} = -1, \quad \frac{\partial \Pi'}{\partial \pi'} = 1, \quad \frac{\partial \sigma}{\partial J} = \frac{1}{2}\cos\frac{J}{2},$$
$$\frac{\partial L'}{\partial \tau'} = -1, \quad \frac{\partial L}{\partial \tau} = -1, \quad \frac{\partial L'}{\partial \varepsilon'} = 1.$$

Substituting the corresponding values of Expression (44) into Expression (42) we obtain

$$\left.\begin{aligned}
&\frac{\partial R}{\partial \Omega'} = -\left(\frac{\partial R}{\partial \Pi'} + \frac{\partial R}{\partial L'}\right)\left(1 - \frac{\sin i \cos N}{\sin J}\right) + \left(\frac{\partial R}{\partial \Pi} + \frac{\partial R}{\partial L}\right)\frac{\sin i' \cos N'}{\sin J} -\\
&\qquad \frac{1}{2}\frac{\partial R}{\partial \sigma}\sin i' \sin N' \cos\frac{J}{2},\\
&\frac{\partial R}{\partial i'} = -\left(\frac{\partial R}{\partial \Pi'} + \frac{\partial R}{\partial L'}\right)\frac{\sin N'}{\sin J}\cos J - \left(\frac{\partial R}{\partial \Pi} + \frac{\partial R}{\partial L}\right)\frac{\sin N'}{\sin J} - \frac{1}{2}\frac{\partial R}{\partial \sigma}\cos N' \cos\frac{J}{2},\\
&\qquad \frac{\partial R}{\partial \pi'} = \frac{\partial R}{\partial \Pi'}, \quad \frac{\partial R}{\partial \varepsilon'} = \frac{\partial R}{\partial L'}.
\end{aligned}\right\} \tag{45}$$

/35

Hence

$$\left.\begin{aligned}
&\frac{di'}{dt} = -\frac{m}{1+m'} n' a' \operatorname{sc}\varphi' \left[\left(\frac{\partial R}{\partial \Pi'} + \frac{\partial R}{\partial L'}\right)\left(\frac{\sin i \cos N}{\sin i' \sin J} - \csc i' + \tan\frac{i'}{2}\right) +\right.\\
&\qquad \left.\left(\frac{\partial R}{\partial \Pi} + \frac{\partial R}{\partial L}\right)\frac{\cos N'}{\sin J} - \frac{1}{2}\frac{\partial R}{\partial \sigma}\sin N' \cos\frac{J}{2}\right],\\
&\sin i' \frac{d\Omega'}{dt} = -\frac{m}{1+m'} n' a' \operatorname{sc}\varphi' \left[\left(\frac{\partial R}{\partial \Pi'} + \frac{\partial R}{\partial L'}\right)\frac{\sin N' \cos J}{\sin J} +\right.\\
&\qquad \left.\left(\frac{\partial R}{\partial \Pi} + \frac{\partial R}{\partial L}\right)\frac{\sin N'}{\sin J} + \frac{1}{2}\frac{\partial R}{\partial \sigma}\cos N' \cos\frac{J}{2}.\right.
\end{aligned}\right\} \tag{46}$$

Let us note that

$$\frac{\partial R}{\partial \Pi'} + \frac{\partial R}{\partial L'} = \frac{\partial R}{\partial W'},$$
$$\frac{\partial R}{\partial \Pi} + \frac{\partial R}{\partial L} = \frac{\partial R}{\partial W},$$

and Equation (46) will be rewritten as

$$\left.\begin{aligned}
\frac{di'}{dt} &= \frac{m}{1+m'}\,\mathrm{sc}\,\varphi' n' \left[a' \frac{\partial R}{\partial W'}\left(\mathrm{ctn}\, i' - \frac{\sin i \cos N}{\sin i' \sin J}\right) - a' \frac{\partial R}{\partial W} \frac{\cos N'}{\sin J} + \right. \\
&\qquad \left. \frac{1}{2} a' \frac{\partial R}{\partial \sigma} \sin N' \cos \frac{J}{2} \right], \\
\sin i' \frac{d\Omega'}{dt} &= -\frac{m}{1+m'}\,\mathrm{sc}\,\varphi' n' \left(a' \frac{\partial R}{\partial W'} \sin N' \mathrm{ctn}\, J + a' \frac{\partial R}{\partial W} \frac{\sin N'}{\sin J} + \right. \\
&\qquad \left. \frac{1}{2} a' \frac{\partial R}{\partial \sigma} \cos N' \cos \frac{J}{2} \right).
\end{aligned}\right\} \tag{47}$$

To determine $\delta i'$ and $\sin i' \delta\Omega'$ we derive the equalities

$$\left.\begin{aligned}
\delta i' &= \frac{m}{1+m'}\,\mathrm{sc}\,\varphi' n' \int \left[a' \frac{\partial R}{\partial W'}\left(\mathrm{ctn}\, i' - \frac{\sin i \cos N}{\sin i' \sin J}\right) - a' \frac{\partial R}{\partial W} \frac{\cos N'}{\sin J} + \right. \\
&\qquad \left. \frac{1}{2} a' \frac{\partial R}{\partial \sigma} \sin N' \cos \frac{J}{2} \right] dt + C_4, \\
\sin i' \delta\Omega' &= -\frac{m}{1+m'}\,\mathrm{sc}\,\varphi' n' \int \left[a' \frac{\partial R}{\partial W'} \sin N' \mathrm{ctn}\, J + a' \frac{\partial R}{\partial W} \frac{\sin N'}{\sin J} + \right. \\
&\qquad \left. \frac{1}{2} a' \frac{\partial R}{\partial \sigma} \cos N' \cos \frac{J}{2} \right] dt + C_5,
\end{aligned}\right\} \tag{48}$$

where C_4 and C_5 are constants of integration.

7. Determining Heliocentric Longitude and Latitude

In the preceding sections we derived equations for determining perturbations of the logarithm of the radius vector, longitude in orbit, longitude of node, and inclination. Knowing these values we may easily derive the perturbed values of heliocentric longitude l' and latitude b'. For this purpose we utilize the formulas

$$\tan(l' - \Omega') = \cos i' \tan\,(w' - \sigma'), \tag{49}$$

$$\sin b' = \sin i' \sin(w' - \sigma'), \tag{50}$$

where σ' is the distance from the node to the readout point in the orbit. /36
Let us assume that $\Omega'_0 = \Omega'_0$, then because of equality $d\sigma' = d\Omega' \cos i'$ we have

$$\sigma' = \Omega'_0 + d\Omega' \cos i'.$$

8. Determining the Constants of Integration

Solution of the equations of planetary motion by the methods set forth above contains seven constants of integration C_1, C_2, C_3, K_1, K_2, C_4, and C_5. Only six of these constants are independent. The constants C_1, C_2, and K_1 are interconnected by the relationship which we are about to derive.

From the integral of kinetic energy of perturbed motion (29),

---

Translator's note: ctn = cot.

$$\left(\frac{dr'}{dt}\right)^2 + r'^2\left(\frac{dw'}{dt}\right)^2 - \frac{2k^2(1+m')}{r'} = 2k^2 m \int D_t' R\, dt + C_0$$

we subtract the integral of kinetic energy of unperturbed motion,

$$\left(\frac{dr_0'}{dt}\right)^2 + r_0'^2\left(\frac{dw_0'}{dt}\right)^2 - \frac{2k^2(1+m')}{r_0'} = -\frac{k^2(1+m')}{a'}. \tag{51}$$

Then, restricting ourselves to terms of the first order of smallness, and taking into consideration that $2k^2mC_1 = C_0 + \frac{k^2(1+m')}{a'}$, after slight transformations we will arrive at the relationship

$$\left(\frac{3a'}{r'} - 1\right)\delta\rho' + e'q\frac{d\delta\rho'}{dn't} + \cos\varphi'\frac{d\delta w'}{dn't} = \frac{m'}{1+m}\left(a'C_1 + \int a' D_t' R\, dt\right). \tag{52}$$

Let us substitute into this equality the corresponding expressions for $\delta\rho'$, $\frac{d\delta'}{dn't}$, $\frac{d\delta w'}{dn't}$

$$\left.\begin{aligned}
&\frac{1+m'}{m}\delta\rho' = \left(\frac{a'}{r_0'}\right)^2\{q[a'K_2 + n'\int p(a'Q + 2a'C_1)\,dt] + \\
&\qquad p[a'K_1 - n_1\int q(a'Q + 2a'C_1)\,dt]\},\\
&\frac{1+m'}{m}\frac{d\delta\rho'}{dn't} = \left(\frac{a'}{r_0'}\right)^2\left\{\left(\frac{dq}{dn't} - \frac{2q}{r_0'}\frac{dr_0'}{dn't}\right)\left[a'K_2 + n'\int p(a'Q + 2a'C_1)\,dt\right] + \right.\\
&\qquad \left.\left(\frac{dp}{dn't} - \frac{2p}{r_0'}\frac{dr_0'}{dn't}\right)[a'K_1 - n'\int q(a'Q + 2a'C_1)\,dt]\right\},\\
&\frac{1+m'}{m}\frac{d\delta w'}{dn't} = \left(\frac{a'}{r_0'}\right)^2\left\{\left(n'\int a'\frac{\partial R}{\partial W'}dt + C_2\right) - 2\cos\varphi'\left(\frac{a'}{r_0'}\right)^2 q[a'K_2 + n'\int p(a'Q + \right.\\
&\qquad \left. 2a'C_1)\,dt] - 2\cos\varphi'\left(\frac{a'}{r_0'}\right)^2 p[a'K_1 - n'\int q(a'Q + 2a'C_1)\,dt]\right\}
\end{aligned}\right\} \tag{53}$$

and we will derive:

$$\left(\frac{a'}{r_0'}\right)^2\cos\varphi'\left[n'\int a'\frac{\partial R}{\partial W'}dt + C_2\right] - \left(\frac{a'}{r_0'}\right)^2 e'\left[a'K_1 - n'\int q(a'Q + 2a'C_1)\,dt\right] = a'C_1 + \int a' D_t' R\, dt$$

or, in other terms,

$$C_2\cos\varphi' - a'K_1e' + a'C_1\left[2e'n'\int q\,dt - \left(\frac{r_0'}{a'}\right)^2\right] = \left(\frac{r_0'}{a'}\right)^2\int a' D_t' R\, dt - \cos\varphi' n'\int a'\frac{\partial R}{\partial W'}dt - e'n'\int qa'Q\,dt. \tag{54}$$

Let us examine separately the expression /37

$$I = \cos\varphi' n'\int a'\frac{\partial R}{\partial W'}dt + e'n'\int \sin E' a'Q\,dt,$$

which contains only periodic terms since its constant part entered into the constants of integration K_1 and C_1

$$I=\int\left(\cos\varphi' n'a'\frac{\partial R}{\partial W'}+e'n'\sin E'a'\frac{\partial R}{\partial\rho'}\right)dt+2e'n'\int\left(\sin E'\int a'D_t'R\,dt\right)dt.$$

Since

$$\frac{\partial w'}{\partial t}=\left(\frac{a'}{r_0'}\right)^2 n'\cos\varphi',$$

$$\frac{\partial\rho'}{dt}=\left(\frac{a'}{r_0'}\right)^2 n'e'\sin E',$$

$$D_t'R=\frac{\partial R}{\partial\rho'}\frac{d\rho'}{dt}+\frac{\partial R}{\partial W'}\frac{dw'}{dt},$$

then

$$I=\int\left(\frac{r_0'}{a'}\right)^2 a'D_t'R\,dt+2e'n'\int\left[\sin E'\int a'D_t'R\,dt\right]dt.$$

Taking into account that

$$\int\left(\frac{r_0'}{a'}\right)^2 a'D_t'R\,dt=\left(\frac{r_0'}{a'}\right)^2\int a'D_t'R\,dt-2e'n'\int\left(\sin E'\int a'D_t'R\,dt\right)dt,$$

we derive

$$I=\left(\frac{r_0'}{a'}\right)^2\int a'D_t'R\,dt-A_0,$$

where A_0 is the constant part of the expansion

$$\left(\frac{r_0'}{a'}\right)^2\int a'D_t'R\,dt$$

in mean anomalies.

Let us examine the expression:

$$II=2e'n'\int q\,dt=2e'n'\int\sin E'\,dt=(1-e'\sin E')^2-\left(1+\frac{3}{2}e'^2\right)=\left(\frac{r_0'}{a'}\right)^2-\left(1+\frac{3}{2}e'^2\right),$$

where $(1+\frac{3}{2}e'^2)$ is the constant part of the expansion $\frac{(\Omega'_0)^2}{a'}$ in mean anomalies.

Substituting the derived values I and II into Equation (54), we arrive at the condition connecting constants K_1, C_1, and C_2:

$$\cos\varphi' C_2-e'a'K_1-\left(1+\frac{3}{2}e'^2\right)a'C_1=A_0. \tag{55}$$

In determining the constants of integration we proceed from the osculating elements, i.e., we determine the constants of integration from the condition

that at moment $t = t_0$ (moment of osculation) the perturbations and derivatives of the perturbations are zero.

Thus from the condition:

$$(\delta\rho')_{t=t_0}=\left(\frac{d\delta\rho'}{dt}\right)_{t=t_0}=\left(\frac{d\delta w'}{dt}\right)_{t=t_0}=(\delta w')_{t=t_0}=0,$$

Equality (52) will give:

$$a'C_1=\left(-\int a'D_t'R\,dt\right)_{t=t_0}. \tag{56}$$

Let us rewrite Equality (36) as follows: /38

$$\frac{1+m'}{m}\left(\frac{r_0'}{a'}\right)^2\delta\rho' = q\left[n'\int p\,(a'Q+2a'C_1)\,dt+a'K_2\right]+$$
$$p\left[-n'\int q\,(a'Q+2a'C_1)\,dt+a'K_1\right],$$

hence,

$$\frac{1+m'}{m}\frac{d}{dt}\left[\left(\frac{r_0'}{a'}\right)^2\delta\rho'\right]=\dot{q}\left[n'\int p\,(a'Q+2a'C_1)\,dt+a'K_2\right]+$$
$$+\dot{p}\left[-n'\int q\,(a'Q+2a'C_1)\,dt+a'K_1\right].$$

When $t = t_0$:

$$\frac{d}{dt}\left[\left(\frac{r_0'}{a'}\right)^2\delta\rho'\right]=0.$$

Let us multiply the first equality by q, the second by q, and subtract the second from the first; then

$$\left\{(pq-q\dot{p})\left[-n'\int q\,(a'Q+2a'C_1)\,dt+a'K_1\right]\right\}_{t=t_0}=0,$$

$$p\dot{q}-q\dot{p}=n',$$

consequently:

$$a'K_1=\left[n'\int q\,(a'Q+2a'C_1)\,dt\right]_{t=t_0}. \tag{57}$$

If the first equality is multiplied by p, the second by p, and the two are added, we obtain

$$a'K_2=-\left[n'\int p\,(a'Q+2a'C_1)\,dt\right]_{t=t_0}. \tag{58}$$

Equation (39) for $t = t_0$ yields

$$C_2=-\left[n'\int a'\frac{\partial R}{\partial W'}\,dt\right]_{t=t_0} \tag{59}$$

From Equations (40) and (48) when $t = t_0$ we derive the values of constants C_3, C_4, and C_5

$$C_3=-\left\{n'\int\left(\frac{a'}{r_0'}\right)^2\left[\frac{m}{1+m'}\left(n'\int a'\frac{\partial R}{\partial W'}\,dt+C_2\right)-2\cos\varphi'\delta\rho'\right]dt\right\}_{t=t_0}; \tag{60}$$

$$C_4 = -\frac{m}{1+m'} \operatorname{sc} \varphi' \left\{ n' \int \left[a' \frac{\partial R}{\partial W'} \left(\operatorname{ctg} i' - \frac{\sin i \cos N}{\sin i' \sin J} \right) - \right.\right.$$

$$\left.\left. - a' \frac{\partial R}{\partial W} \frac{\cos N'}{\sin J} + \frac{1}{2} a' \frac{\partial R}{\partial \sigma} \sin N' \cos \frac{J}{2} \right| dt \right\}_{t=t_0}; \tag{61}$$

$$C_5 = \frac{m}{1+m'} \operatorname{sc} \varphi' \left\{ n' \int \left[a' \frac{\partial R}{\partial W'} \sin N' \operatorname{ctg} J + a' \frac{\partial R}{\partial W} \frac{\sin N'}{\sin J} + \right.\right.$$

$$\left.\left. + \frac{1}{2} a' \frac{\partial R}{\partial \sigma} \cos N' \cos \frac{J}{2} \right] dt \right\}_{t=t_0} \tag{62}$$

CHAPTER III /39

PERTURBATIONS OF THE FIRST ORDER BY JUPITER

9. Elements

Computation of the perturbations of Pluto and Jupiter are based on the nineteenth barycentric system of Plutonian elements according to Bower (1931), and the mean elements of Jupiter according to Hill. The mean elements of Pluto have not been derived up to the present time. Of all the existing systems of elements, Bower's nineteenth system has been determined over the longest arc and partially takes into consideration the perturbations. Since the distances of the heliocentric and barycentric coordinates are magnitudes of the order of the perturbing masses, they already give in the perturbations a second-order discrepancy, and in determination of the perturbations these elements were taken as heliocentric.

The elements are in terms of the ecliptic and equinox of 1950.0.

Elements of Pluto Moment of Osculation 1930 September 20.0	Elements of Jupiter Epoch 1850 January 0 Greenwich mean noon
$T' = 1989$ October 0.0344	$M_0 = 148^\circ.033$
$\Omega' = 109^\circ.634$	$\Omega = 99.779$
$i' = 17.144$	$i = 1.308$
$\omega' = 113.542$	$\omega = 273.525$
$e' = 0.248\,644$	$e = 0.048\,254$
$a' = 39.517\,738$	$a = 5.202\,803$
$n' = 14''.283$	$n = 299''.128.$
	$\frac{1}{m} = 1047.4$

The elements of the mutual disposition of the orbits N, N', and J were derived for verification from the two systems of formulas given below:

$$\left.\begin{aligned}
\sin J \cos N &= \sin i \cos i' - \cos i \sin i' \cos(\Omega - \Omega'),\\
\sin J \sin N &= \sin i' \sin(\Omega - \Omega'),\\
\cos J &= \cos i \cos i' + \sin i \sin i' \cos(\Omega - \Omega'),\\
\sin J \cos N' &= -\sin i' \cos i + \cos i' \sin i \cos(\Omega - \Omega'),\\
\sin J \sin N' &= \sin i \sin(\Omega - \Omega').
\end{aligned}\right\} \qquad (63) \qquad /40$$

$$\left.\begin{aligned}
\sin\frac{J}{2}\sin\frac{N'+N}{2}&=\sin\frac{\Omega-\Omega'}{2}\sin\frac{i+i'}{2},\\
\sin\frac{J}{2}\cos\frac{N'+N}{2}&=\cos\frac{\Omega-\Omega'}{2}\sin\frac{i-i'}{2},\\
\cos\frac{J}{2}\sin\frac{N'-N}{2}&=\sin\frac{\Omega-\Omega'}{2}\cos\frac{i+i'}{2},\\
\cos\frac{J}{2}\cos\frac{N'-N}{2}&=\cos\frac{\Omega-\Omega'}{2}\cos\frac{i-i'}{2}.
\end{aligned}\right\}\qquad(64)$$

For the Pluto-Jupiter case, the values of N, N' $\sigma = \sin\frac{J}{2}$, and of the constant values α, Π, Π', which we need in the following, are:

$$N=190^\circ.6406,\ \sin\frac{J}{2}=0.137935,\ \Pi=82^\circ.8844,\ \alpha=0.131657,$$

$$N'=180.8194,\ \cos\frac{J}{2}=0.990441,\ \Pi'=292.7223.$$

10. Expansion of the Perturbation Function

As already indicated above, when the perturbation function was expanded we used Newcomb's method of operators (with respect to multiple mean anomalies). The computations were manually performed and only later when the computing laboratory was organized at the Institute of Theoretical Astronomy was the expansion of the perturbation function repeated for verification on analytical computers. The method of expanding the perturbation function on analytical computers is set forth in detail in Chapter I, Section 4 of the present monograph.

In both manual and in machine computations, the basic initial data are the table of coefficients of Newcomb's operators given in Appendix II, and the tables of coefficients of expansion of circular motion $a'A_i$, $a'B_i$, . . . and their derivatives with respect to log a and σ.

To compute these values it is sufficient in Formulas (3) and (11) to restrict ourselves to the following terms:

$$a'D^kA_i=D^kc_1^i-\frac{1}{2}\sigma^2D^k\left(c_3^{i-1}+c_3^{i+1}\right)+\frac{3}{8}\sigma^4D^k\left(c_5^{i-2}+4c_5^i+c_5^{i+2}\right),$$

where $i=0, 1, 2 \ldots 7$, $k=0, 1, 2 \ldots 8$;

$$a'D^kB_i=\frac{1}{2}D^kc_3^i-\frac{3}{4}\sigma^2D^k\left(c_5^{i-1}+c_5^{i+1}\right),$$

where $i=0, 1, 2 \ldots 5$, $k=0, 1, 2 \ldots 6$;

$$a'D^kC_i=\frac{3}{8}D^kc_5^i,$$

where $i=0, 1, 2, 3$, $k=0, 1, 2, 3, 4$;

$$a'D_\sigma D^kA_i=-\sigma D^k\left(c_3^{i-1}+c_3^{i+1}\right)+\frac{3}{2}\sigma^3D^k\left(c_5^{i-2}+4c_5^i+c_5^{i+2}\right),$$

where $i=0, 1 \ldots 7$, $k=0, 1 \ldots 8$;

$$a' D_\sigma D^k \sigma^2 B_i = \sigma D^k c_3^i - 3\sigma^3 D^k \left(c_5^{i-1} + c_5^{i+1}\right),$$

where $i=0, 1 \ldots 5$, $k=0, 1 \ldots 6$;

$$a' D_\sigma D^k \sigma^4 C_i = \frac{3}{2} \sigma^3 D^k c_5^i,$$

where $i=0, 1 \ldots 3$, $k=0, 1 \ldots 4$.

TABLE 2 /41

i	c_1^i	Dc_1^i	$D^2c_1^i$	$D^3c_1^i$	$D^4c_1^i$	$D^5c_1^i$	$D^6c_1^i$	$D^7c_1^i$	$D^8c_1^i$
0	2.008753	0.0176779	0.0360569	0.0749697	0.1616778	0.3722419	0.951997	2.80881	9.71189
1	0.1325225	0.1342719	0.1395971	0.1559640	0.2070631	0.3706546	0.915069	2.83220	10.1176
2	0.0130952	0.0263826	0.0535427	0.1102492	0.2334837	0.5209345	1.270273	3.53251	11.5288
3	0.0014373	0.0013339	0.0131136	0.0399073	0.1226072	0.3826264	1.224615	4.07711	14.3903
4	0.0001656	0.0006651	0.0026761	0.0108008	0.0437896	0.1787434	0.737037	3.08506	13.1991
5	0.0000196	0.0000984	0.0004945	0.0024882	0.0125525	0.0635568	0.323450	1.65790	8.58312
6	0.0000024	0.0000143	0.0000858	0.0005175	0.0031250	0.0189148	0.114825	0.699845	4.28832
7	0.0000003	0.0000020	0.0000143	0.0001003	0.0007055	0.0049706	0.035089	0.248318	1.76294

i	c_3^i	Dc_3^i	$D^2c_3^i$	$D^3c_3^i$	$D^4c_3^i$	$D^5c_3^i$	$D^6c_3^i$
0	0.273869	0.295561	0.363027	0.577717	1.28572	3.7473	12.951
1	0.053735	0.111027	0.236646	0.533916	1.32423	3.7609	12.521
2	0.008824	0.027017	0.083833	0.265789	0.87160	3.0060	11.120
3	0.001354	0.005496	0.022477	0.092923	0.39030	1.6772	7.4397
4	0.000200	0.001014	0.005151	0.026346	0.13596	0.71021	3.7720
5	0.000029	0.000176	0.001068	0.006516	0.03999	0.24723	1.5434
6	0.000004	0.000029	0.000206	0.001465	0.01043	0.07464	0.5375
7	0.000001	0.000005	0.000038	0.000307	0.00249	0.02025	0.1655

i	c_5^i	Dc_5^i	$D^2c_5^i$	$D^3c_5^i$	$D^4c_5^i$
0	0.03863	0.08562	0.20646	0.56501	1.8091
1	0.01232	0.03885	0.12638	0.43085	1.5696
2	0.00281	0.01160	0.04870	0.20907	0.9262
3	0.00055	0.00282	0.01460	0.07645	0.4074
4	0.00010	0.00061	0.00374	0.02322	0.1456
5	0.00002	0.00012	0.00086	0.00620	0.0448
6	0.00000	0.00002	0.00018	0.00150	0.0123
7	0.00000	0.00000	0.00004	0.00034	0.0031

TABLE 3

i	$a'A_i$	$a'DA_i$	$a'D^2A_i$	$a'D^3A_i$	$a'D^4A_i$	$a'D^5A_i$	$a'D^6A_i$	$a'D^7A_i$
0	2.0077521	0.0156151	0.0316798	0.0651749	0.1377167	0.3055345	0.713772	1.88477
1	0.1298420	0.1312300	0.1354339	0.1482425	0.1876611	0.3109398	0.686084	1.92307
2	0.0125779	0.0252921	0.0511326	0.1044794	0.2179415	0.4725733	1.080386	2.74282
3	0.0013534	0.0040741	0.0122921	0.0372290	0.1134627	0.3493704	1.082950	3.45472
4	0.0001529	0.0006130	0.0024608	0.0098961	0.0399027	0.1615343	0.651580	2.66461
5	0.0000178	0.0000890	0.0004460	0.0022374	0.0112400	0.0545736	0.282453	1.42594
6	0.0000021	0.0000126	0.0000759	0.0004565	0.0027474	0.0165513	0.098468	0.59408
7	0.0000003	0.0000018	0.0000124	0.0000874	0.0006140	0.0043205	0.029975	0.21120

i	$\sigma^2a'B_i$	$\sigma^2a'DB_i$	$\sigma^2a'D^2B_i$	$\sigma^2a'D^3B_i$	$\sigma^2a'D^4B_i$	$\sigma^2a'D^5B_i$	$\sigma^2a'D^6B_i$
0	0.0025986	0.0027906	0.0033819	0.0052620	0.011379	0.03227	0.1232
1	0.0004999	0.0010298	0.0021820	0.0048690	0.011855	0.03277	0.1191
2	0.0000805	0.0002457	0.0007592	0.0023907	0.007755	0.02630	0.1058
3	0.0000121	0.0000490	0.0001996	0.0008209	0.003422	0.01454	0.0708
4	0.0000018	0.0000088	0.0000418	0.0002282	0.001171	0.00606	0.0359
5	0.0000002	0.0000015	0.0000091	0.0000553	0.000338	0.00207	0.0147

i	$\sigma^4a'C_i$	$\sigma^4a'DC_i$	$\sigma^4a'D^2C_i$	$\sigma^4a'D^3C_i$	$\sigma^4a'D^4C_i$
0	0.0000052	0.0000116	0.0000280	0.0000767	0.0002456
1	0.0000016	0.0000053	0.0000172	0.0000585	0.0002131
2	0.0000004	0.0000016	0.0000066	0.0000284	0.0001257
3	0.0000001	0.0000004	0.0000020	0.0000104	0.0000553

TABLE 4

i	$a'D_\sigma A_i$	$a'DD_\sigma A_i$	$a'D^2D_\sigma A_i$	$a'D^3D_\sigma A_i$	$a'D^4D_\sigma A_i$	$a'D^5D_\sigma A_i$	$a'D^6D_\sigma A_i$	$a'D^7D_\sigma A_i$
0	—0.0141973	—0.0291900	—0.0616492	—0.1367487	—0.3295366	—0.896935	—3.45417	—13.9997
1	—0.0387538	—0.0437191	—0.0590927	—0.1075678	—0.2650718	—0.800159	—3.32019	—13.1821
2	—0.0074028	—0.0155509	—0.0341479	—0.0808550	—0.2142145	—0.652332	—2.75328	—11.4502
3	—0.0011887	—0.0036686	—0.0115432	—0.0373714	—0.1262067	—0.451803	—2.05409	— 9.0244
4	—0.0001785	—0.0007270	—0.0029963	—0.0125215	—0.0533644	—0.233607	—1.23909	— 6.0963
5	—0.0000257	—0.0001308	—0.0006677	—0.0034362	—0.0178698	—0.094247	—0.59444	— 3.3632
6	—0.0000036	—0.0000221	—0.0001349	—0.0008259	—0.0050913	—0.031643	—0.23572	— 1.5336
7	—0.0000005	—0.0000029	—0.0000244	—0.0001723	—0.0012130	—0.008554	—0.07414	— 0.5382

i	$a'D_\sigma\sigma^2B_i$	$a'DD_\sigma\sigma^2B_i$	$a'D^2D_\sigma\sigma^2B_i$	$a'D^3D_\sigma\sigma^2B_i$	$a'D^4D_\sigma\sigma^2B_i$	$a'D^5D_\sigma\sigma^2B_i$	$a'D^6D_\sigma\sigma^2B_i$
0	0.0375873	0.0401565	0.0480842	0.0729032	0.1526307	0.418799	1.78640
1	0.0070885	0.0145491	0.0306329	0.0675513	0.1611221	0.431618	1.72709
2	0.0011180	0.0033985	0.0104536	0.0326676	0.1046587	0.348094	1.53380
3	0.0001647	0.0006620	0.0026875	0.0109885	0.0453975	0.190349	1.02620
4	0.0000233	0.0001167	0.0005888	0.0029833	0.0151927	0.077898	0.52030
5	0.0000032	0.0000193	0.0001164	0.0007041	0.0042720	0.026007	0.21289
6	0.0000004	0.0000080	0.0000213	0.0001506	0.0010611	0.007492	0.07414

TABLE 4 (continued)

i	$a'D_\sigma\sigma^4C_i$	$a'DD_\sigma\sigma^4C_i$	$a'D^2D_\sigma\sigma^4C_i$	$a'D^3D_\sigma\sigma^4C_i$	$a'D^4D_\sigma\sigma^4C_i$
0	0.000150	0.000337	0.000813	0.002224	0.007122
1	0.000047	0.000153	0.000498	0.001696	0.006178
2	0.000010	0.000046	0.000192	0.000823	0.000365
3	0.000002	0.000011	0.000058	0.000301	0.000160

The Laplace coefficients and their derivatives with respect to log a entering into these formulas were computed by two methods, Newcomb's and Innes'. This was done both for a deep check on these basic values and for determining the practical advantages of one method over the other. As has already been indicated, Innes' method has a number of advantages over that of Newcomb in convenience and especially verification of computations. Each stage may be verified and there is no need to repeat computations at second hand. /43

Tables 2, 3, and 4, respectively, give the values:

$$D^k c_1^i, \qquad D^k c_3^i, \qquad D^k c_5^i;$$
$$a'D^k A_i, \qquad a'D^k\sigma^2 B_i, \qquad a'D^k\sigma^4 C_i;$$
$$a'D_\sigma D^k A_i, \qquad a'D_\sigma D^k\sigma^2 B_i, \qquad a'D_\sigma D^k\sigma^4 C_i.$$

Using the tables of operator coefficients (Appendix II) and Tables 3 and 4, we derive the operators $P^{nn'}_{mm'}$, $P'^{nn'}_{mm'}$, $P''^{nn'}_{mm'}$ and their derivatives $DP^{nn'}_{mm'}$, $DP'^{nn'}_{mm'}$, $DP''^{nn'}_{mm'}$, $D_\sigma P^{nn'}_{mm'}$, $D_\sigma P'^{nn'}_{mm'}$, $D_\sigma P''^{nn'}_{mm'}$.

We introduce the corrections for the indirect member of the perturbation function into the values of the operators. In Section 1 of Chapter I, we are given formulas for determining corrections to $a'A_{\mp1}$, $a'\sigma^2B_0$, $a'D_\sigma A_{\mp1}$, $a'D_\sigma\sigma^2B_0$, and to their derivatives with respect to log α. Let us recall these formulas:

$$\delta a'A_{\mp1} = -\alpha^{-2}(1-\sigma^2), \qquad \delta a'\sigma^2B_0 = -\alpha^{-2}\sigma^2,$$
$$\delta a'DA_{\mp1} = 2\alpha^{-2}(1-\sigma^2), \qquad \delta a'D\sigma^2B_0 = 2\alpha^{-2}\sigma^2,$$
$$\delta a'D^kA_{\mp1} = -(-2)^k\alpha^{-2}(1-\sigma^2), \qquad \delta a'D^k\sigma^2B_0 = -(-2)^k\alpha^{-2}\sigma^2,$$
$$\delta a'D_\sigma A_{\mp1} = 2\alpha^{-2}\sigma, \qquad \delta a'D_\sigma\sigma^2B_0 = -2\alpha^{-2}\sigma,$$
$$\delta a'DD_\sigma A_{\mp1} = -4\alpha^{-2}\sigma, \qquad \delta a'DD_\sigma\sigma^2B_0 = 4\alpha^{-2}\sigma,$$
$$\delta a'D^kD_\sigma A_{\mp1} = 2(-2)^k\alpha^{-2}\sigma, \qquad \delta a'D^kD_\sigma\sigma^2B_0 = -2(-2)^k\alpha^{-2}\sigma.$$

It is easy to note that the factor $\alpha^{-2}(1-\sigma^2)$ enters into all derivatives of $\delta a' A_{\mp 1}$ with respect to log a, factor $\alpha^{-2}\sigma^2$ enters into derivatives of $\delta a' \sigma^2 B_0$ and so on. The corefficients in these factors will be fixed numbers independent of the elements of the planets, and they will be the same for the corresponding derivatives of all six corrections. Let us compute the auxiliary operators $\left(P^{nn'}_{mm'}\right)_{\mp 1}$, $\left(P'^{nn'}_{mm'}\right)_0$, of these numbers; for this purpose we multiply the free term of the operator by - 1, the coefficient of D by - 2, coefficient of D^2 by $-(-2)^2$, and the coefficient of D^k by $-(-2)^k$.

The values of the auxiliary operators are given in Appendix III. To derive corrections for the indirect term of the perturbation function in the operators and in their derivatives, it is enough to multiply the auxiliary operator by the corresponding factor.

Thus, for obtaining		$\delta P^{nn'}_{mm'}$	we multiply the aux. operator			$\left(P^{nn'}_{mm'}\right)$ by	$\alpha^{-2}(1-\sigma^2)$,
"	"	$\delta P'^{nn'}_{mm'}$	"	"	"	$\left(P'^{nn'}_{mm'}\right)$ "	$\alpha^{-2}\sigma^2$,
"	"	$\delta D_\sigma P^{nn'}_{mm'}$	"	"	"	$\left(P^{nn'}_{mm'}\right)$ "	$-2\alpha^{-2}\sigma$,
"	"	$\delta D_\sigma P'^{nn'}_{mm'}$	"	"	"	$\left(P'^{nn'}_{mm'}\right)$ "	$2\alpha^{-2}\sigma$,
"	"	$\delta D P^{nn'}_{mm'}$	"	"	"	$\left(P^{nn'}_{mm'}\right)$ "	$-2\alpha^{-2}(1-\sigma^2)$,
"	"	$\delta D P'^{nn'}_{mm'}$	"	"	"	$\left(P'^{nn'}_{mm'}\right)$ "	$-2\alpha^{-2}\sigma^2$.

In the case of the combination of Pluto and Jupiter the values of the vectors will be: /44

$$\alpha^{-2}(1-\sigma^2)=57.59345, \qquad 2\alpha^{-2}\sigma=15.915293,$$

$$\alpha^{-2}\sigma^2=1.097638, \qquad -2\alpha^{-2}(1-\sigma^2)=-115.1869,$$

$$-2\alpha^{-2}\sigma=-15.915293, \qquad -2\alpha^{-2}\sigma^2=-2.195276.$$

After introducing the corrections, we multiply the operators by the eccentricities of Pluto and Jupiter in corresponding powers, and from the system in Appendix I we derive the expansion coefficients a'R, a'DR and a'D_σR with respect to cos [(i + cl.) L' + (- i + cl) L + mM + m'M'].

We obtain the derivative with respect to log a' by using the formula:

$$D'R = -R - DR.$$

Formulas (10) of Chapter I, Section 1, serve for computing the coefficients of the derivatives of R with respect to W, W', and t, since

$$\frac{\partial R}{\partial L'} = \frac{\partial R}{\partial W'}, \qquad \text{and}$$

$$\frac{\partial R}{\partial L} = \frac{\partial R}{\partial W};$$

we derive $a' \frac{\partial R}{\partial W'}$, $a' \frac{\partial R}{\partial W}$, $a' D_t' R$ in the form of series in

$$\sin\ [(i + \text{cl.})\ L' + (-\ i + \text{cl.})\ L + mM + m'M').]$$

By means of the values $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\ [(i + \text{cl.})\,\Pi' + (-\,i + \text{cl.})\,\Pi]$, we apply $a'R$, $a'D'R$, $a'D_oR$, $a' \frac{\partial R}{\partial W'}$, $a' \frac{\partial R}{\partial W}$ $a' D_t' R$, to the expansion in $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}$ (i'M' + iM).

To verify the derivatives $a' \frac{\partial R}{\partial W'}$, $a'D'R$ and $a' D_t' R$, we obtain the time derivative from the relationship

$$D_t' R = \frac{\partial R}{\partial W'} \frac{dw'}{dt} + \frac{\partial R}{\partial \rho'} \frac{d\rho'}{dt},$$

where w' and ρ' may be expanded into a series in mean anomalies:

$$\left.\begin{aligned} w' &= M' + \pi' + \sum H_j \sin jM', \\ \rho' &= \sum K_j \cos jM' \end{aligned}\right\} \qquad (65)$$

where H_j and K_j are polynomials with respect to the degrees of eccentricity.

From Expression (65) we obtain

$$\frac{1}{n'} \frac{dw'}{dt} = 1 + \sum jH_j \cos jM',$$

$$\frac{1}{n'} \frac{d\rho'}{dt} = \sum -jK_j \sin jM'.$$

TABLE 5

		$a'R\cdot 10^4$		$a'D'R\cdot 10^4$		$a'D_\sigma R\cdot 10^4$		$a'\frac{dR}{dW'}\cdot 10^4$		$a'\frac{dR}{dW}\cdot 10^4$		$\frac{1}{n'}a'D_t'R\cdot 10^4$	
i	i'	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos
0	0	—	+10042.9	—	— 10130.3	—	— 80.7	—	—	—	—	—	—
	1	— 45.4	2576.7	+ 93.2	— 2718.3	15.4	— 105.2	— 77.6	— 46.0	+ 80.4	+ 46.2	— 2576.7	— 45.4
	2	— 19.6	652.5	36.9	— 710.0	54.4	— 90.9	— 32.4	— 17.2	40.2	23.8	— 1305.0	— 39.2
	3	— 7.2	185.1	11.3	— 204.4	44.8	— 58.2	— 10.9	+ 4.4	17.1	10.2	— 555.3	— 21.6
	4	— 2.1	54.5	2.8	— 61.8	46.7	— 30.9	— 3.1	— 0.7	6.7	4.3	— 218.0	— 8.4
	5	— 0.7	16.8	+ 0.5	— 18.9	14.0	— 14.5	— 1.0	+ 0.2	2.8	1.8	— 84.0	— 3.5
	6	— 0.1	5.6	— 0.5	— 5.8	7.7	— 7.5	— 0.1	0.3	+ 0.9	+ 0.5	— 33.6	— 06
	7		1.6	— 0.7	— 1.1	+ 3.0	— 2.9	— 0.4	0.4	— 0.4	— 0.4	— 11.2	
1	—8	+ 6	10	+ 8	+ 14		— 2	+ 10	+ 6	— 10	+ 6	+ 82	— 49
	—7	21	37	32	54	— 1	— 9	37	— 20	— 37	21	260	— 146
	—6	72	128	94	159	— 13	— 33	126	— 70	— 126	71	766	— 433
	—5	296	520	365	625	— 72	— 141	518	— 293	— 518	294	2600	— 1481
	—4	1277	2233	1452	2509	— 342	— 603	2230	— 1271	— 2230	1273	8934	— 5107
	—3	5986	10448	6424	11159	— 1662	— 3020	10443	— 5974	— 10442	5976	31343	— 17958
	—2	33040	57607	34020	59255	— 9251	— 16284	57614	— 33023	— 57602	33027	115215	— 66080
	—1	+271939	+473974	+273814	+477263	— 76130	—134535	+474142	—271894	—473969	+271943	473974	—271939
	0	—105967	—178999	—105964	—178977	+ 13550	+108594	—186766	+103829	+178886	—106027	0	0
	1	+ 4987	— 6210	+ 5008	— 6275	40746	—149127	+ 13930	558	6485	+ 5145	6210	+ 4987
	2	518	— 908	534	— 934	4974	— 18072	1539	155	941	538	1816	1036
	3	86	— 178	92	— 191	903	— 3270	265	36	184	89	535	257
	4	18	— 40	20	— 45	192	— 697	54	9	42	18	162	70
	5	4	— 11	5	— 13	46	— 168	14	3	11	4	54	22
	6	1	— 3	+ 2	— 3	11	— 39	3	+ 1	+ 3	1	19	+ 5
	7					4	— 13						
	8					+ 1	— 4						
2	—9	1	+ 1	— 3	— 2	— 2		1	— 2	— 1	2	6	— 10
	—8	3	2	— 10	— 5	— 3	— 1	3	— 7	— 3	7	12	— 25

/45

TABLE 5 (continued)

		$a'R\cdot 10^4$		$a'D'R\cdot 10^4$		$a'D_3R\cdot 10^4$		$a'\frac{dR}{dW'}\cdot 10^4$		$a'\frac{dR}{dW}\cdot 10^4$		$\frac{1}{n'}a'D_t'R\cdot 10^4$	
i	i'	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos
2	−7	+ 6	+ 3	− 20	− 9	− 6	− 2	+ 6	− 13	− 6	+ 13	21	− 43
	−6	22	22	− 32	− 5	− 13	− 8	29	− 36	− 29	36	133	− 130
	−5	61	73	− 50	+ 16	− 29	− 25	88	− 91	− 88	91	364	− 304
	−4	179	251	− 30	140	− 70	− 82	280	234	− 280	234	1003	− 717
	−3	672	1073	+ 344	897	− 218	− 323	1121	758	− 1121	757	3219	− 2014
	−2	3300	5649	2960	5474	− 962	− 1634	5704	− 3391	− 5702	3390	11297	− 6600
	−1	+26258	+45784	+26362	+45898	− 7358	13026	− 45793	− 26211	− 45775	+26242	45784	−26258
	0	− 10226	−17265	−10208	− 17252	+ 1285	+10393	18016	+10013	+17261	− 10235	0	0
	1	+ 483	− 607	+ 491	− 603	3929	−14438	1352	56	627	+ 493	607	+ 483
	2	50	− 89	53	− 88	481	− 1767	150	16	92	50	178	100
	3	9	− 20	10	− 19	88	− 325	28	4	19	9	58	26
	4	2	− 4	2	− 14	19	− 71	6	1	+ 4	+ 2	18	+ 7
	5					5	− 19						
	6					1	− 5						
3	−4	1	+ 21	54	15	+ 9	− 5	23	+ 22	− 23	− 22	84	− 1
	−3	46	92	83	82	− 5	− 23	94	− 34	− 94	+ 34	275	− 139
	−2	270	460	236	419	− 72	− 125	463	− 280	− 463	280	919	− 539
	−1	+ 2143	+ 3737	+ 2145	+ 3739	− 591	− 1053	+ 3737	− 2143	− 3737	+ 2143	3737	− 2143
	0	− 834	− 1408	− 834	− 1408	+ 108	+ 853	− 1470	+ 816	+ 1408	− 833	0	0
	1	+ 40	− 52	+ 40	− 53	321	− 1175	113	6	53	+ 40	53	+ 40
	2	4	− 8	4	− 8	38	− 139	13	+ 2	+ 8	4	16	9
	3					8	− 28						
	4					2	− 6						
4	−2	21	+ 37	20	+ 36	+ 6	− 10	36	− 22	− 36	22	73	− 42
	−1	+ 169	+ 294	+ 169	+ 294	− 48	− 83	+ 294	− 169	− 294	+ 169	294	− 169
	0	− 63	− 110	− 63	− 110	+ 8	+ 65	− 110	+ 63	+ 110	− 63	0	0
	1					26	− 92						
	2					3	− 12						

/46

In our case the values of jH_j and $-jK_j$ are: /47

$$
\begin{aligned}
&H_0=1, \\
&H_1=0.493\,496, && -K_1=0.242\,864, \\
&2H_2=0.151\,098, && -2K_2=0.089\,254, \\
&3H_3=0.048\,076, && -3K_3=0.030\,977, \\
&4H_4=0.015\,536, && -4K_4=0.010\,564, \\
&5H_5=0.005\,062, && -5K_5=0.003\,560, \\
&6H_6=0.001\,650, && -6K_6=0.001\,193, \\
&7H_7=0.000\,542, && -7K_7=0.000\,455, \\
&8H_8=0.000\,207, && -8K_8=0.000\,158. \\
&9H_9=0.000\,074,
\end{aligned}
$$

Table 5 gives the expansion coefficients $a'R$, $a'D'R$, $a'\frac{\partial R}{\partial W'}$, $a'\frac{\partial R}{\partial W}$, $a'D_2R$, $\frac{a'}{n'}D_t'R$.

11. Computing Perturbations ρ' and w'

To determine the perturbations of the logarithm of the radius vector and of longitude in orbit in Section 5, we were given the following formulas:

$$\delta\rho'=\frac{m}{1+m'}\left(\frac{a'}{r_0'}\right)^2\left[n'q\int p\,(a'Q+2a'C_1)\,dt-n'p\int q\,(a'Q+2a'C_1)\,dt+a'K_1p+a'K_2q\right],$$

$$\delta w'=n'\int\left(\frac{a'}{r_0'}\right)^2\left[\frac{m}{1+m'}\left(n'\int a'\frac{\partial R}{\partial W'}\,dt+C_2\right)-2\cos\varphi'\delta\rho'\right]dt+C_3.$$

(a) The Part of the Perturbations Not Depending on Constants

When determining the perturbations of the orbital coordinates we first compute that part of the perturbations which does not depend on constants. Let us designate them by $(\delta\rho')$ and $(\delta w')$. Then,

$$(\delta\rho')=\frac{m}{1+m'}\left(\frac{a'}{r_0'}\right)^2\left[n'q\int pa'Q\,dt-n'p\int qa'Q\,dt\right],$$

$$(\delta w')=n'\int\left(\frac{a'}{r_0'}\right)^2\frac{m}{1+m'}\left[n'\int a'\frac{\partial R}{\partial W'}\,dt-\frac{1+m'}{m}2\cos\varphi'(\delta\rho')\right]dt.$$

As already indicated:

$$a'Q=2\int a'D_t'R\,dt+a'\frac{\partial R}{\partial\rho'},$$

$$p=\sum p_k\cos kM',$$

$$q = \sum q_k \sin kM\,,$$

$$\left(\frac{a'}{r_0'}\right)^2 = \sum C_k^{-2,0} \cos kM'.$$

Tables 6 and 7 give the coefficients P_k, q_k and $C_k^{-2.0}$ in the form of series of powers of the eccentricity (Sharaf, 1953).

TABLE 6 /48

$$p_0 = -\frac{1}{2}e$$

$$p_1 = 1 - \frac{3}{8}e^2 + \frac{5}{192}e^4 - \frac{7}{9216}e^6 + \frac{9}{737280}e^8 - \ldots$$

$$p_2 = -\frac{1}{2}e - \frac{1}{3}e^3 + \frac{1}{16}e^5 - \frac{1}{180}e^7 + \frac{1}{3456}e^9 - \ldots$$

$$p_3 = -\frac{3}{8}e^2 - \frac{45}{128}e^4 + \frac{567}{5120}e^6 - \frac{729}{40960}e^8 + \ldots$$

$$p_4 = -\frac{1}{3}e^3 - \frac{2}{5}e^5 + \frac{8}{45}e^7 - \frac{8}{189}e^9 + \ldots$$

$$p_5 = \frac{125}{384}e^4 - \frac{4375}{9216}e^6 + \frac{15625}{57344}e^8 - \ldots$$

$$p_6 = \frac{27}{80}e^5 - \frac{81}{140}e^7 + \frac{729}{1792}e^9 - \ldots$$

$$p_7 = \frac{16808}{46080}e^6 - \frac{117649}{163840}e^8 + \ldots$$

$$p_8 = \frac{128}{315}e^7 - \frac{512}{567}e^9 + \ldots$$

$$p_9 = \frac{531441}{1146880}e^8 - \ldots$$

$$p_{10} = \frac{78125}{145152}e^9 - \ldots$$

$$q_1 = 1 - \frac{1}{8}e^2 + \frac{1}{192}e^4 - \frac{1}{9216}e^6 + \frac{1}{737280}e^8 - \ldots$$

$$q_2 = \frac{1}{2}e - \frac{1}{6}e^3 + \frac{1}{48}e^5 - \frac{1}{720}e^7 + \frac{1}{17280}e^9 - \ldots$$

$$q_3 = \frac{3}{8}e^2 - \frac{27}{128}e^4 + \frac{243}{5120}e^6 - \frac{243}{40960}e^8 + \ldots$$

$$q_4 = \frac{1}{3}e^3 - \frac{4}{15}e^5 + \frac{4}{45}e^7 - \frac{16}{945}e^9 + \ldots$$

$$q_5 = \frac{125}{384}e^4 - \frac{3125}{9216}e^6 + \frac{78125}{516098}e^8 - \ldots$$

$$q_6 = \frac{27}{80}e^5 - \frac{243}{460}e^7 + \frac{2187}{8960}e^9 - \ldots$$

TABLE 6 (continued)

$$q_7 = \frac{16807}{46080} e^6 - \frac{823543}{1474560} e^8 + \dots$$

$$q_8 = \frac{128}{315} e^7 - \frac{2048}{2835} e^9 + \dots$$

$$q_9 = \frac{531441}{1146880} e^8 - \dots$$

$$q_{10} = \frac{78125}{145152} e^9 - \dots$$

TABLE 7 /49

$$C_0^{-2,0} = 1 + \frac{1}{2} e^2 + \frac{3}{8} e^4 + \frac{5}{16} e^6 + \frac{35}{128} e^8 + \dots$$

$$C_1^{-2,0} = 2e + \frac{3}{4} e^3 + \frac{65}{96} e^5 + \frac{2675}{4608} e^7 + \frac{63499}{122880} e^9 + \dots$$

$$C_2^{-2,0} = \frac{5}{2} e^2 + \frac{1}{3} e^4 + \frac{21}{32} e^6 + \frac{779}{1440} e^8 + \dots$$

$$C_3^{-2,0} = \frac{13}{4} e^3 - \frac{25}{64} e^5 + \frac{393}{512} e^7 + \frac{10047}{20480} e^9 + \dots$$

$$C_4^{-2,0} = \frac{103}{24} e^4 - \frac{129}{80} e^6 + \frac{1673}{1440} e^8 + \dots$$

$$C_5^{-2,0} = \frac{1097}{192} e^5 - \frac{16621}{4608} e^7 + \frac{543513}{258048} e^9 + \dots$$

$$C_6^{-2,0} = \frac{1223}{160} e^6 - \frac{7589}{1120} e^8 + \dots$$

$$C_7^{-2,0} = \frac{47273}{4608} e^7 - \frac{287019}{245760} e^9 + \dots$$

$$C_8^{-2,0} = \frac{556403}{40320} e^8 + \dots$$

$$C_9^{-2,0} = \frac{10661993}{573440} e^9 + \dots$$

For Pluto the values of p_k, q_k, and $C^{-2.0}$ will be:

$$p_0 = 0.372\,966, \qquad C^{-2,0} = 1.032\,423,$$

$$\frac{1}{2} p_1 = 0.488\,458, \qquad \frac{1}{2} q_1 = 0.496\,146, \qquad \frac{1}{2} C_1^{-2,0} = 0.254\,748,$$

$$\frac{1}{2} p_2 = 0.059\,628, \qquad \frac{1}{2} q_2 = 0.060\,890, \qquad \frac{1}{2} C_2^{-2,0} = 0.077\,998,$$

$$\frac{1}{2} p_3 = 0.010\,932, \qquad \frac{1}{2} q_3 = 0.011\,194, \qquad \frac{1}{2} C_3^{-2,0} = 0.024\,818,$$

$$\frac{1}{2}p_4=0.002\,377,\qquad \frac{1}{2}q_4=0.002\,438,\qquad \frac{1}{2}C_4^{-2,0}=0.008\,020,$$

$$\frac{1}{2}p_5=0.000\,568,\qquad \frac{1}{2}q_5=0.000\,583,\qquad \frac{1}{2}C_5^{-2,0}=0.002\,612,$$

$$\frac{1}{2}p_6=0.000\,144,\qquad \frac{1}{2}q_6=0.000\,148,\qquad \frac{1}{2}C_6^{-2,0}=0.000\,856,$$

$$\frac{1}{2}p_7=0.000\,038,\qquad \frac{1}{2}q_7=0,000\,039,\qquad \frac{1}{2}C_7^{-2,0}=0.000\,280,$$

$$\frac{1}{1}p_8=0.000\,010,\qquad \frac{1}{2}q_8=0.000\,010,\qquad \frac{1}{2}C_8^{-2,0}=0.000\,096,$$

$$\frac{1}{2}p_9=0.000\,003,\qquad \frac{1}{2}q_9=0.000\,003,\qquad \frac{1}{2}C_9^{-2,0}=0.000\,034,$$

$$\frac{1}{2}C_{10}^{-2,0}=0.000\,012.$$

In integration we must multiply by the values $\nu_{ii'}=\frac{n'}{in+i'n'}$, whose values /50
for the Pluto-Jupiter system we give below:

PLUTO-JUPITER

i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$
0	0			1	0.0455727		2	0.0227683
	1	1.0000000		2	0.0435863		3	0.0222787
	2	0.5000000		3	0.0417659		4	0.0217932
	3	0.3333333		4	0.0400914	3	—10	0.0189290
	4	0.2500000		5	0.0385461		— 9	0.0185774
	5	0.2000000		6	0.0371154		— 8	0.0182386
	6	0.1666667	2	— 9	0.0304081		— 7	0.0179119
	7	0.1428571		— 8	0.0295108		— 6	0.0155967
1	—8	0.0772620		—7	0.0286648		— 5	0.0172924
	—7	0.0717207		—6	0.0278661		— 4	0.0169984
	—6	0.0669211		—5	0.0271106		— 3	0.0167143
	—5	0.0627236		—4	0.0263950		— 2	0.0164396
	—4	0.0590215		—3	0.0257162		— 1	0.0161737
	—3	0.0557321		—2	0.0250715		0	0.0159162
	—2	0.0527900		—1	0.0244583		1	0.0156669
	—1	0.0501430		0	0.0238744		2	0.0154252
	0	0.0477488		1	0.0233177		3	0.0151909

In order to get a clearer idea of the course of perturbation computations we will trace all the stages in the work for the values i = 0 and all possible values of i' in the given case. (Computations are carried out to the fifth decimal place).

SCHEME 1

i	i'	$\frac{1}{n'} a'D'_t R$		$2\int a'D'_t R dt$		$a'\frac{\partial R}{\partial \rho'}$		$a'Q$	
		sin	cos	sin	cos	sin	cos	sin	cos
0	0	—	—	—	—	—	—101303	—	—101303
	1	—25767	—454	—908	+51534	+932	— 27183	+24	+ 24351
	2	—13050	—392	—392	13050	369	— 7100	—23	5950
	3	— 5553	—216	—144	3702	113	— 2044	—31	1658
	4	— 2180	— 84	— 42	1090	28	— 608	—14	482
	5	— 840	— 35	— 14	336	+ 5	— 189	— 9	147
	6	— 336	— 6	— 2	112	— 5	— 58	— 7	54
	7	— 112			32	— 7	— 11	— 7	21

Since

$$a'Q = \sum a'Q_s \sin(iM + i'M') + \sum a'Q_c \cos(iM + i'M'),$$

then when multiplying $a'Q$ by $p = \sum p_k \cos kM'$ and $q = \sum q_k \sin kM'$ we have:

$$\begin{aligned} pa'Q &= \sum a'Q_s p_k \sin(iM + i'M') \cos kM' + \sum a'Q_c p_k \cos(iM + i'M') \cos kM' = \\ &\frac{1}{2} \sum a'Q_s p_k \{\sin[iM + (i' + k)M'] + \sin[iM + (i' - k)M']\} + \\ &\frac{1}{2} \sum a'Q_c p_k \{\cos[iM + (i' + k)M'] + \cos[iM + (i' - k)M']\} = \\ &\sum H_s \sin(iM + j'M') + \sum H_c \cos(iM + j'M'), \end{aligned} \tag{65}$$

$$\begin{aligned} qa'Q &= \sum a'Q_s q_k \sin(iM + i'M') \sin kM' + \sum a'Q_c q_k \cos(iM + i'M') \sin kM' = \\ &\frac{1}{2} \sum a'Q_s q_k \{-\cos[iM + (i' + k)M'] + \cos[iM + (i' - k)M']\} + \\ &\frac{1}{2} \sum a'Q_c q_k \{\sin[iM + (i' + k)M'] - \sin[iM + (i' - k)M']\} \\ &= \sum F_s \sin(iM + j'M') + \sum F_c \cos(iM + j'M'). \end{aligned}$$

The scheme for computing coefficients F_s is given as an example /52

(Scheme 2).

H_s, H_c, and F_c are computed in a similar manner.

Summing the numbers on each line and assuming that j = i' we obtain the coefficients qa'Q and pa'Q.

SCHEME 2

i	j'	$\frac{1}{2}a'Q_c q_1$ $(j'=i'+1)$	$-\frac{1}{2}a'Q_c q_1$ $(j'=i'-1)$	$\frac{1}{2}a'Q_c q_2$ $(j'=i'+2)$	$-\frac{1}{2}a'Q_c q_2$ $(j'=i'-2)$	$\frac{1}{2}a'Q_c q_3$ $(j'=i'+3)$	$-\frac{1}{2}a'Q_c q_3$ $(j'=i'-3)$	$\frac{1}{2}a'Q_c q_4$ $(j'=i'+4)$	$-\frac{1}{2}a'Q_c q_4$ $(j'=i'-4)$	$\frac{1}{2}a'Q_c q_5$ $(j'=i'+5)$	$-\frac{1}{2}a'Q_c q_5$ $(j'=i'-5)$	$\frac{1}{2}a'Q_c q_6$ $(j'=i'+6)$	$-\frac{1}{2}a'Q_c q_6$ $(j'=i'-6)$	$\frac{1}{2}a'Q_c q_7$ $(j'=i'+7)$	$-\frac{1}{2}a'Q_c q_7$ $(j'=i'-7)$
0	−8														
	−7														−4
	−6												+15		−1
	−5										+59		−4		
	−4								+247		−14		−1		
	−3						+1134		−59		−3				
	−2				+6168		−273		−15		−1				
	−1		+50261		−1483		−67		−4						
	0		−12082		−362		−19		−1						
	1	−50261	−2952		−101		−5								
	2	+12082	−823	−6168	−29		−2								
	3	2952	−239	+1483	9	1134	−1								
	4	823	−73	362	−3	+273		247							
	5	239	−27	101	−1	67		59		59					
	6	73	−10	29		19		15		−14		−15			
	7	27		9		5		4		3		−4		−4	
	8	10		3		2		1		1		1		−1	
	9			1		1									

SCHEME 3

i	i'	$pa'Q$		$qa'Q$		$n'\int pa'Qdt$		$n'\int qa'Qdt$	
		sin	cos	sin	cos	sin	cos	sin	cos
0	0		+ 50051		+11		$-50051n't$		$+11n't$
	1	−23	−103515	102026	−14	−103515	+ 23	−14	102026
	2	+ 5	− 1285	− 819	−28	− 642	− 2	−14	+ 410
	3	+ 6	+ 1832	+ 1980	+ 2	+ 611	− 2	+ 1	− 660
	4	−15	859	903	12	215	+ 4	3	− 226
	5	− 9	317	324	6	63	2	1	− 65
	6	− 5	108	111	2	18	1		− 18
	7	0	35	44	3	5			− 6
	8	− 3	17	17	3	2			− 2

To derive n'p $\int$ qa'Qdt and n'q $\int$ pa'Qdt we use Formulas (65) and a system of computations similar to Scheme 2. This scheme of computations will also be the same when multiplying $qn'\int pa'Q\,dt - pn'\int qa'Q\,dt$ by $\left(\frac{a'}{r_0'}\right)^2$.

SCHEME 4

i	i'	$qn'\int pa'Qdt$		$pn'\int qa'Qdt$		$qn'\int pa'Qdt - pn'\int qa'Qdt$		$\frac{1+m'}{m}(\delta\rho')$	
		sin	cos	sin	cos	sin	cos	sin	cos
0	0		−51390		+49852		−101242		−98100
	1	+ 2	− 6588	+5	−31808	−1	+ 25220	3	−23579
	2	+13	+50515	−4	+50463	−2	+ 52	15	− 8739
	3	− 2	6481	−2	6659	−7	− 178	5	− 3044
	4	− 2	867	0	926	−2	− 59	0	− 1039
	5	+ 2	109	+1	128	+1	− 19	1	− 353
	6	1	+ 10		+ 15	1	− 5		− 121
	7		− 2		0		− 2		− 38
	8		− 2		− 1		− 1		− 14
	9								− 4
	10								− 1

After multiplying $\frac{\delta\rho'}{m}$ by mM=0.00041465834, where m is the mass of Jupiter and M is the conversion modulus from natural to decimal logarithms, we obtain $(\delta \lg r')$.

Knowing $(\delta\rho')$, we may determine $(\delta w')$ (Scheme 5).

SCHEME 5

i	i'	$a'\frac{\partial R}{\partial W'}$ sin	$a'\frac{\partial R}{\partial W'}$ cos	$n\int a'\frac{\partial R}{\partial W'}dt$ sin	$n\int a'\frac{\partial R}{\partial W'}dt$ cos	$-2\cos\varphi'(\delta\rho')$ sin	$-2\cos\varphi'(\delta\rho')$ cos	$\left[n'\int a'\frac{\partial R}{\partial W'}dt-2\cos\varphi'(\delta\rho')\right]$ sin	$\left[n'\int a'\frac{\partial R}{\partial W'}dt-2\cos\varphi'(\delta\rho')\right]$ cos
0	0						190038		190038
	1	−776	−460	−460	776	− 6	45677	−466	46453
	2	−324	−172	− 86	162	−30	16929	−116	17091
	3	−109	− 44	− 15	36	−10	5897	− 25	5933
	4	− 31	− 7	− 2	8	0	2013	− 2	2021
	5	− 10	+ 2		2	− 2	684	− 2	685
	6	− 10	3				234		234
	7	+ 4	4				74		74
	8						27		27
	9						8		8

SCHEME 5 (continued)

i	i'	$\left[n'\int a'\frac{\partial R}{\partial W'}dt-2\cos\varphi'(\delta\rho')\right]\left(\frac{a'}{r_0'}\right)^2$ sin	$\left[n'\int a'\frac{\partial R}{\partial W'}dt-2\cos\varphi'(\delta\rho')\right]\left(\frac{a'}{r_0'}\right)^2$ cos	$1+\frac{m'}{m}(\delta w')$ sin	$1+\frac{m'}{m}(\delta w')$ cos
0	0		209532		$209532n't$
	1	−477	153757	153757	477
	2	−231	62120	31060	115
	3	− 97	24533	8178	32
	4	− 41	9465	2366	10
	5	− 16	3587	717	3
	6	− 5	1340	223	1
	7		491	70	
	8		181	23	
	9		64	7	
	10		22	2	
	11		9	1	

(b) Secular and Mixed Terms

Secular and mixed terms in $\delta\rho'$ appear because of the presence of free terms in pQ and qQ (see Scheme 3). In the intervals $n'\int pa'Qdt$ and $n'\int qa'Qdt$ this gives 0.50051 n't and 0.00011 n't.

SCHEME 6

i	i'	$q0.50051n't$ $n't$ sin	$p0.00011n't$ $n't$ cos	$(q0.50051n't - p0.00011n't)$ $n't$ sin	$(q0.50051n't - p0.00011n't)$ $n't$ cos	$\frac{1}{m}(\delta\rho')$ secular $n't$ sin	$\frac{1}{m}(\delta\rho')$ secular $n't$ cos
0	0		− 4		+ 4		+ 2
	1	49665	+11	49665	−11	48886	−10
	2	6095	1	6095	− 1	17966	− 4
	3	1121		1121		6236	− 1
	4	244		244		2127	
	5	58		58		718	
	6	15		15		242	
	7	4		4		81	
	8	1		1		26	
	9					8	
	10					3	
	11					1	

SCHEME 7

/54

i	i'	$2\cos\varphi'(\delta\varphi')_{\text{sec.}}$ $n't$ sin	$2\cos\varphi'(\delta\varphi')_{\text{sec.}}$ $n't$ cos	$\left(n'\int a'\frac{\partial R}{\partial W'}dt\right)_{\text{sec.}}$ $n't$ sin	$\left(n'\int a'\frac{\partial R}{\partial W'}dt\right)_{\text{sec.}}$ $n't$ cos	$\frac{d(\delta\omega')}{dt}$ secular $n't$ cos	$\frac{d(\delta\omega')}{dt}$ secular sin	$\frac{1}{m}(\delta\omega')$ secular $n't$ sin	$\frac{1}{m}(\delta\omega')$ secular cos
0	0		−5		$\frac{1}{21}$				
	1	94701	19	99334		99334	−99334	21	21
	2	31801	8	−60829	13	30414	−15207	6	3
	3	−12080	2	−29033	3	9678	−3226	1	
	4	−4120		12511	1	3128	−786		
	5	1391		−5092		1018	−204		
	6	469		1999		333	−56		
	7	157		784		112	−16		
	8	−51		287		36	−4		
	9	17		−105		12	−1		
	10	2		−40		4			
	11			−13		1			
	12			−1					

There will be no term in $\frac{d\delta\omega'}{dt}$ containing t outside the sine or cosine sign. In fact such a term might appear because of the presence in $\frac{d\delta\omega'}{dt}$ of the following expressions:

$$m\left(\frac{a'}{r_0'}\right)^2 n'\int a'\frac{\partial R}{\partial W'}dt \quad 2\left(\frac{a'}{r_0'}\right)^4\cos\varphi' pn'\int qa'Q\,dt.$$

Considering that

$$\left(\frac{a'}{r_0'}\right)^2\left[e'n'\int qa'Q\,dt - \cos\varphi' n'\int a'\frac{\partial R}{\partial W'}dt\right] = \int a'D_t R\,dt,$$

we obtain

$$m\left(\frac{a'}{r_0'}\right)^2 n'\int a'\frac{\partial R}{\partial W'}dt + 2\left(\frac{a'}{r_0'}\right)^4\cos\varphi' pn'\int qa'Q\,dt = \frac{1}{\cos\varphi'}\int a'D_t R\,dt$$
$$+\left[-\left(\frac{a'}{r_0'}\right)^2\frac{e'}{\cos\varphi'} \quad 2\left(\frac{a'}{r_0'}\right)^4\cos\varphi' p\right]n'\int qa'Q\,dt. \tag{66}$$

The expansion of $a'D_t'R$ does not contain a constant term; consequently, t in Expression (66) may lie outside the sine or cosine only on the condition that the constant part of the expansion $-\left(\frac{a'}{r_0'}\right)^2\frac{e'}{\cos\varphi'} \quad 2\left(\frac{a'}{r_0'}\right)^4\cos\varphi' p$ is not equal to zero. Let us prove the opposite.

Let us note that

$$p = \cos E' - e' = \frac{\cos^2\varphi'}{e'} - \frac{1}{e'}\left(\frac{a'}{r_0'}\right)^{-1}.$$

Then

$$-\left(\frac{a'}{r_0'}\right)^2 \frac{e'}{\cos\varphi'} + 2\left(\frac{a'}{r_0'}\right)^4 \cos\varphi' p = \frac{1}{\cos\varphi' e'}\left[-\left(\frac{a'}{r_0'}\right)^2 e'^2 + 2\cos^4\varphi'\left(\frac{a'}{r_0'}\right)^4 - \right.$$
$$\left. -2\cos^2\varphi'\left(\frac{a'}{r_0'}\right)^3\right].$$

The free terms of the expansion $\left(\frac{a'}{r_0'}\right)^2, \left(\frac{a'}{r_0'}\right)^4, \left(\frac{a'}{r_0'}\right)^3$ in mean anomalies are

$$C_0^{-2,0} = 1 + \frac{1}{2}e'^2 + \frac{3}{8}e'^4 + \frac{5}{16}e'^6 + \ldots,$$
$$C_0^{-4,0} = 1 + 3e'^2 + \frac{45}{8}e'^4 + \frac{35}{4}e'^6 + \ldots,$$
$$C_0^{-3,0} = 1 + \frac{3}{2}e'^2 + \frac{15}{8}e'^4 + \frac{35}{16}e'^6 + \ldots.$$

We multiply $C_0^{-2.0}$ by e'^2, $C_0^{-4.0}$ by $2(1 - e'^2)^2$, and $C_0^{-3.0}$ by $-2(1 - e'^2)$, add term by term, and obtain zero in the sum. /55

Consequently the underlined number in scheme 7 is the result of accumulation of error in the computations.

(c) Constants of Integration :

In determining the constants of integration for the initial moment t_0 we take the moment of osculation, that is, 1930 September 20.0.

For $t = t_0$ we have

$$M_0' = 275^\circ.169,$$
$$M_{10} = 77^\circ.480,$$

and for different multiples $iM_{10} + i'M_0'$ we compile the tables $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left(iM_{10} + i'M_0'\right)$.

We compute the constants from the following formulas:

$$
\left.
\begin{aligned}
a'C_1 &= -\left[\int a' D_t' R\, dt\right]_{t=t_0}, \\
a'K_1 &= \left[n' \int q\ (a'Q + 2C_1)\, dt\right]_{t=t_0}, \\
a'K_2 &= -\left[n' \int p\,(a'Q + 2C_1)\, dt\right]_{t=t_0}, \\
C_2 &= -\left[n' \int a' \frac{\partial R}{\partial W'}\, dt\right]_{t=t_0}, \\
C_3 &= -\left\{n' \int \left(\frac{a'}{r_0'}\right)^2 \left[\frac{m}{1+m'}\left(n' \int a' \frac{\partial R}{\partial W'}\, dt + C_2\right) - 2\cos\varphi' \delta\rho'\right] dt\right\}_{t=t_0}.
\end{aligned}
\right\} \quad (67)
$$

For verifying the computation of perturbations and constants we use an equation connecting the constants of Integration in Equation (55)

$$\cos\varphi' C_2 - e'a'K_1 - \left(1 + \frac{3}{2}e'^2\right) a'C_1 = A_0.$$

The constants are

$$
\begin{aligned}
a'C_1 &= -2.2758, \\
a'K_1 &= -0.5195, \\
a'K_2 &= -5.0496, \\
C_2 &= -2.7675.
\end{aligned}
$$

The verification gives

$$\cos\varphi' C_2 - e'a'K_1 - \left(1 + \frac{3}{2}e'^2\right) a'C_1 = -0.0649,\ A_0 = -0.0646.$$

In the coordinates of the perturbations we are left with an indeterminate part of the perturbations depending on the constants of integration. Let us designate them by $(\delta\rho')_c$ and $(\delta w')_c$. Then

$$(\delta\rho')_c = \frac{m}{1+m'}\left(\frac{a'}{r_0'}\right)^2 \cdot \left[2a'C_1\left(n'q\int p\, dt - n'p\int q\, dt\right) + a'K_1 p + a'K_2 q\right],$$

$$(\delta w')_c = n' \int \left(\frac{a'}{r_0'}\right)^2 \left[\frac{m}{1+m'} C_2 - 2\cos\varphi' (\delta\rho')_c\right] dt + C_3.$$

We show in Scheme 8 the part of the perturbations which depends on the constants and is computed according to these formulas. /56

Scheme 8

i	i'	$(\delta \lg r')_c\ 10^7$ sin	cos	$n't$ sin	$(\delta \omega')''_c$ sin	cos	$n't$ cos
0	0		—17861			+ 409.80	1331.03
	1	—20451	— 7460	6874	738.01	—1973.72	663.41
	2	— 7516	— 2768	2526	225.26	— 604.29	203.08
	3	— 2609	— 963	877	71.55	— 192.29	64.63
	4	— 890	— 329	299	23.10	— 62.15	[illegible]
	5	— 301	— 112	101	7.52	— 20.25	6.81
	6	— 101	— 37	32	2.46	— 6.62	2.21
	7	— 34	— 13	11	0.81	— 2.17	0.71
	8	— 11	— 4	5	0.26	— 0.71	0.26
	9	— 3	— 1	1	0.10	— 0.24	0.08
	10	— 1			0.02	— 0.08	0.02

12. Determining the Perturbation of Ω' and i'

We conduct the computations by Formulas (48) of Chapter II, Section 6.

$$\sin i'\,\delta\Omega' = -\frac{m}{1+m'}\,\mathrm{sc}\,\varphi' n' \int\left[a'\frac{\partial R}{\partial W'}\sin N'\,\mathrm{ctn}\,J + a'\frac{\partial R}{\partial W}\frac{\sin N'}{\sin J} + \frac{1}{2}a'\frac{\partial R}{\partial \sigma}\cos N'\cos\frac{J}{2}\right]dt + C_3.$$

$$\delta i' = \frac{m}{1+m'}\,\mathrm{sc}\,\varphi' n' \int\left[a'\frac{\partial R}{\partial W'}\left(\mathrm{ctn}\,i' - \frac{\sin i\cos N}{\sin i'\sin J}\right) - a'\frac{\partial R}{\partial W}\frac{\cos N'}{\sin J} + \frac{1}{2}a'\frac{\partial R}{\partial \sigma}\sin N'\cos\frac{J}{2}\right]dt + C_4. \qquad (48)$$

The values of the coefficients entering here, which depend only upon the elements of Pluto and Jupiter, will be

$$-\sin N'\,\mathrm{ctn}\,J = 0.0503451,\quad \mathrm{ctn}\,i' - \frac{\sin i\cos N}{\sin i'\sin J} = 3.520250,$$

$$\frac{-\sin N'}{\sin J} = 0.0523366,\quad -\frac{\cos N'}{\sin J} = 3.659503,$$

$$-\frac{1}{2}\cos N'\cos\frac{J}{2} = 0.4951700,\quad \frac{1}{2}\sin N'\cos\frac{J}{2} = -0.0070817.$$

13. Perturbations of Pluto by Jupiter

Table 8 gives the perturbations of Pluto by Jupiter. The perturbations of the logarithm of the radius vector are given to the seventh decimal place of decimal logarithms, the perturbation of longitude in orbit, longitude of the node, and inclination are given in hundredths of a second of arc.

Translator's note: ctn = cot.

TABLE 8

i	i'	$\delta \log r' \cdot 10^7$ sin	$\delta \log r' \cdot 10^7$ cos	$\delta w'$ sin	$\delta w'$ cos	$\delta \Omega'$ sin	$\delta \Omega'$ cos	$\delta i'$ sin	$\delta i'$ cos
0	0		—21929		+ 408.62″		+ 9.24″		+152.40″
	0				+1743.68$n't$		— 2.76$n't$		
	1	—20451	— 8438	+845.19″	—1972.82	— 3.59″	— 0.54	+0.15″	— 0.43
	1	+ 8901$n't$			+ 859.01$n't$				
	2	— 7516	— 3130	256.48	— 604.06	— 1.55	— 0.95	0.27	— 0.34
	2	+ 3271$n't$			+ 262.98$n't$				
	3	— 2609	— 1089	81.31	— 192.23	— 0.66	— 0.52	0.15	— 0.16
	3	+ 1136$n't$			+ 83.70$n't$				
	4	— 890	— 372	26.22	— 62.13	— 0.26	— 0.23	0.07	— 0.07
	4	+ 387$n't$			+ 27.04$n't$				
	5	— 301	— 127	8.53	— 20.24	— 0.10	— 0.10	0.03	— 0.03
	5	+ 131$n't$			+ 8.81$n't$				
	6	— 101	— 42	2.79	— 6.62	— 0.04	— 0.04	0.01	— 0.01
	6	— 42$n't$			+ 2.87$n't$				
	7	— 34	— 15	0.92	— 2.17	— 0.01	— 0.01		
	7	+ 14$n't$			+ 0.93$n't$				
	8	— 11	— 5	0.30	— 0.71				
	8	+ 6$n't$			+ 0.33$n't$				
	9	— 3	— 1	0.11	— 0.24				
	9	+ 1$n't$			+ 0.10$n't$				
	10	— 1		+ 0.02	— 0.08				
	10				+ 0.03$n't$				
1	— 8			— 0.02					
	— 7			— 0.04	0.02				
	— 6	— 1	— 2	— 0.10	0.06	— 0.07		0.02	+ 0.02
	— 5	— 3	— 6	— 0.32	0.18	— 0.28	+ 0.14	0.06	0.10
	— 4	— 10	— 19	— 0.91	0.53	— 1.17	0.69	0.22	0.37
	— 3	— 32	— 56	— 2.66	1.52	— 5.73	3.24	0.98	1.63
	— 2	— 91	— 160	— 7.62	4.37	— 29.11	17.11	5.08	8.50
	— 1	— 249	— 434	— 20.66	+ 11.86	—228.52	+133.61	+39.75	66.12
	0	+ 34	+ 58	+ 2.84	— 1.58	+176.03	— 21.93	—22.59	+ 2.85
	1		13	— 0.26	— 0.22	—231.21	— 66.70	+20.25	— 67.16
	2		4	— 0.14	— 0.06	— 26.76	— 7.79	2.34	— 7.83
	3		1	— 0.06	— 0.02	— 4.62	— 1.38	0.41	— 1.36
	4			— 0.02		— 0.97	— 0.28	0.08	— 0.28
	5					— 0.21	— 0.07	0.02	— 0.06
	6					— 0.07			— 0.02
2	— 4			— 0.02	+ 0.02	— 0.07	+ 0.07	0.02	+ 0.02
	— 3		— 1	— 0.06	0.04	— 0.28	— 0.21	0.06	0.08
	— 2	— 2	— 5	— 0.18	0.12	— 1.38	+ 0.83	0.24	0.41
	— 1	— 5	— 10	— 0.49	+ 0.30	— 10.76	+ 6.28	+ 1.87	3.11
	0	+ 1	+ 2	+ 0.06	— 0.04	+ 8.42	— 1.03	— 1.10	+ 0.12
	1		1			— 11.45	— 3.31	+ 1.00	— 3.33
	2					— 1.38	— 0.41	0.12	— 0.41
	3					— 0.28	— 0.07	0.02	— 0.08
	4					— 0.07			— 0.02
3	— 2					— 0.07	+ 0.07	0.02	+ 0.02
	— 1			— 0.02	+ 0.02	— 0.55	+ 0.55	+ 0.10	+ 0.16
	0					+ 0.48	— 0.07	— 0.06	— 0.08
	1					— 0.62	— 0.21	+ 0.06	— 0.18
	2					— 0.07			— 0.02

CHAPTER IV /58

PERTURBATIONS OF THE FIRST ORDER BY SATURN

14. Elements

To determine the perturbations of Pluto by Saturn the elements of Pluto given in Chapter III and the mean elements of Saturn given by Hill are used.

The elements of Saturn in terms of ecliptic and equinox of 1950.0 are:

Epoch 1590 January 0 Greenwich mean noon

$$M = 284^\circ.717$$
$$\Omega = 113^\circ.481$$
$$i = 2.488$$
$$\omega = 338.026$$
$$e = 0.0560608$$
$$a = 9.538843$$
$$n = 120''.455$$
$$\frac{1}{m} = 1047.4$$

The elements of the mutual position of the orbits, as in the case of Pluto-Jupiter, are derived for verification from the two systems of Formulas (63) and (64), have the following values:

$$N = 175^\circ.5174, \quad \Pi = 162^\circ.509, \quad \sin\frac{J}{2} = 0.127605, \quad \alpha = 0.2413813,$$
$$N' = 179^\circ.3405, \quad \Pi' = 294^\circ.202, \quad \cos\frac{J}{2} = 0.991825.$$

15. Expansion of the Perturbation Function

For expansion of the perturbation function in the case of Pluto-Saturn, the method of Newcomb's operators was used, as in the case of Pluto-Jupiter. The coefficients of the expansion of the perturbation function were derived on analytical computers. For verification all the computations were repeated either in another computer setup, or with replaced counters. The method of computation has been described in detail in Chapter I, Section 4; therefore we will not consider it here.

TABLE 9

i	c_1^i	Dc_1^i	$D^2c_1^i$	$D^3c_1^i$	$D^4c_1^i$	$D^5c_1^i$	$D^6c_1^i$	$D^7c_1^i$	$D^8c_1^i$
0	2.0301276	0.0623303	0.1333161	0.3034825	0.7685746	2.2760571	8.124392	34.77603	174.02
1	0.2468560	0.2582243	0.2940814	0.4108845	0.8109507	2.2856545	8.282705	35.68428	177.72
2	0.0448020	0.0918994	0.1933510	0.427246	1.031510	2.86761	9.59589	39.037	188.94
3	0.00902333	0.0275558	0.0851780	0.268674	0.876066	3.01081	11.19200	46.268	217.72
4	0.00190725	0.00773475	0.0315911	0.130427	0.547319	2.35346	10.48880	49.180	247.14
5	0.000414548	0.00209619	0.0106487	0.0544567	0.281123	1.47080	7.84173	42.904	243.64
6	0.0000917586	0.000555805	0.00337776	0.0206194	0.126636	0.78419	4.91077	31.222	202.61
7	0.0000205724	0.000145196	0.00102729	0.00729156	0.051973	0.37250	2.68921	19.579	144.66
8	0.0000046564 1	0.0000375227	0.000302941	0.00245173	0.019903	0.16221	1.32861	10.952	91.05
9	0.00000106145	0.00000961775	0.0000872332	0.00079309	0.0072238	0.066021	0.605301	5.558	51.67

i	c_3^i	Dc_3^i	$D^2c_3^i$	$D^3c_3^i$	$D^4c_3^i$	$D^5c_3^i$	$D^6c_3^i$	$D^7c_3^i$	$D^8c_3^i$
0	0.5523056	0.7049659	1.221835	3.09661	10.5684	43.9670	213.40	1185.5	7433
1	0.1956464	0.4367986	1.072057	3.04463	10.4004	42.9004	208.80	1167.0	7342
2	0.0585937	0.1885173	0.633672	2.27484	8.94646	39.3957	196.84	1114.2	7077
3	0.0164384	0.0692006	0.298653	1.33564	6.27444	31.4272	170.40	1010.7	6586
4	0.0044539	0.0231820	0.122604	0.66279	3.68997	21.3303	129.70	836.5	5771
5	0.0011808	0.0073220	0.045924	0.29224	1.89590	12.6006	86.54	617.2	4609
6	0.0003084	0.0022197	0.016119	0.11821	0.87884	6.6217	51.32	407.4	3335
7	0.0000797	0.0006536	0.005388	0.04483	0.37625	3.1970	27.58	242.6	2178
8	0.00002043	0.00018787	0.001736	0.01612	0.15122	1.4090	13.66	133.1	1309
9	0.00000526	0.00005302	0.000546	0.00557	0.05780	0.6017	6.32	67.3	721

i	c_5^i	Dc_5^i	$D^2c_5^i$	$D^3c_5^i$	$D^4c_5^i$	$D^5c_5^i$	$D^6c_5^i$
0	0.1676506	0.457410	1.49389	5.9222	27.967	152.87	945.4
1	0.0913544	0.323168	1.24681	5.3713	26.248	145.67	910.2
2	0.0372198	0.166211	0.77919	3.8926	21.033	124.27	806.2
3	0.0132294	0.071811	0.40191	2.3382	14.320	93.14	647.7
4	0.0043429	0.027809	0.18189	1.2210	8.482	61.40	465.4
5	0.0013521	0.009992	0.07496	0.5708	4.481	36.26	301.9
6	0.0004058	0.003404	0.02881	0.2477	2.168	19.37	176.9
7	0.0001187	0.001111	0.01049	0.0981	0.970	9.77	96.4
8	0.0000337	0.000348	0.003682	0.0383	0.410	4.44	48.3
9	0.0000098	0.000104	0.001233	0.0122	0.162	2.22	23.7

i	c_7^i	Dc_7^i	$D^2c_7^i$	$D^3c_7^i$	$D^4c_7^i$
0	0.0554908	0.23887	1.1650	6.447	40.14
1	0.037816	0.18686	0.9970	5.812	37.22
2	0.018949	0.10960	0.6663	4.298	29.64
3	0.008040	0.05390	0.3737	2.698	20.39
4	0.003074	0.02343	0.1840	1.492	12.45
5	0.001092	0.00940	0.0826	0.744	6.82
6	0.000369	0.00345	0.0342	0.351	3.49
7	0.000120	0.00127	0.0136	0.150	1.61
8	0.000037	0.00035	0.0050	0.076	1.36
9	0.000016	0.00014	0.0017	0.026	0.30

TABLE 10

i	$a'A_i$	$a'DA_i$	$a'D^2A_i$	$a'D^3A_i$	$a'D^4A_i$	$a'D^5A_i$	$a'D^6A_i$	$a'D^7A_i$	$a'D^8A_i$
0	2.02701504	0.0551282	0.116584	0.256888	0.61357	1.66304	5.260	15.774	54.47
1	0.24192817	0.2511134	0.279611	0.369917	0.66564	1.68864	5.460	16.961	59.59
2	0.04310661	0.0878910	0.182650	0.393730	0.90698	2.33321	6.970	21.307	75.55
3	0.00852416	0.0258920	0.079300	0.246199	0.78137	2.57155	8.912	30.386	113.12
4	0.00176911	0.00713864	0.0289316	0.1180211	0.48610	2.03372	8.680	35.926	156.00
5	0.00037757	0.00190013	0.0095866	0.0485392	0.24699	1.26788	6.562	32.777	169.50
6	0.00008207	0.00049481	0.0029883	0.0180844	0.10976	0.66983	4.103	24.222	147.35
7	0.00001807	0.00012696	0.0008929	0.0062877	0.04436	0.31483	2.231	15.180	106.85

i	$a'\sigma^2B_i$	$a'D\sigma^2B_i$	$a'D^2\sigma^2B_i$	$a'D^3\sigma^2B_i$	$a'D^4\sigma^2B_i$	$a'D^5\sigma^2B_i$	$a'D^6\sigma^2B_i$
0	0.0044611	0.0056148	0.009471	0.02319	0.07633	0.3000	1.375
1	0.0015528	0.0034354	0.008294	0.02294	0.07562	0.2942	1.352
2	0.0004567	0.0014587	0.004845	0.01707	0.06534	0.2733	1.293
3	0.0001258	0.0005263	0.002249	0.00992	0.04566	0.2189	1.134
4	0.0000335	0.0001732	0.000908	0.00485	0.02659	0.1479	0.867
5	0.0000087	0.0000537	0.000335	0.00211	0.01349	0.0865	0.577

i	$a'\sigma^4C_i$	$a'D\sigma^4C_i$	$a'D^2\sigma^4C_i$	$a'D^3\sigma^4C_i$	$a'D^4\sigma^4C_i$
0	0.000016	0.000044	0.000140	0.00054	0.00248
1	0.000009	0.000031	0.000117	0.00049	0.00233
2	0.000004	0.000016	0.000072	0.00035	0.00186
3	0.000001	0.000007	0.000037	0.00021	0.00125

TABLE 11

i	$a'D_\sigma A_i$	$a'DD_\sigma A_i$	$a'D^2D_\sigma A_i$	$a'D^3D_\sigma A_i$	$a'D^4D_\sigma A_i$	$a'D^5D_\sigma A_i$	$a'D^6D_\sigma A_i$	$a'D^7D_\sigma A_i$	$a'D^8D_\sigma A_i$
0	—0.04765305	—0.1049572	—0.251305	—0.6859039	—2.219608	—8.26809	—36.4764	—297.8301	—1873.75
1	—0.07653286	—0.1089603	—0.2171896	—0.6009348	—2.079042	—8.07720	—36.1459	—293.4532	—1851.54
2	—0.02609167	—0.0611337	—0.1608236	—0.4934275	—1.787659	—7.26743	—33.9401	—277.8854	—1777.28
3	—0.00760714	—0.0251716	—0.0879351	—0.3309236	—1.364982	—6.02076	—29.8157	248.9191	1639.47
4	—0.00208452	—0.0089375	—0.0395079	—0.1819430	0.882516	—4.40492	—23.9208	—[illegible]7.7282	—1428.54
5	—0.00055241	—0.0029119	—0.0156481	—0.0862800	—0.490551	—2.79403	—17.0162	—158.7274	—1161.97
6	—0.00014335	—0.0008972	—0.0056891	—0.0366822	—0.241315	—1.56909	—10.7559	—109.7148	—866.06
7	—0.00003666	—0.0002656	—0.0019446	—0.0114396	—0.108428	—0.78306	—6.0752	—68.9705	—592.69

i	$a'D_\sigma\sigma^2B_i$	$a'DD_\sigma\sigma^2B_i$	$a'D^2D_\sigma\sigma^2B_i$	$a'D^3D_\sigma\sigma^2B_i$	$a'D^4D_\sigma\sigma^2B_i$	$a'D^5D_\sigma\sigma^2B_i$
0	0.06937696	0.0861063	0.141288	0.333497	1.05555	3.79437
1	0.02371874	0.0520030	0.123461	0.332268	1.05392	3.74673
2	0.00684691	0.0217061	0.071219	0.246189	0.91567	3.53853
3	0.00185055	0.0076891	0.032537	0.141347	0.63669	2.85289
4	0.00048288	0.0024831	0.012910	0.068179	0.36708	1.91521
5	0.00012326	0.0007557	0.004667	0.029103	0.18362	1.10435

i	$a'D_\sigma\sigma^4C_i$	$a'DD_\sigma\sigma^4C_i$	$a'D^2D_\sigma\sigma^4C_i$	$a'D^3D_\sigma\sigma^4C_i$
0	0.0005081	0.001354	0.00428	0.01625
1	0.0002706	0.000941	0.00354	0.01470
2	0.0001073	0.000472	0.00217	0.01051
3	0.0000370	0.000198	0.00109	0.00619

TABLE 12

i	i'	$a'R\cdot 10^4$		$a'D'R\cdot 10^4$		$a'D_\sigma R\cdot 10^4$		$a'\frac{\partial R}{\partial W'}\cdot 10^4$		$a'\frac{\partial R}{\partial W}\cdot 10^4$		$\frac{a'}{n'}D'_t R\cdot 10^4$	
		sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos
0	0	—	+10151.7	—	—10466.6	—	— 274.6	—	+ 1.7	—	— 1.0	—	—
	1	+ 150.8	2715.3	— 313.2	— 3098.2	— 6.8	— 240.2	134.6	153.1	131.7	146.5	— 2715.3	+ 150.8
	2	87.2	701.2	— 198.7	— 844.8	+ 123.0	— 249.6	— 46.5	101.6	65.0	— 81.0	— 1408.5	174.5
	3	43.9	201.0	— 108.6	— 244.0	140.4	— 174.8	— 9.0	56.5	26.3	— 35.8	— 603.4	132.3
	4	20.0	59.0	— 52.7	— 68.5	84.0	— 102.4	+ 0.8	27.8	9.8	— 15.7	— 236.4	80.7
	5	9.1	18.2	— 24.0	— 20.9	39.7	— 47.0	1.1	12.8	4.2	— 6.9	— 90.6	44.9
	6	3.5	5.3	— 9.6	— 3.7	21.9	— 31.5	2.5	5.0	+ 1.3	— 1.9	— 31.5	21.1
	7	0.8	1.1	— 2.3	+ 0.1	+ 10.0	— 10.9	+ 1.3	+ 1.3			— 7.6	5.0
1	—8		1	— 9	5	— 1	+ 1.3	— 3		— 1		+ 8	+ 1
	—7	+ 1	5	— 40	15	— 9	30	+ 10	— 4	— 6	+ 5	35	— 8
	—6	— 6	19	— 110	52	— 9	33	28	— 4	— 22	+ 4	116	+ 40
	—5	— 71	87	— 312	197	— 1	+ 44	101	+ 52	— 93	— 52	435	355
	—4	— 420	415	— 1001	757	+ 75	— 15	432	384	— 422	— 385	1662	1682
	—3	— 2323	2129	— 3667	3078	531	— 466	2144	2267	— 2138	— 2266	6387	6970
	—2	— 14000	12530	— 16908	14885	3548	— 3180	12537	13939	— 12538	— 13939	25059	27998
	—1	—120210	+107081	— 125512	+ 111876	+ 31373	— 27635	+ 107082	+120260	—107084	— 120218	+ 107081	120210
	0	+ 45902	— 41985	+ 45866	— 41855	— 28456	+ 8972	— 41689	— 48011	+ 41937	+ 45936	0	0
	1	1712	+ 1216	1802	+ 1315	+ 41631	4667	+ 516	+ 3664	— 1138	1618	— 1216	1712
	2	235	107	299	145	4797	559	+ 26	385	— 98	224	— 213	468
	3	41	14	74	27	783	102		60	— 13	40	— 45	123
	4	7	3	20	7	144	21		10	— 2	8	— 9	31
	5	+ 1	+ 1	8	1	21	10		1	— 1	+ 2	— 2	5
	6			3		4	+ 2						1
2	—9	— 4	— 1			10	— 6	— 3	7	+ 2	— 8		
	—8	— 12	— 5			25	— 5	— 15	24	14	— 26		
	—7	— 24	— 10			31	0	— 30	51	29	— 52		
	—6	— 56	— 13	156	68	56	0	— 42	111	40	— 113	— 75	334
	—5	— 125	— 12	321	117	99	+ 5	— 55	241	53	— 243	— 61	624
	—4	— 271	+ 18	572	203	170	+ 22	— 40	489	+ 38	— 489	+ 69	1081

/61

TABLE 12 (continued)

i	i'	$a'R\cdot 10^4$		$a'D'R\cdot 10^4$		$a'D_5R\cdot 10^4$		$a'\frac{\partial R}{\partial W'}\cdot 10^4$		$a'\frac{\partial R}{\partial W}\cdot 10^4$		$\frac{a'}{n'}D'_tR\cdot 10^4$	
		sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos
2	−3	−630	218	+701	445	302	−12	+158	972	−160	−971	654	1891
	−2	−2008	1435	−625	1630	681	−312	1399	2367	−1405	−2364	2870	4016
	−1	−13562	+12143	−13912	+12212	+3610	−3043	+12142	+13521	−12156	−13508	+12143	13562
	0	+5154	−4686	+5107	−4747	−2986	+1218	−4670	−5374	+4669	+5166	0	0
	1	196	+143	182	+119	+4790	612	+61	+415	−142	193	−143	196
	2	29	15	23	+6	597	93	4	46	−16	29	−30	58
	3	7	3	3	0	116	20	0	8	−3	7	−8	17
	4	2	0	1	−1	26	4	0	+2	0	2	−2	7
	5					+8	+1						+2
3	−10	2	3			−7	−4	9	−4	−9	5		
	−9	3	9			−12	−12	29	−8	−29	9		
	−8	5	15			−10	−17	48	−11	−48	12		
	−7	10	23			−16	−25	75	−28	−76	29		
	−6	19	39			−24	−44	123	−57	−123	58		
	−5	30	57			−27	−66	177	−94	−176	95		
	−4	+28	70	−159	−257	−25	−82	204	−107	−202	108	280	−111
	−3	−15	67	−114	−130	0	−80	150	−32	−145	+34	199	+45
	−2	−186	119	−64	+227	+79	−81	86	+228	−78	−225	238	374
	−1	−1290	+1147	−1312	+1179	+343	−391	+1150	+1286	−1138	−1285	+1147	1290
	0	+489	−447	+485	−442	−290	+84	−414	−510	+449	+490	0	0
	1	18	−13	17	14	+451	48	+5	+18	−12	19	−13	18
	2	3	1			55	6	0	4	−1	3	2	5
	3	1	0			11	1	0	1		1		2
	4					2	0						1
4	−2	−18	12	−3	16	−13	−8	9	22	−11	−24	24	35
	−1	−118	+106	−118	+107	+30	−31	+106	+118	−106	−118	106	118
	0	+43	−39	+43	−40	−25	+8	−39	−42	+40	+43	0	0
	1	3	0	−3	0	+11	6	0	+3	0	3		3
	2					5	0						1

/62

The derivatives of Laplacian coefficients are computed by Innes' method. In determining the coordinates of expansion of circular motion we limited ourselves with Formulas (3) and (11) to the same terms as in the case of Pluto-Jupiter.

Tables 9, 10, and 11 give the values of /63

$$D^k c_1^i,\ D^k c_3^i,\ D^k c_5^i;\ a' D^k A_i,\ a'\sigma^2 D^k B_i,\ a'\sigma^4 D^k C_i,$$
$$a' D^k D_\sigma A_i,\ a' D^k D_\sigma \sigma^2 B_i,\ a' D^k D_\sigma \sigma^4 C_i.$$

The values of the factors for obtaining corrections in the operators for the indirect term in the case of Pluto-Saturn (Chapter III, Section 10) will be:

$$\alpha^{-2}(1-\sigma^2)=16.883526, \qquad 2\alpha^{-2}\sigma=4.380167,$$
$$\alpha^{-2}\sigma^2=0.279465, \qquad -2\alpha^{-2}(1-\sigma^2)=-33.767052,$$
$$-2\alpha^{-2}\sigma=-4.380167, \qquad -2\alpha^{-2}\sigma^2=0.558930.$$

Table 12 gives the coefficients of expansion of the perturbation function a'R and of its derivatives

$$a' D' R,\ a' D_\sigma R,\ a' \frac{\partial R}{\partial W'},\ a' \frac{\partial R}{\partial W},\ \frac{a'}{n'} D_t' R.$$

16. Determining the Perturbations of the Logarithm of the Radius Vector, Longitude in Orbit, Node and Inclination

The computations were conducted according to the formulas and systems given in Chapter III.

The values of the integrating factors are:

PLUTO-SATURN

i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$
0	1	+1.0000000	1	—6	+0.4109393	2	—9	+0.1271149	3	—10	+0.0653580
	2	0.5000000		—5	0.2912522		—8	0.1127790		— 9	0.0613484
	3	0.3333333		—4	0.2255580		—7	0.1013490		— 8	0.0578023
	4	0.2500000		—3	0.1840450		—6	0.0920226		— 7	0.0546438
	5	0.2000000		—2	0.1554376		—5	0.0842680		— 6	0.0518125
	6	0.1666667		—1	0.1345270		—4	0.0777188		— 5	0.0492602
	7	0.1428571		0	0.1185754		—3	0.0721142		— 4	0.0469476
	8	0.1250000		1	0.1060058		—2	0.0672635		— 3	0.0448423
	9	0.1111111		2	0.0958456		—1	0.0630243		— 2	0.0429178
	10	+0.1000000		3	0.0874627		0	0.0592877		— 1	0.0411517
				4	0.0804282		1	0.0559694		0	0.0395251
1	—11	—0.3896280		5	0.0744410		2	0.0530029		1	0.0380223
	—10	—0.6383451		6	0.0692835		3	0.0503350		2	0.0366296
	— 9	—1.7650674		7	0.0647943		4	0.0479228		3	0.0353352
	— 8	+2.3070740		8	0.0608515		5	0.0457312		4	0.0341293
	— 7	0.6976179					6	0.0437313			

TABLE 13

/64

i	i'	$\delta \lg r' 10^7$ sin	$\delta \lg r' 10^7$ cos	$\delta w'$ sin	$\delta w'$ cos	$\delta \Omega'$ sin	$\delta \Omega'$ cos	$\delta i'$ sin	$\delta i'$ cos
0	0	—	+3307	—	— 61.″61	—	— 5.″63	—	—30.″80
	0				—216.53 $n't$		+ 2.81 $n't$		
	1	+2887	1242	—121.55	+277.56	— 2.46	+ 0.07	+ 0.03	— 0.04
	1	—1124 $n't$			-108.48 $n't$				
	2	+1061	461	— 37.28	+ 85.05	— 1.29	— 0.62	0.20	— 0.24
	2	— 413 $n't$			— 33.20 $n't$				
	3	+ 368	160	— 11.86	+ 27.07	— 0.60	— 0.47	0.15	— 0.14
	3	— 144 $n't$			— 10.57 $n't$				
	4	+ 126	54	— 3.84	+ 8.75	— 0.26	— 0.21	0.07	— 0.06
	4	— 49 $n't$			— 3.41 $n't$				
	5	+ 42	18	— 1.25	+ 2.85	— 0.10	— 0.08	0.03	— 0.03
	5	— 17 $n't$			— 1.11 $n't$				
	6	+ 14	6	— 0.41	+ 0.94	— 0.05	— 0.04	0.01	— 0.02
	6	— 6 $n't$			— 0.36 $n't$				
	7	+ 5	2	— 0.14	+ 0.31	— 0.02	— 0.01		
	7	— 2 $n't$			— 0.11 $n't$				
	8		1	— 0.05	+ 0.10				
	8	— 1 $n't$			— 0.04 $n't$				
	9			— 0.02	+ 0.03				
	9				— 0.01 $n't$				
	10			— 0.01	+ 0.01				
	10				— 0.01 $n't$				
1	—10			— 0.02	— 0.01				
	— 9		+ 2	+ 0.09	— 0.08				
	— 8	— 3	— 5	+ 0.92	— 0.14	+ 0.29			— 0.10
	— 7	+ 8	0	— 0.06	+ 0.58	0.21	+ 0.06	+ 0.02	— 0.06
	— 6	6	0	— 0.12	0.31	0.14	+ 0.04	0	— 0.05
	— 5	7	— 2	— 0.18	+ 0.08	+ 0.12	0	— 0.01	— 0.04
	— 4	12	— 7	— 0.41	— 0.27	— 0.02	— 0.17	— 0.07	+ 0.02
	— 3	29	— 24	— 1.13	— 1.14	— 0.87	— 1.01	— 0.33	0.28
	— 2	78	— 68	— 3.21	— 3.53	— 4.99	— 5.69	— 1.71	1.51
	— 1	+ 207	— 184	— 8.70	— 9.74	—37.55	—43.61	—12.65	+11.13
	0	— 27	+ 25	+ 1.17	+ 1.38	+10.93	+34.70	— 1.42	— 4.46
	1	— 6	+ 1	0.10	— 0.07	4.54	—45.20	+13.17	+ 1.47
	2	— 2		+ 0.02	— 0.05	0.50	— 4.70	1.37	0.15
	3				— 0.02	0.08	— 0.70	0.21	0.02
	4				— 0.01	0.02	— 0.12	0.04	0.01
	5						— 0.02	+ 0.01	
2	— 9						— 0.02	— 0.01	
	— 8				— 0.01		— 0.02	— 0.01	
	— 7				— 0.01		— 0.04	— 0.01	
	— 6				— 0.02		— 0.06	— 0.01	
	— 5				— 0.03		— 0.10	— 0.02	
	— 4			— 0.01	— 0.05	+ 0.02	— 0.14	— 0.03	
	— 3		— 1	— 0.02	— 0.08	0	— 0.23	— 0.05	0.01
	— 2	+ 1	— 2	— 0.09	— 0.14	— 0.21	— 0.47	— 0.12	0.08
	— 1	+ 6	— 5	— 0.25	— 0.28	— 1.94	— 2.35	— 0.65	+ 0.61
	0	— 1	+ 1	— 0.03	— 0.04	+ 0.74	+ 1.82	— 0.04	— 0.21
	1				— 0.01	0.33	— 2.74	+ 0.80	+ 0.10
	2					0.04	— 0.33	0.09	0.01
	3					+ 0.02	— 0.06	0.02	+ 0.01
	4						— 0.02	+ 0.01	
3	— 2					— 0.04	— 0.04	— 0.01	— 0.01
	— 1					— 0.17	— 0.14	— 0.04	— 0.02
	0					+ 0.04	+ 0.12	— 0.01	+ 0.02
	1					0.01	— 0.19	+ 0.03	— 0.01

To determine the constants of integration at the initial moment, the moment of osculation 1930 September 20.0 was also taken. The values of the constants determined by Formulas (67) are given below:

$$
\begin{aligned}
a' C_1 &= 1.92450, \\
a' K_1 &= 0.14707, \\
a' K_2 &= 2.38300, \\
C_2 &= 2.14096, \\
C_3 &= -61''.61.
\end{aligned}
$$

Verification:

$$A_0 = -0.068,$$

$$(A_0) = -0.066.$$

Table 13 gives the perturbations of the logarithm of the radius vector, longitude in orbit, longitude of the node, and inclination.

CHAPTER V

/65

PERTURBATIONS OF THE FIRST ORDER BY URANUS

17. Elements

The initial elements of Uranus taken according to Newcomb are:

Epoch 1900 January 0 Greenwich mean noon

$$M_0 = 74^\circ.314$$

$$\left.\begin{aligned} \Omega &= 73.711 \\ i &= 0.774 \\ \omega &= 96.036 \end{aligned}\right\} \text{ecliptic and equinox 1950.0}$$

$$e = 0.047\,044$$

$$a = 19.190\,979$$

$$n = 42''.234\,34$$

$$\frac{1}{m} = 22869$$

Formulas (63) and (64) give:

$$N = 217^\circ.451, \quad \Pi = 238^\circ.585, \quad \sin\frac{J}{2} = 0.143694, \quad \alpha = 0.485\,630,$$

$$N' = 181.597, \quad \Pi' = 291.945, \quad \cos\frac{J}{2} = 0.989\,622.$$

18. Expansion of the Perturbation Function

The chief difficulty in expansion of perturbation function in the case of Pluto-Uranus, results from the comparatively large value of α and σ. At the given values of α and σ the series determining $a'A_i$, $a'\sigma^2 B_i$ . . . , converge very slowly. In addition, the integrating factor with a commensurability of 1: - 3, as will be shown in Section 19, assumes the value $\nu = -23.236119$, which also demands increased accuracy of expansion of the perturbation function.

In order to insure the seventh place after the decimal point in the logarithm of the perturbed radius vector and $0''.01$ in the perturbed values of longitude in orbit, longitude of the node, and inclination, we must have:

TABLE 14 /66

i	c_1^i	Dc_1^i	$D^2c_1^i$	$D^3c_1^i$	$D^4c_1^i$	$D^5c_1^i$	$D^6c_1^i$	$D^7c_1^i$	$D^8c_1^i$
0	2.1366956	0.3186380	0.8561020	2.862381	12.56207	71.20731	500.9088	4209.598	41140.85
1	0.5362484	0.6561340	1.106737	3.034856	12.78669	72.18991	506.1082	4240.284	41359.97
2	0.1975359	0.4444023	1.125410	6.486879	14.19339	76.54974	523.9531	4339.654	42058.70
3	0.08040439	0.2623894	0.9104109	3.493250	15.57728	84.28910	561.8992	4545.847	43413.58
4	0.03428644	0.1464652	0.6495686	3.053888	15.66875	90.79976	612.5677	4881.338	45724.11
5	0.01502110	0.07927485	0.4290948	2.411390	14.31611	91.85906	653.7239	5279.734	49011.38
6	0.006698260	0.04207681	0.2691760	1.767281	12.03926	86.36991	664.8005	5601.207	52693.91
7	0.003024434	0.02203281	0.1627302	1.224887	9.464438	75.81430	637.7918	5724.095	55742.57
8	0.001378351	0.01142300	0.09569058	0.8132281	7.046596	62.67757	577.4532	5573.959	57139.90
9	0.0006326955	0.005877400	0.05507136	0.5218812	5.019691	49.24082	495.7746	5165.851	56280.23
10	0.0002920880	0.003005916	0.03115431	0.3258408	3.448150	37.04926	406.0601	4567.001	53115.66
11	0.0001354747	0.001529852	0.01737853	0.1988918	2.297896	26.87141	319.1404	3866.666	48065.17

i	c_3^i	Dc_3^i	$D^2c_3^i$	$D^3c_3^i$	$D^4c_3^i$	$D^5c_3^i$	$D^6c_3^i$	$D^7c_3^i$	$D^8c_3^i$
0	1.762871	4.141592	15.8215	84.9766	578.298	4746.39	45600.3	501788	6220929
1	1.174740	3.723484	15.4295	83.7694	572.116	4710.51	45350.4	499692	6200417
2	0.690374	2.829324	13.6081	78.9069	552.725	4602.01	44588.0	493307	6138209
3	0.384597	1.945874	10.9278	69.8219	515.342	4404.31	43254.6	482434	6032182
4	0.207948	1.254988	8.14560	57.9474	459.261	4096.28	41216.6	466032	5877728
5	0.110305	0.774152	5.73126	45.3908	389.993	3677.91	38354.1	443283	5666389
6	0.0577368	0.462239	3.85466	33.8335	316.100	3177.69	34679.4	413235	5387614
7	0.0299259	0.269231	2.50115	24.1834	245.521	2641.47	30376.5	376032	5034062
8	0.0153947	0.153780	1.57643	16.6851	183.598	2116.29	25750.3	333097	4607218
9	0.00787231	0.0864627	0.970103	11.1717	132.776	1638.63	21137.2	286849	4119824
10	0.00400617	0.0479866	0.585150	7.29121	93.243	1229.95	16826.4	240112	3594174
11	0.00203055	0.0263444	0.347017	4.65492	63.813	897.645	13016.1	195509	3057510

i	c_5^i	Dc_5^i	$D^2c_5^i$	$D^3c_5^i$	$D^4c_5^i$	$D^5c_5^i$	$D^6c_5^i\cdot 10^{-1}$	$D^7c_5^i\cdot 10^{-2}$	$D^8c_5^i\cdot 10^{-3}$
0	2.12810	11.0221	72.876	586.96	5562.3	60560	74446	101946	153859
1	1.82638	10.2794	70.181	572.75	5465.6	59766	73684	101108	152820
2	1.34494	8.5398	62.589	531.11	5180.9	57420	71422	98615	149725
3	0.90588	6.5070	52.037	467.54	4728.6	53630	67738	94531	144636
4	0.57575	4.6480	40.734	391.47	4150.2	48611	62771	88968	137661
5	0.35135	3.1604	30.316	313.01	3504.0	42707	56747	82103	128962
6	0.20807	2.0675	21.630	240.17	2850.3	36352	49986	74192	118773
7	0.12040	1.3114	14.897	177.68	2239.5	29996	42876	65574	107412
8	0.068413	0.81110	9.9580	127.31	1704.5	24025	35811	56644	95281
9	0.038299	0.49126	6.4893	88.692	1260.3	18709	29142	47808	82840
10	0.021179	0.29233	4.1374	60.279	907.84	14193	23129	39433	70562
11	0.011591	0.17135	2.5883	40.084	638.70	10509	17924	31801	58879

i	c_7^i	Dc_7^i	$D^2c_7^i$	$D^3c_7^i$	$D^4c_7^i\cdot 10^{-1}$	$D^5c_7^i\cdot 10^{-2}$	$D^6c_7^i\cdot 10^{-3}$
0	3.0723	24.895	235.9	2563	3143	4291	6458
1	2.8452	23.748	228.5	2504	3087	4230	6383
2	2.3418	20.783	207.8	2334	2924	4052	6164
3	1.7692	16.916	178.4	2079	2672	3771	5812
4	1.2547	12.975	145.4	1773	2357	3409	5351
5	0.8480	9.4794	113.2	1453	2008	2993	4808
6	0.5520	6.6539	84.77	1147	1656	2554	4216
7	0.3487	4.5174	61.32	876.4	1324	2121	3608
8	0.2149	2.9819	43.06	649.8	1029	1715	3015
9	0.1297	1.9218	29.46	469.0	778.6	1352	2462
10	0.0770	1.2134	19.70	330.5	575.1	1041	1966
11	0.04503	0.75255	12.91	227.9	415.4	784.2	1537

TABLE 14 (continued)

i	c_9^i	Dc_9^i	$D^2c_9^i$	$D^3c_9^i$	$D^4c_9^i \cdot 10^{-2}$
0	4.800	52.87	653.1	8955	1351
1	4.569	51.02	635.9	8771	1329
2	3.986	46.08	587.8	8241	1263
3	3.233	39.14	517.1	7437	1162
4	2.469	31.57	434.9	6459	1034
5	1.798	24.33	351.0	5412	892
6	1.258	18.03	273.1	4384	747
7	0.852	12.92	205.7	3443	608
8	0.561	8.996	150.3	2631	182
9	0.361	6.106	107.2	1959	373
10	0.227	4.055	74.7	1425	282
11	0.141	2.641	50.9	1014	209

TABLE 15

i	$a'A_i$	$a'DA_i$	$a'D^2A_i$	$a'D^3A_i$	$a'D^4A_i$	$a'D^5A_i$	$a'D^6A_i$	$a'D^7A_i$	$a'D^8A_i$
0	2.114093	0.250110	0.593627	1.571546	4.642976	13.7447	21.505	— 274.25	— 4363
1	0.512395	0.592342	0.857119	1.766579	4.937802	14.9753	27.452	— 242.71	— 4200
2	0.182617	0.392871	0.902105	2.295233	6.600466	20.2639	48.626	— 138.90	— 3643
3	0.0719685	0.225724	0.726997	2.429370	8.456797	29.9052	94.143	+ 88.48	— 2473
4	0.0297277	0.122415	0.510320	2.159166	9.256941	39.5888	159.120	477.19	— 242
5	0.0126210	0.0643401	0.329925	1.701772	8.805939	45.2234	223.782	985.10	+ 3258
6	0.00545591	0.0331578	0.202037	1.233262	7.521026	45.4991	268.591	1497.37	7675
7	0.00238897	0.0168591	0.119071	0.840797	5.922958	41.3937	284.484	1893.13	12283

i	$a'\sigma^2 B_i$	$a'\sigma^2 DB_i$	$a'\sigma^2 D^2B_i$	$a'\sigma^2 D^3B_i$	$a'\sigma^2 D^4B_i$	$a'\sigma^2 D^5B_i$	$a'\sigma^2 D^6B_i$	$a'\sigma^2 D^7B_i$
0	0.017137	0.037025	0.126304	0.593849	3.45295	23.535	181.457	4374.41
1	0.011117	0.032997	0.123599	0.588345	3.43239	23.486	181.509	4351.56
2	0.006338	0.024566	0.108405	0.557831	3.35903	23.317	181.832	4286.21
3	0.0034230	0.0164748	0.085861	0.493830	3.17339	22.818	181.540	4176.83
4	0.0017935	0.0103384	0.062791	0.407142	2.85055	21.688	178.864	4021.67
5	0.0009220	0.0061978	0.043201	0.315008	2.42368	19.798	171.842	3814.22
6	0.00046772	0.00359406	0.028352	0.230955	1.95244	17.270	159.520	3548.90
7	0.00023496	0.00203249	0.017928	0.161896	1.50316	14.469	159.273	3226.00

i	$a'\sigma^4 C_i$	$a'\sigma^4 DC_i$	$a'\sigma^4 D^2C_i$	$a'\sigma^4 D^3C_i$	$a'\sigma^2 D^4C_i$
0	0.000298	0.001428	0.008584	0.061927	0.518392
1	0.000252	0.001322	0.008244	0.060382	0.509486
2	0.000181	0.001081	0.007288	0.055795	0.482976
3	0.000118	0.000805	0.005971	0.048752	0.440115
4	0.000073	0.000561	0.004587	0.040352	0.384592

TABLE 16

i	$\sigma a'D_\sigma A_i$	$\sigma a'DD_\sigma A_i$	$\sigma a'D^2D_\sigma A_i$	$\sigma a'D^3D_\sigma A_i$	$\sigma a'D^4D_\sigma A_i$	$\sigma a'D^5D_\sigma A_i$	$\sigma a'D^6D_\sigma A_i$	$\sigma a'D^7D_\sigma A_i$
0	—0.042154	—0.121150	—0.430680	—1.906054	—10.090664	—60.13036	—362.758	—1462.88
1	—0.044998	—0.113152	—0.408246	—1.867238	—10.046902	—60.29012	—367.764	—1509.76
2	—0.027688	—0.090770	—0.364982	—1.761670	— 9.819258	—60.38192	—376.438	—1643.18
3	—0.0153560	—0.063746	—0.298364	—1.578714	— 9.327518	—59.77172	—386.832	—1845.00
4	—0.0081346	—0.041150	—0.224462	—1.327788	— 8.493428	—57.72142	—393.700	—2079.20
5	—0.0041964	—0.0251086	—0.157894	—1.048712	— 7.348630	—53.72082	—390.824	—2296.52
6	—0.00212792	—0.0147204	—0.105376	—0.783142	— 6.038456	—47.83746	—374.554	—2445.70
7	—0.00106626	—0.0083784	—0.067450	—0.557372	— 4.721308	—40.67576	—343.956	—2485.28

TABLE 16 (continued)

i	$\sigma a' D_\sigma \sigma^2 B_i$	$\sigma a' D D_\sigma \sigma^2 B_i$	$\sigma a' D^2 D_\sigma \sigma^2 B_i$	$\sigma a' D^3 D_\sigma \sigma^2 B_i$	$\sigma a' D^4 D_\sigma \sigma^2 B_i$	$\sigma a' D^5 D_\sigma \sigma^2 B_i$
0	0.032342	0.064066	0.191794	0.754160	3.35016	13.852
1	0.020398	0.056526	0.188678	0.754074	3.36974	14.150
2	0.011254	0.041112	0.164428	0.723828	3.40088	14.974
3	0.058708	0.026770	0.128004	0.642454	3.31262	15.904
4	0.029658	0.016258	0.091364	0.525208	3.03156	16.242
5	0.014696	0.009411	0.061042	0.399350	2.59012	15.606

i	$\sigma a' D_\sigma \sigma^4 C_i$	$\sigma a' D D_\sigma \sigma^4 C_i$	$\sigma a' D^2 D_\sigma \sigma^4 C_i$	$\sigma a' D^3 D_\sigma \sigma^4 C_i$
0	0.001116	0.005144	0.029388	0.199618
1	0.000936	0.004742	0.028178	0.194588
2	0.000662	0.003846	0.024788	0.179496
3	0.000422	0.002822	0.020140	0.156220

$a' D A_i$, $a' \sigma^2 D B_i \ldots$, $a' D_\sigma A_i$, $a' D_\sigma \sigma^2 B_i$ . .	with accuracy to		10^{-4}
$a' D^2 A_i$	»	»	10^{-3}
$a' D^3 A_i$	»	»	10^{-2}
$a' D^4 A_i$	»	»	10^{-1}
$a' D^5 A_i$	»	»	10
$a' D^6 A_i$	»	»	10^{-}
$a' D^7 A_i$	»	»	10^{3}
$a' D^8 A_i$	»	»	10^{4}

The expansion of $D^k a' A_i$ was extended to σ^{12} but further computations showed that it could be restricted to σ^8 and terms of the second class. In computing the derivatives of the Laplace coefficients, Innes' method was used.

Tables 14, 15, and 16 give the values $D^\kappa c_n^i$, $a' D^\kappa A_i$, $a' \sigma^2 D^\kappa B_i$, $a' \sigma^4 D^\kappa C_i$, $\sigma a' D^\kappa D_\sigma A_i$, $\sigma a' D^\kappa D_\sigma \sigma^2 B_i$, $\sigma a' D^\kappa D_\sigma \sigma^4 C_i$.

Corrections for the indirect term of the perturbation function to the operators was obtained by each of the following factors:

$$\alpha^{-2}(1-\sigma^2) = 4.152683, \qquad 2\alpha^{-2}\sigma = 1.218592,$$
$$\alpha^{-2}\sigma^2 = 0.087552, \qquad -2\alpha^{-2}(1-\sigma^2) = -8.305366,$$
$$-2\alpha^{-2}\sigma = -1.218592, \qquad -2\alpha^{-2}\sigma^2 = -0.175104.$$

The expansion of the perturbation function was conducted by Newcomb's operator method on analytical computers (see Chapter I, Section 4).

Table 17 contains the coefficients of the expansion of perturbation function a'R and its derivatives $a'D'R$, $a'\sigma D_\sigma R$, $a'\frac{\partial R}{\partial W'}$, $a'\frac{\partial R}{\partial W}$, $\frac{a'}{n'}D'_t R$.

19. Determination of the Logarithm of the Radius Vector, Longitude in Orbit, Longitude of the Node, and Inclination /69

Computation of the perturbations was conducted by formulas and systems presented in the preceding chapters.

The integrating factors have values given below.

PLUTO-URANUS

i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$	i	i'	$\nu_{ii'}$
0	0										
	1	1.000000	2	−15	−0.110059	3	−11	−0.469680	4	−3	0.113278
	2	0.500000		−14	−0.123669		−10	−0.885654		−2	0.101752
	3	0.333333		−13	−0.141122		−9	−7.745373		−1	0.092354
	4	0.250000		−12	−0.164310		−8	+1.148250		0	0.084546
	5	0.200000		−11	−0.196615		−7	0.534505		1	0.077955
	6	0.166667		−10	−0.244731		−6	0.348324		2	0.072318
	7	0.142857		−9	−0.324036		−5	0.258338		3	+0.067441
				−8	−0.179370		−4	0.205301			
1	13	0.099571		−7	−0.920748		−3	0.170332	5	−15	−4.647224
	−12	−0.110582		−6	−11.618059		−2	0.145542		−14	+1.274181
	−11	0.124331		−5	+1.094179		−1	0.127050		−13	0.560281
	−10	−0.141984		−4	0.522486		0	0.112728		−12	0.359090
	−9	0.165180		−3	0.343179		1	0.101308		−11	0.264214
	−8	0.198293		−2	0.255498		2	0.091989		−10	0.208994
	−7	0.247339		−1	0.203503		3	+0.084240		−9	0.172866
	−6	0.328619		0	0.169092		4	−0.077695		−8	0.147388
	−5	−0.489468		1	0.144636					−7	0.128455
	−4	−0.958739		2	0.126360	4	−14	−0.460374		−6	0.113833
	−3	23.236119		3	0.112184		−13	−0.853136		−5	0.102199
	−2	1.044972		4	0.100868		−12	−5.809029		−4	0.092723
	−1	0.516996		5	+0.091626		11	+1.207942		−3	0.084855
	0	0.338185					−10	0.517090		−2	0.078218
	1	0.252719	3	−17	−0.123015		−9	0.353625		−1	0.072544
	2	0.201736		16	−0.140270		−8	0.261243		0	0.067637
	3	0.167871		−15	−0.163156		−7	0.207131			
	4	0.143741		−14	−0.194966		−6	0.171590			
	5	0.125676		−13	−0.242183		−5	0.146459			
	6	0.111645		−12	−0.319804		−4	0.127749			

As previously, the initial moment 1930 September 20.0 was taken in determining the constants. The constants determined are Formulas (67):

$$a'C_1 = 2.5109,$$
$$a'K_1 = -4.0125,$$
$$a'K_2 = -28.665,$$
$$C_2 = 1.7044,$$
$$C_3 = -842''.90,$$
$$C_4 = 0.7025,$$
$$C_5 = -1.1989.$$

TABLE 17

i	i'	$a'R\cdot 10^4$		$a'D'R\cdot 10^4$		$\sigma a'D_iR\cdot 10^4$		$a'\frac{\partial R}{\partial W'}\cdot 10^4$		$a'\frac{\partial R}{\partial W}\cdot 10^4$		$\frac{a'}{n'}D'_tR\cdot 10^4$	
		sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos
0	0	—	10639	—	—12057	—	— 251	—	19	—	— 12	—	—
	1	302	2753	— 728	— 3641	— 38	— 176	205	325	— 239	— 310	— 2753	303
	2	226	609	— 692	— 679	+110	— 202	270	302	— 136	— 140	— 1221	455
	3	128	111	— 463	+ 62	108	— 160	208	181	— 63	— 44	— 340	382
	4	62	+ 3	— 230	158	61	— 96	127	84	— 26	— 6	— 12	248
	5	25	— 16	— 88	131	23	— 52	73	+ 25	— 10	+ 4	+ 83	129
	6	10	— 13	— 23	79	1	— 29	35	— 3	— 4	+ 2	72	57
	7	+ 5	— 6	— 17	+ 38	+ 3	— 14	+ 17	+ 5	0	0	39	37
1	—8	— 8	+ 5	+ 60	— 7	— 23	— 6	— 1	35	— 5	— 8	41	62
	—7	— 22	11	120	0	— 34	— 24	— 14	76	— 7	— 33	74	149
	—6	— 26	28	164	— 60	— 60	— 20	+ 1	118	— 13	— 50	170	156
	—5	— 16	87	+ 117	— 309	—102	— 15	87	178	— 78	— 70	431	+ 82
	—4	+ 10	181	— 252	— 875	—146	— 35	216	242	— 193	— 107	720	— 36
	—3	— 85	+ 176	— 1906	— 2294	—141	— 72	+ 250	381	— 218	— 284	+ 531	+ 258
	—2	— 2098	— 1347	— 7942	— 6596	— 31	— 83	— 1252	2303	+ 1267	— 2324	— 2691	4196
	—1	—28180	—21018	—41192	—30403	+963	+ 721	—20985	+28167	+20991	—28149	—21018	+28180
	0	+12466	+ 8896	+11895	+ 8976	—693	—1095	+ 9659	—12571	— 8970	+12490	0	0
	1	— 75	+ 473	— 250	1199	+211	+1336	— 791	+ 311	— 517	— 77	— 473	— 75
	2	— 6	— 19	+ 22	507	— 17	+ 4	+ 4	— 17	+ 14	+ 6	+ 37	— 11
	3	— 10	— 33	55	229	— 29	— 49	26	— 29	37	— 1	101	— 29
	4	— 7	— 19	49	95	— 21	— 21	+ 7	— 18	21	— 5	71	— 32
	5	— 6	— 6	36	28	— 14	— 5	— 3	— 16	8	— 3	31	— 31
	6	— 3	— 4	20	10	— 6	0	— 1	— 6	+ 4	— 2	21	— 14
2	—9	+ 10	12		— 101	— 23	31	+ 61	+ 1	— 25	+ 19	111	— 88
	—8	14	34	28	— 187	— 45	25	122	+ 29	— 85	14	269	— 119
	—7	44	35	— 49	— 207	— 52	23	133	— 10	— 95	53	251	— 310
	—6	152	31	— 412	— 247	— 74	50	177	— 221	— 110	260	195	— 913
	—5	372	+ 7	— 1132	— 230	— 122	55	+ 186	— 663	— 114	693	+ 36	— 1857
	—4	770	— 109	— 2495	+ 144	—180	38	— 36	— 1487	+ 86	1494	— 438	— 3081
	—3	1276	— 365	— 4353	912	—253	33	— 585	— 2600	588	2577	— 1096	— 3827
	—2	+ 1170	— 750	— 5223	+ 1351	—274	73	— 1247	— 2724	1288	+ 2698	— 1503	— 238
	—1	— 3174	— 2181	— 2478	— 2313	+ 41	+ 43	— 2098	+ 3291	+ 2175	— 3378	— 2183	+ 3174
	0	+ 1050	+ 766	+ 1634	+ 1145	—280	— 201	+ 886	— 1193	— 693	+ 949	0	0
	1	— 66	29	259	199	— 75	+ 111	— 103	+ 326	+ 7	— 120	— 29	— 66
	2	— 25	3	127	+ 41	— 39	15	— 24	3	+ 10	— 50	— 2	— 49
	3	— 9	3	56	— 4	— 14	9	— 13	4	0	— 21	— 6	— 30
	4	— 3	2	18	— 12	— 2	6	— 7	2	— 2	— 7	— 6	— 13
	5	— 1	+ 1	+ 7	— 8	— 1	6	— 4	+ 1	— 2	0	— 4	— 6

TABLE 17 (continued)

i	i'	$a'R\cdot 10^4$		$a'D'R\cdot 10^4$		$\sigma a'D\sigma R\cdot 10^4$		$a'\frac{\partial R}{\partial W'}\cdot 10^4$		$a'\frac{\partial R}{\partial W}\cdot 10^4$		$\frac{a'}{n'}D'_t R\cdot 10^4$	
		sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos
3	−10	+20	−10	−116	−22	+10	43	+12	−81	+30	+62	−100	−205
	−9	48	−3	−224	−101	−14	51	80	−161	−24	157	−22	−436
	−8	59	−23	−253	−16	−25	42	+23	−175	+17	176	−186	−473
	−7	106	−117	−476	+384	−19	73	−253	−341	296	322	−819	−739
	−6	183	−255	−820	950	−47	107	−660	−604	689	576	−1528	−1095
	−5	249	−458	−1080	1847	−79	126	−1286	−803	1304	773	−2284	−1251
	−4	271	−625	−1090	2587	−101	152	−1849	−838	1840	810	−2498	−1088
	−3	+173	−489	−618	+2016	−92	134	−1479	−522	+1475	+460	−1466	−517
	−2	−69	+121	+278	−622	−90	20	+379	+100	−411	−213	+239	+140
	−1	−320	−146	158	−341	−116	71	−159	+325	+92	−473	−146	+320
	0	+62	+83	251	−12	−50	20	+80	−62	−106	+9	0	0
	1	−10	8	47	−46	−7	11	5	+16	−22	−29	−8	−10
	2	0	3	+3	−25	+3	6	+1	8	−11	−2	−6	−1
	3	0	2	−4	−10	5	3		4	−5	+1	−5	+1
	4	0	0	−4	−6	3	1		+2	−2		−3	
4	−11	−2	−30	−56	143	+41	22	−111	−36	+121	−7	−333	−19
	−10	+31	−58	−290	278	38	35	−216	−229	229	+172	−578	−311
	−9	+22	−60	−210	285	13	30	−213	−164	219	+139	−548	−196
	−8	−36	−126	+132	622	30	52	−472	+98	476	−119	−1011	+289
	−7	−74	−206	325	1045	37	90	−815	238	805	−258	−1446	520
	−6	−125	−272	635	1375	30	104	−1088	456	+1074	−471	−1630	755
	−5	−150	−263	802	1304	28	108	−1062	591	−911	+676	−1317	750
	−4	−79	−121	+462	+549	+9	88	−493	+329	+440	−358	−475	+319
	−3	+49	+79	−214	−488	−32	56	+266	−204	−347	+177	+241	−146
	−2	−16	42	+88	−262	−14	70	80	+46	−163	−64	87	+31
	−1	−3	12	+15	−81	−5	18	25	+7	−46	−11	+12	3
	0	+7	7	−3	−18	+3	+3	13	−5	−15	+11	0	0
	1	2	1	−12	−4	4	−1	4	+1	−3	5	−1	2
	2	1	0	−5	−0	2	0	+2		−2	2	−1	2
	3	+1	−1	−5	+1	1	−1			+2	+3	+2	2
5	−12	−47	−3	+287	16	22	+1	−13	+235	13	−235	−31	563
	−11	−50	−42	277	288	20	20	−244	219	244	−219	−455	552
	−10	−27	−41	120	276	13	19	−233	88	233	−88	−414	271
	−9	−92	−25	539	141	42	12	−111	433	111	−435	−226	832
	−8	−139	−35	826	209	71	26	−190	687	177	−676	−284	1118
	−7	−151	−31	910	180	65	35	−179	745	157	−742	−231	1058
	−6	−119	−13	724	+37	55	33	−71	603	+40	−574	−79	717
	−5	−26	0	+155	−69	33	36	−5	+154	−39	−132	+7	+133

TABLE 17 (continued)

i	i'	$a'R\cdot 10^4$		$a'D'R\cdot 10^4$		$\sigma a'D_\sigma R\cdot 10^4$		$a'\frac{\partial R}{\partial W'}\cdot 10^4$		$a'\frac{\partial R}{\partial W}\cdot 10^4$		$\frac{a'}{n'}D'_t R\cdot 10^4$	
		sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos
5	−4	+52	+10	−337	−106	16	37	+24	−230	−67	+264	45	−208
	−3	10	21	−72	−135	20	22	66	−23	−96	48	61	−31
	−2	0	+3	−17	−19	1	+3	11	−2	−13	5	+6	−1
	−1	1	−1	−9	+5	1	−2	+2	−4	+3	+7	−1	−1
	0	0	−2	−3	9	0	−3	−2	0	+10	0	0	0
	1	0	0	0	4	0	0						
	2	0	0	0	1	0	0						
6	−13	−30	36	216	−256	16	−18	+217	+181	−217	−181	+469	+391
	−12	−58	+15	425	−67	30	−5	+47	360	−47	−360	+178	700
	−11	−41	−17	307	+170	23	+9	−154	260	+154	−260	−187	453
	−10	−55	+36	370	−248	27	−19	+209	311	−209	−311	+359	511
	−9	−64	58	421	−408	46	−27	349	379	−347	−361	518	576
	−8	−45	42	299	−298	48	−24	258	287	−255	−260	346	363
	−7	−26	+24	+137	−183	40	−16	+157	+285	−154	−313	+175	+385
	−6	+8	−5	−85	+15	33	−1	0	−24	+1	+62	−9	−47
	−5	23	−15	−189	+111	24	+8	−89	−107	+88	140	−72	414
	−4	11	+4	−83	−33	14	1	+24	−41	−26	63	+18	−45
	−3	2	2	−15	−9	+2	+2	7	−5	−9	10	7	−6
	−2	0	0	−5	+1	0	0						
	−1	0	0	−2	0	0	0						
7	−14	10	43	−83	−343	−6	−24	299	−72	−299	72	599	−141
	−13	15	62	−121	−503	−8	−34	436	−105	−436	105	809	−192
	−12	+5	21	−43	−177	−4	−15	150	−36	−150	36	257	−62
	−11	−5	34	+28	−268	0	−21	231	+24	−231	−24	381	+54
	−10	−5	54	+23	−417	0	−27	362	+17	−362	−17	540	+41
	−9	+1	32	−13	−257	−2	−18	221	−13	−221	+13	291	−11
	−8	−4	+14	+22	−109	+2	−8	+88	+18	−88	−18	+106	+30
	−7	−4	0	+34	+13	7	0	−7	32	+7	−28	−4	+35
	−6	0	−7	−4	+54	6	+3	−49	+3	+49	+1	−42	−2
	−5	4	+1	−28	−8	3	−1	+10	−21	−10	25	+6	20
	−4	1	1	−4	−1	1	0	5	−5	−4	7	3	−4
8	−12	3	14	−27	−113	−1	−6	95	−23	−95	23	163	39
	−11	4	18	−33	−138	2	−8	123	−29	−123	29	193	46
	−10	+1	3	−5	−20	0	−2	20	−5	−20	+6	29	−7
	−9	−1	2	+10	−13	+1	−2	9	+8	−9	−8	14	−12
	−8	−1	+3	7	−26	1	−2	+16	6	−16	−6	+20	+8
	−7	0	0	+1	+3	1	−1	0	0	0	0	0	0
	−6	0	−2	0	+10	0	0	−14	+3	+14	−3	−12	+3
	−5	0	+1	2	2	0	0	+7	−2	−7	+2	+5	−1

TABLE 18

i	i'	$\delta \log r' \cdot 10^7$ sin	$\delta \log r' \cdot 10^7$ cos	$\delta\omega'$ sin	$\delta\omega'$ cos	$\delta\Omega'$ sin	$\delta\Omega'$ cos	$\delta i'$ sin	$\delta i'$ cos
					″		″		″
0	0		+792		−842.90		−38.03		+6.56
	0				− 46.38 $n't$		− 2.72 $n't$		
	1	5318	− 491	′ + 49.72	−513.64	″ − 1.91	+ 0.43	″ +0.02	+0.14
	1	− 246 $n't$	+ 1 $n't$	− 0.14 $n't$	− 23.78 $n't$				
	2	1953	− 183	+ 14.96	−157.18	− 1.07	− 0.62	0.25	− 0.20
	2	91 $n't$	+ 1 $n't$	− 0.04 $n't$	− 7.28 $n't$				
	3	− 677	− 64	+ 4.74	− 49.99	− 0.57	− 0.41	0.15	− 0.15
	3	− 31 $n't$		− 0.01 $n't$	− 2.32 $n't$				
	4	− 231	− 22	+ 1.54	− 16.14	− 0.26	− 0.17	0.06	− 0.08
	4	− 11 $n't$			− 0.75 $n't$				
	5	− 78	− 8	0.49	− 5.26	− 0.11	− 0.05	+0.02	−0.04
	5	− 4 $n't$			− 0.24 $n't$				
	6	− 26	− 3	0.16	− 1.72	− 0.05			−0.02
	6	− 1 $n't$			− 0.07 $n't$				
	7	− 9	− 1	0.06	− 0.57	− 0.02	− 0.01		0.01
	7				− 0.03 $n't$				
	8	− 3		0.02	− 0.19				
	8				− 0.01 $n't$				
	9	− 1		0.01	− 0.06				
	10				− 0.02				
	11				− 0.01				
1	−13			0.01					
	−12			0.02	+ 0.01				
	−11	− 1	+ 1	0.06	0.04				
	−10	− 2	3	0.19	0.12				
	− 9	− 6	9	0.58	0.36				
	− 8	− 17	26	1.74	1.11	+ 0.01	− 0.05	−0.02	
	− 7	− 49	77	5.35	3.41	0.06	− 0.09	−0.03	0.02
	− 6	− 143	225	16.55	10.55	0.07	− 0.21	−0.07	0.01
	− 5	− 413	648	51.98	33.18	0.07	− 0.54	−0.16	+0.01
	− 4	− 1121	1766	169.53	108.48	0.33	− 1.52	−0.39	0.06
	− 3	− 249	+ 475	679.44	459.39	+17.62	−35.45	−6.34	+1.95
	− 2	+ 990	−2135	210.72	90.93	− 0.98	+ 0.33	− 0.39	−0.23
	− 1	434	− 737	66.13	25.80	+ 3.83	− 5.49	−1.95	−1.41
	0	118	− 281	19.92	9.73	− 3.98	+ 2.52	+0.52	−0.34
	1	43	− 96	6.61	3.02	+ 3.69	− 0.48	+0.15	+1.06
	2	14	− 33	2.16	0.98	+ 0.01	+ 0.04	−0.01	−0.01
	3	5	− 11	0.71	0.32	− 0.09	0.05	−0.01	−0.03
	4	2	− 4	0.23	0.11	− 0.03	0.03	−0.01	−0.01
	5	1	− 1	0.08	0.04	− 0.01	0.02	−0.01	
	6			0.03	+ 0.01		+ 0.01		
	7			0.01					
2	−15				− 0.01				
	−14			0.01	− 0.03				
	−13	1		0.02	− 0.09				
	−12	5	+ 1	0.05	− 0.28				
	−11	13	2	0.15	− 0.85				
	−10	37	7	0.48	− 2.60				
	− 9	109	20	1.50	− 8.03	− 0.11	− 0.08	−0.02	+0.04
	− 8	315	59	4.68	− 25.21	− 0.14	− 0.23	−0.07	0.05
	− 7	862	160	15.34	− 82.17	− 0.24	− 0.51	−0.13	0.10
	− 6	+ 478	+ 85	64.44	−326.94	− 6.97	− 9.14	−1.77	+2.34
	− 5	−1448	− 182	17.81	−144.26	+ 0.67	+ 1.43	+0.20	−0.24
	− 4	− 495	− 73	5.85	− 40.95	0.23	1.01	0.11	−0.10

TABLE 18 (continued)

/74

i	i'	$\delta \log r' \cdot 10^7$ sin	$\delta \log r' \cdot 10^7$ cos	$\delta\omega'$ sin	$\delta\omega'$ cos	$\delta\Omega'$ sin	$\delta\Omega'$ cos	$\delta i'$ sin	$\delta i'$ cos
				″	″	″	″	″	″
2	— 3	—155	—30	+2.12	—12.01	+0.13	+0.94	+0.09	—0.04
	— 2	— 46	—12	0.83	— 3.56	0.21	+0.76	+0.07	—0.08
	— 1	— 16	— 3	0.32	— 1.26	+0.08	—0.10	—0.15	—0.11
	0	— 6	— 2	0.08	— 0.38	—0.38	+0.51	—0.11	—0.09
	1	— 2	— 1	0.04	— 0.13	+0.17	0.12	—0.05	+0.04
	2	— 1		+0.01	— 0.05	0.02	0.05	—0.02	
	3				— 0.02	0.01	0.02	—0.01	
	4				— 0.01	0.01			
	5					+0.01			
3	—17				— 0.01				
	—16				— 0.02				
	—15	+ 1			— 0.06				
	—14	3			— 0.18				
	—13	8		—0.01	— 0.56				
	—12	23		—0.02	— 1.72				
	—11	67	— 1	—0.06	— 5.41				
	—10	184	— 2	—0.24	—17.66	—0.41	+0.11	+0.05	0.12
	— 9	+139	—16	—5.73	—75.65	—4.31	—1.03	—0.03	+1.35
	— 8	—275	+58	—5.88	—27.29	+0.53	+0.30	+0.03	—0.15
	— 7	— 92	+10	—0.98	— 7.62	0.42	0.10	—0.01	—0.09
	— 6	— 28	— 2	+0.05	— 2.16	0.40	0.17	—0.01	—0.07
	— 5	— 7	— 5	0.25	— 0.57	0.35	0.21		—0.06
	— 4	— 1	— 5	0.22	— 0.13	0.34	0.22		—0.05
	— 3		— 3	0.12	— 0.03	0.25	0.17	—0.02	—0.03
	— 2			0.03	— 0.02	0.03	0.15	—0.06	+0.02
	— 1			0.01	— 0.01	0.09	0.16	—0.07	0.02
	0					0.02	0.06	—0.02	0.01
	1					0.01	+0.01		0.01
	2					0.01			
4	—13		— 2	0.06	— 0.01				
	—12	+ 1	— 9	+3.93	+ 0.05				
	—11	— 5	+62	—7.18	— 0.41	0.27	—0.55	—0.17	—0.05
	—10	+ 3	17	—1.58	+ 0.19	0.20	—0.23	—0.09	—0.04
	— 9	+ 2	+ 4	—0.35	+ 0.09	0.11	—0.05	—0.02	—0.02
	— 8	0	— 1	—0.01		0.15	—0.09	—0.02	—0.02
	— 7	— 1	— 2	+0.08	— 0.03	0.20	—0.09	—0.02	—0.01
	— 6	— 1	— 2	0.08	— 0.04	0.19	—0.06	—0.02	—0.02
	— 5	— 1	— 1	0.06	— 0.04	0.23	+0.04	+0.59	+0.92
	— 4			0.03	— 0.02	0.12	—0.01	—0.02	0.01
	— 3					0.07	+0.04	—0.01	0.03
	— 2					0.08	0.02	—0.01	0.03
	— 1					0.02	+0.01		+0.01
5	—14	+ 1			+ 0.05				
	—13	— 1		0.02	— 0.06				
	—12	— 2		0.04	— 0.13		—0.09	—0.01	
	—11	— 2	— 1	0.05	— 0.08	0.06	—0.06	—0.01	—0.01
	—10	— 1	— 1	0.04	— 0.05	0.04	—0.03		—0.01
	— 9	— 1		0.02	— 0.05	0.02	—0.08	—0.01	
	— 8	— 1		0.01	— 0.05	0.04	—0.11	—0.01	
	— 7	— 1		0.01	— 0.03	0.05	—0.09	—0.01	+0.01
	— 6				— 0.02	0.04	—0.07	—0.01	0.01
	— 5				— 0.01	0.04	—0.03		0.01
	— 4					+0.03	—0.02	+0.01	0.01
	— 3					—0.02	—0.02	+0.01	0.01

Verification:

$$A_0 = -0.07,$$
$$(A_0) = -0.09.$$

Table 18 contains the perturbations of the logarithm of the radius vector, longitude in orbit, longitude of the node, and inclination.

CHAPTER VI /75

PERTURBATIONS BY NEPTUNE

20. Perturbations of Pluto by Neptune

Difficulties in determining perturbations of Pluto by Neptune are the result of the peculiarities of the motion of these planets. The major semiaxes of Pluto and Neptune and the eccentricity of Pluto are such that the ratio of the radius vectors takes on values close to unity.

Under these conditions the ordinary methods of expansion of the perturbation function cease to have effect; therefore we had to have recourse to numerical integration.

In determining the perturbations of Pluto by Neptune we decided on the Cowell method of integrating the equations of motion.

We limited ourselves to

$$\left.\begin{aligned} x(t_0+\omega n) &= f_n^{-2}+\frac{1}{12}f_n-\frac{1}{240}f_n^2, \\ y(t_0+\omega n) &= g_n^{-2}+\frac{1}{12}g_n-\frac{1}{240}g_n^2, \\ z(t_0+\omega n) &= h_n^{-2}+\frac{1}{12}h_n-\frac{1}{240}h_n^2 \end{aligned}\right\} \tag{68}$$

as the operating formulas.

The integration interval was 160 days and even the second difference rarely exerted enough effect.

In Formulas (68), as usu l, the integration interval is designated by ω and

$$\left.\begin{aligned} f &= -\omega^2k^2\frac{x'}{r'^3}+k^2\omega^2m_H\frac{x_H-x'}{\Delta^3}-k^2\omega^2m_H\frac{x_H}{r_H^3}, \\ g &= -\omega^2k^2\frac{y'}{r'^3}+k^2\omega^2m_H\frac{y_H-y'}{\Delta^3}-k^2\omega^2m_H\frac{y_H}{r_H^3}, \\ h &= -\omega^2k^2\frac{z'}{r'^3}+k^2\omega^2m_H\frac{z_H-z'}{\Delta^3}-k^2\omega^2m_H\frac{z_H}{r_H^3}. \end{aligned}\right\} \tag{69}$$

In computing f, g, and h, we use Comrie's tables (1933).

TABLE 19

No. of integration step	Date			x	y	z	$160\frac{dx}{dt}$	$160\frac{dy}{dt}$	$160\frac{dz}{dt}$
−40	1913	Mar	13	+ 0.196280	42.787975	13.465372	−0.36010044	−0.10674613	0.07517201
38	1914	Jan	27	− 0.523914	42.567285	13.613438	−0.36007323	−0.11395111	0.07288630
36	1914	Dec	13	− 1.243931	42.332143	13.756884	−0.35992274	−0.12119699	0.07055042
34	1915	Oct	29	− 1.963521	42.082470	13.895606	−0.35964629	−0.12848283	0.06816378
−32	1916	Sep	13	− 2.682430	41.818186	14.029504	−0.35924103	−0.13580760	0.06572588
−30	1917	Jul	30	− 3.400397	41.539214	14.158475	−0.35870408	−0.14317021	0.06323618
28	1918	June	15	− 4.117157	41.245181	14.282414	−0.35803245	−0.15056945	0.06069116
−26	1919	May	1	− 4.832435	40.936913	14.401217	−0.35722310	−0.15800409	0.05809931
−24	1920	Mar	16	− 5.545955	40.613442	14.514776	−0.35627286	−0.16547271	0.05545109
22	1921	Jan	30	− 6.257431	40.275000	14.622985	−0.35517849	−0.17297386	0.05274900
20	1921	Dec	16	− 6.966571	39.921526	14.725736	−0.35393667	−0.18050593	0.04999253
−18	1922	Nov	1	− 7.673077	39.552957	14.822919	−0.35254400	−0.18806722	0.04718119
−16	1923	Sep	17	− 8.376644	39.169238	14.914424	−0.35099696	−0.19565591	0.04431451
−14	1924	Aug	2	− 9.076959	38.770317	15.000140	−0.34929195	−0.20327002	0.04139208
−12	1925	June	19	− 9.773704	38.356143	15.079955	−0.34742532	−0.21090747	0.03841336
−10	1926	May	5	−10.466550	37.926673	15.153755	−0.34539330	−0.21856598	0.03537794
− 8	1927	Mar	21	−11.155164	37.481866	15.221428	−0.34319205	−0.22624316	0.03228545
− 6	1928	Feb	4	−11.839203	37.021689	15.282859	−0.34081761	−0.23393647	0.02913549
− 4	1928	Dec	20	−12.518317	36.546112	15.337932	−0.33826597	−0.24164312	0.02592771
− 2	1929	Nov	5	−13.192146	36.055110	15.386531	−0.33553301	−0.24936020	0.02266183
0	1930	Sep	20	−13.860325	35.548666	15.428540	−0.33261456	−0.25708458	0.01933754
+ 2	1931	Aug	7	−14.522478	35.026769	15.463842	−0.32950637	−0.26481293	0.01595460
4	1932	June	22	−15.178221	34.489414	15.492319	−0.32620407	−0.27254173	0.01251281
6	1933	May	8	−15.827162	33.936604	15.513854	−0.32270330	−0.28026718	0.00901206
8	1934	Mar	24	−16.468899	33.368350	15.528328	−0.31899959	−0.28798528	0.00545227
10	1935	Feb	7	−17.103022	32.784671	15.535624	−0.31508836	−0.29569178	+0.00183339
12	1935	Dec	24	−17.729111	32.185594	15.535622	−0.31096507	−0.30338214	−0.00184452
14	1936	Nov	8	−18.346738	31.571156	15.528206	−0.30662509	−0.31105160	−0.00558136
16	1937	Sep	24	−18.955464	30.941405	15.513258	−0.30206377	−0.31869502	−0.00937692
18	1938	Aug	10	−19.554842	30.296397	15.490660	−0.29727645	−0.32630705	−0.01323090
20	1939	June	26	−20.144416	29.636201	15.460295	−0.29225839	−0.33388199	−0.01714293
22	1940	May	11	−20.723719	23.960898	15.422049	−0.28700490	−0.34141380	−0.02111253
24	1941	Mar	27	−21.292275	28.270579	15.375807	−0.28151131	−0.34889613	−0.02513909
26	1942	Feb	10	−21.849601	27.565351	15.321456	−0.27577298	−0.35632221	−0.02922193
28	1942	Dec	27	−22.395201	26.845332	15.258883	−0.26978524	−0.36368500	−0.03336017
30	1943	Nov	12	−22.928573	26.110658	15.187979	−0.26354358	−0.37097701	−0.03755283
32	1944	Sep	27	−23.449203	25.361477	15.108636	−0.25704351	−0.37819036	−0.04179877
34	1945	Aug	13	−23.956571	24.597954	15.020749	−0.25028065	−0.38531678	−0.04609672
36	1946	June	29	−24.450148	23.820273	14.924215	−0.24325079	−0.39234759	−0.05044521
38	1947	May	15	−24.929394	23.028634	14.818935	−0.23594982	−0.39927367	−0.05484257
40	1948	Mar	30	−25.393764	22.223255	14.704814	−0.22837385	−0.40608543	−0.05928698
42	1949	Feb	13	−25.842703	21.404375	14.581758	−0.22051917	−0.41277289	−0.06377638
44	1949	Dec	30	−26.275652	20.572253	14.449680	−0.21238239	−0.41932559	−0.06830851
46	1950	Nov	15	−26.692043	19.727170	14.308497	−0.20396031	−0.42573260	−0.07288086
48	1951	Oct	1	−27.091301	18.869428	14.158131	−0.19525012	−0.43198253	−0.07749072
50	1952	Aug	16	−27.472849	17.999352	13.998511	−0.18624935	−0.43806353	−0.08213506
52	1953	July	2	−27.836104	17.117294	13.829570	−0.17695594	−0.44396332	−0.08681064
54	1954	May	18	−28.180477	16.223628	13.651250	−0.16736829	−0.44966911	−0.09151389
56	1955	Apr	3	−28.505380	15.318756	13.463499	−0.15748528	−0.45516764	−0.09624096

TABLE 20

/77

No. of integration step	Date			$\delta \lg_{10} a' \cdot 10^8$*	$\delta e' \; 10^8$	$\delta \lambda'$	$\delta \pi'$	$\delta \Omega'$	$\delta i'$	$\delta \lg_{10} r' \cdot 10^8$*	$\delta \omega'$
						″	″	″	″		″
40	1913	Mar	13	—5655	—4491	—34.83	+53.86	—0.41	+0.60	— 587	—0.23
—38	1914	Jan	27	—5268	—4311	—32.45	49.62	—0.36	0.57	— 529	—0.22
—36	1914	Dec	13	—4893	—4126	—30.16	45.62	—0.33	0.55	— 474	—0.19
—34	1915	Oct	29	—4528	—3933	—27.92	41.77	—0.28	0.52	— 420	—0.15
—32	1916	Sep	13	—4174	—3731	—25.77	38.10	—0.25	0.49	— 370	—0.12
30	1917	Jul	30	—3829	—3526	—23.70	34.57	—0.23	0.46	— 323	—0.10
28	1918	June	15	—3495	—3318	—21.69	31.22	—0.20	0.42	— 281	—0.08
—26	1919	May	1	—3175	—3102	—19.77	28.00	—0.18	0.38	— 241	—0.06
—24	1920	Mar	16	—2867	—2881	—17.92	24.97	—0.15	0.34	— 204	—0.05
—22	1921	Jan	30	—2569	—2656	—16.14	22.06	—0.12	0.31	— 172	—0.04
—20	1921	Dec	16	—2279	—2428	—14.39	19.32	—0.09	0.27	— 140	—0.03
18	1922	Nov	1	—2001	—2198	—12.72	16.71	—0.07	0.23	— 114	—0.02
—16	1923	Sep	17	—1737	—1963	—11.11	14.28	—0.06	0.20	— 90	—0.01
—14	1924	Aug	2	—1485	—1724	— 9.55	12.00	—0.04	0.16	— 69	—0.01
—12	1925	June	19	—1246	—1481	— 8.03	9.90	—0.02	0.13	— 52	0.00
—10	1926	May	5	—1015	—1236	— 6.58	7.92	0.00	0.10	— 36	0.00
— 8	1927	Mar	21	— 789	— 992	— 5.18	6.07	+0.01	0.07	— 22	0.00
— 6	1928	Feb	4	— 573	— 746	— 3.82	4.32	0.01	0.05	— 12	0.00
— 4	1928	Dec	20	— 368	— 498	— 2.50	2.73	0.01	0.03	— 4	0.00
— 2	1929	Nov	5	— 177	— 249	— 1.23	+ 1.29	0.02	+0.01	0	0.00
0	1930	Sep	20	0	0	0.00	0.00	0.00	0.00	0	0.00
+ 2	1931	Aug	7	+ 167	+ 250	+ 1.18	— 1.15	0.00	—0.01	— 1	0.00
4	1932	June	22	330	501	2.35	— 2.21	0.02	—0.03	— 6	0.00
6	1933	May	8	491	750	3.47	— 3.19	0.01	—0.04	— 12	0.00
8	1934	Mar	24	644	995	4.53	— 4.07	0.00	—0.05	— 22	—0.01
10	1935	Feb	7	785	1240	5.57	— 4.81	0.00	—0.05	— 33	—0.01
12	1935	Dec	24	917	1485	6.60	— 5.43	0.02	—0.04	— 47	+0.01
14	1936	Nov	8	1037	1730	7.61	— 5.95	0.02	—0.04	— 67	0.03
16	1937	Sep	24	1147	1972	8.57	— 6.23	0.00	—0.05	— 89	0.04
18	1938	Aug	10	1253	2208	9.50	— 6.62	0.01	—0.04	— 111	0.07
20	1939	June	26	1356	2443	10.40	— 6.79	0.03	—0.03	— 133	0.10
22	1940	May	11	1447	2678	11.29	— 6.79	0.04	—0.02	— 159	0.13
24	1941	Mar	27	1532	2906	12.16	— 6.80	0.07	—0.01	— 188	0.18
26	1942	Feb	10	1612	3129	13.00	— 6.81	0.04	0.00	— 220	0.24
28	1942	Dec	27	1681	3352	13.80	— 6.59	0.04	+0.01	— 258	0.28
30	1943	Nov	12	1743	3571	14.58	— 6.35	0.07	0.03	— 298	0.32
32	1944	Sep	27	1799	3786	15.35	— 6.04	0.09	0.05	— 341	0.38
34	1945	Aug	13	1856	3994	16.11	— 5.60	0.11	0.06	— 384	0.48
36	1946	June	29	1910	4198	16.85	— 5.19	0.14	0.08	— 428	0.57
38	1947	May	15	1951	4397	17.55	— 4.67	0.16	0.10	— 477	0.66
40	1948	Mar	30	1985	4590	18.24	— 4.11	0.19	0.12	— 529	0.77
42	1949	Feb	13	2015	4780	18.92	— 3.33	0.23	0.14	— 582	0.88
44	1949	Dec	30	2045	4963	19.59	— 2.66	0.26	0.17	— 638	1.02
46	1950	Nov	15	2077	5142	20.24	— 1.91	0.32	0.19	— 693	1.17
48	1951	Oct	1	2098	5319	20.88	— 1.07	0.36	0.21	— 754	1.33
50	1952	Aug	16	2109	5485	21.51	— 0.20	0.41	0.23	— 819	1.50
52	1953	Jul	2	2116	5653	22.14	+ 0.77	0.46	0.25	— 886	1.70
54	1954	May	18	2124	5819	22.74	1.61	0.47	0.27	— 956	1.91
56	1955	Apr	3	2144	5972	23.32	2.65	0.52	0.29	—1024	2.13

*(lg = log)

As the initial moment of integration, 1930 September 21.0 was chosen. For this and adjacent moments the coordinates and velocities of Pluto were calculated from the formulas of elliptic motion, and the initial terms of the columns of the sums were determined from the following formulas: /78

$$\left.\begin{aligned} f_{-\frac{1}{2}}^{-1} &= \omega\dot{x} - \frac{1}{2} f_0 + \frac{1}{12} f_0' - \frac{11}{720} f_0^{3}, \\ f_0^{-2} &= x_0 - \frac{1}{12} f_0 + \frac{1}{240} f_0^{2} \end{aligned}\right\} \qquad (70)$$

and in similar fashion for the remaining two coordinates.

Table 19 gives the values of the perturbed coordinates and velocities at intervals of 320 days.

From the given values of the coordinates and velocities the elements of Pluto were computed for every moment and the perturbations of the elements were determined.

From the perturbation of the elements we proceed to the perturbations of the coordinates by means of the formulas:

$$\delta\varpi' = \delta\lambda' + \sum jH_j \cos jM' \,\delta M' + \sum \frac{\partial H_j}{\partial e'} \sin jM' \delta e',$$

$$\delta \lg r' = \delta a' + \sum -jK_j \sin jM' \,\delta M' + \sum \frac{\partial K_j}{\partial e'} \cos jM' \delta e'.$$

In Table 20 the perturbations of the elements are given for the period from 1913 to 1955 at intervals of 320 days.

CHAPTER VII

IMPROVING THE ELEMENTS OF PLUTO

All earlier derived elements of Pluto were determined in 1930-1931. In the best case they were computed from observations embracing the period from 1915 to 1931. If it is taken into consideration that the earlier observations were not very reliable, and that the arc traversed by Pluto in mean longitude was no more than 22° at that moment, then it is clear that these elements will in time give increasingly greater discrepancies.

Therefore, it was decided, after determination of the perturbations of the first order of Pluto by the four large planets, to make a preliminary improvement in its elements. The nineteenth system of elements by Bower was taken as the osculating orbit, with the center of the sun as a reference point (the masses of the four inner planets are added to the mass of the sun.) The values of the elements are:*

Osculation and epoch 1930 September 20.0, Universal Time

$$M_0 = 275^\circ\ 10'\ 9''.30$$

$$\left.\begin{aligned} i &= 17\ \ 6\ 33.44 \\ \Omega &= 109\ \ 37\ 56\ 96 \\ \omega &= 112\ \ 43\ 50.56 \\ a &= 39.656022 \end{aligned}\right\}\ \text{ecliptic and equinox 1950.0}$$

$$e = 0.24699758$$

$$n = 14''.208367$$

21. Observations of Pluto and Compilation of Normal Positions

At the moment that the orbit of Pluto was corrected approximately 500 accurate observations had been collected,** comprising the period from 1914 to 1951, and obtained in 28 oppositions.

The observations of Pluto were stated in terms of the equator and the equinox, 1950.0, and corrected for parallax. The observations were not reduced to a single system of stellar positions.

In compiling the normal positions of the following, ephemerides of Pluto were used: for the oppositions of 1930-1931, the unperturbed ephemeris of

*In the following, the notation of the elements and coordinates of Pluto is given without primes.

**Forty of these observations were made at the Main Astronomical Observatory in Pulkovo. The results of the measurements of the photographic plates were kindly placed at our disposal even before they were published by the Senior Scientific Worker of the Main Astronomical Observatory, V. V. Lavdovskiy.

TABLE 21

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
1	1914	Jan	23.7904	$6^h00^m00^s.75$	$+17°37'26''.7$	$-0^s.22$	$+1''.4$	Heidelberg	
2*			23.7905	00 01.05	37 25.5	+0.08	+0.2	"	
3		Nov	12.3260	10 07.73	17 51 55.9	−0.21	+0.8	Harvard	*L. O. B.*, 15, 437.
4	1915	Mar	19.2639	02 39.69	18 04 37.1	−0.15	−2.2	Flagstaff	
5		Apr	7.1674	03 04.23	18 07 49.7	−0.08	−4.4	"	
6	1919	Dec	28.1861	30 19.31	19 20 55.2	+0.04	−1.3	Mt. Wilson	
7			28.2861	30 18.46	20 57.3	−0.29	0.0	"	*L. O. B.*, 15, 427.
8			29.4285	30 12.89	21 05.3	+0.05	−0.1	"	
9			30.3660	30 07.91	21 10.2	−0.07	−2.0	"	
10*	1921	Jan	29.0896	6 32 33.09	19 42 18.4	−0.39	−1.4	Yerkes	*A. N.*, 239, 5719.
11	1925	Dec	23.2823	7 01 08.84	20 54 20.1	−0.08	+3.1	Mt. Wilson	*A. N.*, 243, 5809.
12			23.3441	01 08.43	20 54 20.3	−0.17	+2.6	"	
13	1927	Jan	6.2500	05 14.59	21 11 11.7	−0.31	−0.9	Yerkes	*A. N.*, 239, 5719.
14			27.8942	03 22.16	15 39.7	+0.02	+3.2	Ukkel	
15	1930	Aug	21.39954	28 23.14	52 28.7	−0.06	−0.1	Yerkes	*A. J.*, 41, 961.
16			22.39863	28 28.66	52 21.4	−0.07	+0.6	"	
17			30.1088	29 09.36	51 24.9	+0.08	+1.3	Heidelberg	*B. Z.*, 35, 1930.
18		Sep	1.40697	29 20.58	51 08.4	0.00	+0.2	Yerkes	*A. J.*, 41, 961.
19			5.1076	29 37.76	50 44.8	−0.23	+0.1	Heidelberg	*B. Z.*, 36, 1930.
20			22.36857	30 44.35	49 26.1	−0.07	−1.1	Yerkes	*A. J.*, 41, 961.
21			23.43542	30 47.51	49 23.3	−0.14	−0.9	"	
22			25.1267	30 52.72	49 19.6	+0.18	−0.3	"	*U. A. I.*, 300.
23			26.06576	30 55.21	49 17.2	+0.05	−0.6	Helwan	*L. O. B.*, 15, 437.
24			27.1381	30 57.97	49 15.2	−0.07	−0.6	Bergdorf	*A. N.*, 242, 5804.
25			28.1215	31 00.52	49 13.1	−0.06	−0.9	Paris	*J. O.*, 14, 1.
26			28.40141	31 01.23	49 13.6	−0.06	+0.1	Yerkes	*A. J.*, 41, 961.
27		Oct	1.4906	31 08.56	49 09.1	+0.01	−0.5	Mt. Wilson	*L. O. B.*, 15, 437.
28			2.1528	31 09.91	49 09.6	−0.04	+0.5	Paris	*J. O.*, 14, 1.
29			2.4984	31 10.69	49 08.1	−0.02	−0.7	Mt. Wilson	*L. O. B.*, 15, 437.
30			4.173	31 14.00	49 07.4	−0.08	−0.5	Algiers	*J. O.*, 14, 5.
31			7.180	31 19.39	49 06.1	−0.03	−1.7	"	
32			23.0864	31 32.07	49 41.4	−0.02	−0.5	Bergdorf	*A. N.*, 242, 5804.
33			24.1348	31 35.51	49 38.3	+0.01	−0.1	Berlin-Babels.	*A. N.*, 243, 5824.
34*			25.42265	31 31.16	51 42.7	−0.49	−9.2	Yerkes	*A. J.*, 41, 961.
35		Nov	1.1264	31 31.04	50 18.1	+0.16	−1.7	Berlin-Babels.	*A. N.*, 243, 5824.
36			23.27083	30 40.81	53 33.9	−0.01	−2.1	Yerkes	*A. J.*, 41, 961.
37			24.1141	30 38.22	53 43.7	+0.07	−1.4	Toulouse	*J. O.*, 14, 5.
38			29.9664	30 17.99	21 54 51.0	+0.09	−1.0	Yerkes	*A. J.*, 42, 971.
39	1931	Jan	1.46936	27 45.61	22 02 23.3	+0.17	−1.2	"	
40			27.25333	25 26.54	08 52.0	+0.39	−0.2		

TABLE 21 (continued)

Position No.	Universal Time		$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
41	1931 Feb	8.9103	$7^h24^m21^s.13$	+22°11′45″.2	+0^s.45	1″.7	Heidelberg	A. N., 242, 5796.
42		9.9329	24 19.48	12 00.2	+0.52	0.0	"	
43		10.17429	24 18.15	12 00.5	+0.30	−2.8	Yerkes	A. J., 41, 961.
44		15.1151	23 56.57	13 02.3	+0.43	−3.1	Cordova	L. O. B., 16, 33.
45		18.8775	23 41.26	13 49.1	+0.61	−1.2	Heidelberg	A. N., 242, 5796.
46		21.0902	23 32.48	14 13.3	+0.44	−2.3	Cordova	L. O. B., 16, 33.
47		22.8119	23 26.00	14 33.1	+0.38	−1.7	Bergdorf	A. N., 242, 5804.
48	Mar	6.8128	22 47.98	16 32.0	+0.44	−2.2	Berlin-Babels.	A. N., 243, 5824.
49		12.0462	22 35.48	17 15.5	+0.45	−2.2	Cordova	L. O. B., 16, 33.
50		13.7675	22 31.78	17 27.7	+0.30	−3.3	Berlin-Babels.	A. N., 243, 5824.
51		13.8851	22 31.56	17 28.9	+0.32	−2.9	Heidelberg	A. N., 242, 5796.
52		14.0433	22 31.42	17 31.0	+0.49	−2.0	Cordova	L. O. B., 16, 33.
53		15.8504	22 27.98	17 42.6	+0.42	−3.5	Berlin-Babels.	A. N., 243, 5824.
54		15.8872	22 27.74	17 43.8	+0.24	−2.6	Heidelberg	A. N., 242, 5796.
55		15.9896	22 27.73	17 43.5	+0.41	−3.5	Berlin-Babels.	A. N., 243, 5824.
56		16.8781	22 26.36	17 48.9	+0.56	−4.3	"	
57		16.9738	22 26.03	17 50.4	+0.39	−3.4	"	
58		17.7593	22 24.80	17 56.3	+0.43	−2.8	Pulkovo	Izv. GAO, 19, 3, 150.*
59		17.8079	22 24.75	17 57.6	+0.46	−1.9	Bergdorf	A. N., 242, 5804.
60		17.9619	22 24.48	17 57.3	+0.43	−3.3	Berlin-Babels.	A. N., 243, 5824.
61		18.8156	22 23.19	18 03.4	+0.43	−2.7	Bergdorf	A. N., 242, 5804.
62		18.8205	22 23.44	18 03.7	+0.69	−2.4	Berlin-Babels.	A. N., 243, 5824.
63		18.9651	22 23.02	18 04.3	+0.49	−2.8		
64		19.8006	22 21.86	18 08.9	+0.50	−3.4	"	
65		20.7705	22 20.47	18 15.6	+0.38	−2.7	Pulkovo	Izv. GAO, 19, 3, 150.
66		20.8096	22 20.52	18 15.3	+0.48	−3.3	Berlin-Babels.	A. N., 243, 5824.
67		20.8195	22 20.44	18 16.4	+0.41	−2.3	Bergdorf	A. N., 242, 5804.
68		20.8592	22 20.52	18 14.6	+0.54	−4.3	Berlin-Babels.	A. N., 243, 5824.
69		21.8130	22 19.26	18 22.6	+0.44	−1.9	Bergdorf	A. N., 242, 5804.
70		21.9014	22 19.14	18 21.8	+0.43	−3.2	Berlin-Babels.	A. N., 243, 5824.
71		22.7658	22 18.07	18 26.5	+0.32	−3.5	Cimiez	A. N., 242, 5798.
72		23.7583	22 17.15	18 32.8	+0.40	−2.8	"	
73		23.9100	22 17.04	18 32.8	+0.44	−3.6	Berlin-Babels.	A. N., 243, 5824.
74	Apr	7.8577	22 14.65	19 31.5	+0.45	−2.1	"	
75		8.8330	22 15.27	19 33.4	+0.41	−2.3	"	
76		8.8719	22 15.46	19 32.7	+0.57	−3.1	Heidelberg	A. N., 242, 5796.
77*		12.12028	22 18.37	19 47.0	+0.55	+5.6	Yerkes	A. J., 42, 974.
78*		13.11123	22 19.46	19 48.9	+0.53	+6.3	"	
79		15.9214	22 23.13	19 43.4	+0.45	−1.7	Berlin-Babels.	A. J., 41, 961.
80		19.8635	22 29.67	19 43.6	+0.40	−2.2	"	
81		19.8962	22 29.76	19 45.0	+0.43	−0.8	"	

*Reports of the Main Astronomical Observatory

/81

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
82	1931	May	4.10042	$7^h23^m06.^s52$	$+22°19'21.''1$	$+0.^s49$	$-1.''5$	Yerkes	
83			13.11428	23 39.57	18 46.0	+0.50	−2.1	"	A. J., 42, 974.
84			18.12405	24 00.91	18 20.2	+0.45	−2.7	"	
85		Sep	24.41680	36 22.98	22 00 07.6	−0.01	+0.7	"	A. J., 42,982.
86		Oct	9.38050	36 58.01	21 59 52.6	+0.06	−0.1	"	
87			19.5189	37 08.52	22 00 13.6	−0.06	−0.3	Mt. Hamilton (Lick Observa.)	U. A. I., 348.
88		Nov	6.41707	37 00.79	01 52.1	+0.01	−1.0	Yerkes	A. J., 42, 982.
89		Dec	5.20729	35 41.51	07 03.7	−0.06	−0.2	"	
90			6.4438	35 36.58	07 20.4	−0.01	−0.4	Mt. Hamilton	U. A. I., 348.
91			8.18833	35 29.40	07 43.8	+0.03	−1.0	Yerkes	A. J., 42, 982.
92			30.9400	33 39.29	13 29.3	−0.01	+0.1	Bucharest	
93	1932	Jan	5.9585	33 06.87	15 05.8	−0.01	0.0	"	J. O., 15, 9.
94			7.9502	32 56.02	15 37.9	−0.01	+0.1	"	
95			9.4473	32 47.76	16 02.1	−0.06	+0.1	Mt. Hamilton	L. O. B., 16, 33.
96			15.9151	32 12.30	17 45.8	−0.03	+0.3	Bucharest	J. O., 15, 9.
97			28.17280	31 06.24	20 56.9	−0.07	+1.0	Yerkes	A. J., 42, 982.
98		Feb	5.14558	30 25.65	22 51.5	−0.03	−1.2	"	
99			12.3097	29 51.69	24 30.4	−0.07	−0.3	Mt. Hamilton	L. O. B., 16, 33.
100			13.0160	29 48.50	24 39.5	−0.07	−0.5	Bergdorf	A. N., 251, 6014.
101			15.9092	29 35.72	25 17.3	−0.11	+0.2	"	
102			26.8128	28 53.10	27 24.0	−0.11	0.0	Pulkovo	
103		Mar	5.7952	28 27.95	28 42.9	−0.12	+0.1	"	
104			7.81262	28 22.52	29 00.5	−0.08	0.0	"	Izv. GAO, 19, 3, 150.
105			12.81249	28 10.65	29 40.5	−0.11	−0.5	"	
106			28.79359	27 49.53	31 13.6	−0.15	−0.1	"	
107*		Sep	28.44036	42 14.89	10 05.4	−0.03	+0.7	Yerkes	A. J.,44, 1017.
108*		Nov	2.47362	42 48.25	11 43.8	−0.18	+0.3	"	
109		Dec	21.24767	40 11.00	21 56.0	−0.13	−0.4	Washington	A. J., 48, 1115.
110			21.9588	40 10.55	22 06.3	−0.05	−1.7	Bergdorf	A. N., 251, 6014.
111			26.8814	39 45.43	23 29.0	−0.10	−0.4	"	
112	1933	Jan	24.971829	37 07.47	31 37.4	−0.03	+1.3	Ukkel	U. A. I., 422.
113			25.949015	37 02.27	31 52.6	0.00	+0.8	"	
114*			26.9340	36 57.14	32 07.8	+0.11	+0.4	Bergdorf	A. N., 251, 6014.
115*			27.0885	36 56.37	32 10.3	+0.17	+0.4	"	
116			27.935418	36 51.69	32 23.4	−0.01	+0.1	Ukkel	U. A. I., 422.
117		Feb	2.99089	36 20.21	33 57.2	−0.08	+0.7	"	
118			15.8443	35 19.35	36 55.8	−0.02	−1.0	Stara Dala	B. Z., 13, 1933.
119			16.8623	35 14.97	37 08.6	+0.03	−1.4	"	
120			18.17569	35 09.31	37 27.9	−0.05	+1.1	Yerkes	A. J., 44, 1017.
121			21.80785	34 54.44	38 11.1	−0.11	+0.1	Pulkovo	Izv. GAO, 19, 3, 150.

/82

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
122	1933	Feb	22.1958	$7^h34^m53^s.01$	22°38′16″.2	$-0^s.01$	+0″.6	Mt. Hamilton	*L. O. B.*, 16, 452.
123			23.14421	34 49.31	38 27.8	0.00	−1.0	Yerkes	*A. J.*, 44, 1017.
124		Mar	1.07862	34 27.75	39 31.2	+0.19	−0.7	Washington	*A. J.*, 48, 1115.
125			2.77639	34 22.22	39 49.3	−0.14	+0.2	Pulkovo	Izv. GAO, 19, 3, 150.
126			21.90847	33 38.05	42 17.5	+0.09	−1.1	Bergdorf	*A. N.*, 251, 6014.
127			22.2775	33 37.40	42 21.5	+0.04	+0.8	Mt. Hamilton	*L. O. B.*, 16, 452.
128			24.9003	33 34.29	42 33.0	0.00	−1.2	Bergdorf	*A. N.*, 251, 6014.
129		Apr	13.8675	33 34.19	43 22.6	+0.02	0.0	"	
130			14.8627	33 35.26	43 22.5	−0.01	0.0	"	
131			14.9063	33 35.34	43 22.2	+0.02	−0.3	"	
132*		Oct	17.45937	48 34.77	19 47.5	+0.04	−0.3	Yerkes	*A. J.*, 44, 1017.
133*			18.41543	48 35.56	19 50.6	−0.06	−0.5	"	
134	1934	Feb	10.20006	41 38.02	46 28.3	−0.16	+0.5	"	
135			12.2635	41 28.30	46 58.0	−0.04	+0.4	Mt. Hamilton	*L. O. B.*, 17, 6.
136			12.2767	41 28.22	46 58.2	−0.05	+0.4	"	
137			13.16065	41 24.07	47 10.9	−0.09	+0.6	Yerkes	*A. J.*, 44, 1017.
138			13.93072	41 20.55	47 21.1	−0.03	0.0	Ukkel	*B. A. B.*, 1, 13, 295.
139*		Mar	10.12407	39 50.60	52 00.0	+0.11	+3.5	Yerkes	*A. J.*, 44, 1017.
140*			10.86349	39 48.36	52 08.7	−0.15	+6.0	Ukkel	*B. A. B.*, 1, 13, 295.
141*			11.10704	39 48.02	52 07.4	+0.15	−2.6	Yerkes	*A. J.*, 44, 1017.
142			11.82125	39 45.95	52 10.7	−0.08	0.0	Pulkovo	Izv. GAO, 19, 3, 150.
143			13.77008	39 41.12	52 26.1	−0.13	−0.1	"	
144		Apr	3.80623	39 14.02	22 54 11.2	−0.13	+0.1	"	
145	1935	Feb	25.19931	46 29.17	23 00 02.6	−0.04	+0.8	Mt. Hamilton	*L. O. B.*, 17, 125.
146			25.23194	46 29.04	00 02.9	−0.05	+0.8	"	
147		Mar	4.8507	46 01.18	01 27.7	−0.04	0.0	Pulkovo	Izv. GAO, 19, 3, 150.
148			8.8416	45 48.53	02 06.8	0.05	+0.3	"	
149			25.8107	45 11.78	04 04.9	−0.06	−0.4	"	
150			27.07890	45 10.20	04 11.0	−0.06	+0.8	Yerkes	*A. J.*, 45, 1046.
151			29.11148	45 08.03	04 18.1	−0.03	−0.3	"	
152		Apr	7.11157	45 03.65	23 04 41.3	−0.08	+0.8	"	
153*		Dec	24.6507	57 55.55	22 51 01.3	−0.09	−2.5	Tokyo	*A. N.*, 260, 6236.
154*	1936	Jan	17.5838	55 47.30	58 52.8	+0.06	2.9	"	
155*			19.5889	55 36.49	22 59 32.3	+0.45	−2.0	"	
156*		Feb	10.85007	53 36.32	23 06 08.8	0.00	1.6	Bergdorf	*A. N.*, 262, 6267.
157*			15.13590	53 15.46	07 17.4	+0.07	+0.3	Yerkes	*A. J.*, 45, 1046.
158*			20.04950	52 52.69	08 30.2	0.00	+1.9	"	
159*		Mar	19.85389	7 51 16.09	23 13 22.0	−0.27	+0.1	Bergdorf	*A. N.*, 262, 6267.
160*		Nov	11.35934	8 06 21.82	22 47 21.7	−0.02	+0.6	Yerkes	*A. J.*, 46, 1068.
161	1937	Jan	13.24518	02 15.83	23 06 06.9	0.00	+0.3	"	
162		Feb	5.10691	00 09.57	13 29.4	+0.11	−0.2		

/83

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
				$h\ m\ s$	$°\ '\ ''$	s	$''$		
163	1937	Mar	7.17137	7 57 55.60	+23 20 37.0	+0.06	+0.9	Yerkes	*A. J.*, 46, 1068.
164		Apr	10.83333	56 58.00	23 30.4	+0.06	+1.9	Toulouse	
165			22.87778	57 07.84	23 01.6	+0.02	+0.4	"	
166			22.90278	57 07.87	23 01.7	+0.01	+0.6	"	
167			23.90972	57 09.45	22 58.1	+0.06	+1.1	"	*J. O.*, 21, 2.
168			24.87986	57 11.02	22 53.4	+0.04	+0.8	"	
169			29.86528	7 57 20.73	22 27.6	+0.07	+1.3	"	
170*	1938	Jan	27.9396	8 07 08.46	18 40.3	−0.21	+0.5	Bergdorf	*A. N.*, 268, 6428.
171		Feb	1.80347	06 41.73	20 16.4	−0.05	0.0	Wurzburg	*A. N.*, 270, 53.
172*			6.0079	06 18.78	21 34.7	−0.26	−0.9	Bergdorf	*A. N.*, 268, 6428.
173*			19.9233	05 09.01	25 28.9	−0.07	+1.4		
174*			20.84236	05 04.98	25 42.3	+0.16	−1.9	Wursburg	*A. N.*, 270, 53.
175*			20.9401	05 04.31	25 44.2	−0.06	−1.4	Bergdorf	*A. N.*, 268, 6428.
176			21.8523	05 00.17	25 59.9	−0.03	+0.8	Pulkovo	Izv, GAO, 19, 3, 150.
177			23.8910	04 51.12	26 28.2	+0.02	+0.3	Bergdorf	*A. N.*, 268, 6428.
178			23.9216	04 50.90	26 29.8	−0.06	+0.9	Vienna	*A. N.*, 268, 6380.
179			24.8852	04 46.72	26 41.6	−0.03	+0.7	Bergdorf	*A. N.*, 268, 6428.
180			25.8012	04 42.77	26 55.5	−0.05	+0.6	Pulkovo	Izv. GAO, 19, 3, 150.
181			25.8883	04 42.39	26 57.0	−0.06	+0.9	Vienna	
182			27.8400	04 34.23	27 22.7	−0.05	+0.7	"	*A. N.*, 266, 6380.
183		Mar	5.8680	04 10.86	28 35.2	−0.06	+0.6	"	
184			5.8780	04 10.98	28 35.0	+0.09	+0.3	"	*A. N.*, 271, 28.
185			18.8468	03 31.37	30 32.7	+0.01	+0.1	Pulkovo	Izv. GAO, 19, 3, 150.
186			23.7939	03 20.50	31 03.8	0.00	+0.5	Vienna	*A. N.*, 266, 6380.
187		Apr	19.8469	03 06.91	31 28.0	−0.05	+0.4	"	*A. N.*, 271, 28.
188*			21.8648	03 08.91	31 21.9	−0.23	+1.8	Pulkovo	Izv. GAO, 13, 3, 150.
189	1939	Jan	19.9671	14 14.31	22 40.5	−0.05	+1.3	"	
190			21.9448	14 03.10	23 24.0	−0.09	+1.4	Vienna	*A. N.*, 271, 25.
191*		Feb	8.7993	12 24.50	29 29.6	+0.27	+1.9	"	
192*			8.8340	12 24.08	29 31.7	+0.03	+3.4	Pulkovo	Izv. GAO, 19, 3, 150.
193			23.8652	11 09.46	33 43.4	−0.05	+1.0	Washington	
194		Mar	14.0779	09 59.66	37 20.9	+0.01	+0.2	"	
195			14.1460	09 59.43	37 21.6	−0.02	+0.3	"	
196			14.1531	09 59.40	37 21.6	−0.03	+0.2	"	
197			14.1602	09 59.39	37 22.1	−0.02	+0.7	"	*A. J.*, 50, 1141.
198			15.1395	09 56.41	37 31.4	−0.04	+1.3	"	
199			15.1506	09 56.36	37 31.1	−0.05	+1.0	"	
200			15.1697	09 56.30	37 31.2	−0.06	+0.9	"	
201*			16.7962	09 51.79	37 45.6	+0.13	+1.7	Vienna	*A. N.*, 271, 25.
202		Nov	11.42406	25 02.61	04 47.6	+0.06	+1.1	Yerkes	*A. J.*, 50, 1148.
203			13.40885	25 01.12	05 12.5	+0.08	+0.8	"	

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
204	1939	Dec	3.99853	$8^h24^m20^s.51$	$+23°10'47''.35$	$-0^s.02$	$0''.0$	Vienna	
205			7.16666	24 10.66	11 50.93	+0.11	+1.4	"	*A. N.*, 271, 25.
206*			21.36991	23 12.99	17 06.17	−2.15	+12.4	"	
207	1940	Jan	3.8852	22 09.49	22 07.90	+0.03	0.0	"	
208			10.9146	21 31.80	24 55.7	0.00	+1.1	"	*A. N.*, 271, 237.
209		Feb	11.10131	18 37.96	36 20.8	−0.02	+0.4	Yerkes	*A. J.*, 50, 1148.
210		Mar	11.8331	16 25.58	43 42.2	+0.02	+0.7	Pulkovo	
211			14.8536	16 15.87	44 09.2	−0.01	−0.6	"	Izv. GAO, 19, 3, 150.
212			25.7821	15 48.12	45 25.1	0.00	+0.4	"	
213*			29.8778	15 40.92	45 44.3	+0.11	+2.9	Vienna	
214		Apr	1.8587	15 36.81	45 50.5	+0.21	+0.9	"	
215			2.8278	15 35.66	45 51.8	+0.22	+0.2	"	*A. N.*, 275, 187.
216			7.8210	15 31.19	45 56.4	+0.14	+0.3	"	
217	1941	Mar	21.1340	22 25.51	50 39.9	+0.12	+0.2	Mt. Wilson	
218			21.2510	22 25.19	50 40.8	+0.10	+0.4	"	*A. J.*, 50, 1147.
219			29.7968	22 06.21	51 26.3	−0.03	+0.9	Pulkovo	
220		Apr	2.8004	22 00.02	51 37.2	−0.01	+0.6	"	Izv. GAO, 19, 3, 150.
221		Nov	18.24065	37 52.05	13 35.2	+0.14	+0.7	Yerkes	
222			19.28396	37 50.82	13 52.2	+0.13	+1.2	"	
223*			27.25431	37 37.81	16 11.1	+0.27	+1.8	"	
224		Dec	14.20280	36 48.32	22 05.6	+0.15	−0.6	"	
225			21.28392	36 20.02	24 56.3	+0.15	+1.4	"	
226	1942	Jan	13.16575	34 26.30	34 37.6	+0.09	+1.4	"	
227			15.14200	34 15.30	35 28.8	+0.01	+1.9	"	
228			16.17631	34 09.57	35 54.8	+0.04	+1.5	"	
229			17.15817	34 04.16	36 19.7	+0.11	+1.4	"	
230			22.21241	33 35.57	38 27.5	+0.06	+2.0	"	
231		Feb	12.13650	31 37.99	46 28.6	+0.07	+0.7	"	
232			14.12633	31 27.40	47 09.3	+0.14	+0.9	"	
233			18.06998	31 06.78	48 25.8	+0.20	+0.3	"	*A. J.*, 50, 1148.
234			20.14745	30 56.13	49 03.7	+0.15	−0.6	"	
235		Mar	10.28510	29 35.07	53 39.3	+0.14	+0.6	"	
236			19.17847	29 04.69	55 08.4	+0.17	+0.4	"	
237			22.20299	28 56.01	55 31.4	+0.14	+0.3	"	
238		Apr	4.13335	28 29.27	56 28.4	+0.08	+1.1	"	
239			11.13953	28 22.23	56 29.9	+0.14	+0.6	"	
240			13.10509	28 21.24	56 26.3	+0.16	+0.1	"	
241			15.11545	28 20.63	56 22.0	+0.15	+0.5	"	
242		May	8.13896	28 45.61	53 38.8	+0.12	+0.2	"	
243			9.11962	28 47.97	53 27.9	+0.13	+0.4	"	
244			19.10108	29 17.50	51 18.2	+0.18	+0.9	"	

/85

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
245*	1943	Feb	7.07477	$8^h38^m51^s.04$	$+23^\circ47'50''.7$	$+0^s.12$	$+4''.1$	Yerkes	
246*			8.09282	38 45.27	48 13.1	+0.06	+3.2	"	
247*		Mar	5.12412	36 38.26	56 07.0	+0.14	+1.5	"	A. J., 50, 1148.
248*		May	22.11021	35 56.64	54 11.8	+0.27	+0.9	"	
249*			25.12507	36 07.91	53 21.7	+0.28	+3.6	"	
250*			28.12083	36 19.93	52 33.8	+0.26	+4.7	"	
251	1944	Jan	18.06383	47 36.13	41 07.2	+0.10	+2.6	"	
252			21.10469	47 19.25	42 29.9	+0.22	+1.9	"	
253			22.04936	47 13.98	42 55.7	+0.27	+2.0	"	
254			24.08035	47 02.37	43 51.1	+0.19	+2.3	"	A. J., 51, 1152.
255*		Feb	4.42459	45 57.44	48 41.5	+0.14	2.3	"	
256			17.08058	44 47.01	23 53 39.5	+0.21	+1.8	Vienna	
257		Apr	12.8784	41 37.64	24 03 09.1	+0.13	+1.9		A. N., 275, 187.
258*	1945	Feb	1.06902	53 08.38	23 48 37.4	+0.20	+3.2	Yerkes	
259*		Mar	10.11931	49 55.18	24 01 31.4	+0.15	+0.2	"	
260*			11.31545	8 49 51.22	24 01 45.6	+1.18	−2.0	"	A. J., 54, 1176.
261*		Oct	16.36317	9 04 02.04	23 10 38.0	+0.24	+2.2	"	
262*		Nov	15.47345	04 58.37	14 36.5	+5.20	+1.3	"	
263*		Dec	12.0236	04 13.90	23 47.3	+0.28	+2.9	Kazan	
264*	1946	Jan	24.8736	00 52.21	44 52.7	−0.28	+2.4	"	Astr. Ts., 49.**
265*			29.09564	9 00 29.04	46 43.5	+0.68	−8.2	Yerkes	A. J., 54, 1176.
266*		Feb	11.93794	8 59 09.05	52 47.8	+0.09	−15.8	Padua	
267*			13.93890	58 57.88	53 39.9	+0.16	−13.3	"	Contr. Astrofisico Univ. Padova in Asiago, 12.
268*			22.88405	58 09.08	57 02.3	+0.22	15.7	"	
269*			28.23301	57 41.56	23 59 09.9	+0.26	+3.7	Yerkes	
270*		Mar	3.05887	57 27.57	24 00 02.4	+0.22	+3.7	"	A. J., 54, 1176.
271*			19.90629	56 16.08	03 43.5	+0.24	−14.2	Padua	Contr. Astrofisico Univ. Padova in Asiago, 12.
272*			20.10140	56 15.31	04 02.6	+0.17	+3.1	Yerkes	A. J., 54, 1176.
273*			21.82962	56 09.34	04 02.1	+0.20	14.3	Padua	Contr. Astrofisico Univ. Padova in Asiago, 12.
274*			22.7163	56 14.71	06 21.8	+8.55	+117.4	Kazan	
275*			23.8708	56 10.60	06 34.2	+8.21	+119.9	"	Astr. Ts., 49.
276*			30.8052	55 51.16	07 22.7	+8.71	+122.7	"	
277*		Apr	3.92500	55 33.25	05 20.4	+0.38	−15.4	Padua	Contr. Astrofisico Univ. Padova in Asiago, 12.
278*			5.8236	8 55 37.69	24 07 24.2	+8.65	+103.9	Kazan	Astr. Ts., 49.
279		Dec	4.42795	9 11 34.70	23 18 18.5	+0.24	+1.1	Yerkes	
280			22.48830	10 43.45	26 13.1	+0.22	+1.2	"	A. J., 54, 1176.

** - Astronomical Circular

/86

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
				$h\ m\ s$	° ′ ″	s	″		
349	1950	Mar	14.84731	9 25 24.46	+23 53 03.6	+0.31	+2.1	Kazan	Astr. Ts., 112.
350			21.79743	24 54.42	54 44.2	+0.24	+2.6	"	
351*		Apr	15.79971	23 42.44	56 53.0	+0.13	−0.3	"	
352			17.85175	23 39.53	56 50.0	+0.27	+3.3	Vienna	Mitteilungen der Wiener-Sternwarte, 5, 1952.
353			20.88561	23 35.77	56 37.0	+0.16	+2.0	"	
354			21.85606	23 34.85	56 32.2	+0.20	+2.7	"	
355*			24.76318	23 33.29	56 20.7	+0.87	+8.1	Kazan	Astr. Ts., 112.
356		May	11.84699	23 39.00	52 57.2	+0.27	+2.2	Vienna	Mitteilungen der Wiener-Sternwarte, 5, 1952.
357			14.89896	23 43.58	52 06.4	+0.24	+2.0	"	
358			18.86806	23 51.15	50 53.2	+0.25	+1.3	"	
359			20.91078	23 55.63	23 50 14.0	+0.16	+2.1	"	
360		Oct	20.15422	39 18.89	22 47 34.3	+0.31	+2.1	"	
361			22.14097	39 26.68	47 33.2	+0.27	+2.4	"	
362		Nov	16.11214	40 28.95	50 36.2	+0.32	+1.7	"	
363			19.10990	40 31.56	22 51 23.2	+0.25	+2.0	"	
364*		Dec	13.02819	40 15.54	23 00 22.0	+0.47	+0.9	Kazan	Astr. Ts., 112.
365*			17.01451	40 06.42	02 23.9	+0.37	+7.6	"	
366	1951	Mar	5.93225	33 37.13	44 13.0	+0.28	+1.9	Pulkovo	Izv. GAO, 19, 3, 150.
367			6.95775	33 31.76	44 36.0	+0.27	+2.4	"	
368			27.87919	31 57.52	50 10.7	+0.40	+2.7	Algiers	*U. A. I.*, 1316.
369			28.89171	31 53.81	50 20.5	+0.39	+2.6	"	
370			30.94618	31 46.44	50 37.8	+0.25	+1.9	Vienna	Mitteilungen der Wiener-Sternwarte, 5, 1952.
371		Apr	2.01213	31 39.63	50 53.5	+0.33	+1.9	"	
372			3.90318	31 33.68	51 05.9	+0.33	+2.4	Pulkovo	Izv. GAO, 19, 3, 150.
373			4.95694	31 30.56	51 11.6	+0.36	+2.2	Vienna	Mitteilungen der Wiener Sternwarte, 5, 1952.
374			7.80367	31 22.47	51 23.8	+0.27	+2.3	Pulkovo	Izv. GAO, 19, 3, 150.
375			7.84792	31 22.38	51 23.8	+0.30	+2.1	Vienna	Mitteilungen der Wiener-Sternwarte, 5, 1952.
376			9.84222	31 17.28	51 29.3	+0.31	+2.2	"	
377			12.96736	31 10.12	51 33.6	+0.31	+2.5	"	
378			25.98546	30 51.67	50 47.1	+0.34	+2.6	"	
379			26.98264	30 50.95	50 39.7	+0.27	+2.8	"	
380		May	7.92632	30 51.36	48 38.6	+0.26	+1.9	"	
381			8.95022	30 52.14	48 25.0	+0.31	+3.0	"	

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
281	1947	Jan	17.05504	$9^h08^m44^s.50$	$+23°39'12''.9$	$+0^s.22$	$+2''.1$	Ukkel	*M. P. C.*, 13.
282			17.23093	08 43.52	39 17.1	+0.19	+0.8	Yerkes	*A. J.*, 54, 1176.
283			22.29199	08 15.67	41 53.1	+0.18	+0.7	"	
284			24.04757	08 05.79	42 46.5	+0.14	+0.5	Ukkel	*M. P. C.*, 13.
285		Feb	20.12973	05 31.61	55 14.2	+0.33	+2.8	Yerkes	*A. J.*, 54, 1176.
286			21.88723	05 21.95	55 53.2	+0.28	+1.0	Algiers	*J. O.*, 31, 5.
287			22.92985	05 16.34	56 16.9	+0.30	+1.1	"	
288*			23.14786	05 15.28	56 25.5	+0.41	+4.8	Yerkes	*A. J.*, 54, 1176.
289			27.86904	04 50.35	23 58 02.9	+0.33	+1.0	Algiers	*J. O.*, 31, 5.
290		Mar	11.82414	03 53.05	24 01 35.3	+0.34	+0.4	"	
291			12.82456	03 48.69	24 01 50.1	+0.34	+0.4	"	
292	1948	Jan	7.29225	16 47.47	23 30 57.1	+0.27	+3.0	Yerkes	*A. J.*, 54, 1176.
293			13.97816	16 14.12	34 33.3	+0.26	+4.3	Kazan	Astr. Ts., 88
294			14.20501	16 12.95	34 40.3	+0.27	+3.9	Yerkes	*A. J.*, 54, 1176.
295			17.95718	15 53.11	36 41.3	+0.22	+4.2	Kazan	Astr. Ts., 88.
296			19.93608	15 42.61	37 44.3	+0.42	+3.7	"	
297			31.9525	14 34.88	43 59.8	+0.30	+3.0	Algiers	*J. O.*, 31, 5.
298		Feb	6.22523	14 04.40	46 36.5	+0.29	+2.7	Yerkes	*A. J.*, 54, 1176.
299*			6.89763	14 00.28	46 55.3	+0.06	+1.9	San Fernando	*U. A. I.*, 1149.
300*			6.93156	14 00.00	46 56.2	−0.02	+1.8	"	
301			8.89556	13 49.01	47 53.5	+0.40	+2.5	Kazan	Astr. Ts., 88
302			9.9441	13 42.83	48 23.3	+0.25	+2.5	Algiers	*J. O.*, 31, 5.
303			13.94090	13 19.97	50 12.8	+0.38	+1.8	Kazan	Astr. Ts., 88.
304			18.06333	12 56.44	52 01.3	+0.28	+2.0	Ukkel	*M. P. C.*, 111.
305		Mar	4.8888	11 32.36	57 56.6	+0.47	+2.9	Algiers	*J. O.*, 31, 5.
306*			8.80484	11 13.63	59 07.0	+0.60	+2.6	Kazan	Astr. Ts., 88.
307			8.83041	11 13.35	59 06.4	+0.44	+1.6	"	
308			9.0162	11 12.44	59 11.4	+0.41	+3.4	Toulouse	*J. O.*, 31, 12.
309*			9.9645	11 08.24	23 59 25.0	+0.60	+1.1	"	
310			17.9273	10 33.45	24 01 20.9	+0.19	+1.1	Algiers	*J. O.*, 31, 5.
311			26.74033	10 01.55	02 52.4	+0.25	+2.8	Kazan	Astr. Ts., 88.
312			30.83892	09 49.14	03 19.7	+0.27	+2.2	"	
313		Apr	2.78752	09 41.29	03 34.1	+0.34	+2.2	"	
314			6.75883	09 31.92	03 46.4	+0.26	+2.7	"	
315			6.87716	09 31.59	03 46.4	+0.18	+2.5	San Fernando	*U. A. I.*, 1149.

TABLE 21 (continued)

Position No.	Universal Time			$\alpha_{1950.0}$	$\delta_{1950.0}$	Obs.-Calc. $\Delta\alpha$	Obs.-Calc. $\Delta\delta$	Place of Observation	Sources
316	1948	Apr	6.91105	$9^h09^m31^s.50$	$+24^\circ03'46''.9$	$+0^s.17$	$+3''.0$	San Fernando	U. A. I., 1149.
317*			7.86889	09 29.44	24 03 47.8	+0.09	+2.3	"	
318	1949	Jan	29.9394	22 00.14	23 39 23.9	+0.31	+1.5	Kazan	Astr. Ts., 101–102.
319*		Feb	1.91419	21 42.75	40 58.6	+0.02	+2.3	San Fernando	U. A. I., 1212.
320			3.88235	21 31.77	42 00.2	+0.43	+2.8	Kazan	Astr. Ts., 101–102.
321			17.89488	20 11.07	48 44.1	+0.53	+2.8	Ukkel	B. A. B., 4, 4.
322			18.8682	20 05.28	49 09.4	+0.24	+2.2	Pulkovo	Izv. GAO, 19, 3, 150.
323			26.8793	19 21.02	52 26.3	+0.19	+2.0	"	
324		Mar	1.97161	19 04.79	53 34.5	+0.34	+1.5	Algiers	U. A. I., 1210, 1499.
325			4.85656	18 49.97	54 34.2	+0.36	+1.0	Kazan	Astr. Ts., 101–102.
326			5.9187	18 44.53	54 56.3	+0.28	+2.0	Pulkovo	Izv. GAO, 19, 3, 150.
327*			6.89466	18 44.13	55 16.0	+4.74	+2.7	Kazan "	Astr. Ts., 101–102.
328*			22.86014	17 29.32	59 19.3	+0.06	+4.8		
329			24.956	17 21.92	59 41.8	+0.37	+5.7	Posen	U. A. I., 1210.
330			25.924	17 18.35	23 59 48.9	+0.23	+3.7	"	
331			27.83672	17 11.92	24 00 06.5	+0.32	+4.7	Kazan	Astr. Ts., 101–102.
332			30.907	17 01.99	00 26.1	+0.15	+2.0	Posen	U. A. I., 1210.
333		Apr	3.83859	16 50.98	00 49.3	+0.29	+4.4	Kazan	Astr. Ts., 101–102.
334			21.90625	16 20.61	00 32.1	+0.36	+2.4	Vienna	Mitteilungen der Wiener Sternwarte, 5, 1952.
335			23.83176	16 19.48	24 00 21.1	+0.36	+3.5	Kazan	Astr. Ts., 101–102.
336		May	3.86422	16 20.35	23 58 44.9	+0.31	+1.8	Vienna	Mitteilungen der Wiener Sternwarte, 5, 1952.
337			16.85866	16 38.47	55 29.7	+0.30	+1.9	"	
338		Dec	18.07012	32 39.20	10 20.0	+0.36	+4.0	Kazan	Astr. Ts., 101–102.
339			18.99324	32 36.55	10 44.6	+0.35	+1.2	"	
340*			20.01066	32 32.53	11 14.8	−0.67	+0.8	"	
341	1950	Jan	22.03867	30 09.06	29 32.6	+0.25	+1.5	Ukkel	U. A. I., 1259.
342		Feb	9.92524	28 21.78	39 49.8	+0.37	+1.9	Algiers	U. A. I., 1260.
343			15.92880	27 46.95	42 47.3	+0.33	+1.7	"	
344			18.93745	27 29.66	44 13.0	+0.31	+2.8	Kazan	Astr. Ts., 112.
345			20.97584	27 18.08	45 08.7	+0.33	+3.3	"	
346			24.875334	26 56.10	46 48.6	+0.18	+2.0	Pulkovo	Izv. GAO, 19, 3, 150.
347		Mar	11.837315	25 38.47	52 11.7	+0.23	+1.9	"	
348			14.807696	25 24.58	53 03.2	+0.24	+2.4	"	

/88

Bower; for the oppositions of 1931-1934, the perturbed ephemeris of Bower; for the oppositions from 1935 to 1951, the ephemeris in BJ.

For the early years (from 1914 to 1927) comparison with the observations was made by taking into consideration the perturbations obtained from the above explained theory of Plutonian motion. /90

For the opposition of 1929-1930, we used the normal position obtained by Bower from 103 observations (Bower, 1931).

Table 21 gives a resume of the observations of Pluto.

The distribution of the columns is the following: the first column gives the ordinal number of the observation; the second column, universal time of the observation; the third and fourth columns, the geocentric coordinates of Pluto referred to the axes of 1950.0; the fifth and sixth columns, results of comparison with the above mentioned ephemerides; the seventh column, place of observation; and the eighth column, sources from which the observations were taken.

The observations of Pluto were combined into the normal positions. Taking into consideration the slowness of Plutonian motion, the comparatively good quality of the ephemerides, and the small number of observations, one normal position was obtained for each opposition. The apparently regressive observations (noted in Table 21 by asterisks) were not taken into consideration.

Thus, no normal positions could be compiled for the oppositions of 1921, 1935, 1945, and 1946. In all 24 normal positions were obtained.

In compiling the normal positions, the formulas used were

$$\Delta\alpha \cos\delta = a + bt, \quad \Delta\delta = a' + b't.$$

Corrections were introduced into all normal positions for that part of the aberration depending on eccentricity of the earth's orbit. The formulas used for computing the corrections were

$$\left.\begin{aligned}(\Delta\alpha) &= h_0 \sin(H_0 + \alpha) \sec\delta,\\ (\Delta\delta) &= h_0 \cos(H_0 + \alpha) \sin\delta + i_0 \cos\delta,\end{aligned}\right\} \qquad (71)$$

where for h_0, H_0, and i_0, are taken the values

$$h_0 = 0''.342, \quad H_0 = 348^\circ.9, \quad i_0 = -0^\circ.028.$$

Table 22 gives a resume of the normal positions. Distribution of the columns is as follows: first column contains the ordinal number of the normal position; the second column, universal time of the moment of normal position; third and fourth columns, coordinates of Pluto; the fifth and sixth columns, /91

corrections for the constant part of the aberration; and the seventh and eighth columns, number of observations used in compiling the given normal position.

TABLE 22

Ordinal No.	Universal Time		$\alpha_{1950.0}$	$\delta_{1950.0}$	$(\Delta\alpha)$	$(\Delta\delta)$	No. of Observations on α	No. of Observations on δ
1	1914 Jan.	23.7904	90° 00′ 11.″25	17° 37′ 26.″70	−0.″352	0.″007	1	1
2	Nov.	12.3260	92 31 55.95	17 51 55.90	−0.355	0.011	1	1
3	1915 April	7.1674	90 46 02.94	18 07 50.81	−0.354	0.008	2	2
4	1919 Dec.	29.4285	97 33 11.51	19 21 04.55	−0.362	0.019	4	4
5	1925 Dec.	23.2823	105 17 11.93	20 54 19.87	−0.365	0.035	2	2
6	1927 Jan.	6.2500	106 18 41.32	21 11 13.73	−0.365	0.037	2	2
7	1930 April	24.1151	109 17 46.59	22 07 06.39	−0.365	0.014	103	103
8	1931 Feb.	2.0000	111 14 17.13	22 10 11.88	−0.364	0.049	67	67
9	1932 Feb.	4.0000	112 37 48.97	22 22 36.28	−0.362	0.052	22	22
10	1933 March	15.0000	113 27 27.43	22 41 34.52	−0.362	0.054	21	21
11	1934 March	8.0000	114 59 04.92	22 51 37.83	−0.360	0.058	9	8
12	1935 March	18.0000	116 21 16.98	23 03 20.38	−0.359	0.061	8	8
13	1937 March	7.0000	119 29 02.13	23 20 36.55	−0.353	0.069	9	9
14	1938 March	11.0000	120 58 21.02	23 29 28.12	−0.351	0.072	15	13
15	1939 Feb.	18.0000	122 54 16.18	23 32 11.56	−0.346	0.077	10	10
16	Dec.	11.0000	125 59 18.76	23 13 08.83	−0.338	0.062	13	13
17	1941 March	27.0000	125 32 53.92	23 51 14.54	−0.340	0.083	4	4
18	1942 March	3.0000	127 31 12.62	23 52 03.52	−0.335	0.088	23	23
19	1944 Feb.	28.0000	130 57 43.72	23 57 12.87	−0.325	0.095	7	6
20	1947 Jan.	22.0000	137 04 20.19	23 41 44.74	−0.302	0.107	12	12
21	1948 Feb.	18.0000	138 14 12.06	23 52 00.56	−0.298	0.110	21	21
22	1949 March	12.0000	139 33 48.93	23 56 47.52	−0.293	0.112	17	17
23	1950 Feb.	2.0000	142 16 53.74	23 35 39.99	−0.281	0.116	19	19
24	1951 March	22.0000	143 05 09.70	23 49 01.60	−0.278	0.118	20	20

22. Improvement of the Center of Pluto's Elements

One of the basic stages in the work of correcting elements is representation of the normal positions by theory. In the preceding chapters the perturbation series of Pluto by Jupiter, Saturn and Uranus were used (Tables 8, 13, and 18). Having values of the average anomalies of Pluto, Jupiter, Saturn and Uranus for every moment of the normal position, we derive perturbations of the logarithm of the radius vector, of the longitude in orbit, of the longitude of node, and of the inclination by using the given series. Analytical computers were used to calculate these values. The perturbations of Pluto by Neptune were derived by interpolation from Table 20.

Table 23 gives the values of the perturbations for these moments.

We obtain the unperturbed coordinates of Pluto from the usual formulas of elliptic motion for the moment of the normal position corrected for planetary aberration:

$$w_0 = v_0 + \Omega_0 + \omega_0, \qquad \Omega = \Omega_0 + \delta\Omega,$$

$$w = w_0 + \delta w, \qquad i = i_0 + \delta i,$$

$$\log r = \log r_0 + \delta \log r, \qquad \sigma = \Omega_0 + \delta\Omega \cos i.$$

TABLE 23

Ordinal Number	Universal Time of Normal Position (Corrected for Aberration)		$\delta \log r \cdot 10^8$	δw	$\delta \Omega$	δi
1	1914 January	23.5373	— 7279	+141.″21	—159.″81	+257.″82
2	November	12.0732	— 4956	130.44	—124.70	227.54
3	1915 April	6.9094	— 3637	126.17	— 96.30	+201.78
4	1919 December	29.1831	— 4969	128.92	+155.19	— 52.28
5	1925 December	23.0442	— 5669	54.04	— 40.97	+299.14
6	1927 January	6.0133	— 4670	32.43	— 29.99	260.47
7	1930 April	23.8756	+ 815	0.74	— 2.30	+ 9.67
8	1931 February	1.7684	+ 815	0.40	— 0.05	— 4.58
9	1932 February	3.7697	— 386	+ 0.27	+ 2.21	+ 11.46
10	1933 March	14.7687	— 3090	— 2.51	13.00	66.70
11	1934 March	7.7708	— 6291	— 9.24	31.38	133.13
12	1935 March	17.7715	— 9938	— 21.53	56.57	203.81
13	1937 March	6.7754	—15666	— 59.21	92.20	282.81
14	1938 March	10.7766	—16709	— 81.76	83.13	267.15
15	1939 February	17.7796	—16902	—100.95	+ 46.20	212.88
16	December	10.7796	—16155	—113.82	— 6.20	144.60
17	1941 March	26.7800	—14587	—124.31	—103.15	+ 32.70
18	1942 March	2.7834	—14141	—123.96	—155.79	— 21.85
19	1944 February	27.7866	—16611	—109.24	—154.63	— 23.19
20	1947 January	21.7916	—31140	— 92.18	+ 71.51	+141.44
21	1948 February	17.7932	—37870	— 95.23	159.19	197.34
22	1949 March	11.7939	—44158	—103.73	205.85	225.35
23	1950 February	1.7962	—48651	—113.44	194.95	219.45
24	1951 March	21.7964	—52920	—125.22	105.36	173.54

To determine the perturbed values of longitude in ecliptic l, and of latitude b, we use the familiar formulas

$$\tan(l-\Omega) = \cos i \tan(w-\sigma), \quad \sin b = \sin i \sin(w-\sigma).$$

From l, b, r by the formulas

$$\rho \cos\alpha \cos\delta = r \cos l \cos b + X_{\odot},$$
$$\rho \sin\alpha \cos\delta = r \sin l \cos b \cos\varepsilon - r \sin b \sin\varepsilon + Y_{\odot},$$
$$\rho \sin\delta = r \sin l \cos b \sin\varepsilon + r \sin b \cos\varepsilon + Z_{\odot}$$

we proceed to α, δ, and ρ.

The coordinates of the sun $X_{\odot}$, $Y_{\odot}$, $Z_{\odot}$ we derive from the Annuals for the moments of the normal positions.

To correct the sixth ecliptical elements of Pluto, we use Eckert's modified version (Samoylova-Yakahontova, 1945):

TABLE 24 /92

Ordinal No.	$\frac{1}{42}\delta M_0$	$\frac{1}{19}\delta i$	$\frac{1}{45}\delta\Omega$	$\frac{1}{43}\delta\omega$	$\frac{1}{32}\frac{\delta a}{a}$	$\frac{1}{80}\delta e$	$10^{-3}\rho\cos\delta\Delta\alpha$	v
1	+0.7778802	+0.2206026	+0.9882669	+0.9986981	+0.6577472	—0.7686285	—0.″13546	—1.″10
2	0.7883255	0.2004947	0.9851538	0.9990572	0.5972166	—0.7899598	—0.12885	—0.69
3	0.7808704	0.2024142	0.9836769	0.9946021	0.6269953	—0.7803821	—0.07141	+0.54
4	0.8099721	0.1227774	0.9638389	0.9835577	0.4295475	—0.8472111	—0.04135	1.78
5	0.8442248	0.0439589	0.9331233	0.9643116	0.1911684	—0.9124815	—0.07249	0.71
6	0.8488281	0.0331584	0.9274471	0.9601909	0.1556791	—0.9205951	—0.08032	0.36
7	0.8619557	+0.0030268	0.9085082	0.9458188	+0.0474372	—0.9421711	—0.04576	+0.67
8	0.8708871	—0.0030432	0.9031067	0.9437847	—0.0040934	—0.9519438	—0.08889	—0.73
9	0.8771236	—0.0099695	0.8966236	0.9396170	—0.0477319	—0.9591041	—0.05242	—0.04
10	0.8806119	—0.0171384	0.8898413	0.9340733	—0.0810519	—0.9635928	—0.03625	+0.11
11	0.8874352	—0.0220705	0.8831584	0.9299509	—0.1274666	—0.9700975	—0.07469	—1.30
12	0.8934539	—0.0265068	0.8761871	0.9252298	—0.1714494	—0.9754140	—0.05576	—1.16
13	0.9072179	—0.0312968	0.8620398	0.9162579	—0.2670644	—0.9852889	+0.02109	+0.07
14	0.9136512	—0.0324010	0.8547098	0.9113293	—0.3139550	—0.9889285	—0.00142	—1.05
15	0.9220460	—0.0309689	0.8472080	0.9070000	—0.3695131	—0.9929455	+0.00315	—1.51
16	0.9349531	—0.0243111	0.8385358	0.9034860	—0.4478191	—0.9980486	0.07616	—0.18
17	0.9330031	—0.0291858	0.8319447	0.8956928	—0.4582553	—0.9956838	0.10356	+0.25
18	0.9413007	—0.0240979	0.8241560	0.8909693	—0.5151438	—0.9972752	0.17927	1.78
19	0.9552276	—0.0125679	0.8081751	0.8799558	—0.6204340	—0.9967037	0.23121	1.91
20	0.9787081	+0.0181726	0.7827893	0.8628112	—0.7984638	—0.9893209	0.27291	0.62
21	0.9829676	0.0259953	0.7751584	0.8562758	—0.8409266	—0.9847834	0.29841	0.64
22	0.9876488	0.0355168	0.7672818	0.8496435	—0.8867194	—0.9794510	0.31712	+0.39
23	0.9972526	0.0542637	0.7580511	0.8436649	—0.9599712	—0.9727475	0.31598	—0.77
24	0.9998457	+0.0624284	+0.7508122	0.8366205	—0.9957572	—0.9660289	0.32569	—1.24
1*	0.2245824	—0.7650053	—0.0011751	0.2880086	+0.1878913	—0.2221402	0.06185	+1.97
2*	0.2147655	—0.7324068	—0.0158916	0.2735823	0.1712472	—0.2142925	+0.03593	+1.31
3*	0.2230750	—0.7127326	—0.0071887	0.2826016	0.1698525	—0.2239002	—0.14880	—2.84
4*	0.1969274	—0.5041463	—0.0505549	0.2395328	0.1067347	—0.2058359	—0.03659	—0.78
5*	0.1637350	—0.2259884	—0.0983156	0.1875877	0.0401462	—0.1769506	+0.11682	+2.19
6*	0.1593131	—0.1765174	—0.1043187	0.1803612	+0.0300203	—0.1727849	+0.04399	+0.30
7*	0.1466405	—0.0180026	—0.1205518	0.1592033	—0.0010359	—0.1599619	—0.02449	—1.79
8*	0.1358198	+0.0196068	—0.1315184	0.1465040	—0.0042844	—0.1483018	—0.00522	—1.48
9*	0.1286183	0.0683605	—0.1388562	0.1370495	—0.0108916	—0.1404291	+0.00519	—1.37
10*	0.1251645	0.1221784	—0.1424200	0.1311602	—0.0200122	—0.1364022	0.03769	—0.71
11*	0.1168440	0.1698632	—0.1501696	0.1209416	—0.0247094	—0.1271268	0.07311	+0.04
12*	0.1094757	0.2197847	—0.1566871	0.1117237	—0.0296931	—0.1187658	0.12119	1.12
13*	0.0915686	0.3155642	—0.1715356	0.0909740	—0.0349006	—0.0985931	0.15455	1.74
14*	0.0829240	0.3644489	—0.1781038	0.0811395	—0.0367941	—0.0887722	0.15252	1.58
15*	0.0708738	0.4099763	—0.1872351	0.0685658	—0.0344853	—0.0755405	0.15866	1.67
16*	0.0497564	0.4492379	—0.2031140	0.0489493	—0.0192484	—0.0537448	0.11725	+0.60
17*	0.0553429	0.5104216	—0.1968296	0.0513709	—0.0365100	—0.0576461	0.05376	—1.30
18*	0.0424969	0.5547121	—0.2054038	0.0387921	—0.0308819	—0.0437895	0.02742	—2.07
19*	+0.0200776	0.6476989	—0.2182842	+0.0171447	—0.0202888	—0.0196169	0.06288	—1.24
20*	—0.0224081	0.7790447	—0.2408062	—0.0201356	+0.0161984	+0.0231006	0.07803	—0.76
21*	—0.0296219	0.8262621	—0.2423187	—0.0269128	0.0192378	0.0310666	0.14917	+1.18
22*	—0.0383200	0.8721353	—0.2444053	—0.0345044	0.0258566	0.0400489	0.13865	0.89
23*	—0.0586669	0.9095826	—0.2545024	—0.0503395	0.0525125	0.0582131	0.11269	+0.38
24*	—0.0631426	0.9566726	—0.2527378	—0.0544693	0.0536253	0.0634308	0.08468	—0.45

TABLE 25 /93

$\frac{1}{42}\delta M_0$	$\frac{1}{43}\delta\omega$	$\frac{1}{32}\frac{\delta a}{a}$	$\frac{1}{80}\delta e$	$\frac{1}{19}\delta i$	$\frac{1}{45}\delta\Omega$	L
19.875888	20.198542	—5.342443	—20.826524	0.010771	18.441160	1.412200
+20.198542	+20.863055	—4.058482	—21.197821	+0.000390	+19.053132	+1.048404
— 5.342443	— 4.058482	+7.018984	+ 5.467512	—0.058600	— 3.677478	—1.949838
—20.826524	—21.197821	+5.467512	+21.861453	—0.001155	—19.357114	—1.393189
+ 0.010771	+ 0.000390	—0.058600	— 0.001155	+7.610174	— 0.988666	+0.826939
18.441160	19.053132	—3.677478	—19.357114	—0.988666	+18.931896	0.525360

$$
\left.\begin{array}{cccccc}
\delta M_0 & \delta i & \delta\Omega & \delta\omega & \frac{\delta a}{a} & \delta e \\
\delta x=\frac{\dot{x}}{n}, & N_y z-N_z y, & -z\sin\varepsilon-y\cos\varepsilon, & R_y z-R_z y, & x-\frac{3}{2}t\dot{x}, & Hx+K\frac{\dot{x}}{n}, \\
\delta y=\frac{\dot{y}}{n}, & N_z x-N_x z, & x\cos\varepsilon, & R_z x-R_x z, & y-\frac{3}{2}t\dot{y}, & Hy+K\frac{\dot{y}}{n}, \\
\delta z=\frac{\dot{z}}{n}, & N_x y-N_y x, & y\sin\varepsilon, & R_x y-R_y x, & z-\frac{3}{2}t\dot{z}, & Hz+K\frac{\dot{z}}{n},
\end{array}\right\} \quad (72)
$$

$$
\left\{\begin{array}{c}\rho\cos\delta\,\Delta\alpha \\ \rho\,\Delta\delta \\ \\ \end{array}\right\}=\left\{\begin{array}{c}\delta x \\ \delta y \\ \delta z\end{array}\right\}\left\{\begin{array}{cc}-\sin\alpha, & -\sin\delta\cos\alpha \\ \cos\alpha, & -\sin\delta\sin\alpha \\ 0 & \cos\delta\end{array}\right\},
$$

where

$$
\begin{array}{lll}
N_x=\cos\Omega, & R_x=\sin i\sin\Omega, & H=-(\cos E+e)\sec^2\varphi, \\
N_y=\sin\Omega\cos\varepsilon, & R_y=-\sin i\cos\Omega\cos\varepsilon-\cos i\sin\varepsilon, & K=\left(1+\frac{r}{p}\right)\sin E. \\
N_z=\sin\Omega\sin\varepsilon, & R_z=-\sin i\cos\Omega\sin\varepsilon+\cos i\cos\varepsilon, &
\end{array}
$$

Table 24 gives the system of conventional equations which participate in correction of the elements.

In the first column the number of the conventional equation is given, corresponding to the number of the normal position. The numbers without asterisks refer to equations for the right ascension; the numbers with asterisks to the declination. In the last column the values of the discrepancies expressed in seconds of arc are given.

The system of normal equations obtained from the given conditional equations is given in Table 25. Here the unknown values are arranged in order of their increasing weights.

The determinant of the system is very small because of the smallness of the arc traversed by Pluto. Therefore, the corrections are determined with an accuracy of only two or three significant places. Experiments conducted on replacing some variables by others, and a change in the order of excluding the unknown values in the given case, gave no appreciable improvements.

The systems of normal correction equations for the initial system of Plutonian elements obtained after solution, are:

$$
\begin{aligned}
\delta M_0 &= 111''.84 \pm 162''.76, \\
\delta\omega &= -\ 67''.81 \pm 148''.60, \\
\delta\Omega &= -\ 5''.07 \pm 0''.96, \\
\delta i &= 4''.04 \pm 1''.02,
\end{aligned}
$$

$$\delta e = 0.00007590 \pm 0.00002096,$$
$$\delta a = 0.003106 \pm 0.009592.$$

TABLE 26 /94

Ordinal No.	Universal Time		Before Correction		After Correction	
			$\Delta\alpha$	$\Delta\delta$	$\Delta\alpha$	$\Delta\delta$
			″	″	″	″
1	1914 Jan.	23.7904	−0.99	+0.81	−1.11	+1.95
2	Nov.	12.3260	−3.08	+0.82	−0.70	+1.30
3	1915 April	7.1674	−1.17	−4.46	+0.53	−2.87
4	1919 Dec.	29.4285	+0.80	−0.11	1.74	−0.79
5	1925 Dec.	23.2823	−1.21	+3.06	0.67	+2.19
6	1927 Jan.	6.2500	−4.58	−0.88	0.33	+0.38
7	1930 April	24.1151	−1.20	−0.59	+0.66	−1.79
8	1931 Feb.	2.0000	−0.32	+0.18	−0.73	−1.48
9	1932 Feb.	4.0000	−0.99	0.13	−0.05	−1.37
10	1933 March	15.0000	−0.15	0.87	+0.09	−0.71
11	1934 March	8.0000	−1.47	1.84	−1.31	+0.03
12	1935 March	18.0000	−1.11	2.54	−1.21	1.11
13	1937 March	7.0000	+0.95	4.14	+0.05	1.74
14	1938 March	11.0000	−0.12	3.52	−1.05	1.58
15	1939 Feb.	18.0000	+0.95	4.24	−1.54	1.67
16	Dec.	11.0000	2.57	3.00	−0.23	+0.59
17	1941 March	27.0000	3.56	1.27	+0.22	−1.30
18	1942 March	3.0000	5.80	0.30	1.72	−2.07
19	1944 Feb.	28.0000	7.36	1.68	1.91	−1.24
20	1947 Jan.	22.0000	8.80	1.98	0.59	−0.77
21	1948 Feb.	18.0000	9.38	3.83	0.61	+1.17
22	1949 March	12.0000	9.41	3.86	+0.32	0.90
23	1950 Feb.	2.0000	9.90	3.01	−0.77	+0.38
24	1951 March	22.0000	10.61	2.34	−0.89	−0.46

Table 26 presents the normal positions by theory before and after correction of the elements.

Below are given the new heliocentric elements of Pluto:

Epoch and Osculation 1930 September 20.0

$$M_0 = 275^\circ 12' 1''.14$$
$$\left.\begin{aligned} \omega &= 112\ 42\ 42.75 \\ \Omega &= 109\ 37\ 51.89 \\ i &= 17\ 6\ 37.48 \end{aligned}\right\} \text{ecliptic and equinox 1950.0}$$
$$e = 0.247\ 07348$$
$$a = 39.659\ 128$$
$$n = 14''.206\ 698$$

APPENDIX

I. SYMBOLIC EXPANSION OF PERTURBATION FUNCTION

1. Terms of Zero Class

$$
\begin{aligned}
&\frac{1}{2}\left(P_0^0 + e^2P_0^2 + e'^2P_{0,0}^{0,2} + e'^4P_{0,0}^{0,4} + e'^6P_{0,0}^{0,6} + e^2e'^2P_{0,0}^{2,2} + \ldots\right)\cos V_i \\
&+\left(ee'P_{-1,1}^{1,1} + ee'^3P_{-1,1}^{1,3} + ee'^5P_{-1,1}^{1,5} + e^3e'P_{-1,1}^{3,1} + \ldots\right)\cos(V_i - M + M') \\
&+\left(e^2e'^2P_{-2,2}^{2,2} + \ldots\right)\cos(V_i - 2M + 2M') \\
&+\ldots \\
&+\left(e^2e'P_{2,-1}^{2,1} + e^2e'^3P_{2,-1}^{2,3} + \ldots\right)\cos(V_i + 2M - M') \\
&+\left(eP_1^1 + e^3P_1^3 + ee'^2P_{1,0}^{1,2} + ee'^4P_{1,0}^{1,4} + \ldots\right)\cos(V_i + M) \\
&+\left(e'P_{0,1}^{0,1} + e'^3P_{0,1}^{0,3} + e'^5P_{0,1}^{0,5} + e'^7P_{0,1}^{0,7} + e^2e'P_{0,1}^{2,1} + e^2e'^3P_{0,1}^{2,3} + \ldots\right)\cos(V_i + M') \\
&+\left(ee'^2P_{-1,2}^{1,2} + ee'^4P_{-1,2}^{1,4} + \ldots\right)\cos(V_i - M + 2M') \\
&+\left(e^2e'^3P_{-2,3}^{2,3} + \ldots\right)\cos(V_i - 2M + 3M') \\
&+\ldots \\
&+\left(e^3e'P_{3,-1}^{3,1} + \ldots\right)\cos(V_i + 3M - M') \\
&+\left(e^2P_2^2 + e^2e'^2P_{2,0}^{2,2} + \ldots\right)\cos(V_i + 2M) \\
&+\left(ee'P_{1,1}^{1,1} + ee'^3P_{1,1}^{1,3} + ee'^5P_{1,1}^{1,5} + e^3e'P_{1,1}^{3,1} + \ldots\right)\cos(V_i + M + M') \\
&+\left(e'^2P_{0,2}^{0,2} + e'^4P_{0,2}^{0,4} + e'^6P_{0,2}^{0,6} + e^2e'^2P_{0,2}^{2,2} + \ldots\right)\cos(V_i + 2M') \\
&+\left(ee'^3P_{-1,3}^{1,3} + ee'^5P_{-1,3}^{1,5} + \ldots\right)\cos(V_i - M + 3M') \\
&+\ldots \\
&+\left(e^3P_3^3 + \ldots\right)\cos(V_i + 3M) \\
&+\left(e^2e'P_{2,1}^{2,1} + e^2e'^3P_{2,1}^{2,3} + \ldots\right)\cos(V_i + 2M + M') \\
&+\left(ee'^2P_{1,2}^{1,2} + ee'^4P_{1,2}^{1,4} + \ldots\right)\cos(V_i + M + 2M') \\
&+\left(e'^3P_{0,3}^{0,3} + e'^5P_{0,3}^{0,5} + e'^7P_{0,3}^{0,7} + e^2e'^3P_{0,3}^{2,3} + \ldots\right)\cos(V_i + 3M') \\
&+\left(ee'^4P_{-1,4}^{1,4} + \ldots\right)\cos(V_i - M + 4M') \\
&+\ldots \\
&+\left(e^3e'P_{3,1}^{3,1} + \ldots\right)\cos(V_i + 3M + M') \\
&+\left(ee'^3P_{1,3}^{1,3} + ee'^5P_{1,3}^{1,5} + \ldots\right)\cos(V_i + M + 3M') \\
&+\left(e^2e'^2P_{2,2}^{2,2} + \ldots\right)\cos(V_i + 2M + 2M') \\
&+\left(e'^4P_{0,4}^{0,4} + e'^6P_{0,4}^{0,6} + \ldots\right)\cos(V_i + 4M')
\end{aligned}
$$

$$+\left(ee'^5P^{1,5}_{-1,5}+\ \ldots\right)\cos\left(V_i-M+5M'\right)$$
$$+\ \ldots$$
$$+\left(e^2e'^3P^{2,3}_{2,3}+\ \ldots\right)\cos\left(V_i+2M+3M'\right)$$
$$+\left(ee'^4P^{1,4}_{1,4}+\ \ldots\right)\cos\left(V_i+M+4M'\right)$$
$$+\left(e'^5P^{0,5}_{0,5}+e'^7P^{0,7}_{0,5}+\ \ldots\right)\cos\left(V_i+5M'\right)$$
$$+\ \ldots$$
$$+\left(ee'^5P^{1,5}_{1,5}+\ \ldots\right)\cos\left(V_i+M+5M'\right)$$
$$+\left(e'^6P^{0,6}_{0,6}+\ \ldots\right)\cos\left(V_i+6M'\right)$$
$$+\left(e'^7P^{0,7}_{0,7}+\ \ldots\right)\cos\left(V_i+7M'\right)$$
$$+\ \ldots$$

2. Terms of First Class /96

$$+\left(e^2e'^2P'^{2,2}_{2,-2}+\ \ldots\right)\cos\left(V'_i+2M-2M'\right)$$
$$+\left(ee'P'^{1,1}_{1,-1}+ee'^3P'^{1,3}_{1,-1}+e^3e'P'^{3,1}_{1,-1}+ee'^5P'^{1,5}_{1,-1}+\ \ldots\right)\cos\left(V'_i+M-M'\right)$$
$$+\left(P'^0_0+e^2P'^2_0+e'^2P'^{0,2}_{0,0}+e'^4P'^{0,4}_{0,0}+e'^6P'^{0,6}_{0,0}+e^2e'^2P'^{2,2}_{0,0}+\ \ldots\right)\cos V'_i$$
$$+\left(ee'P'^{1,1}_{-1,1}+ee'^3P'^{1,3}_{-1,1}+e^3e'P'^{3,1}_{-1,1}+ee'^5P'^{1,5}_{-1,1}+\ \ldots\right)\cos\left(V'_i-M+M'\right)$$
$$+\left(e^2e'^2P'^{2,2}_{-2,2}+\ \ldots\right)\cos\left(V'_i-2M+2M'\right)$$
$$+\ \ldots$$
$$+\left(e^2e'P'^{2,1}_{2,-1}+e^2e'^3P'^{2,3}_{2,-1}+\ \ldots\right)\cos\left(V'_i+2M-M'\right)$$
$$+\left(eP'^1_1+ee'^2P'^{1,2}_{1,0}+e^3P'^3_1+ee'^4P'^{1,4}_{1,0}+\ \ldots\right)\cos\left(V'_i+M\right)$$
$$+\left(e'P'^{0,1}_{0,1}+e^2e'P'^{2,1}_{0,1}+e'^3P'^{0,3}_{0,1}+e'^5P'^{0,5}_{0,1}+e'^7P'^{0,7}_{0,1}+e^2e'^3P'^{2,3}_{0,1}+\ \ldots\right)\cos\left(V'_i+M'\right)$$
$$+\left(ee'^2P'^{1,2}_{-1,2}+ee'^4P'^{1,4}_{-1,2}+\ \ldots\right)\cos\left(V'_i-M+2M'\right)$$
$$+\left(e^2e'^3P'^{2,3}_{-2,3}+\ \ldots\right)\cos\left(V'_i-2M+3M'\right)$$
$$+\ \ldots$$
$$+\left(e^2e'P'^{2,1}_{-2,1}+e^2e'^3P'^{2,3}_{-2,1}+\ \ldots\right)\cos\left(V'_i-2M+M'\right)$$ /97
$$+\left(eP'^1_{-1}+ee'^2P'^{1,2}_{-1,0}+e^3P'^3_{-1}+ee'^4P'^{1,4}_{-1,0}+\ \ldots\right)\cos\left(V'_i-M\right)$$
$$+\left(e'P'^{0,1}_{0,-1}+e^2e'P'^{2,1}_{0,-1}+e'^3P'^{0,3}_{0,-1}+e'^5P'^{0,5}_{0,-1}+e'^7P'^{0,7}_{0,-1}+e^2e'^3P'^{2,3}_{0,-1}+\ \ldots\right)\cos\left(V'_i-M'\right)$$
$$+\left(ee'^2P'^{1,2}_{1,-2}+ee'^4P'^{1,4}_{1,-2}+\ \ldots\right)\cos\left(V'_i+M-2M'\right)$$

$$+\left(e^2e'^3P'^{2,3}_{2,-3}+\ldots\right)\cos\left(V_i'+2M-3M'\right)$$
$$+\ldots$$
$$+\left(e^3e'P'^{3,1}_{3,-1}+\ldots\right)\cos\left(V_i'+3M-M'\right)$$
$$+\left(e^2P'^2_2+e^2e'^2P'^{2,2}_{2,0}+\ldots\right)\cos\left(V_i'+2M\right)$$
$$+\left(ee'P'^{1,1}_{1,1}+ee'^3P'^{1,3}_{1,1}+e^3e'P'^{3,1}_{1,1}+ee'^5P'^{1,5}_{1,1}+\ldots\right)\cos\left(V_i'+M+M'\right)$$
$$+\left(e'^2P'^{0,2}_{0,2}+e'^4P'^{0,4}_{0,2}+e'^6P'^{0,6}_{0,2}+e^2e'^2P'^{2,2}_{0,2}+\ldots\right)\cos\left(V_i'+2M'\right)$$
$$+\left(ee'^3P'^{1,3}_{-1,3}+ee'^5P'^{1,5}_{-1,3}+\ldots\right)\cos\left(V_i'-M+3M'\right)$$
$$+\ldots$$
$$+\left(e^3e'P'^{3,1}_{-3,1}+\ldots\right)\cos\left(V_i'-3M+M'\right)$$
$$+\left(e^2P'^2_{-2}+e^2e'^2P'^{2,2}_{-2,0}+\ldots\right)\cos\left(V_i'-2M\right)$$
$$+\left(ee'P'^{1,1}_{-1,-1}+ee'^3P'^{1,3}_{-1,-1}+e^3e'P'^{3,1}_{-1,-1}+ee'^5P'^{1,5}_{-1,-1}+\ldots\right)\cos\left(V_i'-M-M'\right)$$
$$+\left(e'^2P'^{0,2}_{0,-2}+e'^4P'^{0,4}_{0,-2}+e'^6P'^{0,6}_{0,-2}+e^2e'^2P'^{2,2}_{0,-2}+\ldots\right)\cos\left(V_i'-2M'\right)$$
$$+\left(ee'^3P'^{1,3}_{1,-3}+ee'^5P'^{1,5}_{1,-3}+\ldots\right)\cos\left(V_i'+M-3M'\right)$$
$$+\ldots$$
$$+\left(e^3P'^3_3+\ldots\right)\cos\left(V_i'+3M\right)$$
$$+\left(e^2e'P'^{2,1}_{2,1}+e^2e'^3P'^{2,3}_{2,1}+\ldots\right)\cos\left(V_i'+2M+M'\right)$$
$$+\left(ee'^2P'^{1,2}_{1,2}+ee'^4P'^{1,4}_{1,2}+\ldots\right)\cos\left(V_i'+M+2M'\right)$$
$$+\left(e'^3P'^{0,3}_{0,3}+e'^5P'^{0,5}_{0,3}+e'^7P'^{0,7}_{0,3}+e^2e'^3P'^{2,3}_{0,3}+\ldots\right)\cos\left(V_i'+3M'\right)$$
$$+\left(ee'^4P'^{1,4}_{-1,4}+\ldots\right)\cos\left(V_i'-M+4M'\right)$$
$$+\ldots$$
$$+\left(e^3P'^3_{-3}+\ldots\right)\cos\left(V_i'-3M\right)$$
$$+\left(e^2e'P'^{2,1}_{-2,-1}+e^2e'^3P'^{2,3}_{-2,-1}+\ldots\right)\cos\left(V_i'-2M-M'\right)$$
$$+\left(ee'^2P'^{1,2}_{-1,-2}+ee'^4P'^{1,4}_{-1,-2}+\ldots\right)\cos\left(V_i'-M-2M'\right)$$ /98
$$+\left(e'^3P'^{0,3}_{0,-3}+e'^5P'^{0,5}_{0,-3}+e'^7P'^{0,7}_{0,-3}+e^2e'^3P'^{2,3}_{0,-3}+\ldots\right)\cos\left(V_i'-3M'\right)$$
$$+\left(ee'^4P'^{1,4}_{1,-4}+\ldots\right)\cos\left(V_i'+M-4M'\right)$$
$$+\ldots$$

$$+\left(e^3e'P'^{3,1}_{3,1}+\ldots\right)\cos\left(V_i'+3M+M'\right)$$
$$+\left(e^2e'^2P'^{2,2}_{2,2}+\ldots\right)\cos\left(V_i'+2M+2M'\right)$$
$$+\left(ee'^3P'^{1,3}_{1,3}+ee'^5P'^{1,5}_{1,3}+\ldots\right)\cos\left(V_i'+M+3M'\right)$$
$$+\left(e'^4P'^{0,4}_{0,4}+e'^6P'^{0,6}_{0,4}+\ldots\right)\cos\left(V_i'+4M'\right)$$
$$+\left(ee'^5P'^{1,5}_{-1,5}+\ldots\right)\cos\left(V_i'-M+5M'\right)$$
$$+\ldots$$
$$+\left(e^3e'P'^{3,1}_{-3,-1}+\ldots\right)\cos\left(V_i'-3M-M'\right)$$
$$+\left(e^2e'^2P'^{2,2}_{-2,-2}+\ldots\right)\cos\left(V_i'-2M-2M'\right)$$
$$+\left(ee'^3P'^{1,3}_{-1,-3}+ee'^5P'^{1,5}_{-1,-3}+\ldots\right)\cos\left(V_i'-M-3M'\right)$$
$$+\left(e'^4P'^{0,4}_{0,-4}+e'^6P'^{0,6}_{0,-4}+\ldots\right)\cos\left(V_i'-4M'\right)$$
$$+\left(ee'^5P'^{1,5}_{1,-5}+\ldots\right)\cos\left(V_i'+M-5M'\right)$$
$$+\ldots$$
$$+\left(e^2e'^3P'^{2,3}_{2,3}+\ldots\right)\cos\left(V_i'+2M+3M'\right)$$
$$+\left(ee'^4P'^{1,4}_{1,4}+\ldots\right)\cos\left(V_i'+M+4M'\right)$$
$$+\left(e'^5P'^{0,5}_{0,5}+e'^7P'^{0,7}_{0,5}+\ldots\right)\cos\left(V_i'+5M'\right)$$
$$+\ldots$$
$$+\left(e^2e'^3P'^{2,3}_{-2,-3}+\ldots\right)\cos\left(V_i'-2M-3M'\right)$$
$$+\left(ee'^4P'^{1,4}_{-1,-4}+\ldots\right)\cos\left(V_i'-M-4M'\right)$$
$$+\left(e'^5P'^{0,5}_{0,-5}+e'^7P'^{0,7}_{0,-5}+\ldots\right)\cos\left(V_i'-5M'\right)$$
$$+\ldots$$
$$+\left(ee'^5P'^{1,5}_{1,5}+\ldots\right)\cos\left(V_i'+M+5M'\right)$$
$$+\left(e'^6P'^{0,6}_{0,6}+\ldots\right)\cos\left(V_i'+6M'\right)$$
$$+\ldots$$ /99
$$+\left(ee'^5P'^{1,5}_{-1,-5}+\ldots\right)\cos\left(V_i'-M-5M'\right)$$
$$+\left(e'^6P'^{0,6}_{0,-6}+\ldots\right)\cos\left(V_i'-6M'\right)$$
$$+\ldots$$

$$+\left(e'^7 P'^{0,7}_{0,7}+\ldots\right)\cos\left(V'_i+7M'\right)$$
$$+\ldots$$
$$+\left(e'^7 P'^{0,7}_{0,-7}+\ldots\right)\cos\left(V'_i-7M'\right)$$
$$+\ldots$$

3. Terms of Second Class

$$+\left(ee'P''^{1,1}_{1,-1}+\ldots\right)\cos\left(V''_i+M-M'\right)$$
$$+\left(P''^{0}_{0}+e'^2P''^{0,2}_{0,0}+\ldots\right)\cos V''_i$$
$$+\left(ee'P''^{1,1}_{-1,1}+\ldots\right)\cos\left(V''_i-M+M'\right)$$
$$+\ldots$$
$$+\left(eP''^{1}_{1}+\ldots\right)\cos\left(V''_i+M\right)$$
$$+\left(e'P''^{0,1}_{0,1}+e'^3P''^{0,3}_{0,1}+\ldots\right)\cos\left(V''_i+M'\right)$$
$$+\ldots$$
$$+\left(eP''^{1}_{-1}+\ldots\right)\cos\left(V''_i-M\right)$$
$$+\left(e'P''^{0,1}_{0,-1}+e'^3P''^{0,3}_{0,-1}+\ldots\right)\cos\left(V''_i-M'\right)$$
$$+\ldots$$
$$+\left(ee'P''^{1,1}_{1,1}+\ldots\right)\cos\left(V''_i+M+M'\right)$$
$$+\left(e'^2P''^{0,2}_{0,2}+\ldots\right)\cos\left(V''_i+2M'\right)$$
$$+\ldots$$
$$+\left(ee'P''^{1,1}_{-1,-1}+\ldots\right)\cos\left(V''_i-M-M'\right)$$
$$+\left(e'^2P''^{0,2}_{0,-2}+\ldots\right)\cos\left(V''_i-2M'\right)$$
$$+\ldots$$
$$+\left(e'^3P''^{0,3}_{0,3}+\ldots\right)\cos\left(V''_i+3M'\right)$$
$$+\ldots$$
$$+\left(e'^3P''^{0,3}_{0,-3}+\ldots\right)\cos\left(V''_i-3M'\right)$$
$$+\ldots$$

II. TABLES OF COEFFICIENTS OF NEWCOMB'S OPERATORS /100

1. Operators of Zero Class

i	Π_0^0	Π_0^2			$\Pi_{0,0}^{0,2}$		
			D	D^2		D	D^2
−7	+1.000000	−49.000000	+0.250000	+0.250000	−49.000000	+0.250000	+0.250000
−6	1.000000	−36.000000	0.250000	0.250000	−36.000000	0.250000	0.250000
−5	1.000000	−25.000000	0.250000	0.250000	−25.000000	0.250000	0.250000
−4	1.000000	−16.000000	0.250000	0.250000	−16.000000	0.250000	0.250000
−3	1.000000	− 9.000000	0.250000	0.250000	− 9.000000	0.250000	0.250000
−2	1.000000	− 4.000000	0.250000	0.250000	− 4.000000	0.250000	0.250000
−1	1.000000	− 1.000000	0.250000	0.250000	− 1.000000	0.250000	0.250000
0	1.000000	0.000000	0.250000	0.250000	0.000000	0.250000	0.250000
1	1.000000	− 1.000000	0.250000	0.250000	− 1.000000	0.250000	0.250000
2	1.000000	− 4.000000	0.250000	0.250000	− 4.000000	0.250000	0.250000
3	1.000000	− 9.000000	0.250000	0.250000	− 9.000000	0.250000	0.250000
4	1.000000	−16.000000	0.250000	0.250000	−16.000000	0.250000	0.250000
5	1.000000	−25.000000	0.250000	0.250000	−25.000000	0.250000	0.250000
6	1.000000	−36.000000	0.250000	0.250000	−36.000000	0.250000	0.250000
7	1.000000	−49.000000	0.250000	0.250000	−49.000000	0.250000	0.250000

i	$\Pi_{0,0}^{0,4}$				
		D	D^2	D^3	D^4
−7	+587.234375	−12.156250	−5.953125	+0.093750	+0.015625
−6	314.437500	− 8.906250	−4.328125	0.093750	0.015625
−5	149.609375	− 6.156250	−2.953125	0.093750	0.015625
−4	59.750000	− 3.906250	−1.828125	0.093750	0.015625
−3	17.859375	− 2.156250	−0.953125	0.093750	0.015625
−2	+ 2.937500	− 0.906250	−0.328125	0.093750	0.015625
−1	− 0.015625	− 0.156250	+0.046875	0.093750	0.015625
0	0.000000	+ 0.093750	0.171875	0.093750	0.015625
1	− 0.015625	− 0.156250	+0.046875	0.093750	0.015625
2	+ 2.937500	− 0.906250	−0.328125	0.093750	0.015625
3	17.859375	− 2.156250	−0.953125	0.093750	0.015625
4	59.750000	− 3.906250	−1.828125	0.093750	0.015625
5	149.609375	− 6.156250	−2.953125	0.093750	0.015625
6	314.437500	− 8.906250	−4.328125	0.093750	0.015625
7	587.234375	−12.156250	−5.953125	0.093750	0.015625

i	$\Pi_{0,0}^{0,6}$						
		D	D^2	D^3	D^4	D^5	D^6
−7	−2971.475694	+141.246094	+43.568142	−1.944010	−0.218316	+0.006510	+0.000434
−6	−1137.437500	74.536458	22.290799	−1.402344	−0.150608	0.006510	0.000434
−5	− 358.767361	34.589844	9.786892	−0.944010	−0.093316	0.006510	0.000434
−4	− 83.861112	13.156250	3.306424	−0.569010	−0.046441	0.006510	0.000434
−3	− 11.406250	3.485677	+ 0.599392	−0.277344	−0.009983	0.006510	0.000434
−2	− 0.381944	+ 0.328125	− 0.084201	−0.069010	+0.016059	0.006510	0.000434
−1	− 0.059028	− 0.066406	+ 0.005642	+0.055990	0.031684	0.006510	0.000434
0	0.000000	+ 0.052083	0.118924	0.097656	0.036892	0.006510	0.000434
1	− 0.059028	− 0.066406	+ 0.005642	+0.055990	0.031684	0.006510	0.000434
2	− 0.381944	+ 0.328125	− 0.084201	−0.069010	+0.016059	0.006510	0.000434
3	− 11.406250	3.485677	+ 0.599392	−0.277344	−0.009983	0.006510	0.000434
4	− 83.861112	13.156250	3.306424	−0.569010	−0.046441	0.006510	0.000434
5	− 358.767361	34.589844	9.786892	−0.944010	−0.093316	0.006510	0.000434
6	−1137.437500	74.536458	22.290799	−1.402344	−0.150608	0.006510	0.000434
7	−2971.475694	141.246094	43.568142	−1.944010	−0.218316	0.006510	0.000434

i	$\Pi_{0,0}^{2,2}$	D	D^2	D^3	D^4	$\Pi_{-1,1}^{1,1}$	D	D^2
—7	+2401.000000	—24.500000	—24.437500	+0.125000	+0.062500	+45.500000	—0.250000	—0.250000
—6	1296.000000	—18.000000	—17.937500	0.125000	0.062500	33.000000	—0.250000	—0.250000
—5	625.000000	—12.500000	—12.437500	0.125000	0.062500	22.500000	—0.250000	—0.250000
—4	256.000000	— 8.000000	— 7.937500	0.125000	0.062500	14.000000	—0.250000	—0.250000
—3	81.000000	— 4.500000	— 4.437500	0.125000	0.062500	7.500000	—0.250000	—0.250000
—2	16.000000	— 2.000000	— 1.937500	0.125000	0.062500	3.000000	—0.250000	—0.250000
—1	1.000000	— 0.500000	— 0.437500	0.125000	0.062500	0.500000	—0.250000	—0.250000
0	0.000000	0.000000	+ 0.062500	0.125000	0.062500	0.000000	—0.250000	—0.250000
1	1.000000	— 0.500000	— 0.437500	0.125000	0.062500	1.500000	—0.250000	—0.250000
2	16.000000	— 2.000000	— 1.937500	0.125000	0.062500	5.000000	—0.250000	—0.250000
3	81.000000	— 4.500000	— 4.437500	0.125000	0.062500	10.500000	—0.250000	—0.250000
4	256.000000	— 8.000000	— 7.937500	0.125000	0.062500	18.000000	—0.250000	—0.250000
5	625.000000	—12.500000	—12.437500	0.125000	0.062500	27.500000	—0.250000	—0.250000
6	1296.000000	—18.000000	—17.937500	0.125000	0.062500	39.000000	—0.250000	—0.250000
7	2401.000000	—24.500000	—24.437500	0.125000	0.062500	52.500000	—0.250000	—0.250000

i	$\Pi_{-1,1}^{1,3}$	D	D^2	D^3	D^4
—7	— 915.250000	+16.437500	+10.218750	—0.125000	—0.031250
—6	— 469.875000	11.843750	7.250000	—0.125000	—0.031250
—5	— 210.625000	8.000000	4.781250	—0.125000	—0.031250
—4	— 76.750000	4.906250	2.812500	—0.125000	—0.031250
—3	— 19.500000	2.562500	1.343750	—0.125000	—0.031250
—2	— 2.125000	0.968750	+ 0.375000	—0.125000	—0.031250
—1	+ 0.125000	0.125000	— 0.093750	—0.125000	—0.031250
0	0.000000	0.031250	— 0.062500	—0.125000	—0.031250
1	— 1.750000	0.687500	+ 0.468750	—0.125000	—0.031250
2	— 16.375000	2.093750	1.500000	—0.125000	—0.031250
3	— 67.125000	4.250000	3.031250	—0.125000	—0.031250
4	— 189.250000	7.156250	5.062500	—0.125000	—0.031250
5	— 430.000000	10.812500	7.593750	—0.125000	—0.031250
6	— 848.625000	15.218750	10.625000	—0.125000	—0.031250
7	—1516.375000	20.375000	14.156250	—0.125000	—0.031250

i	$\Pi_{-1,1}^{1,5}$	D	D^2	D^3	D^4	D^5	D^6
—7	+ 5857.906250	—256.166667	— 96.364588	+3.759115	+0.569007	—0.014323	—0.001302
—6	2085.921900	—129.751305	— 46.938803	2.649742	0.386718	—0.014323	—0.001302
—5	591.979175	— 56.817710	— 19.192708	1.727866	0.235678	—0.014323	—0.001302
—4	116.156250	— 19.740886	— 5.751304	0.993490	0.115888	—0.014323	—0.001302
—3	+ 10.656250	— 4.395833	— 0.739584	0.446613	+0.027343	—0.014323	—0.001302
—2	— 0.192708	— 0.157553	+ 0.217448	+0.087240	—0.029948	—0.014323	—0.001302
—1	+ 0.062500	+ 0.098958	— 0.005208	—0.084635	—0.055989	—0.014323	—0.001302
0	0.000000	— 0.001302	— 0.032552	—0.069010	—0.050781	—0.014323	—0.001302
1	0.322917	— 0.333333	+ 0.010416	+0.134114	—0.014323	—0.014323	—0.001302
2	13.859376	— 4.272136	— 1.501303	0.524740	+0.053384	—0.014323	—0.001302
3	125.562500	— 18.692708	— 7.692708	1.102866	0.152344	—0.014323	—0.001302
4	612.510424	— 53.970053	— 23.188803	1.868489	0.282552	—0.014323	—0.001302
5	2123.906275	—123.979165	— 54.114586	2.821614	0.444008	—0.014323	—0.001302
6	5921.078125	—246.095055	—108.095052	3.962242	0.636720	—0.014323	—0.001302
7	14177.479167	—441.192713	—194.255210	5.290364	0.860676	—0.014323	—0.001302

/102

i	$\Pi_{-1,1}^{3,1}$	D	D^2	D^3	D^4
—7	— 921.375000	+4.250000	+10.406250	0.000000	—0.031250
—6	— 474.375000	2.906250	7.437500	0.000000	—0.031250
—5	— 213.750000	1.812500	4.968750	0.000000	—0.031250
—4	— 78.750000	0.968750	3.000000	0.000000	—0.031250
—3	— 20.625000	0.375000	1.531250	0.000000	—0.031250
—2	— 2.625000	+0.031250	0.562500	0.000000	—0.031250
—1	0.000000	—0.062500	0.093750	0.000000	—0.031250
0	0.000000	+0.093750	0.125000	0.000000	—0.031250
1	— 1.875000	0.500000	0.656250	0.000000	—0.031250
2	— 16.875000	1.156250	1.687500	0.000000	—0.031250
3	— 68.250000	2.062500	3.218750	0.000000	—0.031250
4	— 191.250000	3.218750	5.250000	0.000000	—0.031250
5	— 433.125000	4.625000	7.781250	0.000000	—0.031250
6	— 853.125000	6.281250	10.812500	0.000000	—0.031250
7	—1522.500000	8.187500	14.343750	0.000000	—0.031250

i	$\Pi_{-2,2}^{2,2}$	D	D^2	D^3	D^4
—7	+344.640625	—4.343750	—4.328125	+0.031250	+0.015625
—6	167.437500	—3.000000	—2.984375	0.031250	0.015625
—5	69.140625	—1.906250	—1.890625	0.031250	0.015625
—4	22.000000	—1.062500	—1.046875	0.031250	0.015625
—3	4.265625	—0.468750	—0.453125	0.031250	0.015625
—2	0.187500	—0.125000	—0.109375	0.031250	0.015625
—1	0.015625	—0.031250	—0.015625	0.031250	0.015625
0	0.000000	—0.187500	—0.171875	0.031250	0.015625
1	2.390625	—0.593750	—0.578125	0.031250	0.015625
2	15.437500	—1.250000	—1.234375	0.031250	0.015625
3	53.390625	—2.156250	—2.140625	0.031250	0.015625
4	136.500000	—3.312500	—3.296875	0.031250	0.015625
5	291.015625	—4.718750	—4.703125	0.031250	0.015625
6	549.187500	—6.375000	—6.359375	0.031250	0.015625
7	949.265625	—8.281250	—8.265625	0.031250	0.015625

i	$\Pi_{2,-1}^{2,1}$	D	D^2	D^3
—7	+216.562500	—14.625000	—1.000000	+0.062500
—6	141.375000	—11.062500	—0.875000	0.062500
—5	85.937500	— 8.000000	—0.750000	0.062500
—4	47.250000	— 5.437500	—0.625000	0.062500
—3	22.312500	— 3.375000	—0.500000	0.062500
—2	8.125000	— 1.812500	—0.375000	0.062500
—1	1.687500	— 0.750000	—0.250000	0.062500
0	0.000000	— 0.187500	—0.125000	0.062500
1	+ 0.062500	— 0.125000	0.000000	0.062500
2	— 1.125000	— 0.562500	+0.125000	0.062500
3	— 6.562500	— 1.500000	0.250000	0.062500
4	— 19.250000	— 2.937500	0.375000	0.062500
5	— 42.187500	— 4.875000	0.500000	0.062500
6	— 78.375000	— 7.312500	0.625000	0.062500
7	—130.812500	—10.250000	0.750000	0.062500

i	$\Pi^{2,3}_{2,i-1}$	D	D^2	D^3	D^4	D^5
—7	—6255.046875	+476.679688	+54.085938	—4.125000	—0.101562	+0.007812
—6	—3076.265625	276.164062	35.289062	—3.171875	—0.085938	0.007812
—5	—1343.750000	146.664062	21.429688	—2.343750	—0.070312	0.007812
—4	— 496.781250	69.054688	11.757812	—1.640625	—0.054688	0.007812
—3	— 142.640625	27.210938	5.523438	—1.062500	—0.039062	0.007812
—2	— 26.609375	8.007812	1.976562	—0.609375	—0.023438	0.007812
—1	— 1.968750	1.320312	+ 0.367188	—0.281250	—0.007812	0.007812
0	0.000000	+ 0.023438	— 0.054688	—0.078125	+0.007812	0.007812
1	+ 0.015625	— 0.007812	— 0.039062	—0.000000	0.023438	0.007812
2	0.796875	+ 0.101562	— 0.335938	—0.046875	0.039062	0.007812
3	17.062500	2.226562	— 1.695312	—0.218750	0.054688	0.007812
4	105.531250	11.242188	— 4.867188	—0.515625	0.070312	0.007812
5	394.921875	35.023438	—10.601562	—0.937500	0.085938	0.007812
6	1115.953125	84.445312	—19.648438	—1.484375	0.101562	0.007812
7	2631.343750	173.382812	—32.757812	—2.156250	0.117188	0.007812

i	Π^1_1	D	Π^3_1	D	D^2	D^3
—7	—7.000000	—0.500000	—203.000000	+14.625000	+0.937500	—0.062500
—6	6.000000	—0.500000	—131.250000	11.062500	0.812500	—0.062500
—5	5.000000	—0.500000	— 78.750000	8.000000	0.687500	—0.062500
—4	4.000000	—0.500000	— 42.500000	5.437500	0.562500	—0.062500
—3	3.000000	—0.500000	— 19.500000	3.375000	0.437500	—0.062500
—2	2.000000	—0.500000	— 6.750000	1.812500	0.312500	—0.062500
—1	+1.000000	—0.500000	— 1.250000	0.750000	0.187500	—0.062500
0	0.000000	—0.500000	0.000000	0.187500	+0.062500	—0.062500
1	—1.000000	—0.500000	0.000000	0.125000	—0.062500	—0.062500
2	—2.000000	—0.500000	+ 1.750000	0.562500	—0.187500	—0.062500
3	—3.000000	—0.500000	8.250000	1.500000	—0.312500	—0.062500
4	—4.000000	—0.500000	22.500000	2.937500	—0.437500	—0.062500
5	—5.000000	—0.500000	47.500000	4.875000	—0.562500	—0.062500
6	—6.000000	—0.500000	86.250000	7.312500	—0.687500	—0.062500
7	—7.000000	—0.500000	141.750000	10.250000	—0.812500	—0.062500

i	$\Pi^{1,2}_{1,0}$	D	D^2	D^3
—7	—343.000000	+26.250000	+1.625000	—0.125000
—6	—216.000000	19.500000	1.375000	—0.125000
—5	—125.000000	13.750000	1.125000	—0.125000
—4	— 64.000000	9.000000	0.875000	—0.125000
—3	— 27.000000	5.250000	0.625000	—0.125000
—2	— 8.000000	2.500000	0.375000	—0.125000
—1	— 1.000000	0.750000	+0.125000	—0.125000
0	0.000000	0.000000	—0.125000	—0.125000
1	+ 1.000000	0.250000	—0.375000	—0.125000
2	8.000000	1.500000	—0.625000	—0.125000
3	27.000000	3.750000	—0.875000	—0.125000
4	64.000000	7.000000	—1.125000	—0.125000
5	125.000000	11.250000	—1.375000	—0.125000
6	216.000000	16.500000	—1.625000	—0.125000
7	343.000000	22.750000	—1.875000	—0.125000

i	$\Pi_{1,0}^{1,4}$					
		D	D^2	D^3	D^4	D^5
—7	+4110.640625	—378.710938	—35.593750	+3.632812	+0.062500	—0.007812
—6	1886.625000	—210.656250	—21.515625	2.726562	0.046875	—0.007812
—5	748.046875	—105.585938	—11.687500	1.945312	0.031250	—0.007812
—4	239.000000	— 45.500000	— 5.359375	1.289062	+0.015625	—0.007812
—3	53.578125	— 15.398438	— 1.781250	0.757812	0.000000	—0.007812
—2	+ 5.875000	— 3.281250	— 0.203125	0.351562	—0.015625	—0.007812
—1	— 0.015625	— 0.148438	+ 0.125000	+0.070312	—0.031250	—0.007812
0	0.000000	0.000000	— 0.046875	—0.085938	—0.046875	—0.007812
1	+ 0.015625	+ 0.164062	+ 0.031250	—0.117188	—0.062500	—0.007812
2	— 5.875000	+ 0.343750	1.109375	—0.023438	—0.078125	—0.007812
3	— 53.578125	— 2.460938	3.937500	+0.195312	—0.093750	—0.007812
4	— 239.000000	— 14.250000	9.265625	0.539062	—0.109375	—0.007812
5	— 748.046875	— 44.023438	17.843750	1.007812	—0.125000	—0.007812
6	—1886.625000	—103.781250	30.421875	1.601562	—0.140625	—0.007812
7	—4110.640625	—208.523438	47.750000	2.320312	—0.156250	—0.007812

i	$\Pi_{0,1}^{0,1}$		$\Pi_{0,1}^{0,3}$			
		D		D	D^2	D^3
—7	—6.500000	+0.500000	+130.750000	—11.687500	—0.625000	+0.062500
—6	—5.500000	0.500000	78.312500	— 8.500000	—0.500000	0.062500
—5	—4.500000	0.500000	42.125000	— 5.812500	—0.375000	0.062500
—4	—3.500000	0.500000	19.187500	— 3.625000	—0.250000	0.062500
—3	—2.500000	0.500000	6.500000	— 1.937500	—0.125000	0.062500
—2	—1.500000	0.500000	+ 1.062500	— 0.750000	0.000000	0.062500
—1	—0.500000	0.500000	— 0.125000	— 0.062500	+0.125000	0.062500
0	+0.500000	0.500000	— 0.062500	+ 0.125000	0.250000	0.062500
1	1.500000	0.500000	— 1.750000	— 0.187500	0.375000	0.062500
2	2.500000	0.500000	— 8.187500	— 1.000000	0.500000	0.062500
3	3.500000	0.500000	— 22.375000	— 2.312500	0.625000	0.062500
4	4.500000	0.500000	— 47.312500	— 4.125000	0.750000	0.062500
5	5.500000	0.500000	— 86.000000	— 6.437500	0.875000	0.062500
6	6.500000	0.500000	—141.437500	— 9.250000	1.000000	0.062500
7	7.500000	0.500000	—216.625000	—12.562500	1.125000	0.062500

i	$\Pi_{0,1}^{0,5}$					
		D	D^2	D^3	D^4	D^5
—7	— 836.843750	+96.369793	+ 6.882813	—1.028646	—0.007812	+0.002604
—6	— 347.653650	50.596355	3.606771	—0.742188	—0.002604	0.002604
—5	— 118.395835	23.203125	1.518229	—0.497396	+0.002604	0.002604
—4	— 29.039063	8.565104	+ 0.367188	—0.294271	0.007812	0.002604
—3	— 3.552083	2.057292	— 0.096354	—0.132812	0.013021	0.002604
—2	+ 0.096354	+ 0.054688	— 0.122396	—0.013021	0.018229	0.002604
—1	— 0.062500	— 0.067708	+ 0.039062	+0.065104	0.023437	0.002604
0	+ 0.002604	+ 0.065104	+ 0.138021	0.101562	0.028646	0.002604
1	0.322917	— 0.171875	— 0.075521	0.096354	0.033854	0.002604
2	6.929688	— 0.403646	— 0.851563	+0.049479	0.039062	0.002604
3	41.854167	+ 0.744792	— 2.440104	—0.039062	0.044271	0.002604
4	153.127604	5.648438	— 5.091146	—0.169271	0.049479	0.002604
5	424.781250	17.682292	— 9.054688	—0.341146	0.054687	0.002604
6	986.846354	41.221355	—14.580729	—0.554687	0.059896	0.002604
7	2025.354167	81.640626	—21.919271	—0.809896	0.065104	0.002604

$\Pi_{0,1}^{0,7}$

i		D	D^2	D^3	D^4	D^5	D^6	D^7
−7	+2457.750000	−383.273437	− 22.579970	+5.974989	−0.029839	−0.029731	+0.000434	+0.000054
−6	680.875922	−139.925347	− 6.398383	2.960015	−0.060493	−0.020128	0.000543	0.000054
−5	136.036458	− 40.028863	− 0.608073	1.203830	−0.064453	−0.011827	0.000651	0.000054
−4	14.927246	− 7.350586	+ 0.499946	+0.323622	−0.049533	−0.004829	0.000760	0.000054
−3	0.324653	− 0.349829	+ 0.197157	−0.000922	−0.023546	+0.000868	0.000868	0.000054
−2	+ 0.096191	+ 0.008898	− 0.057454	−0.027615	+0.005697	0.005263	0.000977	0.000054
−1	− 0.039062	− 0.056641	+ 0.007595	+0.048231	0.030382	0.008355	0.001085	0.000054
0	− 0.000054	+ 0.041124	+ 0.101291	0.093804	0.042697	0.010145	0.001194	0.000054
1	− 0.083333	− 0.052951	− 0.004883	+0.038791	+0.034831	0.010634	0.001302	0.000054
2	− 1.952637	+ 0.550781	+ 0.148058	−0.124620	−0.001031	0.009820	0.001411	0.000054
3	− 30.878472	3.674262	2.331597	−0.341743	−0.072700	0.007704	0.001519	0.000054
4	− 214.227702	9.259223	10.254720	−0.495388	−0.187988	+0.004286	0.001628	0.000054
5	− 951.203125	+ 10.054687	30.188910	−0.405870	−0.354709	−0.000434	0.001736	0.000054
6	−3208.833062	− 16.195529	71.593153	+0.169000	−0.580675	−0.006456	0.001845	0.000054
7	−8979.210938	−117.065321	147.738932	1.533908	−0.873698	−0.013780	0.001953	0.000054

$\Pi_{0,1}^{2,1}$

i		D	D^2	D^3
−7	+318.500000	26.125000	−1.500000	+0.125000
−6	198.000000	−19.375000	−1.250000	0.125000
−5	112.500000	−13.625000	−1.000000	0.125000
−4	56.000000	− 8.875000	−0.750000	0.125000
−3	22.500000	− 5.125000	−0.500000	0.125000
−2	6.000000	− 2.375000	−0.250000	0.125000
−1	+ 0.500000	− 0.625000	0.000000	0.125000
0	0.000000	+ 0.125000	+0.250000	0.125000
1	− 1.500000	− 0.125000	0.500000	0.125000
2	− 10.000000	− 1.375000	0.750000	0.125000
3	− 31.500000	− 3.625000	1.000000	0.125000
4	− 72.000000	− 6.875000	1.250000	0.125000
5	−137.500000	−11.125000	1.500000	0.125000
6	−234.000000	−16.375000	1.750000	0.125000
7	−367.500000	−22.625000	2.000000	0.125000

$\Pi_{0,1}^{2,3}$

i		D	D^2	D^3	D^4	D^5
−7	− 6406.750000	+605.375000	+60.390625	−6.140625	−0.140625	+0.015625
−6	− 2819.250000	325.578125	35.453125	−4.500000	−0.109375	0.015625
−5	− 1053.125000	155.843750	18.453125	−3.109375	−0.078125	0.015625
−4	− 307.000000	62.796875	7.890625	−1.968750	−0.046875	0.015625
−3	− 58.500000	19.062500	2.265625	−1.078125	−0.015625	0.015625
−2	− 4.250000	3.265625	+ 0.078125	−0.437500	+0.015625	0.015625
−1	+ 0.125000	+ 0.031250	− 0.171875	−0.046875	0.046875	0.015625
0	0.000000	− 0.015625	+ 0.015625	+0.093750	0.078125	0.015625
1	+ 1.750000	− 0.250000	− 0.859375	−0.015625	0.109375	0.015625
2	32.750000	+ 1.953125	− 4.296875	−0.375000	0.140625	0.015625
3	201.375000	15.218750	− 11.796875	− 0.984375	0.171875	0.015625
4	757.000000	54.171875	− 24.859375	−1.843750	0.203125	0.015625
5	2150.000000	139.437500	− 44.984375	−2.953125	0.234375	0.015625
6	5091.750000	297.640625	− 73.671875	−4.312500	0.265625	0.015625
7	10614.625000	561.406250	−112.421875	−5.921875	0.296875	0.015625

/106

i	$\Pi_{-1,2}^{1,2}$			
		D	D^2	D^3
—7	—119.875000	+11.562500	+0.562500	—0.062500
—6	— 70.500000	8.375000	0.437500	—0.062500
—5	— 36.875000	5.687500	0.312500	—0.062500
—4	— 16.000000	3.500000	0.187500	—0.062500
—3	— 4.875000	1.812500	+0.062500	—0.062500
—2	— 0.500000	+ 0.625000	—0.062500	—0.062500
—1	+ 0.125000	— 0.062500	—0.187500	—0.062500
0	0.000000	— 0.250000	—0.312500	—0.062500
1	2.125000	+ 0.062500	—0.437500	—0.062500
2	9.500000	0.875000	—0.562500	—0.062500
3	25.125000	2.187500	— 0.687500	—0.062500
4	52.000000	4.000000	—0.812500	—0.062500
5	93.125000	6.312500	—0.937500	—0.062500
6	151.500000	9.125000	—1.062500	—0.062500
7	230.125000	12.437500	—1.187500	—0.062500

i	$\Pi_{-1,2}^{1,4}$					
		D	D^2	D^3	D^4	D^5
—7	+1270.062500	—152.604175	—11.401042	+1.739583	+0.020833	—0.005208
—6	499.750000	— 77.104167	— 5.708333	1.223958	+0.010417	—0.005208
—5	156.562500	— 33.458333	— 2.203125	0.791667	0.000000	—0.005208
—4	33.000000	— 11.291667	— 0.385417	0.442708	—0.010417	—0.005208
—3	+ 2.562500	— 2.229167	+ 0,241792	+0.177083	—0.020833	—0.005208
—2	— 0.250000	+ 0.104167	+ 0.187500	—0.005208	—0.031250	—0.005208
—1	+ 0.062500	0.083333	— 0.057292	—0.104167	—0.041667	—0.005208
0	0.000000	0.083333	+ 0.010417	—0.119791	—0.052083	—0.005208
1	— 2.937500	+ 0.479167	0.890625	—0.052083	—0.062500	—0.005208
2	— 30.250000	— 0.354167	3.083333	+0.098958	—0.072917	—0.005208
3	— 142.437500	— 6.041667	7.088542	0.333333	—0.083333	—0.005208
4	— 459.000000	— 22.208333	13.406250	0.651042	—0.093750	—0.005208
5	—1178.437500	— 56.479167	22.536458	1.052083	—0.104167	—0.005208
6	—2598.250000	—118.479167	34.979166	1.536458	—0.114583	—0.005208
7	—5134.937500	—219.833333	51.234375	2.104167	—0.125000	—0.005208

i	$\Pi_{-2,3}^{2,3}$					
		D	D^2	D^3	D^4	D^5
—7	— 467.067708	+ 57.867195	+ 4.768230	—0.723965	—0.013021	+0.002604
—6	— 171.890625	27.710942	2.304688	—0.494796	—0.007812	0.002604
—5	— 48.437500	11.091148	0.841146	—0.307295	—0.002604	0.002604
—4	— 8.364582	3.257815	+ 0.127605	—0.161460	+0.002604	0.002604
—3	— 0.328125	+ 0.460938	— 0.085937	—0.057293	0.007812	0.002604
—2	+ 0.015625	— 0.049479	— 0.049479	+0.005208	0.013021	0.002604
—1	0.010417	— 0.023438	— 0.013021	0.026042	0.018229	0.002604
0	0.000000	— 0.210938	— 0.226 63	+0.005208	0.023437	0.002604
1	3.328125	— 0.361978	— 0.940105	—0.057292	0.028646	0.002604
2	26.338542	+ 0.773438	— 2.403646	—0.161459	0.033854	0.002604
3	108.375000	5.445315	— 4.867188	—0.307294	0.039062	0.002604
4	321.781250	16.903650	— 8.580730	—0.494795	0.044271	0.002604
5	781.901042	39.398442	—13.794272	—0.723964	0.049479	0.002604
6	1657.078125	78.179695	—20.757814	—0.994799	0.054687	0.002604
7	3178.656250	139.497418	—29.721356	—1.307301	0.059896	0.002604

i	$\Pi^{3,\ 1}_{3,\ -1}$	D	D^2	D^3	D^4
—7	+686.875000	—84.833333	—0.739583	+0.375000	—0.010417
—6	401.375000	—56.739583	—0.645833	0.333333	—0.010417
—5	215.416667	—35.645833	—0.552083	0.291667	—0.010417
—4	102.750000	—20.552083	—0.458333	0.250000	—0.010417
—3	41.125000	—10.458333	—0.364583	0.208333	—0.010417
—2	12.291667	— 4.364583	—0.270833	0.166667	—0.010417
—1	2.000000	— 1.270833	—0.177083	0.125000	—0.010417
0	0.000000	— 0.177083	—0.083333	0.083333	—0.010417
1	+ 0.041667	— 0.083333	+0.010417	+0.041667	—0.010417
2	— 0.125000	+ 0.010417	0.104167	0.000000	—0.010417
3	+ 1.250000	1.104167	0.197917	—0.041667	—0.010417
4	9.916667	4.197917	0.291667	—0.083333	—0.010417
5	35.625000	10.291667	0.385417	—0.125000	—0.010417
6	92.125000	20.385417	0.479167	—0.166667	—0.010417
7	197.166667	35.479167	0.572917	—0.208333	—0.010417

i	Π^2_2	D	D^2	$\Pi^{2,\ 2}_{2,\ 0}$	D	D^2	D^3	D^4
—7	+28.875000	—3.875000	+0.125000	—1414.875000	+197.093750	+0.125000	—0.937500	+0.031250
—6	21.750000	—3.375000	0.125000	— 783.000000	126.937500	0.093750	—0.812500	0.031250
—5	15.625000	—2.875000	0.125000	— 390.625000	75.781250	0.062500	—0.687500	0.031250
—4	10.500000	—2.375000	0.125000	— 168.000000	40.625000	+0.031250	—0.562500	0.031250
—3	6.375000	—1.875000	0.125000	— 57.375000	18.468750	0.000000	—0.437500	0.031250
—2	3.250000	—1.375000	0.125000	— 13.000000	6.312500	—0.031250	—0.312500	0.031250
—1	+ 1.125000	—0.875000	0.125000	— 1.125000	+ 1.156250	—0.062500	—0.187500	0.031250
0	0.000000	—0.375000	0.125000	0.000000	0.000000	—0.093750	—0.062500	0.031250
1	— 0.125000	+0.125000	0.125000	+ 0.125000	— 0.156250	—0.125000	+0.062500	0.031250
2	+ 0.750000	0.625000	0.125000	— 3.000000	— 2.312500	—0.156250	0.187500	0.031250
3	2.625000	1.125000	0.125000	— 23.625000	— 9.468750	—0.187500	0.312500	0.031250
4	5.500000	1.625000	0.125000	— 88.000000	— 24.625000	—0.218750	0.437500	0.031250
5	9.375000	2.125000	0.125000	— 234.375000	— 50.781250	—0.250000	0.562500	0.031250
6	14.250000	2.625000	0.125000	— 513.000000	— 90.937500	—0.281250	0.687500	0.031250
7	20.125000	3.125000	0.125000	— 986.125000	—148.093750	—0.312500	0.812500	0.031250

i	$\Pi^{1,\ 1}_{1,\ 1}$	D	D^2	$\Pi^{1,\ 3}_{1,\ 1}$	D	D^2	D^3	D^4
—7	—45.500000	+6.750000	—0.250000	+ 915.250000	—147.187500	+1.468750	+0.750000	—0.031250
—6	—33.000000	5.750000	—0.250000	469.875000	— 90.156250	1.250000	0.625000	—0.031250
—5	—22.500000	4.750000	—0.250000	210.625000	— 50.125000	1.031250	0.500000	—0.031250
—4	—14.000000	3.750000	—0.250000	76.750000	— 24.093750	0.812500	0.375000	—0.031250
—3	— 7.500000	2.750000	—0.250000	19.500000	— 9.062500	0.593750	0.250000	—0.031250
—2	— 3.000000	1.750000	—0.250000	+ 2.125000	— 2.031250	0.375000	+0.125000	—0.031250
—1	— 0.500000	+0.750000	—0.250000	— 0.125000	0.000000	+0.156250	0.000000	—0.031250
0	0.000000	—0.250000	—0.250000	0.000000	+ 0.031250	—0.062500	—0.125000	—0.031250
1	— 1.500000	—1.250000	—0.250000	+ 1.750000	1.062500	—0.281250	—0.250000	—0.031250
2	— 5.000000	—2.250000	—0.250000	16.375000	6.093750	—0.500000	—0.375000	—0.031250
3	—10.500000	—3.250000	—0.250000	67.125000	18.125000	—0.718750	—0.500000	—0.031250
4	—18.000000	—4.250000	—0.250000	189.250000	40.156250	—0.937500	—0.625000	—0.031250
5	—27.500000	—5.250000	—0.250000	430.000000	75.187500	—1.156250	—0.750000	—0.031250
6	—39.000000	—6.250000	—0.250000	848.625000	126.218750	—1.375000	—0.875000	—0.031250
7	—52.500000	—7.250000	—0.250000	1516.375000	196.250000	—1.593750	—1.000000	—0.031250

/108

i	$\Pi_{1,1}^{1,5}$	D	D^2	D^3	D^4	D^5	D^6
−7	−5857.906327	+1093.010432	−0.005205	−10.641928	+0.459639	+0.022135	−0.001302
−6	−2085.921900	477.404955	−3.657552	−6.256514	0.355470	0.016927	−0.001302
−5	−591.979175	175.213542	−4.010418	−3.246094	0.261718	0.011719	−0.001302
−4	−116.156250	48.779948	−2.813800	−1.360678	0.178384	0.006510	−0.001302
−3	−10.656250	7.947918	−1.317708	−0.350259	0.105469	+0.001302	−0.001302
−2	+0.192708	+0.061199	−0.272136	+0.035154	+0.042968	−0.003906	−0.001302
−1	−0.062500	−0.036458	+0.072916	+0.045573	−0.009115	−0.009115	−0.001302
0	0.000000	−0.001302	−0.032552	−0.069010	−0.050781	−0.014323	−0.001302
1	−0.322917	+0.010416	+0.161458	−0.058594	−0.082031	−0.019531	−0.001302
2	−13.859376	−2.657552	1.904949	+0.326824	−0.102864	−0.024740	−0.001302
3	−125.562500	−23.161460	6.947916	1.337238	−0.113282	−0.029948	−0.001302
4	−612.510424	−99.157555	17.540365	3.222657	−0.113280	−0.035156	−0.001302
5	−2123.906275	−300.802088	36.432294	6.233074	−0.102862	−0.040365	−0.001302
6	−5921.078202	−740.751314	66.873696	10.618486	−0.082032	−0.045573	−0.001302
7	−14177.479330	−1584.161477	112.614584	16.628908	−0.050780	−0.050781	−0.001302

i	$\Pi_{1,1}^{3,1}$	D	D^2	D^3	D^4
−7	+1319.500000	−196.562500	+1.218750	+0.875000	−0.031250
−6	721.875000	−126.468750	1.062500	0.750000	−0.031250
−5	354.375000	−75.375000	0.906250	0.625000	−0.031250
−4	148.750000	−40.281250	0.750000	0.500000	−0.031250
−3	48.750000	−18.187500	0.593750	0.375000	−0.031250
−2	10.125000	−6.093750	0.437500	0.250000	−0.031250
−1	0.625000	−1.000000	0.281250	+0.125000	−0.031250
0	0.000000	+0.093750	+0.125000	0.000000	−0.031250
1	0.000000	0.187500	−0.031250	−0.125000	−0.031250
2	4.375000	2.281250	−0.187500	−0.250000	−0.031250
3	28.875000	9.375000	−0.343750	−0.375000	−0.031250
4	101.250000	24.468750	−0.500000	−0.500000	−0.031250
5	261.250000	50.562500	−0.656250	−0.625000	−0.031250
6	560.625000	90.656250	−0.812500	−0.750000	−0.031250
7	1063.125000	147.750000	−0.968750	−0.875000	−0.031250

i	$\Pi_{0,2}^{0,2}$	D	D^2	$\Pi_{0,2}^{0,4}$	D	D^2	D^3	D^4
−7	+17.125000	−2.875000	+0.125000	−181.437500	+34.760417	−0.854167	−0.187500	+0.010417
−6	11.750000	−2.375000	0.125000	−83.291667	19.791667	−0.697917	−0.145833	0.010417
−5	7.375000	−1.875000	0.125000	−31.312500	9.822917	−0.541667	−0.104167	0.010417
−4	4.000000	−1.375000	0.125000	−8.250000	3.854167	−0.385417	−0.062500	0.010417
−3	1.625000	−0.875000	0.125000	−0.854167	+0.885417	−0.229167	−0.020833	0.010417
−2	+0.250000	−0.375000	0.125000	+0.125000	−0.083333	−0.072917	+0.020833	0.010417
−1	−0.125000	+0.125000	0.125000	−0.062500	−0.052083	+0.083333	0.062500	0.010417
0	+0.500000	0.625000	0.125000	−0.166667	−0.020833	0.239583	0.104167	0.010417
1	2.125000	1.125000	0.125000	−2.937500	−0.989583	0.395833	0.145833	0.010417
2	4.750000	1.625000	0.125000	−15.125000	−3.958333	0.552083	0.187500	0.010417
3	8.375000	2.125000	0.125000	−47.479167	−9.927083	0.708333	0.229167	0.010417
4	13.000000	2.625000	0.125000	−114.750000	−19.895833	0.864583	0.270833	0.010417
5	18.625000	3.125000	0.125000	−235.687500	−34.864583	1.020833	0.312500	0.010417
6	25.250000	3.625000	0.125000	−433.041667	−55.833333	1.177083	0.354167	0.010417
7	32.875000	4.125000	0.125000	−733.562500	−83.802083	1.333333	0.395833	0.010417

i	$\Pi^{0,6}_{0,2}$	D	D^2	D^3	D^4	D^5	D^6
−7	+ 660.265299	−154.273112	+ 4.082031	+1.536133	−0.108724	−0.002930	+0.000326
−6	193.177083	− 58.532552	2.792969	0.768555	−0.079753	−0.001628	0.000326
−5	39.026367	− 16.900716	1.561849	0.292643	−0.053385	−0.000326	0.000326
−4	+ 3.447917	− 2.776042	0.623047	+0.045898	−0.029622	+0.000977	0.000326
−3	− 0.142253	+ 0.068034	+ 0.085937	−0.034180	−0.008464	0.002279	0.000326
−2	+ 0.078125	− 0.016927	− 0.065104	−0.010091	+0.010091	0.003581	0.000326
−1	− 0.037435	− 0.054362	+ 0.029297	+0.055664	0.026042	0.004883	0.000326
0	+ 0.020833	+ 0.057292	+ 0.103516	0.100586	0.039388	0.006185	0.000326
1	1.043945	0.044596	− 0.233073	+0.062174	0.050130	0.007487	0.000326
2	14.729167	1.759115	− 1.496094	−0.122070	0.058268	0.008789	0.000326
3	89.305012	11.677409	− 4.326172	−0.514648	0.063802	0.010091	0.000326
4	350.156250	43.401041	− 9.488932	−1.178060	0.066732	0.011393	0.000326
5	1055.448893	120.156576	−17.875000	−2.174805	0.067057	0.012695	0.000326
6	2666.755208	277.295573	−30.500000	−2.567383	0.064779	0.013997	0.000326
7	5934.678711	564.794596	−48.504557	−5.418294	0.059896	0.015299	0.000326

i	$\Pi^{2,2}_{0,2}$	D	D^2	D^3	D^4
−7	− 839.125000	+145.156250	−2.562500	−0.687500	+ 0.031250
−6	− 423.000000	88.437500	−2.156250	−0.562500	0.031250
−5	− 184.375000	48.718750	−1.750000	−0.437500	0.031250
−4	− 64.000000	23.000000	−1.343750	−0.312500	0.031250
−3	− 14.625000	8.281250	−0.937500	−0.187500	0.031250
−2	− 1.000000	+ 1.562500	−0.531250	−0.062500	0.031250
−1	+ 0.125000	− 0.156250	−0.125000	+0.062500	0.031250
0	0.000000	+ 0.125000	+0.281250	0.187500	0.031250
1	− 2.125000	− 0.593750	0.687500	0.312500	0.031250
2	− 19.000000	− 5.312500	1.093750	0.437500	0.031250
3	− 75.375000	− 17.031250	1.500000	0.562500	0.031250
4	− 208.000000	− 38.750000	1.906250	0.687500	0.031250
5	− 465.625000	− 73.468750	2.312500	0.812500	0.031250
6	− 909.000000	−124.187500	2.718750	0.937500	0.031250
7	−1610.875000	−193.906250	3.125000	1.062500	0.031250

i	$\Pi^{1,3}_{-1,3}$	D	D^2	D^3	D^4
−7	+162.458333	−33.750000	+1.135417	+0.166667	−0.010417
−6	72.375000	−18.968750	0.916667	0.125000	−0.010417
−5	25.833333	− 9.187500	0.697917	0.083333	−0.010417
−4	6.083333	− 3.406250	0.479167	+0.041667	−0.010417
−3	+ 0.375000	− 0.625000	0.260417	0.000000	−0.010417
−2	− 0.041667	+ 0.156250	+0.041667	−0.041667	−0.010417
−1	+ 0.083333	− 0.062500	−0.177083	−0.083333	−0.010417
0	0.000000	− 0.281250	−0.395833	−0.125000	−0.010417
1	2.958333	+ 0.500000	−0.614583	−0.166667	−0.010417
2	16.208333	3.281250	−0.833333	−0.208333	−0.010417
3	51.000000	9.062500	−1.052083	−0.250000	−0.010417
4	122.583333	18.843750	−1.270833	−0.291667	−0.010417
5	250.208333	33.625000	−1.489583	−0.333333	−0.010417
6	457.125000	54.406250	−1.708333	−0.375000	−0.010417
7	770.583333	82.187500	−1.927083	−0.416667	−0.010417

i	$\Pi_{-1,3}^{1,5}$	D	D^2	D^3	D^4	D^5	D^6
—7	— 958.708333	+ 240.013012	— 7.778642	— 2.466796	+0.185548	+0.005859	—0.000651
—6	— 258.539058	86.009764	— 5.044920	— 1.159506	0.133461	0.003255	—0.000651
—5	— 44.739580	22.466147	— 2.638020	— 0.383463	0.086589	+0.000651	—0.000651
—4	— 2.192708	+ 2.804035	— 0.932942	— 0.013674	0.044920	—0.001953	—0.000651
—3	+ 0.281250	— 0.304686	— 0.054688	+ 0.074871	+0.008464	—0.004557	—0.000651
—2	— 0.075520	+ 0.061850	+ 0.121744	+ 0.007160	—0.022788	—0.007162	—0.000651
—1	+ 0.041667	0.075520	— 0.028646	— 0.091797	—0.048828	—0.009766	—0.000651
0	0.000000	+ 0.158203	+ 0.119140	— 0.097005	—0.069662	—0.012370	—0.000651
1	— 4.770833	— 0.018230	1.440104	+ 0.116536	—0.085286	—0.014974	—0.000651
2	— 52.778644	— 6.531901	5.059246	0.673828	—0.095702	—0.017578	—0.000651
3	— 276.468744	— 36.210937	12.351612	1.699868	—0.100912	—0.020182	—0.000651
4	— 992.223932	— 121.633462	24.942058	3.319663	—0.100910	—0.022786	—0.000651
5	— 2820.364510	— 316.127594	44.705730	5.658202	—0.095704	—0.025391	—0.000651
6	— 6835.148262	— 698.771466	73.767581	8.840495	—0.085284	—0.027995	—0.000651
7	—14744.770453	—1379.393194	114.502597	12.991536	—0.069662	—0.030599	—0.000651

i	Π_3^3	D	D^2	D^3	$\Pi_{2,1}^{2,1}$	D	D^2	D^3
—7	+91.583333	—17.416667	+1.062500	—0.020833	—187.687500	+39.625000	—2.750000	+0.062500
—6	61.750000	—13.479167	0.937500	—0.020833	—119.625000	29.437500	—2.375000	0.062500
—5	39.166667	—10.041667	0.812500	—0.020833	— 70.312500	20.750000	— 2.000000	0.062500
—4	22.833333	— 7.104167	0.687500	—0.020833	— 36.750000	13.562500	—1.625000	0.062500
—3	11.750000	— 4.666667	0.562500	—0.020833	— 15.937500	7.875000	—1.250000	0.062500
—2	4.916667	— 2.729167	0.437500	—0.020833	— 4.875000	3.687500	—0.875000	0.062500
—1	+ 1.333333	— 1.291667	0.312500	—0.020833	— 0.562500	+ 1.000000	—0.500000	0.062500
0	0.000000	— 0.354167	0.187500	—0.020833	0.000000	— 0.187500	—0.125000	0.062500
1	— 0.083333	+ 0.083333	+0.062500	—0.020833	— 0.187500	+ 0.125000	+0.250000	0.062500
2	+ 0.083333	+ 0.020833	—0.062500	—0.020833	+ 1.875000	1.937500	0.625000	0.062500
3	— 0.500000	— 0.541667	—0.187500	—0.020833	9.187500	5.250000	1.000000	0.062500
4	— 2.833333	— 1.604167	—0.312500	—0.020833	24.750000	10.062500	1.375000	0.062500
5	— 7.916667	— 3.166667	—0.437500	—0.020833	51.562500	16.375000	1.750000	0.062500
6	—16.750000	— 5.229167	—0.562500	—0.020833	92.625000	24.187500	2.125000	0.062500
7	—30.333333	— 7.791667	—0.687500	—0.020833	150.937500	33.500000	2.500000	0.062500

i	$\Pi_{2,1}^{2,3}$	D	D^2	D^3	D^4	D^5
—7	+3775.406250	—844.132812	+43.585938	+2.765625	—0.320312	+0.007812
—6	1703.296875	—449.179688	27.601562	1.984375	—0.273438	0.007812
—5	658.203125	—211.929688	16.117188	1.328125	—0.226562	0.007812
—4	201.468750	— 83.632812	8.382812	0.796875	—0.179688	0.007812
—3	41.437500	— 24.539062	3.648438	0.390625	—0.132812	0.007812
—2	+ 3.453125	— 3.898438	1.164062	+0.109375	—0.085938	0.007812
—1	— 0.140625	+ 0.039062	+ 0.179688	—0.046875	—0.039062	0.007812
0	0.000000	+ 0.023438	— 0.054688	—0.078125	+0.007812	0.007812
1	+ 0.218750	— 0.195312	— 0.289062	+0.015625	0.054688	0.007812
2	— 6.140625	— 5.867188	— 1.273438	0.234375	0.101562	0.007812
3	— 58.734375	— 31.242188	— 3.757812	0.578125	0.148438	0.007812
4	— 260.218750	— 99.570312	— 8.492188	1.046875	0.195312	0.007812
5	— 806.250000	—243.101562	—16.226562	1.640625	0.242188	0.007812
6	—2015.484375	—503.085938	—27.710938	2.359375	0.289062	0.007812
7	—4359.578125	—929.773438	—43.695312	3.203125	0.335938	0.007812

/111

i	$\Pi^{1,2}_{1,2}$	D	D^2	D^3
—7	+119.875000	—28.687500	+2.312500	—0.062500
—6	70.500000	—20.125000	1.937500	—0.062500
—5	36.875000	—13.062500	1.562500	—0.062500
—4	16.000000	— 7.500000	1.187500	—0.062500
—3	4.875000	— 3.437500	0.812500	—0.062500
—2	+ 0.500000	— 0.875000	0.437500	—0.062500
—1	— 0.125000	+ 0.187500	+0.062500	—0.062500
0	0.000000	— 0.250000	—0.312500	—0.062500
1	— 2.125000	— 2.187500	—0.687500	—0.062500
2	— 9.500000	— 5.625000	—1.062500	—0.062500
3	— 25.125000	—10.562500	—1.437500	—0.062500
4	— 52.000000	—17.000000	—1.812500	—0.062500
5	— 93.125000	—24.937500	—2.187500	—0.062500
6	—151.500000	—34.375000	—2.562500	—0.062500
7	—230.125000	—45.312500	—2.937500	—0.062500

i	$\Pi^{1,4}_{1,2}$	D	D^2	D^3	D^4	D^5
—7	—1270.062500	+334.041667	—23.359375	—0.885416	+0.166667	—0.005208
—6	— 499.750000	160.395833	—14.083333	—0.526041	0.135417	—0.005208
—5	— 156.562500	64.770833	— 7.619792	—0.250000	0.104167	—0.005208
—4	— 33.000000	19.541667	— 3.468750	—0.057292	0.072917	—0.005208
—3	— 2.562500	+ 3.083333	— 1.130208	+0.052083	0.041667	—0.005208
—2	+ 0.250000	— 0.229167	— 0.104167	0.078125	+0.010417	—0.005208
—1	— 0.062500	— 0.020833	+ 0.109375	+0.020833	—0.020833	—0.005208
0	0.000000	+ 0.083333	0.010417	—0.119792	—0.052083	—0.005208
1	+ 2.937500	2.458333	0.098958	—0.343750	—0.083333	—0.005208
2	30.250000	15.479167	0.875000	—0.651042	—0.114583	—0.005208
3	142.437500	53.520833	2.838542	—1.041667	—0.145833	—0.005208
4	459.000000	136.958333	6.489583	—1.515625	—0.177083	—0.005208
5	1178.437500	292.166667	12.328125	—2.072917	—0.208333	—0.005208
6	2598.250000	551.520833	20.854167	—2.713542	—0.239583	—0.005208
7	5134.937500	953.395833	32.567708	—3.437500	—0.270833	—0.005208

i	$\Pi^{0,3}_{0,3}$	D	D^2	D^3
—7	— 23.208333	+ 6.479167	—0.625000	+0.020833
—6	— 12.062500	4.166667	—0.500000	0.020833
—5	— 5.166667	2.354167	—0.375000	0.020833
—4	— 1.520833	1.041667	—0.250000	0.020833
—3	— 0.125000	+ 0.229167	—0.125000	0.020833
—2	+ 0.020833	— 0.083333	0.000000	0.020833
—1	— 0.083333	+ 0.104167	+0.125000	0.020833
0	+ 0.562500	0.791667	0.250000	0.020833
1	2.958333	1.979167	0.375000	0.020833
2	8.104167	3.666667	0.500000	0.020833
3	17.000000	5.854167	0.625000	0.020833
4	30.645833	8.541667	0.750000	0.020833
5	50.041667	11.729167	0.875000	0.020833
6	76.187500	15.416667	1.000000	0.020833
7	110.083333	19.604167	1.125000	0.020833

/112

i	$\Pi^{0,5}_{0,3}$	D	D^2	D^3	D^4	D^5
−7	+ 136.958333	− 44.070311	+4.259114	+0.048177	−0.029948	+0.001302
−6	43.089843	− 17.925781	2.334635	−0.001302	−0.022135	0.001302
−5	8.917916	− 5.388021	1.066406	−0.029948	−0.014323	0.001302
−4	+ 0.548177	− 0.769531	+ 0.329427	−0.037760	−0.006510	0.001302
−3	− 0.093750	+ 0.117187	− 0.001302	−0.024740	+ 0.001302	0.001302
−2	+ 0.037760	− 0.010365	− 0.050781	+0.009115	0.009115	0.001302
−1	− 0.041667	− 0.054687	+0.055990	0.063802	0.016927	0.001302
0	− 0.316406	− 0.238281	0.194010	0.139323	0.024740	0.001302
1	− 4.770833	− 2.403646	0.238281	0.235677	0.032552	0.001302
2	− 26.389322	− 9.863281	+0.063803	0.352865	0.010365	0.001302
3	− 92.156248	− 27.429687	− 0.454427	0.490885	0.048177	0.001302
4	− 248.055983	− 61.415363	−1.441406	0.649740	0.055990	0.001302
5	− 564.072917	−119.632812	−3.022135	0.829427	0.063802	0.001302
6	−1139.191407	−211.394531	−5.321614	1.029948	0.071615	0.001302
7	−2106.395833	−347.513021	−8.464844	1.251302	0.079427	0.001302

i	$\Pi^{0,7}_{0,3}$	D	D^2	D^3	D^4	D^5	D^6	D^7
−7	− 273.876042	+ 107.977214	−12.780338	− 0.297884	+0.169271	−0.008529	−0.000391	+0.000033
−6	− 45.644238	25.108593	− 4.491113	+ 0.045540	0.075684	−0.006608	−0.000195	0.000033
−5	− 2.780208	+ 2.878125	− 1.013737	0.097624	+0.020508	−0.004427	0.000000	0.000033
−4	+ 0.137728	− 0.152669	− 0.014225	+ 0.038053	−0.004069	−0.001986	+0.000195	0.000033
−3	− 0.070312	+ 0.062109	+ 0.028906	− 0.015983	−0.005859	+0.000716	0.000391	0.000033
−2	+ 0.032975	− 0.009766	− 0.050358	− 0.009798	+0.007324	0.003678	0.000586	0.000033
−1	− 0.029167	− 0.041146	+ 0.019466	+ 0.048796	0.027669	0.006901	0.000781	0.000033
0	+ 0.071191	+ 0.091992	+ 0.072363	+ 0.089486	0.047363	0.010384	0.000977	0.000033
1	2.435417	0.748047	− 0.370182	− 0.020540	0.058594	0.014128	0.001172	0.000033
2	29.157064	7.457292	− 1.974186	− 0.476595	0.053548	0.018132	0.001367	0.000033
3	178.790625	40.359375	− 5.468164	− 1.536491	+0.024414	0.022396	0.001562	0.000033
4	742.070280	148.140820	−11.518131	− 3.520540	−0.036621	0.026921	0.001758	0.000033
5	2397.878646	426.472526	−20.602604	− 6.811556	−0.137370	0.031706	0.001953	0.000033
6	6510.465528	1040.947266	−32.887597	−11.854850	−0.285645	0.036751	0.002148	0.000033
7	15548.916668	2254.517188	−48.101627	−19.158235	−0.489258	0.042057	0.002344	0.000033

i	$\Pi^{2,3}_{0,3}$	D	D^2	D^3	D^4	D^5
−7	+1137.208333	−323.281250	+26.442708	+0.442725	−0.151042	+0.005208
−6	434.250000	−153.015625	16.026042	+0.166667	−0.119792	0.005208
−5	129.166667	− 60.145833	8.671875	−0.026042	−0.088542	0.005208
−4	24.333333	− 17.046875	3.880208	−0.135417	−0.057292	0.005208
−3	+ 1.125000	− 2.093750	+ 1.151042	−0.161458	−0.026042	0.005208
−2	− 0.083333	+ 0.338542	− 0.015625	−0.104167	+0.005208	0.005208
−1	+ 0.083333	− 0.125000	− 0.119792	+0.036458	0.036458	0.005208
0	0.000000	+ 0.140625	+ 0.338542	0.260417	0.067708	0.005208
1	− 2.958333	− 1.239584	0.859375	0.567708	0.098958	0.005208
2	− 32.416667	− 12.640625	0.942708	0.958333	0.130208	0.005208
3	− 153.000000	− 48.437500	+ 0.088542	1.432292	0.161458	0.005208
4	− 490.333333	−129.005214	− 2.203125	1.989583	0.192708	0.005208
5	−1251.041667	−280.718750	− 6.432292	2.630208	0.223958	0.005208
6	−2742.750000	−535.953125	−13.098958	3.354167	0.255208	0.005208
7	−5394.083333	−933.083333	−22.703125	4.161458	0.286458	0.005208

$\Pi_{-1,4}^{1,4}$

i		D	D^2	D^3	D^4	D^5
−7	−117.851567	+41.384114	−4.627606	+0.035158	+0.026042	−0.001302
6	−35.125000	16.291667	−2.507813	0.063800	0.018230	−0.001302
−5	−6.523410	4.555990	−1.106772	0.071615	0.010417	−0.001302
−4	−0.250000	+0.489582	−0.299480	0.058595	+0.002604	−0.001302
−3	−0.007812	−0.095052	+0.039062	0.024738	−0.005208	−0.001302
−2	0.000000	+0.114583	+0.033854	−0.029948	−0.013021	−0.001302
−1	+0.070312	−0.069011	−0.190104	−0.105468	−0.020833	−0.001302
0	0.000000	−0.333333	−0.507812	−0.201823	−0.028646	−0.001302
1	4.085938	+1.134114	−0.794270	−0.319010	−0.036458	−0.001302
2	26.125000	7.645833	−0.924479	−0.457032	−0.044270	−0.001302
3	94.414065	24.014322	−0.773437	−0.615885	−0.052083	−0.001302
4	256.750000	56.552083	−0.216145	−0.795573	−0.059895	−0.001302
5	585.429700	113.071615	+0.872396	−0.996096	−0.067708	−0.001302
6	1182.250000	202.885421	2.617188	−1.217446	−0.075520	−0.001302
7	2183.507833	336.805996	5.143230	−1.459635	−0.083333	−0.001302

$\Pi_{3,1}^{3,1}$

i		D	D^2	D^3	D^4
−7	−595.291667	+159.000000	−15.614583	+0.666667	−0.010417
−6	339.627000	105.010417	−11.895833	0.583333	−0.010417
−5	−176.250000	64.770833	−8.677083	0.500000	−0.010417
−4	−79.916667	36.281250	−5.958333	0.416667	−0.010417
−3	−29.375000	17.541667	−3.739583	0.333333	−0.010417
−2	−7.375000	6.552083	−2.020833	0.250000	−0.010417
−1	−0.666667	+1.312500	−0.802083	0.166667	−0.010417
0	0.000000	−0.177083	−0.083333	+0.083333	−0.010417
1	−0.125000	+0.083333	+0.135417	0.000000	−0.010417
2	+0.208333	+0.093750	−0.145833	−0.083333	−0.010417
3	−1.750000	−2.145833	−0.927083	−0.166667	−0.010417
4	−12.750000	−8.635417	−2.208333	−0.250000	−0.010417
5	−43.541667	−21.375000	−3.989583	−0.333333	−0.010417
6	108.875000	−42.364583	−6.270833	−0.416667	−0.010417
7	227.500000	−43.604167	−9.052083	−0.500000	−0.010417

$\Pi_{5/2,\frac{3}{2}}^{5/2,\frac{3}{2}}$

i		D	D^2	D^3	D^4
7	494.484375	−149.375000	+16.890625	−0.843750	+0.015625
6	255.562500	−91.312500	12.203125	−0.718750	0.015625
5	115.234375	−50.500000	8.265625	−0.593750	0.015625
4	42.000000	−23.937500	5.078125	−0.468750	0.015625
3	10.359375	−8.625000	2.640625	−0.343750	0.015625
2	0.812500	−1.562500	0.953125	−0.218750	0.015625
1	0.140625	+0.250000	+0.015625	−0.093750	0.015625
0	0.000000	−0.187500	−0.171875	+0.031250	0.015625
1	−0.265625	+0.125000	+0.390625	0.156250	0.015625
2	−3.562500	4.187500	1.703125	0.281250	0.015625
3	21.984375	15.000000	3.765625	0.406250	0.015625
4	71.500000	35.562500	6.578125	0.531250	0.015625
5	174.609375	68.875000	10.140625	0.656250	0.015625
6	359.812500	117.937500	14.453125	0.781250	0.015625
7	661.609375	185.750000	19.515625	0.906250	0.015625

i	$\Pi^{1,3}_{1,3}$	D	D^2	D^3	D^4
−7	−162.458333	+ 56.958333	− 7.614583	+0.458333	−0.010416
−6	− 72.375000	31.031250	− 5.083333	0.375000	−0.010416
−5	− 25.833333	14.354167	− 3.052083	0.291667	−0.010416
−4	− 6.083333	4.927083	− 1.520833	0.208333	−0.010416
−3	− 0.375000	+ 0.750000	− 0.489583	0.125000	−0.010416
−2	+ 0.041667	− 0.177083	+ 0.041667	+0.041667	−0.010416
−1	− 0.083333	+ 0.145833	+ 0.072917	−0.041667	−0.010416
0	0.000000	− 0.281250	− 0.395833	−0.125000	−0.010416
1	− 2.958333	− 3.458333	− 1.364583	−0.208333	−0.010416
2	− 16.208333	− 11.385417	− 2.833333	−0.291667	−0.010416
3	− 51.000000	− 26.062500	− 4.802083	−0.375000	−0.010416
4	−122.583333	− 49.489583	− 7.270833	−0.458333	−0.010416
5	−250.208333	− 83.666667	−10.239583	−0.541667	−0.010416
6	−457.125000	−130.593750	−13.708333	−0.625000	−0.010416
7	−770.583333	−192.270833	−17.677083	−0.708333	−0.010416

i	$\Pi^{1,5}_{1,3}$	D	D^2	D^3	D^4	D^5	D^6
−7	+ 958.708310	− 376.971342	+ 51.848954	−1.792318	−0.233725	+0.024088	− 0.000651
−6	258.539058	− 129.099608	22.970700	−1.175130	−0.132159	0.018880	−0.000651
−5	44.739580	− 31.414063	8.026040	− 0.682943	−0.056641	0.013672	−0.000651
−4	+ 2.192708	− 3.352212	+ 1.702474	−0.315753	−0.007160	0.008463	−0.000651
−3	− 0.281250	+ 0.398436	− 0.062500	−0.073569	+0.016276	+0.003255	−0.000651
−2	+ 0.075520	− 0.099610	− 0.081380	+0.043620	+0.013672	−0.001953	−0.000651
−1	− 0.041667	− 0.033854	+ 0.083333	+0.035807	−0.014974	−0.007162	−0.000651
0	0.000000	+ 0.158203	0.119140	−0.097005	−0.069662	−0.012370	−0.000651
1	+ 4.770833	4.789062	0.963542	−0.354818	−0.150390	−0.017578	−0.000651
2	52.778644	32.921223	4.804035	−0.737632	−0.257162	−0.022787	−0.000651
3	276.468744	128.367185	15.078125	−1.245442	−0.389974	−0.027995	−0.000651
4	992.223932	369.689444	36.473305	−1.878257	−0.548830	−0.033203	−0.000651
5	2820.364510	880.200500	74.927080	−2.636068	−0.733724	−0.038412	−0.000651
6	6835.148262	1837.962880	137.626950	−3.518881	−0.944664	−0.043620	− 0.000651
7	14744.770453	3485.788970	233.010413	−4.526692	−1.181640	−0.048828	− 0.000651

i	$\Pi^{0,4}_{0,4}$	D	D^2	D^3	D^4
−7	+ 16.835938	− 7.114583	+1.169271	−0.088542	+0.002604
−6	5.854167	− 3.203125	0.684896	−0.067708	0.002604
−5	1.304688	− 1.041667	0.325521	−0.046875	0.002604
−4	0.062500	− 0.130208	+0.091146	−0.026042	0.002604
−3	+ 0.002604	+ 0.031250	−0.018229	−0.005208	0.002604
−2	0.000000	− 0.057292	−0.002604	+0.015625	0.002604
−1	− 0.070312	+ 0.104167	+0.138021	0.036458	0.002604
0	+ 0.666667	1.015625	0.403646	0.057292	0.002604
1	4.085938	3.177083	0.794271	0.078125	0.002604
2	13.062500	7.088542	1.309896	0.098958	0.002604
3	31.471354	13.250000	1.950521	0.119792	0.002604
4	64.187500	22.161458	2.716146	0.140625	0.002604
5	117.085938	34.322917	3.606771	0.161458	0.002604
6	197.041667	50.234375	4.622396	0.182292	0.002604
7	311.929688	70.395833	5.763021	0.203125	0.002604

i	$\Pi^{0,6}_{0,4}$	D	D^2	D^3	D^4	D^5	D^6
—7	— 52.515625	+ 26.431900	— 4.938932	+0.328776	+0.014974	—0.003255	+0.000130
—6	— 8.658333	6.218750	— 1.685677	0.180339	+0.003255	—0.002214	0.000130
—5	— 0.257292	+ 0.540625	— 0.319141	0.073568	—0.003255	—0.001172	0.000130
—4	+ 0.062500	— 0.102083	+ 0.035677	+0.008464	—0.004557	—0.000130	0.000130
—3	— 0.011458	+ 0.040234	+ 0.003776	—0.014974	—0.000651	+0.000911	0.000130
—2	+ 0.020833	— 0.032292	— 0.039844	+0.003255	+0.008464	0.001953	0.000130
—1	— 0.028125	— 0.069661	+ 0.029818	0.063151	0.022786	0.002995	0.000130
0	— 0.533333	— 0.571875	+ 0.087760	0.164714	0.042318	0.004036	0.000130
1	— 7.557291	— 4.788932	— 0.241016	0.307943	0.067057	0.005078	0.000130
2	— 44.350000	— 20.720833	— 1.581510	0.492839	0.097005	0.006120	0.000130
3	— 168.848958	— 63.117577	— 4.808724	0.719401	0.132161	0.007161	0.000130
4	— 497.179167	— 155.479167	—11.047656	0.987630	0.172526	0.008203	0.000130
5	—1233.153125	— 332.055599	—21.673307	1.297526	0.218099	0.009245	0.000130
6	—2703.770833	— 639.846875	—38.310676	1.649088	0.268880	0.010286	0.000130
7	—5400.719792	—1140.602995	—62.834766	2.042318	0.324870	0.011328	0.000130

i	$\Pi^{1,5}_{-1,5}$	D	D^2	D^3	D^4	D^5	D^6
—7	+ 42.422919	— 23.464064	+ 4.970835	—0.430335	—0.001950	+0.002738	—0.000130
—6	6.329688	— 5.111590	1.616014	—0.224608	+0.005857	0.001695	—0.000130
—5	0.156250	— 0.341145	+ 0.257293	—0.081378	0.008462	+0.000653	—0.000130
—4	+ 0.019792	+ 0.034766	— 0.042840	—0.000652	+0.005859	—0.000389	—0.000130
—3	— 0.025000	— 0.046352	+ 0.028126	+0.017578	—0.001953	—0.001431	—0.000130
—2	+ 0.014062	+ 0.102995	+ 0.032682	—0.026693	—0.014974	—0.002473	—0.000130
—1	0.066667	— 0.079688	— 0.216667	—0.133463	—0.033202	—0.003515	—0.000130
0	0.000000	— 0.406901	— 0.657422	—0.302734	—0.056640	—0.004557	—0.000130
1	5.618750	+ 2.058854	— 0.977084	—0.534506	—0.085286	—0.005599	—0.000130
2	40.665104	15.005078	— 0.613150	—0.828776	—0.119140	—0.006641	—0.000130
3	164.818750	52.869272	+ 1.246874	—1.185546	—0.158204	—0.007683	—0.000130
4	497.696880	138.838934	5.665495	—1.604818	—0.202475	—0.008725	—0.000130
5	1248.854185	306.851565	13.955206	—2.086590	—0.251955	—0.009767	—0.000130
6	2753.782848	603.594695	27.678514	—2.630861	—0.306638	—0.010810	—0.000130
7	5515.912570	1090.505765	48.647919	—3.237633	—0.366534	—0.011852	—0.000130

i	$\Pi^{2,3}_{2,3}$	D	D^2	D^3	D^4	D^5
—7	— 670.140625	+277.018229	—46.054689	+3.833333	—0.158854	+0.002604
—6	— 262.359375	131.335938	—26.445314	2.661458	—0.132812	0.002604
—5	— 80.729167	51.638021	—13.273438	1.697917	—0.106771	0.002604
—4	— 15.968750	14.549479	— 5.289063	0.942708	—0.080729	0.002604
—3	— 0.796875	+ 1.695312	— 1.242188	0.395833	—0.054687	0.002604
—2	+ 0.067708	— 0.299478	+ 0.117187	+0.057291	—0.028646	0.002604
—1	— 0.093750	+ 0.190104	+ 0.039062	—0.072917	—0.002604	0.002604
0	0.000000	— 0.210938	— 0.226563	+0.005208	+0.023437	0.002604
1	— 0.369792	+ 0.122396	+ 0.570312	0.291667	0.049479	0.002604
2	+ 6.078125	7.815105	3.679688	0.786458	0.075521	0.002604
3	44.625000	34.492188	10.351563	1.489582	0.101562	0.002604
4	168.552083	96.778646	21.835938	2.401042	0.127604	0.002604
5	469.140625	216.299479	39.382813	3.520833	0.153646	0.002604
6	1085.671875	419.679688	64.242188	4.848958	0.179688	0.002604
7	2215.427083	738.544271	97.664063	6.385417	0.205729	0.002604

i	$\Pi_{1,4}^{1,4}$	D	D^2	D^3	D^4	D^5
−7	+ 117.851567	− 58.220052	+11.742188	−1.204430	+0.062500	−0.001302
−6	35.125000	− 22.145833	5.710938	−0.748696	0.049479	−0.001302
−5	6.523440	− 5.860677	2.148438	−0.397135	0.036458	−0.001302
−4	0.250000	− 0.552082	+ 0.429688	−0.149741	0.023438	−0.001302
−3	+ 0.007812	+ 0.092448	− 0.070312	−0.006510	+0.010417	−0.001302
−2	0.000000	− 0.114584	+ 0.023438	+0.032552	−0.002604	−0.001302
−1	− 0.070312	+ 0.139323	+ 0.085937	−0.032552	−0.015625	−0.001302
0	0.000000	− 0.333333	− 0.507812	−0.201823	−0.028646	−0.001302
1	− 4.085938	− 5.220052	− 2.382812	−0.475260	−0.041667	−0.001302
2	− 26.125000	− 20.708333	− 6.164063	−0.852864	−0.054687	−0.001302
3	− 94.414065	− 55.485680	−12.476563	−1.334636	−0.067708	−0.001302
4	− 256.750000	−120.739586	−21.945313	−1.920573	−0.080729	−0.001302
5	− 585.429700	−230.157555	−35.195313	−2.610675	−0.093750	−0.001302
6	−1182.250000	−399.927091	−52.851564	−3.404950	−0.106771	−0.001302
7	−2183.507833	−648.735667	−75.539063	−4.303385	−0.119792	−0.001302

i	$\Pi_{0,5}^{0,5}$	D	D^2	D^3	D^4	D^5
−7	− 6.060417	+ 3.784896	− 0.980469	+0.131510	−0.009115	+0.000260
−6	− 1.054948	0.939844	− 0.347656	0.066406	−0.006510	0.000260
−5	− 0.031250	+ 0.071354	− 0.058594	+0.022135	−0.003906	0.000260
−4	− 0.004948	− 0.008073	+ 0.011719	−0.001302	−0.001302	0.000260
−3	+ 0.008333	+ 0.014062	− 0.011719	−0.003906	+0.001302	0.000260
−2	− 0.007031	− 0.049740	− 0.003906	+0.014323	0.003906	0.000260
−1	− 0.066667	+ 0.113021	+ 0.160156	0.053385	0.006510	0.000260
0	+ 0.813802	1.314844	0.605469	0.113281	0.009115	0.000260
1	5.618750	4.868229	1.457031	0.194010	0.011719	0.000260
2	20.332552	12.585677	2.839844	0.295573	0.014323	0.000260
3	54.939584	26.779688	4.878906	0.417969	0.016927	0.000260
4	124.424220	50.262761	7.699219	0.561198	0.019531	0.000260
5	249.770833	86.347397	11.425781	0.725260	0.022135	0.000260
6	458.963802	138.846094	16.183594	0.910156	0.024740	0.000260
7	787.987500	212.071354	22.097657	1.115885	0.027344	0.000260

i	$\Pi_{0,5}^{0,7}$	D	D^2	D^3	D^4	D^5	D^6	D^7
−7	+ 7.647916	− 6.306597	+ 2.092687	−0.335731	+0.020074	+0.001345	−0.000260	+0.000011
−6	+ 0.033735	− 0.299609	+ 0.247103	−0.081391	0.011122	+0.000011	−0.000152	0.000011
−5	− 0.039062	+ 0.074870	− 0.041840	+0.003787	+0.002821	−0.000543	−0.000043	0.000011
−4	− 0.000098	− 0.021376	+ 0.010362	+0.005740	−0.002224	−0.000315	+0.000065	0.000011
−3	− 0,000347	+ 0.026562	+ 0.000716	−0.010428	−0.001411	+0.000694	0.000174	0.000011
−2	+ 0.014941	− 0.025781	− 0.036274	−0.000445	+0.007867	0.002485	0.000282	0.000011
−1	− 0.016667	− 0.094748	− 0.003602	+0.059125	0.028212	0.005056	0.000391	0.000011
0	− 0.847710	− 1.081055	− 0.116764	0.170888	0.062229	0.008409	0.000499	0.000011
1	− 11.743750	− 8.682291	− 0.278754	0.316612	0.112522	0.012543	0.000608	0.000011
2	− 72.556280	− 39.705424	− 5.455067	0.457237	0.181695	0.017459	0.000716	0.000011
3	− 297.045660	− 130.879297	− 16.048697	0.532867	0,272352	0.023155	0.000825	0.000011
4	− 944.454004	− 350.167122	− 38.275139	0.462771	0.387099	0.029633	0.000933	0.000011
5	− 2526.328124	− 810.578993	− 79.537388	+0.145388	0.528537	0.036892	0.001042	0.000011
6	− 5955.592437	−1686.484375	−149.800944	−0.541677	0.699273	0.044933	0.001150	0.000011
7	−12744.871875	−3232.424609	−261.968793	−1.741656	0.901910	0.053754	0.001259	0.000011

$\Pi^{1,5}_{1,5}$

i		D	D^2	D^3	D^4	D^5	D^6
−7	− 42.422919	+ 29.524480	− 8.755731	+ 1.410805	−0.129560	+0.006378	−0.000130
−6	− 6.329688	6.166538	− 2.553858	0.572264	−0.072263	0.004815	−0.000130
−5	− 0.156250	+ 0.372395	− 0.328647	+ 0.139972	−0.030598	0.003253	−0.000130
−4	− 0.019792	− 0.029818	+ 0.050912	− 0.011068	−0.004557	0.001691	−0.000130
−3	+ 0.025000	+ 0.038020	− 0.042188	− 0.005858	+0.005859	+0.000129	−0.000130
−2	− 0.014062	− 0.095965	+ 0.017058	+ 0.030599	+0.000650	−0.001433	−0.000130
−1	− 0.066667	+ 0.146355	+ 0.103645	− 0.026693	−0.020182	−0.002995	−0.000130
0	0.000000	− 0.406901	− 0.657422	− 0.302735	−0.056640	−0.004557	−0.000130
1	− 5.618750	− 7.677604	− 3.891145	− 0.922525	−0.108725	−0.006119	−0.000130
2	− 40.665104	− 35.337630	− 11.972526	− 2.011068	−0.176432	−0.007681	−0.000130
3	− 164.818750	− 107.808856	− 28.026562	− 3.693360	−0.259765	−0.009243	−0.000130
4	− 497.696880	− 263.263154	− 55.928256	− 6.094402	−0.358723	−0.010805	−0.000130
5	−1248.854185	− 556.622405	−100.302604	− 9.339190	−0.473305	−0.012367	−0.000130
6	−2753.782848	−1062.558500	−166.524614	−13.552733	−0.603518	−0.013930	−0.000130
7	−5515.912570	−1878.493275	−260.719279	−18.860025	−0.749350	−0.015492	−0.000130

$\Pi^{0,6}_{0,6}$

i		D	D^2	D^3	D^4	D^5	D^6
−7	+ 0.816688	− 0.791298	+ 0.330165	−0.075629	+ 0.009983	−0.000716	+0.000022
−6	0.015625	− 0.038281	+ 0.035243	−0.015950	0.003798	−0.000456	0.000022
−5	+ 0.003277	+ 0.000543	− 0.005122	+0.002062	+0.000217	−0.000195	0.000022
−4	− 0.004861	− 0.000347	+ 0.005946	−0.000760	−0.000760	+0.000065	0.000022
−3	+ 0.008789	+ 0.008529	− 0.009679	−0.003581	+0.000868	0.000326	0.000022
−2	− 0.011111	− 0.048351	− 0.005122	+0.014431	0.005100	0.000586	0.000022
−1	− 0.067817	+ 0.128494	+ 0.191493	0.074110	0.011936	0.000846	0.000022
0	+ 1.012500	1.713542	0.877040	0.196289	0.021376	0.001107	0.000022
1	7.705751	7.256272	2.473394	0.401801	0.033420	0.001367	0.000022
2	30.914931	21.181163	5.527431	0.711480	0.048069	0.001628	0.000022
3	92.095117	50.287695	10.711024	1.146159	0.065321	0.001888	0.000022
4	228.378472	104.250347	18.821050	1.726671	0.085178	0.002148	0.000022
5	498.699240	196.118598	30.779384	2.473850	0.107639	0.002409	0.000022
6	990.918750	342.816927	47.632900	3.408529	0.132704	0.002669	0.000022
7	1829.950412	565.644813	70.553472	4.551541	0.160373	0.002930	0.000022

$\Pi^{0,7}_{0,7}$

i		D	D^2	D^3	D^4	D^5	D^6	D^7
−7	− 0.007812	+ 0.020257	− 0.020356	+ 0.010493	− 0.003038	+0.000499	−0.000043	+0.000002
−6	− 0.001691	+ 0.001073	+ 0.00'682	− 0.001335	+0.000163	+0.000076	−0.000022	0.000002
−5	+ 0.001935	− 0.001835	− 0.001671	+ 0.001053	+0.000109	−0.000087	0.000000	0.000002
−4	− 0.003641	+ 0.001442	+ 0.004091	− 0.000575	−0.000597	+0.000011	+0.000022	0.000002
−3	+ 0.008730	+ 0.006021	− 0.009028	− 0.003613	+0.000651	0.000369	0.000043	0.000002
−2	− 0.014532	− 0.050273	− 0.006521	+ 0.015375	0.006456	0.000987	0.000065	0.000002
−1	− 0.072321	+ 0.150595	+ 0.233615	0.100662	0.019423	0.001866	0.000087	0.000002
0	+ 1.276573	2.244367	1.245888	0.317350	0.042155	0.003006	0.000109	0.000002
1	10.518884	10.621993	4.002300	0.751378	0.077257	0.004405	0.000130	0.000002
2	46.222279	34.367132	10.037359	1.509515	0.127333	0.006066	0.000152	0.000002
3	149.870784	89.693650	21.573068	2.719368	0.194987	0.007986	0.000174	0.000002
4	402.550191	202.883125	41.643934	4.529373	0.282823	0.010167	0.000195	0.000002
5	947.433482	413.722340	74.221962	7.108800	0.393446	0.012609	0.000217	0.000002
6	2019.696244	779.440786	124.341656	10.647754	0.529460	0.015310	0.000239	0.000002
7	3984.652083	1379.148165	198.225022	15.357172	0.693468	0.018273	0.000260	0.000002

2. Operators of First Class

/118

i	$\Pi'^{1,1}_{1,-1}$			$\Pi'^{1,3}_{1,-1}$				
		D	D^2		D	D^2	D^3	D^4
−5	+27.000000	+0.750000	−0.250000	−283.875000	−1.093750	+6.562500	0.000000	−0.031250
−4	17.500000	0.750000	−0.250000	−111.875000	−0.375000	4.281250	0.000000	−0.031250
−3	10.000000	0.750000	−0.250000	− 32.750000	+0.093750	2.500000	0.000000	−0.031250
−2	4.500000	0.750000	−0.250000	− 5.250000	0.312500	1.218750	0.000000	−0.031250
−1	+ 1.000000	0.750000	−0.250000	− 0.125000	+0.281250	0.437500	0.000000	−0.031250
0	− 0.500000	0.750000	−0.250000	− 0.125000	0.000000	0.156250	0.000000	−0.031250
1	0.000000	0.750000	−0.250000	0.000000	−0.531250	0.375000	0.000000	−0.031250
2	+ 2.500000	0.750000	−0.250000	− 6.500000	−1.312500	1.093750	0.000000	−0.031250
3	7.000000	0.750000	−0.250000	− 38.375000	−2.343750	2.312500	0.000000	−0.031250
4	13.500000	0.750000	−0.250000	−126.375000	−3.625000	4.031250	0.000000	−0.031250
5	22.000000	0.750000	−0.250000	−313.250000	−5.156250	6.250000	0.000000	−0.031250

i	Π'^{0}_{0}	Π'^{2}_{0}			$\Pi'^{0,2}_{0,0}$		
			D	D^2		D	D^2
−5	+1.000000	−36.000000	+0.250000	+0.250000	−16.000000	+0.250000	+0.250000
−4	1.000000	−25.000000	0.250000	0.250000	− 9.000000	0.250000	0.250000
−3	1.000000	−16.000000	0.250000	0.250000	− 4.000000	0.250000	0.250000
−2	1.000000	− 9.000000	0.250000	0.250000	− 1.000000	0.250000	0.250000
−1	1.000000	− 4.000000	0.250000	0.250000	0.000000	0.250000	0.250000
0	1.000000	− 1.000000	0.250000	0.250000	− 1.000000	0.250000	0.250000
1	1.000000	0.000000	0.250000	0.250000	− 4.000000	0.250000	0.250000
2	1.000000	− 1.000000	0.250000	0.250000	− 9.000000	0.250000	0.250000
3	1.000000	− 4.000000	0.250000	0.250000	−16.000000	0.250000	0.250000
4	1.000000	− 9.000000	0.250000	0.250000	−25.000000	0.250000	0.250000
5	1.000000	−16.000000	0.250000	0.250000	−36.000000	0.250000	0.250000

i	$\Pi'^{0,4}_{0,0}$				
		D	D^2	D^3	D^4
−5	+ 59.750000	−3.906250	−1.828125	+0.093750	+0.015625
−4	17.859375	−2.156250	−0.953125	0.093750	0.015625
−3	+ 2.937500	−0.906250	−0.328125	0.093750	0.015625
−2	− 0.015625	−0.156250	+0.046875	0.093750	0.015625
−1	0.000000	+0.093750	0.171875	0.093750	0.015625
0	− 0.015625	−0.156250	+0.046875	0.093750	0.015625
1	+ 2.937500	−0.906250	−0.328125	0.093750	0.015625
2	17.859375	−2.156250	−0.953125	0.093750	0.015625
3	59.750000	−3.906250	−1.828125	0.093750	0.015625
4	149.609375	−6.156250	−2.953125	0.093750	0.015625
5	314.437500	−8.906250	−4.328125	0.093750	0.015625

i	$\Pi'^{1,1}_{-1,1}$			$\Pi'^{1,3}_{-1,1}$				
		D	D^2		D	D^2	D^3	D^4
−5	+21.000000	−1.250000	−0.250000	−115.125000	+12.156250	+3.312500	−0.250000	−0.031250
−4	12.500000	−1.250000	−0.250000	− 32.500000	6.437500	1.593750	−0.250000	−0.031250
−3	6.000000	−1.250000	−0.250000	− 4.250000	2.468750	+0.375000	−0.250000	−0.031250
−2	+ 1.500000	−1.250000	−0.250000	+ 0.375000	+ 0.250000	−0.343750	−0.250000	−0.031250
−1	− 1.000000	−1.250000	−0.250000	0.125000	− 0.218750	−0.562500	−0.250000	−0.031250
0	− 1.500000	−1.250000	−0.250000	+ 1.750000	+ 1.062500	−0.281250	−0.250000	−0.031250
1	0.000000	−1.250000	−0.250000	0.000000	4.093750	+0.500000	−0.250000	−0.031250
2	+ 3.500000	−1.250000	−0.250000	− 22.375000	8.875000	1.781250	−0.250000	−0.031250
3	9.000000	−1.250000	−0.250000	− 94.625000	15.406250	3.562500	−0.250000	−0.031250
4	16.500000	−1.250000	−0.250000	−258.000000	23.687500	5.843750	−0.250000	−0.031250
5	26.000000	−1.250000	−0.250000	−565.750000	33.718750	8.625000	−0.250000	−0.031250

/119

i	$\Pi'^{2,1}_{2,-1}$	D	D^2	D^3	Π'^{1}_{1}	D
−5	+97.875000	−4.312500	−1.125000	+0.062500	+6.000000	−0.500000
−4	54.687500	−2.250000	−1.000000	0.062500	5.000000	−0.500000
−3	26.250000	−0.687500	−0.875000	0.062500	4.000000	−0.500000
−2	9.562500	+0.375000	−0.750000	0.062500	3.000000	−0.500000
−1	+1.625000	0.937500	−0.625000	0.062500	2.000000	−0.500000
0	−0.562500	1.000000	−0.500000	0.062500	+1.000000	−0.500000
1	0.000000	+0.562500	−0.375000	0.062500	0.000000	−0.500000
2	+0.312500	−0.375000	−0.250000	0.062500	−1.000000	−0.500000
3	−2.625000	−1.812500	−0.125000	0.062500	−2.000000	−0.500000
4	−11.812500	−3.750000	0.000000	0.062500	−3.000000	−0.500000
5	−30.250000	−6.187500	+0.125000	0.062500	−4.000000	−0.500000

i	$\Pi'^{1,2}_{1,0}$	D	D^2	D^3
−5	−96.000000	+9.500000	+1.375000	−0.125000
−4	−45.000000	5.750000	1.125000	−0.125000
−3	−16.000000	3.000000	0.875000	−0.125000
−2	−3.000000	1.250000	0.625000	−0.125000
−1	0.000000	0.500000	0.375000	−0.125000
0	−1.000000	0.750000	+0.125000	−0.125000
1	0.000000	2.000000	−0.125000	−0.125000
2	+9.000000	4.250000	−0.375000	−0.125000
3	32.000000	7.500000	−0.625000	−0.125000
4	75.000000	11.750000	−0.875000	−0.125000
5	144.000000	17.000000	−1.125000	−0.125000

i	$\Pi'^{0,1}_{0,1}$	D	$\Pi'^{0,3}_{0,1}$	D	D^2	D^3
−5	−3.500000	+0.500000	+19.187500	−3.625000	−0.250000	+0.062000
−4	−2.500000	0.500000	6.500000	−1.937500	−0.125000	0.062000
−3	−1.500000	0.500000	+1.062500	−0.750000	0.000000	0.062000
−2	−0.500000	0.500000	−0.125000	−0.062500	+0.125000	0.062000
−1	+0.500000	0.500000	−0.062500	+0.125000	0.250000	0.062000
0	1.500000	0.500000	−1.750000	−0.187500	0.375000	0.062000
1	2.500000	0.500000	−8.187500	−1.000000	0.500000	0.062000
2	3.500000	0.500000	−22.375000	−2.312500	0.625000	0.062000
3	4.500000	0.500000	−47.312500	−4.125000	0.750000	0.062000
4	5.500000	0.500000	−86.000000	−6.437500	0.875000	0.062000
5	6.500000	0.500000	−141.437500	−9.250000	1.000000	0.062000

i	$\Pi'^{0,5}_{0,1}$	D	D^2	D^3	D^4	D^5
−5	−29.039063	+8.565104	+0.367188	−0.294271	+0.007812	+0.002604
−4	−3.552083	2.057292	−0.096354	−0.132812	0.013021	0.002604
−3	+0.096354	+0.054688	−0.122396	−0.013021	0.018229	0.002604
−2	−0.062500	−0.067708	+0.039062	+0.065104	0.023437	0.002604
−1	+0.002604	+0.065104	+0.138021	0.101562	0.028646	0.002604
0	0.322917	−0.171875	−0.075521	0.096354	0.033854	0.002604
1	6.929688	−0.403646	−0.851563	+0.049479	0.039062	0.002604
2	41.854167	+0.744792	−2.440104	−0.039062	0.044271	0.002604
3	153.127604	5.648438	−5.091146	−0.169271	0.049479	0.002604
4	424.781250	17.682292	−9.054688	−0.341146	0.054687	0.002604
5	986.846354	41.221355	−14.580729	−0.554687	0.059896	0.002604

/120

i	$\Pi'^{2,1}_{0,1}$	D	D^2	D^3	$\Pi'^{1,2}_{-1,2}$	D	D^2	D^3
−5	+126.000000	−18.875000	−0.750000	+0.125000	−24.000000	6.250000	−0.062500	−0.062500
−4	62.500000	−13.125000	−0.500000	0.125000	−8.125000	3.562500	−0.187500	−0.062500
−3	24.000000	−8.375000	−0.250000	0.125000	−1.000000	+1.375000	−0.312500	−0.062500
−2	+4.500000	−4.625000	0.000000	0.125000	+0.375000	−0.312500	−0.437500	−0.062500
−1	−2.000000	−1.875000	+0.250000	0.125000	−1.000000	−1.500000	−0.562500	−0.062500
0	−1.500000	−0.125000	0.500000	0.125000	−2.125000	−2.187500	−0.687500	−0.062500
1	0.000000	+0.625000	0.750000	0.125000	0.000000	−2.375000	−0.812500	−0.062500
2	−3.500000	+0.375000	1.000000	0.125000	+8.375000	−2.062500	−0.937500	−0.062500
3	−18.000000	−0.875000	1.250000	0.125000	26.000000	−1.250000	−1.062500	−0.062500
4	−49.500000	−3.125000	1.500000	0.125000	55.875000	+0.062500	−1.187500	−0.062500
5	−104.000000	−6.375000	1.750000	0.125000	101.000000	1.875000	−1.312500	−0.062500

i	$\Pi'^{2,1}_{-2,1}$	D	D^2	D^3	Π'^{1}_{-1}	D
−5	−49.875000	−2.062500	+0.875000	+0.062500	−6.000000	−0.500000
−4	−23.437500	−0.625000	0.750000	0.062500	−5.000000	−0.500000
−3	−8.250000	+0.312500	0.625000	0.062500	−4.000000	−0.500000
−2	−1.312500	0.750000	0.500000	0.062500	−3.000000	−0.500000
−1	+0.375000	0.687500	0.375000	0.062500	−2.000000	−0.500000
0	−0.187500	+0.125000	0.250000	0.062500	−1.000000	−0.500000
1	0.000000	−0.937500	+0.125000	0.062500	0.000000	−0.500000
2	+3.937500	−2.500000	0.000000	0.062500	+1.000000	−0.500000
3	14.625000	−4.562500	−0.125000	0.062500	2.000000	−0.500000
4	35.062500	−7.125000	−0.250000	0.062500	3.000000	−0.500000
5	68.250000	−10.187500	−0.375000	0.062500	4.000000	−0.500000

i	$\Pi'^{1,2}_{-1,0}$	D	D^2	D^3	$\Pi'^{0,1}_{0,-1}$	D
−5	+96.000000	+6.500000	−1.625000	−0.125000	+4.500000	+0.500000
−4	45.000000	3.250000	−1.375000	−0.125000	3.500000	0.500000
−3	16.000000	+1.000000	−1.125000	−0.125000	2.500000	0.500000
−2	3.000000	−0.250000	−0.875000	−0.125000	1.500000	0.500000
−1	0.000000	−0.500000	−0.625000	−0.125000	+0.500000	0.500000
0	+1.000000	+0.250000	−0.375000	−0.125000	−0.500000	0.500000
1	0.000000	2.000000	−0.125000	−0.125000	−1.500000	0.500000
2	−9.000000	4.750000	+0.125000	−0.125000	−2.500000	0.500000
3	−32.000000	8.500000	0.375000	−0.125000	−3.500000	0.500000
4	−75.000000	13.250000	0.625000	−0.125000	−4.500000	0.500000
5	−144.000000	19.000000	0.875000	−0.125000	−5.500000	0.500000

i	$\Pi'^{2,1}_{0,-1}$	D	D^2	D^3	$\Pi'^{0,3}_{0,-1}$	D	D^2	D^3
−5	−162.000000	−16.875000	+1.250000	+0.125000	−47.312500	−4.125000	+0.750000	+0.062500
−4	−87.500000	−11.625000	1.000000	0.125000	−22.375000	−2.312500	0.625000	0.062500
−3	−40.000000	−7.375000	0.750000	0.125000	−8.187500	−1.000000	0.500000	0.062500
−2	−13.500000	−4.125000	0.500000	0.125000	−1.750000	−0.187500	0.375000	0.062500
−1	−2.000000	−1.875000	+0.250000	0.125000	−0.062500	+0.125000	0.250000	0.062500
0	+0.500000	−0.625000	0.000000	0.125000	−0.125000	−0.062500	+0.125000	0.062500
1	0.000000	−0.375000	−0.250000	0.125000	+1.062500	−0.750000	0.000000	0.062500
2	2.500000	−1.125000	−0.500000	0.125000	6.500000	−1.937500	−0.125000	0.062500
3	14.000000	−2.875000	−0.750000	0.125000	19.187500	−3.625000	−0.250000	0.062500
4	40.500000	−5.625000	−1.000000	0.125000	42.125000	−5.812500	−0.375000	0.062500
5	88.000000	−9.375000	−1.250000	0.125000	78.312500	−8.500000	−0.500000	0.062500

i	$\Pi'^{0,5}_{0,-1}$	D	D^2	D^3	D^4	D^5
−5	+153.127606	+ 5.648438	−5.091146	−0.169271	+0.049479	+0.002604
−4	41.854167	+ 0.744792	−2.440104	−0.039062	0.044271	0.002604
−3	6.929688	− 0.403646	−0.851563	+0.049479	0.039062	0.002604
−2	0.322917	− 0.171875	−0.075521	0.096354	0.033854	0.002604
−1	+ 0.002604	+ 0.065104	+0.138021	0.101562	0.028646	0.002604
0	− 0.062500	− 0.067708	+0.039062	+0.065104	0.023437	0.002604
1	+ 0.096354	+ 0.054688	−0.122396	−0.013021	0.018229	0.002604
2	− 3.552083	2.057292	−0.096354	−0.132812	0.013021	0.002604
3	− 29.039063	8.565104	+0.367188	−0.294271	0.007812	0.002604
4	−118.395835	23.203125	1.518229	−0.497396	+0.002604	0.002604
5	−347.653650	50.596355	3.606771	−0.742188	−0.002604	0.002604

i	$\Pi'^{1,2}_{1,-2}$	D	D^2	D^3
−5	+78.000000	+9.250000	−0.562500	−0.062500
−4	41.875000	6.437500	−0.437500	−0.062500
−3	19.000000	4.125000	−0.312500	−0.062500
−2	6.375000	2.312500	−0.187500	−0.062500
−1	+ 1.000000	1.000000	−0.062500	−0.062500
0	− 0.125000	+0.187500	+0.062500	−0.062500
1	0.000000	−0.125000	0.187500	−0.062500
2	− 1.625000	+0.062500	0.312500	−0.062500
3	− 8.000000	0.750000	0.437500	−0.062500
4	−22.125000	1.937500	0.562500	−0.062500
5	−47.000000	3.625000	0.687500	−0.062500

i	Π'^{2}_{2}	D	D^2	$\Pi'^{1,1}_{1,1}$	D	D^2
−5	+ 21.750000	−3.375000	+0.125000	−21.000000	+4.750000	−0.250000
−4	15.625000	−2.875000	0.125000	−12.500000	3.750000	−0.250000
−3	10.500000	−2.375000	0.125000	− 6.000000	2.750000	−0.250000
−2	6.375000	−1.875000	0.125000	− 1.500000	1.750000	−0.250000
−1	3.250000	−1.375000	0.125000	+ 1.000000	+0.750000	−0.250000
0	+ 1.125000	−0.875000	0.125000	+ 1.500000	−0.250000	−0.250000
1	0.000000	−0.375000	0.125000	0.000000	−1.250000	−0.250000
2	− 0.125000	+0.125000	0.125000	− 3.500000	−2.250000	−0.250000
3	+ 0.750000	0.625000	0.125000	− 9.000000	−3.250000	−0.250000
4	2.625000	1.125000	0.125000	−16.500000	−4.250000	−0.250000
5	5.500000	1.625000	0.125000	−26.000000	−5.250000	−0.250000

i	$\Pi'^{1,3}_{1,1}$	D	D^2	D^3	D^4	$\Pi'^{0,2}_{0,2}$	D	D^2
−5	+115.125000	− 31.343750	+0.312500	+0.500000	−0.031250	+ 4.000000	−1.375000	+0.125000
−4	32.500000	− 12.937500	0.343750	0.375000	−0.031250	1.625000	−0.875000	0.125000
−3	+ 4.250000	− 3.531250	0.375000	0.250000	−0.031250	+ 0.250000	−0.375000	0.125000
−2	− 0.375000	− 0.125000	0.406250	+0.125000	−0.031250	− 0.125000	+0.125000	0.125000
−1	− 0.125000	+ 0.281250	0.437500	0.000000	−0.031250	+ 0.500000	0.625000	0.125000
0	− 1.750000	0.687500	0.468750	−0.125000	−0.031250	2.125000	1.125000	0.125000
1	0.000000	4.093750	0.500000	−0.250000	−0.031250	4.750000	1.625000	0.125000
2	+ 22.375000	13.500000	0.531250	−0.375000	−0.031250	8.375000	2.125000	0.125000
3	94.625000	31.906250	0.562500	−0.500000	−0.031250	13.000000	2.625000	0.125000
4	258.000000	62.312500	0.593750	−0.625000	−0.031250	18.625000	3.125000	0.125000
5	565.750000	107.718750	0.625000	−0.750000	−0.031250	25.250000	3.625000	0.125000

/122

i	$\Pi'^{0,4}_{0,2}$	D	D^2	D^3	D^4
−5	− 8.250000	+ 3.854167	−0.385417	−0.062500	+0.010417
−4	− 0.854167	+ 0.885417	−0.229167	−0.020833	0.010417
−3	+ 0.125000	− 0.083333	−0.072917	+0.020833	0.010417
−2	− 0.062500	− 0.052083	+0.083333	0.062500	0.010417
−1	− 0.166667	− 0.020833	0.239583	0.104167	0.010417
0	− 2.937500	− 0.989583	0.395833	0.145833	0.010417
1	− 15.125000	− 3.958333	0.552083	0.187500	0.010417
2	− 47.479167	− 9.927083	0.708333	0.229167	0.010417
3	−114.750000	−19.895833	0.864583	0.270833	0.010417
4	−235.687500	−34.864583	1.020833	0.312500	0.010417
5	−433.041667	−55.833333	1.177083	0.354167	0.010417

i	$\Pi'^{1,3}_{-1,3}$	D	D^2	D^3	D^4	Π'^{2}_{-2}	D	D^2
−5	+ 9.125000	− 5.489583	+0.979167	0.000000	−0.010417	+14.250000	+2.625000	+0.125000
−4	+ 0.625000	− 1.083333	0.510416	−0.041667	−0.010417	9.375000	2.125000	0.125000
−3	− 0.083333	+ 0.322917	+0.041667	−0.083333	−0.010417	5.500000	1.625000	0.125000
−2	+ 0.250000	− 0.270833	−0.427083	−0.125000	−0.010417	2.625000	1.125000	0.125000
−1	− 1.125000	− 1.864583	−0.895833	−0.166667	−0.010417	+ 0.750000	0.625000	0.125000
0	− 2.958333	− 3.458333	−1.364583	−0.208333	−0.010417	− 0.125000	+0.125000	0.125000
1	0.000000	− 4.052083	−1.833333	−0.250000	−0.010417	0.000000	−0.375000	0.125000
2	+ 17.000000	− 2.645833	−2.302083	−0.291667	−0.010417	+ 1.125000	−0.875000	0.125000
3	61.291667	+ 1.760417	−2.770833	−0.333333	−0.010417	3.250000	−1.375000	0.125000
4	150.125000	10.166667	−3.239583	−0.375000	−0.010417	6.375000	−1.875000	0.125000
5	304.750000	23.572917	−3.708333	−0.416667	−0.010417	10.500000	−2.375000	0.125000

i	$\Pi'^{1,1}_{-1,-1}$	D	D^2	$\Pi'^{1,3}_{-1,-1}$	D	D^2	D^3	D^4
−5	−27.000000	−5.250000	−0.250000	+283.875000	+48.406250	−2.437500	−0.750000	−0.031250
−4	−17.500000	−4.250000	−0.250000	111.875000	22.750000	−1.968750	−0.625000	−0.031250
−3	−10.000000	−3.250000	−0.250000	32.750000	8.093750	−1.500000	−0.500000	−0.031250
−2	− 4.500000	−2.250000	−0.250000	5.250000	+ 1.437500	−1.031250	−0.375000	−0.031250
−1	− 1.000000	−1.250000	−0.250000	0.125000	− 0.218750	−0.562500	−0.250000	−0.031250
0	+ 0.500000	−0.250000	−0.250000	0.125000	+ 0.125000	−0.093750	−0.125000	−0.031250
1	0.000000	+0.750000	−0.250000	0.000000	− 0.531250	+0.375000	0.000000	−0.031250
2	− 2.500000	1.750000	−0.250000	6.500000	− 5.187500	0.843750	+0.125000	−0.031250
3	− 7.000000	2.750000	−0.250000	38.375000	−16.843750	1.312500	0.250000	−0.031250
4	−13.500000	3.750000	−0.250000	126.375000	−38.500000	1.781250	0.375000	−0.031250
5	−22.000000	4.750000	−0.250000	313.250000	−73.156250	2.250000	0.500000	−0.031250

i	$\Pi'^{0,2}_{0,-2}$	D	D^2	$\Pi'^{0,4}_{0,-2}$	D	D^2	D^3	D^4
−5	+13.000000	+2.625000	+0.125000	−114.750000	−19.895833	+0.864583	+0.270833	+0.010417
−4	8.375000	2.125000	0.125000	− 47.479167	− 9.927083	0.708333	0.229167	0.010417
−3	4.750000	1.625000	0.125000	− 15.125000	− 3.958333	0.552083	0.187500	0.010417
−2	2.125000	1.125000	0.125000	− 2.937500	− 0.989583	0.395833	0.145833	0.010417
−1	+ 0.500000	0.625000	0.125000	− 0.166667	− 0.020833	0.239583	0.104167	0.010417
0	− 0.125000	+0.125000	0.125000	− 0.062500	− 0.052083	+0.083333	0.062500	0.010417
1	+ 0.250000	−0.375000	0.125000	+ 0.125000	− 0.083333	−0.072917	+0.020833	0.010417
2	1.625000	−0.875000	0.125000	− 0.854167	+ 0.885417	−0.229167	−0.020833	0.010417
3	4.000000	−1.375000	0.125000	− 8.250000	3.854167	−0.385417	−0.062500	0.010417
4	7.375000	−1.875000	0.125000	− 31.312500	9.822917	−0.541667	−0.104167	0.010417
5	11.750000	−2.375000	0.125000	− 83.291667	19.791667	−0.697917	−0.145833	0.010417

/123

i	$\Pi'^{1,3}_{1,-3}$	D	D^2	D^3	D^4
−5	+183.875000	+35.927083	+0.229167	−0.250000	−0.010417
−4	85.000000	20.770833	0.197917	−0.208333	−0.010417
−3	32.416667	10.614583	0.166667	−0.166667	−0.010417
−2	8.875000	4.458333	0.135417	−0.125000	−0.010417
−1	+ 1.125000	1.302083	0.104167	−0.083333	−0.010417
0	− 0.083333	+ 0.145833	0.072917	−0.041667	−0.010417
1	0.000000	− 0.010417	0.041667	0.000000	−0.010417
2	+ 0.125000	− 0.166667	+0.010417	+0.041667	−0.010417
3	3.041667	− 1.322917	−0.020833	0.083333	−0.010417
4	15.500000	− 4.479167	−0.052083	0.125000	−0.010417
5	48.250000	−10.635417	−0.083333	0.166667	−0.010417

i	$\Pi'^{2,1}_{2,1}$	D	D^2	D^3	$\Pi'^{1,2}_{1,2}$	D	D^2	D^3
−5	−76.125000	+22.687500	−2.125000	+0.062500	+ 24.000000	−10.250000	+1.437500	−0.062500
−4	−39.062500	15.000000	−1.750000	0.062500	8.125000	− 5.187500	1.062500	−0.062500
−3	−15.750000	8.812500	−1.375000	0.062500	+ 1.000000	− 1.625000	0.687500	−0.062500
−2	− 3.187500	4.125000	−1.000000	0.062500	− 0.375000	+ 0.437500	+0.312500	−0.062500
−1	+ 1.625000	+ 0.937500	−0.625000	0.062500	+ 1.000000	1.000000	−0.062500	−0.062500
0	+ 1.687500	− 0.750000	−0.250000	0.062500	+ 2.125000	+ 0.062500	−0.437500	−0.062500
1	0.000000	− 0.937500	+0.125000	0.062500	0.000000	− 2.375000	−0.812500	−0.062500
2	− 0.437500	+ 0.375000	0.500000	0.062500	− 8.375000	− 6.312500	−1.187500	−0.062500
3	+ 3.375000	3.187500	0.875000	0.062500	− 26.000000	−11.750000	−1.562500	−0.062500
4	14.437500	7.500000	1.250000	0.062500	− 55.875000	−18.687500	−1.937500	−0.062500
5	35.750000	13.312500	1.625000	0.062500	−101.000000	−27.125000	−2.312500	−0.062500

i	$\Pi'^{0,3}_{0,3}$	D	D^2	D^3
−5	− 1.520833	+ 1.041667	−0.250000	+0.020833
−4	− 0.125000	+ 0.229167	−0.125000	0.020833
−3	+ 0.020833	− 0.083333	0.000000	0.020833
−2	− 0.083333	+ 0.104167	+0.125000	0.020833
−1	+ 0.562500	0.791667	0.250000	0.020833
0	2.958333	1.979167	0.375000	0.020833
1	8.104167	3.666667	0.500000	0.020833
2	17.000000	5.854167	0.625000	0.020833
3	30.645833	8.541667	0.750000	0.020833
4	50.041667	11.729167	0.875000	0.020833
5	76.187500	15.416667	1.000000	0.020833

i	$\Pi'^{0,5}_{0,3}$	D	D^2	D^3	D^4	D^5
−5	+ 0.548177	− 0.769531	+0.329427	−0.037760	−0.006510	+0.001302
−4	− 0.093750	+ 0.117187	−0.001302	−0.024740	+0.001302	0.001302
−3	+ 0.037760	− 0.040365	−0.050781	+0.009115	0.009115	0.001302
−2	− 0.041667	− 0.054687	+0.055990	0.063802	0.016927	0.001302
−1	− 0.316406	− 0.238281	0.194010	0.139323	0.024740	0.001302
0	− 4.770833	− 2.403646	0.238281	0.235677	0.032552	0.001302
1	− 26.389322	− 9.863281	+0.063803	0.352865	0.040365	0.001302
2	− 92.156248	− 27.429687	−0.454427	0.490885	0.048177	0.001302
3	− 248.055983	− 61.415363	−1.441406	0.649740	0.055990	0.001302
4	− 564.072917	−119.632812	−3.022135	0.829427	0.063802	0.001302
5	−1139.191407	−211.394531	−5.321614	1.029948	0.071615	0.001302

/124

i	$\Pi'^{2,1}_{-2,-1}$	D	D^2	D^3	$\Pi'^{1,2}_{-1,-2}$	D	D^2	D^3
5	+64.125000	+18.937500	+1.875000	+0.062500	−78.000000	−22.250000	−2.062500	−0.062500
4	32.812500	12.125000	1.500000	0.062500	−41.875000	−14.812500	−1.687500	−0.062500
3	13.750000	6.812500	1.125000	0.062500	−19.000000	− 8.875000	−1.312500	−0.062500
2	3.937500	3.000000	0.750000	0.062500	− 6.375000	− 4.437500	−0.937500	−0.062500
1	0.375000	+ 0.687500	+0.375000	0.062500	− 1.000000	− 1.500000	−0.562500	−0.062500
0	+ 0.062500	− 0.125000	0.000000	0.062500	+ 0.125000	− 0.062500	−0.187500	−0.062500
1	0.000000	+ 0.562500	−0.375000	0.062500	0.000000	− 0.125000	+0.187500	−0.062500
2	− 2.812500	2.750000	−0.750000	0.062500	1.625000	− 1.687500	0.562500	−0.062500
3	−11.375000	6.437500	−1.125000	0.062500	8.000000	− 4.750000	0.937500	−0.062500
4	−28.687500	11.625000	−1.500000	0.062500	22.125000	− 9.312500	1.312500	−0.062500
5	−57.750000	18.312500	−1.875000	0.062500	47.000000	−15.375000	1.687500	−0.062500

i	$\Pi'^{0,3}_{0,-3}$	D	D^2	D^3
−5	+30.645833	+8.541667	+0.750000	+0.020833
−4	17.000000	5.854167	0.625000	0.020833
−3	8.104167	3.666667	0.500000	0.020833
−2	2.958333	1.979167	0.375000	0.020833
−1	+ 0.562500	0.791667	0.250000	0.020833
0	− 0.083333	+0.104167	+0.125000	0.020833
1	+ 0.020833	−0.083333	0.000000	0.020833
2	− 0.125000	+0.229167	−0.125000	0.020833
3	− 1.520833	1.041667	−0.250000	0.020833
4	− 5.166667	2.354167	−0.375000	0.020833
5	−12.062500	4.166667	−0.500000	0.020833

i	$\Pi'^{0,5}_{0,-3}$	D	D^2	D^3	D^4	D^5
−5	−248.055983	−61.415363	−1.441406	+0.649740	+0.055990	+0.001302
−4	− 92.156248	−27.429687	−0.454427	0.490885	0.048177	0.001302
−3	− 26.389322	− 9.863281	+0.063803	0.352865	0.040365	0.001302
−2	− 4.770833	− 2.403646	0.238281	0.235677	0.032552	0.001302
−1	− 0.316406	− 0.238281	0.194010	0.139323	0.024740	0.001302
0	− 0.041667	− 0.054687	+0.055990	0.063802	0.016927	0.001302
1	+ 0.037760	− 0.040365	−0.050781	+0.009115	0.009115	0.001302
2	− 0.093750	+ 0.117187	−0.001302	−0.024740	+0.001302	0.001302
3	+ 0.548177	− 0.769531	+0.329427	−0.037760	−0.006510	0.001302
4	8.947916	− 5.388021	1.066406	−0.029948	−0.014323	0.001302
5	43.089843	−17.925781	2.334635	−0.001302	−0.022135	0.001302

i	$\Pi'^{1,3}_{1,3}$	D	D^2	D^3	D^4
−5	− 9.125000	+ 7.010417	− 2.020833	+0.250000	−0.010417
−4	− 0.625000	+ 1.208333	− 0.739583	0.166667	−0.010417
−3	+ 0.083333	− 0.343750	+ 0.041667	+0.083333	−0.010417
−2	− 0.250000	+ 0.354167	0.322917	0.000000	−0.010417
−1	+ 1.125000	1.302083	+ 0.104167	−0.083333	−0.010417
0	+ 2.958333	+ 0.500000	− 0.614583	−0.166667	−0.010417
1	0.000000	− 4.052083	− 1.833333	−0.250000	−0.010417
2	− 17.000000	−14.354167	− 3.552083	−0.333333	−0.010417
3	− 61.291667	−32.406250	− 5.770833	−0.416667	−0.010417
4	−150.125000	−60.208333	− 8.489583	−0.500000	−0.010417
5	−304.750000	−99.760417	−11.708333	−0.583333	−0.010417

/125

$\Pi'^{0,4}_{0,4}$

i		D	D^2	D^3	D^4
−5	+ 0.062500	− 0.130208	+0.091146	−0.026042	+0.002604
−4	+ 0.002604	+ 0.031250	−0.018229	−0.005208	0.002604
−3	0.000000	− 0.057292	−0.002604	+0.015625	0.002604
−2	− 0.070312	+ 0.104167	+0.138021	0.036458	0.002604
−1	+ 0.666667	1.015625	0.403646	0.057292	0.002604
0	4.085938	3.177083	0.794271	0.078125	0.002604
1	13.062500	7.088542	1.309896	0.098958	0.002604
2	31.471354	13.250000	1.950521	0.119792	0.002604
3	64.187500	22.161458	2.716146	0.140625	0.002604
4	117.085938	34.322917	3.606771	0.161458	0.002604
5	197.041667	50.234375	4.622396	0.182292	0.002604

$\Pi'^{1,3}_{-1,-3}$

i		D	D^2	D^3	D^4
−5	−183.875000	−66.572917	−8.770833	−0.500000	−0.010417
−4	− 85.000000	−37.770833	−6.052083	−0.416667	−0.010417
−3	− 32.416667	−18.718750	−3.833333	−0.333333	−0.010417
−2	− 8.875000	− 7.416667	−2.114583	−0.250000	−0.010417
−1	− 1.125000	− 1.864583	−0.895833	−0.166667	−0.010417
0	+ 0.083333	− 0.062500	−0.177083	−0.083333	−0.010417
1	0.000000	− 0.010417	+0.041667	0.000000	−0.010417
2	− 0.125000	+ 0.291667	−0.239583	+0.083333	−0.010417
3	− 3.041667	2.843750	−1.020833	0.166667	−0.010417
4	− 15.500000	9.645833	−2.302083	0.250000	−0.010417
5	− 48.250000	22.697917	−4.083333	0.333333	−0.010417

$\Pi'^{0,4}_{0,-4}$

i		D	D^2	D^3	D^4
−5	+64.187500	+22.161459	+2.716146	+0.140625	+0.002604
−4	31.471355	13.250000	1.950521	0.119792	0.002604
−3	13.062500	7.088542	1.309896	0.098958	0.002604
−2	4.085938	3.177083	0.794271	0.078125	0.002604
−1	+ 0.666667	1.015625	0.403646	0.057292	0.002604
0	− 0.070312	+ 0.104167	+0.138021	0.036458	0.002604
1	0.000000	− 0.057292	−0.002604	+0.015625	0.002604
2	+ 0.002604	+ 0.031250	−0.018229	−0.005208	0.002604
3	0.062500	− 0.130208	+0.091146	−0.026042	0.002604
4	1.304688	− 1.041667	0.325521	−0.046875	0.002604
5	5.854167	− 3.203125	0.684896	−0.067708	0.002604

$\Pi'^{0,5}_{0,5}$

i		D	D^2	D^3	D^4	D^5
−5	− 0.004948	− 0.008073	+ 0.011719	−0.001302	−0.001302	+0.000260
−4	+ 0.008333	+ 0.014062	− 0.011719	−0.003906	+0.001302	0.000260
−3	− 0.007031	− 0.049740	− 0.003906	+0.014323	0.003906	0.000260
−2	− 0.066667	+ 0.113021	+ 0.160156	0.053385	0.006510	0.000260
−1	+ 0.813802	1.314844	0.605469	0.113281	0.009115	0.000260
0	5.618750	4.868229	1.457031	0.194010	0.011719	0.000260
1	20.332552	12.585677	2.839844	0.295573	0.014323	0.000260
2	54.939584	26.779688	4.878906	0.417969	0.016927	0.000260
3	124.424220	50.262761	7.699219	0.561198	0.019531	0.000260
4	249.770833	86.347397	11.425781	0.725260	0.022135	0.000260
5	458.963802	138.846094	16.183594	0.910156	0.024740	0.000260

/126

i	$\Pi'^{0,5}_{0,-5}$					
		D	D^2	D^3	D^4	D^5
−5	+124.424220	+50.262761	−7.699219	+0.561198	+0.019531	+0.000260
−4	54.939584	26.779688	4.878906	0.417969	0.016927	0.000260
−3	20.332552	12.585677	2.839844	0.295573	0.014323	0.000260
−2	5.618750	4.868229	1.457031	0.194010	0.011719	0.000260
−1	+ 0.813802	1.314844	0.605469	0.113281	0.009115	0.000260
0	− 0.066667	+ 0.113021	+0.160156	0.053385	0.006510	0.000260
1	− 0.007031	− 0.049740	−0.003906	+0.014323	0.003906	0.000260
2	+ 0.008333	+ 0.014062	−0.011719	−0.003906	+0.001302	0.000260
3	− 0.004948	− 0.008073	−0.011719	−0.001302	−0.001302	0.000260
4	− 0.031250	+ 0.071354	−0.058594	+0.022135	−0.003906	0.000260
5	− 1.054948	0.939844	−0.347656	0.066406	−0.006510	0.000260

3. Operators of Second Class

i	$\Pi''^{1,1}_{1,-1}$			Π''^{0}_{0}
		D	D^2	
−3	+7.500000	+1.750000	−0.250000	+1.000000
−2	+2.000000	1.750000	−0.250000	1.000000
−1	−1.500000	1.750000	−0.250000	1.000000
0	−3.000000	1.750000	−0.250000	1.000000
1	−2.500000	1.750000	−0.250000	1.000000
2	0.000000	1.750000	−0.250000	1.000000
3	+4.500000	1.750000	−0.250000	1.000000

i	$\Pi''^{0,2}_{0,0}$			$\Pi''^{1,1}_{-1,1}$		
		D	D^2		D	D^2
−3	− 1.000000	+0.250000	+0.250000	+2.500000	−2.250000	−0.250000
−2	0.000000	0.250000	0.250000	−2.000000	−2.250000	−0.250000
−1	− 1.000000	0.250000	0.250000	−4.500000	−2.250000	−0.250000
0	− 4.000000	0.250000	0.250000	−5.000000	−2.250000	−0.250000
1	− 9.000000	0.250000	0.250000	−3.500000	−2.250000	−0.250000
2	−16.000000	0.250000	0.250000	0.000000	−2.250000	−0.250000
3	−25.000000	0.250000	0.250000	+5.500000	−2.250000	−0.250000

i	Π''^{1}_{1}		$\Pi''^{0,1}_{0,1}$	
		D		D
−3	+5.000000	−0.500000	−0.500000	+0.500000
−2	4.000000	−0.500000	+0.500000	0.500000
−1	3.000000	−0.500000	1.500000	0.500000
0	2.000000	−0.500000	2.500000	0.500000
1	+1.000000	−0.500000	3.500000	0.500000
2	0.000000	−0.500000	4.500000	0.500000
3	−1.000000	−0.500000	5.500000	0.500000

/127

i	$\Pi''^{0,3}_{0,1}$	D	D^2	D^3	Π''^{1}_{-1}	D
−3	− 0.125000	−0.062500	+ 0.125000	+0.062500	−5.000000	−0.500000
−2	− 0.062500	+0.125000	0.250000	0.062500	−4.000000	−0.500000
−1	− 1.750000	−0.187500	0.375000	0.062500	−3.000000	−0.500000
0	− 8.187500	−1.000000	0.500000	0.062500	−2.000000	−0.500000
1	−22.375000	−2.312500	0.625000	0.062500	−1.000000	−0.500000
2	− 47.312500	−4.125000	0.750000	0.062500	0.000000	−0.500000
3	−86.000000	−6.437500	0.875000	0.062500	+1.000000	−0.500000

i	$\Pi''^{0,1}_{0,-1}$	D	$\Pi''^{0,3}_{0,-1}$	D	D^2	D^3
−3	+1.500000	+0.500000	− 1.750000	−0.187500	+0.375000	+0.062500
−2	+0.500000	0.500000	− 0.062500	+0.125000	0.250000	0.062500
−1	−0.500000	0.500000	− 0.125000	−0.062500	+0.125000	0.062500
0	−1.500000	0.500000	+ 1.062500	−0.750000	0.000000	0.062500
1	−2.500000	0.500000	6.500000	−1.937500	−0.125000	0.062500
2	−3.500000	0.500000	19.187500	−3.625000	−0.250000	0.062500
3	−4.500000	0.500000	42.125000	−5.812500	−0.375000	0.062500

i	$\Pi''^{1,1}_{1,1}$	D	D^2	$\Pi''^{0,2}_{0,2}$	D	D^2
−3	−2.500000	+2.750000	−0.250000	− 0.125000	+0.125000	+0.125000
−2	+2.000000	1.750000	−0.250000	+ 0.500000	0.625000	0.125000
−1	4.500000	+0.750000	−0.250000	2.125000	1.125000	0.125000
0	5.000000	−0.250000	−0.250000	4.750000	1.625000	0.125000
1	+3.500000	−1.250000	−0.250000	8.375000	2.125000	0.125000
2	0.000000	−2.250000	−0.250000	13.000000	2.625000	0.125000
3	−5.500000	−3.250000	−0.250000	18.625000	3.125000	0.125000

i	$\Pi''^{1,1}_{-1,-1}$	D	D^2	$\Pi''^{0,2}_{0,-2}$	D	D^2
−3	−7.500000	−3.250000	−0.250000	+2.125000	+1.125000	+0.125000
−2	−2.000000	−2.250000	−0.250000	+0.500000	0.625000	0.125000
−1	+1.500000	−1.250000	−0.250000	−0.125000	+0.125000	0.125000
0	3.000000	−0.250000	−0.250000	+0.250000	−0.375000	0.125000
1	+2.500000	+0.750000	−0.250000	1.625000	−0.875000	0.125000
2	0.000000	1.750000	−0.250000	4.000000	−1.375000	0.125000
3	−4.500000	2.750000	−0.250000	7.375000	−1.875000	0.125000

i	$\Pi''^{0,3}_{0,3}$	D	D^2	D^3	$\Pi''^{0,3}_{0,-3}$	D	D^2	D^3
−3	− 0.083333	+0.104167	+0.125000	+0.020833	+2.958333	+1.979167	+0.375000	+0.020833
−2	+ 0.562500	0.791667	0.250000	0.020833	+0.562500	0.791667	0.250000	0.020833
−1	2.958333	1.979167	0.375000	0.020833	−0.083333	+0.104167	+0.125000	0.020833
0	8.104167	3.666667	0.500000	0.020833	+0.020833	−0.083333	0.000000	0.020833
1	17.000000	5.854167	0.625000	0.020833	−0.125000	+0.229167	−0.125000	0.020833
2	30.645833	8.541667	0.750000	0.020833	−1.520833	1.041667	−0.250000	0.020833
3	50.041667	11.729167	0.875000	0.020833	−5.166667	2.354167	−0.375000	0.020833

III. PARTIAL VALUES OF OPERATORS FOR COMPUTING CORRECTIONS FOR SECOND TERM

(Operators from -1, +2, -4, +8, -16, +32, -64, +128)

/128

	i			i			i			i	
Π_0^2	−1	0.500000000	$\Pi_{0,1}^{0,1}$	−1	1.500000000	$\Pi_{0,2}^{0,2}$	−1	−0.125000000	$\Pi_{-1,4}^{1,4}$	−1	0.000000000
	+1	0.500000000		+1	−0.500000000		+1	−0.375000000		+1	−0.651041667
$\Pi_{0,0}^{0,2}$	−1	0.500000000	$\Pi_{0,1}^{0,3}$	−1	0.000000000	$\Pi_{0,2}^{0,4}$	−1	−0.041666667	$\Pi_{3,1}^{3,1}$	−1	8.000000000
	+1	0.500000000		+1	0.375000000		+1	0.375000000		+1	−0.083333333
$\Pi_{0,0}^{0,4}$	−1	0.015625000	$\Pi_{0,1}^{0,5}$	−1	0.000000000	$\Pi_{0,2}^{0,6}$	−1	−0.024414062	$\Pi_{2,2}^{2,2}$	−1	−0.421875000
	+1	0.015625000		+1	−0.052083333		+1	−0.108398438		+1	−0.046875000
$\Pi_{0,0}^{0,6}$	−1	0.025173611	$\Pi_{0,1}^{0,7}$	−1	0.000000000	$\Pi_{0,2}^{2,2}$	−1	0.062500000	$\Pi_{1,3}^{1,3}$	−1	−0.083333333
	+1	0.025173611		+1	0.013888889		+1	0.187500000		+1	0.000000000
$\Pi_{0,0}^{2,2}$	−1	−0.250000000	$\Pi_{0,1}^{2,1}$	−1	−0.750000000	$\Pi_{-1,3}^{1,3}$	−1	0.000000000	$\Pi_{1,3}^{1,5}$	−1	−0.020833333
	+1	−0.250000000		+1	0.250000000		+1	−0.666666667		+1	0.000000000
$\Pi_{-1,1}^{1,1}$	−1	0.000000000	$\Pi_{0,1}^{2,3}$	−1	0.000000000	$\Pi_{-1,3}^{1,5}$	−1	0.000000000	$\Pi_{0,4}^{0,4}$	−1	−0.023437500
	+1	−0.100000000		+1	−0.187500000		+1	0.833333333		+1	−0.325520833
$\Pi_{-1,1}^{1,3}$	−1	0.000000000	$\Pi_{-1,2}^{1,2}$	−1	0.000000000	Π_3^3	−1	−5.333333333	$\Pi_{0,4}^{0,6}$	−1	−0.002343750
	+1	0.750000000		+1	−0.750000000		+1	−0.166666667		+1	0.488281250
$\Pi_{-1,1}^{1,5}$	−1	0.000000000	$\Pi_{-1,2}^{1,4}$	−1	0.000000000	$\Pi_{2,1}^{2,1}$	−1	5.062500000	$\Pi_{-1,5}^{1,5}$	−1	0.000000000
	+1	−0.104166667		+1	0.750000000		+1	−0.062500000		+1	−0.675000000
$\Pi_{1,1}^{3,1}$	−1	0.000000000	$\Pi_{-2,3}^{2,3}$	−1	−0.005208333	$\Pi_{2,1}^{2,3}$	−1	0.000000000	$\Pi_{2,3}^{2,3}$	−1	−0.140625000
	+1	0.750000000		+1	−1.125000000		+1	0.046875000		+1	−0.041666667
$\Pi_{-2,2}^{2,2}$	−1	−0.015625000	$\Pi_{3,-1}^{3,1}$	−1	−2.666666667	$\Pi_{1,2}^{1,2}$	−1	−0.250000000	$\Pi_{1,4}^{1,4}$	−1	−0.046875000
	+1	−1.265625000		+1	0.250000000		+1	0.000000000		+1	0.000000000
$\Pi_{2,-1}^{2,1}$	−1	−1.687500000	Π_2^2	−1	−3.375000000	$\Pi_{1,2}^{1,4}$	−1	−0.083333333	$\Pi_{0,5}^{0,5}$	−1	−0.016666666
	+1	0.187500000		+1	−0.125000000		+1	0.000000000		+1	−0.337500000
$\Pi_{2,-1}^{2,3}$	−1	1.265625000	$\Pi_{2,0}^{2,2}$	−1	1.687500000	$\Pi_{0,3}^{0,3}$	−1	−0.041666667	$\Pi_{0,5}^{0,7}$	−1	0.001388889
	+1	0.000000000		+1	0.062500000		+1	−0.333333333		+1	0.590625000
Π_1^1	−1	−2.000000000	$\Pi_{1,1}^{1,1}$	−1	3.000000000	$\Pi_{0,3}^{0,5}$	−1	−0.010416667	$\Pi_{1,5}^{1,5}$	−1	−0.033333333
	+1	0.000000000		+1	0.000000000		+1	0.416666667		+1	0.000000000
Π_1^3	−1	1.500000000	$\Pi_{1,1}^{1,3}$	−1	0.000000000	$\Pi_{0,3}^{0,7}$	−1	−0.008333333	$\Pi_{0,6}^{0,6}$	−1	−0.013563368
	+1	0.000000000		+1	0.000000000		+1	−0.179166666		+1	−0.364735243
$\Pi_{1,0}^{1,2}$	−1	1.000000000	$\Pi_{1,1}^{1,5}$	−1	0.000000000	$\Pi_{0,3}^{2,3}$	−1	0.020833333	$\Pi_{0,7}^{0,7}$	−1	−0.012053571
	+1	0.000000000		+1	0.000000000		+1	0.166666667		+1	−0.406349206
$\Pi_{1,0}^{1,4}$	−1	0.031250000	$\Pi_{1,1}^{3,1}$	−1	−2.250000000						
	+1	0.000000000		+1	0.000000000						

/129

	i			i			i			i	
$\Pi'^{1,1}_{1,-1}$	0	3.000000000	$\Pi'^{0,5}_{0,1}$	0	—0.052083333	$\Pi'^{1,3}_{1,1}$	0	0.750000000	$\Pi'^{0,3}_{0,3}$	0	—0.333333333
$\Pi'^{1,3}_{1,-1}$	0	0.000000000	$\Pi'^{2,1}_{0,1}$	0	0.250000000	$\Pi'^{0,2}_{0,2}$	0	—0.375000000	$\Pi'^{0,5}_{0,3}$	0	0.416666666
$\Pi'^{0,0}_{0,0}$	0	—1.000000000	$\Pi'^{1,2}_{-1,2}$	0	0.000000000	$\Pi'^{0,4}_{0,2}$	0	0.375000000	$\Pi'^{2,1}_{-2,-1}$	0	0.187500000
$\Pi'^{2,0}_{0,0}$	0	0.500000000	$\Pi'^{2,1}_{-2,1}$	0	—0.062500000	$\Pi'^{1,3}_{-1,3}$	0	0.000000000	$\Pi'^{1,2}_{-1,-2}$	0	0.000000000
$\Pi'^{0,2}_{0,0}$	0	0.500000000	$\Pi'^{1,0}_{-1,0}$	0	0.000000000	$\Pi'^{2,0}_{-2,0}$	0	—0.125000000	$\Pi'^{0,3}_{0,-3}$	0	—0.041666666
$\Pi'^{0,4}_{0,0}$	0	—0.015625000	$\Pi'^{1,2}_{-1,0}$	0	0.000000000	$\Pi'^{1,1}_{-1,-1}$	0	0.000000000	$\Pi'^{0,5}_{0,-3}$	0	—0.010416667
$\Pi'^{1,1}_{-1,1}$	0	0.000000000	$\Pi'^{0,1}_{0,-1}$	0	1.500000000	$\Pi'^{1,3}_{-1,-1}$	0	0.000000000	$\Pi'^{1,3}_{1,3}$	0	—0.666666667
$\Pi'^{1,3}_{-1,1}$	0	0.000000000	$\Pi'^{2,1}_{0,-1}$	0	—0.750000000	$\Pi'^{0,2}_{0,-2}$	0	—0.125000000	$\Pi'^{0,4}_{0,4}$	0	—0.325520833
$\Pi'^{2,1}_{2,-1}$	0	5.062500000	$\Pi'^{0,3}_{0,-1}$	0	0.000000000	$\Pi'^{0,4}_{0,-2}$	0	—0.041666667	$\Pi'^{1,3}_{-1,-3}$	0	0.000000000
$\Pi'^{1,0}_{1,0}$	0	2.000000000	$\Pi'^{0,5}_{0,-1}$	0	0.000000000	$\Pi'^{1,3}_{1,-3}$	0	—0.083333333	$\Pi'^{0,4}_{0,-4}$	0	—0.023437500
$\Pi'^{1,2}_{1,0}$	0	1.000000000	$\Pi'^{1,2}_{1,-2}$	0	—0.250000000	$\Pi'^{2,1}_{2,1}$	0	—1.687500000	$\Pi'^{0,5}_{0,5}$	0	—0.337500000
$\Pi'^{0,1}_{0,-1}$	0	—0.500000000	$\Pi'^{2,0}_{2,0}$	0	—3.375000000	$\Pi'^{1,2}_{1,2}$	0	—0.750000000	$\Pi'^{0,5}_{0,-5}$	0	—0.016666667
$\Pi'^{0,3}_{0,1}$	0	0.375000000	$\Pi'^{1,1}_{1,1}$	0	—1.000000000						

IV. CERTAIN ERRORS IN THE EXPRESSIONS FOR NEWCOMB'S OPERATORS

In 1895, Newcomb published symbolic expressions of operators serving to expand the perturbation function over the mean anomalies.

The verifications applied to the operators indicated mistakes in the expressions for operators of the eighth order referring to an outer planet.

These mistakes are the following:

1. In operators $\Pi^{0,8}_{0,2}$, $\Pi^{0,8}_{0,4}$, $\Pi^{0,8}_{0,6}$, $\Pi^{0,8}_{0,8}$ all the printed signs of the numerical coefficients in the odd degrees of subscript 2, with the exception of two cases which will be specially indicated, must be changed to the reverse signs.

2. In operator 147456 $\Pi^{0,8}_{0,0}$ in the expression for the coefficient D, the free term + 5048 must be changed to + 5040.

3. In operator 184320 $\Pi^{0,8}_{0,2}$: (a) in the free term the factor with i - 2832, must be changed to - 1232; (b) in the coefficient D, the factor with i, - 1656, must be changed to - 2408, and the free term 8656, to 6064.

4. In operator 368640 $\Pi^{0,8}_{0,4}$: (a) the coefficient D, the factor with i, + 198060, must be changed to + 185772, and the free term - 34992, to 76464; (b) in the coefficient D^2 the free term 21964 must be changed to 21944; (c) in the coefficient D^7 in the factor with i, + 8, the sign need not be changed.

5. In operator 10321920 $\Pi^{0,8}_{0,8}$: (a) in a free term with the factor i, + 4239088, the sign need not be changed; (b) in the coefficient of D^2 the factor with i^4, + 47130, must be changed to + 47040.

The table of corrected changes for the operators is given below.

$$147456\Pi_{0,0}^{0,8} =$$ /130

$$256i^8 - 3040i^6 + 13321i^4 - 16260i^2 + (-1024i^6 + 9088i^4 - 21052i^2 + 5040)\,D + (-256i^6 + 4432i^4 - 18386i^2 + 13068)\,D^2 + (+960i^4 - 8340i^2 + 13132)\,D^3 + (96i^4 - 2158i^2 + 6769)\,D^4 + (-288i^2 + 1960)\,D^5 + (-16i^2 + 322)\,D^6 + 28\,D^7 + D^8$$

$$184320\Pi_{0,2}^{0,8} =$$

$$-256i^8 - 2496i^7 - 7440i^6 - 4556i^5 + 4968i^4 - 11098i^3 - 20996i^2 - 1232i - 256 + (-256i^7 - 1280i^6 + 1696i^5 + 12720i^4 + 91i^3 - 27328i^2 - 2408i + 6064)\,D + (+128i^6 + 2064i^5 + 7360i^4 + 235i^3 - 21231i^2 - 914i + 16236)\,D^2 + (+192i^5 + 960i^4 - 1980i^3 - 11310i^2 + 55i + 16340)\,D^3 + (-660i^3 - 2855i^2 + 855i + 8549)\,D^4 + (-48i^3 - 288i^2 + 445i + 2492)\,D^5 + (-8i^2 + 75i + 398)\,D^6 + (+4i + 32)\,D^7 + D^8$$

$$368640\Pi_{0,4}^{0,8} =$$

$$256i^8 + 4992i^7 + 38880i^6 + 156328i^5 + 355857i^4 + 491126i^3 + 449504i^2 + 266416i + 65536 + (+512i^7 + 8192i^6 + 49344i^5 + 141120i^4 + 210506i^3 + 209252i^2 + 185772i + 76464)\,D + (+256i^6 + 2208i^5 + 720i^4 - 35230i^3 - 86934i^2 - 36038i + 21964)\,D_z + (-128i^5 - 2880i^4 - 18920i^3 - 43420i^2 - 17614i + 25100)\,D^3 + (-160i^4 - 1720i^3 - 4050i^2 + 5880i + 18849)\,D^4 + (-32i^3 + 96i^2 + 2370i + 5464)\,D^5 + (+16i^2 + 246i + 722)\,D^6 + (+8i + 44)\,D^7 + D^8$$

$$1290240\Pi_{0,6}^{0,8} =$$

$$-256i^8 - 7488i^7 - 91280i^6 - 602532i^5 - 2342872i^4 - 5471214i^3 - 7485924i^2 - 5512464i - 1679616 + (-768i^7 - 19712i^6 - 205856i^5 - 1128400i^4 - 3496703i^3 - 6152384i^2 - 5813688i - 2386512)\,D + (-896i^6 - 19152i^5 - 159040i^4 - 648655i^3 - 1356663i^2 - 1397326i - 613908)\,D^2 + (-448i^5 - 6720i^4 - 31220i^3 - 28350i^2 + 115157i + 181972)\,D^3 + (+1540i^3 + 19985i^2 + 82285i + 106309)\,D^4 + (+112i^3 + 2016i^2 + 11095i + 18844)\,D^5 + (+56i^2 + 609i + 1582)\,D^6 + (+12i + 64)\,D^7 + D^8$$

$$10321920\Pi_{0,8}^{0,3} =$$

$$256i^8 + 9984i^7 + 164640i^6 + 1490384i^5 + 8034537i^4 + 26106892i^3 + 49046156i^2 + 47239088i + 16777216 + (+1024i^7 + 35840i^6 + 520576i^5 + 4045440i^4 + 18031860i^3 + 45627204i^2 + 59755672i + 30461872)\,D + (+1792i^6 + 55104i^5 + 684880i^4 + 4381860i^3 + 15119902i^2 + 26434436i + 18049548)\,D^2 + (+1792i^5 + 47040i^4 + 479920i^3 + 2367820i^2 + 5615428i + 5079116)\,D^3 + (+1120i^4 + 24080i^3 + 188930i^2 + 638400i + 779569)\,D^4 + (+448i^3 + 7392i^2 + 39620i + 68712)\,D^5 + (112i^2 + 1260i + 3458)\,D^6 + (+16i + 92)\,D^7 + D^8$$

REFERENCES /131

1. Orlov, B. A.: Expansion of the Perturbation Function by Newcomb's Method. Trudy Astronomicheskoy Observatorii LGU (Transactions of the Astronomical Observatory of Leningrad State University), Vol. 6, 1936.
2. Orlov, B. A.: Application of the Delaunay-Hill Method to the Case of Commensurability 3:4 (Tule). Izvestiya Glavnoy Astronomicheskoy Observatorii (Proceedings of the Main Astronomical Observatory), Vol. 17, No. 3, 1947.
3. Samoylova-Yakhontova, N. S.: Correction of Elliptical Orbits. Byulletin Astronomicheskogo Instituta (Bulletin of the Astronomical Institute), Vol. 3, p. 55, 1945.

4. Subbotin, M. F.: A Course in Celestial Mechanics, Vol. I (Kurs Nebesnoy Mekh niki, 1), OGIZ, 1941.
5. Subbotin, M. F.: A Course in Celestial Mechanics, Vol. II (Kurs Nebesnoy Mekhaniki, 2), ONTI, 1937.
6. Sharaf, Sh. G.: Expansion of Some Functions of Coordinates of Elliptical Motion into Series to the Ninth Degree of Eccentricity. Byulletin Instituta Teoreticheskoy Astronomii (Bulletin of the Institute of Theoretical Astronomy), Vol. 5, No. 5, p. 68, 1953.
7. Bower, E. C.: On the Orbit and Mass of Pluto with an Ephemeris for 1931-1932. Lick. Observ. Bull., Vol. 15, p. 437, 1931.
8. Bower, E. C. and F. L. Whipple: Preliminary Elements and Ephemeris of the Lowell Observatory Object. Lick. Observ. Bull., Vol. 14, p. 421, 1930a.
9. Bower, E. C. and F. L. Whipple: Elements and Ephemeris of the Lowell Observatory Object (Pluto). Lick. Observ. Bull., Vol. 15, p. 427, 1930b.
10. Brown, E. W. and D. Brouwer: Tables for the Development of the Disturbing Function with Schedules for Harmonic Analysis. Trans. Yala Obs., Vol. 6, p. V., 1932.
11. Comrie, L.: Planetary Coordinates. London, 1933.
12. Eckert, W. J., D. Brouwer, and G. M. Clemence: Coordinates of the Five Outer Planets 1653-2060. Astr. Pap., Vol. 12, 1951.
13. Innes, R. T. A.: Note on Certain Coefficients Appearing in the Algebraical Development of the Perturbative Function. M. N. of the RAS, Vol. 69, 1909.
14. Newcomb, S.: Appearance of a Direct and Easy Method for Accomplishing the Development of the Perturbative Function and Differential Coefficients (Apercu d'une Methode directe et facile pour effectuer le developpment de la function perturbatice et de ses coefficients differentiells.) C. R., Vol. 70, 1870.
15. Newcomb, S.: An Investigation of the Orbit of Uranus, with General Tables of its Motion. Smithsonian Contrib., 1873.
16. Newcomb, S.: A Method of Developing the Perturbative Function of Plantary Motion. Amer. Jour. of Math., Vol. 3, 1880.
17. Newcomb, S.: Development of the Perturbative Function and its Derivatives, in Sines and Cosines of Multiples of the Eccentric Anomalies, and in Powers of the Eccentricities and Inclinations. Astr. Pap., Vol. 3, 1891a.
18. Newcomb, S.: Periodic Perturbations of the Longitudes and Radii Vectores of the Four Inner Planets of the First Order as to the Masses. Astr. Pap., Vol. 3, 1891b.
19. Newcomb, S.: Development of the Perturbative Function in Cosines of Multiples of the Mean Anomalies and of Angles between the Perihelia and Common Node and in Powers of the Eccentricities and Mutual Inclination. Astr. Pap., Vol. 5, 1895a.
20. Newcomb, S.: Secular Variations of the Orbits of the Four Inner Planets. Astr. Pap., Vol. 5, 1895b.
21. Nicolson, S. B., and N. U. Mayall: Position, Orbit and Mass of Pluto. Aph. J., Vol. 73, No. 1, 1931.
22. Roure, H.: Study of an Analytical Plutonian Theory (Essai d'une theorie analytique de Pluton). J. O., Vol. 18, No. 1, p. 3, 6, 1935; Vol. 19, No. 2, p. 5, 1936; Vol. 20, p. 6, 1937.
23. Roure, H.: On a Long Period Inequality of the Average Plutonian Movement Due to the Perturbative Action of Jupiter and Saturn (Sur une inegalite

a longue periode du moyen mouvement de Pluton due aux action perturbatrices de Jupiter et de Saturn). J. O., Vol. 23, No. 3, 1940a.
24. Roure, H.: On a Long Period Inequality of the Average Plutonian Movement Due to the Perturbative Action of Uranus and Neptune (Sur une inegalite a longue periode du moyen mouvement de Pluton due aux action perturbatrices d'Uranus et de Neptune.) J. O., Vol. 23, No. 4, 1940b.
25. Roure, H.: On a Long Period Inequality of Plutonian Orbit Excentricity Due to the Action of Neptune (Sur une inegalite a longue periode de l'excentricite de l'orbite de Pluton due a l'action de Neptune). J. O., Vol. 23, No. 10, 1940b.
26. Zagar, F.: The Plutonian Path with Respect to Disturbances (Die Bahn Pluton mit Berucksightigung der Storungen). A. N., Vol. 240, No. 5755.

Translation prepared for the National Aeronautics and Space Administration by INTERNATIONAL INFORMATION INCORPORATED, 2101 Walnut St., Philadelphia, Pa. 19103 Contract No. NASw-1499.

NASA-Langley, 1968 —— 30 F-490

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546

OFFICIAL BUSINESS

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

[illegible] 06351 00903
AIR FORCE WEAPONS LABORATORY/AFWL/
KIRTLAND AIR FORCE BASE, NEW MEXICO 8711

ATT E. LOU BOWMAN, ACTING CHIEF TECH. LI

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546