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AN APPROXIMATE THEORY FOR TRANSVERSE SHEAR DEFORMATION AND ROTATORY INERTIA EFFECTS IN VIBRATING BEAMS

by D. M. Egle

Prepared by
UNIVERSITY OF OKLAHOMA
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AN APPROXIMATE THEORY FOR TRANSVERSE SHEAR DEFORMATION AND ROTATORY INERTIA EFFECTS IN VIBRATING BEAMS[1]

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Abstract

A beam theory, which includes the effects of transverse shear deformation and rotatory inertia and whose kinetic and potential energy may be written in terms of a single dependent variable, is developed in this paper. This simplification will reduce the computational effort required in the analysis of complex beams or structures composed of a number of beams. This is accomplished by neglecting the coupling between the transverse shear deformation and the rotatory inertia. Comparisons of natural frequencies calculated by this theory with those of the Timoshenko theory show that this coupling is indeed negligible.

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Introduction

The Timoshenko theory of vibrating beams [1][2] has been the basis of a large number of investigations into the effects of transverse shear deformation and rotatory inertia on the dynamics of beams. It has been used extensively because it is simple to formulate and its results compare very well with those obtainable by linear elasticity theory [2,3]. In exchange for the accuracy of this theory as compared to the classical Bernoulli-Euler beam theory, it is necessary to introduce an additional dependent variable, thus complicating the solution. While the addition of this variable is not of serious consequence in many of the simpler beam problems, it can be a considerable disadvantage if an approximate solution by the methods of Ritz or Galerkin is being sought for a complex beam or a structure composed of a number of beams.

In this paper, the energy expressions for a beam theory, which includes the effects of transverse shear deformation and rotatory inertia and which may be written in terms of a single dependent variable, are developed. To accomplish this, it is necessary to neglect the coupling between the transverse shear deformation and the rotatory inertia. The natural frequencies of a simply supported and a cantilevered beam calculated with this theory compare very well to those determined by the Timoshenko theory, implying that the neglect of this coupling

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[2]Numbers in brackets designate references at the end of the paper.

is justified.

Nomenclature

A	beam cross-sectional area
E	Young's modulus
G	shear modulus
h	depth of rectangular beam
I	moment of inertia of beam cross section about the y-axis
k	transverse shear coefficient
ℓ	length of beam
M	bending moment
n	vibration mode (1, 2, 3, etc.)
t	time
T	kinetic energy
V	shear force
w	total deflection of neutral axis
w_b	deflection due to bending moment
w_s	deflection due to shear force
α	EI/kGA
ν	wavelength of vibration
ρ	density of beam material
ω	frequency
δ	$\frac{I}{A}\left(\frac{n\pi}{\ell}\right)^2$

Theory Development

Let w be the total deflection of the beam's neutral axis, see Figure 1. This deflection may be divided into two components, that due to the bending moment and that due to transverse shear

deformation. Thus

$$w = w_b + w_s \tag{1}$$

The shear deflection, w_s, may be related to the shear strain in the beam in several ways (see reference [4] for example) and the value of the transverse shear coefficient is dependent on this relationship. However, for our purposes, it is only necessary to state that

$$V = kGA\, w_{s,x} \tag{3}$$

where k is the appropriate transverse shear coefficient.

The bending moment is given by

$$M = EI\, w_{b,xx} \tag{4}$$

The strain energy due to the bending moment and the shear force is

$$U = 1/2 \int_0^{\ell} \{\frac{M^2}{EI} + \frac{V^2}{kGA}\}\ dx \tag{5}$$

and the translational and rotatory kinetic energy is

$$T = 1/2 \int_0^{\ell} \rho\ \{A\, \dot{w}^2 + I\, \dot{w}_{b,x}^2\}\ dx \tag{6}$$

where the dot above a variable denotes differentation with respect to time.

In the energy formulation of the Timoshenko theory, equations (1, 3, 4) are substituted into equations (5, 6) which gives energy expressions which are functions of two variables, for example, w and w_b.

If it is assumed that, for the purpose of determining the shear force and, hence, the shear deflection, the rotatory inertia is negligible, then

$$V = -\, M,_x \tag{7}$$

Thus, the shear force is related to the bending moment just as it is in static beam theory. Note that the rotatory inertia has not been neglected altogether as it still appears in the second term of equation (6). In fact, only the coupling between the rotatory inertia and the transverse shear force, and hence, the transverse shear deflection, has been omitted.

Combining equations (3, 4, and 7),

$$w_{s,x} = - \alpha \, w_{b,xxx} \tag{8}$$

where $\alpha = EI/kGA$.

Equation (8) may be integrated to determine the shear deflection

$$w_s = - \alpha \, w_{b,xx} + C \, (t) \tag{9}$$

C(t) is determined from the end conditions. However, because C(t) influences the energy only through the sum $w = w_b + w_s$, it will be deleted from equation (9) with the understanding that it may be added to w_b (or w) to satisfy the end conditions.

The energy expressions (5,6) may now be written in terms of the bending deflection, w_b. Thus

$$U = (1/2) \int_o^\ell EI \, \{w_{b,xx}^2 + \alpha \, w_{b,xxx}^2\} \, dx \tag{10}$$

$$T = (1/2) \int_o^\ell \rho \, A \, \{[\dot{w}_b - \alpha \, \dot{w}_{b,xx}]^2 + \frac{I}{A} \dot{w}_{b,x}^2\} \tag{11}$$

The end conditions are taken to be similar to those of the Timoshenko theory. The natural end conditions (those end conditions for which the forces and moments at the end do no work) are either

$$M = EI\, w_{b,xx} = 0 \tag{12-a}$$

or

$$w_{b,x} = 0 \tag{12-b}$$

and, either

$$V = -\, EI\, w_{b,xxx} = 0 \tag{12-c}$$

or

$$w = w_b - \alpha\, w_{b,xx} = 0 \tag{12-d}$$

Application to a Uniform Simply-Supported Beam

The end conditions (12-a,d) are satisfied at $x = 0, \ell$ if

$$w_b = A \sin n\pi x/\ell \sin \omega t \tag{13}$$

With equations (10, 11, and 13), the natural frequencies may be calculated by Rayleigh's method

$$\left(\frac{\omega}{\omega_1}\right)^2 = \frac{1 + \frac{E}{kG}\zeta}{\left(1 + \frac{E}{kG}\zeta\right)^2 + \zeta} \tag{14}$$

where

$\omega_1 = \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho A}}$, natural frequencies of the simply supported Bernoulli-Euler beam

$$\zeta = \frac{I}{A}\left(\frac{n\pi}{\ell}\right)^2$$

The ratio of bending deflection to total deflection may be shown to be

$$\frac{w_b}{w} = \frac{1}{1 + \frac{E}{kG}\zeta} \tag{15}$$

Figure 2 compares the frequencies calculated with the

present analysis to those of the Timoshenko theory (see Appendix) for a rectangular bar of material with Poisson's ratio 0.3. The shear coefficient, k was taken to be 0.85 as recommended by Cowper [4].

The frequencies are plotted over a wide range of the variable ζ which is a measure of the depth of the beam to the wavelength of the vibration. For the rectangular beam, for example,

$$\zeta = \frac{\pi^2}{3} \left(\frac{h}{\nu}\right)^2$$

where h = depth of the beam

$\nu = 2\ell/n$, wavelength

The range of (h/ν) in Figures 3, 4, 5 is from 0.06 to 1.8, the upper limit being much higher than one would expect to encounter in determining the first few frequencies of a beam. However, the upper limit was selected to show the wide range of agreement between the present analysis and the Timoshenko theory.

As is seen in Figure 2, the agreement between the two theories is excellent up to $\zeta = 10$. The values of $(\omega/\omega_1)^2$ calculated by the Timoshenko theory were slightly lower than those of equation (14), the maximum difference being about 1.25% for $\zeta \doteq 1$. The differences for values of $\zeta < 1$ and $\zeta > 1$ are less than 1.25%.

The ratio of deflection due to bending moment to the total deflection for the simply supported beam is shown in Figure 3. The discrepancy between the two theories is more apparent in the deflections but is still not excessive if $\zeta < 0.5$ which correspond to a depth to wavelength ratio of about 0.4. Because the present

analysis underestimates the bending deflection, it will yield values for the shear deflection which are too large. This is verified in Figure 4 which shows the shear force in a simply-supported beam normalized by

$$V_1 = - EI\, w_{b,xxx}$$

which is the shear force corresponding to the Bernoulli-Euler theory and is equal to the shear force predicted by the present theory. As would be expected, the agreement between the present approximation and Timoshenko beam theory is much worse for the shear force than the natural frequencies, but the error is less than 10% if $\zeta < 0.1$ or if the height to wavelength ratio is less than 0.18.

Application to a Uniform Cantilevered Beam

The Rayleigh-Ritz method is applied to the energy expressions, equations (10, 11) to obtain an approximate solution for the natural frequencies of a uniform beam of length ℓ, clamped at $x = 0$. The bending deflection is taken in the form

$$w_b = \sum_{n=1}^{\infty} A_n \left[X_n(x) + \alpha X_n''(0)\right] \sin \omega t \qquad (16)$$

where the $X_n(x)$ are the Bernoulli-Euler eigenfunctions for the clamped-free beam [5] and the $X_n' = dX_n/dx$. Equation (16) satisfies the end conditions (12-b, d) at $x=0$ and (12-a,c) at $x=\ell$.

The maximum kinetic and potential energies may be calculated with eqs. (10, 11, 16).

$$T_{max} = 1/2\ \rho A\omega^2 \int_o^\ell \{(\sum_{n=1}^{\infty} A_n[X_n(x) + \alpha\ (X_n''\ (0) - X_n''\ (x))])^2$$

$$+ \frac{I}{A} (\sum_{n=1}^{\infty} A_n\ X_n'\ (x))^2\}\ dx \tag{17}$$

$$U_{max} = 1/2\ EI \int_o^\ell \{(\sum_{n=1}^{\infty} A_n\ X_n''\ (x))^2 + \alpha\ (\sum_{n=1}^{\infty} A_n\ X_n'''(x))^2\}\ dx \tag{18}$$

Applying the Rayleigh-Ritz technique to eqs. (17,18) with the aid of the beam eigenfunction integrals tabulated by Felgar [6], results in

$$\sum_{n=1}^{\infty} (K_{in} - \lambda\ M_{in})\ A_n = 0 \quad i = 1, 2, \ldots \tag{19}$$

where

$$K_{in} = \beta_n^4 \delta_{in} + \alpha (I_2)_{in}$$

$$M_{in} = \delta_{in} + \alpha\ \{4\frac{\beta_n^2\ \alpha_i}{\ell\beta_i} + 4\frac{\beta_n^2\ \alpha_n}{\ell\beta n} - [(I_1)_{in} + (I_1)_{ni}]\}$$

$$+ \alpha^2\ \{4\beta_i^2\ \beta_n^2 - 4(-1)^{i+1}\ \alpha_i\beta_i\beta_n^2/\ell - 4(-1)^{n+1}\alpha_n\beta_n\beta_i^2/\ell$$

$$+\beta_n^4\ \delta_{in}\} + (I/A)\ (I_3)_{in}$$

$$(I_1)_{nn} = \alpha_n\beta_n\ (2 - \alpha_n\beta_n)/\ell$$

$$(I_1)_{in} = \frac{4\beta_n^2(\alpha_i\beta_i - \alpha_n\beta_n)}{\ell(\beta_i^4 - \beta_{in}^4)}\ [(-1)^{i+n}\ \ \beta_n^2 + \beta_i^2] \qquad i \neq n$$

$$(I_2)_{nn} = \alpha_n\beta_n^5\ (\alpha_n\beta_n\ell + 2)/\ell$$

$$(I_2)_{in} = \frac{4\beta_i^3\ \beta_n^3}{\ell(\beta_n^4 - \beta_i^4)}\ [(-1)^{i+n}\ \beta_i\beta_n\ (\alpha_i\beta_i - \alpha_n\beta_n) + \alpha_i\beta_n^3$$

$$-\alpha_n\beta_i)] \qquad i \neq n$$

$$(I_3)_{nn} = \alpha_n\beta_n\ (\alpha_n\beta_n\ell + 2)/\ell$$

$$(I_3)_{in} = \frac{4\beta_i\beta_n}{\ell(\beta_n^4 - \beta_i^4} [(-1)^{i+n}\ (\alpha_i\beta_n^3 - \alpha_n\beta_i^3) - \beta_i\beta_n\ (\alpha_n\beta_n - \alpha_i\beta_i)] \qquad i \neq n$$

$$\lambda = \frac{\rho A}{EI}\ \omega^2$$

Numerical values of α_n and β_n are given in reference [5].

Equations (19) may be solved approximately by considering only the first few terms in the series. The problem is then a linear eigenvalue problem and may be solved any of the standard numerical techniques.

Table 1 compares the five lowest frequencies of a clamped-free uniform beam for four geometrical configurations calculated with equation (19) to values given by the Timoshenko theory in reference [7]. Seven terms were used in equation (19), which was sufficient for 3-place accuracy in the fifth lowest frequency when compared to a ten-term series.

If we consider only a single term in (19) and neglect the α^2 term in M_{in}, the frequency equation is

$$\left(\frac{\omega}{\omega_2}\right)^2 = \frac{1 + \frac{E}{kG}\ \gamma_n}{1 + (1+2\ \frac{E}{kG})\ \gamma_n} \tag{20}$$

where

$$\gamma_n = \alpha_n\beta_n\ (2 + \alpha_n\beta_n\ell)\ I/A\ell$$

$$\omega_2 = \beta_n^2 \sqrt{\frac{EI}{\rho A}}$$, natural frequency of Bernoulli-Euler clamped-free beam

The term which was neglected in arriving at equation (20) can be shown to be the contribution of the shear deflection to the translational kinetic energy and the terms remaining in the denominator of (20) are the contributions of the bending deflection, and the rotatory inertia. Equation (20) will give values of the frequency larger than those calculated with (19) and the error will increase as γ_n increases.

The point to be made, however, in presenting equation (20) is that it suggests the use of γ_n as a similarity parameter in plotting the frequencies. Figure 5 shows the frequencies calculated with a ten-term series in equation (19) plotted as a function of γ_n for the first four modes of vibration. It is apparent that γ_n serves reasonably well as a similarity parameter since the values tend to lie on a single curve.

CONCLUSION

It is evident, from the excellent agreement between the natural frequencies calculated with this analysis and those of the Timoshenko theory, that the neglect of the coupling between the rotatory inertia and the transverse shear deformation is justified. The energy expressions equations (10, 11), in which the present theory is embodied, are slightly more complex than those of the Timoshenko theory, but, because they are functions of a single dependent variable, a considerable simplification in the solution to problems to which they are applied will result.

The concept used to develop this theory, that is, ignoring the coupling between the transverse shear deformation and the rotatory inertia, would seem to be applicable to plates and shells as well as beams. A study has been initiated to do this and it will be interesting to see if this will agree as well with the Mindlin theory, reference (8), as the present analysis agrees with the Timoshenko theory.

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APPENDIX

The equations governing the motion of a Timoshenko beam are

$$V = kGA\,(w_{,x} - w_{b,x}) \tag{A-1}$$

$$M = EI\,w_{b,xx} \tag{A-2}$$

$$V_{,x} = \rho A\ddot{w} \tag{A-3}$$

$$M_{,x} + V = \rho I\,\ddot{w}_{b,x} \tag{A-4}$$

For the simply supported beam, the deflections are of the form

$$w = C_1 \sin n\pi x/\ell \sin \omega t \tag{A-5}$$

$$w_b = C_2 \sin n\pi x/\ell \sin \omega t \tag{A-6}$$

Substitution of equations (A-5,-6) into (A-1,-4) eventually leads to

$$\left(\frac{\omega}{\omega_1}\right)^2 = \frac{1}{2\xi}\left\{1 + \frac{kG}{E}\left(1 + \frac{1}{\zeta}\right) \pm \left[\left(1 + \frac{kG}{E}\left(1 + \frac{1}{\zeta}\right)\right)^2 - 4\frac{kG}{E}\right]^{\frac{1}{2}}\right\} \tag{A-7}$$

$$\frac{V}{V_1} = \frac{(\omega/\omega_1)^2}{1 - \frac{E\zeta}{kG}\left(\frac{\omega}{\omega_1}\right)^2} \tag{A-8}$$

$$\frac{w}{w_b} = \frac{1}{1 - \frac{E\zeta}{kG}\left(\frac{\omega}{\omega_1}\right)^2} \tag{A-9}$$

where

$$\omega_1 = \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

$$V_1 = -\,EI\,w_{b,xx}$$

Table 1. Comparison of the natural frequencies of a uniform cantilevered beam calculated with the present theory to those of the Timoshenko Theory.

$E/kG = 4,\ I/A\ell^2 = 4\times10^{-4}$

n		1	2	3	4	5
$(\frac{\omega}{\omega_1})^2$	a	.99080	.93925	.86773	.7824	.696
	b	.99081	.93881	.86758	.77710	.69535

$E/kG = 6.25,\ I/A\ell^2 = 4\times10^{-4}$

n		1	2	3	4	5
$(\frac{\omega}{\omega_1})^2$	a	.98671	.91421	.81909	.7128	.612
	b	.98671	.91398	.81901	.71254	.61175

$E/kG = 1.78,\ I/A\ell^2 = 9\times10^{-4}$

n		1	2	3	4	5
$(\frac{\omega}{\omega_1})^2$	a	.98854	.92540	.84144	.7461	.654
	b	.98854	.92504	.84116	.74514	.65205

$E/kG = 2.78,\ I/A\ell^2 = 9\times10^{-4}$

n		1	2	3	4	5
$(\frac{\omega}{\omega_1})^2$	a	.98447	.90121	.79640	.6843	.582
	b	.98447	.90112	.79609	.68318	.58030

[a]Equation (19), 7 terms

[b]Timoshenko theory, reference [7]

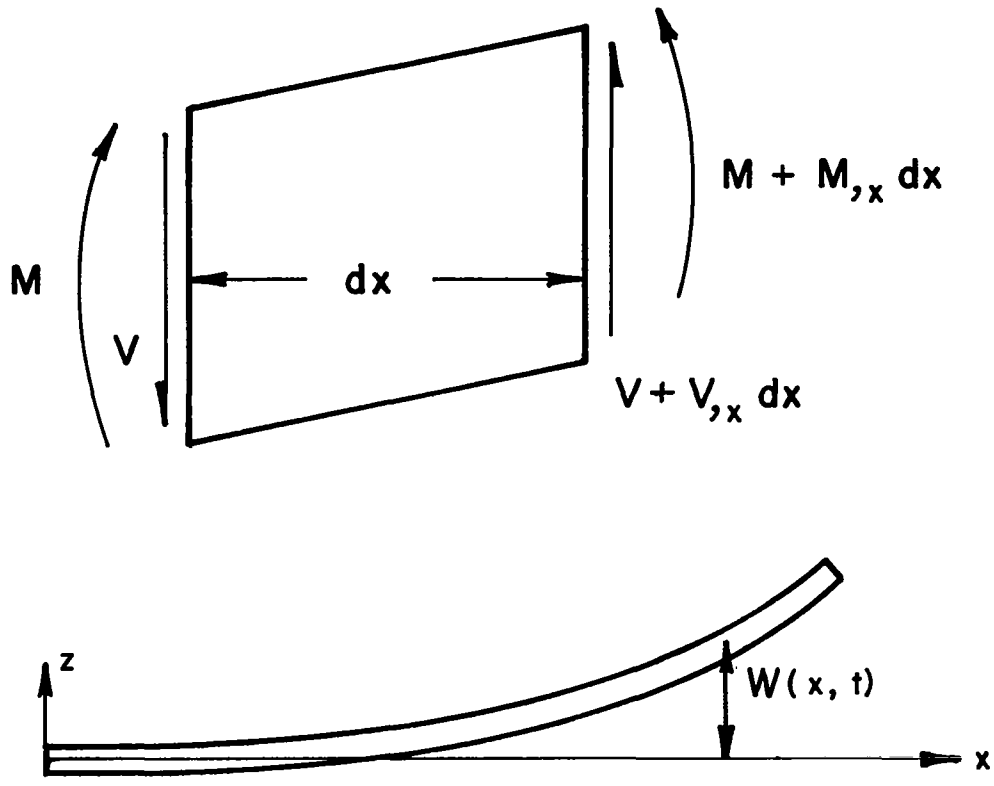

Figure 1. Coordinates and Notation

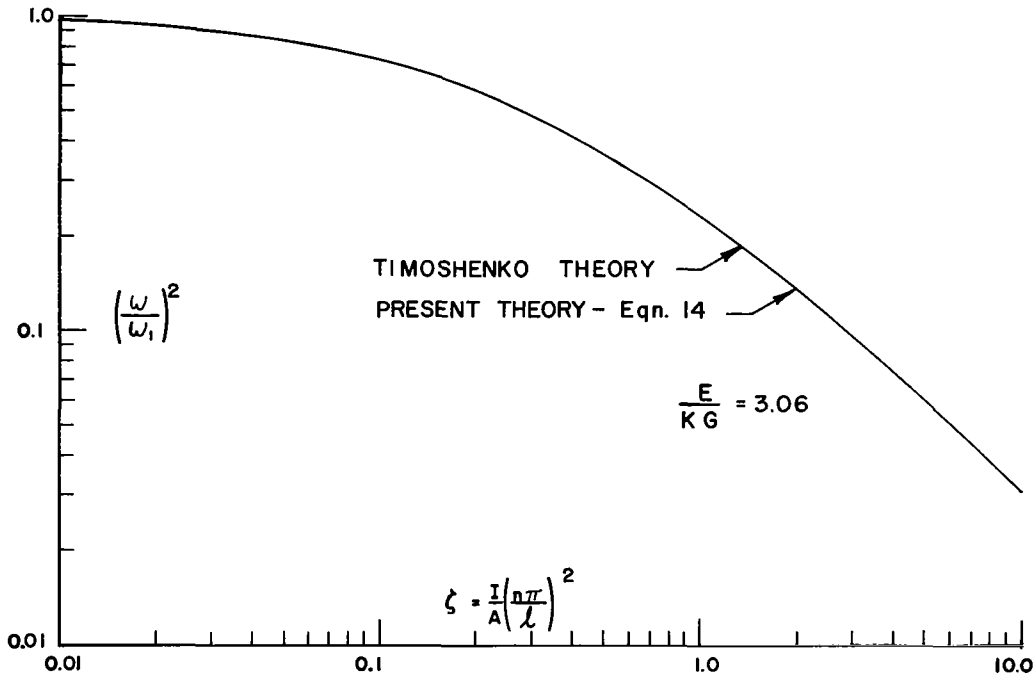

Figure 2. Effect of Shear Deformation and Rotatory Inertia on the Natural Frequency of a Simply Supported Beam.

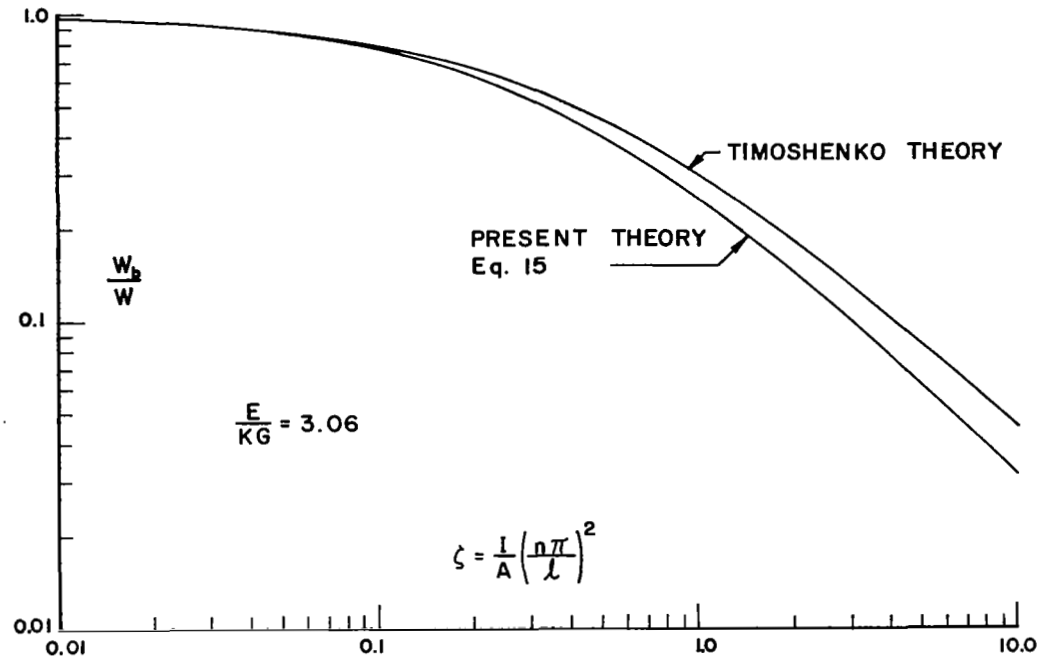

Figure 3. Effect of Shear Deformation and Rotatory Inertia on the Deflection due to Bending Moment in a Simply Supported Beam.

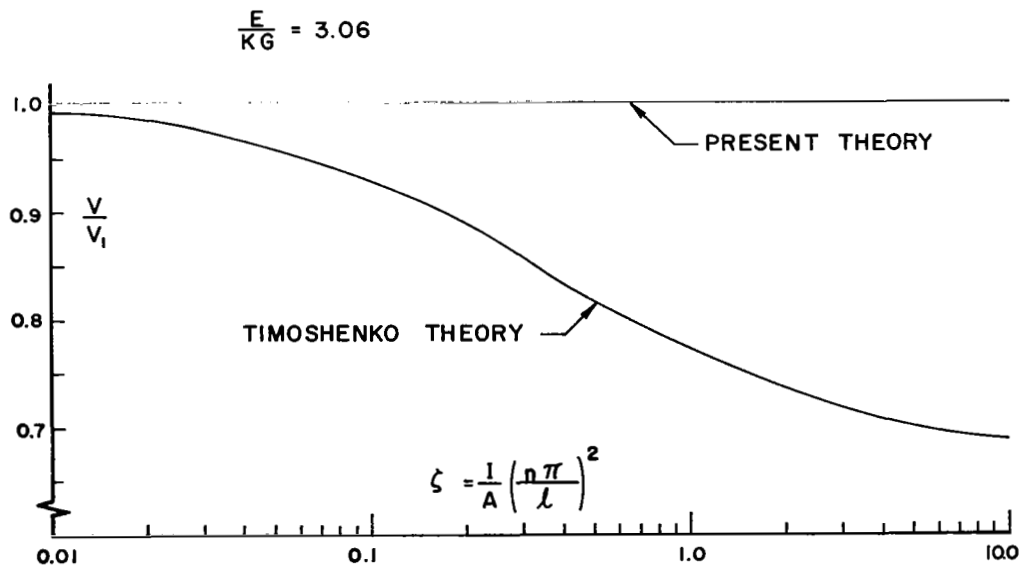

Figure 4. Effect of Shear Deformation and Rotatory Inertia on the Shear Force in a Simply Supported Beam.

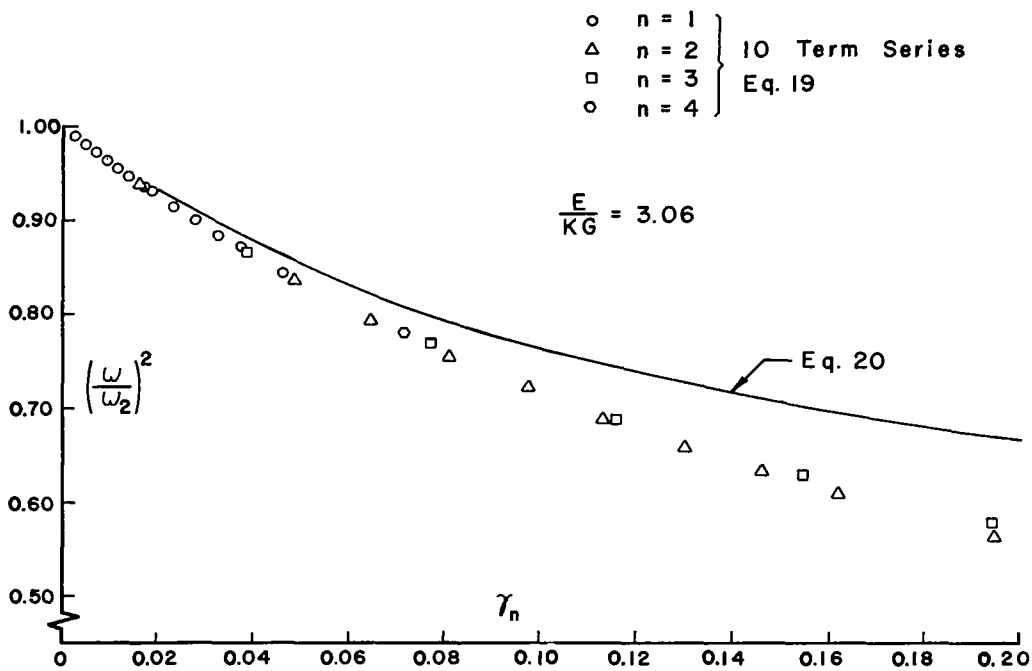

Figure 5. Effect of Shear Deformation and Rotatory Inertia on the Natural Frequency of a Cantilevered Beam.

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