OPTIMUM TRANSFERS BETWEEN HYPERBOLIC ORBITS

(FINITE PLANETARY RADIUS)

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SUMMARY

INTRODUCTION

Given the cost of space experiments, the optimization of orbital maneuvers is of evident interest. For the majority of existing rockets, the minimum-propellant maneuver corresponds to a transfer of minimum characteristic velocity. This is the case for chemical and nuclear rockets and also, for the most part, for nuclear-electric propulsion systems. It is with this point of view that transfers between hyperbolic arcs in the attractive Newtonian field of a massive sphere of finite radius are studied here. The ideal case of zero radius leads to some simple solutions but these are not realizable in practice. The study leads to some impulsive transfers, which can be approximated in practice with a very small loss by means of finite thrust arcs of short duration.

PART I--EXPOSITION OF THE THEORETICAL PROBLEM

Consider a single, spherical, mass-attracting planet of a given radius. At time $t_1 = -\infty$ a vehicle which is infinitely far away approaches the planet on a hyperbolic orbit. At time $t_2 = +\infty$ we want the vehicle to be on another hyperbolic orbit, prescribed in advance (and with prescribed orientation) by the transfer of minimum characteristic velocity which does not result in collision with the planetary surface.

Possibilities of atmospheric braking or other atmospheric maneuvers will not be taken into account.

Remark I

The characteristic velocity is the sum of all the propulsive changes of velocity. It is, therefore, the time integral of the absolute value of thrust acceleration added to the arithmetic sum of all impulsive velocity changes, ΔV. The minimum characteristic velocity transfer corresponds to minimum propellant consumption for fixed-exhaust-velocity rockets and for those for which maximum thrust corresponds to operation at maximum exhaust velocity (chemical rockets, etc.).

Remark II

The study leads to some impulsive transfers which are not realizable in practice. Fortunately, the replacement of an impulse made at a distance r from

the attractive center by an optimal finite thrust arc of duration t leads to a small loss relative to the ideal case. The loss is always less than $\frac{\pi^2}{6}\frac{t^2}{T^2}$, T being the period of a circular orbit of radius r, and with the restriction that the thrust acceleration does not vary very much during the time of thrust application.

Remark III

In the ideal case of a point mass planet the optimum is well known (Ref. 1). It consists of six infinitely small impulses: four at infinity and two at the attractive center. The optimum is therefore not realizable in practice. The practical problem requires that one take the finite radius of the planet into account.

Remark IV

At a large distance from the attracting planet, one can modify the asymptotes of arrival and departure at will (but not the asymptotic directions) by means of some infinitely small impulses separated by sufficiently long intervals of time. Likewise one can advance or retard the motion as much as one wishes with a negligible cost in propellant. We can therefore omit consideration of transfer time and rendezvous.

The posed problem then amounts to:

Given: $\vec{V}_1$ = incoming velocity vector at infinity. } with respect to the planet

$\vec{V}_2$ = outgoing velocity vector at infinity. }

R = radius of the planet.

μ = gravitational constant of the planet = the product of its mass and the universal gravitational constant.

Problem: Find the minimum characteristic velocity transfer between $\vec{V}_1$ and $\vec{V}_2$ for which collision with the planetary surface does not occur. Using some dimensional considerations, one can see that μ and R only come into the problem in terms of the planetary escape speed at the surface. We will define then

$$L = \sqrt{\frac{2\mu}{R}} = \text{escape speed at the surface (L for libération in French)}$$

and also

$$A = \text{angle}\ (\vec{V}_1,\ \vec{V}_2)\ \text{where}\ 0^\circ \leq A \leq 180^\circ$$

$U_1 = \sqrt{V_1^2 + L^2}$ = perivelocity at the surface on the hyperbola with $\vec{V}_1$ at infinity

$A_1 = \text{Arc sin } \dfrac{L^2}{L^2 + 2V_1^2}$

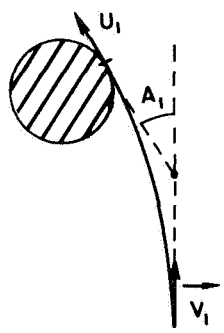
$U_2 = \sqrt{V_2^2 + L^2}$

$A_2 = \text{Arc sin } \dfrac{2}{L^2 + 2V_2^2}$

C = characteristic velocity of the transfer = $\int_{-\infty}^{+\infty} |\vec{\gamma}| dt + \sum |\vec{\Delta V}|$

$\vec{\gamma}$ designates the thrust acceleration and $\vec{\Delta V}$ the impulsive velocity changes.

Fig. 1. Angle A_1 and velocity U_1



PART II--RESULTS

2.1 Evident Cases

2.1.1 The Free Transfer

If $V_1 = V_2$ and $A \leq 2A_1$, $\vec{V}_1$ and $\vec{V}_2$ are the asymptotic velocities of the same hyperbolic orbit which does not intersect the planet. The optimal transfer is then completely determined (Fig. 2) and it costs nothing in terms of propellant.

2.1.2 The Optimal Deviation Transfer

If $A = A_1 + A_2$ the transfer can be made (Fig. 2) by joining two half-branches of hyperbolas which are tangent at perigee, using a tangential impulse, $\Delta V = |U_2 - U_1|$. It is easy to see that this certainly corresponds to the optimal transfer: in effect, this optimal transfer, considered only from the point of view of energy level change from $\frac{V_1^2}{2}$ to $\frac{V_2^2}{2}$, is made optimally by tangential impulses or tangential thrust arcs as near as possible to the center of attraction, and this is the case.

Thus $A = A_1 + A_2$: $C = |U_2 - U_1|$

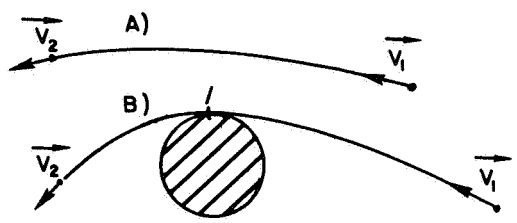
$A \neq A_1 + A_2$: $C \geq |U_2 - U_1|$

<u>Remark</u> $\frac{\partial C}{\partial A} \leq 0$ if $A \leq A_1 + A_2$; $\frac{\partial C}{\partial A} \geq 0$ if $A \geq A_1 + A_2$

Fig. 2

A) free transfer: $V_1 = V_2$; $A \leq 2A_1$

B) "optimal deviation transfer": $A = A_1 + A_2$



2.2 General Cases

The transfers presented above are coplanar. They lie in the plane passing through the center of attraction and containing $\vec{V}_1$ and $\vec{V}_2$. One can demonstrate that the optimal transfer is coplanar for the case of transfers of infinitesimal cost (neighbors of the free transfer) and in a certain number of other cases, but a general demonstration has not been found although this fact appears evident to many in spite of the fact that in certain cases some non-coplanar solutions are equivalent to coplanar ones.

If one accepts this general proposition one obtains the following results.

2.2.1 The Sub-Optimal-Deviation Transfers

These transfers correspond to the case $A < A_1 + A_2$.

They are of five different types:

1. Type F: non-grazing transfers with an impulse at finite distance.
2. Type F_∞ : non-grazing transfers with two impulses, one at finite distance, the other at infinity.
3. & 4. Types RF and RF_∞ : grazing transfers corresponding to the two cases cited above.
5. Type PNP: through the parabolic level. (In French: par le niveau parabolique: PNP)

2.2.1.1 Transfers of Type F

This case has an easy geometric construction (Fig. 3).

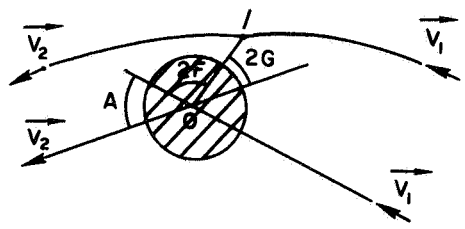
Let us put $S = 45^{\circ} - \frac{A}{4}$

$$D = \text{Arc tan}\left[\frac{V_2 - V_1}{V_2 + V_1} \tan S\right]$$

$$F = S + D$$

$$G = S - D$$

Fig. 3. Non-skimming transfer with one impulse at a finite distance.



F and G determine the direction OI, I being the point where the optimal impulse is made.

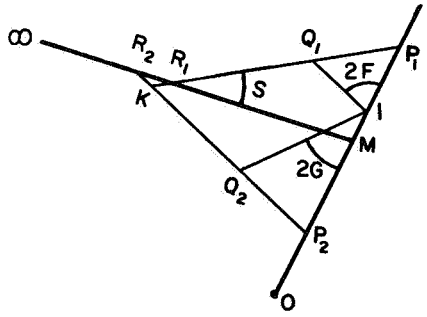
At I the local escape speed is: $L_I = \frac{V_1}{\sin G} \cos D \sqrt{2 \cos^2 S - \cos^2 D}$

which completely determines the point I and the transfer is situated within the re-entrant angle between $\widehat{V_1 O V_2}$.

The cost of this transfer is $C = (V_1 + V_2)\ |\sin D|$

The figure formed in the neighborhood of the point I by the velocities of arrival at and departure from I, $\vec{V}_1'$ and $\vec{V}_2'$, possesses numerous geometric properties (Fig. 4).

Fig. 4. Geometrical properties of the transfer of the figure 3.



$\overrightarrow{IQ_1} = \vec{V_1}$; $\overrightarrow{IR_1} = \vec{V'_1}$;

$\overrightarrow{IQ_2} = \vec{V_2}$; $\overrightarrow{IR_2} = \vec{V'_2}$;

$IP_1 = IQ_1$; $IP_2 = IQ_2$;

$KR_1 = KR_2$; $P_1Q_1 = P_2Q_2$;

$P_1R_1 = P_2R_2$; $MP_1 = MP_2$;

$\widehat{OMR_1} = 90^\circ - D$; $C = R_1R_2 = (V_2 + V_1)\ |\sin D|$.

2.2.1.2 Transfers of Type F∞

If the impulse at infinity is suppressed, a particular case of the preceding transfer is obtained, the case where:

$$\min(V_1, V_2) = \max(V_1, V_2)\left(\frac{1}{5} + \frac{2S^2}{15} + \frac{S^4}{90} + \ldots\ldots\right)$$

(S being expressed in radians). The impulse at infinity is situated on the side of the smaller of the velocities V_1 and V_2. It is an accelerating impulse if $V_1 < V_2$ and a brake if $V_1 > V_2$.

2.2.1.3 Transfers of Type RF

Analogous to Type F, the Type RF transfer is distinguished from it by the fact that in Figure 4, the larger of the two velocities $\vec{V'_1}$ and $\vec{V'_2}$ ends farther away, at a point on the straight line $R_2\infty$ (if $V'_2 > V'_1$) or $R_1\infty$ (if $V'_1 > V'_2$).

With respect to the point of grazing passage, the impulse is on the side of the smaller of the two velocities V_1 and V_2.

2.2.1.4 Transfers of Type RF∞

This is a combination of the Types RF and F∞ .

2.2.1.5 Transfers of Type PNP

These transfers are generalizations of the six-impulse transfers of Ref. 1 to the case of finite planetary radius.

1. A braking impulse of cost $U_1 - L$ at F causes a transfer onto the parabola (or the elongated ellipse) FG (Fig. 5).

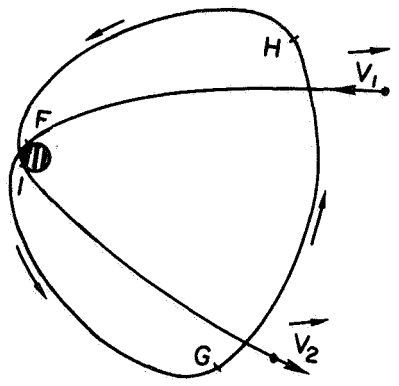
2. Infinitely small impulses at far away points G and H allow one to change the shape and orientation of the parabola.

3. At I, the accelerating impulse $U_2 - L$ permits one to acquire the velocity V_2 at infinity. Only four impulses are seen here. Two more infinitely small impulses applied at a great distance are necessary in order to obtain the given

asymptotes; hence the name six-impulse transfer. The cost of this transfer is then $C = U_1 + U_2 - 2L$.

Remark: This mode of transfer being independent of the orientation, one can write in all cases $|U_2 - U_1| \leq C$ optimal $\leq U_1 + U_2 - 2L$.

Fig. 5. Transfer "through the parabolic level" (PNP).



In practice the PNP transfer is subject to non-coplanar improvements in the greater than optimal-deviation case. $A_1 + A_2 \leq A \leq 180^\circ - |A_2 - A_1|$.

Given V_1, V_2 and A in a sub-optimal deviation case, one must first of all determine to which of the five types the corresponding optimal transfer belongs.

Thus: If: $\min(V_1, V_2) \geq \max(V_1, V_2)\cdot\left(\frac{1}{5} + \frac{2S^2}{15} + \frac{S^4}{90} + \ldots\ldots\right)$

($S = \frac{\pi - A}{4}$ being expressed in radians, the maximum of the expression in brackets is obtained for $S = \pi/4$ and equals 0.28693.) The transfer is of the type F or RF or PNP.

If the transfer of type F does not impact the planet, it is necessary to compare it to the type PNP (the type RF is not then optimal).

Otherwise (this is the case for example if $|V_1 - V_2| \geq L\sqrt{2}$), it is necessary to compare the types RF and PNP.

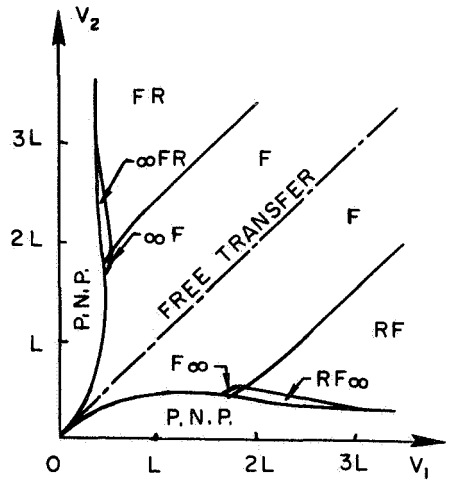
If: $\min(V_1, V_2) < \max(V_1, V_2)\cdot\left(\frac{1}{5} + \frac{2S^2}{15} + \frac{S^4}{90} + \ldots\ldots\right)$

the optimal transfer is not of the type F. It can be of the type RF, RF∞ or PNP. It can even be of the type F∞ in the case where:

$$\min(V_1, V_2) \geq \max(V_1, V_2)\cdot\left(\frac{1}{5} + \frac{56}{845}S^2 + \ldots\ldots\right)$$

Thus it is seen that the erroneous statement, "if there is an impulse at infinity the transfer is necessarily a grazing one," which appeared in the summary last April, is contradicted by type F ∞. This type remains an exception, as one can also see in Figure 6 for the case A = 0.

Fig. 6. Discussion in the case A = 0.



2.2.2 Greater Than Optimal-Deviation Transfers

These transfers correspond to the case $A > A_1 + A_2$. They are always grazing. There are six different types.

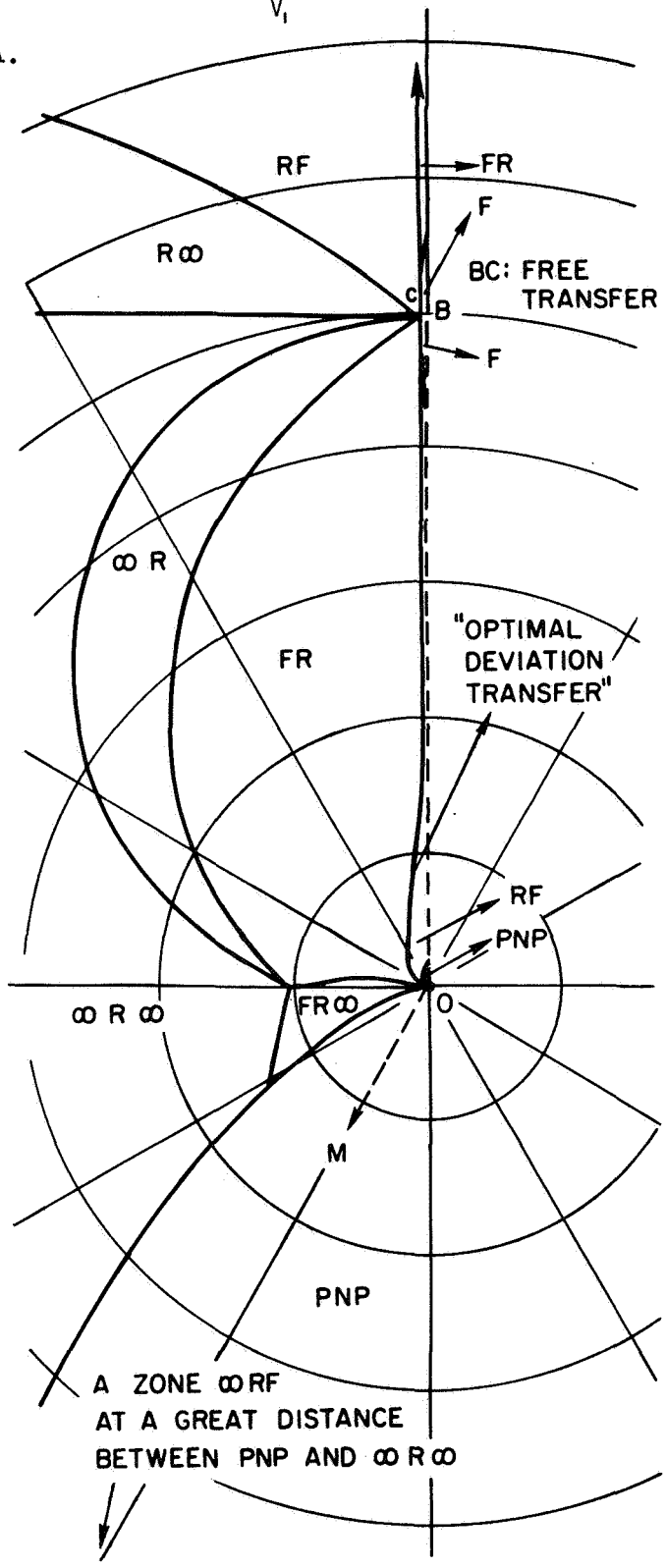
1. Type PNP analogous to that encountered above (always optimal for A = 180°).
2. Type R∞: with an impulse at infinity.
3. Type ∞R∞: with two impulses at infinity, one on each side.
4. Type RF: with an impulse at a finite distance.
5. & 6. Type RF∞ and FR∞: with two impulses, one at a finite distance, the other at infinity, either on the same side as the grazing passage, or on the other side.

These transfers will not be developed. We will make only the following remarks which are equally valid for the cases of sub-optimal-deviation transfer and greater than optimal-deviation transfer.

1. The impulses are always directed in one of the two acute angles between the tangent and the local horizontal (or in one of the two corresponding right angles obtained by continuity for the case of infinite distance).
2. The impulses are always directed at less than 35.264° $\left(= \sin^{-1} \frac{\sqrt{3}}{3}\right)$ from the local horizontal. (This condition, true if the impulses at infinity are executed all at one time, is no longer true if they are broken up into a number of impulses or replaced by arcs of continuous thrust, which can be readily done.)
3. For the sub-optimal-deviation transfers, except the case PNP, the impulses placed before perigee are accelerating thrusts while those placed after are braking. It is the opposite for transfers with greater than optimal deviation.

Figure 7 presents the results in the case where one of the velocities (here V_1) is very much larger than L.

Fig. 7. Discussion in the case $\frac{L}{V_1} \sim 0$, according to the position of M: $\overrightarrow{OB} = \overrightarrow{V_1}$; $\overrightarrow{OM} = \overrightarrow{V_2}$; $\widehat{BOM} = A$.



PART III--UTILIZATION OF THE TRANSFERS BETWEEN HYPERBOLAS

TRIPS IN THE SOLAR SYSTEM

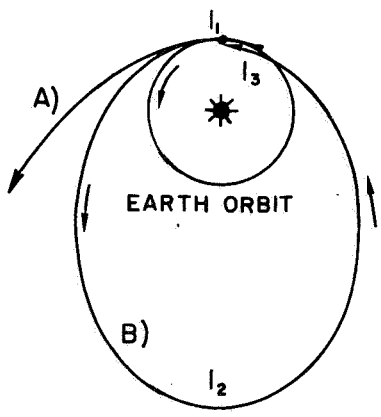
The speed necessary to escape from the solar system leaving from Earth is 16.6 km/sec. But the utilization of the set of impulses shown in Fig. 8 with a braking impulse at I_2 (ΔV = 300 m/s) and an optimal deviation transfer at I_3 (ΔV = 1.1 km/sec) a little less than 4 years after departure, allows one to escape from the solar system with a characteristic velocity of only 15.2 km/sec.

Systematic utilization of the possibilities of hyperbolic transfer offered by the different bodies of the solar system leads to the following for Earth-planet transfers between low parking orbits.

Fig. 8. Escape from the solar system

A) Direct: C = 16.6 km/s

B) I_1: 13.8 km/s
I_2: 0.3 km/s
I_3: 1.1 km/s
} C = 15.2 km/s



Transfer to:	Distance of the Planet from the Sun (Earth = 1)	Characteristic Transfer Velocity Between Low Planetocentric Parking Orbits		Duration of the Hohmann Transfer
		1) Hohmann transfer	2) transfer of long duration	
Mercury	0.387	13.0 km/s	5.6 km/s	0.290 yr
Venus	0.723	7.0	6.8	0.405
Mars	1.524	5.8	5.3	0.71
Minor	2	8.1	5.1	0.92
Planets	3	10.3	5.7	1.42
Jupiter	5.203	24.2	16.6	2.70
Saturn	9.55	18.2	13.2	6.1
Uranus	19.22	14.8	9.8	16
Neptune	30.11	15.8	9.9	31
Pluto	39.52	11.0	5.3	45.5
Escape from the Solar System	∞	8.7	3.5	∞

Hohmann transfers are the classical transfers by means of an ellipse which is tangent to both the orbit of earth and that of the target planet. The "long duration" transfers (50 to 200 years) utilize possibilities offered by the presence of the nine major planets and the larger satellites (Titan, Triton and the four large satellites of Jupiter). The transfer to Mercury utilizes the eccentricity of the orbit of this planet.

From a practical point of view, one can obtain characteristic velocities mid way between the Hohmann transfer and the "long duration" transfer at the price of an increase in trip time of the order of 5 to 10 years with respect to the Hohmann transfer time.

Hence: 1. Transfers between hyperbolic orbits offer little interest for trips to Mars or Venus.

2. The long mission times that they impose render them useless for manned missions, while on the other hand they could be used for the transport of great quantities of supplies or of raw material.

Remark: The above study has been carried out independent of the possibilities of atmospheric braking. The maximal utilization of these possibilities ends in a simple result: the characteristic velocity of a round trip is diminished by half in each case.

CONCLUSIONS

From the point of view of minimum characteristic velocity, optimum transfers between hyperbolic orbits are always composed of impulses and have at most two finite impulses. While these transfers are practically useless for trips to Mars and Venus, their utilization in other planetary missions leads to a saving in characteristic velocity of a few km/sec while requiring an increase in trip time of between 5 and 10 years.

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