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CALCULATION OF THE MOTION OF A LOW-THRUST SPACECRAFT

by V. N. Lebedev

Computer Center, Academy of Sciences USSR
Moscow, 1968

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Translation of "Raschet Dvizheniya Kosmicheskogo
Apparata s Maloy Tyagoy"
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INTRODUCTION

The development of space technology has resulted in the creation of inter-/3* plantetary spacecraft. A spacecraft is an artifical celestial body upon which both Newtonian forces and the force of gravity act. The motion of a spacecraft is the object of investigation of a new branch of mechanics, specifically the mechanics of space flight.

There are two types of engines for spacecraft. Engines of the first type develop thrust which exceeds the Newtonian force, but only for a short period of time. Engines of the second type develop low thrust, but operate for long periods of time.

The dynamics of low-thrust spacecraft are examined in this book. In many cases the force of gravity can be regarded as low in comparison with the Newtonian force, and it can also be regarded as a perturbation. This circumstance enables us to use the classical methods of the perturbation theory of celestial mechanics or other asymptotic methods. The use of these methods makes it possible to obtain approximate analytical solutions for certain problems, while in the other cases it affords graphical physical prerequisites for the numerical solution.

The vital problem concerning the search for the optimum characteristics of a low-thrust spacecraft executing a given maneuver has been stated in the works of such scientists G. L. Grodzovskiy, U. Yu. Ivanov, and V. V. Tokarev. They have shown that the solution of the stated problem can be found in many cases by making a preliminary calculation of the motion of a spacecraft.

This book represents an outline of the methods for calculating the motion of a spacecraft developing low-thrust. Numerous calculations, which can be of independent interest, are made by the methods proposed herein. Several examples which illustrate the methods by which it is possible to determine the optimum characteristics of a spacecraft on the basis of the calculation of its /4 motion are examined in the last chapter.

It is assumed in the solution of all problems contained herein that the spacecraft is equipped with a non-throttleable engine. Moreover, it is always assumed that the spacecraft, considered as a material point, moves within the gravitational field of a single body. Perturbations caused by non-sphericity of a planet, atmospheric drag etc. are not taken into account.

The basic content of the book consists of problems whose solutions were found by the author in the laboratory of General Mechanics and Hydrodynamics at the Computer Center of the USSR Academy of Sciences. The appropriate citations had been made in those cases where the results of other authors have been used. Furthermore, the author would like to take this opportunity to mention that the results of unpublished works of I. A. Vatel', I. Ye. Vol'fson, A. A. Golubeva, and F. D. Turetska were used in the third chapter.

*Numbers in the margin indicate pagination on the foreign text.

The author also expressed his gratitude to N. N. Moiseyev, U. Yu. Yevtushenko, A. F. Kononenko, A. A. Petrov, and S. N. Miroshnik for their dilegent review of the manuscript of this book and for their many helpful remarks.

CHAPTER 1

SIMPLE METHODS OF SPACECRAFT CONTROL

V. N. Lebedev

ABSTRACT. The derivation and analysis of simple methods of control which make it possible to execute certain maneuvers are considered in Chapter 1. The basic requirement imposed upon these control methods is simplicity of analysis and execution. Methods of control which maintain the osculating elements, or their more rapidly changing values, constant are examined. The methods of control described herein, and combinations of them, can be used for the execution of a variety of maneuvers.

One of the basic problems concerning the control of a spacecraft is the determination and analysis of simple control systems by which a given maneuver can be executed. Such methods of control do not necessarily have to be close to optimal. A basic requirement is simplicity of analysis and execution (in comparison with optimal control). In this chapter we will investigate methods of control by which the values of the osculating elements, or their more rapidly changing values, are kept constant. A wide variety of manuvers can be executed by these methods, or of combinations thereof. /5

§1. Differential Relations for Osculating Elements

By regarding the thrust of an engine as a perturbing force, the motion of the craft can be described by the Newton equations for the theory of perturbed motion [1]:

$$\frac{de}{dt} = \sqrt{\frac{p}{\mu}}\left(\sin\vartheta a_r + \frac{e\cos^2\vartheta + 2\cos\vartheta + e}{1+e\cos\vartheta}\, a_\varphi\right); \tag{1.1}$$

$$\frac{dp}{dt} = 2p\sqrt{\frac{p}{\mu}}\,\frac{1}{1+e\cos\vartheta}\, a_\varphi; \tag{1.2}$$

$$\frac{d\omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{\mu}}\left[-\cos\vartheta a_r + \frac{\sin\vartheta(2+e\cos\vartheta)}{1+e\cos\vartheta}\, a_\varphi - \frac{e\sin u\,\mathrm{ctg}\, i}{1+e\cos\vartheta}\, a_z\right]; \tag{1.3}$$

$$\frac{d\Omega}{dt} = \sqrt{\frac{p}{\mu}}\,\frac{\sin u\,\mathrm{cosec}\, i}{1+e\cos\vartheta}\, a_z; \tag{1.4}$$

$$\frac{di}{dt} = \sqrt{\frac{p}{\mu}}\,\frac{\cos u}{1+e\cos\vartheta}\, a_z; \tag{1.5}$$

$$\frac{d\tau}{dt} = \frac{p^2}{e\mu(1+e\cos\vartheta)^2}\left[(eN\sin\vartheta - \cos\vartheta)a_r + \frac{N}{1+e\cos\vartheta}\, a_\varphi\right], \tag{1.6}$$ /6

where $u = \omega + \vartheta$;

$$N = \frac{2}{1+e\cos\vartheta}\int_0^{\vartheta}\frac{\cos\vartheta' d\vartheta'}{(1+e\cos\vartheta')^3}$$

and ϑ is related to t by

$$t-\tau = p\sqrt{\frac{p}{\mu}}\int_0^{\vartheta}\frac{d\vartheta'}{(1+e\cos\vartheta')^2}$$

Here e is eccentricity, p is a parameter, ω is the angular distance from the pericenter to the node: Ω is the longitude of the ascending node: i is the inclination of the orbit: τ is the time of transit through the pericenter: t is time: ϑ is the true anomaly: u is the latitude argument: a_r, a_ψ, a_z are projections of the reactive acceleration (ratios of the force of gravity to the mass of the spacecraft) in the direction of the vector radius, perpendicular to it in the orbital plane, and perpendicular to the orbital plane: $\mu = fM$ is the product of the gravitational constant and the mass of the attracting center.

Instead of the elements e and p, we may use the following values: the distance to the pericenter

$$r_n = \frac{p}{1+e}, \tag{1.7}$$

the distance to the apocenter

$$r_a = \frac{p}{1-e} \tag{1.8}$$

and the large semiaxis

$$A = \frac{p}{1-e^2}. \tag{1.9}$$

By differentiating expressions (1.7)-(1.9) and using (1.1) and (1.2), we obtain

$$\frac{dr_n}{dt} = \frac{p}{(1+e)^2}\sqrt{\frac{p}{\mu}}\left[-\sin\vartheta a_r + \frac{2(1-\cos\vartheta)+e\sin^2\vartheta}{1+e\cos\vartheta}a_\varphi\right]; \tag{1.10}$$

$$\frac{dr_a}{dt} = \frac{p}{(1-e)^2}\sqrt{\frac{p}{\mu}}\left[\sin\vartheta a_r + \frac{2(1+\cos\vartheta)-e\sin^2\vartheta}{1+e\cos\vartheta}a_\varphi\right]; \tag{1.11}$$

$$\frac{dA}{dt} = \frac{2p}{(1-e)^2}\sqrt{\frac{p}{\mu}}\left[e\sin\vartheta a_r + (1+e\cos\vartheta)a_\varphi\right]. \tag{1.12}$$ /7

As the first two equations in the equation system (1.1)-(1.6), we may use any pair of (1.1), (1.2), (1.10)-(1.12) [2].

§2. Simple Methods of Controlling the Thrust of a Spacecraft

Several simple methods of spacecraft control based on the alteration of the orbital parameters can be derived directly from the equation system (1.1)-(1.6). Thus, for example, it is clear from equation (1.2) that if the thrust factor of the engine is directed within the orbital plane perpendicular to the vector radius (in the transversal) then the parameter p will vary at the greatest possible rate for a given amount of thrust. But if the projection of the thrust on the transversal is equal to zero, then the parameter will remain constant. It follows from equations (1.4) and (1.5) that the position of the orbital plane in space is affected only by the thrust component normal to the instantaneous orbital plane, whereupon the monotonic change of the inclination of the orbit requires that the direction of the thrust be reversed at the point where $u = \pm\pi/2$, while the monotonic change of the longitude of the ascending node requires that the reversals occur on the line of the nodes where $u = 0, \pi$. The qualitative and quantitative (based on the method of averages) analyses of the variation of orbital elements under the effect of thrust normal to the orbital plane are given in [3]. The solution of the problem is given in finite form in [4] for the case of a circular orbit and constant reactive acceleration.

Let us consider flat manouvers for spacecraft ($a_z = 0$). Let a be the modulus of reactive acceleration, and λ the angle between the vector radius of the spacecraft and the thrust vector (Figure 1). Then $a_r = a$ cosine λ $a_\phi = a$ sin λ, and the differential relations (1.1)-(1.3) and (1.10)-(1.12) for the elements e, p, ω, r_n, r_a, and A will be written as follows:

$$\frac{de}{dt} = a\sqrt{\frac{p}{\mu}}\left[\sin\vartheta\cos\lambda + \frac{e\cos^2\vartheta + 2\cos\vartheta + e}{1+e\cos\vartheta}\sin\lambda\right]; \tag{1.13}$$

$$\frac{dp}{dt} = 2ap\sqrt{\frac{p}{\mu}}\frac{1}{1+e\cos\vartheta}\sin\lambda; \tag{1.14}$$ /8

$$\frac{d\omega}{dt} = \frac{a}{e}\sqrt{\frac{p}{\mu}}\left[-\cos\vartheta\cos\lambda + \frac{\sin\vartheta(2+e\cos\vartheta)}{1+e\cos\vartheta}\sin\lambda\right]; \tag{1.15}$$

$$\frac{dr_n}{dt} = \frac{ap}{(1+e)^2}\sqrt{\frac{p}{\mu}}\left[-\sin\vartheta\cos\lambda - \frac{2(1-\cos\vartheta)+e\sin^2\vartheta}{1+e\cos\vartheta}\sin\lambda\right]; \tag{1.16}$$

$$\frac{dr_a}{dt} = \frac{a\,p}{(1-e)^2}\sqrt{\frac{p}{\mu}}\left[\sin\vartheta\cos\lambda + \frac{2(1+\cos\vartheta) - e\sin^2\vartheta}{1+e\cos\vartheta}\sin\lambda\right]; \tag{1.17}$$

$$\frac{dA}{dt} = \frac{2a\,p}{(1-e^2)^2}\sqrt{\frac{p}{\mu}}\,[e\sin\vartheta\cos\lambda + (1+e\cos\vartheta)\sin\lambda]. \tag{1.18}$$

All the relations obtained are of the form

$$\frac{dK}{dt} = \frac{a}{\sqrt{\mu}}[f_1(e, p, \vartheta)\cos\lambda + f_2(e, p, \vartheta)\sin\lambda], \tag{1.19}$$

where K is any one of the six elements under consideration; f_1 and f_2 are known functions of e, p, ϑ. By equating expression (1.19) to zero, we find

$$\tan\lambda = -\frac{f_1(e, p, \vartheta)}{f_2(e, p, \vartheta)}, \tag{1.20}$$

which assures that the element K will be constant.

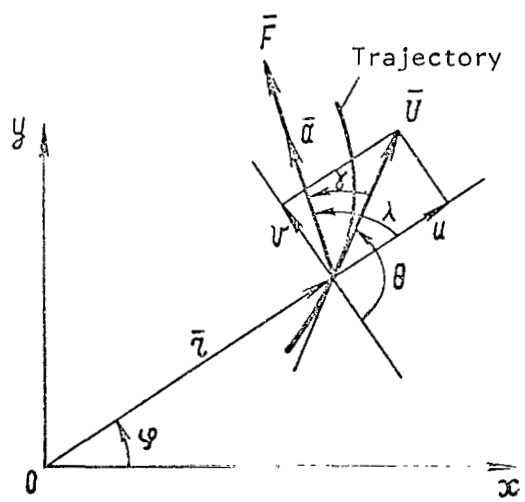

Figure 1

To find the equation which will insure the maximum rate of change of the element K, we will equate the partial derivative for the λ function of (1.19) to zero, assuming that acceleration a clearly is not a function of λ:

$$\frac{\partial}{\partial\lambda}\frac{dK}{dt} = \frac{a}{\sqrt{\mu}}[-f_1(e, p, \vartheta)\sin\lambda + f_2(e, p, \vartheta)\cos\lambda] = 0,$$

from which we find

$$\tan\lambda = \frac{f_2(e, p, \vartheta)}{f_1(e, p, \vartheta)}; \tag{1.21}$$

$$\sin\lambda = \frac{f_2(e, p, \vartheta)}{\pm\sqrt{[f_1(e, p, \vartheta)]^2 + [f_2(e, p, \vartheta)]^2}}; \tag{1.22}$$

$$\cos\lambda = \frac{f_1(e, p, \vartheta)}{\pm\sqrt{[f_1(e, p, \vartheta)]^2 + [f_2(e, p, \vartheta)]^2}}. \tag{1.23}$$

We then substitute the values of sin λ and cosine λ found, into the expression/9 for the second derivative of the function dK/dt for λ and into the value of the derivative of dK/dt:

$$\frac{\partial^2}{\partial\lambda^2}\frac{dK}{dt} = \frac{a}{\sqrt{\mu}}[-f_1(e, p, \vartheta)\cos\lambda - f_2(e, p, \vartheta)\sin\lambda] =$$

$$= \mp\frac{a}{\sqrt{\mu}}\sqrt{[f_1(e, p, \vartheta)]^2 + [f_2(e, p, \vartheta)]^2};$$

$$\frac{dK}{dt} = \pm\frac{a}{\sqrt{\mu}}\sqrt{[f_1(e, p, \vartheta)]^2 + [f_2(e, p, \vartheta)]^2}.$$

Thus the equation

$$\tan\lambda = \frac{f_2(e, p, \vartheta)}{f_1(e, p, \vartheta)}; \quad \sin\lambda = \frac{f_2(e, p, \vartheta)}{\sqrt{[f_1(e, p, \vartheta)]^2 + [f_2(e, p, \vartheta)]^2}}$$

insures the maximum rate of increase for K, while the equation

$$\tan\lambda = \frac{f_2(e, p, \vartheta)}{f_1(e, p, \vartheta)}; \quad \sin\lambda = -\frac{f_2(e, p, \vartheta)}{\sqrt{[f_1(e, p, \vartheta)]^2 + [f_2(e, p, \vartheta)]^2}}$$

corresponds to the maximum rate of reduction of K.

Comparison of formulas (1.20) and (1.21) shows that the thrust vectors which/10 provide constant values of K and the maximum rate of change of K are perpendicular.

By using the procedure outlined above for relations (1.13)-(1.18), we find the following means of control of the spacecraft:

1) control insuring constant eccentricity

$$\tan\lambda = -\frac{\sin\vartheta(1+e\cos\vartheta)}{e\cos^2\vartheta+2\cos\vartheta+e}; \tag{1.24}$$

2) control insuring the constant parameter

$$\lambda = 0, \quad \lambda = \pi \tag{1.25}$$

(radial thrust);

3) control insuring constant position of apsidal line

$$\tan\lambda = \cot\vartheta\frac{1+e\cos\vartheta}{2+e\cos\vartheta}; \tag{1.26}$$

4) control insuring constant distance to the pericenter

$$\tan\lambda = \frac{\sin\vartheta(1+e\cos\vartheta)}{2(1-\cos\vartheta)+e\sin^2\vartheta}; \tag{1.27}$$

5) control insuring constant distance to apocenter

$$\tan\lambda = -\frac{\sin\vartheta(1+e\cos\vartheta)}{2(1+\cos\vartheta)-e\sin^2\vartheta}; \tag{1.28}$$

6) control insuring constant large semiaxis

$$\tan\lambda = -\frac{e\sin\vartheta}{1+e\cos\vartheta} \tag{1.29}$$

(normal thrust);

7) control insuring maximum rate of change of eccentricity

$$\tan\lambda = \frac{e\cos^2\vartheta+2\cos\vartheta+e}{\sin\vartheta(1+e\cos\vartheta)}; \tag{1.30}$$

8) control insuring maximum rate of change of the parameter

$$\lambda = \pm\pi/2 \tag{1.31}$$

(transversal thrust);

9) control insuring maximum rate of rotation of apsidal line /11

$$\tan\lambda = -\tan\vartheta\,\frac{2+e\cos\vartheta}{1+e\cos\vartheta}; \tag{1.32}$$

10) control insuring maximum rate of change of distance to pericenter

$$\tan\lambda = -\frac{2(1-\cos\vartheta)+e\sin^2\vartheta}{\sin\vartheta(1+e\cos\vartheta)}; \tag{1.33}$$

11) control insuring maximum rate of change of distance to apocenter

$$\tan\lambda = \frac{2(1+\cos\vartheta)-e\sin^2\vartheta}{\sin\vartheta(1+e\cos\vartheta)}; \tag{1.34}$$

12) control insuring maximum rate of change of large semiaxis

$$\tan\lambda = \frac{1+e\cos\vartheta}{e\sin\vartheta} \tag{1.35}$$

(tangential thrust).

It follows from formulas (1.24)-(1.35) that the direction of the thrust vector is a function only of eccentricity and true anomaly. These values can be expressed through the coordinates of the spacecraft and its velocity components. By using a polar coordinate system, control becomes a function of the distance between the attracting center and the spacecraft r and of the radial and transversal velocity components u and v. The form of this function can be found by using the following relations:

$$e = \sqrt{(rv^2-1)^2+(ruv)^2};$$

$$\sin\vartheta = \frac{ruv}{\sqrt{(rv^2-1)^2+(ruv)^2}};$$

$$\cos\vartheta = \frac{rv^2-1}{\sqrt{(rv^2-1)^2+(ruv)^2}}.$$

Some of formulas (1.24)-(1.35) are given in [5].

§3. Some Features of Simple Methods of Control /12

Relations (1.24)-(1.35) determine the direction of the thrust vector. If the reactive acceleration is controllable, it should be equal to the maximum value, since the rate of change of the osculating elements is proportional to the magnitude of reactive acceleration. Reactive acceleration is a known function of time for the case of constant thrust and constant per-second fuel consumption.

In the derivation of relations (1.24)-(1.35) it was assumed that there were no perturbing forces. The presence of perturbations is risky only for the group of control procedures which insure a constant osculating element (equations (1.24)-(1.29)). As soon as the perturbing force is no longer small in comparison with the thrust, the osculating element ceases to be constant. Naturally it is possible that certain types of perturbations will produce some small, but tolerable, long-term changes in the element. It is no worry that the vector sum of perturbing forces is known, then it is always possible to find an equation which insures that the osculating element will be constant (assuming of course that the modulus of this vector is less than the thrust force). The direction of thrust should be selected in such a way that the vector equal to the sum of the vectors of the perturbing forces and thrust will form with the vector radius an angle λ determined by the corresponding relation from (1.24)-(1.29).

The presence of perturbing forces, as we will demonstrate, does not hamper the maneuver for the second group of control procedures, specifically for those equations which insure the maximum rate of change of the elements.

Let b be the ratio of the total perturbing force, a function in the general case of the phase coordinates of the spacecraft and of time, to the mass of the spacecraft, and let κ be the angle between the vector radius and the direction of the perturbing force. Then expression (1.19) for the derivative of some osculating element in time will have the form (for brevity we will not show that f_1 and f_2 are functions of the phase coordinates):

$$\frac{dK}{dt} = \frac{1}{\sqrt{\mu}}[f_1(a\cos\lambda + b\cos\varkappa) + f_2(a\sin\lambda + b\sin\varkappa)].$$

Into this expression we will substitute control (1.22) and (1.23), which insures the maximum rate of cnange of the element K: /13

$$\frac{dK}{dt} = \frac{1}{\sqrt{\mu}}\left[\pm a\sqrt{f_1^2+f_2^2}+b(f_1\cos\varkappa+f_2\sin\varkappa)\right]. \tag{1.36}$$

Here the "plus" sign corresponds to an increase and the "minus" sign corresponds to a decrease of the element K. It is not difficult to demonstrate that the first term in expression (1.36) for the modulus is always greater than the second if a > b. In other words, if the perturbing force is less than the thrust, the change in the element will occur in the required direction although not at the maximum possible rate. The smaller the perturbing force, the closer the rate of change of the element will be to the maximum possible.

The execution of certain maneuvers can be impeded by one of the inadequacies of the control methods outlined above. By way of illustration, let us consider the following maneuver: a reduction in osculating eccentricity at a constant distance from the pericenter. We will substitute the value of the tangent of the controlling angle (1.27) into the expression for the derivative of de/dt derived from (1.13). We thus obtain

$$\frac{de}{dt} = a\sqrt{\frac{p}{\mu}}\sin\lambda\left(\frac{\sin\vartheta}{\tan\lambda}+\frac{e\cos^2\vartheta+2\cos\vartheta+e}{1+e\cos\vartheta}\right).$$

With consideration of the function

$$r = \frac{p}{1+e\cos\vartheta}$$

we find

$$\frac{de}{dt} = \frac{2ar(1+e)}{\sqrt{\mu p}}\sin\lambda,$$

from which it follows that in order to reduce eccentricity, the controlling angle should be located in the third or fourth quadrant. Thus, angle λ is determined from the relations

$$\tan\lambda = \frac{\sin\vartheta(1+e\cos\vartheta)}{2(1-\cos\vartheta)+e\sin^2\vartheta}; \tag{1.37}$$

$$\bar{u} \leq \lambda < 2\pi. \tag{1.38}$$

Figure 2a, where it was assumed in the construction of the graph that eccentri-/14 city is constant value, equal to 0.5, gives a representation of the nature of the dependence of the controlling angle on the true anomaly. The figure shows that when $\vartheta = \pi$, the direction of thrust must be reversed. The dependence of the controlling angle on time differs greatly from the graph in Figure 2, since the angular velocity of rotation of the apsidal line may have either sign and may be as large as desired, as follows from (1.15). These two conditions, specifically the violation of the function $\lambda = \lambda(\vartheta, e)$ and the unboundedness of the angular velocity of rotation of apsidal line, also present difficulties in the execution of the above-mentioned maneuver. Let us elaborate on this.

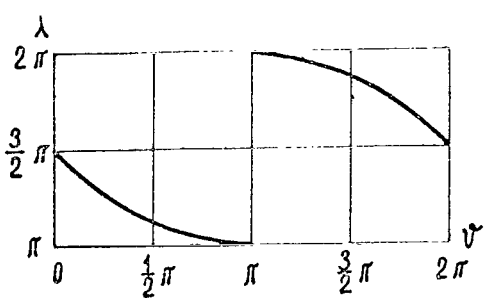

Figure 2

Let ϕ be the polar angle, i. e. the angle taken from some fixed axis in the plane of projectory counter clockwise to the vector radius of the spacecraft (see Figure 1). The angles ϑ, ω and ϕ are related by the relation (assuming that angles ω and ϕ are taken from the same axis)

$$\vartheta = \varphi - \omega,$$

whence we find the value of the derivative of the true anomaly

$$\frac{d\vartheta}{dt} = \frac{d\varphi}{dt} - \frac{d\omega}{dt}. \tag{1.39}$$

but we will compute the value of this derivative in the apocenter. From (1.15) it follows that

$$\left.\frac{d\omega}{dt}\right|_{\vartheta=\pi} = \sqrt{\frac{p}{\mu}}\, a/e\cos\lambda. \tag{1.40}$$

By using the known relations, we find /15

$$\left.\frac{d\varphi}{dt}\right|_{\vartheta=\pi} = \left.\frac{v}{r}\right|_{\vartheta=\pi} = \sqrt{\frac{\mu}{p}}\,\frac{(1-e)^2}{p}. \tag{1.41}$$

Then, by substituting (1.40) and (1.41) into (1.39), we obtain

$$\left.\frac{d\vartheta}{dt}\right|_{\vartheta=\pi} = \sqrt{\frac{\mu}{p}}\,\frac{(1-e)^2}{p} - \sqrt{\frac{p}{\mu}}\,\frac{a}{e}\cos\lambda.$$

A change in the direction of thrust in the apocenter results in a sudden change in the derivative $d\vartheta/dt$ by the value of $2a\sqrt{p/\mu}/e$. If it thus seems that the angular velocity of the spacecraft with respect to the modulus is less than the rotational velocity of the apsidal line, i.e.

$$\sqrt{\frac{\mu}{p}}\,\frac{(1-e)^2}{p} < \sqrt{\frac{p}{\mu}}\,\frac{a}{e}, \tag{1.42}$$

then the derivative $d\vartheta/dt$ changes signs. Let the spacecraft approach the apocenter from the direction of the values of true anomaly less than π. The apocenter is displaced toward the spacecraft, since from (1.15), (1.37) and (1.38) it follows that when ϑ is close to π, the angular rotational velocity of the apsidal line is negative. Let us assume that the inequality

$$\left|\frac{d\omega}{dt}\right| > \left|\frac{d\varphi}{dt}\right|.$$

is satisfied. The spacecraft arrives at the apocenter with a controlling angle $\lambda = \pi$ (see Figure 2). After the direction of thrust is reversed, the angular velocity of the spacecraft does not change, but the angular rotational velocity of the apsidal line changes sign, retaining the same magnitude. The result is that the spacecraft changes its direction of motion relative to the apocenter, i.e. the value of the true anomaly becomes less than π, and the value of the controlling angle should be close to π, but not 2π. Thus, in the example under consideration, it is impossible to find an equation which guarantees the execution of the maneuver (the reduction of eccentricity at a constant distance from the pericenter) after the spacecraft arrives at the apocenter. The simplest escape from this situation is the shutting down of the engine for a period

of time (for example during the time of motion of the spacecraft from the apocenter to the pericenter).

It follows from (1.42) that the obstacle under consideration arises if the/16 inequality

$$a/e \geq \mu/r_a^2,$$

is satisfied at the moment $\vartheta = \pi$, i.e. in cases where either the thrust is great in comparison with the local weight of the spacecraft, or the eccentricity is small.

In concluding this section, it should be noted that in some cases, apparently, the control methods outlined above should be replaced by simpler methods. Thus, if the thrust is small in comparison with the force of attraction, while eccentricity is of the order of unity, but less than unity, we may assume on the basis of equations (1.13) and (1.15) that e and ω will be constants during the time of one, or even of several revolutions. In this case the controlling angle λ will be a function only of the polar angle. The shape of this function should change somewhat through one or through several revolutions of the spacecraft, by determining new values of e and ω.

§4. Analysis of Change in Osculating Elements

In order to accomplish control procedures (1.24)-(1.29), which provide the constancy of some osculating element, it is necessary in each case to know the behavior of the other elements so that we may be able in the required manner to select one of the two controlling angles which satisfy equation (1.24)-(1.29). Thus, in the example described in the preceding section (reduction of eccentricity at a constant distance from the pericenter), the controlling angle was defined uniquely only after the character of change of eccentricity had been analyzed. As regards control methods (1.30)-(1.35), which correspond to the maximum rate of change of the elements, the choice of the direction of change of the element (increase or decrease) enables us to define uniquely the controlling angle. It is necessary in this case, however, to know the behavior of the other osculating elements.

Relations (1.13)-(1.18) enable us, from the qualitative standpoint, to analyze the changes in the elements in one revolution with respect to the true anomaly. It seems that in many cases that certain elements vary in one direction for all values of the true anomaly; this prompts the conclusion that the /17 corresponding elements vary monotonically during the course of the entire maneuver.

As an example, let us consider the behavior of the osculating elements during thrust in the tangential direction.

1. Eccentricity.

Let us substitute the value of the tangent of the controlling angle (1.35) into the expression for the time derivative of eccentricity (1.13):

$$\frac{de}{dt} = a\sqrt{\frac{p}{\mu}}\sin\lambda\left(\frac{\sin\vartheta}{\operatorname{tg}\lambda} + \frac{e\cos^2\vartheta + 2\cos\vartheta + e}{1+e\cos\vartheta}\right) = \frac{2ar(e+\cos\vartheta)}{\sqrt{\mu p}}\sin\lambda.$$

It follows from the expression derived that when $0 \leq \lambda < \pi$, i.e. in the case where thrust is directed with velocity, eccentricity increases in that part of the revolution where $\cos\vartheta > -e$. When $\cos\vartheta < -e$ eccentricity diminishes. If thrust is directed against velocity ($\pi \leq \lambda < 2\pi$), then eccentricity increases if $\cos\vartheta < -e$, but diminishes if $\cos\vartheta > -e$. In the case of hyperbolic motion ($e > 1$), eccentricity always increases when $0 \leq \lambda < \pi$, but diminishes when $\pi \leq \lambda < 2\pi$.

2. Parameter.

$$\frac{dp}{dt} = 2ar\sqrt{\frac{p}{\mu}}\sin\lambda.$$

When $0 \leq \lambda < \pi$, the parameter increases and when $\pi \leq \lambda < 2\pi$, it diminishes.

3. Angular Distance of Pericenter from Node.

$$\frac{d\omega}{dt} = \frac{a}{e}\sqrt{\frac{p}{\mu}}\sin\lambda\left[-\frac{\cos\vartheta}{\tan\lambda} + \frac{\sin\vartheta(2+e\cos\vartheta)}{1+e\cos\vartheta}\right] = \frac{2ar\sin\vartheta}{e\sqrt{\mu p}}\sin\lambda.$$

When $0 \leq \lambda < \pi$, angle ω increases if $0 < \vartheta < \pi$, and diminishes if $\pi < \vartheta < 2\pi$. When $\pi \leq \lambda < 2\pi$, angle ω increases if $\pi < \vartheta < 2\pi$, and diminishes if $0 < \vartheta < \pi$.

4. Distance to Pericenter.

$$\frac{dr_n}{dt} = \frac{ap}{(1+e)^2}\sqrt{\frac{p}{\mu}}\sin\lambda\left[-\frac{\sin\vartheta}{\tan\lambda} + \frac{2(1-\cos\vartheta)+e\sin^2\vartheta}{1+e\cos\vartheta}\right] =$$

$$= \frac{2ar(1-\cos\vartheta)}{(1+e)^2}\sqrt{\frac{p}{\mu}}\sin\lambda.$$

If $0 \leq \lambda < \pi$, the distance to the pericenter increases, and if $\pi \leq \lambda < 2\pi$, it diminishes.

5. Distance to Apocenter. /18

$$\frac{dr_a}{dt} = \frac{ap}{(1-e)^2}\sqrt{\frac{p}{\mu}}\sin\lambda\left[\frac{\sin\vartheta}{\tan\lambda} + \frac{2(1+\cos\vartheta)-e\sin^2\vartheta}{1+e\cos\vartheta}\right] =$$

$$= \frac{2ar(1+\cos\vartheta)}{(1-e)^2}\sqrt{\frac{p}{\mu}}\sin\lambda.$$

The distance to the apocenter increases when $0 \leq \lambda < \pi$ and it diminishes when $\pi \leq \lambda < 2\pi$.

6. The Large Semiaxis.

$$\frac{dA}{dt} = \frac{2ap}{(1-e^2)^2}\sqrt{\frac{p}{\mu}}\sin\lambda\left(\frac{e\sin\vartheta}{\tan\lambda} + 1 + e\cos\vartheta\right) =$$

$$= \frac{2ar[(e\sin\vartheta)^2+(1+e\cos\vartheta)^2]}{(1-e^2)^2}\sqrt{\frac{p}{\mu}}\sin\lambda.$$

When $0 \leq \lambda < \pi$, the large semiaxis increases, but when $\pi \leq \lambda < 2\pi$ it diminishes.

Thus, if the thrust is directed with velocity, the parameter, distances to the pericenter and apocenter, and large semiaxis increase monotonically. When the motion is hyperbolic, eccentricity also increases monotonically. If the thrust is directed counter to the velocity then all these elements will diminish monotonically.

Analogous research was conducted for all methods of control. The results of these analyses are summarized in Table 1 (for elliptical motion) and in Table 2 (for hyperbolic motion). The inequalities for ϑ in the boxes of the tables indicate the ranges where the osculating elements increase when $0 \leq \lambda < \pi$. Thus, for example, for the method of constant direction of the apsidal line when the motion is elliptical, the large semiaxis increases when $\cos\vartheta > 0$ and when $\cos\vartheta < -2e/(1+e^2)$, if $0 \leq \lambda < \pi$. This means that when $-2e/(1+e^2) < \cos\vartheta < 0$ the large semiaxis diminishes. If angle λ is taken within the range π to 2π, then the large semiaxis, on the hand, diminishes if ϑ satisfies the inequalities $\cos\vartheta > 0$ or $\cos\vartheta < -2e/(1+e^2)$ and increases when $-2e/(1+e^2) < \cos\vartheta < 0$.

The empty boxes in the tables signify that the corresponding elements diminish for all ϑ (for the case $0 \leq \lambda < \pi$).

§5. Possible Applications of Simple Methods of Control /19

Of the control methods which we have examined so far, the following have been investigated the most thoroughly tangential thrust, transversal, radial and normal. Much research in particular has been devoted to the application of tangential and transversal thrust for the propulsion of a spacecraft by a low-thrust engine. Some of the other methods can be used for the purposes of adjusting orbits. For example, the rounding of an elliptical orbit can be accomplished by the application of control which insures the maximum rate of change of eccentricity. In reducing eccentricity (not at the maximum rate) it is possible to maintain the constant distances to the pericenter or the apocenter etc.

The method of control which insures increased eccentricity at a constant distance to the pericenter can be used for probing near-earth space. This method is suitable because the spacecraft is nearer the surface of the earth in each revolution, which is advantageous for the transmission of information, while the apocenter gets increasingly farther from the earth as eccentricity is increased.

Combinations of the above-examined control methods can be used for the execution of rather complex maneuvers. The solution of the problem examined in the following section, concerning the positioning of a spacecraft in a circular orbit as it approaches a planet, is a good example.

§6. The Problem of Placing a Spacecraft in a Circular Orbit on Approach to a Planet

We know that during the acceleration of a spacecraft the tangential thrust is close to optimal from the standpoint of fuel consumption. It is natural to assume that deceleration of the spacecraft is also easily accomplished by using tangential thrust.

We will assume that deceleration of a spacecraft begins at the moment that the spacecraft falls into the sphere of influence of the planet. To insure that a circular orbit of a given radius is achieved through tangential thrust during deceleration, the velocity vector of the spacecraft must be completely defined at the initial moment. By solving the opposite problem, i.e. the problem of acceleration of a spacecraft using tangential thrust from a given circular orbit to the sphere of influence, it is possible to determine this /20 vector: $U = \tilde{U}_0$, $\theta = \theta_0$, where U is the modulus of the velocity vector; θ is the angle between the line perpendicular to the vector radius and the velocity vector (see Figure 1). If the values of U_0 and θ_0 differ from those of $\tilde{U}_0$ and $\tilde{\theta}_0$, then a circular orbit of a given radius cannot be achieved by tangential

thrust. It is of interest at this point to consider a specific example of the effect of errors in the values of $\tilde{U}_0$ and $\tilde{\theta}_0$ on a finite orbit, which we may assume to be an ellipse where the distance to the pericenter is equal to the radius of a given circular orbit.

Let us consider the problem of deceleration of the spacecraft with the sphere of influence of the earth (the product of the gravitational constant and the mass of the earth μ = 398,600 km^3/sec^2 and the radius of the sphere of influence ρ^* = 930,000 km). The radius of the circular orbit into which the spacecraft must be placed is assumed to be r_k = 18,920 km. We will assume that reactive acceleration is constant, equal to 3.52 mm/sec^2.

For the problem under consideration, the values of $\tilde{U}_0$ and $\tilde{\theta}_0$ are equal, respectively, to 2.473 km/sec and 4.383 rad.

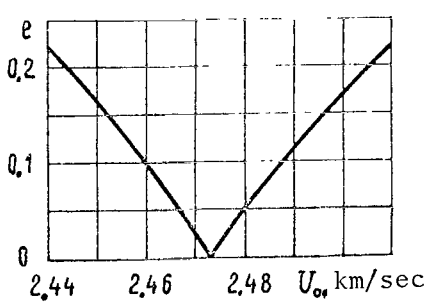

Figure 3

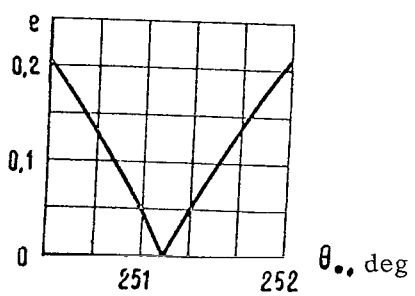

Figure 4

Figures 3 and 4 illustrate the results of the effect of the values of U_0 and θ_0 on the accuracy of execution of the maneuver. The first figure illustrates the dependence of eccentricity of a finite orbit on U_0 for $\theta_0 = \tilde{\theta}_0$, and the second illustrates the dependence of eccentricity of a finite orbit on θ_0 when $U_0 = \tilde{U}_0$. It is clear from the figures that the error in $\tilde{U}_0$ at 30 m/sec of the error $\tilde{\theta}_0$ per degree results in an eccentricity value of 0.2. It is obvious that to obtain an orbit close to the given orbit is practically impossible since a very high precision is required for placing the spacecraft into /21 the sphere of influence of the earth. However, if after deceleration by tangential thrust a method of control is used which insures a reduction in eccentricity at a constant distance to the apocenter, then the given circular orbit can be acheived. If, on the other hand, the osculating distance to the apocenter corresponding to the initial conditions is less than the radius of the given circular orbit, then the proposed control method is inapplicable. In this case the osculating distance to the pericenter can first be increased by using the control method which insures the maximum rate of increase of the osculating

distance, and then by rounding off the orbit to a constant distance to the pericenter.

Thus, the following methods of control are proposed:

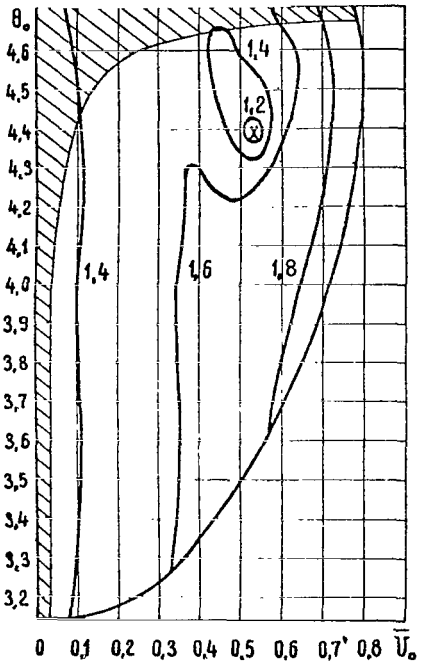

Figure 5

1. The initial distance to the pericenter is greater than the radius of the given circular orbit. The first stage is deceleration by tangential thrust to such moment of time as the osculating distance to the pericenter becomes equal to the radius of the anular orbit. The second stage is the reduction of eccentricity to a constant distance to the pericenter.

2. The initial distance to the pericenter is less than the radius of the given circular orbit. The first stage is an increase in the distance to the pericenter at a maximum rate to such moment of time as this distance becomes equal to the radius of the circular orbit. The second stage is the same as the first case.

Deceleration trajectories were calculated /22 by computer in accordance with the system outlined above for the range of values of U_0 and θ_0 corresponding to the trajectories which do not extend beyond the sphere of influence of the earth. Figure 5 illustrates the ratio of maneuver time to the minimal time, corresponding to the initial values $U_0 = U_0$, $\theta_0 = \theta_0$, as a function of the initial conditions. The hash-marked area corresponds to the initial values at which the distance to the pericenter is less than the radius of the given circular orbit. The values of the dimensionless initial velocity

$$\bar{U}_0 = U_0 \sqrt{\frac{r_k}{\mu}}$$

are arranged along the abscissa.

Table 1

Method of control	Osculating elements		Table 1
	e	p	ω
$\frac{de}{dt} = 0$		$0 < \vartheta < 2\pi$	$0 < \vartheta < \pi$
$\frac{\partial}{\partial\lambda}\frac{de}{dt} = 0$	$\cos\vartheta > \frac{-1+\sqrt{1-e^2}}{e}$	$0 < \vartheta < 2\pi$	$0 < \vartheta < \vartheta_1$ $\vartheta_2 < \vartheta < \pi$ $-\vartheta_2 < \vartheta < -\vartheta_1$ $\vartheta_1 = \arccos\frac{-1+\sqrt{1-e^2}}{e}$ $\vartheta_2 = \arccos\frac{-e^2-3+\sqrt{(9-e^2)(1-e^2)}}{4e}$
$\frac{dp}{dt} = 0$	$0 < \vartheta < \pi$		$\cos\vartheta < 0$
$\frac{\partial}{\partial\lambda}\frac{dp}{dt} = 0$	$\cos\vartheta > \frac{-1+\sqrt{1-e^2}}{e}$	$0 < \vartheta < \pi$	$0 < \vartheta < 2\pi$
$\frac{d\omega}{dt} = 0$	$\cos\vartheta > 0$	$0 < \vartheta < 2\pi$	
$\frac{\partial}{\partial\lambda}\frac{d\omega}{dt} = 0$	$\cos\vartheta > \frac{-e^2-3+\sqrt{(1-e^2)(9-e^2)}}{4e}$	$0 < \vartheta < 2\pi$	$0 < \vartheta < \pi$
$\frac{dr_n}{dt} = 0$	$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$	$0 < \vartheta < \pi$
$\frac{\partial}{\partial\lambda}\frac{dr_n}{dt} = 0$	$\cos\vartheta > \frac{-e^2-3+(e+1)\sqrt{(1-e)(9-e)}}{4e}$	$0 < \vartheta < 2\pi$	$0 < \vartheta < \pi$
$\frac{dr_a}{dt} = 0$		$0 < \vartheta < 2\pi$	$0 < \vartheta < \pi$
$\frac{\partial}{\partial\lambda}\frac{dr_a}{dt} = 0$	$\cos\vartheta > \frac{-e^2-3+(1-e)\sqrt{(1+e)(9+e)}}{4e}$	$0 < \vartheta < 2\pi$	$0 < \vartheta < \vartheta_1$ $\pi < \vartheta < 2\pi - \vartheta_1$ $\vartheta_1 = \arccos\frac{e^2-2e-3-(1-e)\sqrt{e^2-2e+9}}{4e}$
$\frac{dA}{dt} = 0$		$0 < \vartheta < 2\pi$	$0 < \vartheta < \vartheta_1$ $\pi < \vartheta < 2\pi - \vartheta_1$ $\vartheta_1 = \arccos\left(-\frac{2e}{1+e^2}\right)$
$\frac{\partial}{\partial\lambda}\frac{dA}{dt} = 0$	$\cos\vartheta > -e$	$0 < \vartheta < 2\pi$	$0 < \vartheta < \pi$

Table 1 (con't)

Osculating elements		
r_n	r_a	Λ
$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$
$\cos\vartheta > \frac{-e^2-3+(e-1)\sqrt{(1-e)(9-e)}}{4e}$	$\cos\vartheta > \frac{-1+\sqrt{1-e^2}}{e}$	$\cos\vartheta > \frac{-1+\sqrt{1-e^2}}{e}$
$\cos\vartheta < \frac{-1+\sqrt{1-e^2}}{e}$	$\cos\vartheta < \frac{-e^2-3+(1-e)\sqrt{(1+e)(9+e)}}{4e}$	$\cos\vartheta < -e$
$\pi < \vartheta < 2\pi$	$0 < \vartheta < \pi$	$0 < \vartheta < \pi$
$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$
$\cos\vartheta < 0$	$\cos\vartheta > 0$	$\cos\vartheta > 0$ $\cos\vartheta < \frac{-2e}{1+e^2}$
$0 < \vartheta < 2\pi$	$\cos\vartheta > \frac{e^2-2e-3-(1-e)\sqrt{e^2-2e+9}}{4e}$	$0 < \vartheta < 2\pi$
	$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$
$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$
$0 < \vartheta < 2\pi$		$0 < \vartheta < 2\pi$
$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$
$0 < \vartheta < 2\pi$		
$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$	$0 < \vartheta < 2\pi$

Table 2

Method of control	Osculating elements					
	e	p	ω	r_n	r_a	Λ
$\frac{de}{dt}=0$		$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<\pi$		
$\frac{\partial}{\partial\lambda}\frac{de}{dt}=0$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<2\pi$	$\cos\vartheta>\frac{-e^2-3+(e-1)\sqrt{(1+e)(9+e)}}{4e}$	$0<\vartheta<2\pi$
$\frac{dp}{dt}=0$	$0<\vartheta<\pi$		$\cos\vartheta<0$	$\pi<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<\pi$
$\frac{\partial}{\partial\lambda}\frac{dp}{dt}=0$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<\pi$	$0<\vartheta<2\pi$	$\cos\vartheta>\frac{e-2}{e}$	$0<\vartheta<2\pi$
$\frac{d\omega}{dt}=0$	$\cos\vartheta>0$	$0<\vartheta<2\pi$		$\cos\vartheta<0$	$\cos\vartheta>0$	$\cos\vartheta>0$
$\frac{\partial}{\partial\lambda}\frac{d\omega}{dt}=0$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<2\pi$	$\cos\vartheta>\frac{e^2-2e-3-(e-1)\sqrt{e^2-2e+9}}{4e}$	$0<\vartheta<2\pi$
$\frac{dr_n}{dt}=0$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$		$0<\vartheta<2\pi$	$0<\vartheta<2\pi$
$\frac{\partial}{\partial\lambda}\frac{dr_n}{dt}=0$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<2\pi$	$\cos\vartheta>\frac{e^2-3}{2e}$	$0<\vartheta<2\pi$
$\frac{dr_a}{dt}=0$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<2\pi$		$0<\vartheta<2\pi$
$\frac{\partial}{\partial\lambda}\frac{dr_a}{dt}=0$	$\cos\vartheta>\frac{-e^2-3+(e-1)\sqrt{(1+e)(9+e)}}{4e}$	$0<\vartheta<2\pi$	$0<\vartheta<\vartheta_1$ $\pi<\vartheta<2\pi-\vartheta_1$ $\vartheta_1=\arccos\frac{e^2-2e-3-(e-1)\sqrt{e^2-2e+9}}{4e}$	$\cos\vartheta>\frac{e^2-3}{2e}$ $\cos\vartheta<\frac{e-2}{2}$	$\cos\vartheta>\frac{e-2}{e}$	$\cos\vartheta>\frac{e-2}{e}$
$\frac{d\Lambda}{dt}=\Lambda$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<2\pi$		
$\frac{\partial}{\partial\lambda}\frac{d\Lambda}{dt}=0$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<\pi$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$	$0<\vartheta<2\pi$

CHAPTER 2

ANALYSIS OF CERTAIN PROBLEMS IN SPACECRAFT DYNAMICS BY THE METHOD OF AVERAGES

V. N. Lebedev

ABSTRACT. The method of averages is used in Chapter 2 for analyzing the motion of a spacecraft with transversal thrust and for analyzing transition between circular non-coplanar orbits executed in two stages. It is assumed in these analyses that the engine of the spacecraft develops constant thrust at a constant rate of fuel consumption. From the standpoint of fuel consumption, the simultaneous variations of the radius of the orbit and reduction of the angle between the planes of the original and given orbits is considered to be more advantageous. The equations for the execution of these maneuvers are derived and analyzed.

In this chapter we will examine two problems using the method of averages /23
[6]. The first problem (§1-3) is the analysis of the motion of a spacecraft with thrust in the transversal direction. The analysis of this motion is of particular interest because transversal thrust is one of the simplest controls by which it is possible to accelerate the spacecraft to parabolic velocity [7-10]. The second problem (§4-8) is motion between circular noncoplanar orbits. This maneuver can be executed in two steps: transit between circular coplanar orbits and rotation of the plane of a circular orbit. However, the most convenient method from the standpoint of fuel consumption is the simultaneous variation of the radius of the orbit and reduction of the angle between the planes of the initial and given orbits.

In the analysis of both problems, we will assume that the spacecraft is equipped with an engine which produces constant thrust at a constant fuel consumption rate.

§1. Equations of Plane Motion. Parameters of a Constant-Thrust Engine.

Suppose that a spacecraft is accelerated within the gravitational field of a spherical planet; we may disregard perturbations from other planets and the Sun, and also atmospheric drag. Given these assumptions, the motion of the spacecraft in polar coordinates is described by the following equation system:

$$\frac{dr}{dt} = u; \tag{2.1}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{2.2}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + a\cos\lambda; \tag{2.3}$$ /24

$$\frac{dv}{dt} = -\frac{uv}{r} + a\sin\lambda, \tag{2.4}$$

where r and ϕ are polar coordinates (see Figure 1); u and v are the radial and transversal velocity components; t is time; a is reactive acceleration (ratio of thrust to mass of the spacecraft); λ is the angle between the vector radius of the spacecraft and the thrust vector. Here all values are dimensionless. In contrast to the first chapter, the dimensional values will be denoted by an asterisk (*) where they occur. Dimensionless variables are related to the corresponding dimensional variables by the following relations:

$$r^* = r r_0^*; \quad u^* = u\sqrt{\frac{\mu}{r_0^*}}; \quad v^* = v\sqrt{\frac{\mu}{r_0^*}}; \quad t^* = t r_0^* \sqrt{\frac{r_0^*}{\mu}}; \quad a^* = a\frac{\mu}{(r_0^*)^2},$$

where r_0^* is some characteristic distance (in the case of acceleration of the spacecraft from a circular orbit, r_0^* is the radius of this orbit); $\mu = \gamma M$ is the product of the gravitational constant and the mass of the attracting center.

Let the thrust F* of the engine and the mass consumption q* of the working matter per unit time be constant. Then reactive acceleration a* is a function of time. This function is readily expressed in the following manner:

$$a^* = F^*/(m_0^* - q^* t^*) = \left(\frac{F^*}{m_0^*}\right) \Big/ \left(1 - \frac{q^*}{m_0^*} t^*\right).$$

The ratio of thrust to the initial mass m_0^* of the spacecraft is the initial reactive acceleration a_0^*. If we assume that F* = q*V*, where V* is the flow rate of the reactive jet, then we obtain acceleration a* as a function of time and and of two parameters a_0^* and V*:

$$a^* = a_0^* \Big/ \left(1 - \frac{a_0^*}{V^*} t^*\right),$$

or in dimensionless variables:

$$a = a_0 \Big/ \left(1 - \frac{a_0}{V} t\right). \tag{2.5}$$

We assume further that the engine operates without interruption and therefore the same time t which is used in equation system (2.1)-(2.4) will also be used in the formula thus derived. /25

Equation system (2.1)-(2.4) with consideration of (2.5) acquires the form:

$$\frac{dr}{dt} = u; \tag{2.6}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{2.7}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{a_0}{1 - \frac{a_0}{V} t} \cos\lambda; \tag{2.8}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{a_0}{1 - \frac{a_0}{V} t} \sin\lambda. \tag{2.9}$$

With transversal thrust in the direction of motion, sin λ = 1, cosine λ = 0.

§2. Application of Method of Averages.

We can analyze at least the initial stage of acceleration of a spacecraft with a low-thrust engine by the method of averages [11-12]. The method of averages is used in [13] for analyzing motion with constant transversal reactive acceleration. The case of tangential thrust is examined in [14-15].

Let us consider the motion of a spacecraft under the effect of a constant transversal thrust. By substituting the values sin λ = 1 and cosine λ = 0 in equation system (2.6)-(2.9), we obtain

$$\frac{dr}{dt} = u; \tag{2.10}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{2.11}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2}; \tag{2.12}$$ /26

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{a}{1 - \frac{a_0}{V}t}. \tag{2.13}$$

We will use polar angle ϕ as an independent variable. Then equations (2.10)-(2.13) are reduced to the form

$$\frac{dr}{d\varphi} = \frac{ur}{v}; \tag{2.14}$$

$$\frac{du}{d\varphi} = v - \frac{1}{rv}; \tag{2.15}$$

$$\frac{dv}{d\varphi} = -u + \frac{r}{v}\frac{a_0}{1 - \frac{a_0}{V}t}; \tag{2.16}$$

$$\frac{dt}{d\varphi} = \frac{r}{v}. \tag{2.17}$$

Consider the general case of a launch from an elliptical orbit. We will convert to a new variable. Instead of r, u, v, and t we will introduce A, the ratio of the large semiaxis of the osculating ellipse to its initial value,

$$f_1 = e\cos\varphi_n; \tag{2.18}$$

$$f_2 = e\sin\varphi_n \tag{2.19}$$

is the ratio of the Laplace vector component in the rectangular coordinate system xOy (see Figure 1) to the gravitational parameter μ (in (2.18), (2.19) e is eccentricity; ϕ_n is the angular distance of the pericenter); T is the characteristic velocity. Among the variables A, f_1, f_2, T and r, u, v, and t there are the following relations:

$$A = \frac{r}{2 - r(u^2 + v^2)}; \tag{2.20}$$

$$f_1 = rv(v\cos\varphi + u\sin\varphi) - \cos\varphi; \tag{2.21}$$

$$f_2 = rv(v\sin\varphi - u\cos\varphi) - \sin\varphi, \tag{2.22}$$

$$T = \int_0^t a(\tau)d\tau = V\ln\frac{1}{1 - \frac{a_0}{V}t}. \tag{2.23}$$

When reactive acceleration is constant /27

$$T = a_0 t. \tag{2.24}$$

Now we will find the inverse relations, i.e. r, u, v, t as functions of A, f_1, f_2, and T. By using (2.18), (2.19), (2.23) and the following relations:

$$r = \frac{A(1 - e^2)}{1 + e\cos(\varphi - \varphi_n)}; \qquad rv = \sqrt{A(1-e^2)}; \qquad u^2 + v^2 - \frac{2}{r} = -\frac{1}{A},$$

we find

$$\left.\begin{aligned} r &= \frac{A(1-e^2)}{1 + f_1\cos\varphi + f_2\sin\varphi}; \quad u = \frac{f_1\sin\varphi - f_2\cos\varphi}{\sqrt{A(1-e^2)}}; \\ v &= \frac{1 + f_1\cos\varphi + f_2\sin\varphi}{\sqrt{A(1-e^2)}}; \quad t = \frac{1 - \exp\left(-\frac{T}{V}\right)}{a_0}\cdot V. \end{aligned}\right\} \tag{2.25}$$

Assuming that $a_0 \ll 1$ and $V \gg 1$, we will transform the system of equations describing the motion of the spacecraft under the influence of a constant transversal thrust to the standard form of systems with a rapidly rotating phase. For this purpose we will differentiate expressions (2.20)-(2.23) with respect to ϕ, whereupon the derivatives $dr/d\phi$, $du/d\phi$, and $dt/d\phi$ are taken from (2.14)-(2.17), where we will substitute the expressions from (2.25) in the right hand sides in place of r, v, and t. Thus, we have

$$\frac{dA}{d\varphi} = \frac{2a_0 A^3 (1 - f_1^2 - f_2^2) \exp \frac{T}{V}}{1 + f_1 \cos\varphi + f_2 \sin\varphi}; \tag{2.26}$$

$$\frac{df_1}{d\varphi} = a_0 A^2 (1 - f_1^2 - f_2^2)^2 (2\cos\varphi + f_1 \cos^2\varphi + f_2 \sin\varphi \cos\varphi + f_1) \exp \frac{T}{V} \Big/ (1 + f_1 \cos\varphi + f_2 \sin\varphi)^3; \tag{2.27}$$

$$\frac{df_2}{d\varphi} = a_0 A^2 (1 - f_1^2 - f_2^2)^2 (2\sin\varphi + f_2 \sin^2\varphi + f_1 \sin\varphi \cos\varphi + f_2) \exp \frac{T}{V} \Big/ (1 + f_1 \cos\varphi + f_2 \sin\varphi)^3; \tag{2.28}$$

$$\frac{dT}{d\varphi} = \frac{a_0 [A(1 - f_1^2 - f_2^2)]^{3/2} \exp \frac{T}{V}}{(1 + f_1 \cos\varphi + f_2 \sin\varphi)^2}. \tag{2.29}$$

After the integration of this system, we can find time t from relation (2.23): /28

$$t = \frac{V}{a_0} \left[1 - \exp\left(-\frac{T}{V}\right) \right]. \tag{2.30}$$

At a constant reactive acceleration

$$t = T / a_0. \tag{2.31}$$

Assuming that eccentricity of the initial orbit is less than unity, we will take the average of (2.26)-(2.29):

$$\frac{dA}{d\varphi} = \frac{a_0 A^3 (1 - f_1^2 - f_2^2) \exp \frac{T}{V}}{\pi} \int_0^{2\pi} \frac{d\varphi}{1 + f_1 \cos\varphi + f_2 \sin\varphi}; \tag{2.32}$$

$$\frac{df_1}{d\varphi} = \frac{a_0 A^2 (1 - f_1^2 - f_2^2)^2 \exp \frac{T}{V}}{2\pi} \times \int_0^{2\pi} \frac{2\cos\varphi + f_1 \cos^2\varphi + f_2 \sin\varphi \cos\varphi + f_1}{(1 + f_1 \cos\varphi + f_2 \sin\varphi)^3} d\varphi; \tag{2.33}$$

$$\frac{df_2}{d\varphi} = \frac{a_0 A^2 (1 - f_1^2 - f_2^2)^2 \exp \frac{T}{V}}{2\pi} \times$$

$$\times \int_0^{2\pi} \frac{2\sin\varphi + f_2 \sin^2\varphi + f_1 \sin\varphi\cos\varphi + f_2}{(1 + f_1\cos\varphi + f_2\sin\varphi)^3} d\varphi; \tag{2.34}$$

$$\frac{dT}{d\varphi} = \frac{a_0[A(1 - f_1^2 - f_2^2)]^{3/2} \exp\frac{T}{V}}{2\pi} \int_0^{2\pi} \frac{d\varphi}{(1 + f_1\cos\varphi + f_2\sin\varphi)^2}. \tag{2.35}$$

For elliptical motion

$$f_1^2 + f_2^2 < 1. \tag{2.36}$$

We will now integrate in (2.32)-(2.35) with consideration of an equality (2.36).

We introduce the definitions

$$J_1 = \int_0^{2\pi} \frac{d\varphi}{1 + f_1\cos\varphi + f_2\sin\varphi};$$

$$J_2 = \int_0^{2\pi} \frac{2\cos\varphi + f_1\cos^2\varphi + f_2\sin\varphi\cos\varphi + f_1}{(1 + f_1\cos\varphi + f_2\sin\varphi)^3} d\varphi;$$

/29

$$J_3 = \int_0^{2\pi} \frac{2\sin\varphi + f_2\sin^2\varphi + f_1\sin\varphi\cos\varphi + f_2}{(1 + f_1\cos\varphi + f_2\sin\varphi)^3} \cdot d\varphi;$$

$$J_4 = \int_0^{2\pi} \frac{d\varphi}{(1 + f_1\cos\varphi + f_2\sin\varphi)^2}.$$

The integral J_1 is tabular, its value being

$$J_1 = 2\pi/\sqrt{1 - f_1^2 - f_2^2}.$$

In computing this and the following integrals, we will assume, using the condition $a_0 << 1$ and the form of equations (2.33), (2.34) that f_1 and f_2 have constant values. This will enable us to regard the integrals $J_i (i = 1,2,3,4)$ as functions of the parameters f_1 and f_2.

The condition $f_1^2 + f_2^2 < 1$ is simultaneously the condition of differentation under the integral sign:

$$\frac{\partial J_1}{\partial f_1} = -\int_0^{2\pi} \frac{\cos\varphi d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^2};$$

$$\frac{\partial J_1}{\partial f_2} = -\int_0^{2\pi} \frac{\sin\varphi d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^2}.$$

Then J_4 can be written as

$$J_4 = J_1 + f_1 \frac{\partial J_1}{\partial f_1} + f_2 \frac{\partial J_1}{\partial f_2}.$$

By using the value of J_1 thus computed, we will have

$$J_4 = \frac{2\pi}{(1-f_1^2-f_2^2)^{1/2}} + \frac{2\pi f_1^2}{(1-f_1^2-f_2^2)^{3/2}} + \frac{2\pi f_2^2}{(1-f_1^2-f_2^2)^{3/2}} = \frac{2\pi}{(1-f_1^2-f_2^2)^{3/2}}.$$

For the computation of J_2 we will find the expressions for $\partial J_4/\partial f_1$ and $\partial J_4/\partial f_2$:

$$\frac{\partial J_4}{\partial f_1} = -2\int_0^{2\pi} \frac{\cos\varphi d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^3};$$

$$\frac{\partial J_4}{\partial f_2} = -2\int_0^{2\pi} \frac{\sin\varphi d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}.$$

Then J_2 can be written as /30

$$J_2 = \int_0^{2\pi} \frac{\cos\varphi(1+f_1\cos\varphi+f_2\sin\varphi)+\cos\varphi+f_1}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}\,d\varphi =$$

$$= -\frac{\partial J_1}{\partial f_1} - \frac{1}{2}\frac{\partial J_4}{\partial f_1} + f_1 J_4 + \frac{1}{2} f_1^2 \frac{\partial J_4}{\partial f_1} + \frac{1}{2} f_1 f_2 \frac{\partial J_4}{\partial f_2}.$$

Here the relation

$$\int_0^{2\pi} \frac{f_1\,d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^3} = \int_0^{2\pi} \frac{f_1\,d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^2} -$$

$$-f_1\int_0^{2\pi} \frac{f_1\cos\varphi+f_2\sin\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}\,d\varphi = f_1 J_4 + \frac{1}{2} f_1^2 \frac{\partial J_4}{\partial f_1} + \frac{1}{2} f_1 f_2 \frac{\partial J_4}{\partial f_2}.$$

was used. Finally, by using the values of J_1 and J_4 which we have found, we can write

$$J_2 = -\frac{2\pi f_1}{(1-f_1^2-f_2^2)^{1/2}} - \frac{1}{2}\frac{6\pi f_1}{(1-f_1^2-f_2^2)^{5/2}} + \frac{2\pi f_1}{(1-f_1^2-f_2^2)^{1/2}} +$$

$$+\frac{1}{2} f_1^2 \frac{6\pi f_1}{(1-f_1^2-f_2^2)^{5/2}} + \frac{1}{2} f_1 f_2 \frac{6\pi f_2}{(1-f_1^2-f_2^2)^{5/2}} = -\frac{3\pi f_1}{(1-f_1^2-f_2^2)^{1/2}}.$$

Similarly, we compute J_3:

$$J_3 = -3\pi f_2/(1-f_1^2-f_2^2)^{1/2}.$$

By substituting in (2.32)-(2.35) the values of the integrals, we obtain the following system of equations:

$$\frac{dA}{d\varphi} = 2a_0 A^3 \sqrt{1-f_1^2-f_2^2}\,\exp\frac{I}{V}; \qquad (2.37)$$

$$\frac{df_1}{d\varphi} = -\frac{3}{2} a_0 A^2 f_1 \sqrt{1-f_1^2-f_2^2} \exp \frac{T}{V}; \tag{2.38}$$

$$\frac{df_2}{d\varphi} = -\frac{3}{2} a_0 A^2 f_2 \sqrt{1-f_1^2-f_2^2} \exp \frac{T}{V}; \tag{2.39}$$

$$\frac{dT}{d\varphi} = a_0 A^{3/2} \exp \frac{T}{V}. \tag{2.40}$$

At the initial moment of time /31

$$\varphi = 0; \quad A = 1; \quad f_1 = f_{10}; \quad f_2 = f_{20}; \quad T = 0. \tag{2.41}$$

Equation system (2.37)-(2.40) enables us to make certain conclusions concerning the character of motion.

From (2.37) it follows that $(dA/d\phi) > 0$, i.e. the large semiaxis increases.

By dividing (2.38) and (2.39) by (2.37) we obtain

$$\frac{df_1}{dA} = -\frac{3}{4}\frac{f_1}{A}; \qquad \frac{df_2}{dA} = -\frac{3}{4}\frac{f_2}{A}.$$

By integrating these equations we have

$$f_1 = f_{10} A^{-3/4}; \tag{2.42}$$

$$f_2 = f_{20} A^{-3/4}. \tag{2.43}$$

Then, by substituting (2.42) and (2.43) into (2.25) we find

$$e = e_0 A^{-3/4}. \tag{2.44}$$

From this relation it follows that eccentricity diminishes as the large semiaxis increases.

Finally, from (2.42) and (2.43) we have

$$f_1/f_2 = f_{10}/f_{20} = \text{const},$$

and consequently the position of the apsidal line does not vary on the average.

The analogous characteristics of motion for the case of tangential thrust were derived in [14 and 15].

Let us consider in greater detail the case of take-off from circular orbit: $f_{10} = f_{20} = 0$. From (2.42) and (2.43) it follows that in this case $f_1 = f_2 \equiv 0$, and motion is described by

$$\frac{dr}{d\varphi} = 2a_0 r^3 \exp\frac{T}{V}; \tag{2.45}$$

$$\frac{dT}{d\varphi} = a_0 r^{3/2} \exp\frac{T}{V} \tag{2.46}$$

($A \equiv r$ when $e = 0$).

By dividing (2.45) by (2.46) we obtain

$$\frac{dr}{dT} = 2r^{3/2}, \tag{2.47}$$

the solution of which is /32

$$r = (1 - T)^{-2}. \tag{2.48}$$

By substituting (2.23) into (2.48) we find r as a function of t:

$$r = \left(1 - V\ln\frac{1}{1 - \frac{a_0}{V}t}\right)^{-2}. \tag{2.49}$$

When reactive acceleration is constant

$$r = (1 - a_0 t)^{-2}. \tag{2.50}$$

It follows from [16] that formulas (2.49) and (2.50) are valid for the case of tangential thrust as well.

As it was stated at the beginning of this section, the method of averages yields satisfactory results for the initial stage of acceleration of a spacecraft. Therefore formulas (2.49), (2.50) do not guarantee satisfactory precision in the final stage of the maneuver. As an illustration of this /33 fact, let us examine Figure 6, where the precision of the function r(t) obtained on a computer for cases of tangential thrust (curve 1) and transversal thrust (curve 2), and the approximate function (2.49) (curve 3) for $a_0 = 10^{-3}$, V = 1 (time required for achieving parabolic velocity is 565), are shown. It is clear that upon approach toward the point of achieving parabolic velocity, relation (2.49) yields a very large error.

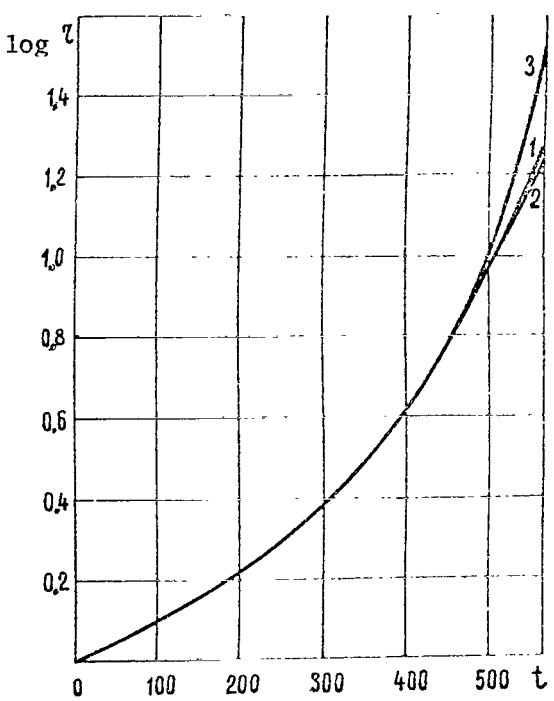

Figure 6

The conclusion was made in [14] that tangential thrust can be used for movement between circular coplanar orbits. The same may be said of transversal thrust. The fact is that if the parameters of the spacecraft and of the orbits between which the movement occurs are such that the conditions of applicability of the method of averages are satisfied: then upon take-off from a circular orbit, eccentricity of a finite orbit will be of the order of magnitude of a small parameter. If the initial orbit is elliptical (for example due to errors in the insertion of the spacecraft into the orbit), then eccentricity will be even less for the case of transition to an orbit of larger radius than the eccentricity of the initial orbit, as follows from (2.44).

§3. Approximation Formulas for Determination of Acceleration Time.

The acceleration trajectory of a spacecraft under a constant tangential or transversal thrust can be calculated precisely only with a computer. As was expected, these calculations showed that the results obtained did not differ, from the qualitative point of view, from the thoroughly investigated case of constant reactive acceleration.

With modern computers, it is possible to calculate the acceleration trajectory of a spacecraft with any given considerations of a_0 and V at the expense of quite reasonable computer times. However, to determine the optimal

relations between the weight components of the engine, including the values of a_0 and V, it is necessary to know the dependence of the time of the maneuver on a_0 and V within a large range of values. A few dozen to several hundred hours of computer time can be devoted to the determination of this function. It therefore becomes necessary to find some simple formulas which will make it possible to determine approximately the time required for the execution of the maneuver.

The approximation formula /34

$$t_k = \frac{1 - \sqrt[8]{20}\sqrt[4]{a_0}}{a_0}, \tag{2.51}$$

was derived in [9] for the case of acceleration with a low constant tangential reactive acceleration. In this formula t_k is the acceleration time to parabolic velocity. An analogous formula was found in [7]:

$$t_k = \frac{1}{a_0(1+\sqrt{2}\sqrt[4]{a_0})},$$

whence for $a_0 \ll 1$

$$t_k = \frac{1-\sqrt{2}\sqrt[4]{a_0}}{a_0}. \tag{2.52}$$

These formulas differ only by the value of one coefficient: in the first case this coefficient is equal to 1.091 and in the second case it is 1.414. The time t_k computed by these formulas differs considerably from the true time, as illustrated in Figure 7, where the precise dependence of the characteristic velocity of the maneuver $T_k = a_0 t_k$ on acceleration (curve 1) and the functions corresponding to formulas (2.51) and (2.52) (curves 2 and 3, respectively), are shown. It was found that if the value of the coefficient in formula (2.51) is assumed to be 0.8082, then it becomes extremely accurate within a large range of values of a_0 (curve 4 in Figure 7).

Now we will derive the relation corresponding to formulas (2.51) and (2.52) for the case of constant thrust.

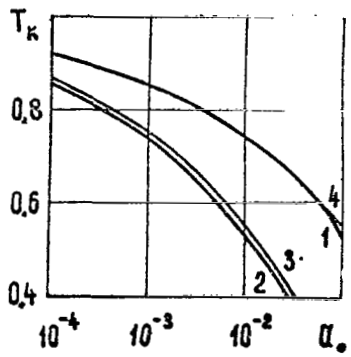

Figure 7

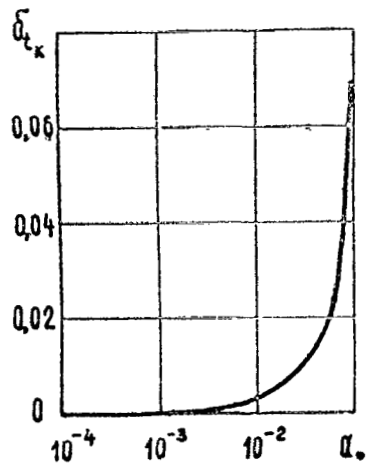

Figure 8

It follows from equation (2.6) and (2.8) that at the moment when parabolic velocity is attained, i.e. when $r(u^2 + v^2) = 2$, the differential relation

$$\frac{d^2r}{dt^2} = \frac{1}{r^2} - \frac{\left(\frac{dr}{dt}\right)^2}{r} \tag{2.53}$$

is satisfied for the case of transversal thrust and

$$\frac{d^2r}{dt^2} = \frac{1}{r^2} - \frac{\left(\frac{dr}{dt}\right)^2}{r} + \frac{a_0}{1-\frac{a_0}{V}t}\sqrt{\frac{r}{2}}\frac{dr}{dt} \tag{2.54}$$

is satisfied for the case of tangential thrust.

Let us assume that relation (2.49) is valid for the entire trajectory of acceleration, including within the vicinity of the point at which parabolic velocity is attained. Then, by substituting r from (2.49), and also dr/dt and d^2r/dt^2 obtained from (2.49) by differentiation, into (2.53) and (2.54), we will have the equation relative to the desired t_k: /35

$$\left(1 - V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right)^8 + C_1\frac{a_0^2}{\left(1-\frac{a_0}{V}t_k\right)^2} + C_2\frac{a_0^2}{\left(V\ 1-\frac{a_0}{V}t_k\right)^2}\left(1 - V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right) = 0, \qquad (2.55)$$

where $C_2 = -2$; $C_1 = -10$ for the case of transversal thrust and $C_1 = -10 + \sqrt{2}$ for the case of tangential thrust.

The values of the coefficients C_1 and C_2 corresponding to the minimal error in the value of t_k determined from equation (2.55) were found by numerical integration of equation system (2.6)-(2.9) for $V \geq 1$ and $a_0 \geq 10^{-4}$: for tangential thrust $C_1 = -0.1820$, and $C_2 = 0.22$; for transversal thrust $C_1 = -0.1062$ and $C_2 = 0.12$.

Figure 8 shows the relative error δ_{t_k} in the value of acceleration time /36 determined by equation (2.55) with the coefficients found for $V \geq 1$ (the value of V has little effect on the error). As seen in the figure, the error diminishes as the initial acceleration is reduced. When $a_0 < 3\cdot10^{-2}$, the error does not exceed 1%, and it becomes less than 0.1% when $a_0 < 3.10^{-3}$.

Equation (2.55) was solved on the computer by dividing into halves the interval $[0, V/a_0]$ in which the left half of the equation, regarded as a function of t_k is monotonic and changes signs if $a_0 \leq 1$ and $V \geq 1$.

An approximate solution for equation (2.55) can be obtained for the case $V >> 1$, which corresponds to a small variation in the mass of the spacecraft during the time of acceleration. By finding the solution in the form

$$t_k = t_{k0} + \frac{t_{k1}}{V} + \frac{t_{k2}}{V^2} + \ldots,$$

Taking the linear term, we then obtain

$$t_k = t_{k0} - \frac{1}{a_0 V}\left\{\frac{1}{8}(1-a_0t_{k0})\left[\frac{C_2}{C_1}(1-a_0t_{k0}) + 2a_0t_{k0}\right] + \frac{1}{2}a_0^2t_{k0}^2\right\},$$

where

$$t_{k0} = \frac{1-\sqrt[8]{-C_1}\sqrt[4]{a_0}}{a_0}.$$

When the values of a_0 are small, the first term in the brackets is much smaller than the second, and therefore, the formula obtained can be simplified:

$$t_k \approx \left[1-\sqrt[8]{-C_1}\sqrt[4]{a_0}-\frac{(1-\sqrt[8]{-C_1}\sqrt[4]{a_0})^2}{2V}\right]\Big/a_0, \tag{2.56}$$

where, as before, $C_1 = -0.1820$ for tangential thrust and $C_1 = -0.1062$ for transversal thrust. Formula (2.56), not withstanding simplicity, is quite accurate; thus, when $a_0 < 10^{-2}$ and $V > 4$, the error in determining the acceleration time does not exceed 1%.

And so, for determining the acceleration time of a spacecraft to parabolic velocity under the effect of constant thrust, the following relations have been obtained.

1. Tangential thrust: /37

$$\left(1-V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right)^8-0{,}1820\frac{a_0^2}{\left(1-\frac{a_0}{V}t_k\right)^2}+$$
$$+0{,}22\frac{a_0^2}{V\left(1-\frac{a_0}{V}t_k\right)^2}\left(1-V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right)=0; \tag{2.57}$$

$$t_k \approx \left[1-0{,}8082\sqrt[4]{a_0}-\frac{(1-0{,}8082\sqrt[4]{a_0})^2}{2V}\right]\Big/a_0; \tag{2.58}$$

at constant reactive acceleration

$$t_k = \frac{1-0{,}8082\sqrt[4]{a_0}}{a_0} \tag{2.59}$$

(this formula can be derived on the basis of the results in [15]).

2. Transversal thrust:

$$\left(1-V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right)^8-0{,}1062\frac{a_0^2}{\left(1-\frac{a_0}{V}t_k\right)^2}+0{,}12\frac{a_0^2}{V\left(1-\frac{a_0}{V}t_k\right)^2}\left(1-V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right)=0; \tag{2.60}$$

$$t_k = \left[1 - 0{,}7555\sqrt[4]{a_0} - \frac{(1-0{,}7555\sqrt[4]{a_0})^2}{2V}\right]\Big/ a_0 ; \tag{2.61}$$

for constant reactive acceleration

$$t_k = \frac{1-0{,}7555\sqrt[4]{a_0}}{a_0} . \tag{2.62}$$

§4. Equations of Three-Dimensional Motion. Possible Means of Control for Motion Between Circular Non-Coplanar Orbits.

An equation system for three-dimensional motion by which it is possible to regard the motion of a spacecraft as a result of combination of two motions is /38 used in [12]. The two motions of a spacecraft are those in some plane and those of the given plane around the attracting center, where the first motion is not a function of the second. This equation system in dimensionless variables has the following form:

$$\frac{dA}{d\varphi} = \frac{2A^3(1-f_1^2-f_2^2)}{1+f_1\cos\varphi+f_2\sin\varphi}\left(a_r\frac{f_1\sin\varphi - f_2\cos\varphi}{1+f_1\cos\varphi+f_2\sin\varphi} + a_\varphi\right); \tag{2.63}$$

$$\frac{df_1}{d\varphi} = \frac{A^2(1-f_1^2-f_2^2)^2}{(1+f_1\cos\varphi+f_2\sin\varphi)^2}\left(a_r\sin\varphi + a_\varphi\frac{2\cos\varphi+f_1\cos^2\varphi+f_2\sin\varphi\cos\varphi+f_1}{1+f_1\cos\varphi+f_2\sin\varphi}\right); \tag{2.64}$$

$$\frac{df_2}{d\varphi} = \frac{A^2(1-f_1^2-f_2^2)^2}{(1+f_1\cos\varphi+f_2\sin\varphi)^2}\left(-a_r\cos\varphi + a_\varphi\frac{2\sin\varphi+f_2\sin^2\varphi+f_1\sin\varphi\cos\varphi+f_2}{1+f_1\cos\varphi+f_2\sin\varphi}\right); \tag{2.65}$$

$$\frac{di}{d\varphi} = \frac{a_z A^2(1-f_1^2-f_2^2)^2\cos(\varphi+\psi)}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}; \tag{2.66}$$

$$\frac{d\Omega}{d\varphi} = \frac{a_z A^2(1-f_1^2-f_2^2)^2\sin(\varphi+\psi)}{(1+f_1\cos\varphi+f_2\sin\varphi)^3\sin i}; \tag{2.67}$$

$$\frac{d\psi}{d\varphi} = -\frac{a_z A^2(1-f_1^2-f_2^2)^2\sin(\varphi+\psi)\cot i}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}. \tag{2.68}$$

The time t is related to angle ϕ by the equation

$$\frac{dt}{d\varphi} = \frac{[A(1-f_1^2-f_2^2)]^{3/2}}{(1+f_1\cos\varphi+f_2\sin\varphi)^2}. \tag{2.69}$$

In equations (2.63)-(2.69) A is the large semiaxis; f_1 and f_2 are the Laplace vector components along the axis, Ox and Oy (Figure 9); i is the inclination of the orbit; Ω is the longitude of the ascending node; a_r, a_ϕ, a_z are the pro- /39 jections of reactive acceleration in the direction of the vector radius, perpendicular to it in the orbital plane, and perpendicular to the orbital plane; ϕ is the angle between axis Ox and the vector radius; ψ is the angle between the node line and axis Ox. The initial value of the large semiaxis was taken as the characteristic dimension.

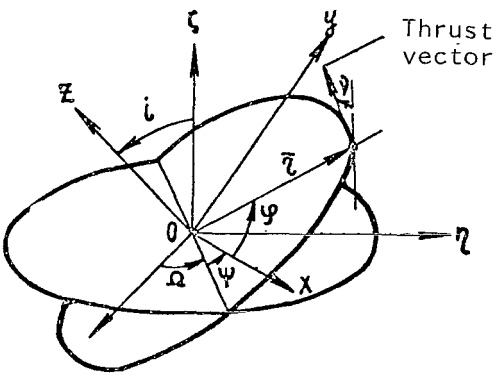

Figure 9

The special selection of the line for calculating angle ϕ causes equations (2.63)-(2.65) to conform in structure to the equations of some planar motion in which angle ϕ behaves as a polar angle.

Let the thrust of the engine be constant and be directed in such a manner that its radial component is equal to 0. Then the direction of the thrust vector can be assigned one value, i.e. the angle ν between the line perpendicular to the vector radius in the orbital plane and the thrust vector (see Figure 9). If relations (2.5) and (2.30) are used, then the components of reactive acceleration can be written in the form

$$a_r = 0;$$

$$a_\varphi = a\cos\nu = \frac{a_0\cos\nu}{1-\frac{a_0}{V}t} = a_0\cos\nu\exp\frac{T}{V};$$

$$a_z = a\sin\nu = \frac{a_0\sin\nu}{1-\frac{a_0}{V}t} = a_0\sin\nu\exp\frac{T}{V}.$$

If we now substitute the values for a_r, a_ϕ, and a_z thus obtained, into equation/40 system (2.63)-(2,68), then when ν = const, equations (2.63)-(2.65) will conform, with an accuracy up to a constant multiple $\cos\nu$, to equation (2.26)-(2.28) for planar motion investigated in §2 of this chapter. By using the method of averages, it was shown that if eccentricity of the initial orbit is equal to zero, then the orbit will become close to circular after a certain length of time, and variations in its average radius can be described approximately by the simple relations (2.45)-(2.46).

As is well-known (this was noted in §2 of the first chapter), it is possible to change the inclination i of the orbit by using the component of reactive acceleration a_z which is normal to the instantaneous plane of the orbit. If at the points where $\phi + \psi = u \pm \pi$ 2 (u is the latitude argument), the direction of thrust is changed to symmetrical relative to the orbital plane, then, as follows from equation (2.66), the inclination of the orbit will vary monotonically. In this case the component of reactive acceleration a_ϕ will not vary.

Thus, by using equations

$$a_r = 0;$$

$$a_\varphi = a_0\cos\nu\exp\frac{T}{V}; \tag{2.70}$$

$$a_z = a_0\sin|\nu|\exp\frac{T}{V}\operatorname{sign}\cos(\varphi+\psi), \tag{2.71}$$

$$|\nu| = \text{const} \quad (-\pi < \nu \leq \pi) \tag{2.72}$$

it is possible to bring about transition from a circular orbit to an orbit close to another given circular orbit. The error in the values of the parameters of the final orbit caused by the application of the averaged equations will be less as the thrust of the engine decreases and as the final orbit converges more closely upon the initial orbit.

Control methods (2.70)-(2.72) insure monotonic increase in the inclination of the orbit. If the execution of a maneuver requires a reduction in inclination, then the sign of a_z should be changed to the opposite sign:

$$a_z = -a_0 \sin|\nu| \exp\frac{T}{V} \operatorname{sign}\cos(\varphi+\psi).$$

From hereon we will assume that transition occurs as the inclination is increased.

§5. Averaging of Equations of Three-Dimensional Motion. /41

By substituting the values of the components of reactive acceleration (2.70)-(2.72) into equations (2.63)-(2.68), and replacing (2.69) with equation (2.29), we find

$$\frac{dA}{d\varphi} = \frac{2a_0 \cos\nu \cdot A^3 (1-f_1^2-f_2^2)\exp\frac{T}{V}}{1+f_1\cos\varphi+f_2\sin\varphi}; \tag{2.73}$$

$$\frac{df_1}{d\varphi} = \left[a_0\cos\nu\cdot A^2(1-f_1^2-f_2^2)^2(2\cos\varphi+f_1\cos^2\varphi + f_2\sin\varphi\cos\varphi+f_1)\exp\frac{T}{V}\right] \Big/ (1+f_1\cos\varphi+f_2\sin\varphi)^3; \tag{2.74}$$

$$\frac{df_2}{d\varphi} = \left[a_0\cos\nu\cdot A^2(1-f_1^2-f_2^2)^2(2\sin\varphi+f_2\sin^2\varphi + f_1\sin\varphi\cos\varphi+f_2)\exp\frac{T}{V}\right] \Big/ (1+f_1\cos\varphi+f_2\sin\varphi)^3; \tag{2.75}$$

$$\frac{di}{d\varphi} = \frac{a_0\sin|\nu| A^2(1-f_1^2-f_2^2)^2\cos(\varphi+\psi)\exp\frac{T}{V}}{(1+f_1\cos\varphi+f_2\sin\varphi)^3} \operatorname{sign}\cos(\varphi+\psi); \tag{2.76}$$

$$\frac{d\Omega}{d\varphi} = \frac{a_0\sin|\nu| A^2(1-f_1^2-f_2^2)^2\sin(\varphi+\psi)\exp\frac{T}{V}}{(1+f_1\cos\varphi+f_2\sin\varphi)^3\sin i} \operatorname{sign}\cos(\varphi+\psi); \tag{2.77}$$

$$\frac{d\psi}{d\varphi} = -\frac{a_0\sin|\nu| A^2(1-f_1^2-f_2^2)^2\sin(\varphi+\psi)\cot i\exp\frac{T}{V}}{(1+f_1\cos\varphi+f_2\sin\varphi)^3} \times \operatorname{sign}\cos(\varphi+\psi); \tag{2.78}$$

$$\frac{dT}{d\varphi} = \frac{a_0[A(1-f_1^2-f_2^2)]^{3/2}\exp\frac{T}{V}}{(1+f_1\cos\varphi+f_2\sin\varphi)^2}. \tag{2.79}$$

The averaged equations (2.73)-(2.75), (2.79), as shown in §2 of this chapter, have the form

$$\frac{dA}{d\varphi} = 2a_0 \cos v \cdot A^3 (1 - f_1^2 - f_2^2) \exp \frac{T}{V}; \tag{2.80}$$

$$\frac{df_1}{d\varphi} = -\frac{3}{2} a_0 \cos v \cdot A^2 f_1 \sqrt{1 - f_1^2 - f_2^2} \exp \frac{T}{V}; \tag{2.81}$$

$$\frac{df_2}{d\varphi} = -\frac{3}{2} a_0 \cos v \cdot A^2 f_2 \sqrt{1 - f_1^2 - f_2^2} \exp \frac{T}{V}; \tag{2.82}$$ /42

$$\frac{dT}{d\varphi} = a_0 A \sqrt{A} \exp \frac{T}{V}. \tag{2.83}$$

When the values of angle i are small, the first halves of equations (2.77) and (2.78) can be as large as desired. Therefore, by averaging these equations we will assume, as suggested in [12], that angle i is close to the value $\pi/2$ (this can always be done by the appropriate choice of the reference plane, the normal to which determines angle i).

By averaging equations (2.76)-(2.78), we obtain

$$\frac{di}{d\varphi} = \frac{a_0 \sin|v| A^2 (1 - f_1^2 - f_2^2)^2 \exp \frac{T}{V}}{2\pi} J_1; \tag{2.84}$$

$$\frac{d\Omega}{d\varphi} = \frac{a_0 \sin|v| A^2 (1 - f_1^2 - f_2^2)^2 \exp \frac{T}{V}}{2\pi \sin i} J_2; \tag{2.85}$$

$$\frac{d\psi}{d\varphi} = -\frac{a_0 \sin|v| A^2 (1 - f_1^2 - f_2^2)^2 \cot i \exp \frac{T}{V}}{2\pi} J_2, \tag{2.86}$$

where

$$J_1 = \int_0^{2\pi} \frac{\cos(\varphi + \psi) \operatorname{sign} \cos(\varphi + \psi)}{(1 + f_1 \cos\varphi + f_2 \sin\varphi)^3} d\varphi; \tag{2.87}$$

$$J_2 = \int_0^{2\pi} \frac{\sin(\varphi + \psi) \operatorname{sign} \cos(\varphi + \psi)}{(1 + f_1 \cos\varphi + f_2 \sin\varphi)^3} d\varphi. \tag{2.88}$$

We will now compute integrals (2.87) and (2.88):

$$J_1 = \int_0^{2\pi} \frac{\cos(\varphi+\psi)\,\mathrm{sign}\cos(\varphi+\psi)}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}\,d\varphi = \int_{-\frac{\pi}{2}-\psi}^{\frac{3\pi}{2}-\psi} \frac{\cos(\varphi+\psi)\,\mathrm{sign}\cos(\varphi+\psi)}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}\,d\varphi =$$

$$= \int_{-\frac{\pi}{2}-\psi}^{\frac{\pi}{2}-\psi} \frac{\cos(\varphi+\psi)\,d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^3} - \int_{\frac{\pi}{2}-\psi}^{\frac{3\pi}{2}-\psi} \frac{\cos(\varphi+\psi)\,d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^3} =$$

/43

$$= \frac{2[2+(f_1\cos\psi-f_2\sin\psi)^2-2(f_1\sin\psi+f_2\cos\psi)^2]}{(1-f_1^2-f_2^2)^2[1-(f_1\sin\psi+f_2\cos\psi)^2]} -$$

$$-\frac{6(f_1\cos\psi-f_2\sin\psi)}{(1-f_1^2-f_2^2)^{5/2}}\,\mathrm{Arctg}\,\frac{(1-f_1^2-f_2^2)^{1/2}}{f_1\cos\psi-f_2\sin\psi}+\frac{3\pi(f_1\cos\psi-f_2\sin\psi)}{(1-f_1^2-f_2^2)^{5/2}};$$

$$J_2 = \int_0^{2\pi} \frac{\sin(\varphi+\psi)\,\mathrm{sign}\cos(\varphi+\psi)}{(1+f_1\cos\varphi+f_2\sin\varphi)^3}\,d\varphi =$$

$$= \int_{-\frac{\pi}{2}-\psi}^{\frac{\pi}{2}-\psi} \frac{\sin(\varphi+\psi)\,d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)^3} - \int_{\frac{\pi}{2}-\psi}^{\frac{3\pi}{2}-\psi} \frac{\sin(\varphi+\psi)\,d\varphi}{(1+f_1\cos\varphi+f_2\sin\varphi)} =$$

$$= 2(f_1\cos\psi-f_2\sin\psi)(f_1\sin\psi+f_2\cos\psi)[5-2(f_1\cos\psi-f_2\sin\psi)^2-$$

$$-5(f_1\sin\psi+f_2\cos\psi)^2]\{(1-f_1^2-f_2^2)^2[1-(f_1\sin\psi+f_2\cos\psi)^2]^2\}^{-1}-$$

$$-\frac{6(f_1\sin\psi+f_2\cos\psi)}{(1-f_1^2-f_2^2)^{5/2}}\,\mathrm{Arctg}\,\frac{(1-f_1^2-f_2^2)^{1/2}}{f_1\cos\psi-f_2\sin\psi}+\frac{3\pi(f_1\sin\psi+f_2\cos\psi)}{(1-f_1^2-f_2^2)^{5/2}}.$$

Then, by substituting the computed values of the integrals into equation (2.84) and (2.86) we obtain

$$\frac{di}{d\varphi}=\frac{a_0\sin|\nu|A^2\exp\frac{T}{V}}{\pi}\Bigg\{[2+(f_1\cos\psi-f_2\sin\psi)^2-$$

$$-2(f_1\sin\psi+f_2\cos\psi)^2][1-(f_1\sin\psi+f_2\cos\psi)^2]^{-1}-$$

$$-\frac{3(f_1\cos\psi-f_2\sin\psi)}{\sqrt{1-f_1^2-f_2^2}}\operatorname{Arctan}\frac{\sqrt{1-f_1^2-f_2^2}}{f_1\cos\psi-f_2\sin\psi}+$$

$$+\frac{3\pi(f_1\cos\psi-f_2\sin\psi)}{2\sqrt{1-f_1^2-f_2^2}}\Bigg\};\qquad(2.89)$$

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$$\frac{d\Omega}{d\varphi}=\frac{a_0\sin|\nu|A^2\exp\frac{T}{V}}{\pi\sin i}\Bigg\{(f_1\cos\psi-f_2\sin\psi)(f_1\sin\psi+$$

$$+f_2\cos\psi)[5-2(f_1\cos\psi-f_2\sin\psi)^2-5(f_1\sin\psi+f_2\cos\psi)^2]\times$$

$$\times[1-(f_1\sin\psi+f_2\cos\psi)^2]^{-1}-\frac{3(f_1\sin\psi+f_2\cos\psi)}{\sqrt{1-f_1^2-f_2^2}}\times$$

$$\times\operatorname{Arctan}\frac{\sqrt{1-f_1^2-f_2^2}}{f_1\cos\psi-f_2\sin\psi}+\frac{3\pi(f_1\sin\psi+f_2\cos\psi)}{2\sqrt{1-f_1^2-f_2^2}}\Bigg\};\qquad(2.90)$$

$$\frac{d\psi}{d\varphi}=-\frac{a_0\sin|\nu|A^2\cot i\exp\frac{T}{V}}{\pi}\Bigg\{(f_1\cos\psi-f_2\sin\psi)(f_1\sin\psi+$$

$$+f_2\cos\psi)[5-2(f_1\cos\psi-f_2\sin\psi)^2-5(f_1\sin\psi+f_2\cos\psi)^2]\times$$

$$\times[1-(f_1\sin\psi+f_2\cos\psi)^2]^{-1}-\frac{3(f_1\sin\psi+f_2\cos\psi)}{\sqrt{1-f_1^2-f_2^2}}\times$$

$$\times\operatorname{Arctan}\frac{\sqrt{1-f_1^2-f_2^2}}{f_1\cos\psi-f_2\sin\psi}+\frac{3\pi(f_1\sin\psi+f_2\cos\psi)}{2\sqrt{1-f_1^2-f_2^2}}\Bigg\}.\qquad(2.91)$$

Here the values of the angle Arctan $\sqrt{1-f_1^2-f_2^2}/(f_1\cos\psi - f_2\sin\psi)$ should be taken within the range of 0 to π.

The equations thus obtained, along with equations (2.80)-(2.83) represent a system of average equations describing the motion of a spacecraft under the effect of the methods of control described above.

If the initial orbit is circular, then as was shown in §2 of this chapter, $f_1 = f_2 \equiv 0$, i.e. the orbit always remains circular. In this case equations (2.89)-(2.91) acquire the simple form:

$$\frac{di}{d\varphi} = \left[2a_0 \sin|v| r^2 \exp\frac{T}{V}\right]\Big/\pi; \quad \frac{d\Omega}{d\varphi} = 0; \qquad \frac{d\psi}{d\varphi} = 0.$$

It follows from the last two equations that the position of the nodal line does not change, and the position of axis Ox relative to the nodal line in the orbital plane also remains constant. /45

Thus, in the case where eccentricity of the initial orbit is equal to zero we have

$$\frac{dr}{d\varphi} = 2a_0 \cos v\, r^3 \exp\frac{T}{V}; \tag{2.92}$$

$$f_1 = 0; \tag{2.93}$$

$$f_2 = 0; \tag{2.94}$$

$$\frac{di}{dt} = \frac{2a_0 \sin|v| r^2 \exp\frac{T}{V}}{\pi}; \tag{2.95}$$

$$\Omega = \text{const}; \tag{2.96}$$

$$\psi = \text{const}; \tag{2.97}$$

$$\frac{dT}{d\varphi} = a_0 r\sqrt{r}\exp\frac{T}{V}. \tag{2.98}$$

§6. Analysis of Averaged Equations.

By omitting the angle ϕ from equations (2.92), (2.95) and (2.98), we obtain

$$\frac{dr}{dT} = 2r\sqrt{r}\cos\nu; \tag{2.99}$$

$$\frac{di}{dT} = \frac{2}{\pi}\sqrt{r}\sin|\nu|. \tag{2.100}$$

If $\nu = 0$, then

$$r = 1/(1-T)^2, \quad i = \text{const}$$

(the radius of the initial orbit is equal to unity), i.e. the plane of the orbit does not change and the radius of the orbit increases. When $\nu = \pi$ the radius diminishes:

$$r = 1/(1+T)^2, \quad i = \text{const}.$$

If the thrust of the engine is directed along the normal to the instantaneous plane of the orbit, i.e. $|\nu| = \pi/2$, then the inclination of the orbit will increase but the radius will remain unchanged:

$$r = 1, \quad i = i + \frac{2}{\pi}T.$$

It follows from equations (2.99), (2.100) that /46

$$\frac{di}{dr} = \frac{\tan|\nu|}{\pi}\frac{1}{r}.$$

By assuming that $i = 0$ when $r = 1$, then the solution of this equation will be

$$i = \frac{\tan|\nu|}{\pi}\ln r. \tag{2.101}$$

Assume that it is required to transfer from an orbit of unit radius to an orbit of radius r_k, the plane of which forms angle i_k with the plane of the orginal orbit. It follows from relation (2.101) that this maneuver can be executed if

$$|\nu| = \mathrm{Arctan}\frac{\pi i_k}{\ln r_k}. \tag{2.102}$$

This relation determines the control by which it is possible to execute transition between two circular non-coplanar orbits.

From the solution of equation system (2.99),(2.100):

$$r = \frac{1}{(1 - T\cos\nu)^2}; \tag{2.103}$$

$$i = \frac{2\tan|\nu|}{\pi}\ln\frac{1}{1 - T\cos\nu}, \tag{2.104}$$

we may determine the characteristic velocity of the maneuver

$$T_k = \frac{1 - \frac{1}{\sqrt{r_k}}}{\cos\nu} \tag{2.105}$$

or

$$T_k = \frac{1 - \exp\left(-\frac{\pi i_k}{2\,\mathrm{Tan}|\nu|}\right)}{\cos\nu}. \tag{2.106}$$

In the partial case where $r_k = 1$, we have $|\nu| = \pi/2$

$$i = \frac{2}{\pi}T; \tag{2.107}$$

$$T_k = \frac{\pi i_k}{2}. \tag{2.108}$$

§7. Variational Problem of Transition Between Circular Non-coplanar Orbits Described by Averaged Equations. /47

In averaging equations (2.73)-(2.78), we assumed that the modulus of angle ν is a constant value. However, the averaged equations thus obtained

$$\frac{dr}{dT} = 2r\sqrt{r}\cos\nu;$$

$$\frac{di}{dT} = \frac{2}{\pi}\sqrt{r}\sin|\nu|$$

are also valid in the case where $|\nu|$ is a slowly changing function of angle ϕ. We may also try to find those functions which enable us to execute the above-mentioned maneuver with less fuel consumption than in the case $|\nu|$ = const.

We will now find the control system which insures transition from the point $r = 1$, $i = 0$ to the point $r = r_k$, $i = i_k$ with the minimum characteristic velocity T_k.

The determination of the optimal control is reduced to the solution of the boundary problem:

$$\frac{dr}{dT} = \frac{2\pi p r^2\sqrt{r}}{\sqrt{1+\pi^2 p^2 r^2}}; \tag{2.109}$$

$$-\frac{di}{dT} = \frac{2\sqrt{r}}{\pi\sqrt{1+\pi^2 p^2 r^2}}; \tag{2.110}$$

$$\frac{dp}{dt} = -\frac{3\pi p^2 r\sqrt{r}}{\sqrt{1+\pi^2 p^2 r^2}} - \frac{1}{\pi\sqrt{r}\sqrt{1+\pi^2 p^2 r^2}}, \tag{2.111}$$

$$r = 1, \quad i = 0 \quad \text{when } T = 0,$$

$$r = r_k, \quad i = i_k \quad \text{when } T = T_k.$$

Here p is the Lagrange factor.

After we have the solution for the boundary problem, optimal control can be determined from the relation

$$\tan|\nu| = 1/\pi p r.$$

System of equations (2.109)-(2.111) is solved analytically. The solution of the boundary problem has the form /48

$$r = \left[1 - \frac{2\left(1 - \frac{\cos\frac{\pi i_k}{2}}{\sqrt{r_k}}\right)}{\sqrt{1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}}} T + T^2 \right]^{-1}; \tag{2.112}$$

$$i = \frac{2}{\pi}\mathrm{Arctan}\left\{\frac{\sin\frac{\pi i_k}{2}}{\sqrt{r_k}} T \left[\sqrt{1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}} - \left(1 - \frac{\cos\frac{\pi i_k}{2}}{\sqrt{r_k}}\right) T\right]^{-1}\right\}. \tag{2.113}$$

The modulus of angle ν as a function of T is expressed by the relation

$$|\nu| = \mathrm{Arctan}\left\{\frac{\sin\frac{\pi i_k}{2}}{\sqrt{r_k}} \left[1 - \frac{\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} - T\sqrt{1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}}\right]^{-1}\right\}. \tag{2.114}$$

The characteristic velocity of the maneuver is

$$T_k = \sqrt{1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}}. \tag{2.115}$$

It follows from expression (2.112) that the dependence of r on T can be non-monotonic. In the case where the radius of the final orbit is greater than the radius of the initial orbit, this function will be non-monotonic if the parameters of the final orbit satisfy the inequality

$$\cos\frac{\pi i_k}{2} < \frac{1}{\sqrt{r_k}}.$$

In this case the radius r at first increases, reaching a maximum when

$$T = \left(1 - \frac{\cos\frac{\pi i_k}{2}}{\sqrt{r_k}}\right)\left(\sqrt{1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}}\right)^{-1},$$

but then diminishes to the given value of r_k. The maximum value of the radius /49

$$r_{max} = \left(1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}\right)\left(\frac{\sin^2\frac{\pi i_k}{2}}{r_k}\right)^{-1}$$

increases as i_k increases, trending toward infinity when $i_k \to 2$. Therefore, if the angle between the orbits is close to 2 radians the method of averages will yield a large error and the parameters of the final orbit will differ greatly from the given values.

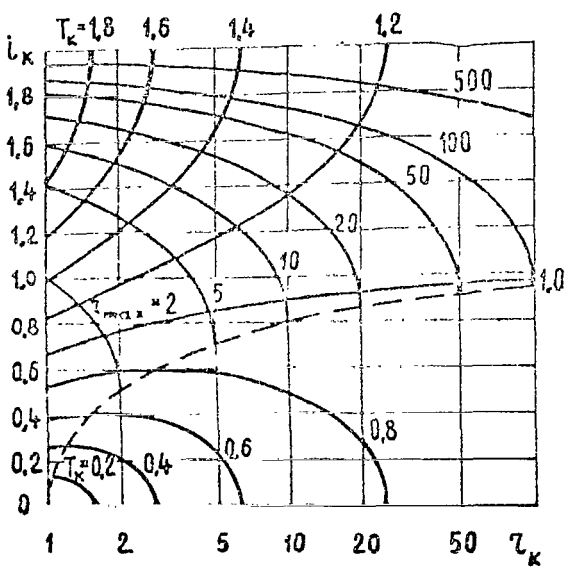

Figure 10

In transitioning to an orbit of smaller radius, the dependence of r on T will be non-monotonic if

$$\cos\frac{\pi i_k}{2} < \sqrt{r_k}.$$

In the case where the radii of the /50 initial and final orbits are identical, the radius increases until such time as $T = T_k/2$ for all angles between the planes of these orbits, but then it diminishes to the initial value. Consequently, an advantage is gained in the value of the characteristic velocity in comparison with the maneuver in which the plane of the orbit is rotated due to thrust normal to the instantaneous orbital plane. For example, when $i_k = \pi/2$, the maximum radius is more than nine times greater than the initial radius. In this case the gain will be 23%.

Figure 10 illustrates the characteristic velocity of the maneuver and maximum radius as functions of r_k and i_k for the case $r_k > 1$. The area bounded

above by the broken line corresponds to maneuvers with a monotonic increase in the radius.

§8. Application of Results to the Problem of Transition to Synchronous Orbit

Let us consider transition from a circular orbit of radius of 6,671 km to synchronous orbit with a radius of 42,240 km. We will assume that the angle between these orbits is 48°.

The dimensionless characteristic velocity of the maneuver for optimal control (2.114) (ratio of characteristic velocity to spacecraft velocity in initial circular orbit) is 0.98. Figure 11 illustrates the controlling angle, angle i and the radius r as functions of characteristic velocity T. It is clear from the figure that the radius becomes maximum at the end of the maneuver, somewhat exceeding the radius of the final orbit.

Figure 12 illustrates the dependence of the time of the maneuver determined from relation (2.30) on the initial reactive acceleration when the velocity of the reactive exhaust is 100 km/sec (curve 1) and 20 km/sec (curve 2).

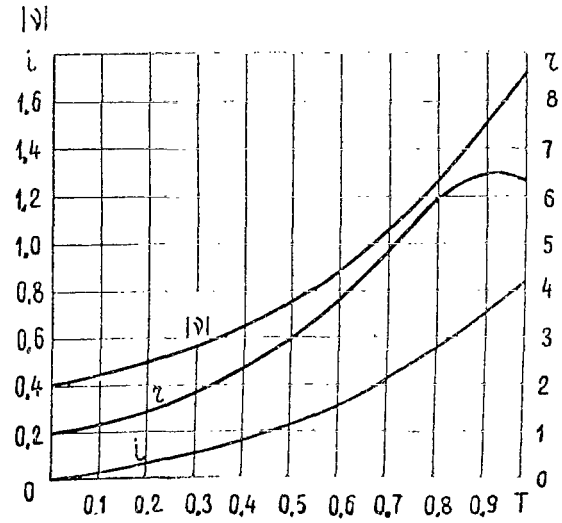

Figure 11

The characteristic velocity of the maneuver when the modulus of the controlling angle is constant (control system (2.101)) is 1.05.

The determination of the characteristic velocity of the maneuver, which is executed in two stages, is of particular interest. The two stages are transitional between circular coplanar orbits with transversal thrust, and then rotation of the plane of the orbit is produced. The characteristic velocity of such

/51

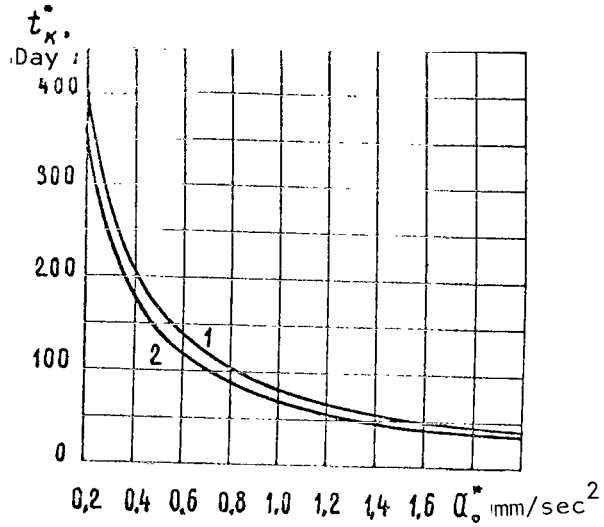

Figure 12

a maneuver is 1.13, which is 14% greater than the characteristic velocity of optimal transition. /53

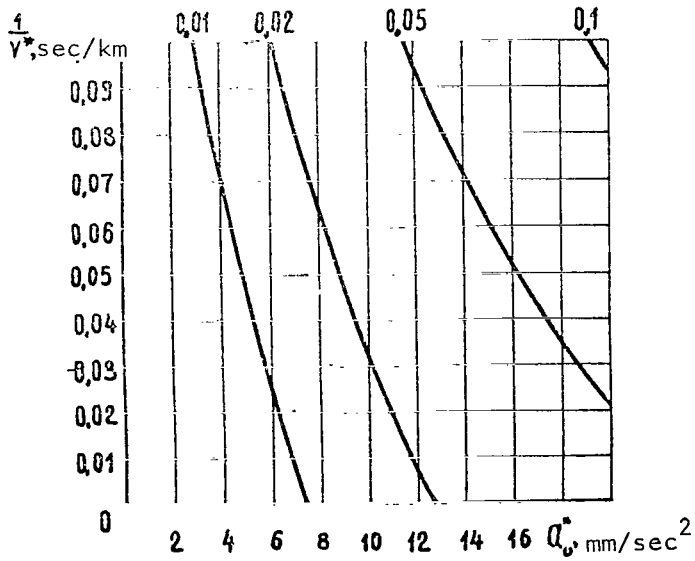

Figure 13

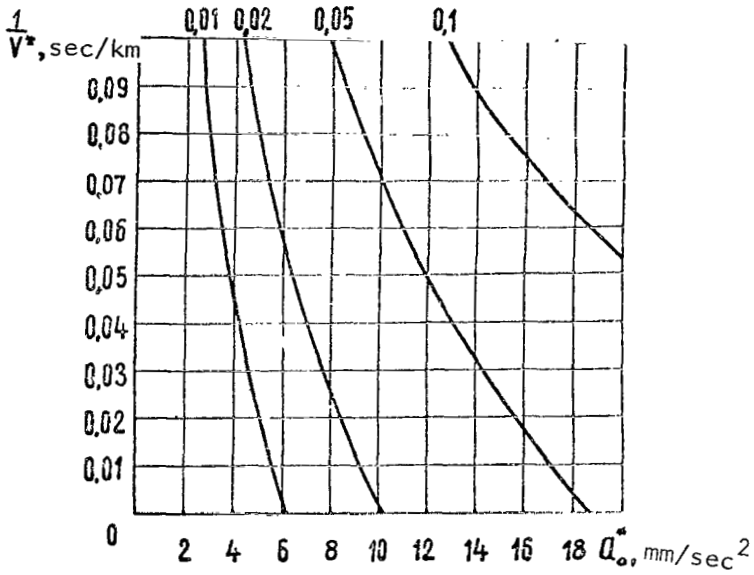

Figure 14

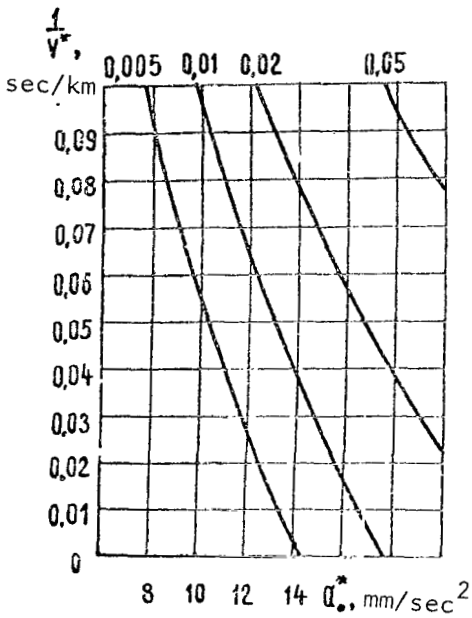

Figure 15

To determine the errors in the parameters of a synchronous orbit caused by the application of averaged equations, the exact equation system (2.73)-(2.79) with optimum control (2.114) was integrated by computer. Integration was terminated for $T = T_k$ (the value of T_k was determined from relation (2.115)). Figures 13-15 illustrate the following values as functions of a_0^* and V^*: eccentricity of the orbit obtained (Figure 13), modulus of the angle between the planes of the obtained and given orbits (Figure 14), and relative error in the average radius (Figure 15). As the figures show, when $a_0^* \leq 2\text{mm/sec}^2$, then $V^* \geq 10$ km/sec, eccentricity of the orbit is less than 0.01, the error in the inclination of the orbit is less than one-half degree, and the relative error in the average is less than one-half per cent.

CHAPTER 3

OPTIMAL TRAJECTORIES FOR SPACECRAFT

V. N. Lebedev

ABSTRACT. The solutions for several variation problems on the dynamics of low-thrust spacecraft are given in this Chapter. The time required for executing the maneuver is minimized in the majority of the problems. Burn time is the minimizing functional in two of the problems, whereupon the trajectories determined have passive segments, in constrast to trajectories which are optimal in the sense of speed. Also examined is the variation problem on the motion of a spacecraft using a solar sail. A modified Newton method is used for the solution of the boundary problem.

In this chapter we will consider the solution for several variation problems in the dynamics of low-thrust spacecraft. In most problems the time required for the execution of a maneuver will be minimized. In two particular problems (§3) the burn time (time of engine operation) is a minimized functional; the trajectories thus obtained have passive segments in contrast to the trajectories which are optimal in the sense of speed. In §9 we will examine a variation problem on motion by a solar sail. /54

The solution of variation problems by the maximum principle of L. S. Pontryagin [18,19]was reduced to the solution of boundary problems for a system of ordinary differential equations. The modified Newton method described in §1 was used for the solution of the boundary problems.

§1. Solution of Two-point Boundary Problem for a System of Ordinary Differential Equations by the Newton Method.

Let be required the solution of the boundary problem

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n), \tag{3.1}$$

for $t = t_0$

$$x_{m+1} = x_{m+1,0}; \quad x_{m+2} = x_{m+2,0}, \ldots, x_n = x_{n,0}; \tag{3.2}$$

for $t = t_k$

$$\left.\begin{array}{l} \varphi_1(t, x_1, x_2, \ldots, x_n) = 0; \\ \varphi_2(t, x_1, x_2, \ldots, x_n) = 0; \\ \ldots\ldots\ldots\ldots \\ \varphi_m(t, x_1, x_2, \ldots, x_n) = 0. \end{array}\right\} \tag{3.3}$$

The values of the variables for $t = t_k$ can be regarded as functions of unknown initial values of the variables $x_1, x_2, \ldots, x_m$: /55

$$x_{1,0}, x_{2,0}, \ldots, x_{m,0}. \tag{3.4}$$

Therefore we may write

$$\varphi_j(t_k, x_{1,k}, x_{2,k}, \ldots, x_{n,k}) = \psi_j(x_{1,0}, x_{2,0}, \ldots, x_{m,0})$$
$$(j = 1, 2, \ldots, m),$$

and the solution of boundary problem (3.1)-(3.3) reduces to the determination of the roots of the equation system

$$\psi_j(x_{1,0}, x_{2,0}, \ldots, x_{m,0}) = 0 \qquad (j = 1, 2, \ldots, m).$$

The use of the Newton method [20] for solving this system of equations leads to the following iteration process:

$$x_{j,0}^{(s+1)} = x_{j,0}^{(s)} + \Delta x_{j,0}^{(s)} \qquad (j = 1, 2, \ldots, m). \tag{3.4'}$$

The increments of $\Delta x_{j,0}^{(s)}$ are found by solving the system of linear algebraic equations

$$\sum_{j=1}^{m} \frac{\partial \psi_i}{\partial x_{j,0}} \Delta x_{j,0}^{(s)} = -\psi_i \qquad (i = 1, 2, \ldots, m). \tag{3.5}$$

Here the values of the functions of ψ and its derivatives are computed at the point $x^{(s)}$.

The nearness of the values of $x_{1,0}^{(s)}, x_{2,0}^{(s)}, \ldots, x_{m,0}^{(s)}$ to the solution will be characterized by the value

$$N = \sum_{j=1}^{m} |\Psi_j|. \tag{3.6}$$

The convergence of iteration process (3.4')-(3.5) can be improved by reducing the increments of Δx_0 found in (3.5) if it seems that $N^{(s+1)} \geq N^{(s)}$. We will substitute relations (3.4') by the following

$$x_{j,0}^{(s+1)} = x_{j,0}^{(s)} + k \times \Delta x_{j,0}^{(s)} \qquad (j = 1, 2, \ldots, m). \tag{3.7}$$

It is clear that when k = 1 relations (3.7) coincide with (3.4). We will start the iteration process, assuming that k = 1. If it seems that

$$N^{(s+1)} \geq N^{(s)}, \tag{3.8}$$

we will assume that k = 1/2 and will once again compute the values of $x_{1,0}^{(s+1)}$, /56
$x_{2,0}^{(s+1)}, \ldots, x_{m,0}^{(s+1)}, N^{(s+1)}$ by formulas (3.5), (3.7). If relation (3.8) appears once more, we will again reduce the value of k by one-half, etc., until we no longer obtain a reduction in the value of N: $N^{(s+1)} < N^{(s)}$ for some value $k = \bar{k}$. To avoid the slowing down of the convergence of the process nearer the solution due to a reduction to the value of k, we will try to double k at each iteration step. Thus, in the case under consideration the values of $x_{1,0}, x_{2,0}, \ldots, x_{m,0}$ at the s + 2 step will be computed by the formulas

$$x_{j,0}^{(s+2)} = x_{j,0}^{(s+1)} + 2\bar{k} \times \Delta x_{j,0}^{(s+1)} \qquad (j = 1, 2, \ldots, m).$$

The boundary problem is considered solved when the value of N becomes less than a given number ε.

The derivatives of $\partial\Psi/\partial x_0$ are found by the m- multiple solution of the Couchy (3.1)-(3.2), (3.4') with consecutive variation of the initial values of (3.4).

It should be noted that the derivatives of $\partial\Psi/\partial x_0$ can be computed by single

integration of m of the system of equation in variations for system (3.1) [21].

The procedure for the solution for the boundary problem using the modified Newton method is outlined below.

1. Given: $s = 0$, $k = 1$.

2. Given: $x_{1,0}, x_{2,0}, \ldots, x_{m,0}$ (first approximation).

3. The Couchy problem (3.1), (3.2), (3.4') is solved and the values of Ψ_j are computed.

4. The value of $N^{(s)}$ is computed by formula (3.6).

5. $N^{(s)}$ and ε are compared:
   a) if $N^{(s)} < \varepsilon$ the problem is considered solved;
   b) if $N^{(s)} \geq \varepsilon$ we move on to step 6.

6. s and 0 are compared:
   a) if $s = 0$ we move on to step 11;
   b) if $s \neq 0$ we move on to step 7.

7. $N^{(s)}$ and $N^{(s-1)}$ are compared:
   a) if $N^{(s)} < N^{(s-1)}$ we proceed to step 9;
   b) if $N^{(s)} \geq N^{(s-1)}$ we proceed to step 8.

8. The value of k is replaced by k/2 and we proceed to step 13.

9. k and 1 are compared:
   a) if $k = 1$ we proceed to step 11;
   b) if $k \neq 1$ we proceed to step 10.

10. The value of k is replaced by 2k. /57

11. The derivatives of $\partial\Psi/\partial x_0$ are computed.

12. The value of s is replaced by $s + 1$.

13. The values of $x_{1,0}, x_{2,0}, \ldots, x_{m,0}$ are computed by formulas (3.5) and (3.7), and then we return to step 3.

§2. Variation Problem on Minimum Acceleration Time to Achievements of Parabolic Velocity [22].

We will assume that the initial orbit is circular and that the spacecraft is propelled by an engine of constant thrust (see Chapter 2, §1).

The motion of the spacecraft is described by the following equation system:

$$\frac{dr}{dt} = u; \tag{3.9}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{3.10}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{a_0}{1 - \frac{a_0}{V} t} \cos\lambda; \tag{3.11}$$

$$\frac{dv}{dt} = \frac{uv}{r} + \frac{a_0}{1 - \frac{a_0}{V} t} \sin\lambda, \tag{3.12}$$

where r and ϕ are polar coordinates of the spacecraft (see Figure 1); u and v are the radial and transversal velocity components; t is time; λ is the angle between the vector radius of the spacecraft and the thrust vector; a_0 is the initial reactive acceleration; V is the velocity of the reactive jet. All values in equations (3.9)-(3.12) are dimensionless. The dimensional variables, which will be denoted by an asterisk as in Chapter 2, can be found by using the following relations:

$$r^* = r r_0^*; \quad u^* = u \sqrt{\frac{\mu}{r_0^*}}; \quad v^* = v \sqrt{\frac{\mu}{r_0^*}}:$$

$$t^* = t r_0^* \sqrt{\frac{r_0^*}{\mu}}; \quad a_0^* = a_0 \frac{\mu}{r_0^{*2}}; \quad V^* = V \sqrt{\frac{\mu}{r_0^*}}.$$

Here r_0^* is the radius of the initial circular orbit; $\mu = \gamma M$ is the product of the gravitational constant and the mass of the attracting body. /58

We are required to find the function $\lambda(t)$ such that the time required for achieving parabolic velocity for the given a_0 and V will be minimal.

But by using the maximum principle, we will find the equations for the pulses p_r, p_ϕ, p_u, and p_v:

$$\frac{dp_r}{dt} = \frac{p_\varphi v}{r^2} + \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v u v}{r^2}; \tag{3.13}$$

$$\frac{dp_\varphi}{dt} = 0; \tag{3.14}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.15}$$

$$\frac{dp_v}{dt} = \frac{p_\varphi}{r} - \frac{2p_u v}{r} + \frac{p_v u}{r}. \tag{3.16}$$

We now construct the function

$$H = p_r u + \frac{p_\varphi v}{r} + \frac{p_u v^2}{r} - \frac{p_u}{r^2} + \frac{a_0 p_u}{1 - \frac{a_0}{V} t} \cos\lambda - \frac{p_v u v}{r} + \frac{a_0 p_v}{1 - \frac{a_0}{V} t} \sin\lambda.$$

The function of H should be maximal with respect to λ for all values of the phase coordinates. The maximum of H is achieved when angle λ satifies the relations

$$\sin\lambda = \frac{p_v}{\sqrt{p_u^2 + p_v^2}}; \qquad \cos\lambda = \frac{p_u}{\sqrt{p_u^2 + p_v^2}}. \tag{3.17}$$

At the final moment of time $t = t_k$ the condition for achieving parabolic velocity

$$r(u^2 + v^2) = 2. \tag{3.18}$$

must be satisfied. Moreover, when $t = t_k$ the phase coordinates should satisfy the conditions of transversality:

$$p_\varphi = 0; \quad p_u = p_r u r^2; \quad p_v = p_r v r^2. \tag{3.19}$$

It follows from (3.6) that p_ϕ is constant. Therefore, considering the first of conditions (3.11), we find

$$p_\varphi \equiv 0. \tag{3.20}$$

By substituting (3.17) into (3.9)-(3.16), recalling (3.20), we obtain /59

$$\frac{dr}{dt} = u; \tag{3.21}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{3.22}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{a_0^{\cdot}}{1-\frac{a_0}{V}t} \frac{p_u}{\sqrt{p_u^2+p_v^2}}; \tag{3.23}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{a_0}{1-\frac{a_0}{V}t} \frac{p_v}{\sqrt{p_u^2+p_v^2}}; \tag{3.24}$$

$$\frac{dp_r}{dt} = \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v uv}{r^2}; \tag{3.25}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.26}$$

$$\frac{dp_v}{dt} = -\frac{2p_u v}{r} + \frac{p_v u}{r}. \tag{3.27}$$

The variable ϕ does not appear in the right hand sides of the equations. Therefore we may exclude equation (3.22) from the system of equations obtained and find ϕ by squaring after we have solved the boundary problem. Equation (3.22) should be integrated along with the other equations when the numerical solution for the problem is being found by computer.

It is easy to see that the form of equations (3.21)-(3.27) and of conditions of (3.19) enable us to assign arbitrarily the value of one of the pulses at the initial moment of time. We will consider that $p_V = 1$ when $t = 0$.

Thus, we are required to find the solution for equation system (3.21)-(3.27) with the following boundary conditions:
when $t = 0$

$$r = 1; \quad \varphi = 0; \quad u = 0; \quad v = 1; \quad p_v = 1; \tag{3.28}$$

and when $t = t_k$

$$r(u^2+v^2)=2;\quad p_u-p_r ur^2=0;\quad p_v-p_r vr^2=0. \tag{3.29}$$

One of the conditions in the end of the trajectory should be used for determining the unknown time t_k, i.e. for terminating integration. For this purpose we should use the condition $r(u^2 + v^2) = 2$, since we do not exclude the possibility that the other two equalities may be repeatedly satisfied in the trajectory. /60

Analysis of the boundary conditions prompts a conclusion that when $t = t_k$ the thrust vector should coincide with the velocity vector, while the derivative of angle γ between these vectors (see Figure 1), in time should be equal to 0.

In order to solve boundary problem (3.21)-(3.29) using the Newton method we have to know the first approximation of the missing initial conditions p_{r_0} and p_{u_0}. In order to simplify this problem we will express p_{r_0} and p_{u_0} through values of a simple physical nature.

From (3.17) we find

$$\tan\lambda = p_v/p_u, \tag{3.30}$$

and, consequently, with consideration of the fact that $p_{v_0} = 1$,

$$p_{u_0} = \cot\lambda_0. \tag{3.31}$$

By differentiating (3.30) with respect to time and substituting the values dp_u/dt and dp_v/dt from (3.26) and (3.27) into the expression obtained, we find

$$p_{r_0} = \frac{1+\cos^2\lambda_0 + \left.\frac{d\lambda}{dt}\right|_{t=0}}{\sin^2\lambda_0}. \tag{3.32}$$

We will assume that the optimal equation at the initial stage is close to transversal thrust: $\lambda \equiv \pi/2$. Then from (3.31) and (3.32) we can find $p_{r_0} = 1$, /61 $p_{u_0} = 0$. These are the values which we shall use as the first approximation in the solution of the problem.

Figures 16 and 17 represent angle γ (in rad) as a function of time and as a function of the number of orbits n around the attracting center at a constant reactive acceleration $a_0 = 10^{-2}$. Here we see that the thrust vector completes its oscillatory motions near the velocity vector. The amplitude of these oscillations increases with the passage of time, while their period is about equal to the time of one orbit.

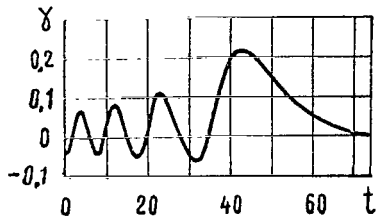

Figure 16

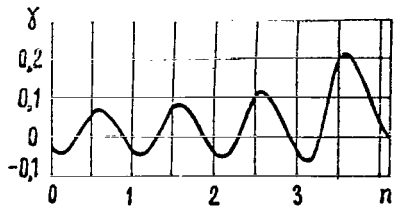

Figure 17

Figure 18 represents the maneuver time t_k as a function of constant acceleration a_0.

Figure 19 illustrates the values $\tau_1 = (t_{E_1} - t_E)/t_E$ (curve 1) and $\tau_2 = (t_{E_2} - t_E)/t_E$ (curve 2), as functions of a constant a_0. Here t_E, t_{E_1} and t_{E_2} are the time of optimal acceleration, the time of acceleration with tangential thrust and the time of acceleration with transversal thrust, respectively. The Figure shows that tangential thrust produces a very slight loss in time (less than 1/2 % when $a_0 < 5 \cdot 10^{-3}$), in comparison with optimal thrust. /62 The loss for transversal thrust is greater, but when the values of a_0 are small it is also quite small.

The calculations made for the case of constant thrust indicate that a reduction in the exhaust velocity V for the same values of a_0 results in a reduction in the values τ_1 and τ_2. This signifies that Figure 19 shows the upper bounds of τ_1 and τ_2 during acceleration at constant thrust.

The calculations made herein indicate that when $V \geq 1$ formulas of the type (2.57)-(2.62) can be used for determining the minimum acceleration time:

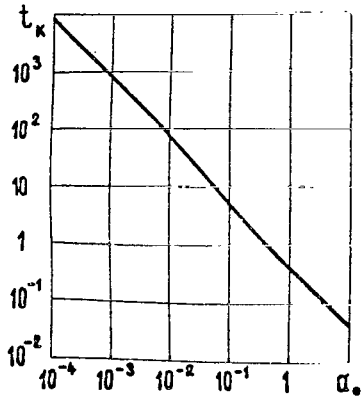

Figure 18

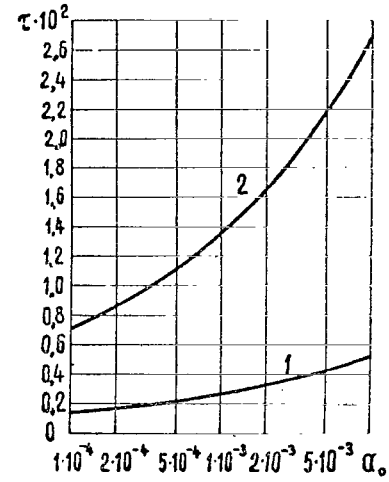

Figure 19

$$\left(1 - V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right)^8 - 0{,}2063\frac{a_0^2}{\left(1-\frac{a_0}{V}t_k\right)^2} + 0{,}25\frac{a_0^2}{V\left(1-\frac{a_0}{V}t_k\right)^2}\left(1 - V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right) = 0. \quad (3.33)$$

The value of t_k can be found from the simple relation

$$t_k = \frac{1-0{,}8209\sqrt[4]{a_0} - (1-0{,}8209\sqrt[4]{a_0})^2/2V}{a_0}. \quad (3.34)$$

with a somewhat greater error. When reactive acceleration is constant

$$t_k = \frac{1-0{,}8209\sqrt[4]{a_0}}{a_0}. \quad (3.35)$$

The error in the determination of t_k from (3.33)-(3.35) is the same as for the cases of tangential and transversal thrust (see Figure 8).

§3. Variation Problem of Minimum Engine Burn Time During Acceleration to Parabolic Velocity.

It is assumed that the initial orbit is circular; reactive acceleration is constant.

The motion of a spacecraft is described by the following equations in the polar coordinate system.

$$\frac{dr}{dt} = u; \quad (3.36)$$ /63

$$\frac{d\varphi}{dt} = \frac{v}{r}; \quad (3.37)$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \delta a \cos\lambda; \tag{3.38}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \delta a \sin\lambda; \tag{3.39}$$

$$\frac{dt_M}{dt} = \delta. \tag{3.40}$$

Here r and ϕ are polar coordinates (see Figure 1); u and v are the radial and transversal velocity components; t is time; t_M is burn time of the engine; λ is the angle between the vector radius of the spacecraft and the reactive acceleration vector; a is reactive acceleration; $\delta = 1$ on the active segments and $\delta = 0$ on the passive segments.

All values in (3.36)-(3.40) are dimensionless. The value of the corresponding dimensional variables can be found by using the relations

$$r^* = r r_0^*; \quad u^* = u\sqrt{\frac{\mu}{r_0^*}}; \quad v^* = v\sqrt{\frac{\mu}{r_0^*}}; \quad t^* = t r_0^* \sqrt{\frac{r_0^*}{\mu}}; \quad a^* = a\frac{\mu}{r_0^{*2}},$$

where r_0^* is the radius of the initial circular orbit; μ is the gravitational parameter of the attracting body.

We are required to find the function $\lambda = \lambda(t)$ and $\delta = \delta(t)$, the minimizing burn time t_M for a given acceleration time t_k.

The pulses p_r, p_ϕ, p_u, p_v, p_M satisfy the following equation system:

$$\frac{dp_r}{dt} = \frac{p_\varphi v}{r^2} + \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v uv}{r^2}; \tag{3.41}$$

$$\frac{dp_\varphi}{dt} = 0; \tag{3.42}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.43}$$

$$\frac{dp_v}{dt} = -\frac{p_\varphi}{r} - \frac{2p_u v}{r} + \frac{p_v u}{r}; \tag{3.44}$$

$$\frac{dp_M}{dt} = 0. \tag{3.45}$$

We will construct the function of H: /64

$$H = p_r u + \frac{p_\varphi v}{r} + \frac{p_u v^2}{r} - \frac{p_u}{r^2} + \delta a \cos\lambda - \frac{p_v u v}{r} + \delta a \sin\lambda + p_M \delta .$$

The function of H is maximum when

$$\sin\lambda = \frac{p_v}{\sqrt{p_u^2 + p_v^2}}; \quad \cos\lambda = \frac{p_u}{\sqrt{p_u^2 + p_v^2}}, \tag{3.46}$$

$$\delta = \begin{cases} 1 & \text{for} \quad \Delta > 0; \\ 0 & \text{for} \quad \Delta < 0, \end{cases} \tag{3.47}$$

where

$$\Delta = a(\sqrt{p_u^2 + p_v^2} - 1),$$

i.e.

$$\delta = \frac{1 - \operatorname{sign}(\sqrt{p_u^2 + p_v^2} - 1)}{2}. \tag{3.48}$$

When $t = t_k$ the conditions of parabolic velocity

$$r(u^2 + v^2) = 2 \tag{3.49}$$

and the conditions of transversality:

$$p_\varphi = 0; \quad p_u = p_r u r^2; \quad p_v = p_r v r^2; \quad p_M = -a. \tag{3.50}$$

must be satisfied. From (3.42), (3.45) and (3.50) it follows that $p_\phi \equiv 0$, $p_M \equiv -a$.

Thus the solution of the variation problem reduces to the solution of the following boundary problem:

$$\frac{dr}{dt} = u; \tag{3.51}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{3.52}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + a\frac{p_u}{\sqrt{p_u^2+p_v^2}}\ \frac{1-\mathrm{sign}(\sqrt{p_u^2+p_v^2}-1)}{2}; \tag{3.53}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + a\frac{p_v}{\sqrt{p_u^2+p_v^2}}\ \frac{1-\mathrm{sign}(\sqrt{p_u^2+p_v^2}-1)}{2}; \tag{3.54}$$

$$\frac{dt_M}{dt} = \frac{1-\mathrm{sign}(\sqrt{p_u^2+p_v^2}-1)}{2}; \tag{3.55}$$

$$\frac{dp_r}{dt} = \frac{p_u v^2}{r} - \frac{2p_u}{r^3} - \frac{p_v uv}{r^2}; \tag{3.56}$$ /65

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.57}$$

$$\frac{dp_v}{dt} = -\frac{2p_u v}{r} + \frac{p_v u}{r}. \tag{3.58}$$

when t = 0

$$r = 1;\quad \varphi = 0;\quad u = 0;\quad v = 1;\quad t_M = 0; \tag{3.59}$$

when $t = t_k$

$$r(u^2+v^2) = 2;\ p_u = p_r u r^2;\ p_v = p_r v r^2. \tag{3.60}$$

The solution of this problem was found for a = 0.01 within the range $100 \leq t_k \leq 160$. It turned out that the solution for the boundary problem (3.51)-(3.60) is ambiguous. Thus, within the range $111 \leq t_k < 125$ we found two

solutions. One of them corresponds to a trajectory with six passive segments and the other corresponds to a trajectory with seven passive segments. For t_k within the range $125 \leq t_k \leq 134$ there were three solutions (trajectories with six, seven, and eight passive segments) and for t_k within the range $134 < t_k \leq 151$ there were two solutions (trajectories with seven and eight passive segments).

Figure 20 illustrates burn time t_{Mk} as a function of acceleration time t_k. Curve 1-1 corresponds to trajectories with six passive segments, curve 1-2 corresponds to trajectories with seven passive segments, and curve 1-3 to trajectories with eight passive segments. As t_k increases the value of t_{Mk} diminishes clearly approaching the value $t_{Mk} = (\sqrt{2} - 1)/a = 41.42$ (slipping condition: an infinitely large number of infinitely small pulses in the pericenters) when the increase in t_k is unbounded.

The calculations show that in the optimal trajectory, and in each orbit around the attracting center, there is one active and one passive segment, and the active segment is located in the area of the pericenter. This is to be expected, since the rate of increase in energy increases as the distance between the spacecraft and the attracting center diminishes.

Figure 21 shows one of the trajectories with six passive segments. The solid lines in the figure indicate the active segments of the trajectory and the broken lines show the passive segments. /67

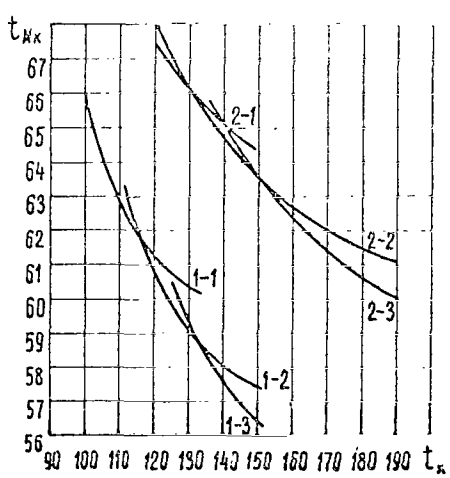

Figure 20

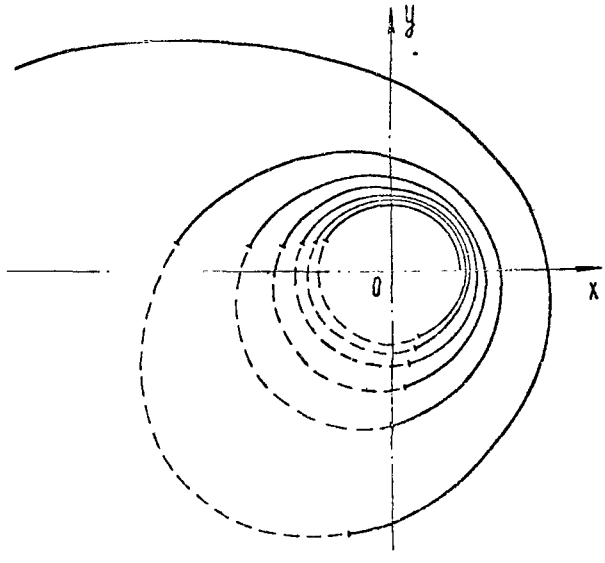

Figure 21

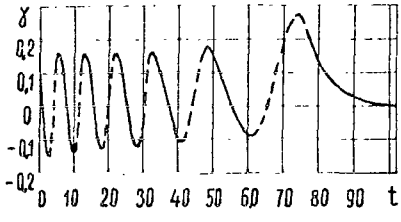

Figure 22

Figure 22 represents angle γ between the velocity vector and thrust vector as a function of time (see Figure 1). The broken line indicates the dependence of the values of angle γ on t computed in the passive segments by formula (3.46) ($\gamma = \lambda - \arctan(v/u)$). As seen in the figure, the values of angle γ osculate nearer zero. The amplitude of osculations increases with the passage of time. The period of osculations is approximately equal to time of one orbit. As in the problem examined in §2, $\gamma = 0$ when $t = t_k$ and the derivative of $d\gamma/dt$ is also equal to zero when $t = t_k$.

In addition to the problem under examination, we also solved the variation problem on the minimum burn time during acceleration with transversal thrust. The corresponding boundary problem is derived from (3.51)-(3.60) by substituting equations (3.53)-(3.55) by the following equations:

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2};$$

$$\frac{dv}{dt} = -\frac{uv}{r} + a\,\frac{1 - \operatorname{sign}(p_v - 1)}{2};$$

$$\frac{dt_M}{dt} = \frac{1-\operatorname{sign}(p_v - 1)}{2}.$$

The solution of this problem was found for the range $120 \leq t_k \leq 190$ (as in the preceding problem, we assumed that a = 0.01). Here the solution of the boundary problem was also ambiguous.

Figure 20 illustrates t_{Mk} as a function of t_k. Curve 2-1 corresponds to trajectories with six passive segments, curve 2-2 to trajectories with seven, and curve 2-3 to trajectories with eight passive segments.

It should be noted that the acceleration time for optimal control without engine burn when a = 0.01 is 74.13. Thus, as seen in Figure 20, the slight increase in the total maneuver time results in the fact that the burn time of /68 acceleration with transversal thrust becomes less than the acceleration time with optimal control when the engine burns constantly.

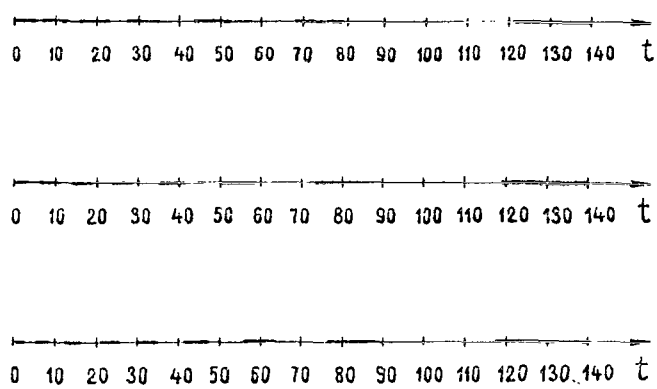

Figure 23

Figure 23 illustrates the distribution of the active segments (heavy lines) for three different solutions when $t_k = 140$.

§4. Variation Problem on Deceleration of Spacecraft Upon Approach to Planet.

In this section we will give the solution to the variation problem of transition from the sphere of influence of a planet to a given circular orbit in a minimum time with constant reactive acceleration.

The motion of a spacecraft is described by the following equation system:

$$\frac{dr}{dt} = U \sin\theta; \tag{3.61}$$

$$\frac{d\varphi}{dt} = -\frac{U\cos\theta}{r}; \tag{3.62}$$

$$\frac{dU}{dt} = -\frac{\sin\theta}{r^2} + a\cos\gamma; \tag{3.63}$$

$$\frac{d\theta}{dt} = \frac{U\cos\theta}{r} - \frac{\cos\theta}{r^2 U} + \frac{a\sin\gamma}{U}. \tag{3.64}$$

Here r and ϕ are polar coordinates; U is the velocity vector modulus (see Figure 1); θ is the angle between the line perpendicular to the vector radius and the velocity vector; t is time; a is reactive acceleration; γ is the angle between the velocity vector and the thrust vector. /69

In equations (3.61)-(3.64) all values are dimensionless. They are related to the corresponding dimensional values by the following relation:

$$r^* = r r_k^*; \quad U^* = U\sqrt{\frac{\mu}{r_k^*}}; \quad t^* = t r_k^* \sqrt{\frac{r_k^*}{\mu}}; \quad a^* = a\frac{\mu}{r_k^{*2}},$$

where r_k^* is the radius of a given circular orbit (into which the spacecraft must be inserted); μ is the gravitational parameter of the planet.

We are required to find an equation which insures transition within minimum time from point $r = \rho$; $\phi = 0$; $U = U_0$; $\theta = \theta_0$, to the surface $r = 1$; $U = 1$; $\theta = \pi$. Here ρ is the ratio of the radius of the sphere of influence to the radius of the given circular orbit.

We will construct a conjugate equation system for the pulses p_r, p_ϕ, p_u and p_θ:

$$\frac{dp_r}{dt} = -\frac{p_\varphi\cos\theta}{r^2} - \frac{2p_u\sin\theta}{r^3} + \frac{p_\theta U\cos\theta}{r^2} - \frac{2p_\theta\cos\theta}{r^3 U}; \tag{3.65}$$

$$\frac{dp_\varphi}{dt} = 0; \tag{3.66}$$

$$\frac{dp_u}{dt} = -p_r \sin\theta + \frac{p_\varphi \cos\theta}{r} - \frac{p_\theta \cos\theta}{r} - \frac{p_\theta \cos\theta}{r^2 U^2} + \frac{p_\theta a \sin\gamma}{U^2}; \tag{3.67}$$

$$\frac{dp_\theta}{dt} = -p_r U \cos\theta - \frac{p_\varphi U \sin\theta}{r} + \frac{p_u \cos\theta}{r^2} + \frac{p_\theta U \sin\theta}{r} - \frac{p_\theta \sin\theta}{r^2 U}. \tag{3.68}$$

We then formulate the function

$$H = p_r U \sin\theta - \frac{p_\varphi U \cos\theta}{r} - \frac{p_u \sin\theta}{r^2} + a p_u \cos\gamma + \frac{p_\theta U \cos\theta}{r} - \frac{p_\theta \cos\theta}{r^2 U} + \frac{a p_\theta \sin\gamma}{U}.$$

The function of H becomes maximum when the values of angle γ satisfy the rela- /70
tions

$$\sin\gamma = \frac{p_\theta}{\sqrt{p_u^2 U^2 + p_\theta^2}}; \quad \cos\gamma = \frac{p_u U}{\sqrt{p_u^2 U^2 + p_\theta^2}}.$$

The value of angle ϕ is not known on the right-hand end of the trajectory. Therefore, considering (3.66) we will find $p_\phi \equiv 0$.

Thus, we are required to find a solution for the following boundary problem:

$$\frac{dr}{dt} = U \sin\theta; \tag{3.69}$$

$$\frac{d\varphi}{dt} = -\frac{U \cos\theta}{r}; \tag{3.70}$$

$$\frac{dU}{dt} = -\frac{\sin\theta}{r^2} + \frac{a p_u U}{\sqrt{p_u^2 U^2 + p_\theta^2}}; \tag{3.71}$$

$$\frac{d\theta}{dt} = \frac{U\cos\theta}{r} - \frac{\cos\theta}{r^2 U} + \frac{a p_\theta}{U\sqrt{p_u^2 U^2 + p_\theta^2}}; \tag{3.72}$$

$$\frac{dp_r}{dt} = -\frac{2p_u \sin\theta}{r^3} + \frac{p_\theta U\cos\theta}{r^2} - \frac{2p_\theta \cos\theta}{r^3 U}; \tag{3.73}$$

$$\frac{dp_u}{dt} = -p_r \sin\theta - \frac{p_\theta \cos\theta}{r} - \frac{p_\theta \cos\theta}{r^2 U^2} + \frac{a p_0^2}{U^2\sqrt{p_u^2 U^2 + p_0^2}}; \tag{3.74}$$

$$\frac{dp_\theta}{dt} = -p_r U\cos\theta + \frac{p_u \cos\theta}{r^2} + \frac{p_\theta U\sin\theta}{r} - \frac{p_0 \sin\theta}{r^2 U}; \tag{3.75}$$

for $t = 0$

$$r = \rho; \quad \varphi = 0; \quad U = U_0; \quad \theta = \theta_0; \tag{3.76}$$

for $t = t_k$

$$r = 1; \quad U = 1; \quad \theta = \pi; \quad p_u = -1. \tag{3.77}$$

Since the time t_k is not known, we will have to use one of the boundary conditions (3.77) to complete integration. This is difficult to do, since in the case of a multiturn maneuver the variables r, U and θ can repeatedly acquire the values of (3.77). Therefore, the equation system is integrated in the opposite direction, and integration is terminated when the condition $r = \rho$ is satisfied. In the sense of the problem, the trajectory of the spacecraft does not leave the sphere of influence of the planet. Therefore, the condition $r = \rho$ can be satisfied only once in the trajectory. The following assumptions were made for the calculations: $\mu = 398{,}600$ km^3/sec^2, $\rho^* = 930{,}000$ km, $r_k^* = 189{,}200$ km, $a^* = 3.52$ mm/sec^2; the values of U_0 and θ_0 varied (see §8 of Chapter 2). /71

Figure 24 illustrates the ratio of the time of the maneuver to the minimal time $t_{kmin} = 344.2$ as a function of the values of θ_0 and U_0. This absolute minimum in deceleration time is found when $U_0 = 0.516$ and $\theta_0 = 4.500$.

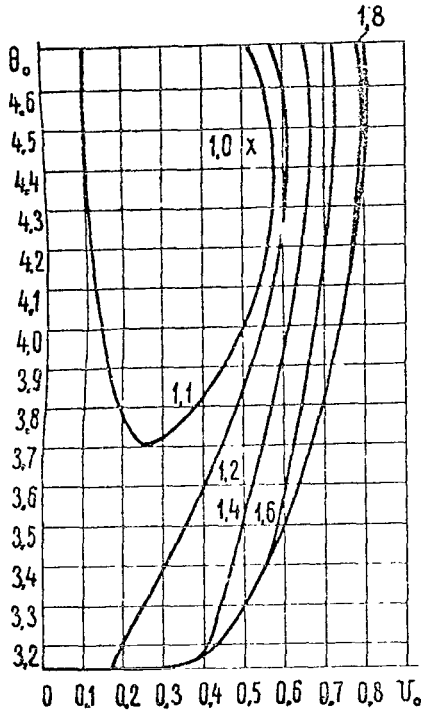

Figure 24.

As regards the nature of control, the maneuver consists of two sections. The duration of the initial section is equal to about the time of one (the first) orbit around the earth (the entire maneuver requires about thirty orbits). In this section the eccentricity of the osculating orbit is reduced to about zero, and the character of control in this case depends strongly on the initial velocity vector. In the second section, which is concluded when the spacecraft is inserted into the given circular orbit, the thrust is close to tangential for all initial conditions. /72

§5. Transition Between Circular Coplanar Orbits

In this section we will examine the variation problem of the minimum transition time from an earth orbit to orbits of other planets. All orbits are regarded as circular. It is assumed that reactive acceleration is constant.

In the solution of the given problem we will assume that the only force acting upon the spacecraft other than reactive forces is the force of attraction of the sun, i.e. we will examine the case where the spacecraft is located beyond the sphere of influence of the planets. The problem of reaching a certain point in the sphere of influence and transitioning from it to a circular orbit around a planet was solved in the preceding section. The motion of the spacecraft is described by the following equations in the polar coordinate system:

$$\frac{dr}{dt} = u; \tag{3.78}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{3.79}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + a\cos\lambda; \tag{3.80}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + a\sin\lambda. \tag{3.81}$$

Here r and ϕ are polar coordinates (see Figure 1); u and v are radial and transversal velocity components; a is reactive acceleration; λ is the angle between the vector radius of the spacecraft and the thrust vector; t is time.

All values in these equations are dimensionless. The corresponding dimensional values, denoted by the asterisk, can be found by formulas

$$r^* = r r_0^*;\ u^* = u\sqrt{\frac{\mu}{r_0^*}};\ v^* = v\sqrt{\frac{\mu}{r_0^*}};\ t^* = t r_0^*\sqrt{\frac{r_0^*}{\mu}};\ a^* = a\frac{\mu}{r_0^{*2}},$$ /73

where r_0^* is the radius of the earth orbit; $\mu = \gamma M$ is the product of the gravitational constant by the mass of the sun.

We are required to find control $\lambda(t)$ which provides transition from the point

$$r = 1;\quad \varphi = 0;\quad u = 0;\quad v = 1$$

to the plane

$$r = r_k;\quad u = 0;\quad v = 1/\sqrt{r_k}$$

in minimum time.

We will formulate the function of H:

$$H = p_r u + \frac{p_\varphi v}{r} + \frac{p_u v^2}{r} - \frac{p_u}{r^2} + a p_u \cos\lambda - \frac{p_v u v}{r} + a p_v \sin\lambda.$$

Here p_r, p_ϕ, p_u and p_v are pulses which satisfy the system of equations

$$\frac{dp_r}{dt} = \frac{p_\varphi v}{r^2} + \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v u v}{r^2}; \tag{3.82}$$

$$\frac{dp_\varphi}{dt} = 0; \tag{3.83}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.84}$$

$$\frac{dp_v}{dt} = -\frac{p_\varphi}{r} - \frac{2p_u v}{r} + \frac{p_v u}{r}. \tag{3.85}$$

From the condition of the maximum of the function of H we find

$$\sin\lambda = \frac{p_v}{\sqrt{p_u^2 + p_v^2}}; \quad \cos\lambda = \frac{p_u}{\sqrt{p_u^2 + p_v^2}}. \tag{3.86}$$

Angle ϕ is not known at the final moment of time $t = t_k$. Therefore, $p_\phi = 0$ when $t = t_k$. Recalling (3.83), we find $p_\phi \equiv 0$. Equations (3.82)-(3.85) are similar in relation to the pulses; this enables us, as in the problem examined in §2, to assume that $p_v = 1$ when $t = 0$.

By substituting (3.86) into equations (3.78)-(3.81), we find

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$$\frac{dr}{dt} = u; \qquad \frac{d\varphi}{dt} = \frac{v}{r};$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{a p_u}{\sqrt{p_u^2 + p_v^2}};$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{a p_v}{\sqrt{p_u^2 + p_v^2}};$$

$$\frac{dp_r}{dt} = \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v uv}{r^2};$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \qquad \frac{dp_v}{dt} = -\frac{2p_u v}{r} + \frac{p_v u}{r}.$$

The boundary conditions are:

$$t = 0; \quad r = 1; \quad \varphi = 0; \quad u = 0; \quad v = 1; \quad p_v = 1;$$
$$t = t_k; \quad r = r_k; \quad u = 0; \quad v = 1/\sqrt{r_k}.$$

It is assumed in the calculations that a = 0.1667; this corresponds approximately to the dimensional value $a^* = 1$ mm/sec^2.

Figure 25 shows the dependence of the heliocentric radius as a function of time for transitions to orbits around Venus, Mars and Jupiter. The relative time $\bar{t} = t/t_k$, where t_k is the time of transition, equal to 2.33 for Venus, /75
3.25 for Mars, and 9.31 for Jupiter, is arranged along the abscissa; the

corresponding time in days is 135, 189 and 540 respectively. The angle of transition ϕ_k is 172, 138 and 185° respectively. The time of transition to the orbits of other planets is 789 days for Saturn, 1,150 for Uranus, 1,480 days for Neptune, and 1,720 days for Pluto.

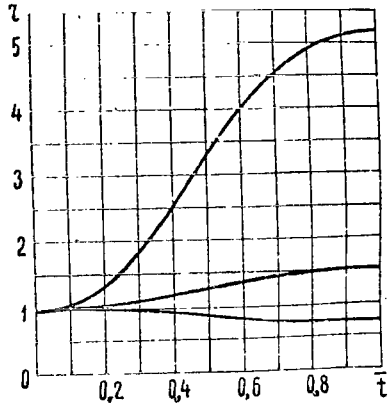

Figure 25

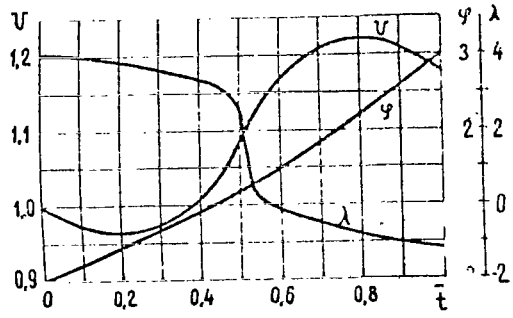

Figure 26

Figure 26 illustrates the graphs of the dependence of $U = \sqrt{u^2 + v^2}$, ϕ and λ on time during transition to an orbit around Venus.

§6. The Rounding Off of an Elliptical Orbit.

In this section we will examine the variation problem of shifting the spacecraft located in an elliptical orbit at the initial moment to a circular orbit of unknown radius. The time required for the maneuver is minimized. Reactive acceleration is assumed to be constant.

The equation system describing the motion of the spacecraft is

$$\frac{dr}{dt} = u; \tag{3.87}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{3.88}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + a\cos\lambda; \tag{3.89}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + a\sin\lambda. \tag{3.90}$$

Here r and ϕ are the polar coordinates (see Figure 1); u and v are the radial /76

and transversal velocity components; t is time; a is reactive acceleration; λ is the angle between the vector radius of the spacecraft and the thrust vector.

The variables in equations (3.87)-(3.90) are dimensionless. The dimensional values of the variables can be found from the following formulas:

$$r^* = r r_0^*; \quad u^* = u\sqrt{\frac{\mu}{r_0^*}}; \quad v^* = v\sqrt{\frac{\mu}{r_0^*}}; \quad t^* = t r_0^* \sqrt{\frac{r_0^*}{\mu}}; \quad a^* = a\frac{\mu}{r_0^{*2}},$$

where r_0^* is the distance to the apocenter of the initial elliptical orbit; μ is the gravitational parameter of the attracting body.

The conjugate equation system for the pulses p_r, p_ϕ, p_u, p_v is

$$\frac{dp_r}{dt} = \frac{p_\varphi v}{r^2} + \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v u v}{r^2}; \tag{3.91}$$

$$\frac{dp_\varphi}{dt} = 0; \tag{3.92}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.93}$$

$$\frac{dp_v}{dt} = -\frac{p_\varphi}{r} - \frac{2p_u v}{r} + \frac{p_v u}{r}. \tag{3.94}$$

We formulate the function

$$H = p_r u + \frac{p_\varphi v}{r} + \frac{p_u v^2}{r} - \frac{p_u}{r^2} - \frac{p_v u v}{r} + a p_u \cos\lambda + a p_v \sin\lambda.$$

The function of H is maximum when

$$\sin\lambda = \frac{p_v}{\sqrt{p_u^2 + p_v^2}}; \qquad \cos\lambda = \frac{p_u}{\sqrt{p_u^2 + p_v^2}}.$$

When $t = t_k$ (at the final moment of time) $rv^2 = 1$, $u = 0$, $2p_r = p_v v$, $p_\phi = 0$. The last of these conditions along with equation (3.92) yields the following result: $p_\phi \equiv 0$.

Thus, the solution of the variation problem reduces to the solution of the following boundary problem:

$$\frac{dr}{dt} = u; \quad \frac{d\varphi}{dt} = \frac{v}{r}; \quad \frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{a p_u}{\sqrt{p_u^2 + p_v^2}};$$

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$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{a p_v}{\sqrt{p_u^2 + p_v^2}}; \quad \frac{dp_r}{dt} = \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v u v}{r^2};$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \quad \frac{dp_v}{dt} = -\frac{2p_u v}{r} + \frac{p_v u}{r}.$$

When t = 0

$$r = \frac{1-e_0}{1+e_0\cos\varphi_0}; \quad \varphi = \varphi_0, \; u = \sqrt{\frac{1}{1-e_0}}\, e_0 \sin\varphi_0;$$

$$v = \sqrt{\frac{1}{1-e_0}}\,(1+e_0\cos\varphi_0); \quad p_v = 1;$$

when $t = t_k$

$$rv^2 = 1; \quad u = 0; \quad 2p_r r = p_v v.$$

Here e_0 is eccentricity of the initial orbit. Angle ϕ is taken from the direction to the pericenter.

The condition $p_v = 1$ is known because of the similarity of equations (3.91)-(3.94) relative to the pulses.

The calculation made for $3\cdot10^{-2} \leq a \leq 10$ and $10^{-2} \leq e_0 \leq 10^{-1}$ indicate that when the values of reactive acceleration a are high, it is more desirable to initiate the maneuver nearer the apocenter. The starting point which insures the minimum time is displaced toward the spacecraft when acceleration is reduced.

For the problem under consideration there is locally optimal control which

is selected in such a way that the rate of reduction of eccentricity will be maximum at each moment of time (see Chapter 1). It seems that the time required for executing the maneuver with locally optimal control is close to the time determined through the solution of the variation problem. The maximal difference is 4.9%, i.e. the optimum control is close to the locally optimal control. This facilitates the selection of the first approximation for the solution of the boundary problem.

§7. Variation Problem of Rendezvous of Spacecraft.

Let two spacecraft (we will denote them by the numbers 1 and 2) move in circular coplanar orbits of radius r_1 and r_2, where $r_1 < r_2$. At the moment /78
$t = 0$ the central angle between the vector radii of the spacecraft is Φ. At this moment the engine of spacecraft 1 is ignited and after time t_k both spacecraft meet in the orbit of radius r_2, where no limitations are imposed on the value of the relative velocity of the spacecraft at the point of rendezvous.

The motion of spacecraft 1 with constant reactive acceleration is described by the following equation system:

$$\frac{dr}{dt} = u; \tag{3.95}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + a\cos\lambda; \tag{3.96}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + a\sin\lambda; \tag{3.97}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}. \tag{3.98}$$

Here r and ϕ are the polar coordinates (see Figure 1); u and v are the radial and transversal components of the velocity vector; t is time; a is reactive acceleration; λ is the angle between the vector radius of the spacecraft and the thrust vector.

All values in (3.95)-(3.98) are dimensionless. The dimensional and dimensionless values are interrelated by the following formulas, where the asterisks denote the dimensional values:

$$\left.\begin{aligned} &r^* = rr_1^*;\quad u^* = u\sqrt{\frac{\mu}{r_1^*}};\quad v^* = v\sqrt{\frac{\mu}{r_1^*}};\\ &t^* = tr_1^*\sqrt{\frac{r_1^*}{\mu}};\qquad a^* = a\frac{\mu}{r_1^{*2}}. \end{aligned}\right\} \tag{3.99}$$

Here r_1^* is the radius of the initial orbit of spacecraft 1; μ is the gravitational parameter of the attracting body.

It is convenient at this point to use a rotating coordinate system connected with spacecraft 2. For this purpose we will make the substitution

$$\psi = \varphi - \omega t, \tag{3.100}$$

where ω is the angular velocity of spacecraft 2 in orbit. If the dimensionless radius of this orbit is denoted through R, then $\omega = 1/\sqrt{R^3}$. As the result of substitution (3.100) equation (3.98) is rewritten as follows:

$$\frac{d\psi}{dt} = \frac{v}{r} - \omega. \tag{3.101}$$ /79

The boundary conditions for system (3.95)-(3.97), (3.101) is as follows: for t = 0

$$r = 1; \quad u = 0; \quad v = 1; \quad \psi = 0; \tag{3.102}$$

for $t = t_k$

$$r = R; \quad \psi = \Phi. \tag{3.103}$$

The problem is now to find the equation $\lambda = \lambda(t)$ such that the solution of system (3.95)-(3.97), (3.101) with boundary conditions (3.102), (3.103) insures the minimum time t_k. We formulate the function of H:

$$H = p_r u + \frac{p_u v^2}{r} - \frac{p_u}{r^2} + a p_u \cos\lambda - \frac{p_v u v}{r} + a p_v \sin\lambda + \frac{p_\psi v}{r} - p_\psi \omega. \tag{3.104}$$

From the condition of the maximum of H with respect to λ, we find the optimal control:

$$\sin\lambda = \frac{p_v}{\sqrt{p_u^2 + p_v^2}}; \qquad \cos\lambda = \frac{p_u}{\sqrt{p_u^2 + p_v^2}}. \tag{3.105}$$

The conjugate system has the following form:

$$\frac{dp_r}{dt} = \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} - \frac{p_v uv}{r^2} + \frac{p_\psi v}{r^2}; \tag{3.106}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.107}$$

$$\frac{dp_v}{dt} = -\frac{2p_u v}{r} + \frac{p_v u}{r}; \tag{3.108}$$

$$\frac{dp_\psi}{dt} = 0. \tag{3.109}$$

Since neither $u(t_k)$ or $v(t_k)$ are known, the following two conditions of transversality must be satisfied:

$$p_u(t_k) = 0; \quad p_v(t_k) = 0. \tag{3.110}$$

The solution of the conjugate system is determined with an accuracy up to a constant factor. Therefore, the initial value of one of the pulses can be regarded as known. Since it follows from (3.109) that p_ψ = const, it is convenient to assume this constant to be equal to some known number. However, the question concerning the choice of the sign of this constant is not trivial and requires special consideration. /29

We will notice that there is some minimum time t_k^0 of intersection of the orbit of spacecraft 2 by spacecraft 1. This time t_k^0 corresponds to the case where angle Ψ is not known when $t = t_k = t_k^0$. Then, in addition to (3.110), we obtain yet another condition of transversality

$$p_\psi(t_k) = 0. \tag{3.111}$$

Let the optimal transition with time t_k^0 correspond at the moment that $t = 0$ to some angle Φ^0 between the vector radii of the spacecraft. We will denote the angles greater than Φ^0 through Φ^+, and the angles less than Φ^0 through Φ^-. The optimal transition times corresponding to these initial angles will be denoted through t_k^+ and t_k^-, respectfully. It is obvious by definition that $t_k^+ > t_k^0$ and $t_k^- > t_k^0$. The following inequaltities will then be valid:

$$\left.\begin{aligned} &\frac{\partial t_k}{\partial \Phi} > 0 \quad \text{for} \quad t_k^+ ; \\ &\frac{\partial t_k}{\partial \Phi} < 0 \quad \text{for} \quad t_k^- . \end{aligned}\right\} \tag{3.112}$$

Based on the results outlined in [18,19] it can be demonstrated that

$$\frac{\partial t_k}{\partial \Phi} = p_\psi , \tag{3.113}$$

and consequently we finally have

$$\left.\begin{aligned} &p_\psi = 1 \quad \text{для} \quad t_k^+ ; \\ &p_\psi = -1 \quad \text{для} \quad t_k^- ; \\ &p_\psi = 0 \quad \text{для} \quad t_k^0 . \end{aligned}\right\} \tag{3.114}$$

Thus, we have the following boundary problem:

$$\frac{dr}{dt} = u ; \tag{3.115}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{a p_u}{\sqrt{p_u^2 + p_v^2}} ; \tag{3.116}$$

(3.117)/81

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{a p_v}{\sqrt{p_u^2 + p_v^2}} ;$$

$$\frac{d\psi}{dt} = \frac{v}{r} - \omega ; \tag{3.118}$$

$$\frac{dp_r}{dt} = \frac{p_u v^2}{r^2} - \frac{2 p_u}{r^3} - \frac{p_v u v}{r^2} + \frac{p_\psi v}{r^2} ; \tag{3.119}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r} ; \tag{3.120}$$

$$\frac{dp_v}{dt} = -\frac{2 p_u v}{r} + \frac{p_v u}{r} ; \tag{3.121}$$

when $t = 0$

$$r = 1; \quad u = 0; \quad v = 1; \quad \psi = 0;$$
$$p_\psi = 1 \text{ for } t_k^+; \quad p_\psi = -1 \text{ for } t_k^- ; \tag{3.122}$$

when $t = t_k$

$$r = R; \ \psi = \Phi; \ p_u = 0; \ p_v = 0. \tag{3.123}$$

In the case t_k^0 in (3.123) the condition $\Psi = \Phi$ must be substituted by the condition

$$p_\psi = 0. \tag{3.124}$$

Moreover, in this case the pulses are once again determined with an accuracy up to a constant factor. Considerations analogous to those outlined above show that we may assume that $p_r(0) = 1$.

In order to find a good first approximation for the solution of the boundary problem, we will construct its approximate solution by using the smallness of reactive acceleration a. We will consider that

$$r = 1 + a r^{(1)} + a^2 r^{(2)} + \dots ; \tag{3.125}$$
$$u = a u^{(1)} + a^2 u^{(2)} + \dots ; \tag{3.126}$$
$$v = 1 + a v^{(1)} + a^2 v^{(2)} + \dots . \tag{3.127}$$

By substituting (3.125)-(3.127) into (3.115)-(3.118) and deleting the terms of the power a^2, we find

$$\frac{dr^{(1)}}{dt} = u^{(1)} ; \tag{3.128}$$

$$\frac{du^{(1)}}{dt} = r^{(1)} + 2v^{(1)} + \cos\lambda; \tag{3.129}$$

$$\frac{dv^{(1)}}{dt} = -u^{(1)} + \sin\lambda; \tag{3.130}$$ /82

$$\frac{d\psi}{dt} = 1 - \omega + a(u^{(1)} - r^{(1)}). \tag{3.131}$$

The conjugate system acquires the following form:

$$\frac{dp_r^{(1)}}{dt} = p_\psi - p_u^{(1)}; \tag{3.132}$$

$$\frac{dp_u^{(1)}}{dt} = -p_r^{(1)} + p_v^{(1)}; \tag{3.133}$$

$$\frac{dp_v^{(1)}}{dt} = -p_\psi - 2p_u^{(1)}. \tag{3.134}$$

Here p_ψ is determined, as before, by (3.114).

The boundary conditions for equations (3.128)-(3.134) for the cases t_k^+ and t_k^- are of the following form: for t = 0

$$r^{(1)} = 0; \quad u^{(1)} = 0; \quad v^{(1)} = 0; \quad \psi = 0; \tag{3.135}$$

for $t = t_k$

$$r^{(1)} = \frac{R-1}{a}; \quad \psi = \Phi; \quad p_u^{(1)} = 0; \quad p_v^{(1)} = 0. \tag{3.136}$$

In the case t_k^0, when $p_\psi = 0$, we have the following boundary conditions: for t = 0

$$r^{(1)} = 0; \quad u^{(1)} = 0; \quad v^{(1)} = 0; \quad \psi = 0; \quad p_r^{(1)} = 1; \tag{3.137}$$

for $t = t_k$

$$r^{(1)} = \frac{R-1}{a}; \quad p_u^{(1)} = 0; \quad p_v^{(1)} = 0. \tag{3.138}$$

We will construct the solution of equations (3.128)-(3.134) for the case t_k^+. The solution of the conjugate system (3.123)-(3.134) has the form

$$p_r^{(1)} = 2\sin x - b\cos x - 3x + 2b; \tag{3.139}$$

$$p_u^{(1)} = -2(1-\cos x) + b\sin x; \tag{3.140}$$

$$p_v^{(1)} = 4\sin x + 2b(1-\cos x) - 3x, \tag{3.141}$$

where $x = t_k - t$; b is an unknown constant. The control $\lambda(t)$ is defined by the formula

$$\tan\lambda = \frac{p_v^{(1)}}{p_u^{(1)}} = \frac{4\sin x + 2b(1-\cos x) - 3x}{b\sin x - 2(1-\cos x)}. \tag{3.142}$$

By substituting λ into (3.128)-(3.131) and integrating the system obtained with consideration of the boundary conditions (3.135), (3.136), we have two equations with two unknowns t_k and b: /83

$$\frac{R-1}{a} = J_1(b, t_k) = \int_0^{t_k} \frac{bc_1 - c_2}{\sqrt{b^2c_1 - 2bc_2 + c_3}}\,dx; \tag{3.143}$$

$$\frac{\Phi - (1-w)t_k}{a} = J_2(b, t_k) = \int_0^{t_k} \frac{-bc_2 + c_3}{\sqrt{b^2c_1 - 2bc_2 + c_3}}\,dx, \tag{3.144}$$

where

$$c_1 = \sin^2 x + 4(1-\cos x)^2;$$
$$c_2 = 6(1-\cos x)(x - \sin x);$$
$$c_3 = 8(1-\cos x) + 3x(3x - 4\sin x).$$

In the case t_k^- in formulas (3.143), (3.144) it is necessary to reverse the sign in front of J_1 and J_2. Finally, in the case t_k^0 formulas (3.142)-(3.144) become

$$\tan\lambda = 2\cot\frac{x}{2}; \qquad \frac{R-1}{a} = \int_0^{t_k} \sqrt{c_1}\,dx;$$

$$\frac{\Phi - (1-\omega)t_k}{a} = -\int_0^{t_k} \frac{c_2}{\sqrt{c_1}}\,dx.$$

Although the integrals in (3.143), (3.144) are not calculated analytically the solution of these equations is considerably simpler than the solution of the original boundary problem.

If the orbits of the spacecraft are similar, i.e. R varies only slightly from unity, equation (3.144) can be approximately substituted by the following:

$$\frac{\Phi}{a} - \frac{3}{2}\frac{R-1}{a} t_k = J_2(b, t_k). \tag{3.145}$$

By comparing (3.143) with (3.145) we see

$$t_k = t_k\left(\frac{\Phi}{a}, \frac{R-1}{a}\right). \tag{3.146}$$

The solution of the accurate boundary problem (3.115)-(3.123) by the Newton method requires three values for the initial conditions: p_{r0}, p_{u0}, p_{v0}. For the cases t_k^+ and t_k^- the first approximation of these initial conditions is taken from the solution of (3.143), (3.144) relative to b and t_k. As follows from (3.139)-(3.141), for the case t_k^+ the following is applicable: /84

$$p_{r0} = 2\sin t_k - b\cos t_k - 3t_k + 2b;$$
$$p_{u0} = -2(1-\cos t_k) + b\sin t_k;$$
$$p_{v0} = 4\sin t_k + 2b(1-\cos t_k) - 3t_k.$$

In the case t_k^- it is necessary to change the signs of these formulas for p_{r0}, p_{u0}, p_{v0} to the opposite sign. In the case t_k^0 the first approximation is found from the formulas

$$p_{u0} = \frac{\sin t_k}{2-\cos t_k}; \qquad p_{v0} = \frac{2(1-\cos t_k)}{2-\cos t_k}.$$

Since the time of transition t_k is unknown, one of the conditions on the right-hand end should be used for determination of integration in the solution of the Couchys problem for the system of differential equations (3.115)-(3.121). In the given case, however, since we have concentrated on obtaining a set of solutions for a wide range of angles Φ and radii R, it is more convenient in the solution of the boundary problem to know the time t_k and to find the angles

Φ (or radii R in the case t_k^0).

The results of the calculations are represented in Figures 27-33.

Figures 27 and 28 represent t_k as a function of R and Φ. Figure 27 corresponds to the case where $a = 10^{-4}$; it was found by calculating the approximate solution. Figure 28 corresponds to the case $a = 10^{-3}$; the approximate solution is represented by the broken line. It differs only from the accurate solution when the values of t_k in the vicinity of the curve for t_k^0 are great.

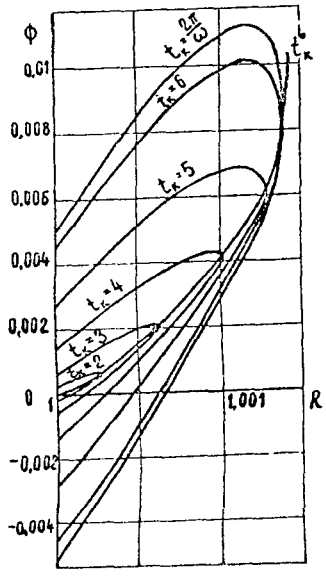

Figure 27

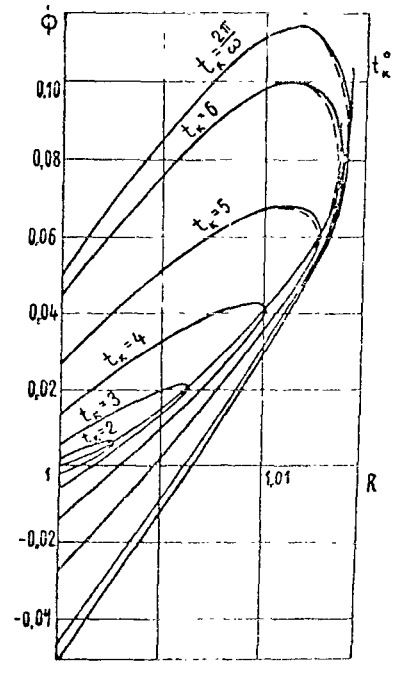

Figure 28

The range of t_k^+ in Figures 27 and 28 is located above the curve for t_k^0; below this curve lies the range of t_k^-.

Figures 27 and 28 differ but slightly from each other, actually only in scale (for $a = 10^{-3}$ it is 10 times greater than for $a = 10^{-4}$). This fact is the obvious result of formula (3.146).

Figure 29 represents time t_k as a function of angle Φ when $a = 10^{-3}$.

Examples of optimal trajectories r(t) and control $\lambda(t)$ are illustrated in Figure 30 ($a = 10^{-2}$; $R = 1.07$; $t_k = 5$, for the case t_k^+) and in Figure 31

($a = 10^{-2}$; $R = 1.05$; $t_k = 5$; for the case t_k^-. For the case t_k^+ the value of $r(t)$ diminishes at first, but then increases sharply to $r = R$. For the case /87
t_k^- the value of $r(t)$ first increases, intersects R, and then diminishes.

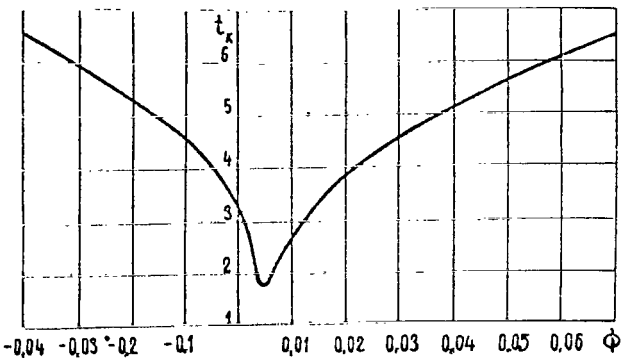

Figure 29

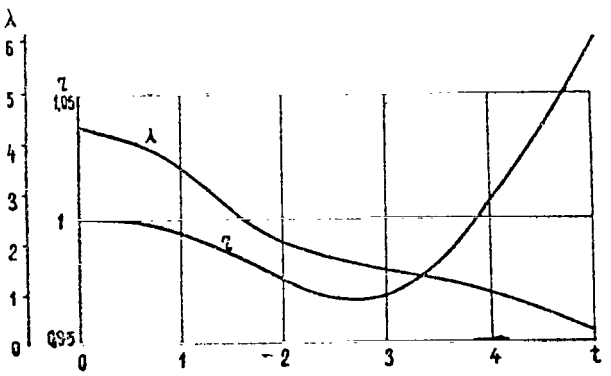

Figure 30

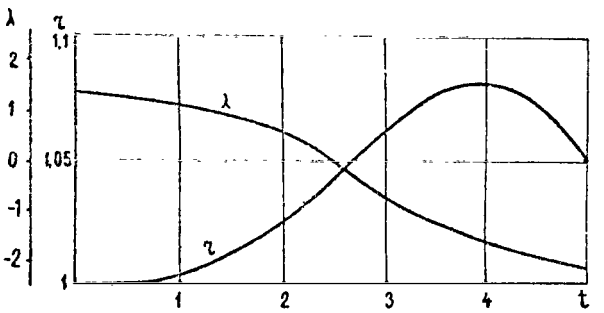

Figure 31

If reactive acceleration is sufficiently great, it might turn out then that transition will be possible for one and the same angle Φ, both in the trajectory of the type t_k^+ and in the trajectory of the type t_k^-, where $t_k^+ = t_k^-$ and $\Phi_k^+ = \Phi_k^-$. Figure 32 shows the graph of the function $t_k = t_k(\Phi)$ for the case $a = 10^{-1}$ and $R = 1$. There is a point on this figure, $\Phi = 3.632$, which corresponds to two optimal trajectories, transition by which is possible in the identical time $t_k = 6.075$. Figure 33 shows both of these trajectories. The letters A and B denote the initial positions of spacecraft 1 and 2, and C denotes the point of their rendezvous.

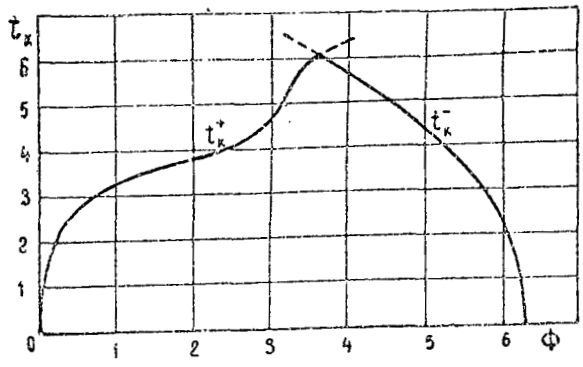

Figure 32

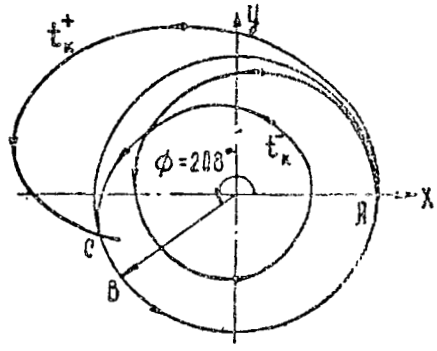

Figure 33

§8. Transition Between Circular Non-coplanar Orbits.

The motion of a spacecraft under constant reactive acceleration is described by the following equations in the spherical coordinate system:

$$\frac{dr}{dt} = u; \tag{3.147}$$

$$\frac{d\varphi}{dt} = \frac{v}{r\cos\theta}; \tag{3.148}$$

$$\frac{d\theta}{dt} = \frac{w}{r}; \tag{3.149}$$

$$\frac{du}{dt} = \frac{v^2+w^2}{r} - \frac{1}{r^2} + a\sin\varkappa\cos\lambda; \tag{3.150}$$ /88

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{vw\,\mathrm{tg}\,\theta}{r} + a\sin\varkappa\sin\lambda; \tag{3.151}$$

$$\frac{dw}{dt} = -\frac{uw}{r} - \frac{v^2\,\mathrm{tg}\,\theta}{r} + a\cos\varkappa. \tag{3.152}$$

here r, ϕ, θ are the spherical coordinates of the spacecraft (Figure 34); u, v, w are the projections of the velocity vector on the directions of the unit vectors $\bar{r}^0$, $\bar{\phi}^0$, and $\bar{\theta}^0$; t is time; a is reactive acceleration; κ and λ are controlling angles (Figure 35).

All values in equations (3.147)-(3.152) are dimensionless. Conversion to dimensional variables is accomplished by formulas

$$r^* = rr_0^*;\quad u^* = u\sqrt{\frac{\mu}{r_0^*}};\quad v^* = v\sqrt{\frac{\mu}{r_0^*}};$$

$$w^* = w\sqrt{\frac{\mu}{r_0^*}}; \quad t^* = t r_0^* \sqrt{\frac{r_0^*}{\mu}}; \quad a^* = a\frac{\mu}{r_0^{*2}},$$

where r_0^* is the radius of the initial circular orbit; μ is the gravitational parameter of the planet (dimensional values are denoted by the asterisk).

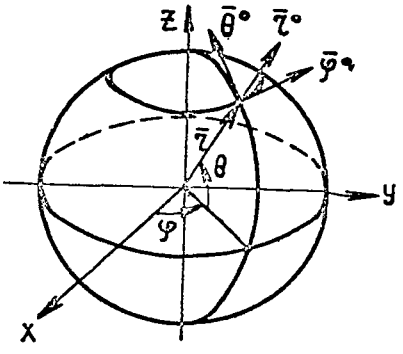

Figure 34

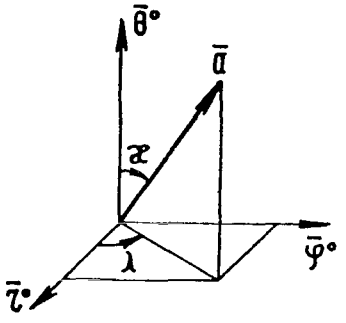

Figure 35

We will assume that the circular orbit into which the spacecraft is to be inserted lies in the plane xOy (see Figure 34)(the radius of this orbit is r_k), /89 the angle between the planes of the initial and final orbits is i_0, the line of intersection of the planes of these orbits (nodal lines) coincides with the Ox axis, and when t = 0 the spacecraft is located on the nodal line.

We are required to find the equation $\kappa = \kappa(t)$, $\lambda = \lambda(t)$ which insures transition from a given point of the circular orbit of unit radius and of inclination i_0 to a circular orbit of radius r_k in the minimum time.

We will formulate the function of H:

$$H = p_r u + \frac{p_\varphi v}{r\cos\theta} + \frac{p_\theta w}{r} + \frac{p_u(v^2+w^2)}{r} - \frac{p_{\cdot u}}{r^2} + a p_u \sin\varkappa\cos\lambda -$$
$$- \frac{p_v u v}{r} + \frac{p_v v w \operatorname{tg}\theta}{r} + a p_v \sin\varkappa\sin\lambda -$$
$$- \frac{p_w u w}{r} - \frac{p_w v^2 \operatorname{tg}\theta}{r} + a p_w \cos\varkappa,$$

where p_r, p_ϕ, p_θ, p_u, p_v and p_w are pulses which satisfy the conjugate equation system:

$$\frac{dp_r}{dt} = \frac{p_\varphi v}{r^2\cos\theta} + \frac{p_\theta w}{r^2} + \frac{p_u(v^2+w^2)}{r^2} - \frac{2p_u}{r^3} - \frac{p_v u v}{r^2} + \frac{p_v v w \operatorname{tg}\theta}{r^2} - \frac{p_w u w}{r^2} - \frac{p_w v^2 \operatorname{tg}\theta}{r^2}; \tag{3.153}$$

$$\frac{dp_\varphi}{dt} = 0; \tag{3.154}$$

$$\frac{dp_\theta}{dt} = -\frac{p_\varphi v \sin\theta}{r\cos^2\theta} - \frac{p_v v w}{r\cos^2\theta} + \frac{p_w v^2}{r\cos^2\theta}; \tag{3.155}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r} + \frac{p_w w}{r}; \tag{3.156}$$

$$\frac{dp_v}{dt} = -\frac{p_\varphi}{r\cos\theta} - \frac{2p_u v}{r} + \frac{p_v u}{r} - \frac{p_v w \operatorname{tg}\theta}{r} + \frac{2p_w v \operatorname{tg}\theta}{r}; \tag{3.157}$$

$$\frac{dp_w}{dt} = -\frac{p_\theta}{r} - \frac{2p_u w}{r} - \frac{p_v v \operatorname{tg}\theta}{r} + \frac{p_w u}{r}. \tag{3.158}$$

We will now find the equation which yields the maximum for the function of H:

$$\sin\varkappa = \frac{\sqrt{p_u^2+p_v^2}}{\sqrt{p_u^2+p_v^2+p_w^2}}; \qquad \cos\varkappa = \frac{p_w}{\sqrt{p_u^2+p_v^2+p_w^2}}; \tag{3.159}$$ /90

$$\sin\lambda = \frac{p_v}{\sqrt{p_u^2+p_v^2}}; \qquad \cos\lambda = \frac{p_u}{\sqrt{p_u^2+p_v^2}}. \tag{3.160}$$

It follows from equation (3.154) that p_ϕ = const. When $t = t_k$, the value of $p_\phi = 0$, since ϕ_k is not known, and therefore $p_\phi \equiv 0$.

By substituting expressions (3.159), (3.160) for controlling angles into equation system (3.147)-(3.152) and combining (3.153)-(3.158) with it, we obtain

$$\frac{dr}{dt} = u; \tag{3.161}$$

$$\frac{d\varphi}{dt} = \frac{v}{r\cos\theta}; \tag{3.162}$$

$$\frac{d\theta}{dt} = \frac{w}{r}; \tag{3.163}$$

$$\frac{du}{dt} = \frac{v^2+w^2}{r} - \frac{1}{r^2} + \frac{a p_u}{\sqrt{p_u^2+p_v^2+p_w^2}}; \tag{3.164}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{vw\,\mathrm{tg}\,\theta}{r} + \frac{a p_v}{\sqrt{p_u^2+p_v^2+p_w^2}}; \tag{3.165}$$

$$\frac{dw}{dt} = -\frac{uw}{r} - \frac{v^2\,\mathrm{tg}\,\theta}{r} + \frac{a p_w}{\sqrt{p_u^2+p_v^2+p_w^2}}; \tag{3.166}$$

$$\frac{dp_r}{dt} = \frac{p_\theta w}{r^2} + \frac{p_u(v^2+w^2)}{r^2} - \frac{2p_u}{r^3} - \frac{p_v uv}{r^2} + \frac{p_v vw\,\mathrm{tg}\,\theta}{r^2} - \frac{p_w uw}{r^2} - \frac{p_w v^2\,\mathrm{tg}\,\theta}{r^2}; \tag{3.167}$$

$$\frac{dp_\theta}{dt} = -\frac{p_v vw}{r\cos^2\theta} + \frac{p_w v^2}{r\cos^2\theta}; \tag{3.168}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r} + \frac{p_w w}{r}; \tag{3.169}$$ /91

$$\frac{dp_v}{dt} = -\frac{2p_u v}{r} + \frac{p_v u}{r} - \frac{p_v w\,\mathrm{tg}\,\theta}{r} + \frac{2p_w v\,\mathrm{tg}\,\theta}{r}; \tag{3.170}$$

$$\frac{dp_w}{dt} = -\frac{p_\theta}{r} - \frac{2p_u w}{r} - \frac{p_v v\,\mathrm{tg}\,\theta}{r} + \frac{p_w u}{r}. \tag{3.171}$$

Equation system (3.161)-(3.171) is uniform with respect to the pulses. Therefore, we may assume that $p_u^2 + p_v^2 = 1$ when $t = 0$.

Thus we must find the solution for equation system (3.161)-(3.171) for the conditions: when $t = 0$

$$\left.\begin{array}{c} r = 1;\quad \varphi = 0;\quad \theta = 0;\quad u = 0; \\ v = \cos i_0;\quad w = \sin i_0;\quad p_u^2 + p_v^2 = 1; \end{array}\right\} \tag{3.172}$$

when $t = t_k$

$$r = r_k; \quad \theta = 0; \quad u = 0; \quad v = 1/\sqrt{r_k}; \quad w = 0. \tag{3.173}$$

The moment t_k is not known before hand. It is impossible to terminate the integration of equation system (3.161)-(3.171) at the moment when one of conditions (3.173) is satisfied, since it is not known before hand whether or not any of the coordinates r, θ, u, v, and w change monotonically. In this problem, therefore, the following procedure is used. To equation system (3.161)-(3.171), we add

$$\frac{dt_k}{dt} = 0. \tag{3.174}$$

The solution of boundary problem (3.161)-(3.174) requires that we select, using the Newton method, such values of p_{r0}, $p_{\theta 0}$, p_{u0}, p_{w0} and t_k, that conditions (3.173) be satisfied. Integration is terminated when $t = t_k$.

The problem is solved for a = 0.005. In choosing the first approximation, the results of the solution of the variation problem for the averaged equations of transition between circular non-coplanar orbits (see Chapter 2) are used.

For this purpose, the selection of the values of p_{r0}, $p_{\theta 0}$, p_{u0}, p_{w0} is reduced to selection for t = 0 of angles α and β and of the derivatives of these angles with respect to time, where α is the angle between perpendicular /92 $\bar{h}^0$, to the instantaneous orbital plane P and acceleration vector $\bar{a}$ (Figure 36); β is the angle between the vector radius of the spacecraft $\bar{r}$ and the projection of $\bar{a}$ on plane P.

Then, in a manner similar to that in which relations (3.30)-(3.31) were derived in §2 of this chapter, we find formulas for calculating p_r, p_θ, p_u, p_v and p_w for t = 0:

$$p_r = \frac{\frac{d\lambda}{dt} + \cos i_0 + \cos i_0 \cos^2\lambda + \sin i_0 \cot\varkappa \sin\lambda}{\sin\lambda};$$

$$p_0 = (1+\cot\varkappa)\frac{d\varkappa}{dt} + p_r \cos\lambda \cot\varkappa + \cos i_0 \cos\lambda \sin\lambda \cot\varkappa - \\ - 2\sin i_0 \cos\lambda - \sin i_0 \cos\lambda \cot\varkappa;$$

$$p_u = \cos\lambda;$$

$$p_v = \sin\lambda;$$

$$p_w = \cot\varkappa,$$

where

$$\sin\lambda = \frac{\sin\alpha\sin\beta\cos i_0 - \sin i_0\cos\alpha}{\sqrt{1-(\cos i_0\cos\alpha+\sin i_0\sin\alpha\sin\beta)^2}};$$

$$\cos\lambda = \frac{\sin\lambda\cos\beta}{\sqrt{1-(\cos i_0\cos\alpha+\sin i_0\sin\alpha\sin\beta)^2}};$$

$$\cot\varkappa = \frac{\sin\alpha\sin\beta\sin i_0 + \cos i_0\cos\alpha}{\sqrt{1-(\cos i_0\cos\alpha+\sin i_0\sin\alpha\sin\beta)^2}};$$

$$\frac{d\varkappa}{dt} = \frac{1}{\sin\varkappa}\left[-\frac{d\alpha}{dt}\sin i_0\cos\alpha\sin\beta + \frac{d\alpha}{dt}\sin\alpha\cos i_0 - \right.$$

$$-\frac{d\beta}{dt}\sin\alpha\cos\beta\sin i_0 + \frac{a}{\sqrt{1+p_w^2}}(-p_v\sin i_0 +$$

$$\left. + p_w\cos i_0)(\cos\alpha\sin i_0 - \sin\alpha\sin\beta\cos i_0)\right];$$

$$\frac{d\lambda}{dt} = \frac{\frac{d\varkappa}{dt}\cos\varkappa\cos\lambda - \frac{d\alpha}{dt}\cos\alpha\cos\beta + \frac{d\beta}{dt}\sin\beta\sin\alpha}{\sin\lambda\sin\varkappa}.$$

The values of α, β, $d\alpha/dt$, $d\beta/dt$, and t_k are assumed to be equal when /93
$t = 0$ (see Chapter 2 § 4-7):

$$\alpha = \frac{\pi}{2} + \mathrm{Arctan}\left[\frac{\sin\frac{\pi i_0}{2}}{\sqrt{r_k}}\left(1 - \frac{\cos\frac{\pi i_0}{2}}{\sqrt{r_k}}\right)^{-1}\right];$$

$$\beta = \frac{\pi}{2}; \qquad \frac{d\alpha}{dt} = 0; \qquad \frac{d\beta}{dt} = 0;$$

$$t_k = \sqrt{1 - \frac{2\cos\frac{\pi i_0}{2}}{\sqrt{r_k}} + \frac{1}{r_k}}\, a^{-1}.$$

The results of the calculations are listed in Table 3. In the three latter cases the value of r varied non-monotonically. At first the radius increases to a value exceeding the radius of the final orbit, but then it diminishes to r_k.

Figure 37 shows the dependence of angles α and β on time when $r_k = 1.13$, /94
$i_0 = 0.14$ rad. The radial component of thrust, as we have assumed, is small, and

this is verified by the nearness of angle β to π/2 through the entire trajectory. The normal components of thrust changes its sign twice in each orbit. The same character of control was observed in all variances. This is evidence of the nearness of the control found in §7 of Chapter 2 to the optimal control.

Table 3*

r_k	i_0	t_k
1,100000	0,100000	31,180
1,117000	0,122340	38,000
1,130000	0,140000	44,000
1,148000	0,178000	53,250
1,159500	0,200070	60,960
1,167250	0,225700	68,480
1,172500	0,254400	76,760
1,174005	0,279455	84,100
1,173250	0,297000	90,050
1,170500	0,318000	95,420
1,167500	0,332000	99,476

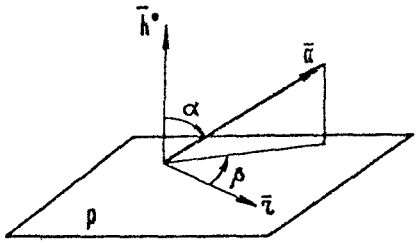

Figure 36

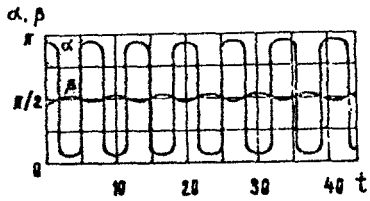

Figure 37

§9. Transition Between Heliocentric Circular Orbits Using a Solar Sail [24].

The variation problem of the minimum time of transition from an earth orbit to orbits of other planets is examined in this section. All orbits are assumed to be circular.

The motion of a spacecraft equipped with a flat, ideally reflecting solar sail is described by the following equation system:

---

*Tr. Note: Commas indicate decimal points.

$$\frac{dr}{dt} = u; \tag{3.175}$$

$$\frac{d\varphi}{dt} = \frac{v}{r}; \tag{3.176}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{a\cos^3\theta}{r^2}; \tag{3.177}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{a\sin\theta\cos^2\theta}{r^2}. \tag{3.178}$$

Here r and ϕ are polar coordinates; u and v are radio and transversal velocity components; t is time; θ is the angle at which the sail is mounted (Figure 39). All variables in equations (3.175)-(3.178) are dimensionless. The dimensionless value of a is calculated by the formula

$$a = a^* r_0^{*2} / \gamma M,$$

where r_0^* is the radius of the earth orbit; γ is the gravitational constant; M /95
is the mass of the sun; a* is the ratio of the force of pressure of the light from the sun to the mass of the spacecraft in earth orbit when $\theta = 0$ (the asterisk denotes dimensional values of the variables).

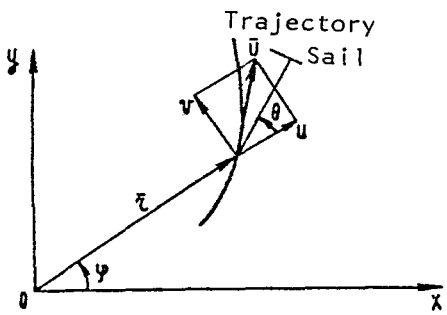

Figure 38

The dimensional values of the remaining variables can be found by using the following relations:

$$r^* = r r_0^*; \quad v^* = v\sqrt{\frac{\gamma M}{r_0^*}};$$

$$u^* = u\sqrt{\frac{\gamma M}{r_0^*}}; \quad t^* = t r_0^* \sqrt{\frac{r_0^*}{\gamma M}}.$$

The conjugate equation system for the pulses has the form

$$\frac{dp_r}{dt} = \frac{p_\varphi v}{r^2} + \frac{p_u v^2}{r^2} - \frac{2p_u}{r^3} + \frac{2ap_u\cos^3\theta}{r^3} - \frac{p_v uv}{r^2} + \frac{2ap_v\sin\theta\cos^2\theta}{r^3}; \tag{3.179}$$

$$\frac{dp_\varphi}{dt} = 0; \tag{3.180}$$

$$\frac{dp_u}{dt} = -p_r + \frac{p_v v}{r}; \tag{3.181}$$

$$\frac{dp_v}{dt} = -\frac{p_\varphi}{r} - \frac{2p_u v}{r} + \frac{p_v u}{r}. \tag{3.182}$$

We formulate the function /96

$$H = p_r u + \frac{p_\varphi v}{r} + \frac{p_u v^2}{r} - \frac{p_u}{r^2} + \frac{a p_u \cos^3\theta}{r^2} - \frac{p_v u v}{r} + \frac{a p_v \sin\theta \cos^2\theta}{r^2}.$$

From the condition of the maximum of the function of H we find

$$\tan\theta = \frac{\sqrt{9p_u^2 + 8p_v^2} - 3p_u}{4p_v}. \tag{3.183}$$

From this relation angle θ is determined uniquely, since the force of pressure from the light can be directed only from the sun.

The final value of angle ϕ is not known. Therefore, $p_\phi = 0$ when $t = t_k$. From (3.180) it follows that p_ϕ is a constant; therefore, $p_\phi \equiv 0$.

By substituting the value of θ from (3.183) into equation (3.177)-(3.179), we obtain a system of seven differential equations.

The right-hand parts of the equations are uniform functions relative to the pulses. Taking this into consideration, we assume that at the initial moment of time $p_r = 1$ when $r_k > 1$ and $p_r = -1$ when $r_k < 1$.

Thus, we are required to find a solution of seven equations (3.175)-(3.179); (3.181), (3.182) for the boundary conditions: when $t = 0$

$$r = 1; \quad \varphi = 0; \quad u = 0; \quad v = 1;$$

when $t = t_k$

$$r = r_k; \quad u = 0; \quad v = \frac{1}{\sqrt{r_k}}; \quad p_r = \pm 1.$$

The trajectories for transition into orbits of all planets of our solar system were determined for $a^* = 2\text{mm/sec}^2$. Figures 39 and 40 represent the values of θ, r, ϕ, u, v as functions of time for transition to Mars. The relative time $\bar{t} = t^*/t_k^*$, where t_k^* is the time of transition, equal to 322 days, is arranged along the abscissa. Figure 41 and 42 represent the same functions for a flight to Venus, the duration of which is 164 days.

The time of transition in years, to orbits of other planets is 0.53 for Mercury, 6.6 for Jupiter, 17 for Saturn, 49 for Uranus, 96 for Neptune, and 145 for Pluto.

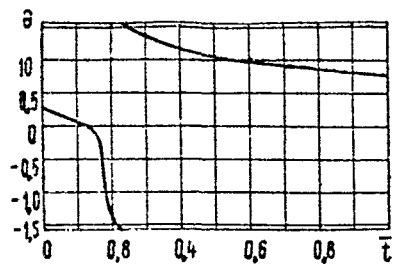

Figure 39

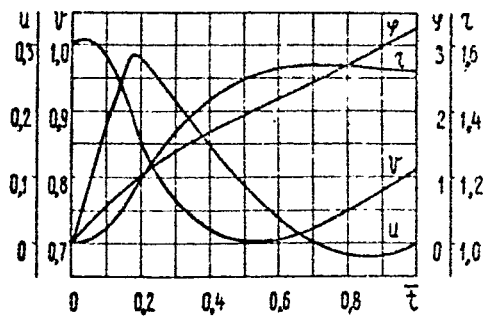

Figure 40 /97

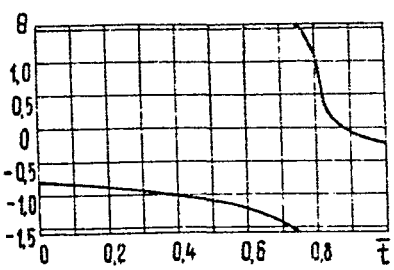

Figure 41

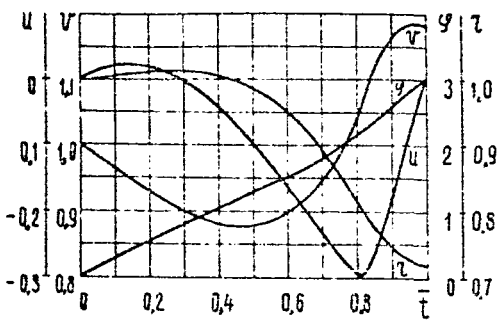

Figure 42

The dependence of flight time on acceleration was investigated for transition to a Mars orbit:

Acceleration in mm/sec^2	Time in Days
1	405
2	322
3	286
4	264
5	248

CHAPTER 4

PROBLEM OF MAXIMUM USEFUL LOAD

V. N. Lebedev

ABSTRACT. In this chapter we will consider the problem of maximum useful load for certain maneuvers in near-plantary space. As in the preceding chapter, we will assume here that the spacecraft is equipped with a non-throttleable propulsion system. It is also assumed that the engine operates through out the entire maneuver time.

In the preceding chapters we have investigated trajectories with simple /99
controls and with optimal controls from the standpoint of speed or minimum engine burn time. These studies simplify the solution of the basic problem of optimalization in the dynamics of spacecraft [25], which problem is to find such control of thrust and to select such values for the parameters of the propulsion system as will make it possible to execute a given maneuver within a specific period of time, insuring maximal useful loads.

In this chapter we will consider the problem of maximum useful load for certain maneuvers in near-plantary space. As in the preceding chapter, we assume here that the spacecraft is equipped with a non-throttleable propulsion system. It is also assumed that the engine operates through out the entire maneuver time.

§1. Formulation of Basic Optimization Problem

We will assume that the total starting weight of a spacecraft G_0^* consists of the following components: the weight of the source of power G_N^*, the weight of the fuel G_m^*, and the weight of the useful load G_n^*:

$$G_0^* = G_N^* + G_m^* + G_n^* \tag{4.1}$$

(as in the preceding two chapters, the asterisk denotes dimensional values).

We will further assume that the entire power N* delivered to the engine from the source is converted into the power of the reactive jet:

$$N^* = \frac{q^* V^{*2}}{2}, \tag{4.2}$$

where q* is the mass flow rate of the fuel per unit time; V* is the velocity of the reactive jet (both of the values are constants in the case of a non-throttleable propulsion system). /100

If, as is usually done, we assume that the weight of the power source is proportional to power, i.e.

$$G_N^* = \alpha^* N^*,$$

where the constant α* is the specific weight of the power source, then, with consideration of the expression (4.2), we find

$$G_N^* = \frac{\alpha^* q^* V^{*2}}{2}.$$

The weight of the fuel expended between the time t = 0 and t = t_k is

$$G_m^* = q^* g^* t_k^*,$$

where g* is the acceleration of the force of gravity.

We know that

$$q^* = \frac{P^*}{V^*},$$

where P* is thrust. Therefore,

$$G_N^* = \frac{\alpha^* P^* V^*}{2}; \qquad G_m^* = \frac{P^* g^* t_k^*}{V^*}.$$

By substituting G_N^* and G_m^* into (4.1), we obtain

$$G_0^* = P^*\left(\frac{g^* t_k^*}{V^*} + \frac{\alpha^* V^*}{2}\right) + G_n^* .$$

Then, by dividing both halves of this equation by G_0^* we find

$$G_n = 1 - a_0^*\left(\frac{t_k^*}{V^*} + \frac{\alpha^* V^*}{2g^*}\right) . \tag{4.3}$$

Here $G_n = G_n^*/G_0^*$ is the relative useful load; $a_0^* = P^*g^*/G^*_0$ is the initial reactive acceleration.

If we assume that for some maneuver executed with a given control we know the dependence of the time required for executing the maneuver t^*_k on the parameters of the propulsion system a^*_0 and V^*:

$$t_k^* = t_k^*(a_0^*, V^*), \tag{4.4}$$ /101

then the basic problem of optimization of this maneuver is formulated in the following manner [26]: we find the conditional maximum of the function of (4.3) of two variables a^*_0 and V^* for a given t_k^* in the presence of relation (4.4) (the coefficient α^* is considered to be known).

§2. Transition to Synchronous Orbit.

Let us consider the following problem. We are required to deliver a maximum relative useful weight into an equatorial synchronous earth orbit (r^*_k= = 42,240 km) in 40 days starting from a circular orbit of radius r^*_0 = 6,671 km and inclination i_k = 48°.

The analytical solution of the variation problem for speedy transition between circular non-coplanar orbits was derived in §7 of Chapter 2 by the method of averages. The optimal trajectory for the case of transition to synchronous orbit was examined in §8 of the same Chapter. For the determination of the characteristic velocity of transition between circular non-coplanar orbits, the following formula (2.115) was derived in §7 of Chapter 2:

$$T_k = \sqrt{1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}}.$$

By substituting this value into relation (2.30)

$$t_k = \frac{V}{a_0}\left[1 - \exp\left(-\frac{T_k}{V}\right)\right], \tag{4.5}$$

we obtain the dependence of the maneuver time on a_0 and V:

$$t_k = \frac{V}{a_0}\left[1 - \exp\left(-\sqrt{1 - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_k}} + \frac{1}{r_k}}\Big/ V\right)\right]. \tag{4.6}$$

All values in this relation are dimensionless. The dimensional values can be found by using the relations

$$r^* = r r_0^*; \quad V^* = V\sqrt{\frac{\mu}{r_0^*}}; \quad t^* = t r_0^* \sqrt{\frac{r_0^*}{\mu}}; \quad a^* = a\frac{\mu}{r_0^{*2}}. \tag{4.7}$$ /102

In dimensional variables, relation (4.6) acquires the following form:

$$t_k^* = \frac{V^*}{a_0^*}\left[1 - \exp\left(-\sqrt{\mu\left(\frac{1}{r_0^*} - \frac{2\cos\frac{\pi i_k}{2}}{\sqrt{r_0^* r_k^*}} + \frac{1}{r_k^*}\right)}\Big/ V^*\right)\right]. \tag{4.8}$$

Thus, we are required to find the maximum of the function (4.3)

$$G_n = 1 - a_0^*\left(\frac{t_k^*}{V^*} + \frac{\alpha^* V^*}{2g^*}\right)$$

for the condition that a_0^* and V^* are tied together by relation (4.8).

In the problem under consideration it is assumed that $\alpha^* = 10$ kg/kw, $\mu = 0.3986 \cdot 10^6$ km^3/sec^2.

The maximum relative useful weight $G_n = 0.51$ is achieved when $a^*_0 = 1.86$ mm/sec^2 and $V^* = 23$ km/sec. In this case the relative weight of the power source $G_N = 0.21$ and the relative weight of the fuel $G_m = 0.28$.

§3. Revolution of Plane of Orbit.

In the case where $r^*_k = r^*_0$ formula (4.8) acquires the form

$$t_k^* = \frac{V^*}{a_0^*}\left[1 - \exp\left(-\frac{2\sqrt{\frac{\mu}{r_0^*}}\sin^2\frac{\pi i_k}{4}}{V^*}\right)\right]. \tag{4.9}$$

We will examine the problem of the maximum relative useful load during revolution of a near-earth circular orbit of radius $r^*_0 = 6{,}671$ km by the angle $i_k = 90°$ in 100 days. This problem reduces to the search for the maximum

$$G_n = 1 - a_0^*\left(\frac{t_k^*}{V^*} + \frac{\alpha^* V^*}{2g^*}\right) \tag{4.10}$$

in the presence of relation (4.9). As in the preceding problem, we assume /103
that $\alpha^* = 10$ kg/kw, $\mu = 0.3986 \cdot 10^6$ km^3/sec^2. The solution of this problem yields the following results: $a^*_0 = 1.37$ mm/sec^2, $V^* = 34$ km/sec; $G_N = 0.23$, $G_m = 0.35$, and $G_n = 0.42$.

It should be noted here that during revolution of the plane of the orbit, the radius does not remain constant (see §7 of Chapter 2). At first the radius increases, reaching a value greater than nine times that of the initial value, but then it diminishes.

We know that the revolution of the plane of an orbit can be accomplished with thrust normal to the instantaneous plane of the orbit (the radius of the orbit in this case remains constant). The characteristic velocity of the maneuver in this case is given by formula (2.108):

$$T_k = \pi i_k / 2.$$

By substituting this relation into (4.5) and converting to dimensional values by formulas (4.7), we find the dependence of the time of revolution of the plane of the orbit on a_*^0 and V^*:

$$t_k^* = \frac{V^*}{a_0^*}\left[1 - \exp\left(-\frac{\pi i_k}{2V^*}\sqrt{\frac{\mu}{r_0^*}}\right)\right]. \tag{4.11}$$

The maximum useful load (4.10) under the condition (4.11) for the case $r^*_0 = 6{,}671$ km, $i_k = 90°$ is achieved when $a^*_0 = 1.65$ mm/sec^2, $V^* = 31$ km/sec. Here $G_N = 0.26$, $G_m = 0.46$, and $G_n = 0.28$.

Thus, the control discussed in §7 of Chapter 2 makes it possible to increase the useful load by a factor of greater than one and one-half during revolution of the plane of a near-earth circular orbit by 90° in comparison with the maneuver using transversal thrust.

§4. Insertion of Maximal Useful Load Into Parabolic Orbit.

In §3 of Chapter 2 and in §2 of Chapter 3, the approximation formulas for the determination of acceleration time of the spacecraft to parabolic velocity were derived:

$$\left(1 - V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right)^8 - c_1\frac{a_0^2}{\left(1-\frac{a_0}{V}t_k\right)^2} + c_2\frac{a_0^2}{V\left(1-\frac{a_0}{V}t_k\right)^2}\left(1 - V\ln\frac{1}{1-\frac{a_0}{V}t_k}\right) = 0, \tag{4.12}$$ /104

where $c_1 = 0.1062$, $c_2 = 0.12$ for acceleration with transversal thrust; $c_1 = 0.1820$, $c_2 = 0.22$ for tangential thrust, and $c_1 = 0.2063$, $c_2 = 0.25$ for thrust which is optimal in the sense of speed. This relation makes it possible to solve the problem of insertion of maximum useful load into a parabolic orbit.

Using formulas (4.7) along with (4.12) we obtain

$$\left(1-V^*\sqrt{\frac{r_0^*}{\mu}}\ln\frac{1}{1-\frac{a_0^*}{V^*}t_k^*}\right)^8 - c_1 a_0^{*2}\frac{r_0^{*4}}{\mu^2}\cdot\frac{1}{\left(1-\frac{a_0^*}{V^*}t_k^*\right)^2} + c_2\frac{a_0^{*2}}{V^*}\frac{r_0^{*3}}{\mu}\sqrt{\frac{r_0^*}{\mu}}\times$$

$$\times\frac{1}{\left(1-\frac{a_0^*}{V^*}t_k^*\right)^2}\left(1-V^*\sqrt{\frac{r_0^*}{\mu}}\ln\frac{1}{1-\frac{a_0^*}{V^*}t_k^*}\right)=0. \qquad (4.13)$$

The problem reduces to a search for a conditional maximum of (4.10) when relation (4.13) is applied to a^*_0 and V^*.

We will assume that $r^*_0 = 6.671$ km, $t^*_k = 40$ days, $\alpha^* = 10$ kg/kw, and $\mu = 0.3986\cdot10^6$ km^3/sec^2.

The results of the solution are listed in Table 4.

Table 4

Control	a_0^*, mm/sec^2	V^*, km/sec	G_N	G_m	G_n
Transversal	1,739	22,59	0,1964	0,2661	0,5375
Tangential	1,728	22,56	0,1950	0,2648	0,5402
Optimal	1,726	22,56	0,1947	0,2644	0,5409

During acceleration with tangential and transversal thrust, the value of useful load is only a fraction of a per-cent less than in the case of the optimal maneuver. This is further confirmation of the advisability of using transversal and tangential thrust for accelerating a spacecraft to parabolic velocity.

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