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A. Sh. Dorfman
and
V. K. Vishnevskiy

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APPLICATIONS OF SELF-SIMILAR SOLUTIONS FOR BOUNDARY LAYER LINEARIZATION EQUATIONS

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Abstract. In earlier studies one of the authors proposed a method for computing a laminar boundary layer based on the joint solution of an equation of motion, linearized using some one-parameter family of profiles, and an integral relation. This new paper gives a solution of the problem of a laminar boundary layer with an arbitrary velocity distribution at the layer boundary.

In [1-3] one of the authors proposed a method for computing the laminar boundary layer, based on joint solution of the equation of motion, linearized using some one-parameter family profiles, and the integral relation. This approach, constituting eventually a combination of the integral computation method and the linearization method, makes it possible to apply the advantages of each of these methods. In particular, due to the simultaneous satisfaction of the linearized equation and the integral relation it is possible to obtain a high computation accuracy. /90*

In addition, the use of a one-parameter set of profiles makes it possible to dispense with a two-layer model of the layer usually employed during linearization [4] and [10]. This considerably simplifies computation, so that they are virtually as simple as computations based on integral methods. However, in contrast to the latter, the proposed method makes it possible to determine not only the fundamental characteristics of the layer, but also the distribution of parameters across the layer. In this sense it is considerably closer than other approximate methods to the method of solving precise equations in series [6, 8]. In addition, linearization of equations considerably simplifies the problem and makes it possible to use well-developed mathematical approaches.

*Numbers in the margin indiate pagination in the foreign text.

This paper gives a solution of the problem for a laminar boundary layer with an arbitrary distribution of the velocities at the layer boundary.

§1. Linearization of the Equation

We will write the Prandtl-Mises equation,

$$\frac{\partial Z}{\partial x} = u\frac{\partial^2 Z}{\partial \psi^2};\ Z = U^2 - u^2;\ \psi = \int_0^y u\,dy;$$
$$\psi = 0;\ Z = U^2(x);\ \psi = \infty\ Z = 0. \tag{1.1}$$

Here all the parameters are dimensionless and related: velocities U and u in the boundary layer and outside the layer are related to the velocity U_∞ distant from the body; the coordinates x and y are related to the characteristic dimension L and the parameter $\sqrt{\nu L/U_\infty}$; the stream function ψ is related to $k\sqrt{U_\infty \nu/L}$ (ν is the kinematic viscosity coefficient).

Introducing the new variables

$$\Phi = \int_0^x U(\xi)\,d\xi;\ \varphi = \frac{\psi}{\sqrt{2\Phi}}, \tag{1.2}$$

in place of expression (1.1) we find /91

$$2\Phi\frac{\partial Z}{\partial \Phi} - \varphi\frac{\partial Z}{\partial \varphi} - \frac{u}{U}\frac{\partial^2 Z}{\partial \varphi^2} = 0. \tag{1.3}$$

For linearizing this equation we use self-similar solutions obtained for $U = cm^m$, as in [2]. Using the notation $(u/U)_{self} = \alpha(\phi, \beta)$, where $\beta = 2m/(m + 1)$ is a parameter in the self-similar solutions [5], we write the following linear equation:

$$2\Phi\frac{\partial Z}{\partial \Phi} - \varphi\frac{\partial Z}{\partial \varphi} - \alpha(\varphi, \beta)\frac{\partial^2 Z}{\partial \varphi^2} = 0; \tag{1.4}$$

$$Z(0) = U^2(\Phi);\ Z(\infty) = 0. \tag{1.5}$$

For finding the β parameter we write an integral expression. Integrating equation (1.3) and (1.4) across the layer and subtracting the second from the first, we obtain

$$\int_0^\infty \left[\frac{u}{U} - \alpha(\varphi, \beta)\right] \frac{\partial^2 Z}{\partial \varphi^2} d\varphi = 0. \tag{1.6}$$

Joint solution of equations (1.4) and (1.6) determines the profile of velocities in the boundary layer.

It is check that the solution of linear equation (1.4) can be represented by a series

$$Z = A_0 S_0 + A_1 S_1 + \ldots + A_k S_k + \ldots = \sum_{k=0}^{\infty} A_k S_k, \tag{1.7}$$

where

$$S_0 = U^2,\ S_1 = \Phi\,(U^2)',\ S_2 = \Phi^2 (U^2)'', \ldots S_k = \Phi^k (U^2)^{(k)} \tag{1.8}$$

are known functions of x, whereas the A_k coefficients are dependent only on ϕ and β parameter and can be tabulated by integrating the equations

$$\alpha(\varphi, \beta)\, \ddot{A}_k + \varphi \dot{A}_k - 2kA_k = 2A_{k-1};\ A_0(0) = 1;\ \dot{A}_k(0) = A_k(\infty) = 0, \tag{1.9}$$

obtained by substituting the series (1.7) into equation (1.4). Here and below the prime denotes differentiation for the variable Φ and the dot denotes differentiation for ϕ.

Using the series (1.7) we find the frictional stress on the wall. Relating it to the parameter $\rho\sqrt{\nu U_\infty^3/L}$, we will have

$$\tau_w = -2^{-3/2}\Phi^{-1/2}\left(\frac{\partial Z}{\partial \varphi}\right)_{\varphi=0} = \Phi^{-1/2}(a_0 S_0 + a_1 S_1 + \ldots$$
$$\ldots + a_k S_k + \ldots) = \Phi^{-1/2} \sum_{k=0}^{\infty} a_k S_k. \tag{1.10}$$

The coefficients $a_k = -2^{-3/2}\dot{A}_k(0, \beta)$ are dependent only on β and also can be tabulated.

§2. Integration of Equations (1.9) /92

It is convenient to find the functions A_k (ϕ, β) in the form of sums

$$A_k = \sum_{i=0}^{i=k} \frac{(-1)^{k+i}}{(k-i)!} f_i, \tag{2.1}$$

making it possible to replace the inhomogeneous equations (1.9) by homogeneous equations [2]

$$\alpha \ddot{f}_i + \varphi \dot{f}_i - 2i f_i = 0;$$
$$f_i(0) = \frac{1}{i!}; \; f_i(\infty) = 0. \tag{2.2}$$

When $i \neq 0$ equation (2.2) has a singularity on the wall ($\phi = 0$) which is characteristic of the Prandtl-Mises equation [9]. Accordingly, numerical integration of (2.2) is possible only beginning with sum $\phi > 0$. The solution can be represented in the form of a series near the wall. Using self-similar solutions [5], the variable ϕ near the wall is written in the form of a series

$$\varphi = b_2\eta^2 + b_3\eta^3 + b_5\eta^5 + \ldots = \sum_{n=2}^{\infty} b_n \eta^n. \tag{2.3}$$

The variable in the self-similar solutions is as follows:

$$\eta = y \sqrt{\frac{(m+1)c}{2} x^{m-1}}.$$

The b_n values are expressed through the parameter β and the coefficient $b_2 = \frac{1}{2}\left(\frac{d^2\phi}{d\eta^2}\right)_{\eta=0}$, determining friction on the wall in self-similar solutions

$$b_3 = -\beta/3!; \; b_4 = 0; \; b_5 = \frac{4}{5!} b_2^2 (2\beta - 1); \; b_6 = \frac{4}{6!} b_2 \beta (2 - 3\beta); \ldots$$

The b_2 values are given in tables as functions of β, for example, in the monograph [5].

Using the expression (2.3) near the wall, the self-similar velocity distribution is written in the form of a series

$$\alpha(\varphi, \beta) = \frac{d\varphi}{d\eta} = d_1\varphi^{1/2} + d_2\varphi + d_3\varphi^{3/2} + \ldots = \sum_{n=1}^{\infty} d_n\varphi^{n/2}, \tag{2.4}$$

whose coefficients are related to b_n by the equations

$$2b_2 = d_1m_1;\ 3b_3 = d_1m_1 + d_2m_1^2;\ 4b_4 = d_1m_3 + 2d_2m_1m_2 + d_3m_1^3;\ \ldots, \tag{2.5}$$

here

$$m_1^2 = b_2;\ 2m_1m_2 = b_3;\ m_2^2 + 2m_1m_3 = b_4;\ \ldots$$

The expansion of (2.4) makes it possible to represent the solution of equation (2.2) in the form of generalized power series

$$f_i = \frac{1}{i!}(q_i - c_ip_i);\ c_i = \frac{q_i(\infty)}{p_i(\infty)}; \tag{2.6}$$

$$q_i = 1 + \sum_{n=1}^{\infty} r_{in}\varphi^{n/2};\ p_i = \varphi\left(1 + \sum_{n=1}^{\infty} R_{in}\varphi^{n/2}\right). \tag{2.7}$$

The substitution of these series into equation (2.2) gives recurrent formulas /93
for the coefficient r_{in}, R_{in}

$$r_{i1} = r_{i2} = R_{i1} = R_{i2} = 0;\ 3r_{i3}d_1 = 8i;\ 8r_{i4}d_1 = 3d_2r_{i3};\ \ldots$$
$$15d_1R_{i3} = 4(2i-1);\ 24d_1R_{i4} = 15d_2R_{i3};\ \ldots$$

Equations (2.7) make it possible to find the q_i and p_i values for some ϕ close to zero. Using these as the initial values, by numerical integration

of equation (2.2) we find the q_i and p_i functions for ϕ and β. Then, using equations (2.6), we determine the constants c_i, the functions f_i, and finally, the coefficients of the series (1.7) and (1.10) necessary for computing the velocity profiles and wall frictional stress. In this case A_k values are determined using equation (2.1) and the a_k coefficients are found using the formula

$$a_k = -2^{-3/2}\dot{A}_k(0,\beta) = 2^{-3/2}\sum_{l=0}^{l=k}\frac{(-1)^{k+l}}{(k-1)!\,l!}c_l, \tag{2.8}$$

which is derived after differentiation of (2.1) and substitution of the $f_i(0)$ value found using expression (2.6) or (2.7).

The results of these computations are given in Tables 1 and 2 and in Figures 1 and 2. It can be seen that the A_k and a_k values rapidly decrease with an increase in no. and have relatively little dependence on β (the solid curves in Figure 1 apply for β = -0.16). In computations of velocity profiles and wall frictional stress, this makes it possible to limit ourselves in the series (1.7) and (1.10) to the first few terms.

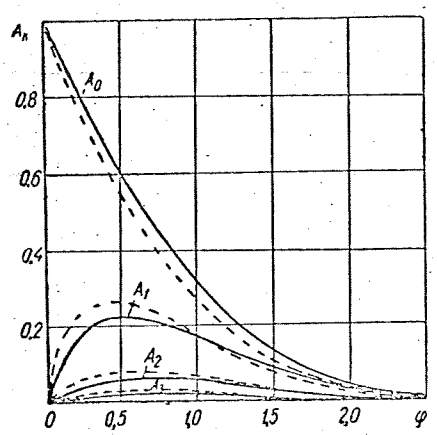

Figure 1.

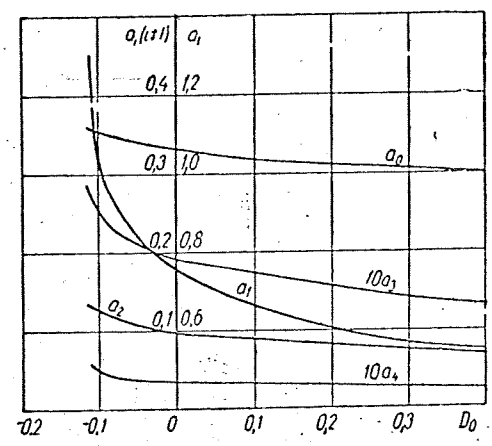

Figure 2.

TABLE 1. /94

	$\beta=-0,16;\ D_0=-0,0855$				$\beta=0;\ D_0=0$			
φ	A_0	$-A_1$	A_2	$-A_3$	A_0	$-A_1$	A_2	$-A_3$
0,1	0,902	0,154	0,032	0,007	0,907	0,130	0,026	0,005
0,3	0,719	0,262	0,070	0,016	0,731	0,237	0,061	0,013
0,5	0,559	0,272	0,083	0,020	0,574	0,255	0,075	0,017
0,7	0,424	0,243	0,082	0,020	0,439	0,253	0,076	0,018
0,9	0,313	0,200	0,072	0,018	0,327	0,197	0,068	0,017
1,2	0,189	0,135	0,052	0,014	0,199	0,136	0,051	0,013
1,6	0,087	0,068	0,028	0,008	0,093	0,070	0,028	0,008
2	0,036	0,029	0,013	0,004	0,038	0,031	0,013	0,004
2,4	0,013	0,011	0,005	0,001	0,014	0,012	0,005	0,002
2,8	0,004	0,003	0,002	0,001	0,004	0,004	0,002	0,001
3,2	0,001	0,001	0,000	0,000	0,001	0,001	0,000	0,000
	$\beta=0,5;\ D_0=0,1870$				$\beta=1;\ D_0=0,2962$			
0,1	0,912	0,112	0,022	0,004	0,914	0,105	0,020	0,004
0,3	0,744	0,215	0,053	0,011	0,749	0,206	0,049	0,010
0,5	0,591	0,240	0,067	0,015	0,598	0,233	0,064	0,014
0,7	0,457	0,227	0,070	0,017	0,464	0,223	0,068	0,016
0,9	0,343	0,195	0,065	0,016	0,350	0,194	0,063	0,015
1,2	0,212	0,138	0,050	0,013	0,217	0,138	0,049	0,013
1,6	0,100	0,073	0,028	0,008	0,103	0,074	0,028	0,008
2	0,041	0,032	0,013	0,004	0,042	0,033	0,013	0,004
2,4	0,015	0,012	0,005	0,002	0,015	0,013	0,005	0,002
2,8	0,005	0,004	0,002	0,001	0,005	0,004	0,002	0,001
3,2	0,001	0,001	0,000	0,000	0,001	0,001	0,000	0,000

TABLE 2.

β	a_0	a_1	$-a_2$	a_3	D_0	$-D_1$	D_2	$-D_3$
2	0,298	0,515	0,067	0,013	0,403	0,291	0,089	0,022
1	0,305	0,564	0,074	0,014	0,296	0,296	0,092	0,022
0,5	0,313	0,616	0,081	0,016	0,187	0,300	0,095	0,023
0	0,332	0,757	0,099	0,019	0,000	0,310	0,100	0,025
—0,16	0,350	0,957	0,121	0,024	—0,089	0,320	0,107	0,027
—0,199	0,360	1,300	0,138	0,028	—0,115	0,327	0,09	0,028

Tr. Note: commas in tables indicate decimal points.

§3. Determining the β Parameter. /93

In computing β we turn to the integral relation (1.6), which gives the condition for equality to zero for the total error arising during substitution

of the self-similar velocity distribution $\alpha(\phi, \beta)$ into equation (1.3) in place of the actual velocity distribution u/U. Since the solution (1.7) determines the parameter Z, dependent on $(u/U)^2$, for simplifying the computations it is convenient to replace this condition by the following:

$$\int_0^\infty \left[\left(\frac{u}{U}\right)^2 - \alpha^2\right] d\varphi = 0. \tag{3.1}$$

This substitution of one integral condition by the other has little effect on the final computation results. Equation (3.1), with the aid of (1.7), is reduced to the form /94

$$D_0 S_0 + D_1 S_1 + D_2 S_2 + \ldots + D_k S_k + \ldots = \sum_{k=0}^{\infty} D_k S_k = 0. \tag{3.2}$$

The D_k coefficients are dependent only on β

$$D_0 = \int_0^\infty (A_0 + \alpha^2 - 1)\, d\varphi; \quad D_k = \int_0^\infty A_k d\varphi.$$

The $D_k(\beta)$ values are given in Table 2. We find that the $D_k = D_k(D_0)$ curves are close to linear. Taking this into account, and solving (3.2) for D_0, we obtain /95

$$D_0 = \frac{0{,}31 S_1 - 0{,}1 S_2 + 0{,}25 S_3 + 0{,}005 S_4}{S_0 + \underset{(0{,}124)}{0{,}046} S_1 - \underset{(0{,}072)}{0{,}027} S_2 + \underset{(0{,}018)}{0{,}009} S_3 - \underset{(0{,}006)}{0{,}002} S_4}. \tag{3.3}$$

Here the coefficients on S_1, S_2, S_3, S_4, present in the denominator without parentheses, pertain to positive D_0 (or β) values, whereas those in parentheses apply to negative D_0 (or β).

Knowing D_0, it is easy to determine the β parameter, unambiguously related to it. However, in computations it is possible to use D_0 in place of β as a parameter.

§4. Transformation in the Plane x, y and Determination of Characteristic Thicknesses

The series (1.7) gives the velocity distribution in the plane x, ϕ. Transformation to the actual plane x, y is by use of the third equation (1.1)

$$y = \frac{\sqrt{2\Phi}}{U} \int_0^{\varphi} \frac{d\varphi}{u/U}. \tag{4.1}$$

Where $\phi \to 0$, $u/U \to 0$ and the integrand increases without limit.

In the expansion (2.7), limiting ourselves to the first term, the velocity distribution (1.7) near the wall ($\phi = 0$) can be represented in the form

$$u^2 = 2^{3/2}\varphi\,(a_0 S_0 + a_1 S_1 + \ldots + a_k S_k + \ldots) = 2^{3/2}\varphi \sum_{k=0}^{\infty} a_k S_k, \tag{4.2}$$

where the a_k values are determined using equation (2.8).

Using expressions (4.2) and (1.10), we represent (4.1) in the form

$$y = (8\Phi)^{1/4}\left(\frac{\varphi_0}{\tau_w}\right)^{1/2} + \frac{(2\Phi)^{1/2}}{U} \int_{\varphi_0}^{\varphi} \frac{d\xi}{u/U}. \tag{4.3}$$

Here ϕ_0 is some ϕ value sufficiently close to zero; ξ is the integration variable.

The expulsion thickness δ^*, momentum losses δ^{**} and energy losses δ^{***} are

$$\delta^* = \int_0^{\infty}\left(1 - \frac{u}{U}\right)dy;\quad \delta^{**} = \int_0^{\infty}\left[\frac{u}{U}\left(1 - \frac{u}{U}\right)\right]dy;\quad \delta^{***} = \int_0^{\infty}\frac{u}{U}\left[1 - \left(\frac{u}{U}\right)^2\right]dy$$

They can be determined by numerical integration using the known velocity distribution in the layer. However, it is possible to compute the thicknesses more simply without recourse to the velocity distributions in the layer. The expression for δ^{***}, using equation (3.1), is reduced to the form

$$\delta^{***} = \frac{\sqrt{2\Phi}}{U} \int_0^\infty \alpha(1-\alpha^2)\, dy = \frac{\sqrt{2\Phi}}{U} I.$$

Since the dependence $I = I(D_0)$ is close to linear, for δ^{***}, we obtain the formula /96

$$\delta^{***} = \frac{\sqrt{2\Phi}}{U} (0{,}735 - 0{,}857 D_0).$$

The other two parameters are determined using the form parameters

$$H = \frac{\delta^*}{\delta^{**}};\ H^* = \frac{\delta^{***}}{\delta^{**}}, \tag{4.4}$$

unambiguously related (Figure 3) to the form parameter $\zeta = \tau_w \frac{\delta^{***}}{U}$, whose value is computed from already known τ_w and δ^{***}.

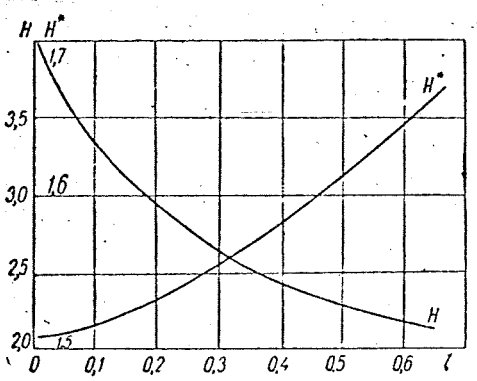

Figure 3.

§5. Computation Examples

On a practical basis the computations are made in the following way. Using the stipulated velocity distribution in potential flow $U(x)$, we find $\Phi(x)$, S_k, D_0. Then we determine the a_k coefficients (Figure 2), wall frictional stress τ_w, and thickness of energy loss δ^{***}. Using known H^* and H, which are taken from Figure 3 as a function of the form parameter ζ, we compute δ^{**} and δ^*. The $A_k(\phi)$ (as a function of D_0) from Table 1 are used for computing the velocity distribution using formula (1.7).

Since A_k values have a relatively weak dependence on D_0 (on β), for those values D_0 which do not coincide with those given in Table 1, the values of the A_k coefficients can be found by linear interpolation.

Now we will give computations of the boundary layer on a round cylinder. In this case the velocity distribution in the potential flow is represented by a sine curve U = sin x. In accordance with formula (1.2) and (1.8) we find

$$\Phi = \int_0^x U dx = 2\sin^2\frac{x}{2};\ S_0 = U^2 = 2\Phi - \Phi^2;$$

$$S_1 = \Phi(U^2)' = 2\Phi - 2\Phi^2;\ S_2 = \Phi^2(U^2)'' = -2\Phi^2;\ S_3 = \ldots = S_k = 0.$$

Using the dependence (3.3), we find

$$D_0 = \frac{0{,}62 - 0{,}42\Phi}{2{,}092 - 1{,}038\Phi} = \frac{0{,}2 + 0{,}42\cos x}{1{,}054 + 1{,}038\cos x},$$

And using Figure 2, we find the a_0, a_1 and a_2 values. Taking expression (1.10) into account, we compute the wall frictional stress

$$\tau_w = \sin\frac{x}{2}\left(K_0 - K_1\sin^2\frac{x}{2}\right);$$

$$K_0 = 2\sqrt{2}(a_0 + a_1);\qquad K_1 = 2\sqrt{2}(a_0 + 2a_1 + 2a_2).$$

For this case, the noted I, the results of the computations (curve 2), in Figure 4 are compared with data obtained in [11] by numerical integration of boundary-layer equations (curve 1). This same figure shows a comparison of data found by numerical integration [7] with computations made on the basis of formula (1.10) for case II, when the velocity and the potential flow changes linearly ($U = d_0 - d_1 x$).

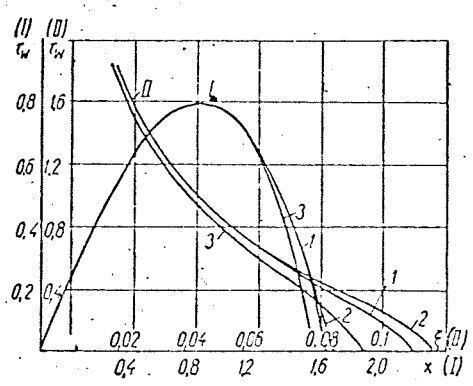

Figure 4.

In addition, as a comparison, Figure 4 shows the results of computations /97
by the one-parameter method (curve 3). It can be seen that the frictional stress determined using formula (1.10) coincides virtually completely with the value determined by numerical integration in all cases other than for a small region near the break, where the solution of even precise equations in series deviations of the same magnitude [6].

Figure 5 gives a comparison of velocity profile (denoted by small crosses) in the boundary layer on a cylinder computed using formulas (1.7) and (4.3) and profiles (solid curve) constructed by mean of numerical integration. In this figure as well the results coincide, other than for the profiles at the breakaway point.

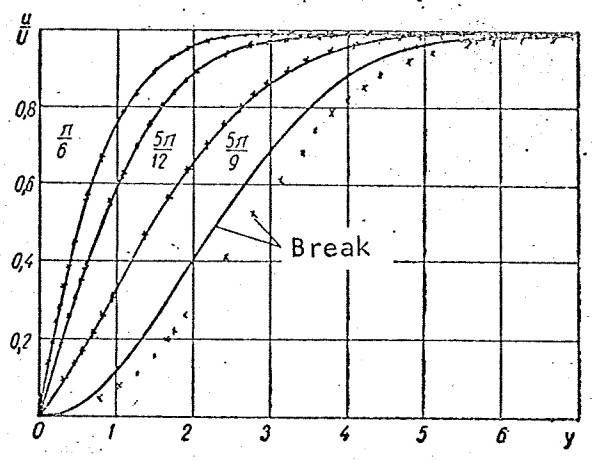

Figure 5.

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