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WALL EFFECT ON A HELICOPTER ROTOR IN A CLOSED CIRCULAR TUNNEL: INCOMPRESSIBLE FLOW AROUND A DOUBLET PLACED IN A CLOSED CIRCULAR TUNNEL

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ABSTRACT: A method of calculating the interaction of a cylindrical wall with a circular straight section with the flow from a doublet placed in any way in this wall is described. This method is designed to be used in testing helicopter rotors. Included are descriptions of conditions to be satisfied by complementary potential $\varphi_c(M)$, calculation principle of complementary potential $\varphi_c(M)=\varphi_c(X,2,\theta)$, development of Fourier series for function $\Phi_d(g,2,\theta)$, and calculations of complementary potential $\varphi_c(X,2,\theta)$, integral $E(u,v,w,\dot{m},m_1,h)$, and velocities induced by doublet D. Annexes contain descriptions of Euler, hypergeometric Gauss and modified Bessel functions, as well as reference citations.

2. Introduction

2.1. The method described in this first installment allows calculation of the interaction of a cylindrical wall with a circular straight section with the flow from a doublet placed in any way whatever inside this wall. /2*

2.2. The presence of the wall imposes a condition at the boundaries on the potential of doublet velocities which it cannot satisfy.

2.2.1. When the wall is prismatic (with straight section most often square or rectangular), the potential of the velocities of the doublet and of its "images," produced by symmetries which are successive with respect to each one of the planes containing the wall components, satisfies this condition at the boundaries.

2.2.2. If the wall is cylindrical, the simplest method of solving the problem consists in adding to the potential of doublet velocities a so-called "complementary" potential, which will be determined by two

* Numbers in the margin indicate pagination in the foreign text.

conditions (see paragraph 4, page 10 [of foreign text]), which, in the case of a circular cylinder, are expressed in a relatively simple manner.

At each point, the gradient of the "complementary" potential represents the difference between the velocities which could be measured according to whether the flow coming from the doublet is carried out or not in the presence of the cylindrical wall: whence the advantage of the "complementary" potential for the calculation of wall interactions during testing in a closed wind tunnel with a circular cross section.

2.3. Although capable of more general applications, the calculation developed below is carried out with a very precise goal in mind, i.e., to be used as a basis for the elaboration of a method of wall correction which could be applied to the testing of a helicopter rotor in wind tunnel S1-MA.

The principle of this correction method will be described in the second installment of this technical memorandum.

3. Conventions and Notations /3

3.1. Axes of Coordinates

Space is related to a system of orthonormalized axes oxyz, whose axis ox coincides with the axis of the cylinder representing the wall.

3.2. Notations (see comment in paragraph 3.3, page 5 [of foreign text])

3.2.1. (see Figure 1 at the end of this installment)

$\mathcal{C}$: Cylinder with straight circular cross-section, representing the wall;

R : Radius of a straight section of $\mathcal{C}$;

$D\begin{cases} x_D \\ y_D = \rho \sin\zeta \\ z_D = \rho\cos\zeta \end{cases}$:Point where doublet D ($\rho \leq R$) is placed;

$\vec{\tau}\ (\alpha, \beta, \gamma)$: Unitary vector defining the axis of doublet D;

$M\begin{cases} x_M = x \\ y_M = r \sin\theta \\ z_M = r\cos\theta \end{cases}$:Stream point of flow;

μ Intensity of doublet D.

3.2.2. Flow

$\varphi_G(M) = \varphi_D(M) + \varphi_C(M)$ Potential of velocities of flow proceeding from doublet D in the presence of wall $\mathcal{C}$;

$\varphi_D(M)$ Potential of velocities of flow proceeding from doublet D, in absence of wall; /4

$\varphi_C(M)$ "Complementary" potential of velocities whose role is to represent the influence of wall $\mathcal{C}$;

Φ_D et Φ_C Images of φ_D and φ_C in the Fourier transform defined in paragraph 5.1, page 13 [of foreign text];

$\vec{V}_G = \vec{V}_D + \vec{V}_C = \overrightarrow{\text{grad}}\,\varphi_G(M)$ Resultant velocity at point M;

$$\vec{V}_G(M) \begin{cases} V_{Gx}(M) = \dfrac{\partial}{\partial x}\varphi_G(M) \\ V_{Gr}(M) = \dfrac{\partial}{\partial r}\varphi_G(M) \\ V_{G\theta}(M) = \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\varphi_G(M) \end{cases}$$

$\vec{V}_D = \overrightarrow{\text{grad}}\,\varphi_D(M)$ Velocity induced by doublet D in the absence of wall;

$\overrightarrow{\Delta V}(M) = \overrightarrow{\text{grad}}\,\varphi_C(M)$ Correction of velocity owing to the presence of wall.

$$\overrightarrow{\Delta V}(M) \begin{cases} \Delta V_x(M) = \dfrac{\partial}{\partial x} \varphi_G(M) \\ \Delta V_r(M) = \dfrac{\partial}{\partial r} \varphi_G(M) \\ \Delta V_\theta(M) = \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \varphi_G(M) \end{cases}$$

3.3. Comment /5

In order to facilitate performance of calculations, the doublet D will occupy a special position with respect to axes oxyz. This position is defined by:

$$D_0 \quad : \quad x_D = y_D = 0 \quad \left(\text{or} \quad \hat{\delta} = 0 \right)$$

It will then be easy to return to the general case, owing to the translation parallel to ox and a rotation around this axis, i.e., by replacing:

$$x = x_M \quad \text{with} \quad x - x_D$$

and

$$\theta \quad \text{with} \quad \theta - \delta$$

The indices C, D, G defined in paragraph 3.22 will, for this special portion D_0 of doublet D, be replaced by c, d, g.

3.4. Special Notations Encountered During Development of Calculations

$$t = \left(r^2 - 2\rho r \cos\theta + \rho^2 \right)^{\frac{1}{2}}$$

$$P_m(g, r, \rho) = m\, K_m(gr)\, I_m(g\rho)$$

$$\mathcal{P}_m(g, r, \rho) = \sum_{n=0}^{\infty} P_{m+2n}(g, r, \rho)$$

$\sigma = gR \qquad u = \operatorname{Arctg}\frac{x}{R} \qquad v = \frac{r}{2R} \qquad w = \frac{\rho}{2R}$ 6

$$E_c(u, v, w, m, m_1, h) =$$

$$\int_0^\infty I_m(2\sigma v)\, I_{m_1}(2\sigma w)\, \frac{K_{m-1}(\sigma) + K_{m+1}(\sigma)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)}\, \sigma^h \cos(\sigma \operatorname{tg} u)\, d\sigma$$

$$E_s(u, v, w, m, m_1, h) =$$

$$\int_0^\infty I_m(2\sigma v)\, I_{m_1}(2\sigma w)\, \frac{K_{m-1}(\sigma) + K_{m+1}(\sigma)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)}\, \sigma^h \sin(\sigma \operatorname{tg} u)\, d\sigma$$

$$E(u, v, w, m, m_1, h) =$$

$$E_c(u, v, w, m, m_1, h) - i\, E_s(u, v, w, m, m_1, h)$$

$$E(u, v, w, m, m_1, h) =$$

$$\int_0^\infty I_m(2\sigma v)\, I_{m_1}(2\sigma w)\, \frac{K_{m-1}(\sigma) + K_{m+1}(\sigma)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)}\, \sigma^h e^{-i\sigma \operatorname{tg} u}\, d\sigma$$

$$W(p, n, m, |m_1|) = \frac{1}{(p-n)!\,(p+|m_1|-n)!\,(m+n)!\,n!}$$

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$$a_p(v, w, m, |m_1|) = \sum_{n=0}^{p} W(p, n, m, |m_1|)\, v^{m+2n}\, w^{|m_1|+2p-2n}$$

$$d_k(m) = \frac{m+2k}{b^k\, k!\,(m+k)!}$$

$$\sum_{p=0}^{\infty} b_p(m)\, \sigma^{2p} = \left(\sum_{k=0}^{\infty} d_k(m)\, \sigma^{2k} \right)^{-1}$$

$$b_p(m) = \frac{(-1)^p}{[d_0(m)]^{p+1}} \begin{vmatrix} d_1(m) & d_0(m) & 0 & \cdots\cdots & 0 & 0 \\ d_2(m) & d_1(m) & d_0(m) & \cdots\cdots & 0 & 0 \\ d_3(m) & d_2(m) & d_1(m) & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ d_{p-1}(m) & d_{p-2}(m) & d_{p-3}(m) & \cdots\cdots & d_1(m) & d_0(m) \\ d_p(m) & d_{p-1}(m) & d_{p-2}(m) & \cdots\cdots & d_2(m) & d_1(m) \end{vmatrix}$$

$$C_q(v, w, m, |m_1|) = \sum_{p=0}^{q} a_p(v, w, m, |m_1|)\, b_{q-p}(m)$$

$$e_m(u,a) = \int_0^\infty e^{-i\sigma \operatorname{tg} u} K_m(\sigma)\, \sigma^{a-1}\, d\sigma$$

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$$s_p(a,b,c) = \frac{\Gamma\left(\frac{1}{2} - b + p\right)}{\Gamma\left(\frac{1}{2} + a + p\right)} \frac{(a-c+p-1)!}{p!}$$

$$\boxed{\varepsilon_m \text{ ou } \varepsilon_n = 2 \quad \text{if} \quad m \text{ ou } n > 0 \quad , \qquad \varepsilon_0 = 1}$$

R(z) = real part of z.

$$C_{a-m}^{p} = \frac{(a-m)!}{p!\,(a-m-p)!}$$

$$H_p(a,m) = \frac{(-1)^m}{2^{a+\varepsilon_m}} \frac{(\varepsilon_m)!}{m!} (a+m-1)! \sum_{q=\varepsilon_p p}^{a-m+p} (-1)^q C_{a-m}^{q-p} s_q(a,m,m)$$

$$H_p^0(a,0) =$$

$$\frac{(a-1)!}{2^a(2a-1)} \sum_{q=\varepsilon_p p}^{a-1+p} (-1)^{q+1} C_{a-1}^{q-p} \left[a\, s_q(a,-1,0) + (a-1)\, s_q(a,0,-1)\right]$$

In the two preceding formulas:

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$$\begin{cases} \varepsilon_\mu = 1 & \text{si } \mu > 0 \\ \varepsilon_\mu = 0 & \text{si } \mu < 0. \end{cases}$$

$$a = 2q + |m_1| + h + 2.$$

$$C_q(v, w, m, |m_1|) = \sum_{\mu=0}^{q} C_{\mu,q}(w; m, |m_1|)\, v^{m+2\mu}.$$

$$C_{\mu,q}(w; m, |m_1|) = \sum_{n=\mu}^{q} b_{\mu,n,q}(m, |m_1|)\, w^{|m_1|+2(n-\mu)}$$

$$b_{\mu,n,q}(m, |m_1|) = \frac{b_{q-n}(m)}{(n-\mu)!\,(n+|m_1|-\mu)!\,(m+\mu)!\,\mu!}$$

$$b = a - 2q = |m_1| + h + 2$$

$$n = \frac{180}{\text{PGCD of } 180 \text{ and of } u'} \qquad (u' \text{ in degrees})$$

$$F_{\mu,q}(b, m, n) = \sum_{k=0}^{\infty} (-1)^k \Big[H_{\mu+nk-2q-b+|m-1|}(2q+b, |m-1|) + H_{\mu+nk-2q-b+m+1}(2q+b, m+1) \Big]$$

$$F^{0}_{\mu,q}(b,1,n) = \sum_{k=0}^{\infty}(-1)^k \left[H^{0}_{\mu+nk-2q-b+1}(2q+b,0) + H_{\mu+nk-2q-b+2}(2q+b,2) \right]$$

4. Conditions Which Should Be Satisfied by Complementary Potential $\varphi_c(M)$ /10

4.1. The total potential of velocities $\varphi_g(M)$ of the flow in presence of wall $\mathcal{C}$ is the sum of the two following potentials:

4.1.1. The potential $\varphi_{o1}(M)$ of doublet D_o in the absence of wall $\mathcal{C}$:

$$\boxed{\varphi_{o1}(M) = \mu \frac{\vec{\tau}\,\overrightarrow{D_oM}}{|D_oM|^3}} \qquad (4.1)$$

4.1.2. The "complementary" potential $\varphi_c(M)$ prescribed by the presence of wall $\mathcal{C}$: its expression is unknown at this time.

4.2. The total potential of velocities $\varphi_g(M)$ should satisfy two conditions which should allow calculation of $\varphi_c(M)$.

4.2.1. $\varphi_g(M)$, potential of velocities of an incompressible flow, should be a harmonic function and therefore satisfy the Laplace equation:

$$\boxed{\Delta = r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + r^2\frac{\partial^2}{\partial x^2} = 0} \qquad (4.2)$$

$\varphi_{o1}(M)$, potential of velocities of a doublet, satisfies this equation: If it is true that

$$\varphi_g(M) = \varphi_{o1}(M) + \varphi_c(M)$$

then $\varphi_c(M)$ alone will have to confirm (4.2): /11

$$\boxed{\Delta\varphi_c(M) = r^2\frac{\partial^2}{\partial r^2}\varphi_c(M) + r\frac{\partial}{\partial r}\varphi_c(M) + \frac{\partial^2}{\partial \theta^2}\varphi_c(M) + r^2\frac{\partial^2}{\partial x^2}\varphi_c(M) = 0}$$

The first condition is therefore that $\varphi_c(M)$ be a harmonic function.

4.2.2. Condition at the Boundaries on Cylinder $\mathcal{C}$

This is the condition prescribing introduction of the function $\varphi_c(M)$.

Since the fluid can neither pass through the wall $\mathcal{C}$, nor separate from it, at any point of this wall, the radial component of the velocity should be zero and may be written:

$$[V_r]_{r=R} = \left[\frac{\partial}{\partial r}\varphi_g(M)\right]_{r=R} = \left[\frac{\partial}{\partial r}\varphi_d(M)\right]_{r=R} + \left[\frac{\partial}{\partial r}\varphi_c(M)\right]_{r=R} = 0$$

or :

$$\left[\frac{\partial}{\partial r}\varphi_c(M)\right]_{r=R} = -\left[\frac{\partial}{\partial r}\varphi_d(M)\right]_{r=R}.$$

4.3. The search for the function $\varphi_c(M)=\varphi_c(X,z,\theta)$ is therefore closely connected to the solution of a Neumann problem for a cylinder, i.e., to the determination of a function: harmonic inside a cylinder with straight circular section, continuous in and on this cylinder, and whose derivative perpendicular to the surface of the latter takes on a fixed value at any point.

In summary, the function $\varphi_c(X,z,\theta)$ is determined by the two relations:

$$r^2\frac{\partial^2\varphi_c}{\partial r^2} + r\frac{\partial\varphi_c}{\partial r} + \frac{\partial^2\varphi_c}{\partial\theta^2} + r^2\frac{\partial^2\varphi_c}{\partial z^2} = 0 \qquad (4.4)$$

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$$\left[\frac{\partial\varphi_c}{\partial r}\right]_{r=R} = -\left[\frac{\partial\varphi_d}{\partial r}\right]_{r=R} \qquad (4.5)$$

5. Principle of Calculation of Complementary Potential: $\varphi_c(M)=\varphi_c(X,2,\theta)$ /13

5.1. Fourier Transforms

Let $\Phi_c(g,2,\theta)$ and $\Phi_d(g,2,\theta)$ be the images of $\varphi_c(X,2,\theta)$ and $\varphi_d(X,2,\theta)$ in the Fourier transforms defined by:

$$\Phi_c(g,r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_c(x,r,\theta)\, e^{igx}\, dx. \tag{5.1}$$

$$\varphi_c(x,r,\theta) = \int_{-\infty}^{+\infty} \Phi_c(g,r,\theta)\, e^{-igx}\, dg \tag{5.2}$$

$$\Phi_d(g,r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_d(x,r,\theta)\, e^{igx}\, dx \tag{5.3}$$

$$\varphi_d(x,r,\theta) = \int_{-\infty}^{+\infty} \Phi_d(g,r,\theta)\, e^{-igx}\, dg \tag{5.4}$$

5.2. Development of Φ_c and Φ_d as a Fourier Series

5.2.1. $\Phi_c(g,2,\theta)$ is written beforehand in the form

$$\Phi_c(g,r,\theta) = \sum_{m=0}^{\infty}\left[\varepsilon_m A_m(g,r)\cos m\theta + 2B_m(g,r)\sin m\theta\right] \tag{5.5}$$ /14

$$\begin{cases} \varepsilon_m = 2 & \text{if } m \neq 0 \\ \varepsilon_0 = 1. \end{cases}$$

5.2.2. $\Phi_d(g,2,\theta)$ will be developed as a Fourier series (see paragraph 6, page 16 [of foreign text]):

$$\boxed{\Phi_d(g,r,\theta) = \sum_{m=0}^{\infty} \left[\varepsilon_m M_m(g,r) \cos m\theta + 2 N_m(g,r) \sin m\theta\right]} \quad (5.6)$$

5.3. The integration of the Laplace equation (4.4) (page 12 [of foreign text]), after replacement of $\varphi_c(X,2,\theta)$ by its Fourier transform defined by (5.2) (page 13 [of foreign text]), gives Am(g,2) and Bm(g,2):

$$\begin{cases} A_m(g,r) = a_m(g)\, I_m(gr) \\ B_m(g,r) = b_m(g)\, I_m(gr) \end{cases}$$

am(g) and bm(g) are functions which will be determined using relation (4.5) (page 12 [of foreign text]).

Im(g,2) is a modified Bessel function of the first class of order n (see Annex III, paragraph 2, page XVIII [of foreign text]).

5.4. The condition at the boundaries (4.5) then allows calculation of am(g) and bm(g).

This condition can indeed be written, taking into consideration (5.2) and (5.4), page 13 [of foreign text]:

$$\left[\frac{\partial}{\partial r} \Phi_c(g,r,\theta)\right]_{r=R} = -\left[\frac{\partial}{\partial r} \Phi_d(g,r,\theta)\right]_{r=R}$$

or, after an easy calculation: /15

$$\begin{cases} a_m(g) = -\left[\dfrac{\frac{\partial}{\partial r} M_m(g,r)}{\frac{\partial}{\partial r} I_m(gr)}\right]_{r=R} & (5.7) \\ b_m(g) = -\left[\dfrac{\frac{\partial}{\partial r} N_m(g,r)}{\frac{\partial}{\partial r} I_m(gr)}\right]_{r=R}. & (5.8) \end{cases}$$

5.5. $\Phi_c(g,z,\theta)$ is determined in this way, since the $A_m(g,z)$ and $B_m(g,z)$ are known.

$\varphi_c(X,z,\theta)$ is finally obtained by formula (5.2).

$$\boxed{\varphi_c(x,r,\theta) = \int_{-\infty}^{+\infty} \sum_{m=0}^{\infty} \left\{ \left[\varepsilon_m a_m(g) \cos m\theta + 2 b_m(g) \sin m\theta \right] I_m(gr) \right\} e^{-igx} dg} \quad (5.9)$$

6. Development as a Fourier Series of the Function $\Phi_d(g,z,\theta)$, Image of the Potential of the Doublet D_o Velocities /16

6.1. Analytic Expression of Potential $\varphi_d(X,z,\theta)$

$$\boxed{\varphi_d(g,r,\theta) = \mu \frac{\vec{\tau} \cdot \overrightarrow{D_oM}}{|D_oM|^3}}$$

The components of vector $D_o\vec{M}$ are (see paragraph 3.2, page 4 [of foreign text]):

$$\overrightarrow{D_oM} \begin{cases} x_M - x_d = x \\ y_M - y_d = r\sin\theta \\ z_M - z_d = -\rho + r\cos\theta \end{cases}$$

Furthermore:

$$|D_oM| = \left[(x_M - x_d)^2 + (y_M - y_d)^2 + (z_M - z_d)^2\right]^{\frac{1}{2}}$$
$$= \left[x^2 + r^2 - 2\rho r\cos\theta + \rho^2\right]^{\frac{1}{2}}$$

and

$$\boxed{\varphi_d(x,r,\theta) = \mu \frac{\alpha x + \beta r\sin\theta - \gamma(\rho - r\cos\theta)}{\left[x^2 + r^2 - 2\rho r\cos\theta + \rho^2\right]^{\frac{1}{2}}}} \quad (6.1)$$

6.2. Calculation of Function $\Phi_d(g,z,\theta)$, Image of $\varphi_{o1}(X,z,\theta)$ /17

6.2.1.

$$\Phi_d(g,z,\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_d(x,r,\theta)\, e^{igx}\, dx$$

Therefore, taking (6.1) into consideration,

$$\boxed{\Phi_d(g,z,\theta) = \frac{\mu}{2\pi}\int_{-\infty}^{+\infty} \frac{\alpha x + \beta r\sin\theta - \gamma(\rho - r\cos\theta)}{\left[x^2 + r^2 - 2\rho r\cos\theta + \rho^2\right]^{3/2}}\, e^{igx}\, dx} \quad (6.2)$$

whence, by granting:

$$t = \left[r^2 - 2\rho r\cos\theta + \rho^2\right]^{\frac{1}{2}}$$

$$E_1 = \int_{-\infty}^{+\infty} (x^2+t^2)^{-\frac{3}{2}} e^{igx}\, dx$$

$$E_2 = \int_{-\infty}^{+\infty} (x^2+t^2)^{-\frac{3}{2}} x\, e^{igx}\, dx.$$

$$\boxed{\Phi_d(g,r,\theta) = \frac{\mu}{2\pi}\left\{\alpha E_2 + \left[\beta r\sin\theta - \gamma(\rho - r\cos\theta)\right] E_1\right\}} \quad (6.3)$$

6.2.2. Calculation of E

$$E_1 = \int_0^{\infty} (x^2+t^2)^{-\frac{3}{2}} e^{igx}\, dx + \int_0^{\infty} (x^2+t^2)^{-\frac{3}{2}} e^{-igx}\, dx$$ /18

$$E_1 = 2\int_0^{\infty} (x^2+t^2)^{-\frac{3}{2}} \frac{e^{igx}+e^{-igx}}{2}\, dx = 2\int_0^{\infty} (x^2+t^2)^{-\frac{3}{2}} \cos gx\, dx$$

The Basset formula (see Annex III, page XXIII [of foreign text]), then provides:

$$\boxed{E_1 = \frac{2g}{t} K_1(gt)} \quad (6.4)$$

6.2.3. Calculation of E_2

$$\frac{\partial E_1}{\partial g} = \frac{\partial}{\partial g}\int_{-\infty}^{+\infty}(x^2+t^2)^{-\frac{3}{2}} e^{igx}\,dx = i\int_{-\infty}^{+\infty}(x^2+t^2)^{-\frac{3}{2}} x e^{igx}\,dx = i E_2$$

Therefore

$$E_2 = -i\frac{\partial E_1}{\partial g} = -\frac{2i}{t}\frac{\partial}{\partial g}\left[g K_1(gt)\right]$$

$$E_2 = -\frac{2i}{t}\left[K_1(gt) + g\frac{\partial}{\partial g}K_1(gt)\right]$$

According to formulas (III,23) and (III,20), of page XXI [of foreign text]:

$$\frac{\partial}{\partial g}K_1(gt) = -\frac{t}{2}\left[K_0(gt) + K_2(gt)\right]$$

$$K_2(gt) = \frac{2}{gt}K_1(gt) + K_0(gt)$$ /19

or:

$$\frac{\partial}{\partial g}K_1(gt) = -t\,K_0(gt) + \frac{1}{g}K_1(gt)$$

and: $$E_2 = -\frac{2i}{t}\left[K_1(gt) - gt\,K_0(gt) - K_1(gt)\right]$$

$$\boxed{E_2 = 2ig\,K_0(gt)} \qquad (6.5)$$

6.2.4. Finally:

$$\boxed{\Phi_d(g,r,\theta) = \frac{\mu g}{2\pi}\left[i\alpha K_0(gt) + \frac{\rho r \sin\theta - \gamma(\rho - r\cos\theta)}{t} K_1(gt)\right]} \quad (6.6)$$

This expression will be generalized later on (see paragraph 6.6, page 30 [of foreign text]), in order to remain valid in the case where t and g have opposite signs:

$$t > 0 \quad , \quad g < 0$$

6.3. Development as a Fourier Series of:

$$\boxed{\frac{K_1(gt)}{t} = \frac{K_1\left[g\left(r^2 - 2\rho r\cos\theta + \rho^2\right)^{\frac{1}{2}}\right]}{\left[r^2 - 2\rho r\cos\theta + \rho^2\right]^{\frac{1}{2}}}}$$

6.3.1. Let: /20

$$\boxed{P_m(g,r,\rho) = m K_m(gr) I_m(g\rho)} \quad (6.7)$$

I_m and K_m being modified Bessel functions of the first and second class, of order n.

Since:

$$gt = \left[g^2 r^2 - 2\rho g \cdot \rho r\cos\theta + g^2\rho^2\right]^{\frac{1}{2}}$$

the formula (III,28) (page XXIII [of foreign text]), in which t is replaced by gt, is written:

$$\boxed{\frac{K_1(gt)}{t} = \frac{r}{\rho r g}\sum_{m=0}^{\infty}{}' P_m \frac{\sin m\theta}{\sin\theta}}$$

in which:

$$\frac{K_1(gt)}{t} = \frac{2}{\rho r g} \sum_{p=0}^{\infty} P_{2p+1} \frac{\sin(2p+1)\theta}{\sin\theta} + P_{2p+2} \frac{\sin(2p+2)\theta}{\sin\theta}$$

6.3.2. The conventional formulas:

$$\begin{cases} \sin(2n+1)\theta - \sin(2n-1)\theta = 2\sin\theta\cos n\theta \\ \sin(2n+2)\theta - \sin 2n\theta = 2\sin\theta\cos(2n+1)\theta \end{cases}$$

give, by summation:

$$\sum_{n=p}^{p} \sin(2n+1)\theta - \sin(2n-1)\theta = \sin(2p+1)\theta + \sin\theta = 2\sin\theta \sum_{n=0}^{p} \cos 2n\theta$$

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$$\sum_{n=0}^{p} \left[\sin(2n+2)\theta - \sin 2n\theta\right] = \sin(2p+2)\theta = 2\sin\theta \sum_{n=0}^{p} \cos(2n+1)\theta$$

therefore:

$$\frac{\sin(2p+1)\theta}{\sin\theta} = \sum_{n=0}^{p} \varepsilon_n \cos 2n\theta$$

$$\frac{\sin(2p+2)\theta}{\sin\theta} = \sum_{n=0}^{p} \cos(2n+1)\theta$$

$$\begin{cases} \varepsilon_n = 2 \text{ if } n \neq 0 \\ \varepsilon_0 = 1. \end{cases}$$

and:

$$\boxed{\frac{K_1(gt)}{t} = \frac{2}{\rho r g} \sum_{p=0}^{\infty} \sum_{n=0}^{p} \left[\varepsilon_n P_{2p+1} \cos 2n\theta + 2 P_{2p+2} \cos(2n+1)\theta\right]}$$

6.3.3. It is easy to confirm that:

$$\sum_{p=0}^{\infty}\sum_{n=0}^{p} \varepsilon_n P_{2p+1} \cos 2n\theta = \sum_{p=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon_p P_{2p+2n+1} \cos 2p\theta$$

$$\sum_{p=0}^{\infty}\sum_{n=0}^{p} P_{2p+2} \cos(2n+1)\theta = \sum_{p=0}^{\infty}\sum_{n=0}^{\infty} P_{2p+2n+2} \cos(2p+1)\theta$$

and it follows that:

$$\frac{K_1(gt)}{t} = \frac{2}{\rho g r} \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\left[\varepsilon_p P_{2p+2n+1} \cos 2p\theta + 2 P_{2p+2n+2} \cos(2p+1)\theta\right]$$ /22

or, more simply, after having granted,

$$\boxed{\mathcal{P}_m(g,r,\rho) = \sum_{n=0}^{\infty} P_{m+2n}(g,r,\rho)}$$

$$\boxed{\frac{K_1(gt)}{t} = \frac{2}{\rho g r} \sum_{m=0}^{\infty} \varepsilon_m \mathcal{P}_{m+1} \cos m\theta} \qquad (6.8)$$

6.3.4. <u>Calculation of $\mathcal{P}_{m+1}(g,r,\rho) = \sum_{n=0}^{\infty} P_{m+2n+1}(g,r,\rho)$</u>

$$\mathcal{P}_m = \sum_{n=0}^{\infty} P_{m+2n} = \sum_{n=0}^{\infty} (m+2n) K_{m+2n}(gr) I_{m+2n}(g\rho)$$

Formulas (III,20) and (III,15) (page XXI [of foreign text]) give:

$$(m+2n)K_{m+2n}(gr) = \frac{gr}{2}\left[K_{m+2n+1}(gr) - K_{m+2n-1}(gr)\right]$$

$$I_{m+2n}(g\rho) = I_{m+2n+2}(g\rho) + \frac{2}{g\rho}(m+2n+1)\, I_{m+2n+1}(g\rho)$$

P_m is written, in this case:

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$$\mathcal{P}_m =$$

$$\frac{gr}{2}\sum_{n=0}^{\infty}\left\{K_{m+2n+1}(gr)\left[I_{m+2n+2}(g\rho) + \frac{2}{g\rho}(m+2n+1)\, I_{m+2n+1}(g\rho)\right] - K_{m+2n+1}(gr)\, I_{m+2n}(g\rho)\right\}$$

in which

$$\mathcal{P}_m =$$

$$\frac{gr}{2}\sum_{n=0}^{\infty}{}'\left[K_{m+2n+1}(gr)\, I_{m+2n+2}(g\rho) - K_{m+2n+1}(gr)\, I_{m+2n}(g\rho)\right] + \frac{r}{\rho}\sum_{n=0}^{\infty}{}'(m+2n+1)\, K_{m+2n+1}(gr)\, I_{m+2n+1}(g\rho)$$

or

$$\boxed{\mathcal{P}_m = -\frac{gr}{2}K_{m-1}(gr)\, I_m(g\rho) + \frac{r}{\rho}\mathcal{P}_{m+1}} \qquad (6.9)$$

By changing m into mH in the preceding formula:

$$\mathcal{P}_{m+1} = -\frac{gr}{2} K_m(gr) I_{m+1}(g\rho) + \frac{2}{\rho} \mathcal{P}_{m+2} \qquad (6.10)$$

However,

$$\mathcal{P}_{m+2} = \sum_{n=0}^{\infty} P_{m+2n+2} = \sum_{n=0}^{\infty} P_{m+2n} - m K_m(gr) I_m(g\rho)$$

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in which:

$$\mathcal{P}_{m+2} = \mathcal{P}_m - m K_m(gr) I_m(g\rho)$$

(6.10) becomes:

$$\mathcal{P}_m = \frac{\rho g}{2} K_m(gr) I_{m+1}(g\rho) + m K_m(gr) I_m(g\rho) + \frac{2}{\rho} \mathcal{P}_{m+1} \qquad (6.11)$$

and, after removal of Pm between (6.9) and (6.11):

$$\frac{2(r^2-\rho^2)}{\rho r g} \mathcal{P}_{m+1} =$$

$$\rho K_m(gr) I_{m+1}(g\rho) + r K_{m-1}(gr) I_m(g\rho) + \frac{2m}{g} K_m(gr) I_m(g\rho)$$

Since:

$$\frac{2m}{g} K_m(gr) = \frac{2mr}{gr} K_m(gr) = r\left[K_{m+1}(gr) - K_{m-1}(gr)\right]$$

(see (III,20), page XXI [of foreign text]).

It is finally true that:

$$\frac{2(r^2-\rho^2)}{\rho r g} \mathcal{P}_{m+1} = \rho K_m(gr) I_{m+1}(g\rho) + r K_{m-1}(gr) I_m(g\rho)$$
$$+ r K_{m+1}(gr) I_m(g\rho) - r K_{m-1}(gr) I_m(g\rho)$$

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$$\boxed{\mathcal{P}_{m+1} = \frac{\rho r g}{2(r^2-\rho^2)}\left[r K_{m+1}(gr) I_m(g\rho) + \rho K_m(gr) I_{m+1}(g\rho)\right]} \tag{6.12}$$

6.3.5. The result sought is therefore:

$$\boxed{(r^2-\rho^2)\frac{K_1(gt)}{t} = \sum_{m=0}^{\infty} \varepsilon_m \left[r K_{m+1}(gr) I_m(g\rho) + \rho K_m(gr) I_{m+1}(g\rho)\right]\cos m\theta} \tag{6.13}$$

$$(\varepsilon_m = 2 \text{ si } m \neq 0, \quad \varepsilon_0 = 1)$$

6.4. Development as a Fourier Series of:

$$\boxed{(\rho - r\cos\theta)\,\frac{K_1(gt)}{t}}$$

6.4.1.

$$\rho - r\cos\theta = \frac{r}{\rho}\left[\frac{\rho^2}{r} - r + r - \rho\cos\theta\right] = \frac{r}{\rho}(r - \rho\cos\theta) - \frac{r^2-\rho^2}{\rho}$$

$$(\rho - r\cos\theta)\frac{K_1(gt)}{t} = \frac{r}{\rho}(r-\rho\cos\theta)\frac{K_1(gt)}{t} - \frac{r^2-\rho^2}{\rho}\,\frac{K_1(gt)}{t}$$

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The formula (III,27) (page XXIII [of foreign text]), in which t is replaced by gt, becomes:

$$(r - \rho\cos\theta)\frac{K_1(gt)}{t} =$$

$$\frac{1}{2}\sum_{m=0}^{\infty} \varepsilon_m \left[K_{m+1}(gr) + K_{m-1}(gr)\right] I_m(g\rho)\cos m\theta$$

6.4.2. $(r^2-\rho^2)\frac{K_1(gt)}{t}$ is given by formula (6.13) of the preceding page:

$$(\rho - r\cos\theta)\frac{K_1(gt)}{t} =$$

$$\frac{r}{2\rho}\sum_{m=0}^{\infty} \varepsilon_m \left[K_{m+1}(gr) + K_{m-1}(gr)\right] I_m(g\rho)\cos m\theta$$

$$-\frac{r}{\rho}\sum_{m=0}^{\infty}\varepsilon_m K_{m+1}(qr)\, I_m(q\rho)\cos m\theta$$

$$-\sum_{m=0}^{\infty}\varepsilon_m K_m(qr)\, I_{m+1}(q\rho)\cos m\theta$$

and, after simplification:

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$$(\rho-2\cos\theta)\frac{K_1(qt)}{t}=$$

$$-\frac{r}{2\rho}\sum_{m=0}^{\infty}\varepsilon_m\left[K_{m+1}(qr)-K_{m-1}(qr)\right] I_m(q\rho)\cos m\theta$$

$$-\sum_{m=0}^{\infty}\varepsilon_m K_m(qr)\, I_{m+1}(q\rho)\cos m\theta$$

taking into consideration (III,20) and (III,15) (page XXI [of foreign text]):

$$K_{m+1}(qr)-K_{m-1}(qr)=\frac{2m}{qr}K_m(qr)$$

$$I_{m+1}(q\rho)=I_{m-1}(q\rho)-\frac{2m}{q\rho}I_m(q\rho)$$

and:

$$(\rho - r\cos\theta)\frac{K_1(gt)}{t} =$$

$$\boxed{-\frac{1}{r}\sum_{m=0}^{\infty} \varepsilon_m \left[I_{m-1}(g\rho) + I_{m+1}(g\rho) \right] K_m(gr) \cos m\theta} \tag{6.14}$$

$$\varepsilon_m = 2 \quad \text{if } m > 0 \qquad \varepsilon_0 = 1$$ /28

6.5. Development of $\Phi_d(g,r,\theta)$ Valid When $g>0$

(6.6) (page 19 [of foreign text]) can be written:

$$\Phi_d(g,r,\theta) =$$

$$\frac{\mu g}{\pi}\left[i\alpha K_0(gt) + \beta \frac{r}{\rho}\rho \sin\theta \frac{K_1(gt)}{t} - \gamma(\rho - r\cos\theta)\frac{K_1(gt)}{t} \right]$$

by replacing t, 2, ρ by gt, g2, gρ in the formulas (III,26) and (III,28) (pages XXII and XXIII [of foreign text]), it follows that:

$$K_0(gt) = \sum_{m=0}^{\infty} \varepsilon_m K_m(gr) I_m(g\rho) \cos m\theta$$

$$\rho \sin\theta \, \frac{K_1(gt)}{t} = \frac{2}{gr} \sum_{m=0}^{\infty} m \, K_m(gr) \, I_m(g\rho) \sin m\theta$$

These two formulas joined together with (6.14) of the preceding page, allow making explicit $\Phi_{01}(g,2,\theta)$: /29

$$\boxed{\Phi_d(g,r,\theta) = \frac{\mu g}{\pi} \sum_{m=0}^{\infty} \varepsilon_m \left\{ i\alpha I_m(g\rho) + \frac{\gamma}{2} \left[I_{m-1}(g\rho) + I_{m+1}(g\rho) \right] \right\} K_m(gr) \cos m\theta + \frac{2\mu}{\pi} \frac{\beta}{\rho} \sum_{m=0}^{\infty} m \, K_m(gr) \, I_m(g\rho) \sin m\theta} \tag{6.15}$$

and, by again taking up the notations of paragraph 5.2.2 (page 14 [of foreign text]), or:

$$\Phi_d(g,r,\theta) = \sum_{m=0}^{\infty} \left[\varepsilon_m M_m(g,r) \cos m\theta + 2 N_m(g,r) \sin m\theta \right]$$

$$\boxed{M_m(g,r) = \frac{\mu g}{\pi} \left\{ i\alpha I_m(g\rho) + \frac{\gamma}{2} \left[I_{m-1}(g\rho) + I_{m+1}(g\rho) \right] \right\} K_m(gr)} \tag{6.16}$$

$$\boxed{N_m(g,r) = \frac{\mu\beta}{\pi\rho} m \, K_m(gr) \, I_m(g\rho) \quad \Big| \quad g > 0} \tag{6.17}$$

6.6. Development of $\Phi_d(g,2,d)$ Valid When $g \gtrless 0$ /30

6.6.1. When $g>0$

$$E_1 = 2\int_0^\infty (x^2+t^2)^{-\frac{3}{2}} \cos gx \, dx = \frac{2g}{t^2} K_1(gt)$$

(see (6.4), page 18 [of foreign text]).

Since the integral above gives a real result, E_1 is real, as is $K_1(gt)$. gt is therefore positive (see Annex III, paragraph 1.4, page XVIII [of foreign text]). Since E_1 is furthermore an even function of g, its expression, valid when y has any sign whatever, is:

$$\boxed{E_1 = \frac{2|g|}{t^2} K_1(|g|t)}$$

6.6.2. Likewise, when $g>0$

$$E_2 = 2i\int_0^\infty (x^2+t^2)^{-\frac{3}{2}} x \sin gx \, dx = 2ig K_0(gt)$$

(see (6.5), page 19 [of foreign text]).

A reasoning similar to the preceding one establishes that E_2 is a /31
pure imaginary function: $K_0(gt)$ should be real, hence gt should be positive. On the other hand, E_2 is an uneven function of y:

$$\boxed{E_2 = 2ig K_0(|g|t)}$$

6.6.3. $\Phi_{01}(g,2,\theta)$ is therefore written as follows when

$$\boxed{\bar{\Phi}_d(g,r,\theta) = \frac{\mu g}{\pi}\left\{i\alpha K_0(|g|t) + [\beta r \sin\theta - \gamma(\rho - r\cos\theta)]\frac{|g|}{g}\frac{K_1(|g|t)}{t}\right\}}$$

(6.18)

and

$$\boxed{\begin{aligned} M_m &= \frac{\mu g}{\pi}\left\{i\alpha I_m(|g|\rho) + \frac{\gamma}{2}\frac{|g|}{g}\left[I_{m-1}(|g|\rho) + I_{m+1}(|g|\rho)\right]\right\}K_m(|g|r) \\ N_m &= \frac{\mu\beta}{\pi\rho} m\, I_m(|g|\rho)\, K_m(|g|r) \end{aligned}}$$

(6.19)

7. Calculation of the Complementary Potential $\varphi_c(X,2,\theta)$ (Part 1) /32

7.1. Determination of $\varphi_c(X,2,\theta)$, Special Solution of the Laplace Equation

7.1.1. The formulas (5.2) (page 13 [of foreign text]) and (5.5) (page 14 [of foreign text]) allow obtaining:

$$\boxed{\varphi_c(x,r,\theta) = \int_{-\infty}^{+\infty}\left\{\sum_{m=0}^{\infty}\left[\varepsilon_m A_m(g,r)\cos m\theta + 2B_m(g,r)\sin m\theta\right]\right\}e^{-igx}\,dg}$$

(7.1)

whence it follows that:

$$\frac{\partial \varphi_c}{\partial r} = \int_{-\infty}^{+\infty} \left\{ \sum_{m=0}^{\infty} \left[\varepsilon_m \frac{\partial A_m}{\partial r} \cos m\theta + i \frac{\partial B_m}{\partial r} \sin m\theta \right] \right\} e^{-i\gamma x} d\gamma$$

$$\frac{\partial^2 \varphi_c}{\partial r^2} = \int_{-\infty}^{+\infty} \left\{ \sum_{m=0}^{\infty} \left[\varepsilon_m \frac{\partial^2 A_m}{\partial r^2} \cos m\theta + i \frac{\partial^2 B_m}{\partial r^2} \sin m\theta \right] \right\} e^{-i\gamma x} d\gamma$$

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$$\frac{\partial^2 \varphi_c}{\partial \theta^2} = - \int_{-\infty}^{+\infty} \left\{ \sum_{m=0}^{\infty} \left[m^2 (\varepsilon_m A_m \cos m\theta + i B_m \sin m\theta) \right] \right\} e^{-i\gamma x} d\gamma$$

$$\frac{\partial^2 \varphi_c}{\partial x^2} = - \int_{-\infty}^{+\infty} \left\{ \sum_{m=0}^{\infty} (\varepsilon_m A_m \cos m\theta + i B_m \sin m\theta) \right\} \gamma^2 e^{-i\gamma x} d\gamma$$

Taking into consideration these values, the equation with partial derivatives (4.4) (page 12 [of foreign text]) becomes:

$$\int_{-\infty}^{+\infty} \left\{ \sum_{m=0}^{\infty} \left[\varepsilon_m \left(r^2 \frac{\partial^2 A_m}{\partial r^2} + r \frac{\partial A_m}{\partial r} - (m^2 + r^2 \gamma^2) A_m \right) \cos m\theta \right. \right.$$

$$\left. \left. + i \left(r^2 \frac{\partial^2 B_m}{\partial r^2} + r \frac{\partial B_m}{\partial r} - (m^2 + r^2 \gamma^2) B_m \right) \sin m\theta \right] \right\} e^{-i\gamma x} d\gamma = 0$$

An equation which will be satisfied if:

$$\begin{cases} r^2 \dfrac{\partial^2 A_m}{\partial r^2} + r \dfrac{\partial A_m}{\partial r} - (m^2 + r^2 g^2) A_m = 0 \\ r^2 \dfrac{\partial^2 B_m}{\partial r^2} + r \dfrac{\partial B_m}{\partial r} - (m^2 + r^2 g^2) B_m = 0 \end{cases}$$

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The partial derivatives of Am(y,2) and Bm(g,2) being taken exclusively with respect to the variable r, the two equations with the preceding partial derivatives are in reality differential equations in which r is the variable and g one parameter.

Am(y,2) and Bm(g,2) are therefore solutions of the equation:

$$\boxed{r^2 \frac{d^2 y}{dr^2} + r \frac{dy}{dr} - (m^2 + r^2 g^2) y = 0} \qquad (7.2)$$

7.1.2. <u>The Changing of Variable</u> $u = rg$

allows simplification of equation (7.2) and closely connecting it to a known type. Indeed:

$$\frac{dy}{dr} = g \frac{du}{dr} \qquad \frac{d^2 y}{dr^2} = g^2 \frac{d^2 u}{dr^2}$$

and

$$\boxed{u^2 \frac{d^2 y}{du^2} + u \frac{dy}{du} - (m^2 + u^2) y = 0} \qquad (7.3)$$

This equation is the modified Bessel equation (see Annex III). Its general solution is:

$$\boxed{y = C_1 I_m(u) + C_2 K_m(u)}$$

in which c_1 and c_2 are independent from the variable u=gr, but are a function of g; hence:

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$$\begin{cases} A_m(g,r) = a_m(g)\, I_m(gr) + c_m(g)\, K_m(gr) \\ B_m(g,r) = b_m(g)\, I_m(gr) + d_m(g)\, K_m(gr) \end{cases}$$

7.1.3. Simplification of Am(g,2) and Bm(y,2)

The function $\varphi_c(X,2,\theta)$ should have a finite value in the case of $x \leq R$. More particularly, it should be bounded on the axis of the cylinder $\mathcal{C}$ (r=0). Since the function Km(y,2) is infinite in the case of r=0 (see Annex III, page XX [of foreign text]), its presence in Am and Bm introduces a singularity in the expression of $\varphi_c(X,2,\theta)$. It therefore should follow:

$$c_m(g) = d_m(g) = 0$$

and

$$\begin{cases} A_m(g,r) = a_m(g)\, I_m(gr) & (7.4) \\ B_m(g,r) = b_m(g)\, I_m(gr) & (7.5) \end{cases}$$

7.1.4. By carrying these two values into (7.1) (page 32 [of foreign text]), it follows that:

$$\varphi_c(x,r,\theta) =$$

$$\boxed{\int_{-\infty}^{+\infty} \sum_{m=0}^{\infty} \left[\varepsilon_m a_m(g)\cos m\theta + 2 b_m(g)\sin m\theta\right] I_m(gr)\, e^{-igx}\, dg} \quad (7.6)$$

and, by granting,

$$\mathcal{A}_m(r) = \int_{-\infty}^{+\infty} A_m(g,r)\, e^{-igx}\, dg = \int_{-\infty}^{+\infty} a_m(g)\, I_m(gr)\, e^{-igx}\, dg \quad (7.7)$$

$$\mathcal{B}_m(r) = \int_{-\infty}^{+\infty} B_m(g,r)\, e^{-igx}\, dg = \int_{-\infty}^{+\infty} b_m(g)\, I_m(gr)\, e^{-igx}\, dg \quad (7.8)$$

$$\boxed{\varphi_c(x,r,\theta) = \sum_{m=0}^{\infty} \left[\varepsilon_m \mathcal{A}_m(r)\cos m\theta + 2\, \mathcal{B}_m(r)\sin m\theta\right]} \quad (7.9)$$

7.1.5. Transformation of Am(2) and Bm(2):

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$$A_m = \int_0^\infty a_m(g)\, I_m(gr)\, e^{-igx}\, dg + \int_0^\infty a_m(-g)\, I_m(-gr)\, e^{igx}\, dg$$

since $e^{\pm igx} = \cos gx \pm i \sin gx$.

$$A_m = \int_0^\infty \left[a_m(g)\, I_m(gr) + a_m(-g)\, I_m(-gr)\right] \cos gx\, dg$$
$$-i \int_0^\infty \left[a_m(g)\, I_m(gr) - a_m(-g)\, I_m(-gr)\right] \sin gx\, dg \qquad (7.10)$$

likewise by changing am(g) into bm(g):

$$B_m = \int_0^\infty \left[b_m(g)\, I_m(gr) + b_m(-g)\, I_m(-gr)\right] \cos gx\, dg \qquad (7.11)$$
$$-i \int_0^\infty \left[b_m g\, I_m(gr) - b_m(-g)\, I_m(-gr)\right] \sin gx\, dg.$$

7.1.6. Calculation of am(y) and bm(y): /38

The condition at the boundaries (4.5) (page 12 [of foreign text]):

$$\left[\frac{\partial}{\partial r}\varphi_c(x,r,\theta)\right]_{r=R} = -\left[\frac{\partial}{\partial r}\varphi_d(x,r,\theta)\right]_{r=R}$$

can be rewritten, taking into account equalities (5.2) and (5.4) (page 13 [of foreign text]):

$$\left[\frac{\partial}{\partial r}\int_{-\infty}^{+\infty}\Phi_c(g,r,\theta)\,e^{-igx}\,dg\right]_{r=R} = -\left[\frac{\partial}{\partial r}\int_{-\infty}^{+\infty}\Phi_d(g,r,\theta)\,e^{-igx}\,dg\right]_{r=R}$$

or:

$$\left[\frac{\partial}{\partial r}\Phi_c(g,r,\theta)\right]_{r=R} = -\left[\frac{\partial}{\partial r}\Phi_d(g,r,\theta)\right]_{r=R}.$$

and, by replacing Φ_c and Φ_d by their developments as Fourier series (see (5.5) and (5.6), page 14 [of foreign text]):

$$\left[\frac{\partial}{\partial r}A_m(g,r)\right]_{r=R} = -\left[\frac{\partial}{\partial r}M_m(g,r)\right]_{r=R}$$

$$\left[\frac{\partial}{\partial r}B_m(g,r)\right]_{r=R} = -\left[\frac{\partial}{\partial r}N_m(g,r)\right]_{r=R}.$$

Since /39

$$\begin{cases} A_m(g,r) = a_m(g)\,I_m(qr) \\ B_m(g,r) = b_m(g)\,I_m(qr) \end{cases}$$

$$M_m(g,r) =$$

$$\frac{\mu}{\pi} g \left\{ i\alpha I_m(|g|\rho) + \frac{\gamma}{2} \frac{|g|}{g} \left[I_{m-1}(|g|\rho) + I_{m+1}(|g|\rho) \right] \right\} K_m(|g|r)$$

$$N_m(g,r) = \frac{\mu \Delta}{\pi \rho} m I_m(|g|\rho) K_m(|g|r)$$

see (6.19), page 31 [of foreign text]

$$\begin{cases} \dfrac{\partial I_m(gr)}{\partial r} = \dfrac{g}{2} \left[I_{m-1}(gr) + I_{m+1}(gr) \right] \\ \dfrac{\partial K_m(|g|r)}{\partial r} = -\dfrac{|g|}{2} \left[K_{m-1}(|g|r) + K_{m+1}(|g|r) \right] \end{cases}$$

see (III,18), page XXI [of foreign text]

see (III,23), page XXI [of foreign text]

am(g) and bm(g) will be expressed as follows:

$$a_m(g) =$$

$$\frac{\mu |g|}{\pi} \left\{ i\alpha I_m(|g|\rho) + \frac{\gamma}{2} \frac{|g|}{g} \left[I_{m-1}(|g|\rho) + I_{m+1}(|g|\rho) \right] \right\} \frac{K_{m-1}(|g|R) + K_{m+1}(|g|R)}{I_{m-1}(gR) + I_{m+1}(gR)}$$

$$b_m(g) = \frac{\mu}{\pi} \frac{\Delta m}{\rho} \frac{|g|}{g} I_m(|g|\rho) \frac{K_{m-1}(|g|R) + K_{m+1}(|g|R)}{I_{m-1}(gR) + I_{m+1}(gR)} \qquad (7.12)$$

7.1.7. Calculation of Am and Bm /40

m, on one hand, m-1 and m+1, on the other hand, being of different parities:

$$\frac{I_m(gr)}{I_{m-1}(gR)+I_{m+1}(gR)} = -\frac{I_m(-gr)}{I_{m-1}(-gR)+I_{m+1}(-gR)}$$

(see Annex III, paragraph 2.2.2, page XX [of foreign text]).

Therefore

$$a_m(g)\,I_m(gr) =$$

$$+\frac{\mu}{\pi}|g|\left\{i\alpha\, I_m(|g|\rho)+\frac{\gamma}{2}\frac{|g|}{g}\left[I_{m-1}(|g|\rho)+I_{m+1}(|g|\rho)\right]\right\} I_m(gr)\,\frac{K_{m-1}(|g|R)+K_{m+1}(|g|R)}{I_{m-1}(gR)+I_{m+1}(gR)}$$

$$a_m(-g)\,I_m(-gr) =$$

$$-\frac{\mu}{\pi}|g|\left\{i\alpha\, I_m(|g|\rho)-\frac{\gamma}{2}\frac{|g|}{g}\left[I_{m-1}(|g|\rho)+I_{m+1}(|g|\rho)\right]\right\} I_m(gr)\,\frac{K_{m-1}(|g|R)+K_{m+1}(|g|R)}{I_{m-1}(gR)+I_{m+1}(gR)}$$

and, since

$$\frac{|g|}{g} = g$$

$$a_m(g)\, I_m(gr) + a_m(-g)\, I_m(-gr) =$$

$$\frac{\mu\gamma}{\pi}\, g\, \left[I_{m-1}(|g|\rho) + I_{m+1}(|g|\rho)\right] I_m(gr)\, \frac{K_{m-1}(|g|R) + K_{m+1}(|g|R)}{I_{m-1}(gR) + I_{m+1}(gR)}$$

$$a_m(g)\, I_m(gr) - a_m(-g)\, I_m(-gr) =$$

$$\frac{2i\mu\alpha}{\pi}\, |g|\, I_m(|g|\rho)\, I_m(gr)\, \frac{K_{m-1}(|g|R) + K_{m+1}(|g|R)}{I_{m-1}(gR) + I_{m+1}(|g|R)}$$

(7.13)

By carrying these values into (7.10) and (7.11), it follows that: /41

$$A_m =$$

$$\frac{\mu\gamma}{\pi} \int_0^\infty \left[I_{m-1}(g\rho) + I_{m+1}(g\rho)\right] I_m(gr)\, \frac{K_{m-1}(gR) + K_{m+1}(gR)}{I_{m-1}(gR) + I_{m+1}(gR)}\, g \cos gx\, dg$$

$$+ \frac{2\mu\alpha}{\pi} \int_0^\infty I_m(g\rho)\, I_m(gr)\, \frac{K_{m-1}(gR) + K_{m+1}(gR)}{I_{m-1}(gR) + I_{m+1}(gR)}\, g \sin gx\, dg$$

(7.14)

Likewise:

$$b_m(g)\, I_m(gr) =$$

$$\frac{\mu\beta}{\pi}\,\frac{m}{\rho}\,\frac{|g|}{g}\; I_m(|g|\rho)\; I_m(gr)\; \frac{K_{m-1}(|g|R) + K_{m+1}(|g|R)}{I_{m-1}(gR) + I_{m+1}(gR)}$$

$$b_m(-g)\, I_m(-gr) = b_m(g)\, I_m(gr)$$

$$\boxed{\mathcal{B}_m = \frac{2\mu\beta}{\pi}\,\frac{m}{\rho}\int_0^\infty I_m(g\rho)\, I_m(gr)\, \frac{K_{m-1}(gR) + K_{m+1}(gR)}{I_{m-1}(gR) + I_{m+1}(gR)}\, \cos gx\, dg} \qquad (7.15)$$

7.1.8. Changing of Notations /42

The new variable σ and the new parameters u, v, w will be defined by:

$$\sigma = gR \qquad g = \frac{\sigma}{R} \qquad dg = \frac{d\sigma}{R}$$

$$\begin{cases} u = \operatorname{Arctg}\dfrac{x}{R} \\ v = \dfrac{r}{2R} \\ w = \dfrac{\rho}{2R} \end{cases} \qquad \begin{cases} gx = \dfrac{\sigma}{R}\, R\,\operatorname{tg} u = \sigma\,\operatorname{tg} u \\ gr = \dfrac{\sigma}{R}\, 2Rv = 2\sigma v \\ g\rho = \dfrac{\sigma}{R}\, 2Rw = 2\sigma w \end{cases}$$

With these new notations Am and Bm are written:

$$\mathcal{A}_m =$$

$$\frac{\mu\gamma}{\pi R^2}\int_0^\infty I_m(2\sigma v)\left[I_{m-1}(2\sigma w)+I_{m+1}(2\sigma w)\right]\frac{K_{m-1}(\sigma)+K_{m+1}(\sigma)}{I_{m-1}(\sigma)+I_{m+1}(\sigma)}\,\sigma\cos(\sigma\,\mathrm{tg}\,u)\,d\sigma$$

$$+\frac{2\mu\alpha}{\pi R^2}\int_0^\infty I_m(2\sigma v)\,I_m(2\sigma w)\,\frac{K_{m-1}(\sigma)+K_{m+1}(\sigma)}{I_{m-1}(\sigma)+I_{m+1}(\sigma)}\,\sigma\sin(\sigma\,\mathrm{tg}\,u)\,d\sigma$$

$$\mathcal{B}_m =$$

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$$\frac{2\mu\beta m}{\pi\rho R}\int_0^\infty I_m(2\sigma v)\,I_m(2\sigma w)\,\frac{K_{m-1}(\sigma)+K_{m+1}(\sigma)}{I_{m-1}(\sigma)+I_{m+1}(\sigma)}\cos(\sigma\,\mathrm{tg}\,u)\,d\sigma$$

Then, by granting

$$E_c(u, v, w, m, m_1, h) =$$

$$\boxed{\int_0^\infty I_m(2\sigma v)\,I_{m_1}(2\sigma w)\,\frac{K_{m-1}(\sigma)+K_{m+1}(\sigma)}{I_{m-1}(\sigma)+I_{m+1}(\sigma)}\,\sigma^h\cos(\sigma\,\mathrm{tg}\,u)\,d\sigma}$$

$$E_s(u, v, w, m, m_1, h) = \int_0^\infty I_m(2\sigma v)\, I_{m_1}(2\sigma w)\, \frac{K_{m-1}(\sigma) + K_{m+1}(\sigma)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)}\, \sigma^h \sin(\sigma \operatorname{tg} u)\, d\sigma$$

(7.16)

$$A_m = \frac{\mu\gamma}{\pi R^2}\left[E_c(u, v, w, m, m-1, 1) + E_c(u, v, w, m, m+1, 1)\right] + \frac{2\mu d}{\pi R^2} E_s(u, v, w, m, m, 1)$$

$$B_m = \frac{2\mu\beta m}{\pi\rho R} E_c(u, v, w, m, m, 0)$$

(7.17)

8. Calculation of the Integral /44

$$E(u, v, w, m, m_1, h) = \int_0^\infty I_m(2\sigma v)\, I_{m_1}(2\sigma w)\, \frac{K_{m-1}(\sigma) + K_{m+1}(\sigma)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)}\, \sigma^h e^{-i\sigma \operatorname{tg} u}\, d\sigma$$

8.1. Calculation of $\boxed{\dfrac{I_m(2\sigma v)\, I_{m_1}(2\sigma w)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)}}$

m is a positive whole number, $m_1 = m-1$, m or m+1.

8.1.1. Calculation of $I_m(2\sigma v)\, I_{m_1}(2\sigma w)$

The Nielsen formula (see (III,31), page XXIV [of foreign text]) gives:

$$J_m(qr)\, J_{m_1}(q\rho) =$$

$$\frac{1}{m!}\left(\frac{qr}{2}\right)^m \left(\frac{q\rho}{2}\right)^{|m_1|} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu\, F(-\mu, -\mu-|m_1|, m+1, r^2/\rho^2)}{\mu!\,(\mu+|m_1|)!} \left(\frac{q\rho}{2}\right)^{2\mu}$$

since

$$J_m(iqx) = i^m I_m(qx)$$
$$(-1)^\mu i^{2\mu} = (-i^2)^\mu = 1$$

(see III,5), page XVI [of foreign text]).

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$$F\left(-p, -p-|m_1|, m+1, r^2/\rho^2\right) =$$

$$p!\,(p+|m_1|)!\,m!\,\sum_{n=0}^{p} W(p, n, m, |m_1|)\left(\frac{r}{\rho}\right)^{2n}$$

(see (II,8), page VIII [of foreign text]).

With:

$$W(p, n, m, |m_1|) = \frac{1}{(p-n)!\,(p+|m_1|-n)!\,(m+n)!\,n!} \tag{8.1}$$

It follows that:

$$I_m(qr)\,I_{m_1}(q\rho) =$$

$$\left(\frac{qr}{2}\right)^m \left(\frac{q\rho}{2}\right)^{|m_1|} \sum_{p=0}^{\infty} \sum_{n=0}^{p} W(p, n, m, |m_1|)\left(\frac{r}{\rho}\right)^{2n} \left(\frac{q\rho}{2}\right)^{2p}.$$

Let, with the notations defined in paragraph 7.1.8 (page 42 [of foreign text]) and by granting in addition:

$$a_\mu(v, w, m, |m_1|) = \sum_{n=0}^{\mu} W(\mu, n, m, |m_1|)\, v^{m+2n}\, w^{|m_1|+2\mu-2n} \tag{8.1'}$$

$$\boxed{I_m(2\sigma v)\, I_{m_1}(2\sigma w) = \sum_{\mu=0}^{\infty} a_\mu(v, w, m, |m_1|)\, \sigma^{m+|m_1|+2\mu}} \tag{8.2}$$

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8.1.2. Development of $\left[I_{m-1}(\sigma) + I_{m+1}(\sigma)\right]^{-1}$ as a Whole Series of σ.

8.1.2.1.

$$I_{m-1}(\sigma) + I_{m+1}(\sigma) = 2\frac{d}{d\sigma} I_m(\sigma)$$

(see (III,16), page XXI [of foreign text]).

And

$$I_m(\sigma) = \sum_{k=0}^{\infty} \frac{1}{k!\,(m+k)!} \left(\frac{\sigma}{2}\right)^{m+2k}$$

(see (III,10), page XIX [of foreign text]).

$$\boxed{I_{m-1}(\sigma)+I_{m+1}(\sigma)=\left(\frac{\sigma}{2}\right)^{m-1}\sum_{k=0}^{\infty}\frac{m+2k}{4^k\,k!\,(m+k)!}\,\sigma^{2k}} \tag{8.3}$$

If it is granted:

$$d_k(m)=\frac{m+2k}{4^k\,k!\,(m+k)!}. \tag{8.4}$$

$$\left[\sum_{k=0}^{\infty} d_k(m)\,\sigma^{2k}\right]^{-1}=\sum_{\mu=0}^{\infty} b_\mu(m)\,\sigma^{2\mu} \tag{8.5}$$

(8.3) becomes: /47

$$\boxed{\left[I_{m-1}(\sigma)+I_{m+1}(\sigma)\right]^{-1}=\left(\frac{2}{\sigma}\right)^{m-1}\sum_{\mu=0}^{\infty} b_\mu(m)\,\sigma^{2\mu}.} \tag{8.6}$$

8.1.2.2. Calculation of $b_\mu(m)$

$b_\mu(m)$ is produced by the solution of a system of p linear equations with p unknowns and deduced from the equality:

$$(d_0+d_1\sigma^2+d_2\sigma^4+\cdots)(b_0+b_1\sigma^2+b_2\sigma^4+\cdots)=1$$

Or:

$$
\begin{aligned}
&d_0 b_0 + 0.b_1 + 0.b_2 + \cdots + 0.b_{p-1} + 0.b_p = 1 \\
&d_1 b_0 + d_0 b_1 + 0.b_2 + \cdots + 0.b_{p-1} + 0.b_p = 0, \\
&\text{- - - - - - - - - - - - - - - -} \\
&d_{p-1} b_0 + d_{p-2} b_1 + d_{p-3} b_2 + \cdots + d_0 b_{p-1} + 0.b_p = 0 \\
&d_p b_0 + d_{p-1} b_1 + d_{p-2} b_2 + \cdots + d_1 b_{p-1} + d_0 b_p = 0.
\end{aligned}
$$

whose solution gives:

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$$
b_p(m) = \frac{\begin{vmatrix} d_0 & 0 & 0 & \cdots & 0 & 1 \\ d_1 & d_0 & 0 & \cdots & 0 & 0 \\ d_2 & d_1 & d_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ d_{p-1} & d_{p-2} & d_{p-3} & \cdots & d_0 & 0 \\ d_p & d_{p-1} & d_{p-2} & \cdots & d_1 & 0 \end{vmatrix}}{\begin{vmatrix} d_0 & 0 & 0 & \cdots & 0 & 0 \\ d_1 & d_0 & 0 & \cdots & 0 & 0 \\ d_2 & d_1 & d_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ d_{p-1} & d_{p-2} & d_{p-3} & \cdots & d_0 & 0 \\ d_p & d_{p-1} & d_{p-2} & \cdots & d_1 & d_0 \end{vmatrix}}.
$$

Then, taking into account the obvious simplifications:

$$(-1)^{\mu} d_0^{\mu+1} b_\mu(m) = \begin{vmatrix} d_1 & d_0 & 0 & \cdots\cdots & 0 & 0 \\ d_2 & d_1 & d_0 & \cdots\cdots & 0 & 0 \\ d_3 & d_2 & d_1 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ d_{\mu-1} & d_{\mu-2} & d_{\mu-3} & \cdots\cdots & d_1 & d_0 \\ d_\mu & d_{\mu-1} & d_{\mu-2} & \cdots\cdots & d_2 & d_1 \end{vmatrix} \tag{8.7}$$

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8.1.3. The consolidation of (8.2), page 45 [of foreign text] and (8.6), page 47 [of foreign text] gives:

$$\frac{I_m(2\sigma v) I_{m_1}(2\sigma w)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)} =$$

$$2^{m-1} \sigma^{|m_1|+1} \left\{ \sum_{\mu=0}^{\infty} a_\mu(v, w, m, |m_1|) \sigma^{2\mu} \right\} \left\{ \sum_{\mu=0}^{\infty} b_\mu(m) \sigma^{2\mu} \right\}$$

or:

$$\boxed{\frac{I_m(2\sigma v) I_{m_1}(2\sigma w)}{I_{m-1}(\sigma) + I_{m+1}(\sigma)} = 2^{m-1} \sum_{q=0}^{\infty} c_q(v, w, m, |m_1|) \sigma^{2q+|m_1|+1}} \tag{8.8}$$

with:

$$\boxed{C_q(v,w,m,|m_1|) = \sum_{p=0}^{q} a_p(v,w,m,|m_1|)\, b_{q-p}(m)} \tag{8.9}$$

8.2. If it is granted: /50

$$\boxed{e_m(u,a) = \int_0^{\infty} e^{-i\sigma \operatorname{tg} u} \cdot K_m(\sigma)\, \sigma^{a-1}\, d\sigma} \tag{8.10}$$

E(u,v,w,m,m_1,h) becomes, taking into account (8.8), page 49 [of foreign text]:

$$\boxed{\begin{aligned} &E(u,v,w,m,m_1,h) = \\ &2^{m-1} \sum_{q=0}^{\infty} C_q(v,w,m,|m_1|) \Big[e_{m-1}(u, 2q+|m_1|+h+2) \\ &\qquad + e_{m+1}(u, 2q+|m_1|+h+2) \Big] \end{aligned}} \tag{8.11}$$

More particularly, taking into account (III,19), page XXI [of foreign text]:

$$E(u,v,w,0,m_1,h) = \sum_{q=0}^{\infty}{}' \, C_q(v,w,0,|m_1|)\, e_1(u, 2q+|m_1|+h+2)$$

and

$$E(u,v,w,1,m_1,h) =$$

$$\sum_{q=0}^{\infty}{}' \, C_q(v,w,1,|m_1|)\left[e_0(u, 2q+|m_1|+h+2) + e_2(u, 2q+|m_1|+h+2)\right]$$

8.3. Calculation of the Integral /51

$$\boxed{e_m(u,a) = \int_0^{\infty} e^{-i\sigma \operatorname{tg} u} K_m(\sigma)\, \sigma^{a-1}\, d\sigma}$$

8.3.1. In the formula (III,32), page XXV [of foreign text], the change of parameter:

$$\alpha = \operatorname{Arg\,sh}(\operatorname{tg} u) + i\frac{\pi}{2}$$

or:

$$\begin{cases} \operatorname{ch}\alpha = i \operatorname{tg} u \\ \operatorname{sh}\alpha = \dfrac{i}{\cos u} \end{cases}$$

gives:

$$e_m(u,a) =$$

$$\sqrt{\frac{\pi}{2}}\, \frac{\Gamma(a-m)\Gamma(a+m)}{\Gamma(\frac{1}{2}+a)} \left[\frac{i}{\cos u}\right]^{\frac{1}{2}-a} \left[\frac{1+i\operatorname{tg} u}{1-i\operatorname{tg} u}\right]^{\frac{1}{2}-\frac{a}{2}} F\left(\frac{1}{2}-m, \frac{1}{2}+m, \frac{1}{2}+a, \frac{1-i\operatorname{tg} u}{2}\right)$$

Since:

$$\left[\frac{1+i\,tg\,u}{1-i\,tg\,u}\right]^{\frac{1}{2}} = \left[\frac{e^{iu}}{e^{i(\pi-u)}}\right]^{\frac{1}{2}} = \frac{e^{iu}}{e^{i\frac{\pi}{2}}} = \frac{e^{iu}}{i}$$

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$$\left[\frac{i}{\cos u}\right]^{\frac{1}{2}-a} \left[\frac{1+i\,tg\,u}{1-i\,tg\,u}\right]^{\frac{1}{2}-\frac{a}{2}} = \left[\frac{i}{\cos u}\,\frac{e^{iu}}{i}\right]^{\frac{1}{2}-a} = \left[\frac{e^{iu}}{\cos u}\right]^{\frac{1}{2}-a}$$

and:

$$\boxed{e_m(u,a) = \sqrt{\frac{\pi}{2}}\,\frac{\Gamma(a-m)\,\Gamma(a+m)}{\Gamma\left(\frac{1}{2}+a\right)}\left[\frac{e^{iu}}{\cos u}\right]^{\frac{1}{2}-a} F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ \frac{1}{2}+a,\ \frac{1-i\,tg\,u}{2}\right)} \qquad (8.12)$$

8.3.2. x=Rtgu, tgu can therefore vary from $-\infty$ to $+\infty$, and the point:

$$z = \frac{1-i\,tg\,u}{2}$$

describes, in the complex plane z, the straight line Δ of the equation:

$$\Delta:\ R(z) = \frac{1}{2}$$

8.3.3. Function $F\left(\frac{1}{2}-m, \frac{1}{2}+m, \frac{1}{2}+a, \frac{1-i\,tg\,u}{2\sqrt{2}}\right)$ can, on the other hand, be developed as a series according to increasing powers of $z = \frac{1-i\,tg\,u}{2\sqrt{2}}$. This series is only absolutely convergent, notwithstanding the values of m and a, when:

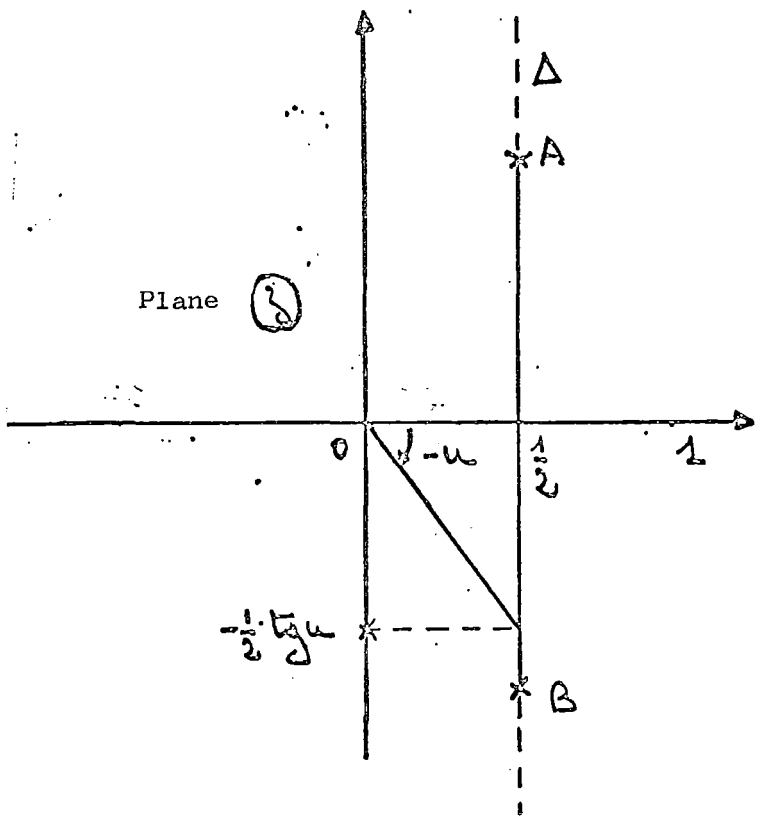
$$\left|\frac{1-i\,tg\,u}{2\sqrt{2}}\right| < 1$$ /53

(see Annex II, paragraph 1.2, page V [of foreign text]).

The point z should therefore be located on the segment AB of Δ inside the circle $|z| = 1$, from which it follows that:

$$|tg\,u| < \sqrt{3} \quad \text{or} \quad |u| < \frac{\pi}{3}$$

When $|u| > \pi/3$ (u can vary from $-\pi/2$ to $+\pi/2$), the development in the aforementioned series diverges. It is then necessary to carry out the analytical extension of $F\left(\frac{1}{2}-m, \frac{1}{2}+m, \frac{1}{2}+a, \frac{1-i\,tg\,u}{2\sqrt{2}}\right)$ of the domain $|z| < 1$ in the domain $|z| > 1$.



8.3.4. A simpler solution consists in setting up a consistent representation between plane z and a plane Z, using the transform:

$$\boxed{Z = \frac{z}{z-1}} \qquad (8.13)$$ /54

which causes correspondence of the straight line Δ (R(z)=1/2) of plane z and circle (c) of equation $|Z|=1$ of plane Z.

8.3.5. The first transform formula of Euler (II,12), page XII [of foreign text]:

$$F(\alpha,\beta,\gamma,z) = (1-z)^{-\alpha} F\left(\alpha,\gamma-\beta,\gamma,\frac{z}{z-1}\right)$$

is written:

$$\boxed{F\left(\frac{1}{2}-m,\frac{1}{2}+m,\frac{1}{2}+a,\frac{1-i\,tg\,u}{2}\right) = \left[\frac{e^{iu}}{2\cos u}\right]^{m-\frac{1}{2}} F\left(\frac{1}{2}-m,a-m,\frac{1}{2}+a,-e^{-2iu}\right)} \qquad (8.14)$$

for:

$$Z = \frac{1-i\,tg\,u}{2}\cdot\frac{1}{\frac{1-i\,tg\,u}{2}-1} = -\frac{1-i\,tg\,u}{1+i\,tg\,u} = -\frac{\cos u - i\sin u}{\cos u + i\sin u} = -e^{-2iu}$$

The function $F(\alpha,\beta,\gamma,Z)$ can then be developed as a series according to the increasing powers of $Z=-e^{-2iu}$. This series will be absolutely convergent, although $|Z|=1$, on the condition that: (see Annex II, paragraph 1.2, page V [of foreign text])

$$\boxed{\gamma-\alpha-\beta = a+\frac{1}{2}-\left(\frac{1}{2}-m\right)-(a-m) = 2m > 0.} \qquad (8.15)$$ /55

8.3.6. Case m=0

The function $F\left(\frac{1}{2}, a, \frac{1}{2}+a, -e^{-2iu}\right)$ can always be developed as a series. However, this series converges without converging absolutely. It is then impossible to modify the order of these terms so as to simplify summation for the sum produced will be a function of the order selected (see reference 1, volume 1, paragraph 19, page 37).

This difficulty can be surmounted by replacing the aforementioned F function by two of its contiguous functions (see Annex II, paragraph 2.4, page VIII [of foreign text]), which, for their part, can be developed as absolutely convergent series.

According to formula (II,15), page XIV [of foreign text],

$$\left(\frac{1}{2}-a\right) F\left(\frac{1}{2}, a, \frac{1}{2}+a, -e^{-2iu}\right) = \frac{1}{2} F\left(\frac{3}{2}, a, \frac{1}{2}+a, -e^{-2iu}\right) - a F\left(\frac{1}{2}, 1+a, \frac{1}{2}+a, -e^{-2iu}\right). \tag{8.16}$$

For the two functions appearing in the second member of (8.16): /56

$$\gamma-\alpha-\beta=-1$$

The third Euler transform formula (II,14), page XII [of foreign text]:

$$F(\alpha,\beta,\gamma,z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, z)$$

will allow replacement of these two functions by two others for which $\gamma-\alpha-\beta$ will be positive because equal to 1:

$$F\left(\frac{3}{2}, a, \frac{1}{2}+a, -e^{-2iu}\right) = \left(1+e^{-2iu}\right)^{-1} F\left(-1+a, \frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right)$$

$$F\left(\frac{1}{2}, 1+a, \frac{1}{2}+a, -e^{-2iu}\right) = \left(1+e^{-2iu}\right)^{-1} F\left(a, -\frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right)$$

and finally:

$$\boxed{F\left(\frac{1}{2}, a, \frac{1}{2}+a, -e^{-2iu}\right) = \frac{1}{2a-1}\,\frac{e^{iu}}{2\cos u}\left[2a\,F\left(a, -\frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right) - F\left(-1+a, \frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right)\right]} \tag{8.17}$$

8.3.7. Formula (8.13) on page 54 [of foreign text], taking (8.14) and (8.17) found on pages 54 and 56 respectively [of foreign text], becomes: /57

$$\boxed{e_m(u,a) = \sqrt{\pi}\,2^{-m}\,\frac{\Gamma(a-m)\,\Gamma(a+m)}{\Gamma\left(\frac{1}{2}+a\right)}\,(\cos u)^{a-m}\,e^{-i(a-m)u}\,F\left(\frac{1}{2}-m, a-m, \frac{1}{2}+a, -e^{-2iu}\right)}$$

$$\boxed{e_0(u,a) = \frac{\sqrt{\pi}}{2(2a-1)}\,\frac{\left[\Gamma(a)\right]^2}{\Gamma\left(\frac{1}{2}+a\right)}\,(\cos u)^{a-1}\,e^{-i(a-1)u}\left[2a\,F\left(a, -\frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right) - F\left(-1+a, \frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right)\right]}$$

Formula (II,7), page VII [of foreign text] allows development in series of the hypergeometric functions contained in the two preceding expressions.

By granting:

$$\mathcal{A}_\mu(a,b,c) = \frac{\Gamma\left(\frac{1}{2}-b+\mu\right)\Gamma(a-c+\mu)}{\Gamma\left(\frac{1}{2}+a+\mu\right)\quad \mu!}$$

it follows that:

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$$F\left(\frac{1}{2}-m, a-m, \frac{1}{2}+a, -e^{-2iu}\right) =$$

$$\frac{\Gamma\left(\frac{1}{2}+a\right)}{\Gamma\left(\frac{1}{2}-m\right)\Gamma(a-m)} \sum_{\mu=0}^{\infty}(-1)^\mu \mathcal{A}_\mu(a,m,m)\, e^{-2i\mu u}$$

$$F\left(a, -\frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right) =$$

$$\frac{\Gamma\left(\frac{1}{2}+a\right)}{\Gamma(a)\Gamma\left(-\frac{1}{2}\right)} \sum_{\mu=0}^{\infty}(-1)^\mu \mathcal{A}_\mu(a,-1,0)\, e^{-2i\mu u}$$

$$F\left(-1+a, \frac{1}{2}, \frac{1}{2}+a, -e^{-2iu}\right) =$$

$$\frac{\Gamma\left(\frac{1}{2}+a\right)}{\Gamma(a-1)\Gamma\left(\frac{1}{2}\right)} \sum_{\mu=0}^{\infty}(-1)^\mu \mathcal{A}_\mu(a,0,-1)\, e^{-2i\mu u}$$

and since:

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$ (see (I,7), page II [of foreign text]);

$$\Gamma\left(\frac{1}{2}-m\right) = \frac{(-1)^m \pi}{\Gamma\left(\frac{1}{2}+m\right)}$$ (see (I,8), page II [of foreign text]).

$$(a-1)\,\Gamma(a-1) = \Gamma(a)$$

em(u,a) and eo(u,a) are written:

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$$\boxed{\begin{aligned} e_m(u,a) &= \frac{(-2)^{-m}}{\sqrt{\pi}}\,\Gamma(a+m)\,\Gamma\left(\frac{1}{2}+m\right)(\cos u)^{a-m}\sum_{\mu=0}^{\infty}(-1)^{\mu}\,sp(a,m,m)\,e^{-i(2\mu+a-m)u} \\ e_o(u,a) &= \\ &\frac{\Gamma(a)}{2(2a-1)}(\cos u)^{a-1}\sum_{\mu=0}^{\infty}(-1)^{\mu+1}\left[a\,sp(a,-1,0)+(a-1)\,sp(a,0,-1)\right]e^{-i(2\mu+a-1)u} \end{aligned}}$$

but

$$(\cos u)^{a-m} = \left[\frac{1+e^{2iu}}{2e^{iu}}\right]^{a-m} = 2^{m-a}\,e^{-i(a-m)u}\sum_{q=0}^{a-m} C_{a-m}^{q}\,e^{2iqu}$$

Since C_{a-m}^{q} is the number of combinations of a-m objects taken q to q,

$$C_{a-m}^{q} = \frac{(a-m)!}{\mu!\,(a-m-\mu)!}$$

and

$$e_m(u,a) = \frac{(-1)^m 2^{-a}}{\sqrt{\pi}}$$

$$\Gamma\left(\frac{1}{2}+m\right)\Gamma(a+m)\, e^{-2i(a-m)u}\left(\sum_{q=0}^{a-m} C_{a-m}^{q}\, e^{2iqu}\right)\left(\sum_{\mu=0}^{\infty}(-1)^{\mu} s_\mu(a,m,m)\, e^{-2i\mu u}\right)$$

(8.19)

It is easy to confirm that: /60

$$\left(\sum_{q=0}^{n} a_q x^q\right)\left(\sum_{\mu=0}^{\infty} b_\mu x^{-\mu}\right) = \sum_{q=-n}^{+\infty}\ \sum_{\mu=\varepsilon_q q}^{n+q} a_{\mu-q}\, b_\mu\, x^{-q}$$

$$\begin{cases} \varepsilon_q = 1 & \text{si} \quad q > 0 \\ \varepsilon_q = 0 & \text{si} \quad q < 0 \end{cases}$$

Furthermore,

$$\Gamma\left(\frac{1}{2}+m\right) = \frac{\sqrt{\pi}}{4^m}\,\frac{(2m)!}{m!}$$

(see (I,6), page II [of foreign text]);

$$\Gamma(a+m) = (a+m-1)!$$

(see (I,5), page II [of foreign text)].

Therefore:

$$\boxed{e_m(u,a) = \sum_{\mu=m-a}^{+\infty} H_\mu(a,m)\, e^{-2i(\mu+a-m)u}} \tag{8.20}$$

Whence: /61

$$\boxed{H_\mu(a,m) = \frac{(-1)^m}{2^{a+2m}} \frac{(2m)!}{m!}(a+m-1)! \sum_{q=\varepsilon_\mu|\mu}^{a-m+\mu} (-1)^q\, C_{a-m}^{q-\mu}\, \Delta_q(a,m,m)} \tag{8.21}$$

Likewise:

$$\boxed{e_0(u,a) = \sum_{\mu=1-a}^{+\infty} H_\mu^0(a,0)\, e^{-2i(\mu+a-1)u}} \tag{8.22}$$

$$\boxed{H_\mu^0(a,0) = \frac{(a-1)!}{2^a(2a-1)} \sum_{q=\varepsilon_\mu|\mu}^{a-1+\mu} (-1)^{q+1}\, C_{a-1}^{q-\mu}\left[a\,\Delta_q(a,-1,0) + (a-1)\,\Delta_q(a,0,-1)\right]} \tag{8.23}$$

$$\begin{cases} \varepsilon_\mu = 1 & \text{si} \quad \mu > 0 \\ \varepsilon_\mu = 0 & \text{si} \quad \mu < 0 \end{cases}$$

9. Calculation of Complementary Potential $\varphi_c(X,2,\theta)$ (Part 2) /62

9.1. In summary, the different formulas allowing calculation of the complementary potential $\varphi_c(X,2,\theta)$ (case in which doublet D is placed as D_0. See paragraph 3.3, page 6 [of foreign text]) may be summarized as follows:

9.1.1.

$$\boxed{\varphi_c(x,r,\theta) = \sum_{m=0}^{\infty} \left[\varepsilon_m A_m \cos m\theta + 2 B_m \sin m\theta\right]} \tag{9.1}$$

$$(\varepsilon_m = 2 \text{ si } m > 0 \ , \ \varepsilon_0 = 1)$$

$$x = R \operatorname{tg} u \qquad r = 2Rv \qquad \rho = 2Rw \tag{9.2}$$

$$\begin{cases} A_m = \dfrac{\mu\gamma}{\pi R^2} \left[E_c(u,v,w,m,m-1,1) + E_c(u,v,w;m,m+1,1) \right. \\ \qquad + \dfrac{2\mu\alpha}{\pi R^2} E_A(u,v,w,m,m,1) \end{cases} \tag{9.3}$$

$$B_m = \frac{2\mu\beta m}{\pi\rho R} E_c(u,v,w,m,m,0) \tag{9.4}$$

(9.5)

$$\boxed{E_c(u,v,w,m,m_1,h) = 2^{m-1}\sum_{q=0}^{\infty} C_q(v,w,m,|m_1|)\left\{\sum_{\mu=|m-1|-a}^{+\infty} H_\mu(a,|m-1|)\cos 2(\mu+a-|m-1|)u + \sum_{\mu=m+1-a}^{+\infty} H_\mu(a,m+1)\cos 2(\mu+a-m-1)u\right\}}$$

Formula in which:

$$\boxed{a = 2q+|m_1|+h+2} \qquad (9.6)$$

In particular:

$$\boxed{E_c(u,v,w,0,m_1,h) = \sum_{q=0}^{+\infty}\sum_{\mu=1-a}^{+\infty} C_q(v,w,0,|m_1|)\,H_\mu(a,1)\cos 2(\mu+a-1)u}$$

(9.7)

and

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$$E_c(u,v,w,1,m_1,h) = \sum_{q=0}^{+\infty} C_q(v,w,1,|m_1|) \left\{ \sum_{\mu=1-a}^{+\infty} H^0_\mu(a,0) \cos l(\mu+a-1)u + \sum_{\mu=l-a}^{+\infty} H_\mu(a,l) \cos l(\mu+a-l)u \right\}$$

(9.8)

The values of E_s are produced by replacing the cosines by sines in E_c.

9.1.3.

$$\boxed{C_q(v,w,m,|m_1|) = \sum_{\mu=0}^{q} \sum_{n=0}^{\mu} \frac{v^{m+2n}\, w^{|m_1|+2\mu-2n}\, b_{q-\mu}(m)}{(\mu-n)!\,(\mu+|m_1|-n)!\,(m+n)!\,n!}}$$

(9.9)

$b_{p-m}(m)$ is defined by the relations:

$$b_0(m) = 1$$

and

(9.10)

$$\sum_{\mu=0}^{q} b_{q-\mu}(m)\, d_\mu(m) = 0 \qquad (m>0)$$

$$d_\mu(m) = \frac{m+2\mu}{4^\mu\,\mu!\,(m+\mu)!} \tag{9.11}$$ /65

9.1.4.

$$\boxed{H_\mu(a,m) = \frac{(-1)^m}{2^{a+2m}}\,\frac{(2m)!}{m!}\,(a+m-1)!\sum_{q=\varepsilon_\mu\mu}^{a-m+\mu}(-1)^q\,C_{a-m}^{q-\mu}\,A_q(a,m,m)} \tag{9.12}$$

$$\boxed{H_\mu^0(a,0) = \frac{(a-1)!}{2^a(2a-1)}\sum_{q=\varepsilon_\mu\mu}^{a+1-\mu}(-1)^{q+1}\,C_{a-1}^{q-\mu}\left[a\,A_q(a,-1,0)+(a-1)\,A_q(a,0,-1)\right]} \tag{9.13}$$

$$\begin{cases} \varepsilon_\mu = 1 & \text{si} \quad \mu > 0 \\ \varepsilon_\mu = 0 & \text{si} \quad \mu < 0. \end{cases}$$

$$C_{a-m}^{q-\mu} = \frac{(a-m)!}{(q-\mu)!\,(a-m+\mu-q)!} \tag{9.14}$$

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$$\boxed{A_q(a,b,c) = \frac{\Gamma\left(\frac{1}{2} - b + q\right)}{\Gamma\left(\frac{1}{2} + a + q\right)} \frac{(a-c+q-1)!}{q!}} \tag{9.15}$$

9.2. Expression of φ_c Valid for Any Position of Doublet D

(see paragraph 3.3, page 5 [of foreign text]).

In order to obtain this expression, it is enough to replace in the formulas of paragraph 9.1:

x	by	$x - x_D$
$u = \mathrm{Arctg}\,\frac{x}{R}$	by	$u' = \mathrm{Arctg}\,\frac{x - x_D}{R}$
θ	by	$\theta - \rho$

Paragraph 9.1 therefore becomes:

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$$\boxed{\varphi_c(x, r, \theta) = \varphi_c(x - x_D, r, \theta - \rho) = \sum_{m=0}^{\infty} \varepsilon_m A'_m \cos m(\theta - \rho) + B'_m \sin m(\theta - \rho)} \tag{9.17}$$

($\varepsilon_m = 2$ if $m > 0$; $\varepsilon_0 = 1$)

$$x - x_D = R\,\mathrm{tg}\,u' \qquad r = 2Rw \qquad \rho = 2Rw \tag{9.18}$$

$$\boxed{\begin{aligned} \mathcal{A}'_m &= \frac{\mu\gamma}{\pi R^2}\left[E_c(u',v,w,m,m-1,1)+E_c(u',v,w,m,m+1,1)\right] \\ &\quad + \frac{2\mu\alpha}{\pi R^2}\,E_s(u',v,w,m,m,1) \\ \mathcal{B}'_m &= \frac{2\mu\beta m}{\pi\rho R}\,E_c(u',v,w,m,m,0) \end{aligned}}$$

(9.19)

The other formulas (paragraphs 9.12 to 9.14) remain valid on condition that u'Au is substituted therein.

10. Calculation of Velocities Induced by Doublet D /68

10.1. These velocities are given by:

$$\Delta V_x(M) = \frac{\partial}{\partial x}\varphi_c(x-x_D, r, \theta-\rho)$$

$$\Delta V_r(M) = \frac{\partial}{\partial r}\varphi_c(x-x_D, r, \theta-\rho)$$

$$\Delta V_\theta(M) = \frac{1}{r}\frac{\partial}{\partial \theta}\varphi_c(x-x_D, r, \theta-\rho)$$

or:

$$\boxed{\Delta V_x(M) = \sum_{m=0}^{\infty} \varepsilon_m \frac{\partial}{\partial x}\mathcal{A}'_m \cos m(\theta-\rho) + 2\frac{\partial}{\partial x}\mathcal{B}'_m \sin m(\theta-\rho)}$$

$$\Delta V_r(M) = \sum_{m=0}^{\infty} \varepsilon_m \frac{\partial}{\partial r} A'_m \cos m(\theta-\delta) + 2\frac{\partial}{\partial r} B'_m \sin m(\theta-\delta)$$

$$\Delta V_\theta(M) = \frac{1}{r} \sum_{m=0}^{\infty} m\left[-\varepsilon_m A'_m \sin m(\theta-\delta) + 2 B'_m \cos m(\theta-\delta)\right]$$

(10.1)

Since: /69

$$x - x_D = R\,\mathrm{tg}\,u' \longrightarrow \frac{\partial u'}{\partial x} = \frac{R}{\cos^2 u'}$$

$$r = 2Rv \longrightarrow \frac{dv}{dr} = \frac{1}{2R}$$

$$\frac{\partial}{\partial x} = \frac{\partial u'}{\partial x} \cdot \frac{\partial}{\partial u'} = \frac{R}{\cos^2 u'} \frac{\partial}{\partial u'}$$

$$\frac{\partial}{\partial r} = \frac{dv}{dr} \cdot \frac{\partial}{\partial r} = \frac{1}{2R} \frac{\partial}{\partial v}$$

Therefore,

$$\frac{\partial}{\partial x} A'_m = \frac{\mu\gamma}{\pi R \cos^2 u'}\left[\frac{\partial}{\partial u'} E_c(u', v, w, m, m-1, 1) + \frac{\partial}{\partial u'} E_c(u', v, w, m, m+1, 1)\right] + \frac{2\mu\alpha}{\pi R \cos^2 u'} \frac{\partial}{\partial u'} E_s(u', v, w, m, m, 1)$$

(10.2)

$$\frac{\partial}{\partial x} B'_m = \frac{2\mu\beta m}{\pi\rho\cos^2 u'} \frac{\partial}{\partial u'} E_c(u', v, w, m, m, 0)$$

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$$\frac{\partial}{\partial z} A'_m =$$

$$\frac{\mu\gamma}{2\pi R^3}\left[\frac{\partial}{\partial v} E_c(u', v, w, m, m-1, 1) + \frac{\partial}{\partial v} E_c(u', v, w, m, m+1, 1)\right]$$

$$+ \frac{\mu\alpha}{\pi R^3} \frac{\partial}{\partial v} E_s(u', v, w, m, m, 1) \tag{10.3}$$

$$\frac{\partial}{\partial z} B'_m = \frac{\mu\beta m}{\pi\rho R^2} \frac{\partial}{\partial v} E_c(u', v, w, m, m, 0)$$

10.2. Transformation of $C_q(v, w, m, |m_1|)$

According to (8.1), (8.1'), page 45 [of foreign text] and (8.9), page 49 [of foreign text]:

$$C_q(v, w, m, |m_1|) = \sum_{\mu=0}^{q} \sum_{n=0}^{\mu} b_{n,\mu,q}(m, |m_1|)\, v^{m+2n}\, w^{|m_1|+2\mu-2n} \tag{10.4}$$

$$b_{n,p,q} = \frac{b_{q-p}(m)}{(p-n)!\,(p+|m_1|-n)!\,(m+n)!\,n!} \tag{10.5}$$

The series (10.4), giving $Cq(v,w,m,|m_1|)$ can be ordered according to the increasing powers of v, using the following identity: /71

$$\sum_{p=0}^{q}\sum_{n=0}^{p} b_{n,p,q}\, w^{2p-2n} v^{2n} = \sum_{p=0}^{q}\left\{\sum_{n=p}^{q} b_{p,n,q}\, w^{2(n-p)}\right\} v^{2p} \tag{10.6}$$

or:

$$\boxed{Cq(v,w;m,|m_1|) = \sum_{p=0}^{q} c_{p,q}(w;m,|m_1|)\, v^{m+2p}.} \tag{10.7}$$

$$\boxed{c_{p,q}(w;m,|m_1|) = \sum_{n=p}^{q} b_{p,n,q}(m,|m_1|)\, w^{|m_1|+2(n-p)}} \tag{10.8}$$

$b_{p,n,q}$ is deduced from $b_{n,p,q}$ (see 10.5) by permuting n and p.

$$b_{p,n,q}(m,|m_1|) = \frac{b_{q-n}(m)}{(n-p)!\,(n+|m_1|-p)!\,(m+p)!\,p!} \tag{10.9}$$

10.3. Calculation of $\frac{\partial}{\partial u'} E_c(u', v, w, m, m_1, h)$ /72

10.3.1. A mere modification of subscript allows formula (9.5), page 63 [of foreign text] to be written in the form:

$$\boxed{\begin{aligned} E_c(u', v, w, m, m_1, h) = 2^{m-1} \sum_{q=0}^{+\infty} \sum_{p=0}^{+\infty} C_q(v, w, m, |m_1|) \\ \left[H_{p-2q-b+|m-1|}(2q+b, |m-1|) + H_{p-2q-b+m+1}(2q+b, m+1)\right] \cos 2pu' \end{aligned}} \tag{10.10}$$

In which: $\boxed{b = a - 2q = |m_1| + h + 2}$ (10.11)

10.3.2. The calculation of $\frac{\partial}{\partial u'} E_c(u', v, w, m, m_1, h)$ gives immediately:

$$\boxed{\begin{aligned}\frac{\partial}{\partial u'} E_c(u', v, w, m, m_1, h) = & -2^m \sum_{q=0}^{+\infty} \sum_{\mu=0}^{+\infty} \mu\, C_q(v, w, m, |m_1|) \\ & \left[H_{\mu-2q-b+|m-1|}(2q+b, |m-1|) + H_{\mu-2q-b+m+1}(2q+b, m+1)\right] \sin 2\mu u'\end{aligned}}$$

(10.12)

In particular, /73

$$\boxed{\frac{\partial}{\partial u'} E_c(u', v, w, 0, m_1, h) = -2 \sum_{q=0}^{+\infty} \sum_{\mu=0}^{+\infty} \mu\, C_q(v, w, 0, |m_1|)\, H_{\mu-2q-b+1}(2q+b, 1) \sin 2\mu u'}$$

(10.13)

$$\boxed{\begin{aligned}\frac{\partial}{\partial u'} E_c(u', v, w, 1, m_1, h) = & -2 \sum_{q=0}^{+\infty} \sum_{\mu=0}^{+\infty} \mu\, C_q(v, w, 1, |m_1|) \\ & \left[H_{\mu-2q-b+1}(2q+b, 0) + H_{\mu-2q-b+2}(2q+b, 2)\right] \sin 2\mu u'\end{aligned}}$$

(10.14)

10.4. Calculation of $\frac{\partial}{\partial w'} E_s(u', v, w, m, m_1, h)$

The corresponding expressions are produced beginning from (10.11), (10.12) and (10.13) by changing the sign of the righthand side in these formulas and also replacing the sines by cosines.

10.5. Calculation of $\frac{\partial}{\partial v} E_c(u', v, w, m, m_1, h)$

It follows directly, beginning from (10.10), page 72 [of foreign text]: /74

$$\boxed{\begin{aligned}\frac{\partial}{\partial v} E_c(u', v, w, m, m_1, h) = 2^{m-1} \sum_{q=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{\partial}{\partial v} C_q(v, w, m, |m_1|) \\ \left[H_{\mu-2q-b+|m-1|}(2q+b, |m-1|) + H_{\mu-2q-b+|m+1|}(2q+b, |m+1|)\right] \cos 2\mu u'\end{aligned}} \tag{10.15}$$

$$\boxed{\frac{\partial}{\partial v} C_q(v, w, m, |m_1|) = \sum_{\mu=0}^{q} (m+2\mu)\, C_{\mu,q}(w; m, |m_1|)\, v^{m+2\mu-1}} \tag{10.16}$$

and

$$\boxed{\frac{\partial}{\partial v} E_c(u', v, w, 0, m_1; h) = \sum_{q=0}^{+\infty}{}' \sum_{\mu=0}^{+\infty}{}' \frac{\partial}{\partial v} C_q(v, w, 0, |m_1|)\, H_{\mu-2q-b+1}(2q+b, 1) \cos 2\mu u'} \tag{10.17}$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial v} E_c(u', v, w, 1, m_1, h) = \sum_{q=0}^{+\infty}{}' \sum_{\mu=0}^{+\infty}{}' \frac{\partial}{\partial v} C_q(v, w, 1, |m_1|) \\ \left[H^{0}_{\mu-2q-b+1}(2q+b, 0) + H_{\mu-2q-b+2}(2q+b, 2)\right] \cos 2\mu u'\end{aligned}} \tag{10.18}$$

10.6. Calculation of $\frac{\partial}{\partial v} E_s(u', v, w, m, m_1, h)$ /75

It is enough, in order to obtain the above, to replace the cosines by sines in formulas (10.15), (10.17) and (10.18).

10.7. Comment with Respect to Numerical Calculations

Since the series show $e_m(u,a)$ ((8.20), page 60 [of foreign text]), and $e_o(u,a)$ ((8.22), page 61 [of foreign text]), as being absolutely convergent (see paragraph 8.3.3 through paragraph 8.3.6), it is allowed to modify the order of component terms.

Independently from the simplifications which will appear during numerical calculations for certain noteworthy values of u', this modification based on the periodicity of trigonometric lines will allow grouping the terms corresponding to one same value of the sine or cosine.

Indeed, it is easy to confirm that, when:

$$n = \frac{180}{\text{PGCD of 180 and of } u'} \qquad (u' \text{ in degrees}) \tag{10.19}$$

$$\boxed{\sum_{\mu=0}^{+\infty} \lambda_\mu e^{-2i\mu u'} = \sum_{\mu=0}^{n-1} \left\{ \sum_{k=0}^{+\infty} (-1)^k \lambda_{\mu+nk} \right\} e^{-2i\mu u'}} \tag{10.20}$$

In this way, by granting: /76

$$F_{\mu,q}(b,m,n) = \sum_{k=0}^{\infty} (-1)^k \left[H_{\mu+nk-2q-b+|m-1|}(2q+b, |m-1|) + H_{\mu+nk-2q-b+m+1}(2q+b, m+1) \right] \tag{10.21}$$

$$F^{o}_{\mu,q}(b,1,n) = \sum_{k=0}^{\infty} (-1)^k \left[H^{o}_{\mu+nk-2q-b+1}(2q+b, 0) + H_{\mu+nk-2q-b+2}(2q+b, 2) \right] \tag{10.22}$$

The preceding formulas can be replaced by:

$$\frac{\partial}{\partial u'} E_c(u', v, w, m, m_1, h) = -2^m \sum_{q=0}^{+\infty} \sum_{p=0}^{n-1} p\, C_q(v, w, m, |m_1|)\, F_{p,q}(b, m, n) \sin 2pu' \tag{10.23}$$

$$\frac{\partial}{\partial u'} E_c(u', v, w, 0, m_1, h) = -2 \sum_{q=0}^{\infty} \sum_{p=0}^{n-1} p\, C_q(v, w, 0, |m_1|)\, F_{p,q}(b, 0, n) \sin 2pu' \tag{10.24}$$

$$\frac{\partial}{\partial u'} E_c(u', v, w, 1, m_1, h) = -2 \sum_{q=0}^{\infty} \sum_{p=0}^{n-1} p\, C_q(v, w, 1, |m_1|)\, F^{\circ}_{p,q}(b, 1, n) \sin 2pu' \tag{10.25}$$

and:

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$$\frac{\partial}{\partial v} E_c(u', v, w, m, m_1, h) = 2^{m-1} \sum_{q=0}^{\infty} \sum_{p=0}^{n-1} \frac{\partial}{\partial v} C_q(v, w, m, |m_1|)\, F_{p,q}(b, m, n) \cos 2pu' \tag{10.26}$$

$$\frac{\partial}{\partial v} E_c(u', v, w, 0, m_1, h) = \sum_{q=0}^{\infty} \sum_{\mu=0}^{n-1} \frac{\partial}{\partial v} C_q(v, w, 0, m, 1)\, F_{\mu,q}(b, 0, n) \cos 2\mu u' \tag{10.27}$$

$$\frac{\partial}{\partial v} E_c(u', v, w, 1, m_1, h) = \sum_{q=0}^{\infty} \sum_{\mu=0}^{n-1} \frac{\partial}{\partial v} C_q(v, w, 1, m, 1)\, F^{0}_{\mu,q}(b, 1, n) \cos 2\mu u' \tag{10.28}$$

These formulas are only advantageous when n is small. This is the case when it is possible to select values of u' for the purpose of studying the development of velocities induced as a function of abscissa X.

For example:

u'	15	18	20	30	36	40	45	54	60	72	75	80	90
$\frac{x-x_D}{R}$	0,268	0,325	0,364	0,577	0,727	0,839	1,000	1,376	1,732	3,078	3,732	5,671	∞
n	12	10	9	6	5	9	4	10	3	5	12	9	2

(10.29)

10.8. The second installment of this technical memorandum will be partially /78 devoted to the calculation of velocities induced by a continuous linear distribution of doublets. This calculation will be carried out beginning from the total potential of this distribution.

Since the knowledge of velocities induced by an isolated doublet is of no utility, the development will not be continued beyond the formulas that allow calculation of these velocities.

13 October 1967

P. Michel

Euler Functions

1. First Class Euler Functions: $\Gamma(x)$ (x is a real number)

1.1. $x>0$

1.1.1. $\Gamma(x)$ is defined by the integral

$$\boxed{\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du} \qquad (I,1)$$

if $x=1$:

$$\Gamma(1) = \int_0^\infty e^{-u} du = \left[-e^{-u}\right]_0^\infty = 1$$

1.1.2.

$$\boxed{\Gamma(x+1) = x\,\Gamma(x)} \qquad (I,2)$$

1.1.3. $0<x<1$.

$$\boxed{\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}} \qquad (I,3)$$

More particularly: /II

$$\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} \longrightarrow \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}} \qquad (I,4)$$

1.1.4. When n is a positive integer:

(I,2) becomes:

$$\boxed{\Gamma(n+1) = n!} \qquad (I,5)$$

and

$$\boxed{\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3\ldots(2n-1)\sqrt{\pi}}{2^n} = \frac{(2n)!}{4^n\, n!}\sqrt{\pi}} \qquad (I,6)$$

1.2. $\boxed{x<0}$

The integral (I,1) has no direction. Nevertheless, $\Gamma(x)$ can then be defined by the recursion formula (I,2), page I [of foreign text].

1.2.1. $\Gamma(x)$ is infinite for negative whole values of x and for x=0.

1.2.2. If n is a positive integer:

$$\boxed{\Gamma\left(\frac{1}{2}-n\right) = \frac{(-1)^n\, 2^{2n-1}\,(n-1)!}{(2n-1)!}\sqrt{\pi}} \qquad (I,7)$$

(I,6) and (I,7) give:

$$\boxed{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right) = (-1)^n\,\pi} \qquad (I,8)$$

2. Second Class Euler Functions: B(p,q) (p and q are real numbers) /III

2.1. B(p,q) is defined by the integral:

$$\boxed{B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\,du} \qquad (I,9)$$

This integral has only one direction when :

$$\begin{cases} p>0 \\ q>0 \end{cases}$$

2.2. The change of variable t=1-u shows that:

$$\underline{B(p,q) = B(q,p)} \qquad \text{(I,10)}$$

2.3. Recursion Formulas:

$$B(p,q) = B(p,q+1) + B(p+1,q) \qquad \text{(I,11)}$$

$$B(p+1,q) = \frac{p}{q} B(p,q+1) \qquad \text{(I,12)}$$

(I,11) and (I,12) give:

$$\boxed{B(p+1,q) = \frac{p}{p+q} B(p,q)} \qquad \text{(I,13)}$$

in which: /IV

$$\boxed{B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}} \qquad \text{(I,14)}$$

These two formulas allow decreasing the magnitude of p and q at the time of calculation of B(p,q). This calculation is performed more easily using formula (I,15):

2.4. B(p,q) can be expressed using function

$$\boxed{B(p,q+1) = \frac{q}{p+q} B(p,q)} \qquad \text{(I,15)}$$

Hypergeometric Gauss Functions: $F(\alpha,\beta,\gamma,z)$

1. The Hypergeometric Gauss Series: $F(\alpha,\beta,\gamma,z)$

1.1. This series may be expanded:

$$\boxed{F(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)\,1.2\dots n} z^n} \qquad \text{(II,1)}$$

α, β, γ are parameters, with the variable z. These four quantities can be real or complex.

1.1.1. $F(\alpha,\beta,\gamma,z)$ is not defined when γ is equal to zero or to a negative integer.

1.1.2. α and β play identical roles, hence:

$$F(\alpha,\beta,\gamma,z) = F(\beta,\alpha,\gamma,z) \qquad \text{(II,2)}$$

1.2. Convergence of the Series $F(\alpha,\beta,\gamma,z)$

The ratio of the coefficients of terms of rank nM and N may be written:

$$\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(1+n)}$$

This ratio tends toward 1 when n tends toward infinity. /VI

The hypergeometric series is therefore absolutely convergent when $|z|<1$.

This absolute convergence is preserved on the circle $|z|=1$, on the condition that the real part of $(\gamma-\alpha-\beta)$ is positive (see reference 1, Volume II, paragraph 112, page 238).

1.3. P. Appell introduced the symbol:

$$\boxed{(\mu,n) = \mu(\mu+1)\dots(\mu+n-1)} \qquad \text{(II,3)}$$

(see reference 4, page 1.)

n is a positive integer or zero.

1.3.1. The formula (I,2), page I [of foreign text] allows writing:

$$\Gamma(p+n) = (p+n-1)\Gamma(p+n-1) = (p+n-1)(p+n-2)\,\Gamma(p+n-2)$$
$$= (p+n-1)(p+n-2)\cdots(p+1)\,p\,.\,\Gamma(p)$$

or, taking into account (II,3):

$$\boxed{(p,n) = \frac{\Gamma(p+n)}{\Gamma(p)}} \qquad \text{(II,4)}$$

This formula is only valid when p is different from zero of from a negative integer.

1.3.2. When p is a positive integer:

1.3.2.1. (I,4), page II [of foreign text] gives: /VII

$$\boxed{(p,n) = \frac{(p+n-1)!}{(p-1)!}} \qquad \text{(II,5)}$$

More particularly:

$$(p,1) = \frac{p!}{(p-1)!} = p$$

$$(1,n) = n! \qquad \text{(since 0!=1 by convention)}$$

$$(p,0) = \frac{(p-1)!}{(p-1)!} = 1$$

1.3.2.2. Calculation of (-p,n)

$$(-p,n) = (-p)(-p+1)\cdots\cdots(-p+n-1)$$

It may be seen that:

$$(-p, n) = 0 \quad \text{if } n > p.$$

$$(-p, n) = (-1)^n (p+n-1)(p+n-2) \ldots (p-1)\, p$$
$$= (-1)^n (p-n+1, n)$$

Since: $(p-n+1, n) = \dfrac{p!}{(p-n)!}$

(see (II,5))

it follows that:

$$\boxed{(-p, n) = \frac{(-1)^n p!}{(p-n)!}} \qquad \text{(II,6)}$$

/VIII

1.4. Taking results from section 1.3 into consideration:

$$\boxed{F(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)(1, n)} z^n.}$$

1.4.1.

$$\boxed{F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)\, n!} z^n.} \qquad \text{(II,7)}$$

1.4.2. If $\alpha = -p$ (p is a positive integer)

$$\boxed{F(-p, \beta, \gamma, z) = p! \frac{\Gamma(\gamma)}{\Gamma(\beta)} \sum_{n=0}^{p} \frac{(-1)^n}{(p-n)!} \frac{\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{1}{n!} z^n}$$

n can indeed not be greater than p (see paragraph 1.3.2.2., page VII [of foreign text]), and the hypergeometric series is reduced to a polynomial of degree p in z.

1.4.3. If $\alpha=-p$ and $\beta=-y$, $p<y$ (p and q are positive integers)

$$\boxed{F(-p,-q,\gamma,z) = p!\,q!\,\Gamma(\gamma)\sum_{n=0}^{p}\frac{z^n}{(p-n)!\,(q-n)!\,\Gamma(\gamma+n)\,n!}} \qquad \text{(II,8)}$$

and if, in addition, $\gamma=r+1$ (r is an integer greater than zero) /IX

$$\boxed{F(-p,-q,r+1,z) = p!\,q!\,r!\sum_{n=0}^{p}\frac{z^n}{(p-n)!\,(q-n)!\,(r+n)!\,n!}} \qquad \text{(II,9)}$$

2. The Hypergeometric Gauss Function

2.1. Definition (see reference 4, page 2)

2.1.1. The analytical extension of the hypergeometric series $F(\alpha,\beta,\gamma,z)$ to the outside of its circle of convergence $|z|=1$, defines a new function likewise designated by the notation $F(\alpha,\beta,\gamma,z)$.

2.1.2. Since point $z=1$ is the only critical point with finite distance of this function, the latter can be made uniform in the whole plane of variable z on condition of suppressing in the latter the part of the real axis going from +1 to $+\infty$.

2.2. Representation of the Hypergeometric Function by a Clearly Defined Integral (see reference 2, section 15, page 36)

When the real parts of β and $\gamma-\beta$ are positive, the function $F(\alpha,\beta,\gamma,z)$ can be represented by an integral of Jacobi:

$$\boxed{F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-uz)^{-\alpha}\,du} \qquad \text{(II,10)}$$

The integral of the righthand side is indeterminate.

If the critical point $u=z^{-1}$ is located on the integration path (0,1), /X
z can therefore take no real value greater than 1.

If this condition is fulfilled (cutoff $(+1,+\infty)$ in the plane of the variable z), the integral is uniform (see, for example, reference 3, Volume 1, Chapter II, paragraph 1, page 155)

The integral and function $F(\alpha,\beta,\gamma,z)$ are therefore defined and uniform in the same domain (see paragraph 2.1.2, page IX [of foreign text]).

2.2.1. <u>If $z<1$</u>

Since $0<u<1$: $|uz|<1$, and:

$$(1-uz)^{-\alpha}=\sum_{\mu=0}^{\infty}\frac{\alpha(\alpha+1)\dots(\alpha+\mu-1)}{\mu!}(uz)^{\mu}=\sum_{\mu=0}^{\infty}\frac{(\alpha,\mu)}{(1,\mu)}(uz)^{\mu}$$

(this formula shows, in passing, that: $(1-z)^{-\beta}=F(\alpha,\beta,\alpha,z)$.)

Still on the hypothesis that $|z|<1$:

$$\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-uz)^{-\alpha}du=\sum_{\mu=0}^{\infty}\frac{(\alpha,\mu)}{(1,\mu)}z^{\mu}\int_0^1 u^{\beta+\mu-1}(1-u)^{\gamma-\beta-1}du$$

(I,15) of Annex I becomes:

$$\int_0^1 u^{\beta+\mu-1}(1-u)^{\gamma-\beta-1}du=B(\beta+\mu,\gamma-\beta)=\frac{\Gamma(\beta+\mu)\Gamma(\gamma-\beta)}{\Gamma(\gamma+\mu)}$$

Therefore: /XI

$$\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-uz)^{-\alpha}du=\frac{\Gamma(\gamma-\beta)\Gamma(\beta)}{\Gamma(\gamma)}\sum_{\mu=0}^{\infty}\frac{(\alpha,\mu)(\beta,\mu)}{(\gamma,\mu)(1,\mu)}z^{\mu}$$

$$=\frac{\Gamma(\gamma-\beta)\Gamma(\beta)}{\Gamma(\gamma)}F(\alpha,\beta,\gamma,z)$$

Formula (II,10) is thus confirmed in the case in which $|z|<1$, i.e., when the hypergeometric function is reduced to the hypergeometric series.

2.2.2. <u>If z=1</u>

(II,10), page IX [of foreign text] becomes:

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\cdot\int_0^1 u^{\beta-1}(1-u)^{\gamma-\alpha-\beta-1}\,du$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\,B(\beta,\gamma-\alpha-\beta)$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\,\frac{\Gamma(\beta)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)}$$

$$\boxed{F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\,\Gamma(\gamma-\alpha)}} \qquad \text{(II,11)}$$

This formula allows calculation of the value of the hypergeometric series when z assumes the critical value 1. /XII

2.3. <u>Transformation of the Hypergeometric Function</u>

The integral:

$$f(z)=\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-uz)^{-\alpha}\,du$$

does not change form when one of the following three changes of variable are carried out on it:

$$u = 1-v \qquad u = \frac{v}{1-x(1-v)} \qquad u = \frac{1-v}{1-vx}.$$

These changes of variable give the three Euler transform formulas:

$$\boxed{\begin{aligned} F(\alpha,\beta,\gamma,z) &= (1-z)^{-\alpha} \cdot F\left(\alpha,\gamma-\beta,\gamma,\frac{z}{z-1}\right) && \text{(II,12)} \\ &= (1-z)^{-\beta} F\left(\gamma-\alpha,\beta,\gamma,\frac{z}{z-1}\right) && \text{(II,13)} \\ &= (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma,z) && \text{(II,14)} \end{aligned}}$$

2.3.1. The hypergeometric functions which appear in the right-hand side of the first two transform formulas are hypergeometric series of variable $Z=z(z-1)^{-1}$. They can therefore be developed as absolutely convergent series of this variable Z, on the condition that z be located in domain D_1 defined by: /XIII

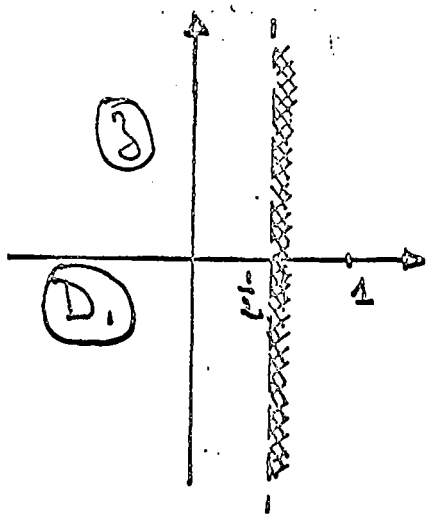
$$\left|\frac{z}{z-1}\right| < 1$$

This inequality becomes, when ρ_0 and ρ_1 designate the moduli of z and z^{-1}:

$$\rho_0 < \rho_1.$$

Domain D_1 is therefore the half-plane located at the left of the straight line:

$$R(z) = \frac{1}{2}. \qquad \text{(R(z)=real part of z)}$$



2.3.2. The transform (II,14) has the advantage of allowing comparison of one hypergeometric series for which $R(\gamma-\alpha-\beta)$ is negative with the case of another series for which $R(\gamma-\alpha-\beta)$ is positive.

This characteristic will be useful for study of a hypergeometric series on its circle of convergence (see section 1.2, page V [of foreign text]).

2.4. Contiguous Hypergeometric Functions (see reference 4, page 3)

2.4.1. The hypergeometric functions $F(\alpha,\beta,\gamma,z)$ and $F(\alpha',\beta',\gamma',z')$ are called "contiguous" when one of the three differences $\alpha'-\alpha$, $\gamma'-\gamma$, or $\beta'-\beta$ is equal to ± 1, the two others being zero. /XIV

$F(\alpha,\beta,\gamma,z)$ has, therefore, six contiguous functions.

$F(\alpha,\beta,\gamma,z)$ and two of its contiguous functions are connected by a linear relation. These relations are therefore in the number of:

$$C_6^2 = \frac{6.5}{1.2} = 15$$

2.4.2. Example of relation between $F(\alpha,\beta,\gamma,z)$ and two of its contiguous functions.

$$\boxed{(\alpha-\beta)F(\alpha,\beta,\gamma,z) = \alpha F(\alpha+1,\beta,\gamma,z) - \beta F(\alpha,\beta+1,\gamma,z)} \quad \text{(II,15)}$$

ANNEX III

Modified Bessel Functions

1. The Bessel Equation

The second order linear differential equation:

$$\boxed{t^2 z'' + t z' - (\nu^2 - t^2) z = 0} \qquad \text{(III,1)}$$

where t is a real or complex variable and ν is a real number, is called the Bessel equation with index ν.

1.2. If ν is not an integer, the general solution to this equation is:

$$\boxed{z = c_1 J_\nu(t) + c_2 J_{-\nu}(t)} \qquad \text{(III,2)}$$

c_1 and c_2 are constants; $J_\nu(t)$ and $J_{-\nu}(t)$ are first class Bessel functions with subscript ν and $-\nu$.

$$\boxed{J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{t}{2}\right)^{\nu+2k}} \qquad \text{(III,3)}$$

1.3. If ν is an integer: $\nu = n$. /XVI

In this case:

$$\Gamma(k+n+1) = (k+n)! \quad \text{(see Annex I)}$$

and:

$$\boxed{J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{t}{2}\right)^{2k+n}} \qquad \text{(III,4)}$$

Likewise:

$$J_{-n}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k-n)!} \left(\frac{t}{2}\right)^{2k-n}$$

$\frac{1}{(k-n)!}$ is zero for k<n (see section 1.2.1, page II [of foreign text]).

$$J_{-n}(t) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\,(k-n)!} \left(\frac{t}{2}\right)^{2k-n}$$

And, replacing k with k+n:

$$J_{-n}(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)!\,k!} \left(\frac{t}{2}\right)^{2k+n}$$

Therefore:

$$\boxed{J_{-n}(t) = (-1)^n J_n(t)} \qquad \text{(III,5)}$$

$J_{-n}(t)$ and $J_n(t)$ are equal practically to the sign, therefore proportional. /XVII
It is then necessary to seek another special solution of the equation in order to make clear its general solution.

The function

$$Y_n(t) = \lim_{a \to n} \frac{J_a(t) \cos a\pi - J_{-a}(t)}{\sin a\pi}$$

confirms this equation, and

$$\boxed{z = C_1 J_n(t) + C_2 Y_n(t)} \qquad \text{(III,6)}$$

is the general solution sought after.

$$\boxed{\begin{aligned} Y_n(t) = {} & \frac{2}{\pi} J_n(t) \ln \frac{t}{2} - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k-n} \\ & + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left[2\gamma - \psi(k) - \psi(k+n)\right] \left(\frac{t}{2}\right)^{2k+n} \end{aligned}} \qquad \text{(III,7)}$$

γ is the Euler-Mascheroni constant.

$$\gamma = \lim_{p \to \infty} \left[1 + \frac{1}{2} + \cdots + \frac{1}{p} - \ln p\right] = 0{,}57722$$

and

$$\psi(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$$

(by convention, $\psi(\rho)=0$)

1.4. <u>If t is a real number</u>: /XVIII

1.4.1. <u>$t>0$</u>

$J_v(t)$ is real regardless of v (integer or not).

$Y_n(t)$ is real regardless of value of integral n.

1.4.2. <u>$t<0$</u>

$J_v(t)$ is only real when v is an integer. In this case:

$$J_{2p}(t) = J_{2p}(-t)$$

$$J_{2p+1}(t) = -J_{2p+1}(-t)$$

$Y_n(t)$ is complex no matter value of integral n.

2. The Modified Bessel Equation

This equation is derived from the conventional Bessel equation by a mere change of sign:

$$\boxed{t^2 z'' + t z' - (\nu^2 + t^2) z = 0} \qquad \text{(III,8)}$$

The study will be limited to the case in which v is an integer (v=n). The general solution of this equation is then written:

/XIX

$$\boxed{z = C_1 I_n(t) + C_2 K_n(t)} \qquad \text{(III,9)}$$

I_nIt) and K_n(t) are first and second class Bessel functions of order n.

$$\boxed{I_n(t) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} \left(\frac{t}{2}\right)^{2k+n}} \qquad \text{(III,10)}$$

$$\boxed{\begin{aligned} (-1)^{n+1} K_n(t) &= I_n(t) \ln \frac{t}{2} - \frac{1}{2} \sum_{k=0}^{n-1} (-1)^{k-n} \frac{(n-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k-n} \\ &+ \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} \left[2\gamma - \psi(k) - \psi(k+n)\right] \left(\frac{t}{2}\right)^{2k+n} \end{aligned}} \qquad \text{(III,11)}$$

(see the significance of γ and ψ in section 1.3, page XVII [of foreign text]).

2.1. Relation between $J_n(t)$ and $I_n(t)$

$$J_n(it) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{it}{2}\right)^{2k+n} = i^n \sum_{k=0}^{\infty} \frac{(-1)^k i^{2k}}{k!\,(k+n)!} \left(\frac{t}{2}\right)^{2k+n}$$

$$(-1)^k i^{2k} = (-1)^k (i^2)^k = (-1)^k \cdot (-1)^k = (-1)^{2k} = 1 .$$

Therefore:

$$\boxed{J_n(it) = i^n I_n(t)} \qquad \text{(III,12)}$$

2.2. If t is a real integer /XX

2.2.1. $t>0$

$I_n(t)$ and $K_n(t)$ are real regardless of value of integral n.

2.2.2. $t<0$

$I_n(t)$ is real regardless of integral n, and

$$I_{2p}(t) = I_{2p}(-t)$$

$$I_{2p+1}(t) = -I_{2p+1}(-t)$$

$K_n(t)$ is complex regardless of value of integral n.

2.3. Function $K_n(t)$ is infinite for $t=0$.

This singularity is due to the presence of logarithmic (for $n=0$) or negative powers of t (for $n>0$).

The solutions of (III,8), finite in the case of $t=0$, are hence in the form:

$$\boxed{y = c\, I_n(t)} \qquad \text{(III,13)}$$

c is a constant.

3. Properties of Modified Bessel Functions

3.1. Recursion Formulas

3.1.1. $I_n(t)$ /XXI

$$I_n(t) = I_{-n}(t) \tag{III,14}$$

$$2n\, I_n(t) = t\left[I_{n-1}(t) - I_{n+1}(t)\right] \tag{III,15}$$

$$\frac{d}{dt} I_n(t) = \frac{1}{2}\left[I_{n-1}(t) + I_{n+1}(t)\right] \tag{III,16}$$

$$\frac{d}{dt} I_0(t) = I_1(t) \tag{III,17}$$

$$\frac{\partial}{\partial t} I_n(qt) = \frac{q}{2}\left[I_{n-1}(qt) + I_{n+1}(qt)\right] \tag{III,18}$$

3.1.2. $K_n(t)$

$$K_n(t) = K_{-n}(t) \tag{III,19}$$

$$2n\, K_n(t) = t\left[K_{n-1}(t) - K_{n+1}(t)\right] \tag{III,20}$$

$$\frac{d}{dt} K_n(t) = -\frac{1}{2}\left[K_{n-1}(t) + K_{n+1}(t)\right] \tag{III,21}$$

$$\frac{d}{dt} K_0(t) = -K_1(t) \tag{III,22}$$

$$\frac{\partial}{\partial t} K_n(qt) = -\frac{q}{2}\left[K_{n-1}(qt) + K_{n+1}(qt)\right] \tag{III,23}$$

3.2. Formulas of addition in the special case in which:

$$\boxed{t = \left[r^2 - 2\rho r \cos\theta + \rho^2\right]^{\frac{1}{2}}}$$

In this case, the functions $I_n(t)$ and $K_n(t)$ are developed according to the following formulas: /XXII

$$\boxed{I_n(t) \begin{matrix}\cos\\ \sin\end{matrix} n\psi = \sum_{m=-\infty}^{+\infty} (-1)^m I_{m+n}(r) I_m(\rho) \begin{matrix}\cos\\ \sin\end{matrix} m\theta} \qquad \text{(III,24)}$$

$$\boxed{K_n(t) \begin{matrix}\cos\\ \sin\end{matrix} n\psi = \sum_{m=-\infty}^{+\infty} K_{m+n}(r) I_m(\rho) \begin{matrix}\cos\\ \sin\end{matrix} m\theta} \qquad \text{(III,25)}$$

with
$$\begin{cases} \sin\psi = \dfrac{\rho \sin\theta}{t} \\ \cos\psi = \dfrac{r-\rho\cos\theta}{t} \end{cases}$$

(see reference 5, page 361)

More particularly:

3.2.1.

$$K_0(t) = \sum_{m=-\infty}^{+\infty} K_m(r) I_m(\rho) \cos m\theta$$

where:

$$\boxed{K_0(t) = \sum_{m=0}^{\infty} \varepsilon_m K_m(r) I_m(\rho) \cos m\theta} \qquad \text{(III,26)}$$

$$\varepsilon_m = 2 \text{ si } m > 0 \qquad \varepsilon_0 = 1.$$

3.2.2. /XXIII

$$\frac{r-\rho\cos\theta}{t} K_1(t) = \sum_{m=-\infty}^{+\infty} K_{m+1}(r) I_m(\rho) \cos m\theta$$

$$\boxed{\frac{r-\rho\cos\theta}{t} K_1(t) = \frac{1}{2} \sum_{m=0}^{+\infty} \varepsilon_m \left[K_{m-1}(r) + K_{m+1}(r)\right] I_m(\rho) \cos m\theta} \qquad \text{(III,27)}$$

3.2.3.

$$\frac{\rho \sin\theta}{t} K_1(t) = \sum_{m=-\infty}^{+\infty} K_{m+1}(r)\, I_m(\rho) \sin m\theta$$

$$\frac{\rho \sin\theta}{t} K_1(t) = \sum_{m=0}^{\infty} \left[K_{m+1}(r) - K_{m-1}(r)\right] I_m(\rho) \sin m\theta$$

and, taking into account the formula (III,20), page XXI [of foreign text]:

$$\boxed{\frac{\rho \sin\theta}{t} K_1(t) = \frac{2}{r} \sum_{m=0}^{\infty} m\, K_m(r)\, I_m(\rho) \sin m\theta} \qquad \text{(III,28)}$$

3.3. Basset Formula

This formula is written (see reference 5, page 172):

$$\boxed{K_n(gt) = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left[\frac{2t}{g}\right]^n \int_0^{\infty} \frac{\cos gu}{\left(u^2+t^2\right)^{n+\frac{1}{2}}}\, du} \qquad \text{(III,29)}$$

This expression is valid for positive values of n, g, and t. /XXIV

When m=1.

$$K_1(gt) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{2t}{g} \int_0^{\infty} \frac{\cos gu}{\left(u^2+t^2\right)^{3/2}}\, du$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

(see Annex I)

$$\boxed{K_1(gt) = \frac{t}{g} \int_0^{\infty} \frac{\cos gu}{\left(u^2+t^2\right)^{3/2}}\, du} \qquad \text{(III,30)}$$

3.4. Nielsen Formula (see reference 5, page 148)

$$J_\mu(az)\,J_\nu(bz) = \frac{1}{\Gamma(\nu+1)}\left(\frac{1}{2}az\right)^{\mu}\left(\frac{1}{2}bz\right)^{\nu}\sum_{m=0}^{\infty}(-1)^m\,\frac{F\left(-m,-\mu-m,\nu+1,b^2/a^2\right)}{m!\,\Gamma(\mu+m+1)}\left(\frac{1}{2}az\right)^{m} \qquad \text{(III,31)}$$

3.5. Lipschitz-Hankel Formula (see reference 5, page 388) /XXV

$$e_\nu^1(\alpha,\mu) = \int_0^\infty e^{-t\,\mathrm{ch}\alpha}\,K_\nu(t)\,t^{\mu-1}\,dt = \sqrt{\frac{\pi}{2}}\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu)\,(\mathrm{sh}\,\alpha)^{\mu-\frac{1}{2}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\mathrm{ch}\,\alpha) \qquad \text{(III,32)}$$

This formula is only valid if: $\begin{cases} \mathcal{R}(\mu) > \mathcal{R}(\nu) \\ \mathcal{R}(\mathrm{ch}\,\alpha) > -1 \end{cases}$

is a generalized Legendre function (see reference 6, paragraph 15-6, page 325).

$$P_q^r(z) = \frac{1}{\Gamma(1-r)}\left[\frac{z+1}{z-1}\right]^{\frac{r}{2}} F\left(-q,q+1,1-r,\frac{1-z}{2}\right) \qquad \text{(III,33)},$$

Therefore:

$$e_\nu^1(\alpha,\mu) = \sqrt{\frac{\pi}{2}}\,\frac{\Gamma(\mu-\nu)\,\Gamma(\mu+\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)}\,(\mathrm{sh}\,\alpha)^{\frac{1}{2}-\mu}\left[\frac{1+\mathrm{ch}\,\alpha}{-1+\mathrm{ch}\,\alpha}\right]^{\frac{1}{4}-\frac{\mu}{2}} F\left(\frac{1}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2}+\mu,\frac{1-\mathrm{ch}\,\alpha}{2}\right) \qquad \text{(III,34)}$$

4. Asymptotic Values of $I_n(t)$ and $K_n(t)$ (see reference 5, page 202) /XXVI

When t is very large, the values of $I_n(t)$ and $K_n(t)$ are represented by:

$$I_n(t) = \frac{e^t}{\sqrt{2\pi t}} \left[1 - \frac{4n^2-1^2}{1!\,8t} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8t)^2} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!\,(8t)^3} + \cdots \right]$$

$$K_n(t) = \sqrt{\frac{\pi}{2t}}\, e^{-t} \left[1 + \frac{4n^2-1^2}{1!\,8t} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8t)^2} + \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!\,(8t)^3} + \cdots \right]$$

These formulas can be rewritten:

$$I_n(t) = \frac{e^t}{\sqrt{2\pi t}} \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma(n+p+\frac{1}{2})}{\Gamma(n-p+\frac{1}{2})} \cdot \frac{1}{p!} \frac{1}{(2t)^p} \qquad \text{(III, 35)}$$

$$K_n(t) = \sqrt{\frac{\pi}{2t}}\, e^{-t} \sum_{p=0}^{\infty} \frac{\Gamma(n+p+\frac{1}{2})}{\Gamma(n-p+\frac{1}{2})} \frac{1}{p!} \frac{1}{(2t)^p} \qquad \text{(III,36)}$$

ANNEX IV /XXVII

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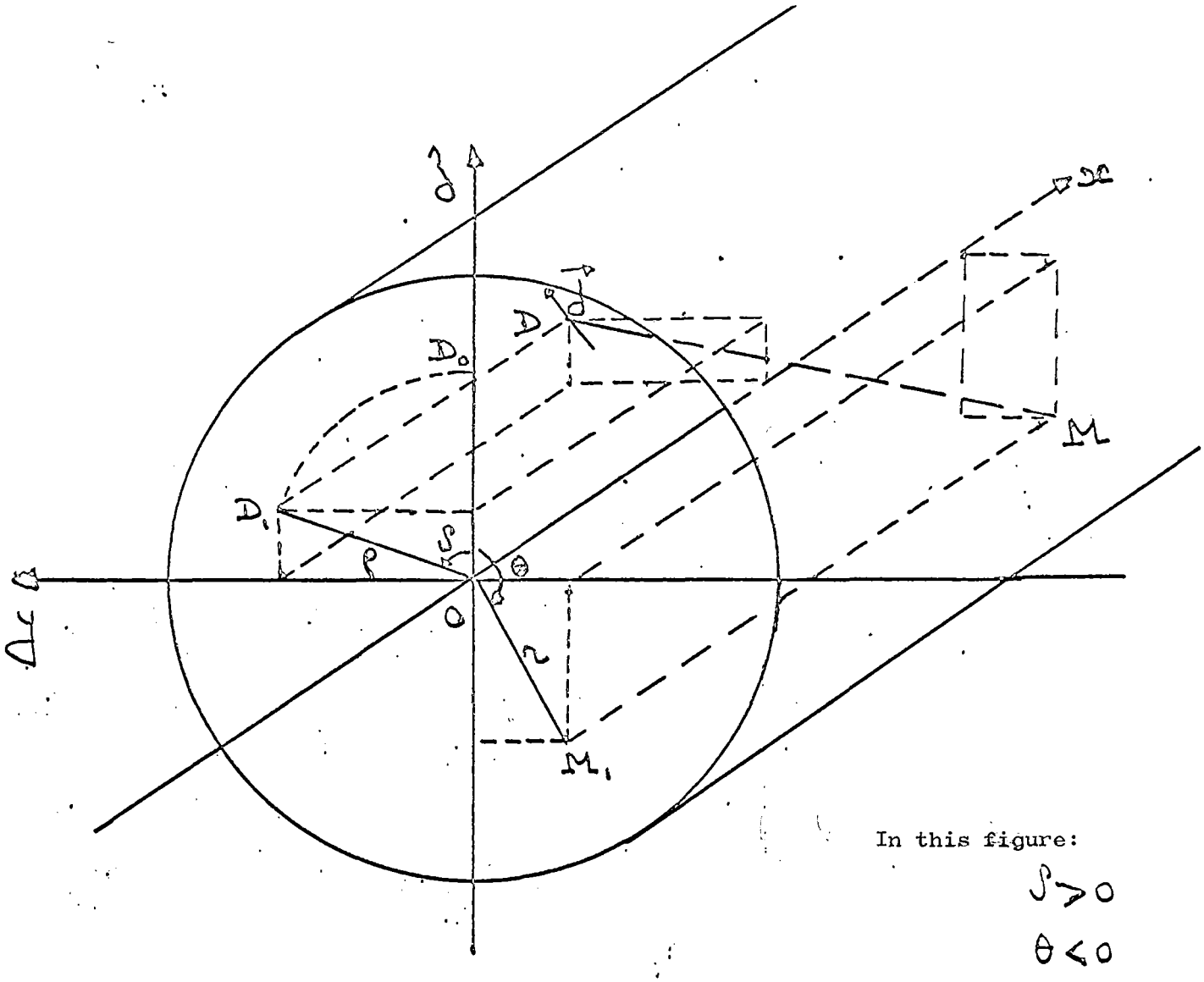

In this figure:

$\delta > 0$

$\theta < 0$

$OD_1 = \rho$

$D_1D = x_D$

$\widehat{Oz, OD_1} = \delta$

$OM_1 = r$

$M_1M = x_M = x$

$\widehat{Oz, OM_1} = \theta$

$$D \begin{cases} x_D \\ y_D = \rho \sin\delta \\ z_D = \rho \cos\delta \end{cases}$$

$$M \begin{cases} x_M = x \\ y_M = r \sin\theta \\ z_M = r \cos\theta \end{cases}$$