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AN ANALYTICAL INVESTIGATION OF FLOW PROPERTIES OF RAREFIED GASES

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CONTENTS

	Page
SUMMARY	1
INTRODUCTION	1
NOMENCLATURE	2
APPROACH 1	4
General Description	4
Required Equations	7
Computation Procedure	13
APPROACH 2	17
General Description	17
Required Equations	18
Computation Procedure	32
CONCLUDING REMARKS	35
RECOMMENDATION	36
REFERENCES	36

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SUMMARY

An investigation was made to develop a new analytical procedure for obtaining flow rates through tubes in the transition regime. Two approaches were investigated, one using a collision model superimposed on a previously obtained free molecular solution, and the other using a collision model based on a grey media radiation analogy in which the molecules are assumed to be scattered isotropically while passing through a media of previously wall scattered molecules. Effort on the former approach has been suspended due to the lack of a sound theoretical basis. The latter approach is currently being programmed for a computer solution.

INTRODUCTION

The flow rate of gases down a tube in a free molecular gas stream has been investigated and reported in Refs. 1-3. These investigations were directed toward the interpretation of impact pressure probe measurements at high altitudes.

Recent investigations (Refs. 4-7) have extended these results into the transition regime where intermolecular collisions must be considered. The investigators of Refs. 4 and 5 utilized a Monte Carlo technique which assumed a constant mean-free-path throughout the tube. The investigators of Ref. 6 obtained experimental measurements which were compared with theoretical predictions based on a first collision modification of free molecular flow theory. The investigators of Ref. 7 obtained an approximate solution to the Boltzmann equation. The purpose of the present investigation was to develop a new analytical procedure for obtaining flow rates through tubes in the transition regime. Two approaches were investigated, one using a collision model superimposed on a previously obtained free molecular solution, and the other using a collision model based on a grey media radiation analogy in which the molecules are assumed to be scattered isotropically while passing through a media of previously wall scattered molecules.

NOMENCLATURE

d	distance from a volume element
D	tube diameter
f	velocity distribution function
I	intensity
k	Boltzmann constant
Kn	Knudsen number (Eq. 8)

L	tube length
m	intermolecular collision index, also, molecular mass
M	terminal value for intermolecular collisions
n	tube wall collision index, also, number collisions as defined by Eq. (10).
N	terminal value for tube wall collisions, also, number collisions as defined by Eq. (15)
q	molecular flux
Q	number scattered molecules as defined by Eq. (36).
r	radial position coordinate
R	tube radius
S	speed ratio (Eq. 3)
T	temperature
U	external stream velocity
u	molecular velocity
v_m	most probable velocity in free stream
V	volume
x	axial position coordinate
y	rectangular coordinate perpendicular to x
z	rectangular coordinate perpendicular to x and y
α	thermal accommodation coefficient
θ	angle
ϕ	spherical coordinate (angular displacement between velocity vector and tube axis), also, as defined by Eq. (26).
ω	solid angle, also persistence of velocity (Eq. 11)
ψ	molecular beam attenuating factor
Λ	dimensionless molecular speed (Eq. 3)
χ	defined by Eq. (34)

σ molecular diameter

γ ratio of specific heats of gas

ρ number density

Subscripts:

E exit plane

g refers to molecules scattered from a volume element

I refers to molecules entering first volume element (Eq. 5)

m intermolecular collision index

n tube wall collision index

s tube wall

w refers to tube wall scattered molecules.

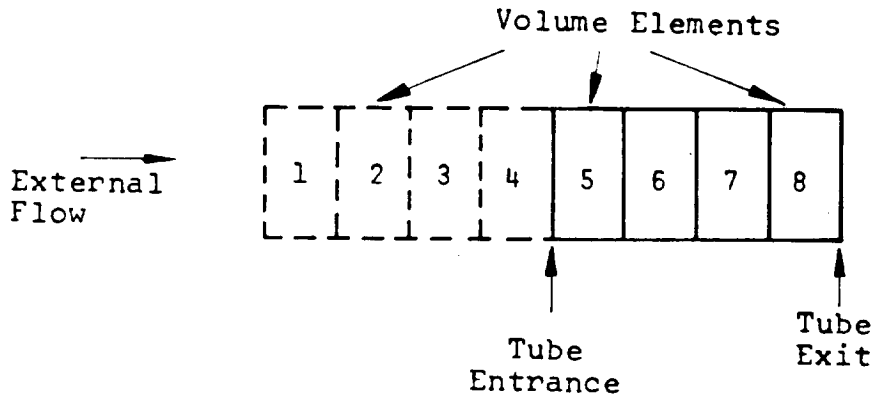
APPROACH 1.

Developed by P. F. Goolsby

General Description

This approach is essentially a collision model superimposed on a free molecular solution previously obtained and reported in Ref. 8. All scattering effects of the wall are assumed to be built into the free molecular solution, and the effects of intermolecular collisions simply change the respective magnitudes. The volume within and immediately in front of the tube is divided into a number of finite volume elements, the dimensions of each volume element being chosen sufficiently small that multiple collisions within the element may be neglected. A typical arrangement of

elements is as follows:



The initial density in each volume element is assumed to be the free molecular density obtained from Ref. 8. The molecules within each volume element are assumed to be a Maxwellian gas at rest. A number of molecules from the external stream with speeds in a certain small range are then allowed to enter the first volume element and undergo collisions within the volume element. The number of molecules from the external stream with speeds in the small range of speeds which enter the volume element is determined from the distribution of velocities in the external stream. The number of these molecules which undergo collision in the volume element is determined from the collision frequency equation. The velocities of both the incoming molecules and the target molecules after collision are determined from the velocity persistence equation and the conservation of momentum.

Some of the incoming molecules which undergo collision are assumed to remain in the volume element after collision

and become part of the density within the volume element. The criteria for determining whether the incoming molecules remain in the volume element after collision is based on the following assumption. If the speed of the molecules after collision, as determined from the persistence of the velocity equation, is equal to or less than the most probable velocity in the volume element, the molecules would be scattered isotropically and would appear, if observed, to be a part of the target molecules in the volume element. On the other hand, if the speed of the molecules after collision is greater than the most probable velocity in the volume element, then the molecules would not be scattered isotropically but would continue directed into the next volume element. A similar determination is made concerning the target molecules after collision. The velocity of the target molecules after collision is determined from the velocity of the incoming molecules before and after collision and the conservation of momentum. The target molecules are assumed to have zero velocity, on the average, prior to collision. As with the incoming molecules, the target molecules involved in collisions are assumed to remain in, or be removed from, the volume element according to whether the speed after collision is less than or greater than the most probable velocity in the volume element. Those target molecules removed from the volume element pass on into the next element along with those incoming molecules involved in collisions which did not remain in the volume element, and those incoming molecules which did not

undergo collisions in the volume element.

The above described process is repeated for all speeds of the incoming molecules. This will result in a new density within the volume element and a new distribution of incoming molecules into the next volume element where the process is repeated. The process is repeated over all the volume elements in turn until the density in each volume element ceases to change significantly. It is then assumed that the collision processes have reached steady-state.

The transmission probability for flow through the tube is obtained by ratioing the number of molecules passing over the exit plane of the tube to the number of molecules entering the first volume element. The number of molecules crossing the exit plane is obtained by integrating over all speeds the distribution of molecules passing out of the last volume element at steady-state. Likewise, the number entering the first volume element is obtained by integrating over all speeds the distribution of the incoming molecules. It was found that this ratio approaches unity for large Knudsen numbers. It was therefore assumed that this ratio represents the transmission probability normalized to the free molecular transmission probability.

Required Equations

Distribution of Incoming Molecular Speeds. - It was necessary to find the distribution of the incoming molecules

from the external stream with respect to their absolute speed. This was necessary because the collision frequency of the incoming molecules when colliding with a Maxwellian gas at rest is a function of the absolute speed of the incoming molecules. The distribution function with respect to speed was obtained by integrating the Maxwellian distribution of the molecules in the free stream. The distribution of velocities in the free stream is assumed to be Maxwellian superimposed on the stream velocity. The Maxwellian distribution function may then be represented by (Ref. 9)

$$f_{x,y,z} = \frac{1}{\pi^{3/2} v_m^3} \exp\{-[(v_x - U)^2 + v_y^2 + v_z^2]/v_m^2\} \quad (1)$$

where $v_m = (2kT_\infty/m)^{1/2}$ is the most probable thermal velocity in the external stream; k is the Boltzmann constant; m is the mass of a molecule; T_∞ is the static temperature and U is the velocity of the external stream; and v_x, v_y, and v_z are the velocity components in the x, y, and z directions, respectively. The x-y plane is parallel to the tube entrance plane and the z axis is parallel to the tube axis and is directed positively from the entrance to the exit.

The fraction of molecules df in an element of volume with velocity components between v_x and $v_x + dv_x$, v_y and $v_y + dv_y$, and v_z and $v_z + dv_z$ is

$$df = f_{x,y,z}\, dv_x\, dv_y\, dv_z \quad (2)$$

The volume element of velocity space $dv_x dv_y dv_z$ may be written in terms of spherical coordinates as $2\pi v^2 \sin\phi \, d\phi \, dv$ where $v^2 = v_x^2 + v_y^2 + v_z^2$ and ϕ is the angle between the velocity vector and the positive z axis. Eq. (2) may then be written as:

$$df = \frac{2}{\sqrt{\pi}} \Lambda^2 \exp\{-[\Lambda^2 - 2S\Lambda \cos\phi + S^2]\} \sin\phi \, d\phi \, d\Lambda \qquad (3)$$

where $\Lambda = v/v_m$ and $S = U/v_m$.

The distribution function f_Λ with respect to Λ is defined by $df = f_\Lambda \, d\Lambda$. The value of f_Λ is obtained by integrating Eq. (3) between proper limits of ϕ. In this study, we are interested only in those molecules which enter the first volume element from one side (traveling in the positive z direction). Therefore, we integrate ϕ from 0 to $\pi/2$:

$$f_\Lambda = \frac{2}{\sqrt{\pi}} \int_0^{\pi/2} \Lambda^2 \exp\{-[\Lambda^2 - 2S\Lambda\cos\phi + S^2]\} \sin\phi \, d\phi$$

$$= \frac{\Lambda}{\sqrt{\pi} S} \exp[-(S^2 + \Lambda^2)][\exp(2S\Lambda) - 1] \qquad (4)$$

Another quantity used in the mathematical treatment involved in this investigation is the total number ρ_I/ρ_∞ of these molecules (all molecules which are traveling in the positive axial direction) per unit volume as a ratio to the total number per unit volume in the free stream.

This is obtained by integrating f_Λ with respect to Λ over the range of Λ from zero to infinity:

$$\frac{\rho_I}{\rho_\infty} = \frac{1}{\sqrt{\pi}S}\int_0^\infty \exp[-(S^2 + \Lambda^2)][\exp(2S\Lambda) - 1]d\Lambda$$

$$= \frac{1}{2}(1 + \mathrm{erf}S) \quad (5)$$

Collision Frequency. - To obtain the number of the incoming molecules having collisions in the volume element, the collision frequency equation of Ref. 10 was used:

$$\nu = \rho\sigma^2 \left(\frac{2\pi kT}{m}\right)^{1/2} \left\{e^{-x^2} + \frac{\sqrt{\pi}}{2}\left(\frac{2x^2 + 1}{x}\right) \mathrm{erf}\, x\right\} \quad (6)$$

where $x = v\,(m/2kT)^{1/2}$, v is the speed of the incoming molecules, ρ is the number density of molecules in the volume element, σ is the diameter of the molecules (assuming hard spheres), and T is the temperature of the molecules in the volume element (assuming a Maxwellian gas at rest).

From Ref. 9, the mean-free-path λ in the static free stream may be related to the molecular diameter σ and the number density ρ_∞ in the free stream by:

$$\lambda = \frac{1}{\sqrt{2}\pi\ \rho_\infty\ \sigma^2} \quad (7)$$

Defining the Knudsen number Kn by:

$$Kn = \lambda/D \quad (8)$$

where D is the tube diameter, and making the proper substitutions in Eq. (6) results in:

$$\nu = \frac{(L/D)}{\sqrt{2\pi}\ Kn\ L} (\rho/\rho_\infty) \left(\frac{2kT}{m}\right)^{1/2} \left\{ e^{-x^2} + \frac{\sqrt{\pi}}{2} \left(\frac{2x^2 + 1}{x}\right) \text{erf } x \right\} \tag{9}$$

where $x = \Lambda/(T/T_\infty)^{1/2}$ and L is the tube length.

Since we are concerned with the total number of collisions encountered by the incoming molecules during a traverse of the volume element, the collision frequency ν of Eq. (9) should be multiplied by the length of time required for the traverse. The assumption will be made that the absolute speed v is nearly equal to the axial component of the velocity, and the time of traverse is $\Delta x/v$, where Δx is the axial length of the volume element. The accuracy of this assumption should increase at the higher speed ratios S. The total number of collisions n encountered by each of the incoming molecules of absolute speed ratio Λ during a total traverse of the volume element is therefore:

$$n = \frac{(L/D)(\Delta x/L)(\rho/\rho_\infty)}{\sqrt{2\pi}\ Kn\ x} \left\{ e^{-x^2} + \frac{\sqrt{\pi}}{2} \left(\frac{2x^2 + 1}{x}\right) \text{erf } x \right\} \tag{10}$$

<u>Persistence of Velocity</u>. - In order to obtain the change in the velocity of the incoming molecules after a

collision, the following equation from Ref. 10 was used:

$$\omega = \frac{1}{2}\left\{1 - \frac{1}{2x^2}\,\frac{\left[e^{-x^2} - \frac{\sqrt{\pi}}{2}\left(\frac{1 - 2x^2}{x}\right)\operatorname{erf} x\right]}{\left[e^{-x^2} + \frac{\sqrt{\pi}}{2}\left(\frac{1 + 2x^2}{x}\right)\operatorname{erf} x\right]}\right\} \qquad (11)$$

where ω is the ratio of the velocity component after collision to the velocity before collision, with the component being in the direction of motion prior to the collision. Here again, we make the assumption that we are obtaining axial components of velocity after collision. This is based on the premise that this would be true "on the average" for a large number of separate collisions. This is somewhat inconsistent with the earlier use of absolute speeds in obtaining the number of collisions. As previously mentioned, however, the accuracy of this assumption should increase at the higher speed ratios.

The dimensionless speed Λ' after collisions is then:

$$\Lambda' = \omega\Lambda \qquad (12)$$

where Λ is the speed prior to collision.

In obtaining the velocity of the target molecules after collision, the conservation of momentum was applied with the assumption that the target molecule was at rest prior to the collision. This is not actually true; however, since the target molecules are assumed to be a Maxwellian gas at rest,

the molecular motion is isotropic. The assumption that the target molecules are at rest, then, is based on the premise that this would be true "on the average" for a large number of collisions. Once again, we make the assumption that we are dealing with axial components.

The dimensionless speed Λ'' of the target molecule after collision is then:

$$\begin{aligned} \Lambda'' &= \Lambda - \Lambda' \\ &= \Lambda(1 - \omega) \end{aligned} \tag{13}$$

Computation Procedure

The required inputs for the computation procedure are the tube dimensions L/R, the dimensions of the volume elements $\Delta x/L$, the speed ratio, S, the initial density ρ/ρ_∞ and temperature T/T_∞ distribution of target molecules, and the Knudsen number Kn.

<u>Collision Process</u>. - A small range of speeds from Λ to $\Lambda + \Delta\Lambda$ of the incoming molecules from the external stream is chosen, and the number of molecules per unit volume $\Delta\rho$ ratioed to the free stream density in this speed range is determined from:

$$\Delta\rho/\rho_\infty = f_\Lambda \, \Delta\Lambda \tag{14}$$

The number N of these molecules colliding in the first

volume element is then found by multiplying Eqs. (10) and (14):

$$N = n\ \Delta\rho/\rho_\infty \tag{15}$$

The velocity of these colliding molecules after collision is obtained from Eq. (12).

The same number of target molecules will be involved in the collisions and the velocity of these molecules after collision is obtained from Eq. (13).

The speeds of the incoming molecules and the target molecules after collision are then compared with the most probable velocity of the molecules in the volume element. This amounts to the same thing as comparing the dimensionless speeds Λ' and Λ'' (Eq. (12) and (13)) with $(T/T_\infty)^{1/2}$. If Λ' is less than $(T/T_\infty)^{1/2}$, then these molecules are added to the density in the volume element. The new density ρ'/ρ_∞ in the volume element is

$$\rho'/\rho_\infty = \rho/\rho_\infty + N \tag{16}$$

Since these molecules are gained by the volume element, they are lost by the incoming stream. The new distribution function f'_Λ at Λ of the molecules passing into the next volume element is obtained from the balance:

$$f'_\Lambda\ \Delta\Lambda = f_\Lambda\ \Delta\Lambda - N$$

or

$$f'_\Lambda = f_\Lambda - \frac{N}{\Delta\Lambda} \tag{17}$$

If Λ' is greater than $(T/T_\infty)^{1/2}$, no change occurs.

If Λ'' is greater than $(T/T_\infty)^{1/2}$, these target molecules are lost from the volume element and added to the stream of molecules passing into the next volume element. The density ρ'/ρ_∞ in the volume element after collision is then:

$$\rho'/\rho_\infty = \rho/\rho_\infty - N \tag{18}$$

and the distribution function f'_Λ of molecules passing into the next volume element is

$$f'_\Lambda = f_\Lambda + \frac{N}{\Delta\Lambda} \tag{19}$$

If Λ'' is less than $(T/T_\infty)^{1/2}$, no change occurs.

This process is repeated for all $\Delta\Lambda$'s over the distribution function f_Λ for the first volume element to establish a new density ρ/ρ_∞ in the first volume element and a new distribution function f_Λ for molecules passing into the next volume element.

The above process is then repeated for each volume element in turn down the tube. This establishes a new distribution of densities in the volume elements.

Using the new distribution of densities as initial conditions, the entire process is repeated until the density distribution no longer changes appreciably, and the collision process is assumed to have reached steady-state.

Transmission Probability. - After the collision process has reached steady-state, the normalized transmission probability K' (normalized to the free molecular transmission probability) is calculated. This is accomplished by first finding the ratio ρ_E/ρ_∞ of the total number of molecules per unit volume passing out of the last volume element to the free stream density. This value is obtained by integrating the distribution function f_Λ over all speeds Λ for molecules passing out of the last volume element:

$$\frac{\rho_E}{\rho_\infty} = \int_0^\infty f_\Lambda \, d\Lambda \tag{20}$$

We then ratio this number to the density of molecules entering the first volume element ρ_I/ρ_∞ (Eq. 5) to obtain the normalized transmission probability K':

$$K' = \frac{\rho_E/\rho_\infty}{\rho_I/\rho_\infty} \tag{21}$$

The actual transmission probability K is found by multiplying K' by the free molecular transmission probability K_∞ which may be found from Refs. 3 or 8.

$$K = K' \, K_\infty \tag{22}$$

APPROACH 2

Developed by S. J. Robertson

General Description

The approach taken here is an extension of the earlier treatment of free molecular flow as reported in Ref. 8. This treatment used form factor relationships analogous to thermal radiation transfer. In extending these studies into the transition regime, it was assumed that the molecules which had been previously scattered from the tube wall acted as a scattering media for other molecules passing through. The following simplifying assumptions were made: (1) the molecules may be considered as hard spheres, (2) the molecules constituting the scattering media are taken to be stationary when compared to the velocity of the incoming molecules, (3) the molecules are scattered isotropically by the stationary spheres without energy exchange and with a Maxwellian velocity distribution, and (4) as in Ref. 8, the molecules are scattered diffusely from the tube wall with an energy exchange determined by the thermal accommodation coefficient.

In order to account for intermolecular collisions in the flow of molecules through the tube, the distribution of those molecules scattered from the tube wall must be known as an initial condition. The exact distribution can be determined, however, only when the intermolecular collisions are considered. It is necessary, therefore, to start with a first

approximation to the distribution of wall scattered molecules and use this distribution as the initial conditions for determining a second approximation. This process is repeated until the successive distributions no longer change appreciably, and convergence is achieved. The first approximation is taken to be the free molecular solution obtained by the method of Ref. 8.

The transmission probability is obtained in the same manner as in Ref. 8 after convergence is achieved.

Required Equations

Attenuated Beams. - The equations required for the determination of wall and exit plane fluxes, total flow rates (transmission probabilities), and density distributions due to molecules arriving at a point directly from the free stream or after a wall collision without undergoing intermolecular collisions in transit are nearly the same as those of Ref. 8 for the free molecular case. The only difference is that the equations of Ref. 8 are multiplied by an exponential factor ψ to account for the attenuation of the beam due to scattering along the beam path. This attenuating factor is based on the assumption that all molecules in the beam that undergo collisions are scattered and removed from the beam. The collision cross section S for hard sphere molecules for a beam of molecules passing through a media of stationary molecules is (Ref. 9):

$$S = \pi \sigma^2 \tag{23}$$

where σ is the molecular diameter.

If the beam intensity is I, the decrease in beam intensity when passing through an elementary distance $d\ell$ in the scattering media is:

$$dI = - \rho_w \pi\sigma^2 I d\ell \tag{24}$$

where ρ_w is the density of tube wall scattered molecules at that point. The fractional decrease in intensity ψ over a distance ℓ is therefore:

$$\psi = \exp \left[- \pi\sigma^2 \int_0^{\ell} \rho_w \, d\ell \right] \tag{25}$$

This attenuation is due entirely to scattering from the beam and not to other effects, such as divergence of the beam. The attenuating factor ψ may be expressed in terms of the Knudsen number Kn through use of Eqs. (7-8):

$$\psi = \exp \left[- \frac{(L/D)}{\sqrt{2}\, Kn \cos\phi} \int_0^{x} (\rho_w/\rho_\infty)(dx'/L) \right] \tag{26}$$

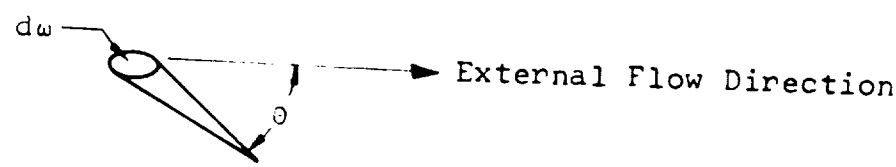
where ϕ is the angle between the beam path and the tube axis and x is the projection of the beam path on the tube axis.

From Ref. 8 and Eq. (26), the density $\rho_{o,o}$ at a point due to molecules streaming directly from the external stream without a prior collision is obtained by integrating

the following equation over the solid angle enclosed by the tube entrance:

$$\frac{d\rho_{o,o}}{\rho_\infty} = \exp(-S^2) \quad \{\exp(S^2\cos^2\theta)[1 + 2S^2\cos^2\theta + 2S^2\cos^2\theta \ \mathrm{erf}(S\cos\theta) + \mathrm{erf}(S\cos\theta)] + \frac{2}{\sqrt{\pi}} S\cos\theta\} \ \psi \frac{d\omega}{4\pi} \tag{27}$$

where θ is the angle between the external flow velocity vector and the solid angle element $d\omega$.

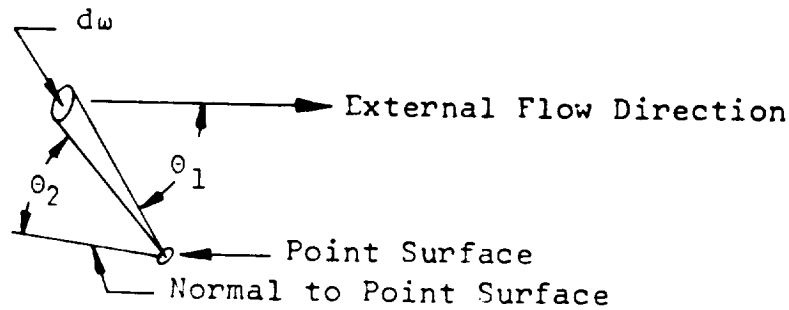


The subscripts of $\rho_{n,m}$ denote n previous collisions with the tube wall and m previous intermolecular collisions since the last tube wall collision.

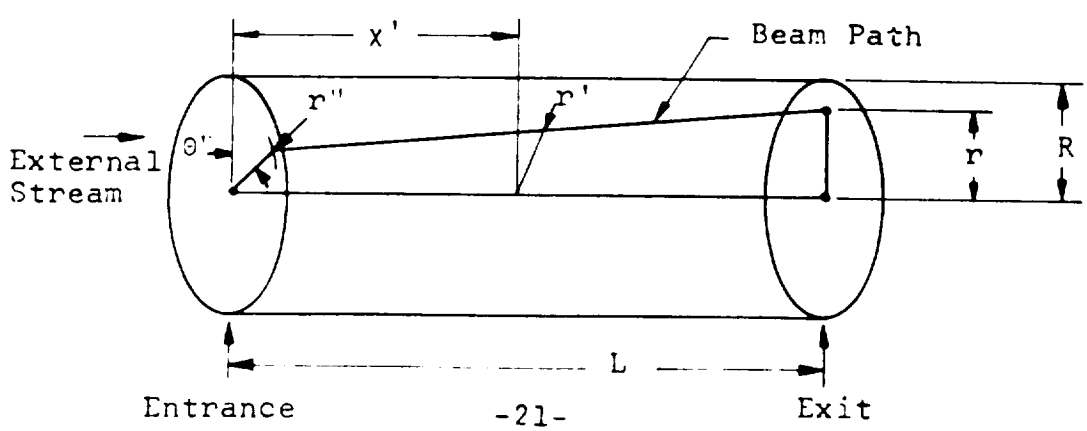
The molecular flux $q_{o,o}$ on or through a point surface (on the tube wall or through the exit plane) due to molecules streaming directly from the external stream without molecular collisions in transit is (Ref. 8 and Eq. 26):

$$\frac{dq_{o,o}}{q_\infty} = \left(1 + S^2\cos^2\theta_1\right) \exp(-S^2) \left\{1 + \frac{3\sqrt{\pi}}{2} S\cos\theta_1 \frac{1 + (2/3)S^2\cos^2\theta_1}{1 + S^2\cos^2\theta_1} \exp(S^2\cos^2\theta_1)[1 + \mathrm{erf}(S\cos\theta_1)]\right\} \psi \frac{d\omega}{\pi} \cos\theta_2 \tag{28}$$

where $(q_\infty = (1/2\sqrt{\pi})(\rho_\infty v_m)$ (Ref. 9) is the flux that would exist across the entrance plane into the tube for a zero stream velocity, θ_1 is the angle between the external flow velocity vector and the solid angle element $d\omega$, and θ_2 is the angle between the normal to the point surface and $d\omega$.



The integration with ψ (Eq. 26) in both Eqs. (27) and (28) is over the beam path between the point and some point in the external stream where collisions are assumed negligible. Since the density ρ_w/ρ_o in ψ is a function of both the axial and radial positions, and since the integration is made along the projection of the beam path on the tube axis, it is necessary to express the beam path in the form of a relation between the axial and radial coordinates. Consider the beam path represented in the following sketch:



The beam is passing from the external stream to the point on the exit plane at radial position r. The beam path is defined by the point where it crosses the entrance plane at the radial position r'' and the angular position θ''. The relation between the radial distance r' and the axial distance x' is found to be:

$$\frac{r'}{R} = \left[\left(\frac{x'}{L}\right)^2 \left(\frac{r}{R}\right)^2 + 2\left(\frac{x'}{L}\right)\left(1 - \frac{x'}{L}\right)\left(\frac{r''}{R}\right) \cos\theta'' + \left(1 - \frac{x'}{L}\right)^2 \left(\frac{r''}{R}\right)^2\right]^{1/2} \tag{29}$$

$\cos\phi$ in Eq. (26) for this case is

$$\cos\phi = \frac{(L/R)}{[(L/R)^2 + (r/R)^2 + (r''/R)^2 - 2(r/R)(r''/R)\cos\theta'']^{1/2}} \tag{30}$$

At each point x' along the tube axis, a radial position r' is determined from Eq. (29). From these x' and r' coordinates, a density (ρ_w/ρ_∞) is found from the known density distribution.

The integration along the beam path and over the solid angle enclosed by the tube entrance in Eqs. (27) and (28) is accomplished by numerical techniques.

The density $\rho_{n,o}$ at a point and the molecular flux $q_{n,o}$ on or through a point surface due to molecules streaming

directly to the point after n collisions with the tube wall, and without intermolecular collisions in transit from the last wall collision, is obtained from (Ref. 8 and Eq. 26):

$$\frac{d\rho_{n,o}}{\rho_\infty} = \frac{1}{[(T_w)_{n-1}/T_\infty]^{1/2}} \frac{(q_w)_{n-1}}{q_\infty} \psi \frac{d\omega}{4\pi} \tag{31}$$

and

$$\frac{dq_{n,o}}{q_\infty} = \frac{(q_w)_{n-1}}{q_\infty} \psi \frac{d\omega\cos\theta}{\pi} \tag{32}$$

where $(q_w)_{n-1}$ is the molecular flux at the element of tube wall enclosed by the solid angle element $d\omega$ for molecules arriving after $n-1$ previous wall collisions, $(T_w)_{n-1}$ is the temperature of these molecules after rebounding from the tube wall element, and θ is the angle between the point surface and $d\omega$. It should be pointed out that the molecules in $(q_w)_{n-1}$ include molecules having previous intermolecular collisions in transit from the last wall collision, as well as the $n-1$ previous wall collisions. The single subscript in the $(q_w)_{n-1}$ and $(T_w)_{n-1}$ nomenclature denotes this fact. In a later section, the procedure for accounting for intermolecular collisions will be described.

The temperature $(T_w)_{o,o}$ of molecules rebounding from an element of tube wall surface after arriving from the external stream without previous wall collisions or intermolecular collisions in transit is (Ref. 8, with the above mentioned

change in nomenclature, and Eq. 25):

$$\frac{(T_w)_{o,o}}{T_\infty} = (1 - \alpha)\left\{\frac{2(\gamma - 1)}{\gamma + 1}\,\frac{\chi}{(q_w)_o/q_\infty} + \frac{5 - 3\gamma}{\gamma + 1}\right\} + \alpha\frac{T_s}{T_\infty} \qquad (33)$$

where:

$$d\chi = 2\exp(-S^2)\left\{1 + \frac{9}{4}S^2\cos^2\theta_1 + \frac{1}{2}S^4\cos^4\theta_1 + \frac{15\sqrt{\pi}}{8}\exp(S^2\cos^2\theta_1)S\cos\theta_1\left(1 + \frac{4}{3}S^2\cos^2\theta_1 + \frac{4}{15}S^4\cos^4\theta_1\right)[1 + \mathrm{erf}(S\cos\theta_1)]\right\}\psi\frac{d\omega}{\pi}\cos\theta_2 \qquad (34)$$

and where, as in Eq. (28), θ_1 is the angle between the external flow velocity and $d\omega$, and θ_2 is the angle between the normal to the point surface and $d\omega$. Also, γ is the ratio of specific heats of the gas, and T_s is the tube wall temperature. Once again, the integration of Eq. (34) is over the solid angle enclosed by the tube entrance.

The temperature $(T_w)_{n,o}$ of molecules rebounding from an element of tube wall surface after n previous wall collisions without intermolecular collisions in transit from

the last wall collision is (Ref. 3 and Eq. 26):

$$\frac{(T_w)_{n,o}}{T_\infty} = (1 - \alpha) \frac{\int_\omega \frac{(q_w)_{n-1}}{q_\infty} \frac{(T_w)_{n-1}}{T_\infty} \psi \frac{d\omega \cos\theta}{\pi}}{(q_w)_n/q_\infty} + \alpha \frac{T_s}{T_\infty} \quad (35)$$

where the integration is over the solid angle ω enclosing the interior of the tube wall surface with apex at the location of $(T_w)_{n,o}$ and the wall flux $(q_w)_{n-1}$ and temperature $(T_w)_{n-1}$ are at the positions on the tube wall surface enclosed by the solid angle element $d\omega$, θ is the angle between $d\omega$ and the normal to the tube wall at the location of $(T_w)_{n,o}$, and α is the thermal accommodation coefficient.

Scattered Beams. - Those molecules that are attenuated from the beam are assumed to be scattered isotropically. The volume elements where attenuation occurs may then be considered as sources of new molecular beams. Referring to Eq. (24), if a narrow beam of intensity I passes through an elementary right cylinder volume element of cross section dA and length $d\ell$ with the axis of the cylinder oriented in the beam direction, the total number dQ of molecules attenuated from the beam in the volume element per unit time is:

$$dQ = - dI\, dA = \rho_w \pi \sigma^2 I\, dV \quad (36)$$

where $dV = d\ell\, dA$ is the volume of the elementary cylinder.

These molecules will be scattered isotropically so that the flux dq' on or through a surface at a distance d from the volume element is:

$$dq' = \frac{\rho_w \sigma^2 I \, dV}{4 d^2} \psi \cos\theta' \tag{37}$$

where θ' is the angle between the normal to the surface and the straight line between dV and the point on the surface.

Now the shape of the volume element dV actually makes no difference and could just as well be a small sphere. If a number of different beams are passing through the elementary sphere from different directions, we are concerned only with the intensity of each elementary beam and not the flux across some fixed surface contributed by the beam. Therefore, rather than the total flux q across a point surface, we are interested in an integrated beam intensity I which is obtained in the same way as the flux q except that the $\cos\theta$ factor is removed from the integration, θ being the angle between the beam direction and the normal to a point surface. Eq. (37) then becomes, when expressed in terms of the Knudsen number Kn (Eqs. 7-8):

$$dq' = \frac{(L/D)(\rho_w/\rho_\infty) \, I \, \psi \, (dV/L^3) \cos\theta'}{4\sqrt{2} \, \pi Kn \, (d/L)^2} \tag{38}$$

The intensity $I_{o,o}$ at a point due to molecules streaming directly from the external stream without intermolecular collisions in transit is obtained by integrating

Eq. (8) in Ref. 8 modified by dropping the $\cos\theta_2$ factor and including ψ:

$$\frac{dI_{o,o}}{q_\infty} = (1 + S^2\cos^2\theta)\exp(-S^2)\left\{1 + \frac{3\sqrt{\pi}}{2} S\cos\theta \frac{1 + (2/3)S^2\cos^2\theta}{1 + S^2\cos^2\theta}\exp(S^2\cos^2\theta)[1 + \mathrm{erf}(S\cos\theta)]\right\}\psi\frac{d\omega}{\pi} \tag{39}$$

where θ is the angle between the external flow direction and $d\omega$.

The intensity $I_{n,o}$ at a point due to molecules streaming directly to the point after n collisions with the tube wall and without intermolecular collisions in transit from the last wall collision is obtained from a modification of Eq. (32):

$$\frac{dI_{n,o}}{q_\infty} = \frac{(q_w)_{n-1}}{q_\infty}\psi\frac{d\omega}{\pi} \tag{40}$$

The intensity $I_{n,m}$ at a point due to molecules arriving at the point after n collisions with the tube wall and after m intermolecular collisions in transit from the last wall collision is obtained from a modification of Eq. (38):

$$\frac{dI_{n,m}}{q_\infty} = \frac{(L/D)(\rho_w/\rho_\infty)(I_{n,m-1}/q_\infty)\psi(dV/L^3)}{4\sqrt{2}\,\pi\,Kn(d/L)^2} \tag{41}$$

The intensity I_n at a point due to all molecules arriving at the point after n tube wall collisions is obtained by summing $I_{n,m}$ over all possible number of intermolecular collisions m. In practice, the summation is terminated at some M beyond which the contributions to the summation are considered negligible:

$$I_n = \sum_{m=o}^{M} I_{n,m} \tag{42}$$

Now referring back to Eq. (38) and the subsequent development for the intensity I, the molecular flux $q_{n,m}$ on or through a point surface due to molecules arriving at the point after n collisions with the tube wall and after m intermolecular collisions in transit from the last wall collision is:

$$\frac{dq_{n,m}}{q_\infty} = \frac{(L/D)(\rho_w/\rho_\infty)(I_{n,m-1}/q_\infty)\psi(dV/L^3)\cos\theta'}{4\sqrt{2}\ \pi\ Kn(d/L)^2} \tag{43}$$

The molecular flux q_n on or through a point surface due to all molecules arriving at the point after n collisions with the tube wall is:

$$q_n = \sum_{m=o}^{M} q_{n,m} \tag{44}$$

In order to obtain the density ρ at a point due to molecules arriving at the point after being scattered from a

volume element, it is necessary to know the average velocity of the scattered molecules. In order to obtain this average velocity, the assumptions were made that the molecules scattered in the volume element did not exchange energy with the target molecules during collision, and that the distribution of velocities of the scattered molecules after collision were Maxwellian.

From a modification of the theory presented in Ref. 8 for the interchange of energy at the tube wall, the temperature $(T_g)_{o,o}$ of molecules scattered from a volume element after arriving at the volume element from the external stream without intermolecular collisions in transit is:

$$\frac{(T_g)_{o,o}}{T_\infty} = \frac{2(\gamma - 1)}{\gamma + 1} \frac{\chi'}{I_{o,o}/q_\infty} + \frac{5 - 3\gamma}{\gamma + 1} \tag{45}$$

where χ' is obtained from a modification of Eq. (34):

$$d\chi' = 2\exp(-S^2) \left\{ 1 + \frac{9}{4} S^2 \cos^2\theta + \frac{1}{2} S^4 \cos^4\theta + \frac{15\sqrt{\pi}}{8} \exp(S^2\cos^2\theta) S\cos\theta \left(1 + \frac{4}{3} S^2\cos^2\theta + \frac{4}{15} S^4\cos^4\theta\right)[1 + \mathrm{erf}(S\cos\theta)] \right\} \psi \frac{d\omega}{\pi} \tag{46}$$

where θ is the same as θ_1 in Eq. (34).

For molecules arriving at the point after n collisions with the tube wall and without any intermolecular collisions

in transit from the last tube wall collision, the temperature of the molecules is:

$$\frac{(T_g)_{n,o}}{T_\infty} = \frac{\int_\omega \frac{(q_w)_{n-1}}{q_\infty} \frac{(T_w)_{n-1}}{T_\infty} \psi \frac{d\omega}{\pi}}{I_{n,o}/q_\infty} \tag{47}$$

where $(q_w)_{n-1}$ and $(T_w)_{n-1}$ are the fluxes and temperatures at the element of tube wall surface enclosed by $d\omega$ and the integration is over the solid angle enclosed by the interior of the tube with apex at the location of $(T_g)_{n,o}$.

For molecules arriving at the point after n collisions with the tube wall and after m intermolecular collisions in transit from the last tube wall collision, the temperature is:

$$\frac{(T_g)_{n,m}}{T_\infty} = \frac{\int \frac{(T_g)_{n,m-1}}{T_\infty} \frac{d\, I_{n,m}}{q_\infty}}{I_{n,m}/q_\infty} \tag{48}$$

where $dI_{n,m}$ is the contribution to $I_{n,m}$ by a volume element dV as defined in Eq. (41), $(T_g)_{n,m-1}$ is the temperature at dV, and the integration is over the volume containing the scattering molecules.

The average temperature $(T_g)_n$ for all molecules arriving at the point after n tube wall collisions is:

$$(T_g)_m = \frac{\sum_{m=o}^{M} I_{n,m} (T_g)_{n,m}}{I_n} \tag{49}$$

Now, the density $d\rho$ in a narrow beam of molecules passing through a point surface is related to the flux dq as follows:

$$d\rho = dq/\bar{v} \cos\theta' \tag{50}$$

where $\bar{v}$ is the average velocity of the molecules and θ' is the angle between the beam and the normal to the point surface. Using Eqs. (43) and (50) and the relation (Ref. 9)

$$q_\infty = (1/2\sqrt{\pi})\rho_\infty v_m \tag{51}$$

the density $\rho_{n,m}$ at a point due to molecules arriving at the point after n collisions with the tube wall and after m intermolecular collisions in transit from the last wall collision is:

$$\frac{d\rho_{n,m}}{\rho_\infty} = \frac{(L/D)(\rho_w/\rho_\infty)(I_{n,m-1}/q_\infty)\psi(dV/L^3)}{16\sqrt{2}\ \pi\ Kn(d/L)^2[(T_g)_{n,m-1}/T_\infty]^{1/2}} \tag{52}$$

The density ρ_n at a point due to all molecules arriving at the point after n tube wall collisions is:

$$\rho_n = \sum_{m=o}^{M} \rho_{n,m} \tag{53}$$

The temperature $(T_w)_{n,m}$ of molecules rebounding from an element of tube wall surface after n previous wall collisions and

after m intermolecular collisions in transit from the last wall collision is:

$$\frac{(T_w)_{n,m}}{T_\infty} = \frac{(1-\alpha)\int \frac{(T_g)_{n,m-1}}{T_\infty} \frac{d(q_w)_{n,m}}{q_\infty}}{(q_w)_{n,m}} + \alpha \frac{T_s}{T_\infty} \tag{54}$$

where $d(q_w)_{n,m}$ is the contribution to $(q_w)_{n,m}$ by a volume element dV as defined in Eq. (43), $(T_g)_{n,m-1}$ is the temperature at dV, and the integration is over the volume containing the scattering molecules.

The average temperature $(T_w)_n$ for all molecules rebounding from the tube wall surface after n tube wall collisions is:

$$\frac{(T_w)_n}{T_\infty} = \frac{\sum_{m=0}^{M} (q_w)_{n,m} (T_w/T_\infty)_{n,m}}{(q_w)_n} \tag{55}$$

Computation Procedure

The required inputs for the computation procedure are the tube dimensions L/R, the speed ratio S, the ratio T_s/T_∞ of the tube wall temperature to the free stream temperature, the thermal accommodation coefficient α, and the Knudsen number Kn.

The first step is to obtain the free molecular distribution of tube wall scattered molecules ρ_w/ρ_∞ by the method

of Ref. 8. This distribution is then used as the first approximation to ρ_w/ρ_∞ for use as inputs to the computation for the finite Knudsen number case.

<u>n=o, m=o</u>. - Using all of the above inputs, the density distribution $\rho_{o,o}/\rho_\infty$ throughout the tube is calculated by numerically integrating Eq. (27). The flux $(q_w)_{o,o}/q_\infty$ along the tube wall and the flux $(q_E)_{o,o}/q_\infty$ over the tube exit are obtained by numerically integrating Eq. (28). The intensity $I_{o,o}/q_\infty$ distribution throughout the tube is obtained from Eq. (39). The temperature $(T_g)_{o,o}/T_\infty$ distribution throughout the tube is obtained by integrating Eq. (46). The temperature $(T_w)_{o,o}/T_\infty$ is obtained from Eq. (33) and (34).

<u>n=o, m=m</u>.- Then $\rho_{o,m}\ \rho_\infty$, $(q_w)_{o,m}/q_\infty$ and $(q_E)_{o,m}/q_\infty$, $I_{o,m}/q_\infty$, $(T_g)_{o,m}/T_\infty$, and $(T_w)_{o,m}/T_\infty$ are obtained from Eqs. (52), (43), (41), (48), and (54), respectively, for each successive m from 1 to some chosen value M.

<u>n=o</u>. - Then ρ_o/ρ_∞, $(q_w)_o/q_\infty$ and $(q_E)_o/q_\infty$, I_o/q_∞, $(T_g)_o/T_\infty$, and $(T_w)_o/T_\infty$ are obtained from Eqs. (53), (44), (42), (49), and (55), respectively.

<u>n=n, m=o</u>. - For each succeeding value of n, $\rho_{n,o}/\rho_\infty$, $(q_w)_{n,o}/q_\infty$, $(T_g)_{n,o}/T_\infty$, and $(T_w)_{n,o}/T_\infty$ are obtained from Eqs. (31), (32), (40), (47), and (35), respectively.

<u>n=n, m=m</u>. - For each value of n, prior to passing on to the next value of n, $\rho_{n,m}/\rho_\infty$, $(q_w)_{n,m}/q_\infty$ and $(q_E)_{n,m}/q_\infty$,

$I_{n,m}/q_\infty$, $(T_g)_{n,m}/T_\infty$, and $(T_w)_{n,m}/T_\infty$ are obtained from the same equations as in n=o, m=m above for each successive m from 1 to M.

<u>n=n</u>. - For each n, prior to passing on to the next n, ρ_n/ρ_∞, $(q_w)_n/q_\infty$ and $(q_E)_n/q_\infty$, I_n/q_∞, $(T_g)_n/T_\infty$, and $(T_w)_n/T_\infty$ are obtained from the same equations as in n=o above.

The above process is repeated for each successive n up to some chosen value N such that the calculated quantities became negligibly small.

Next, ρ/ρ_∞, q_w/q_∞, and q_E/q_∞ are obtained from:

$$\rho/\rho_\infty = \sum_{n=o}^{N} \rho_n/\rho_\infty \tag{56}$$

$$q/q_\infty = \sum_{n=o}^{N} q_n/q_\infty \tag{57}$$

The total flow rate through the tube is obtained by numerically integrating q_E/q_∞ over the exit plane.

The distribution of tube wall scattered molecules ρ_w/ρ_∞ is obtained from

$$\rho_w/\rho_\infty = \rho/\rho_\infty - \rho_o/\rho_\infty \tag{58}$$

This new distribution of ρ_w/ρ_∞ is then taken as the second approximation and used as inputs into another calculation. The above described process is repeated until the succeeding

approximations of ρ_w/ρ_∞ do not change appreciably and convergence is achieved.

CONCLUDING REMARKS

In comparing the two approaches, it is seen that Approach 1 is somewhat simpler and requires fewer calculations than Approach 2. However, the rigor of Approach 1 is not at all obvious and appears to be totally lacking in certain areas. The criterion for determining whether colliding molecules are retained in the volume element or passed onto the next volume element is not clearly based on sound reasoning. Neither is the assumption that wall collisions are "built into" the free molecular solution. The fact that the calculated transmission probability approaches unity for high Knudsen numbers and, therefore, must be considered normalized to the free molecular transmission probability is highly questionable. Due to the lack of a sound theoretical basis, effort on this approach has been suspended.

The collision model of Approach 2 is based primarily on the following assumptions: (1) scattering of a beam of molecules is due only to collisions between molecules in the beam and molecules which have experienced previous wall collisions, (2) the scattering molecules, those that have had previous wall collisions, are assumed to be at rest compared to the molecules in the beam, (3) the scattering of the molecules in the beam is isotropic, and, (4) no energy is

exchanged during the intermolecular collisions. Assumptions (1) and (2) are probably more accurate at the higher speed ratios; however assumptions (3) and (4) are probably more accurate at the lower speed ratios. Considering all assumptions, Approach 2 is probably most accurate, therefore, in an intermediate range of speed ratios. Approach 2 is currently being programmed for computer solution.

RECOMMENDATION

It is recommended that the programming of Approach 2, as applied to tube flow, be carried to completion. It is further recommended that this approach be applied to the calculation of aerodynamic drag for various simple shapes such as spheres and cylinders, both convex and concave.

REFERENCES

1. Pond, H. L., J. Aerospace Sci. 29, 917 (1962).
2. Rothe, D. E. and DeLeeuw, J. H., AIAA J. 1, 220 (1963).
3. Hughes, P. C., "Theory for the Free Molecule Flow Impact Probe at an Arbitrary Angle of Attack", University of Toronto Institute for Aerospace Studies Report No. 103, May 1965.
4. Ballance, J. O., "Transmission Probability Determination With Directed Mass Motion and with Mean Free Path Considerations", Presented at Third International Vacuum Congress, Stuttgard, Germany June 28-July 2, 1965.
5. Morrison, L. K., "Monte Carlo Study of Molecular Flow Through a Cylindrical Duct", TM/20-45, Lockheed Missiles and Space Co., Huntsville Research and Engineering Center, Huntsville, Alabama.

6. Wainwright, J. B. and Rogers, K. W., "Impact Pressure Probe Response Characteristics in High Speed Flows, with Transition Knudsen Numbers", NASA Contractor Report CR-61119, February 18, 1966.

7. Edwards, R. H. and Quan, V., Phys. Fluids 9, 663 (1966).

8. Robertson, S. J., "Density Variations Through a Circular Tube in a Free Molecular Gas Stream", HTL-TR-30, Heat Technology Laboratory, Huntsville, Alabama.

9. Kennard, E. H., Kinetic Theory of Gases, McGraw-Hill Book Company, New York and London, 1938.

10. Chapman, S. and Cowling, T. G., The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, London, 1960.