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NASA PROJECT APOLLO WORKING PAPER NO. 1006

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AN ANALYSIS OF THE ERRORS IN POSITION GIVEN BY

AN ONBOARD LUNAR NAVIGATION SYSTEM USING

OBSERVATIONS OF CELESTIAL BODIES

N70-75842
(ACCESSION NUMBER)
35
(PAGES)
TMX-65210
(NASA CR OR TMX OR AD NUMBER)
(THRU)
None
(CODE)
(CATEGORY)
FACILITY FORM 602

[redacted]

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

SPACE TASK GROUP

Langley Field, Va.

December 16, 1960

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AN ANALYSIS OF THE ERRORS IN POSITION GIVEN BY AN ONBOARD LUNAR NAVIGATION SYSTEM USING OBSERVATIONS OF CELESTIAL BODIES

By Benjamine J. Garland

SUMMARY

A system of navigation in the earth-moon system using observations of the earth, the moon, and two stars have been considered and the effects of errors in the observations on the measured position evaluated. The analysis indicated that the largest error in the measured position resulted from errors in the measurement of the included angle of the earth and the moon. In errors in angle measurement of 10 seconds of arc, there is a region between the earth and the moon in which the navigation system is not effective because of the large errors in the measured position. Any improvements in the accuracy of other quantities involved will be of little benefit unless the error in the measurement of the included angle can be reduced to approximately 1 second of arc.

INTRODUCTION

The problem of navigation in space has been considered by various authors. Some of the techniques which have been considered are earth-based radar tracking (ref. 1), the measurement of solar radiation (ref. 2), the detection of differences in gravity (ref. 3), and the observation of the position of celestial bodies (refs. 1, 4, 5, and 6). In Apollo, there is a requirement for an onboard navigation system which is completely independent of ground observations. The earth-based radar tracking is not considered for this reason. The second and third techniques are of questionable value. The solar radiation varies in intensity with time and is not known well enough to be used to find position. The detection of differences in gravity would require delicate instrumentation even if the gravity fields of the earth and moon were well known.

The technique of navigation using the observed positions of celestial bodies is discussed in this paper. These observations may be used in connection with an inertial platform in the vehicle or the observations may be used alone. The inertial platform is used to establish a reference coordinate system. If the inertial platform is used, it is necessary to observe two bodies, but if the observations alone are used, it is necessary to observe four bodies. At any point, each system actually observes one body more than is necessary, but this body is used as a check on the position.

In this study the use of the inertial platform is not considered. The four bodies which are observed are the earth, the moon, and two stars. The two stars are used to locate a reference set of axes in the body so that the observations of the earth and moon may be easily used. The quantities which are measured are the azimuth and elevation of each of the four bodies in a set of axes fixed in the vehicle, the included angle of the earth, and the included angle of the moon. The last two quantities give the distance between the vehicle and the earth and the distance between the vehicle and the moon.

The errors which occur in the determination of the position of the vehicle are of as much interest as the actual technique used. In either system all the possible errors must be considered. If the inertial platform is used the possible errors in the alinement of the platform must be considered as well as errors in the celestial observations.

LIST OF SYMBOLS

a	the azimuth of a body in the axes fixed in the vehicle (x',y',z')
D	the distance between the centers of the earth and the moon
d	the distance between the vehicle and the center of the earth or the moon
e	the elevation of a body in the axes fixed in the vehicle (x',y',z')
i	the inclination of the plane determined by the earth, the moon, and the vehicle and the plane of the moon's orbit
$\bar{i}, \bar{j}, \bar{k}$	unit vectors in the x, y, z system
R	the radius of the earth or of the moon
$\bar{S}_1$	an unit vector located in the vehicle and pointing toward the first star
$\bar{S}_2$	an unit vector located in the vehicle and pointing toward the second star

$\bar{S}_3$	the vector product of $\bar{S}_1$ and $\bar{S}_2$, (that is, $\bar{S}_1 \times \bar{S}_2$)
X, Y, Z	a rotating set of axes located at the center of the earth as defined in figure 1
x, y, z	a set of axes located at the vehicle and always parallel to the XYZ system
x',y',z'	a set of axes fixed in the vehicle and having a general orientation
α, β, γ	direction cosines in the x'y'z' set of axes
η	the included angle of either the earth or the moon
θ	the angle between the earth-moon axis (X axis) and the line connecting the vehicle to the center of the earth or the center of the moon
λ	the angle between the z axis and the line connecting the vehicle to the center of the moon
v	the angle between the z axis and the line connecting the vehicle to the center of the earth
ϕ	the angle between the lines connecting the vehicle to the center of the earth and the center of the moon
Φ	the angle between the vector $\bar{S}_1$ and the x, y, or z axis
ψ	the angle between the vector $\bar{S}_2$ and the x, y, or z axis
Λ	the angle between the vector $\bar{S}_3$ and the x, y, or z axis

SUBSCRIPTS

E	earth
M	moon
x, y, z	refer to the x, y, z axes
1, 2, 3	refer to the vectors $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$

ANALYSIS

Analysis of Position Determination

The vehicle operates in the three-dimensional space around the earth and moon. The coordinate system which is used is shown in figure 1.

The main set of axes is the XYZ system which is located so that the origin is at the earth, the X axis passes through the moon, the Y axis is positive in the direction of the velocity of the moon and the Z axis follows from the right-hand rule. The secondary set of axes (x, y, z) has its origin at the vehicle and remains parallel to the XYZ axes.

The navigation system requires that four bodies be observed. These bodies are the earth, the moon and two stars. Six individual measurements must be made in order to locate the vehicle. These measurements are:

1-4 the azimuth and elevation of each of the bodies in the vehicle's body axes.

5 the included angle of the earth, and

6 the included angle of the moon.

The body axes are shown in figure 2.

The azimuth of the line-of-sight is $\underline{a}$ and the elevation is $\underline{e}$. The direct cosines of the line-of-sight to the body are

$$\left.\begin{aligned} \cos \alpha_i &= \cos a_i \cos e_i \\ \cos \beta_i &= \sin a_i \cos e_i \\ \cos \gamma_i &= \sin e_i \end{aligned}\right\} \qquad (1)$$

In figure 2 the other axes are in fact shown. The location of the two stars are used to determine the direction of the z axis in the body axes (x',y',z'). The method of determining this is discussed in appendix A so that only the general considerations will be mentioned here. It is assumed that the distances involved in the flight are such that the line-of-sight to each of the stars is always parallel to the line-of-sight located at the origin of the XYZ system. Since the XYZ system is not fixed in space, but rotates,

the location of the star in this system varies with time, but this variation is known as a function of time. The same scheme is used to determine the orientation of the x and y axes.

When the orientation of the z axis in the x'y'z' system has been determined, the geometry of the problem becomes that shown in figure 3.

The angles ν, λ, and ϕ are given by the equations:

$$\left.\begin{aligned} \cos\nu &= \cos\alpha_E \cos\alpha_z + \cos\beta_E \cos\beta_z + \cos\gamma_E \cos\gamma_z \\ \cos\lambda &= \cos\alpha_M \cos\alpha_z + \cos\beta_M \cos\beta_z + \cos\gamma_M \cos\gamma_z \\ \cos\phi &= \cos\alpha_E \cos\alpha_M + \cos\beta_E \cos\beta_M + \cos\gamma_E \cos\gamma_M \end{aligned}\right\} \quad (2)$$

The distance from the earth $\left(d_E\right)$ and the distance from the moon $\left(d_M\right)$ are determined using the known radii of the earth and moon and their included angle. The equations for this and the errors in d_E and d_M are discussed in appendix B.

With the information which is now known, it is possible to locate the vehicle in the XYZ system. Figure 4 is used to obtain the following equations:

$$\left.\begin{aligned} X &= d_E \cos\theta_E \\ Y &= d_E \sin\theta_E \cos i \\ Z &= -d_E \cos\nu \end{aligned}\right\} \quad (3)$$

When the values of $\cos\theta_E$, $\sin\theta_E$ and $\cos i$ are substituted from appendix C, these equations become:

$$X = \frac{d_E^2 \sin^2\phi}{D} - d_E \cos\phi \left[1 - \frac{\left(d_E \sin\phi\right)^2}{D}\right]^{\frac{1}{2}}$$

$$Y = d_E \left\{\sin^2\phi \left[\left(1 - \left[\frac{d_E \sin\phi}{D}\right]^2\right) + 2\,\frac{d_E}{D}\cos\phi \left(1 - \left[\frac{d_E \sin\phi}{D}\right]^2\right)^{\frac{1}{2}} + \frac{d_E^2 \cos^2\phi}{D^2}\right] - \cos^2\nu\right\}^{\frac{1}{2}} \quad (4)$$

$$Z = -d_E \cos\nu$$

The corresponding equations based on d_M, ϕ and λ are:

$$\left.\begin{aligned}
X &= D - \frac{d_M^{\,2}\sin^2\phi}{D} + d_M \cos\phi\left[1 - \left(\frac{d_M\sin\phi}{D}\right)^2\right]^{\frac{1}{2}} \\
Y &= d_M\left\{\sin^2\phi\left[1 - \left(\frac{d_M\sin\phi}{D}\right)^2\right] + 2\,\frac{d_M}{D}\cos\phi\left(\left[1 - \left(\frac{d_M\sin\phi}{D}\right)^2\right]^{\frac{1}{2}} + \frac{d_M^{\,2}\cos^2\phi}{D^2}\right)\right. \\
&\quad \left. - \cos^2\lambda\right\}^{\frac{1}{2}} \\
Z &= -d_M \cos\lambda
\end{aligned}\right] \quad (5)$$

Analysis of Errors in Position

Equations (4) and (5) give the position of the vehicle obtained from two separate sets of data. In an initial study, the errors in the measured position are of more interest than the position of the vehicle. These errors are found using the first two terms of a Taylor's series, which is equivalent to assuming that the errors in position are linear functions of errors in the measured quantities. The errors in X, Y, and Z based on d_E, D, ϕ and ν are:

$$\left.\begin{aligned}
\Delta X &= \frac{\partial X}{\partial d_E}\Delta d_E + \frac{\partial X}{\partial D}\Delta D + \frac{\partial X}{\partial \phi}\Delta\phi + \frac{\partial X}{\partial \nu}\Delta\nu \\
\Delta Y &= \frac{\partial Y}{\partial d_E}\Delta d_E + \frac{\partial Y}{\partial D}\Delta D + \frac{\partial Y}{\partial \phi}\Delta\phi + \frac{\partial Y}{\partial \nu}\Delta\nu \\
\Delta Z &= \frac{\partial Z}{\partial d_E}\Delta d_E + \frac{\partial Z}{\partial D}\Delta D + \frac{\partial Z}{\partial \phi}\Delta\phi + \frac{\partial Z}{\partial \nu}\Delta\nu
\end{aligned}\right\} \quad (6)$$

It was shown in appendix B that:

$$\Delta d_E = \frac{\partial d_E}{\partial R_E}\Delta R_E + \frac{\partial d_E}{\partial \eta_E}\Delta\eta_E \qquad (7)$$

Inspection of equations (4) show that:

$$\frac{\partial X}{\partial \nu} = \frac{\partial Z}{\partial D} = \frac{\partial Z}{\partial \phi} = 0$$

so that now equations (6) become:

$$\left.\begin{aligned}
\Delta X &= \frac{\partial X}{\partial d_E}\frac{\partial d_E}{\partial R_E}\Delta R_E + \frac{\partial X}{\partial d_E}\frac{\partial d_E}{\partial \eta_E}\Delta \eta_E + \frac{\partial X}{\partial D}\Delta D + \frac{\partial X}{\partial \phi}\Delta \phi \\
\Delta Y &= \frac{\partial Y}{\partial d_E}\frac{\partial d_E}{\partial R_E}\Delta R_E + \frac{\partial Y}{\partial d_E}\frac{\partial d_E}{\partial \eta_E}\Delta \eta_E + \frac{\partial Y}{\partial D}\Delta D + \frac{\partial Y}{\partial \phi}\Delta \phi + \frac{\partial Y}{\partial \nu}\Delta \nu \\
\Delta Z &= \frac{\partial Z}{\partial d_E}\frac{\partial d_E}{\partial R_E}\Delta R_E + \frac{\partial Z}{\partial d_E}\frac{\partial d_E}{\partial \eta_E}\Delta \eta_E + \frac{\partial Z}{\partial \nu}\Delta \nu
\end{aligned}\right\} \quad (8)$$

These equations have meaning only for an individual case, but the general results may be found by a statistical method. Then the expressions for the root-mean-square values of ΔX, ΔY, and ΔZ are:

$$\left.\begin{aligned}
\Delta X &= \sqrt{\left(\frac{\partial X}{\partial d_E}\frac{\partial d_E}{\partial R_E}\Delta R_E\right)^2 + \left(\frac{\partial X}{\partial d_E}\frac{\partial d_E}{\partial \eta_E}\Delta \eta_E\right)^2 + \left(\frac{\partial X}{\partial D}\Delta D\right)^2 + \left(\frac{\partial X}{\partial \phi}\Delta \phi\right)^2} \\
\Delta Y &= \sqrt{\left(\frac{\partial Y}{\partial d_E}\frac{\partial d_E}{\partial R_E}\Delta R_E\right)^2 + \left(\frac{\partial Y}{\partial d_E}\frac{\partial d_E}{\partial \eta_E}\Delta \eta_E\right)^2 + \left(\frac{\partial Y}{\partial D}\Delta D\right)^2 + \left(\frac{\partial X}{\partial \phi}\Delta \phi\right)^2 + \left(\frac{\partial Y}{\partial \nu}\Delta \nu\right)^2} \\
\Delta Z &= \sqrt{\left(\frac{\partial Z}{\partial d_E}\frac{\partial Z}{\partial R_E}\Delta R_E\right)^2 + \left(\frac{\partial Z}{\partial d_E}\frac{\partial d_E}{\partial \eta_E}\Delta \eta_E\right)^2 + \left(\frac{\partial Z}{\partial \nu}\Delta \nu\right)^2}
\end{aligned}\right\} \quad (9)$$

The corresponding equations for the information based on d_M, ϕ, D and λ are:

$$\left.\begin{aligned}
\Delta X &= \sqrt{\left(\frac{\partial X}{\partial d_M}\,\frac{\partial d_M}{\partial R_M}\,\Delta R_M\right)^2 + \left(\frac{\partial X}{\partial d_M}\,\frac{\partial d_M}{\partial \eta_M}\,\Delta \eta_M\right)^2 + \left(\frac{\partial X}{\partial D}\,\Delta D\right)^2 + \left(\frac{\partial X}{\partial \phi}\Delta\phi\right)^2} \\
\Delta Y &= \sqrt{\left(\frac{\partial Y}{\partial d_M}\,\frac{\partial d_M}{\partial R_M}\,\Delta R_M\right)^2 + \left(\frac{\partial Y}{\partial d_M}\,\frac{\partial d_M}{\partial \eta_M}\,\Delta \eta_M\right)^2 + \left(\frac{\partial Y}{\partial D}\,\Delta D\right)^2 + \left(\frac{\partial Y}{\partial \phi}\,\Delta\phi\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\Delta\lambda\right)^2} \\
\Delta Z &= \sqrt{\left(\frac{\partial Z}{\partial d_M}\,\frac{\partial d_M}{\partial R_M}\,\Delta R_M\right)^2 + \left(\frac{\partial Z}{\partial d_M}\,\frac{\partial d_M}{\partial \eta_M}\,\Delta \eta_M\right)^2 + \left(\frac{\partial Z}{\partial \lambda}\,\Delta\lambda\right)^2}
\end{aligned}\right\} \quad (10)$$

The partial derivatives $\frac{\partial d_E}{\partial R_E}$, $\frac{\partial d_M}{\partial R_M}$, $\frac{\partial d_E}{\partial \eta_E}$ and $\frac{\partial d_M}{\partial \eta_M}$ are developed in appendix B and the remaining partial derivatives are developed in appendix D.

If the known errors (ΔR_E, ΔR_M, $\Delta \eta_E$, $\Delta \eta_M$, ΔD, $\Delta\phi$, $\Delta\nu$ and $\Delta\lambda$) used in equations (9) and (10) are given in terms of three standard deviations, the final errors (ΔX, ΔY and ΔZ) will also be three standard deviation values.

The errors in position were found along a portion of the reference trajectory which occurred entirely in the earth-moon plane as shown in figure 5. The errors in the measured position were based on errors in the astronomical constants and in the measured angles which were considered to be realistic. The errors in the astronomical constants are given in reference 7. The errors which were used are listed below.

Quantity	Error
Radius of earth	±0.1 mile
Radius of moon	±0.1 mile
Distance between earth and moon	±1.0 mile
Measured angles	10 seconds of arc

The errors in the X, Y and Z resulting from each method are presented in figure 6. The errors when the position is based on d_E and ϕ are small when the time is small (i.e., d_E is small) and increases as the time increases (i.e., d_E increases). The errors when the position is based on d_M and ϕ follows the opposite trend with time and is large when the time is small (i.e., d_M is large) and decreases as the time increases (i.e., d_M decreases). The errors are of the same order at 45 hours after insertion which is approximately one-half the time required to reach the moon. The results of one additional trajectory which are not presented show that the errors are of the same order at approximately one-half the time required to reach the moon. The largest errors in position which may be expected due to the error in measurements are:

$$\Delta X = 280 \text{ statute miles}$$

$$\Delta Y = 200 \text{ statute miles}$$

$$\Delta Z = 9 \text{ statute miles}$$

The errors in the measured position may become sufficiently large so that any corrections to the trajectory which were based upon this measured position might not decrease the error in the final position significantly. If this occurs there is a region between the earth and the moon in which the navigation system is not effective.

The relative importance of the errors in the astronomical constants and measured angles are given in the following table.

The time is 45 hours after insertion.

Position based on	Coordinate	Quantity causing error				
		R_E	D	n_E	ϕ	ν
d_E, ϕ	X	6.18	0.18	272.00	2.45	0
	Y	3.98	0.03	174.00	9.16	0
	Z	0	0	0	0	8.69
		R_M	D	n_M	ϕ	λ
d_M, ϕ	X	10.50	0.71	282.00	3.17	0
	Y	7.64	0.02	204.00	3.85	0
	Z	0	0	0	0	5.32

It is seen that the error in position caused by the error in the included angle of the earth and the moon is at least one order of magnitude greater than any other error. This same relation is true at other times also. Therefore, it would be of little use to improve the accuracy of the other quantities unless the error in the measured included angle of the earth and the moon can be reduced to approximately 1 second of arc.

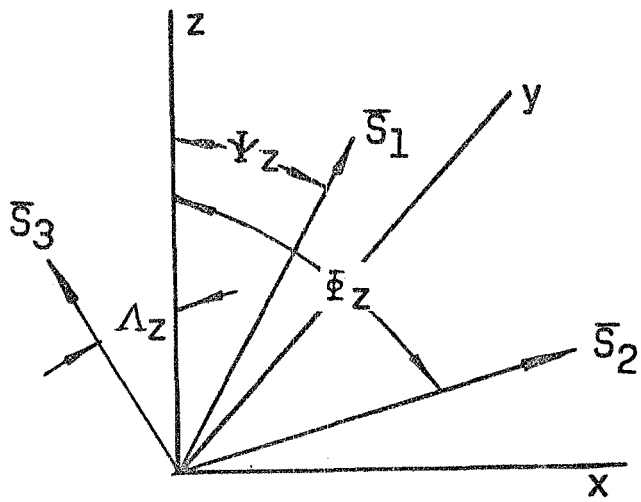
CONCLUSIONS

It is possible to determine the position of the vehicle in the earth-moon system by observing the earth, the moon, and two stars. The position is given by a combination of the distance from the earth and the angle between the lines joining the vehicle to the center of the earth and to the center of the moon or by a combination of the distance from the moon and the same angle. The errors in the position given by each combination are approximately equal at one-half the time required to reach the moon. The combination using the distance from the earth is the most accurate before this time, while the combination using the distance from the moon is the most accurate after this time. There will be a region between the earth and the moon in which the errors in position will be so large that effectively the navigation system is not effective in this region.

It was found that an error of 10 seconds of arc in the measurement of the included angle of the earth and of the moon caused an error in the position which was at least one order of magnitude greater than the errors in position caused by errors in the other quantities. It will be of no use to improve the accuracy of these quantities unless the error in the measurement of the included angle can be reduced to approximately 1 second of arc.

APPENDIX A
DETERMINATION OF ORIENTATION OF xyz AXES

Since it is assumed that the line-of-sight from the vehicle to each of the stars will always be parallel to the line-of-sight from the origin of the set of space axes to the star, there are certain angles which are not functions of the vehicle's position but only of time. This variation with time results from the rotation of the space axes as the moon travels around the earth. The angles which will be used are shown below.



The vectors $\bar{S}_1$ and $\bar{S}_2$ represent the line-of-sight to the stars from the vehicle. The vector $\bar{S}_3$ is the vector product of $\bar{S}_1$ and $\bar{S}_2$. The angles Φ_z, ψ_z, and Λ_z are the angles between the z axes and $\bar{S}_1$, $\bar{S}_2$, and $\bar{S}_3$. If $\bar{S}_1$ and $\bar{S}_2$ are considered to be unit vectors they may be written as:

$$\bar{S}_1 = (\cos \alpha_1)\bar{i} + (\cos \beta_1)\bar{j} + (\cos \gamma_1)\bar{k} \tag{A1}$$

and

$$\bar{S}_2 = (\cos \alpha_2)\bar{i} + (\cos \beta_2)\bar{j} + (\cos \gamma_2)\bar{k}. \tag{A2}$$

The vector product $\bar{S}_3$ is

$$\bar{S}_3 = x_3\bar{i} + y_3\bar{j} + z_3\bar{k} \tag{A3}$$

where

$$x_3 = \cos\beta_1 \cos\gamma_2 - \cos\beta_2 \cos\gamma_1$$

$$y_3 = \cos\alpha_2 \cos\gamma_1 - \cos\alpha_1 \cos\gamma_2$$

$$z_3 = \cos\alpha_1 \cos\beta_2 - \cos\alpha_2 \cos\beta_1 .$$

The direction cosines of $\bar{S}_3$ are

$$\left.\begin{aligned} \cos\alpha_3 &= x_3 \Big/ \sqrt{x_3^2 + y_3^2 + z_3^2} \\ \cos\beta_3 &= y_3 \Big/ \sqrt{x_3^2 + y_3^2 + z_3^2} \\ \cos\gamma_3 &= z_3 \Big/ \sqrt{x_3^2 + y_3^2 + z_3^2} \end{aligned}\right\} \tag{A4}$$

The direction cosines of the z axis in the set of body axes $(x'y'z')$ may now be solved. The angle between any two lines is:

$$\left.\begin{aligned} \cos\Phi_z &= \cos\alpha_1 \cos\alpha_z + \cos\beta_1 \cos\beta_z + \cos\gamma_1 \cos\gamma_z \\ \cos\psi_z &= \cos\alpha_2 \cos\alpha_z + \cos\beta_2 \cos\beta_z + \cos\gamma_2 \cos\gamma_z \\ \cos\Lambda_z &= \cos\alpha_3 \cos\alpha_z + \cos\beta_3 \cos\beta_z + \cos\gamma_3 \cos\gamma_z \end{aligned}\right\} \tag{A5}$$

This set of equations may be rewritten in matrix form to give

$$\left.\begin{bmatrix} \cos\Phi_z \\ \cos\psi_z \\ \cos\Lambda_z \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{bmatrix} \begin{bmatrix} \cos\alpha_z \\ \cos\beta_z \\ \cos\gamma_z \end{bmatrix}\right\} \tag{A6}$$

The inverse of the first matrix on the right side is found and used to solve for $\cos \alpha_z$, $\cos \beta_z$ and $\cos \gamma_z$. Denote this inverse matrix by

$$\left[A_1\right] = \begin{bmatrix} \cos \alpha_1 & \cos \beta_1 & \cos \gamma_1 \\ \cos \alpha_2 & \cos \beta_2 & \cos \gamma_2 \\ \cos \alpha_3 & \cos \beta_3 & \cos \gamma_3 \end{bmatrix}^{-1} \tag{A7}$$

so that

$$\begin{bmatrix} \cos \alpha_z \\ \cos \beta_z \\ \cos \gamma_z \end{bmatrix} = \left[A_1\right] \begin{bmatrix} \cos \Phi_z \\ \cos \psi_z \\ \cos \Lambda_z \end{bmatrix} \tag{A8}$$

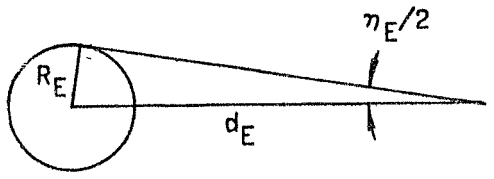
The orientation of the x and z axes may be found in a similar manner.

$$\begin{bmatrix} \cos \alpha_x \\ \cos \beta_x \\ \cos \gamma_x \end{bmatrix} = \left[A_1\right] \begin{bmatrix} \cos \Phi_x \\ \cos \psi_x \\ \cos \Lambda_x \end{bmatrix} \tag{A9}$$

$$\begin{bmatrix} \cos \alpha_y \\ \cos \beta_y \\ \cos \gamma_y \end{bmatrix} = \left[A_1\right] \begin{bmatrix} \cos \Phi_y \\ \cos \psi_y \\ \cos \Lambda_y \end{bmatrix} \tag{A10}$$

APPENDIX B
DISTANCE OF VEHICLE FROM THE EARTH AND THE MOON

The distance between the vehicle and the earth or the moon is determined by the known quantities R_E and R_M and by the measured quantities η_E and η_M. The following sketch illustrates the geometry of the problem:



The distance d_E is

$$d_E = R_E \Big/ \sin \frac{\eta_E}{2} \tag{B1}$$

This may be rewritten as

$$d_E = R_E \left(\frac{2}{1-\cos \eta_E} \right)^{\frac{1}{2}}$$

and similarly (B2)

$$d_M = R_M \left(\frac{2}{1-\cos \eta_M} \right)^{\frac{1}{2}}$$

The errors in d_E and d_M are

$$\Delta d_E = \frac{\partial d_E}{\partial R_E} \Delta R_E + \frac{\partial d_E}{\partial \eta_E} \Delta \eta_E$$

$$\Delta d_M = \frac{\partial d_M}{\partial R_M} \Delta R_M + \frac{\partial d_M}{\partial \eta_M} \Delta \eta_M \tag{B3}$$

These partial derivatives are found using equations (B2) and are:

$$\frac{\partial d_E}{\partial R_E} = \left(\frac{2}{1-\cos\ \eta_E}\right)^{\frac{1}{2}} = \frac{d_E}{R_E}$$

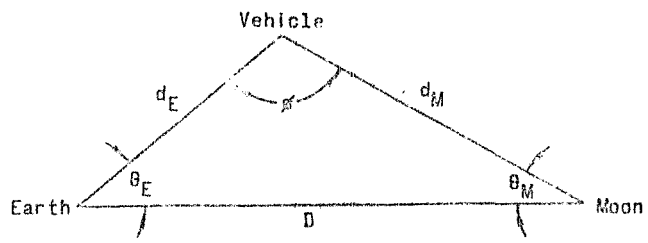
$$\frac{\partial d_E}{\partial \eta_E} = -\left(\frac{d_E}{R_E}\right)^2 \frac{d_E \sin\ \eta_E}{4}$$

$$\frac{\partial d_M}{\partial R_M} = \frac{d_M}{R_M} \qquad \text{(B4)}$$

$$\frac{\partial d_M}{\partial \eta_M} = -\left(\frac{d_M}{R_M}\right)^2 \frac{d_M \sin\ \eta_M}{4}$$

APPENDIX C
DETERMINATION OF THE ANGLES θ_E, θ_M and i

In the plane containing the earth, the moon, and the vehicle, these three points form the triangle shown below:



The first case which will be considered first is when the measured quantities are d_E and ϕ. The angle θ_E is:

$$\theta_E = \pi - \left(\phi + \theta_M\right) \tag{C1}$$

The angle θ_M is found from the law of sines to be:

$$\sin \theta_M = \frac{d_E \sin \phi}{D} \tag{C2}$$

and

$$\cos \theta_M = 1 - \left(\frac{d_E \sin \phi}{D}\right)^2 \tag{C3}$$

Now

$$\sin \theta_E = \sin \phi \cos \theta_M + \cos \phi \sin \theta_M \tag{C4}$$

and

$$\cos \theta_E = \sin \phi \sin \theta_M - \cos \phi \cos \theta_M \tag{C5}$$

When equations (C2) and (C3) are used, equations (C4) and (C5) become

$$\sin \theta_E = \left[1 - \left(\frac{d_E \sin \phi}{D}\right)^2\right]^{\frac{1}{2}} \sin \phi + \frac{d_E \sin \phi \cos \phi}{D}$$

and

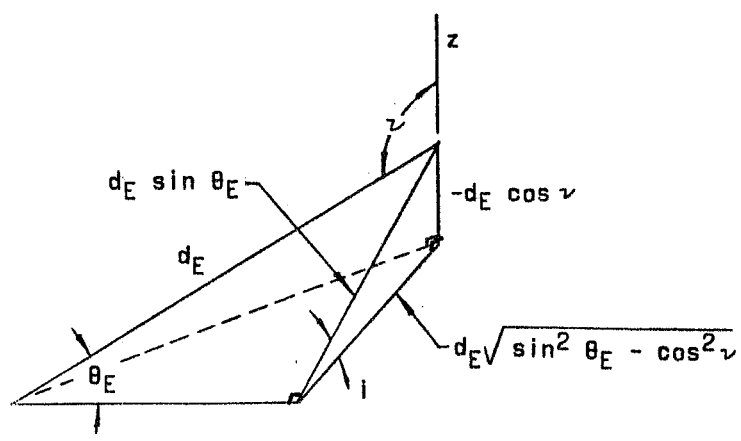
$$\cos \theta_E = \frac{d_E \sin^2 \phi}{D} - \left[1 - \left(\frac{d_E \sin \phi}{D}\right)^2\right]^{\frac{1}{2}} \cos \phi \tag{C6}$$

Similarly,

$$\sin\theta_M = \left[1-\left(\frac{d_M \sin\phi}{D}\right)^2\right]^{\frac{1}{2}} \sin\phi + \frac{d_M \sin\phi\cos\phi}{D}$$

$$\cos\theta_M = \frac{d_M \sin^2\phi}{D} - \left[1-\left(\frac{d_M \sin\phi}{D}\right)^2\right]^{\frac{1}{2}} \cos\phi \tag{C7}$$

The evaluation of i is accomplished using the following figure:



$$\cos i = \frac{d_E\sqrt{\sin^2\theta_E - \cos^2\nu}}{d_E \sin\theta_E} \tag{C8}$$

$$\cos i = \frac{1}{\sin\theta_E}\sqrt{\sin^2\theta_E - \cos^2\nu} \tag{C9}$$

APPENDIX D
DEVELOPMENT OF PARTIAL DERIVATIVES

When the measured quantities are d_E, ϕ and ν, the equations for X, Y, and Z are:

$$\left.\begin{aligned}
X &= \frac{d_E^{\,2}\sin^2\phi}{D} - d_E\,(\cos\phi)\left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^{2-\frac{1}{2}} \\
Y &= d_E\left\{\left[\sin^2\phi\left\{\left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^2 + \frac{2d_E}{D}(\cos\phi)\left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^{2-\frac{1}{2}} + \right.\right.\right. \\
&\qquad \left.\left.\left.\frac{d_E^{\,2}\cos^2\phi}{D^2}\right\} - \cos^2\nu\right]\right\}^{\frac{1}{2}} \\
Z &= -d_E\cos\nu
\end{aligned}\right] \quad (4)$$

The partial derivatives of X are

$$\frac{\partial X}{\partial d_E} = 2\,\frac{d_E\sin^2\phi}{D} - (\cos\phi)\left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^{2-\frac{1}{2}} + \frac{d_E^{\,2}\cos\phi\sin\phi}{D^2}\times \left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^{2-\frac{1}{2}} \qquad \text{(D1)}$$

$$\frac{\partial X}{\partial D} = -\frac{d_E^{\,2}\sin^2\phi}{D^2} - \frac{d_E^{\,3}\cos\phi\sin^2\phi}{D^3}\left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^{2-\frac{1}{2}} \qquad \text{(D2)}$$

$$\frac{\partial X}{\partial \phi} = 2\,\frac{d_E^{\,2}\sin\phi\cos\phi}{D} + d_E\sin\phi\left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^{2-\frac{1}{2}} + \frac{d_E^{\,3}\cos^3\phi\sin\phi}{D^2}\times \left[1-\left(\frac{d_E\sin\phi}{D}\right)\right]^{2-\frac{1}{2}} \qquad \text{(D3)}$$

The partial derivatives of Y are

$$\frac{\partial Y}{\partial d_E} = \frac{Y}{d_E} + \frac{d_E^{\,2} \sin \theta_E \sin \phi}{Y\,D} \left[\cos \phi - \frac{d_E \sin^2 \phi}{D} \left[1 - \left(\frac{d_E \sin \phi}{D} \right)^2 \right]^{-\frac{1}{2}} \right] \tag{D4}$$

$$\frac{\partial Y}{\partial D} = - \frac{d_E^{\,3} \sin \theta_E \sin \phi}{Y\,D^2} \left[\frac{d_E \sin^2 \phi}{D} \left[1 - \left(\frac{d_E \sin \phi}{D} \right)^2 \right]^{-\frac{1}{2}} + \cos \phi \right] \tag{D5}$$

$$\frac{\partial Y}{\partial \phi} = \frac{d_E^{\,2} \sin \theta_E}{Y} \left[\frac{d_E \left(\cos^2 \phi - \sin^2 \phi \right)}{D} + \left[1 - \left(\frac{d_E \sin \phi}{D} \right)^2 \right]^{-\frac{1}{2}} \cos \phi - \frac{d_E^{\,2} \sin^2 \phi \cos \phi}{D^2} \left[1 - \left(\frac{d_E \sin \phi}{D} \right)^2 \right]^{-\frac{1}{2}} \right] \tag{D6}$$

$$\frac{\partial Y}{\partial \nu} = \frac{d_E^{\,2} \cos \nu \sin \nu}{Y} \tag{D7}$$

The partial derivatives of Z are

$$\frac{\partial Z}{\partial d_E} = - \cos \nu \tag{D8}$$

$$\frac{\partial Z}{\partial \nu} = d_E \sin \nu \tag{D9}$$

If the measured quantities are d_M, ϕ and λ the equations for X, Y and Z are:

$$X = D - \frac{d_M^{\,2} \sin^2 \phi}{D} + d_M(\cos \phi) \left[1 - \left(\frac{d_M \sin \phi}{D}\right)^2\right]^{\frac{1}{2}}$$

$$Y = d_M \left[\sin^2 \phi \left\{\left[1 - \left(\frac{d_M \sin \phi}{D}\right)^2\right] + \frac{2d_M}{D} (\cos \phi) \left[1 - \left(\frac{d_M \sin \phi}{D}\right)^2\right]^{\frac{1}{2}} + \frac{d_M^{\,2} \cos^2 \phi}{D^2}\right\} - \cos^2 \lambda\right]^{\frac{1}{2}} \tag{5}$$

$$Z = - d_M \cos \lambda$$

The partial derivatives of X are

$$\frac{\partial X}{\partial d_M} = 2 \frac{d_M \sin^2 \phi}{D} - (\cos \phi) \left[1 - \left(\frac{d_M \sin \phi}{D}\right)^2\right]^{\frac{1}{2}} + \frac{d_M^{\,2} \cos \phi \sin \phi}{D^2} \times \left[1 - \left(\frac{d_M \sin \phi}{D}\right)^2\right]^{-\frac{1}{2}} \tag{D10}$$

$$\frac{\partial X}{\partial D} = 1 + \frac{d_M^{\,2} \sin^2 \phi}{D^2} + \frac{d_M^{\,2} \cos \phi \sin^2 \phi}{D^3} \left[1 - \left(\frac{d_M \sin \phi}{D}\right)^2\right]^{-\frac{1}{2}} \tag{D11}$$

$$\frac{\partial X}{\partial \phi} = 2 \frac{d_M^{\,2} \sin \phi \cos \phi}{D} + d_M (\sin \phi) \left[1 - \left(\frac{d_M \sin \phi}{D}\right)^2\right]^{\frac{1}{2}} + \frac{d_M^{\,3} \cos^3 \phi \sin \phi}{D^2} \times \left[1 - \left(\frac{d_E \sin \phi}{D}\right)^2\right]^{-\frac{1}{2}} \tag{D12}$$

The partial derivatives of Y are

$$\frac{\partial Y}{\partial d_M} = \frac{Y}{d_M} + \frac{d_M^{\,2} \sin \theta_M \sin \phi}{Y\,D}\left[\cos \phi - \frac{d_M \sin^2\phi}{D}\left[1-\left(\frac{d_M \sin \phi}{D}\right)^2\right]^{-\frac{1}{2}}\right] \tag{D13}$$

$$\frac{\partial Y}{\partial D} = -\frac{d_M^{\,3} \sin \theta_M \sin \phi}{Y\,D^2}\left[\frac{d_M \sin^2\phi}{D}\left[1-\left(\frac{d_M \sin \phi}{D}\right)^2\right]^{-\frac{1}{2}} + \cos \phi\right] \tag{D14}$$

$$\frac{\partial Y}{\partial \phi} = \frac{d_M^{\,2} \sin \theta_M}{Y}\left[\frac{d_M\left(\cos^2\phi - \sin^2\phi\right)}{D} + \left[1-\left(\frac{d_M \sin \phi}{D}\right)^2\right]^{\frac{1}{2}} \cos \phi - \frac{d_M^{\,2} \sin^2\phi \cos \phi}{D^2}\left[1-\left(\frac{d_M \sin \phi}{D}\right)^2\right]^{-\frac{1}{2}}\right] \tag{D15}$$

$$\frac{\partial Y}{\partial \lambda} = \frac{d_M^{\,2} \cos \lambda \sin \lambda}{Y} \tag{D16}$$

The partial derivatives of Z are

$$\frac{\partial Z}{\partial d_M} = -\cos \lambda \tag{D17}$$

$$\frac{\partial Z}{\partial \lambda} = d_M \sin \lambda \tag{D18}$$

REFERENCES

1. Hibbs, Albert R.: Exploration of the Moon, the Planets and Interplanetary Space. Rep. No. 30-1, Jet Propulsion Labs., Calif. Inst. of Tech., Apr. 30, 1959.

2. Press, S. James: An Application of Solar Radiation to Space Navigation. Aero/Space Eng., vol. 17, no. 11, Nov. 1958, pp. 51-54.

3. Carroll, Joel J. and Savet, Paul H.: Space Navigation and Exploration by Gravity Difference Detection. Paper No. 59-91, Presented at the Inst. Aero. Sci. Nat. Summer Meeting, Los Angeles, Calif., Jun. 16-19, 1959.

4. Battin, Richard H., PhD.: Inertial Guidance - Terrestrial and Interplanetary, Interplanetary Trajectory Analysis, Navigation and Control. Dept. of Aeronautics and Astronautics, Mass. Inst. of Tech., vol. 3, pp. IX-1–IX-57, Jul. 1960.

5. Carroll, Joel J.: Interplanetary Navigation by Optical Resection and Inertial Systems. Aero/Space Eng., vol. 18, no. 3, Mar. 1959, pp. 53-55, 75.

6. Larmore, Lewis: Celestial Observations for Space Navigation. Aero/Space Eng., vol. 19, no. 1, Jan. 1959, pp. 37-42.

7. Baker, Robert M. L., Jr., and Makemson, Maud W.: An Introduction to Astrodynamics. Academic Press, 1960.

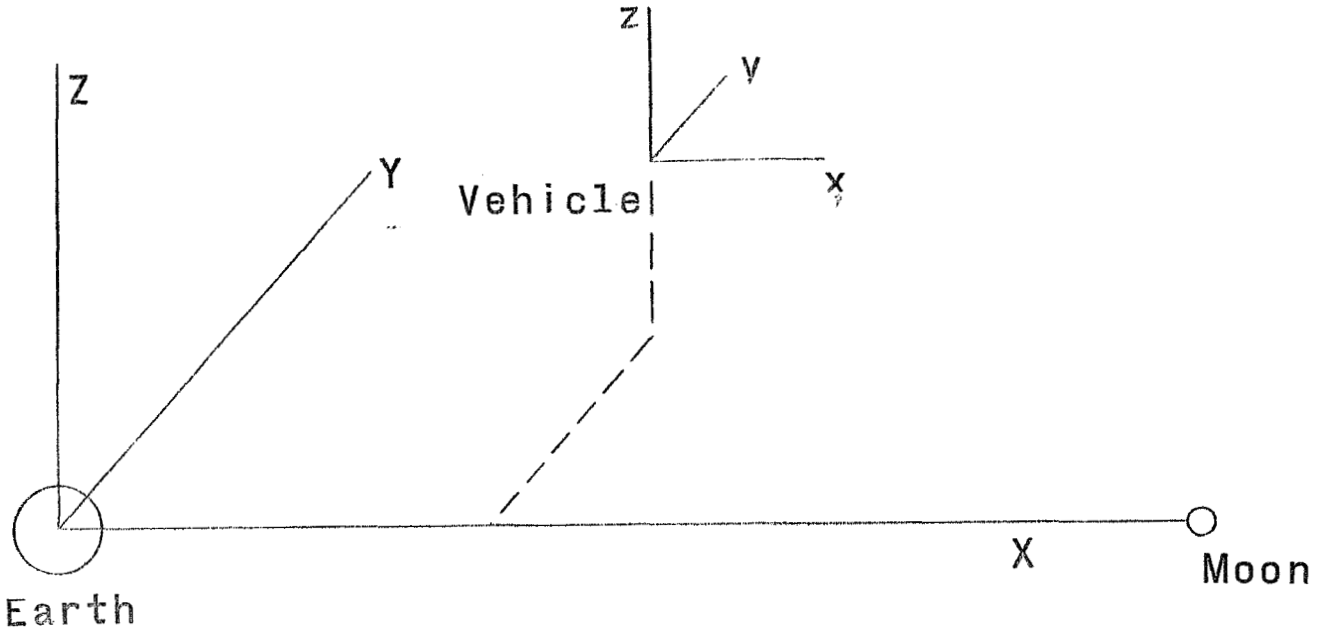


Figure 1.- XYZ and xyz coordinate systems.

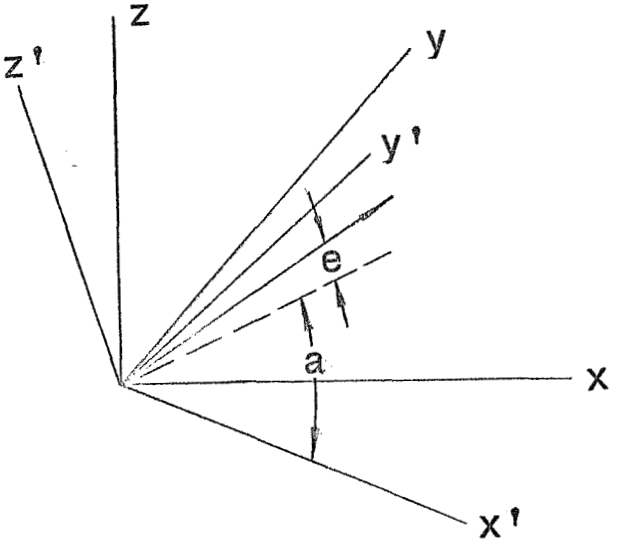


Figure 2.- Body axes.

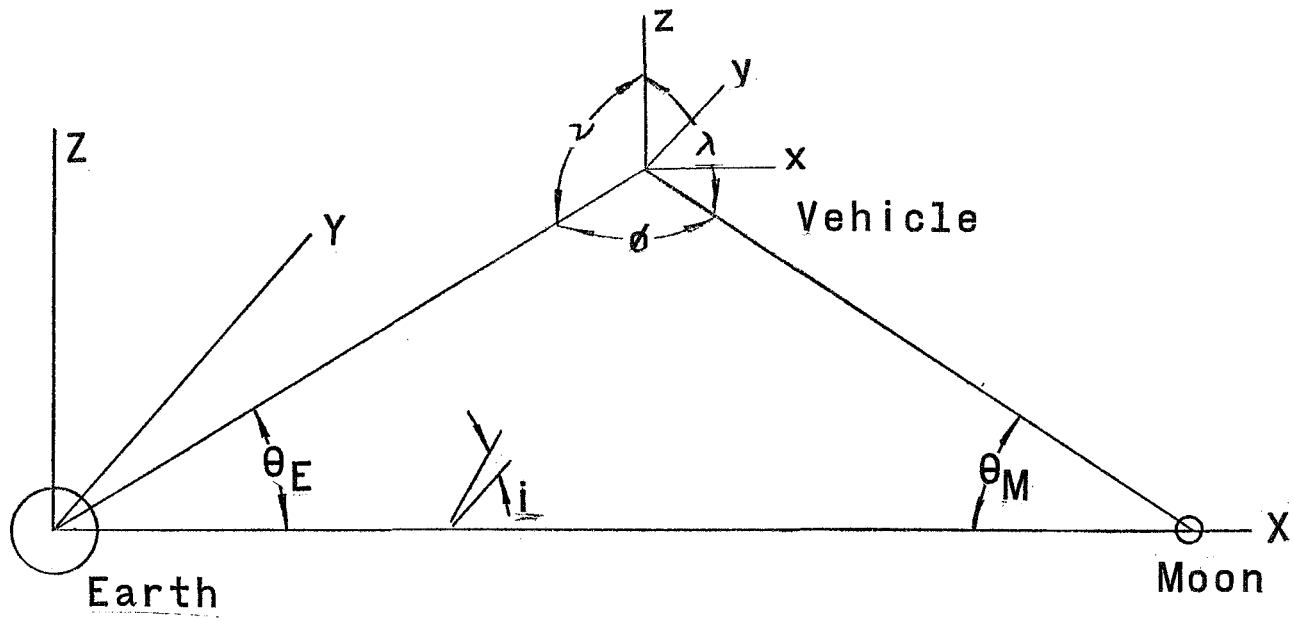


Figure 3.- Geometry of navigation system in three dimensions.

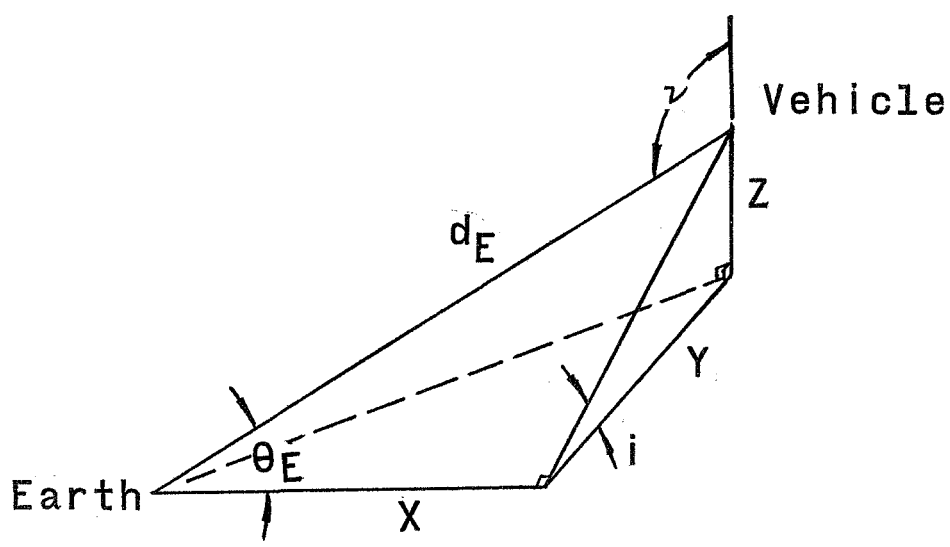


Figure 4.- Geometry used to determine X, Y and Z.

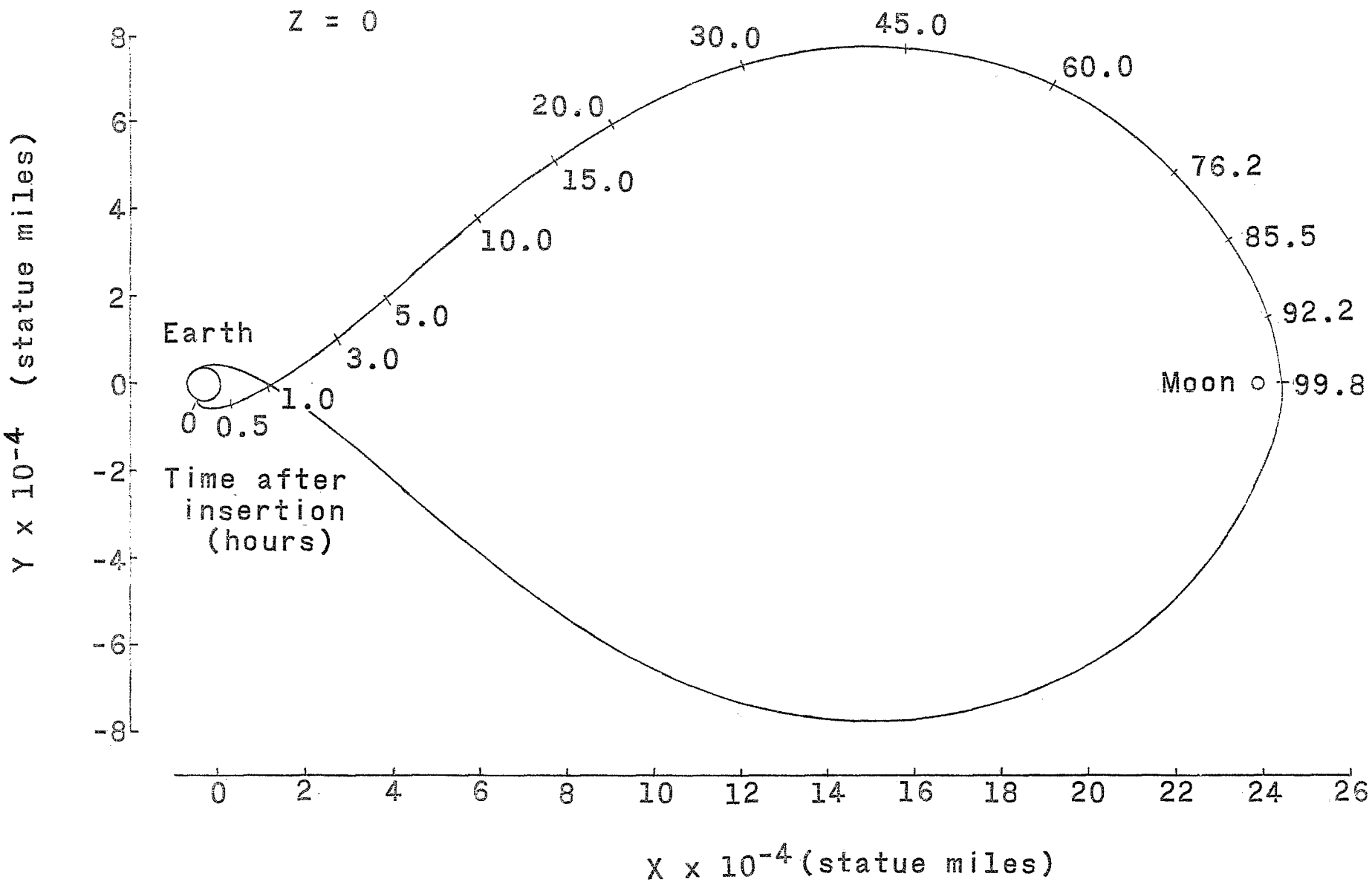
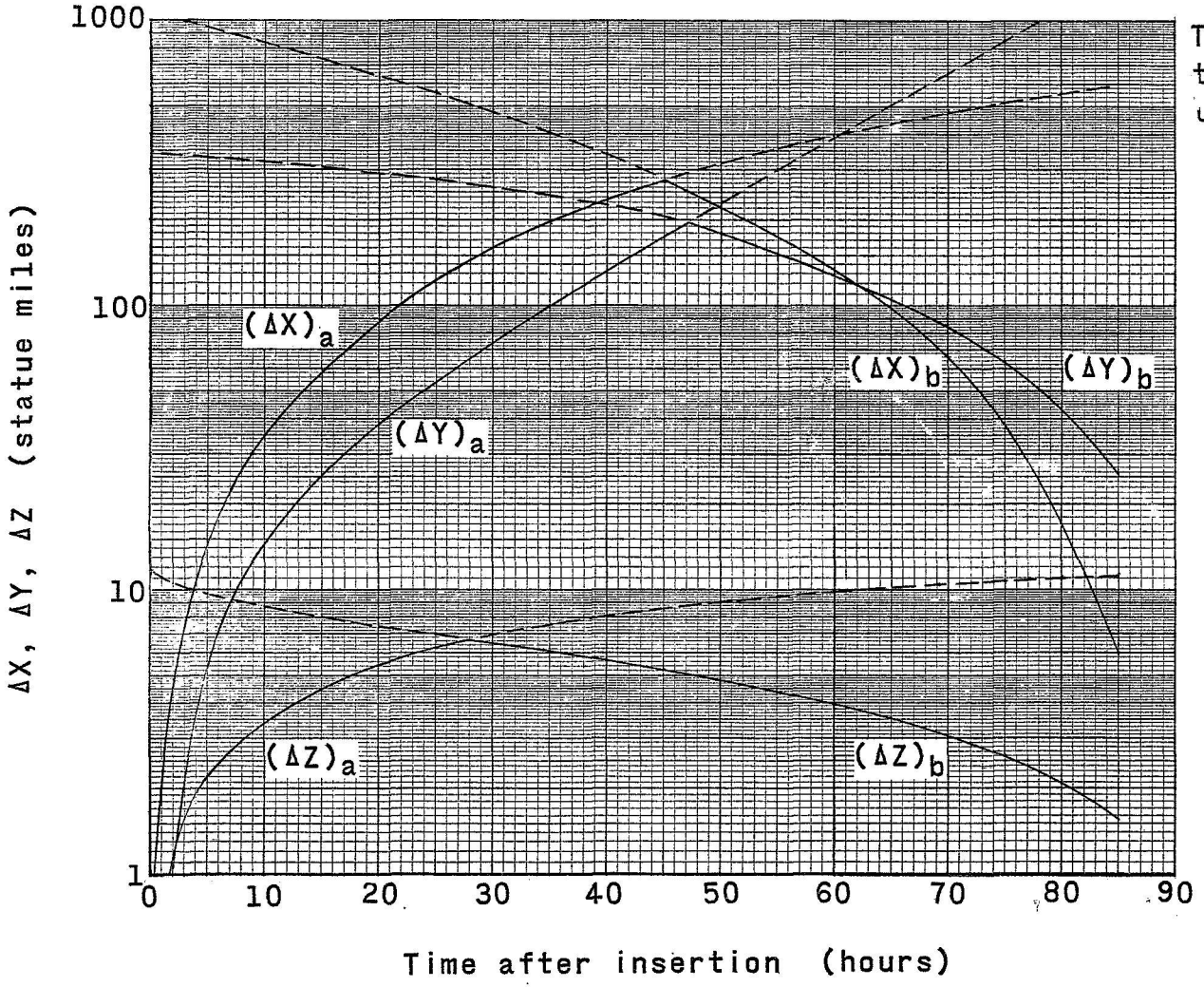
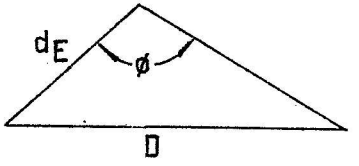


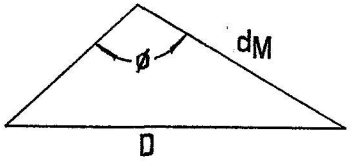
Figure 5.- Reference trajectory.



The solid lines represent the minimum error obtained using either method.



Method a



Method b

Figure 6.- Errors in X,Y, and Z as functions of time.