Univ. of Houston

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EXTENSIONS OF LOCALLY COMPACT ABELIAN GROUPS

Ronald O. Fulp and Phillip A. Griffith[1]

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This paper is concerned with homological algebra in the category of locally compact abelian groups and with certain of its applications. In the first part of the paper we lay down the necessary homological framework for the work which is to follow. This framework provides for the development of a functor Ext which generalizes the usual functor Ext as is defined in (discrete) abelian group theory and also generalizes the functor Ext discussed by Moskowitz [9]. We show that Ext has the properties it should have with the one exception that we have not yet determined whether or not Ext is right exact (we do show that in many cases Ext is right exact).

The second part of the paper (roughly from section 3 on) deals with some applications. In particular, we completely determine the structure of those connected locally compact abelian groups G having the property that whenever H is locally compact abelian and H has G as its component, then G is a direct summand of H. We actually prove a much better theorem (Theorem 6.3). Other theorems of interest are obtained along the way. For example, the injectives and projectives of the category of totally disconnected locally compact abelian groups are obtained. Also slight improvement of one of the Noether isommorphism theorems is made.

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We know of only a few papers dealing with the general subject under investigation. There are two basic papers on extensions of Lie groups by Hochschild [4] and [5]. Some results on the lower dimensional cohomology groups by Moore [7] and [8] bear on our subject. In all of these papers the topological groups investigated have rather severe topological restrictions which we do not have. On the other hand we work only with abelian groups whereas their work includes groups which are not necessarily abelian.

Actually our work was inspired by the paper [9] due to Moskowitz. Certain of the ideas initiated by Moskowitz on homological algebra in the category of locally compact abelian groups will be basic to our work. Thus we shall assume some familiarity with his paper and with basic facts from the theory of topological groups as are set forth in Pontryagin [10] or Hewitt and Ross [3]. Moreover our development of the Ext functor requires a rather detailed knowledge of the corresponding collection of ideas in the theory of discrete abelian groups as is exposed in Chapter 3 of the book [6] by MacLane.

The category with which we deal is the category of locally compact abelian groups with continuous homomorphisms as morphisms. We shall denote this category by $\mathcal{L}$. Actually most of our results deal only with those morphisms of $\mathcal{L}$ which are open onto their image. Such a morphism will be called a <u>proper</u> morphism. The class of all proper morphisms of $\mathcal{L}$ is not a subcategory of $\mathcal{L}$, but it does have several nice properties which we outline below.

(1) If $\varphi : A \to B$ is a proper morphism of $\mathcal{L}$ then $\operatorname{Im} \varphi = \varphi(A)$ is a closed subgroup of B.

(2) A proper morphism of $\mathcal{L}$ is monic in $\mathcal{L}$ if and only if it is a monomorphism.

(3) A proper morphism of $\mathcal{L}$ is epic in $\mathcal{L}$ if and only if it is an epimorphism.

(4) If $\varphi: A \to B$ and $\theta: B \to C$ are proper morphisms of $\mathcal{L}$, then $\theta\varphi$ is a proper morphism if either φ is epic or θ is monic.

We now lay down some notation and terminology to be used throughout the paper. If $\varphi : A \to B$ and $\theta: B \to C$ are morphisms in $\mathcal{L}$, then the sequence $A \xrightarrow{\varphi} B \xrightarrow{\theta} C$ is <u>exact at</u> B if and only if $\operatorname{Im}\varphi = \ker\theta$, <u>i.e.</u>, if and only if $\varphi(A) = \theta^{-1}(0)$. A longer sequence of morphisms of $\mathcal{L}$

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n$$

is <u>exact</u> if and only if for each $2 \leq i \leq n-1$, the sequence $A_{i-1} \to A_i \to A_{i+1}$ is exact at A_i. A sequence $A \xrightarrow{\varphi} B \xrightarrow{\theta} C$ is <u>short exact</u> if and only if it is exact, φ is a monomorphism, and θ is an epimorphism. In general, if $\varphi: A \to B$ is a morphism of $\mathcal{L}$, we write $A \overset{\varphi}{\rightarrowtail} B$ whenever φ is a monomorphism and $A \overset{\varphi}{\twoheadrightarrow} B$ whenever φ is an epimorphism. Any undefined homological notation or terminology is taken from MacLane [6].

An exact sequence of morphisms of

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} A_n$$

is said to be <u>proper exact</u> if and only if φ_i is proper for each i. More generally, if D is a diagram with morphisms in $\mathcal{L}$, then D is a <u>proper</u>

diagram if and only if each morphism of D is a proper morphism. Thus we speak of proper pullbacks, proper pushouts, etc.

Throughout the paper we repeatedly use the duality theory of locally compact abelian groups. If G is a locally compact abelian group, then the dual of G, denoted $G^\wedge$, is the topological group of all continuous characters of G. If $\varphi : G \to H$ is a morphism in $\mathcal{L}$, then the map $\varphi^\wedge : H^\wedge \rightarrowtail G^\wedge$ defined by $\varphi^\wedge(\chi) = \chi \circ \varphi$ is also a morphism of and will be called the dual of φ. If $E: A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is a short exact sequence in $\mathcal{L}$, then the dual of E, denoted $E^\wedge$, is the short exact sequence $C^\wedge \overset{\theta^\wedge}{\rightarrowtail} B^\wedge \overset{\varphi^\wedge}{\twoheadrightarrow} A^\wedge$. In general, if D is a diagram of $\mathcal{L}$, then the dual of D is the diagram obtained by taking duals of each of the morphisms of D.

The additive groups of real numbers, rational numbers, integers, and real numbers modulo 1 will be denoted by R, Q, Z, and T, respectively. If A is a subset of a topological space, A^- will denote the closure of A. If G is a topological group, G_o will denote the identity component of G.

2. Homological Algebra in $\mathcal{L}$.

Our first result in this section establishes a criteria for determining proper morphisms in $\mathcal{L}$ and is the foundation upon which our discussion of homological algebra in $\mathcal{L}$ rests.

Theorem 2.1. A morphism ψ in $\mathcal{L}$ is proper if and only if there are proper morphisms α and β in $\mathcal{L}$ such that

(i) $\psi\alpha$ and $\beta\psi$ are defined and are proper

(ii) $\operatorname{Im} \psi\alpha = \ker \beta$.

Proof. First suppose that ψ is a morphism in $\mathcal{L}$ and that there

are proper morphisms α and β satisfying (i) and (ii) above. We show that ψ is open onto its range. Observe that (i) and (ii) yield the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "g"'] & B \arrow[d, "\psi"] \arrow[dr, "f"] & \\
& C \arrow[r, "\beta"] & D
\end{tikzcd}$$

where α, β, f and g are proper and $\operatorname{Im} g = \ker \beta$. By Proposition 2.2 [9], $\operatorname{Im}\alpha$, $\operatorname{Im}\beta$, $\operatorname{Im} f$ and $\operatorname{Im} g$ are closed in their respective containing groups. We show that $\psi(U)$ has nonvoid interior in $\operatorname{Im}\psi$ and hence that ψ is proper. Choose U_1 a neighborhood of zero in B such that $U_1 + U_1 \subseteq U$. Since f, g and α are proper, there are neighborhoods S_1 and W_1 in D and C, respectively, such that $S_1 \cap \operatorname{Im} f = f(U_1)$ and $W_1 \cap \operatorname{Im} g = g\alpha^{-1}(U_1 \cap \operatorname{Im}\alpha)$. Noting that $W_1 \cap \beta^{-1}(S_1)$ is a neighborhood of zero in C, we choose a neighborhood V_1 of zero such that $V_1 + (-V_1) \subseteq W_1 \cap \beta^{-1}(S_1)$. Since ψ is continuous, $U_2 = U_1 \cap \psi^{-1}(V_1)$ is a neighborhood of zero in B. As before, there are neighborhoods S_2 and W_2 in D and C, respectively, such that $S_2 \subseteq S_1$ and $W_2 \subseteq W_1$ and such that $S_2 \cap \operatorname{Im} f = f(U_2)$ and $W_2 \cap \operatorname{Im} g = g\alpha^{-1}(U_2 \cap \operatorname{Im}\alpha)$. It follows that $V_2 = V_1 \cap W_2 \cap \beta^{-1}(S_2)$ is a neighborhood of zero in C. Let $\psi(b) \in V_2$. Then $\beta\psi(b) = f(b) \in S_2 \cap \operatorname{Im} f = f(U_2)$ and hence $\beta\psi(b) = f(u_2) = \beta\psi(u_2)$ for some $u_2 \in U_2$. Observing that $\psi(u_2) \in V_1$, we have that $\psi(b) - \psi(u_2) \in [V_1 + (-V_1)] \cap \operatorname{Ker}\beta \subseteq W_1 \cap \operatorname{Im} g = g\alpha^{-1}(U_1 \cap \operatorname{Im}\alpha)$. Therefore $\psi(b) - \psi(u_2) = g(a)$ where $\alpha(a) \in U_1$. It follows that $u_2 + \alpha(a) \in U_1 + U_1 \subseteq U$ and that $\psi(u_2 + \alpha(a)) = \psi(u_2) + \psi\alpha(a) = \psi(u_2) + g(a) = \psi(b)$. Thus $V_2 \cap \operatorname{Im}\psi \subseteq \psi(U)$ and $\psi(U)$ has nonvoid interior

in $\mathrm{Im}\psi$.

To establish the necessity, let $\psi: A \to X$ be a proper morphism in $\mathcal{L}$. Since $\mathrm{Im}\psi$ is necessarily closed in X, the natural projection β of X onto $X/\mathrm{Im}\psi$ is proper. Let α be the identity morphism from A onto A. To see that (i) and (ii) are satisfied is elementary.

<u>Corollary 2.2.</u> Assume that

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads, "\delta"'] & B \arrow[r] \arrow[d, "\psi"] & C \arrow[d, "\theta"] \\ A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd}$$

is a commutative diagram in $\mathcal{L}$ with proper exact rows. If δ is a proper epimorphism and θ is a proper monomorphism, then ψ is proper.

<u>Proposition 2.3.</u>. Let $\alpha: A \to X$ and $\beta: B \to X$ be morphisms in $\mathcal{L}$ and suppose that the diagram

$$\begin{tikzcd} P \arrow[r, "\lambda"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & X \end{tikzcd}$$

is a pullback for α and β. If α is proper, then so is δ. The dual statement for pushouts is also true.

<u>Proof.</u> Let M be the subgroup of $A \oplus B$ defined by $M = \{(a,b) \mid \alpha(a) = \beta(b)\}$. Then M is closed in $A \oplus B$ and hence M is in $\mathcal{L}$. The diagram

$$\begin{tikzcd} M \arrow[r, "\gamma"] \arrow[d, "\mu"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & X \end{tikzcd}$$

is a pullback for α and β where μ: $(a,b) \to b$ and γ: $(a,b) \to a$. Henceforth, we shall refer to this diagram as the <u>standard pullback</u> for α and β. Consider a neighborhood $W = (U \times V) \cap M$ of zero in M where U and V are neighborhoods of zero in A and B, respectively. Since α is proper, there is a neighborhood E of X such that $E \cap \mathrm{Im}\alpha = \alpha(U)$. Then $S = \beta^{-1}(E \cap \mathrm{Im}\beta) \cap V$ is a neighborhood of zero in B and $\mathrm{Im}\mu \cap S \subseteq \mu(W)$. Thus μ is proper. If the diagram

$$\begin{tikzcd} P \arrow[r, "\lambda"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & X \end{tikzcd}$$

is also a pullback for α and β, we obtain a commutative diagram

$$\begin{tikzcd} P \arrow[dr, "\theta"] \arrow[drr, "\lambda"] \arrow[ddr, "\delta"'] & & \\ & M \arrow[r, "\gamma"] \arrow[d, "\mu"] & A \arrow[d, "\alpha"] \\ & B \arrow[r, "\beta"] & X \end{tikzcd}$$

where θ is an isomorphism in $\mathcal{L}$. It easily follows from the commutativity of the above diagram that δ is proper. Pontryagin duality yields the dual statement for pushouts.

<u>Corollary 2.4.</u> Let A, B and X be in $\mathcal{L}$. If $\alpha: A \to X$ and $\beta: B \to X$ are proper morphisms in $\mathcal{L}$, then every pullback for α and β is a proper pullback. Dually, if $\gamma: X \to A$ and $\delta: X \to B$ are proper morphisms in $\mathcal{L}$, then every pushout for γ and δ is a proper pushout.

We note that it is not always possible to construct pushouts in $\mathcal{L}$ in the "canonical fashion". In particular, the pushout for the pair of morphisms $\alpha, \beta: R \to R/Z$ defined by $\alpha(x) = r_1 x + Z$ and $\beta(x) = r_2 x + Z$ for distinct fixed irrational numbers r_1 and r_2 may not be obtained in the standard way. However, when this construction is possible we shall refer to it as the standard pushout. The proof of the following proposition is straightforward and is omitted.

<u>Proposition 2.5.</u> If $\alpha: X \to A$ and $\beta: X \to B$ are morphisms in $\mathcal{L}$ and if $N = \{(-\alpha(x), \beta(x)) \mid x \in X\}$ is closed in $A \oplus B$, then α and β have a standard pushout. Moreover, if either α or β is a proper monomorphism, then N is closed in $A \oplus B$.

<u>Proposition 2.6.</u> Assume that

$$\begin{tikzcd}
A_1 \arrow[r, tail, "\varphi_1"] \arrow[d, "\alpha_1"'] & B_1 \arrow[r, two heads, "\theta_1"] \arrow[d, "\beta_1"] & C_1 \arrow[d, "\gamma_1"] \\
A \arrow[r, tail, "\varphi"] & B \arrow[r, two heads, "\theta"] & C
\end{tikzcd}$$

is a commutative diagram in $\mathcal{L}$ with proper short exact rows. If α_1 is proper and γ_1 is a proper monomorphism, then β_1 is proper.

<u>Proof.</u> Consider the commutative diagram

$$\begin{tikzcd}
E_1: & A_1 \arrow[r, tail, "\varphi_1"] \arrow[d, "\alpha_1"'] & B_1 \arrow[r, two heads, "\theta_1"] \arrow[d, "\beta_2"] & C_1 \arrow[d, equal] \\
\alpha_1 E_1: & A \arrow[r, tail, "\varphi_2"] \arrow[d, equal] & B_1' \arrow[r, two heads, "\theta_2"] \arrow[d, "\beta_3"] & C_1 \arrow[d, equal] \\
E\gamma_1: & A \arrow[r, tail, "\varphi_3"] \arrow[d, equal] & B' \arrow[r, two heads, "\theta_3"] \arrow[d, "\beta"] & C_1 \arrow[d, "\gamma_1"] \\
E: & A \arrow[r, tail, "\varphi"] & B \arrow[r, two heads, "\theta"] & C
\end{tikzcd}$$

where $\alpha_1 E_1$ is the standard pushout of φ_1 and α_1 and $E\gamma_1$ is the standard pullback of θ and γ_1. Recall that $B_1' = (A \oplus B_1)/N$ where $N = \{(-\alpha(a_1), \varphi_1(a_1)) : a_1 \in A_1\}$. Define $\beta_3 : B_1' \to B'$ by

$$\beta_3 : (a, b_1) + N \to \varphi_3(a) + (\beta_1(b_1), \theta_1(b_1)).$$

The map β_3 is a continuous one-to-one, surjective homomorphism. By Corollary 2.2, it follows that β_3 is an isomorphism of $\mathcal{L}$ and that β is a proper monomorphism in $\mathcal{L}$. Since $\beta\beta_3$ is a proper monomorphism in $\mathcal{L}$, $\beta_1 = (\beta\beta_3)\beta_2$ is a proper morphism in $\mathcal{L}$.

Applying Pontryagin duality we obtain

<u>Proposition 2.7.</u> Assume that

$$\begin{tikzcd}
A_1 \arrow[r, tail, "\varphi_1"] \arrow[d, two heads, "\alpha_1"'] & B_1 \arrow[r, two heads, "\theta_1"] \arrow[d, "\beta_1"] & C_1 \arrow[d, "\gamma_1"] \\
A \arrow[r, tail, "\varphi"] & B \arrow[r, two heads, "\theta"] & C
\end{tikzcd}$$

is a commutative diagram in $\mathcal{L}$ with proper short exact rows. If α_1 is a proper epimorphism and γ_1 is proper, then β_1 is proper.

Our concluding result on commutative diagrams in $\mathcal{L}$ establishes a "Five Lemma" with respect to proper morphisms.

<u>Proposition 2.8.</u> Assume that the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\lambda"] \arrow[d, "\psi_1"] & B \arrow[r, "\varphi"] \arrow[d, "\psi_2"] & C \arrow[r, "\theta"] \arrow[d, "\psi_3"] & D \arrow[r, "\sigma"] \arrow[d, "\psi_4"] & E \arrow[d, "\psi_5"] \\
A' \arrow[r, "\lambda'"] & B' \arrow[r, "\varphi'"] & C' \arrow[r, "\theta'"] & D' \arrow[r, "\sigma'"] & E'
\end{tikzcd}$$

has proper exact rows. If ψ_1, ψ_2, ψ_4 and ψ_5 are proper isomorphisms, so is ψ_3. More precisely,

(i) if ψ_1 is a proper epimorphism and ψ_2 and ψ_4 are proper monomorphisms, then ψ_3 is a proper monomorphism.

(ii) if ψ_5 is a proper monomorphism and ψ_2 and ψ_4 are proper epimorphisms, then ψ_3 is a proper epimorphism.

<u>Proof.</u> Since (ii) can be obtained from (i) by Pontryagin duality, it suffices to establish (i). Hence assume that ψ_1 is a proper epimorphism and

that ψ_2 and ψ_4 are proper monomorphisms. We need only show that ψ_3 is proper. We first show that $\varphi'\psi_2$ is proper. Let U be a neighborhood of zero in B. Since ψ_2 is proper, there is a neighborhood V of B' such that $V \cap \mathrm{Im}\psi_2 = \psi_2(U)$. Let c' be in the neighborhood $\varphi'(V) \cap \mathrm{Im}\varphi'\psi_2$ of $\mathrm{Im}\varphi'\psi_2$. Then there are elements $v \in V$ and $b \in B$ such that $c' = \varphi'(v) = \varphi'\psi_2(b)$ which implies that $v - \psi_2(b) \in \ker \varphi' = \mathrm{Im}\lambda'$. Since ψ_1 is surjective, we have that $\mathrm{Im}\lambda' \subseteq \mathrm{Im}\psi_2$. Hence there is an element $b_o \in B$ such that $v - \psi_2(b) = \psi_2(b_o)$, that is, $v = \psi_2(b_o + b) \in V \cap \mathrm{Im}\psi_2 = \psi_2(U)$. Since ψ_2 is a monomorphism, $b_o + b$ must be in U. Therefore, $c' \in \varphi' \psi_2(U)$. Since clearly $\varphi'(V) \cap \mathrm{Im}\varphi'\psi_2 \supseteq \varphi' \psi_2(U)$, it follows that $\varphi'(V) \cap \mathrm{Im}\, \varphi'\psi_2 = \varphi'\psi_2(U)$. Thus $\varphi'\psi_2$ is proper. Note that $\psi_4\theta$ is a proper morphism since ψ_4 is monic. Thus ψ_3 is a morphism such that $\psi_3\varphi$ and $\theta'\psi_3$ are proper. Since also $\mathrm{Im}\, \varphi = \ker \varphi'\psi_3$ we may apply the dual of Theorem 2.1, to obtain that ψ_3 is proper.

Moskowitz [9] has shown that $\mathcal{L}$ has neither enough projectives nor enough injectives. However, Moskowitz does show that if a group G in $\mathcal{L}$ has a projective or injective resolution, then G has a short projective or injective resolution, respectively. Our next result shows that one may approximate projective and injective resolutions for an arbitrary group in $\mathcal{L}$. These resolutions aid in certain computations of the Ext functor.

<u>Theorem 2.9.</u> Let G be a group in $\mathcal{L}$. For each compact subgroup K of G such that G/K has no small subgroups, there exist proper short exact sequences

(1) $\quad G \rightarrowtail I \oplus G/K \twoheadrightarrow J$ where I and J are compact injectives, and

(2) $P \rightarrowtail K \oplus F \twoheadrightarrow G$ where P and F are free projectives in $\mathcal{L}$.

<u>Remark.</u> Suppose G is any group in $\mathcal{L}$. By Moskowitz's Theorem 2.7 [9], for each neighborhood U of zero in G, there is a compact subgroup $K \subseteq U$ such that G/K has no small subgroups. Thus the latter theorem applies to arbitrarily small subgroups K of any G in $\mathcal{L}$.

Proof of the theorem. Clearly it suffices to prove (1). Since K is compact, K has a short injective resolution $K \overset{\lambda}{\rightarrowtail} I \twoheadrightarrow L$. We now obtain the commutative diagram

$$\begin{tikzcd}
K \arrow[r, tail, "\delta"] \arrow[d, tail, "\lambda"'] & G \arrow[r, two heads] \arrow[d, tail, "\varphi"] & G/K \arrow[d, equal] \\
I \arrow[r, tail, "\rho"] \arrow[d, two heads] & E \arrow[r, two heads] \arrow[d, two heads, "\theta"] & G/K \\
L \arrow[r, two heads, "\sigma"] & J &
\end{tikzcd}$$

where the second row is a proper pushout for λ and δ and the morphism θ is the natural projection of E onto $E/\mathrm{Im}\,\varphi$. The morphism σ is induced by ρ and δ. Since L is compact, it follows that σ is proper (see Hewitt and Ross [3]). Hence J is a compact injective. Furthermore, since I is injective, the second row splits and thus $E \cong I \oplus G/K$.

If A and C are in $\mathcal{L}$, then an extension of A by C is a proper short exact sequence E: $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ where φ and θ are morphisms in $\mathcal{L}$. If E: $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ and E': $A \overset{\varphi'}{\rightarrowtail} B' \overset{\theta'}{\twoheadrightarrow} C'$ are extensions of A by C, then E

and E' are said to be congruent if and only if there is a morphism $\psi: B \to B'$ in $\mathcal{L}$ such that the diagram

$$\begin{tikzcd}
 & B \arrow[dr, "\theta"] \arrow[dd, "\psi"] & \\
A \arrow[ur, tail, "\varphi"] \arrow[dr, tail, "\varphi'"'] & & C \\
 & B' \arrow[ur, "\theta'"'] &
\end{tikzcd}$$

is commutative. It is a consequence of Corollary 2.2, that ψ is an isomorphism of $\mathcal{L}$. Denote by Ext(C,A) the set of all congruence classes of extensions of A by C. We intend to show that Ext is a functor from $\mathcal{L} \times \mathcal{L}$ into the category of discrete abelian groups which is contravariant in its first argument and covariant in the second. In view of the preceding results and Pontryagin duality, it suffices to establish Lemma 1.3 [6] since one may then follow MacLane's exposition to obtain the desired result.

<u>Proposition 2.10.</u> Assume that $\Gamma_1 = (\alpha_1, \beta_1, \gamma_1): E_1 \longrightarrow E$ is a proper morphism from the extension $E_1: A_1 \overset{\varphi_1}{\rightarrowtail} B_1 \overset{\theta_1}{\twoheadrightarrow} C_1$ to the extension $E: A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$. Then Γ_1 has a unique factorization $E_1 \xrightarrow{(\alpha_1, \beta', 1)} E\gamma_1 \xrightarrow{(1, \beta, \gamma_1)} E$ where $(\alpha_1, \beta', 1)$ and $(1, \beta, \gamma_1)$ are proper morphisms.

Proof. Consider the commutative diagram of the proof of Proposition 2.6 (in the case we are considering here γ_1 is not necessarily monic). By Corollary 2.2, β_3 is an isomorphism in $\mathcal{L}$. By Corollary 2.4, β and β_2 are proper morphisms in $\mathcal{L}$. Since β_3 is an isomorphism, it follows that $\beta' = \beta_3 \beta_2$ is a proper morphism in $\mathcal{L}$. Thus $(\alpha_1, \beta', 1)$ and $(1, \beta_1, \gamma_1)$ are proper morphisms. The uniqueness follows from the uniqueness of proper pullbacks.

For each extension E of A by B, let $[E]$ denote the class of all extensions E' of A by B such that E' is congruent to E.

Theorem 2.11. Assume that A and B are in $\mathcal{L}$. Then the class $\mathrm{Ext}(B,A)$ of all congruence classes of extensions of A by B is an abelian group with respect to the operation defined by

$$[E_1] + [E_2] = [\nabla_A (E_1 \oplus E_2) \Delta_B]$$

where E_1 and E_2 are extensions of A by B and Δ_B and ∇_A are the diagonal and codiagonal morphisms, respectively. The correspondence Ext is an additive functor from $\mathcal{L} \times \mathcal{L}$ into the category of discrete abelian groups which is contravariant in the first variable and covariant in the second.

Proof. See MacLane [6] Chapter 3.

We now obtain further properties of the functor Ext. Observe that our functor Ext generalizes both the Ext functor as defined for discrete abelian groups and as defined by Moskowitz [9].

Theorem 2.12. If A and B are in $\mathcal{L}$, then

$$\mathrm{Ext}\,(A,B) \cong \mathrm{Ext}\,(B^\wedge, A^\wedge).$$

The isomorphism is natural.

Proof. Let ψ: $\mathrm{Ext}(A,B) \longrightarrow \mathrm{Ext}(B^\wedge, A^\wedge)$ be the function induced by $E \longrightarrow E^\wedge$ where $E = (B \overset{\varphi}{\rightarrowtail} X \overset{\theta}{\twoheadrightarrow} A)$ is an extension of B by A and $E^\wedge = (A^\wedge \overset{\theta^\wedge}{\rightarrowtail} X^\wedge \overset{\varphi^\wedge}{\twoheadrightarrow} B^\wedge)$. The function ψ is a natural isomorphism.

Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a family of groups in $\mathcal{L}$. We make $\oplus_{\lambda\in\Lambda} G_\lambda$ into a topological group by taking as a basis for the topology all sets of the form $\Pi_{\lambda\in\Lambda} U_\lambda \cap \oplus_{\lambda\in\Lambda} G_\lambda$ where U_λ is open in G_λ. If $\{G_\lambda\}_{\lambda\in\Lambda}$ is a family in $\mathcal{L}$, then $\Pi_\lambda G_\lambda$ is in $\mathcal{L}$ if and only if G_λ is compact for all but a finite number of $\lambda \in \Lambda$. Thus, if one wishes to topologize $\oplus_\lambda G_\lambda$ such that $\oplus_\lambda G_\lambda$ is in $\mathcal{L}$ and such that $(\oplus_\lambda G_\lambda)^\wedge \cong \Pi_\lambda G_\lambda^\wedge$, then G_λ must be discrete for all but a finite number of $\lambda \in \Lambda$. Conversely, the topology defined on $\oplus_\lambda G_\lambda$ above yields that $\oplus_\lambda G_\lambda$ is in $\mathcal{L}$ and that $(\oplus_\lambda G_\lambda)^\wedge \cong \Pi_\lambda G_\lambda^\wedge$ whenever G_λ is discrete for all but a finite number of $\lambda \in \Lambda$.

Theorem 2.13. Assume that $\{G_\lambda\}_{\lambda\in\Lambda}$ is a family in $\mathcal{L}$ and that H is in $\mathcal{L}$.

(1) If $\oplus_\lambda G_\lambda$ is in $\mathcal{L}$, then $\mathrm{Ext}(\oplus_\lambda G_\lambda, H) \cong \Pi_\lambda \mathrm{Ext}(G_\lambda, H)$

(2) If $\Pi_\lambda G_\lambda$ is in $\mathcal{L}$, then $\mathrm{Ext}(H, \Pi_\lambda G_\lambda) \cong \Pi_\lambda \mathrm{Ext}(H, G_\lambda)$

Proof. In view of Theorem 2.12 we need only establish (1): since Ext is an additive functor on $\mathcal{L} \times \mathcal{L}$, both (1) and (2) hold when Λ is finite. Hence, recalling the discussion above, it suffices to assume that each G_λ is discrete. Let π_λ be the natural injection of G_λ into $\oplus_\lambda G_\lambda$ and define σ: $\mathrm{Ext}(\oplus_\lambda G_\lambda, H) \longrightarrow \Pi_\lambda \mathrm{Ext}(G_\lambda, H)$ by $\sigma([E]) = \langle [E\pi_\lambda] \rangle_{\lambda\in\Lambda}$.

It is easily seen that σ is a monomorphism. Let $\langle [E_\lambda] \rangle_{\lambda \in \Lambda}$ be in $\Pi_\lambda \operatorname{Ext}(G_\lambda, H)$. Stripping H of its topology, we form the extension $H \rightarrowtail X \twoheadrightarrow \oplus_\lambda G_\lambda$ of discrete groups so that the diagram

$$\begin{tikzcd}
H \arrow[r, tail] \arrow[d, equal] & X_\lambda \arrow[r, two heads] \arrow[d, tail] & G_\lambda \arrow[d, tail, "\pi_\lambda"] \\
H \arrow[r, tail] & X \arrow[r, two heads] & \oplus_\lambda G_\lambda
\end{tikzcd}$$

is commutative as a diagram of discrete abelian groups. We make X into a topological group by taking as a base for its topology the collection of all sets of the form $x + U$ where U is an open set in H and $x \in X$. Let $E = (H \rightarrowtail X \twoheadrightarrow \oplus_\lambda G_\lambda)$. Clearly $\sigma([E]) = \langle [E_\lambda] \rangle$.

Let $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ be a proper short exact sequence in $\mathcal{L}$, let G be in $\mathcal{L}$ and let $f \in \operatorname{Hom}(A,G)$. Construct the pushout diagram for f and φ

$$\begin{tikzcd}
 & A \arrow[r, tail, "\varphi"] \arrow[d, tail, "f"'] & B \arrow[r, two heads, "\theta"] \arrow[d, tail] & C \arrow[d, equal] \\
E: & G \arrow[r, tail, "\lambda"] & X \arrow[r, two heads, "\pi"] & C
\end{tikzcd}$$

By Proposition 2.3, λ is proper since φ is proper. It is easily checked that π is also proper. Thus, the homomorphism $\partial^* : \operatorname{Hom}(A,G) \longrightarrow \operatorname{Ext}(C,G)$ may be defined in the usual way, i.e., $\partial(f) = [E]$. A similar argument holds for the homomorphism $\partial_* : \operatorname{Hom}(G,C) \longrightarrow \operatorname{Ext}(G,A)$. The following

theorem now follows from Proposition 2.10 and an argument analagous to the proof of Theorem 3.2 [6].

<u>Theorem 2.14.</u> Let $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ be a proper short exact sequence in $\mathcal{L}$ and let G be in $\mathcal{L}$. Then the two sequences

$$(1)\quad \mathrm{Hom}\,(C,G) \rightarrowtail \mathrm{Hom}\,(B,G) \longrightarrow \mathrm{Hom}\,(A,G) \xrightarrow{\partial^*} \mathrm{Ext}\,(C,G) \longrightarrow \mathrm{Ext}\,(B,G) \longrightarrow \mathrm{Ext}\,(A,G)$$

$$(2)\quad \mathrm{Hom}\,(G,A) \rightarrowtail \mathrm{Hom}\,(G,B) \longrightarrow \mathrm{Hom}\,(G,C) \xrightarrow{\partial_*} \mathrm{Ext}(G,A) \longrightarrow \mathrm{Ext}(G,B) \longrightarrow \mathrm{Ext}(G,C)$$

are exact.

<u>Proposition 2.15.</u> (Notation is same as that of 2.14). If B has a projective resolution or G has an injective resolution, then the sequence (1) is exact on the right end. Dually, if B has an injective resolution or G has a projective resolution, then the sequence (2) is exact on the right.

The proof of 2.15 is a standard one (see MacLane [6]).

Further work regarding right exactness of Ext appears in section 4.

<u>Remark.</u> For each positive integer n, one can define a functor Ext^n from $\mathcal{L} \times \mathcal{L}$ into the category of discrete abelian groups. This functor is contravariant in its first argumant and covariant in the second. Moreover, it reduces to the usual functor Ext^n when restricted to discrete abelian groups and is the functor Ext we have already described in case $n = 1$.

The functor Ext^n is obtained in a manner analogous to the corresponding development of the functor Ext^n in the theory of (discrete) abelian groups. We do not give the details of the construction, but instead refer the

reader to the relevant chapters of MacLane [6]. It should be pointed out that modifications of MacLane's proofs must be made, but these modifications are minor in light of the homological algebra already developed in this paper. One consequence of this development is the following theorem.

Theorem 2.16. Assume G is a group in $\mathcal{L}$ and that $A \rightarrowtail B \twoheadrightarrow C$ is a proper exact sequence in $\mathcal{L}$. Then the sequence

$$\mathrm{Hom}(C,G) \rightarrowtail \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G) \longrightarrow \mathrm{Ext}(C,G) \longrightarrow \mathrm{Ext}(B,G)$$

$$\longrightarrow \mathrm{Ext}(A,G) \longrightarrow \mathrm{Ext}^2(C,G) \longrightarrow \mathrm{Ext}^2(B,G) \longrightarrow \cdots$$

is exact. A dual result holds with G in the first argument.

3. The Second Noether Isomorphism Theorem in $\mathcal{L}$. It is well-known that the Second Isomorphism Theorem does not hold in the category of abelian topological groups (for example, see [3]). However, it is an easy matter to see that it also fails for $\mathcal{L}$. Let A denote the discrete group R/Z and let φ be a one-to-one, surjective homomorphism of A onto T. Let $G = A \oplus T$ and let $B = \{(a, \varphi(a)) : a \in A\}$. It is easily verified that B is a closed discrete subgroup of G. Furthermore, $A \cap B = 0$ and $G = A + B$. However $(A + B)/A = G/A = T$ and $B/(B \cap A) = B$. Before sharpening the version of the Second Isomorphism Theorem as given in [3], we prove the following technical lemma.

Lemma 3.1. Let G be in $\mathcal{L}$ and let A and B be closed subgroups of G such that $G = A + B$ and $A \cap B = 0$. Then $G = A \oplus B$ if and only if there is a closed subgroup $C \subseteq B$ such that the following two

conditions hold:

(i) $A + C$ is closed in G.

(ii) The morphisms $\lambda: A \oplus C \longrightarrow A + C$ and

$\eta: B/C \longrightarrow G/(A + C)$ are proper, where

$\lambda: (a,c) \longrightarrow a + c$ and $\eta: b + C \longrightarrow b + (A + C)$.

<u>Proof.</u> The necessity is clear. Hence suppose that C satisfies the hypothesis above and consider the commutative diagram

$$\begin{tikzcd}
A \oplus C \arrow[r, tail] \arrow[d, two heads, "\lambda"'] & A \oplus B \arrow[r, two heads] \arrow[d, "\sigma"'] & B/C \arrow[d, two heads] \\
A + C \arrow[r, tail] & G \arrow[r, two heads] & G/(A + C)
\end{tikzcd}$$

where $\sigma: (a,b) \longrightarrow a + b$. By Theorem 2.1, σ is proper. Thus σ is an isomorphism in $\mathcal{L}$.

<u>Corollary 3.2.</u> Assume G is in $\mathcal{L}$ and that A and B are closed subgroups of G such that $G = A + B$ and $A \cap B = 0$. If B is σ-compact, then $G \cong A \oplus B$.

<u>Proof.</u> Apply Lemma 3.1 with $C = 0$.

<u>Theorem 3.3.</u> Let G be in $\mathcal{L}$ and suppose that A and B are closed subgroups of G such that $A + B$ is in $\mathcal{L}$. If B is σ-compact,

then

(i) $(A + B)/A \cong B/(A \cap B)$ and

(ii) $(A + B)/B \cong A/(A \cap B)$.

Proof. The first isomorphism is just a restatement of Theorem 5.33 [3]. Since $A + B$ is in $\mathcal{L}$, it follows that $A + B$ is closed in G. We also have that $(A + B)/(A \cap B) = A/(A \cap B) + B/(A \cap B)$, $A/(A \cap B) \cap B/(A \cap B) = 0$ and that $B/(A \cap B)$ is σ-compact. By Corollary 3.2, $(A + B)/(A \cap B) = A/(A \cap B) \oplus B/(A \cap B)$. Let θ be the natural homomorphism of $A + B$ onto $(A + B)/(A \cap B)$ and let π be the natural projection of $(A + B)/(A \cap B)$ onto $A/(A \cap B)$. Then $\pi\theta$ is proper since π is proper and θ is a proper epimorphism. Thus $(A + B)/B = (A + B)/\mathrm{Ker}\,\pi\theta \cong A/(A \cap B)$.

We now establish two consequences of the above results which will be needed in the next section.

Theorem 3.4. Let D be a discrete group in $\mathcal{L}$. Then $\mathrm{Ext}\,(X,D) = 0$ for all totally disconnected groups X in $\mathcal{L}$ if and only if D is divisible.

Proof. The necessity is clear. Therefore suppose that D is a discrete divisible group and that $D \overset{i}{\rightarrowtail} A \twoheadrightarrow X$ represents an element of $\mathrm{Ext}(X,D)$. We may assume that i is the inclusion homomorphism. It is elementary to see that A is necessarily totally disconnected. Since D is a discrete subgroup of A, there is a neighborhood U of zero in A such that $D \cap U = 0$. By Theorem 2.7 [9], there is a compact subgroup K of A contained in U such that A/K is discrete. The compactness of K implies that $D + K$ is closed in A. Since A/K is discrete, we have that $A/K = (D + K)/K \oplus H/K$ for some subgroup H of A. It follows

that $D \cap H = 0$ and that $A = D + H$. Let $\lambda: D \oplus K \longrightarrow D + K$ and $\eta: H/K \longrightarrow A/(D + K)$ be defined as in Lemma 3.1. By Corollary 3.2, λ is an isomorphism. Since $A/(D + K) \cong (A/K)/(D + K)/K$ is discrete, it follows that η is proper. By Lemma 3.1, $A = D \oplus H$ and $D \overset{i}{\rightarrowtail} A \twoheadrightarrow X$ splits.

<u>Theorem 3.5.</u> Let $A \overset{\varphi}{\rightarrowtail} G \overset{\theta}{\twoheadrightarrow} C$ be an extension of a discrete group A by a connected group C. Then $G = G_o \oplus B$ where B is discrete and $\theta(G_o) = C$.

<u>Proof.</u> Since G_o is connected and since G/G_o is totally disconnected, it is easily seen that $\varphi(G_o) = C$. Therefore $G = \varphi(A) + G_o$. Since G_o is necessarily σ-compact, we have by Theorem 3.3 (ii) that $G/G_o = (\varphi(A) + G_o)/G_o \cong \varphi(A)/(\varphi(A) \cap G_o)$. Therefore G/G_o is discrete. By a theorem of Braconnier [1], $G = G_o \oplus B$ where B is discrete.

<u>Corollary 3.6.</u> A group G in $\mathcal{L}$ is an extension of a discrete group by a connected group if and only if G is a topological direct sum of a discrete group and a connected group.

4. Right Exactness of Ext.

In this section we give some sufficient conditions in order that Ext yield a right exact sequence when applied to a short exact sequence.

<u>Theorem 4.1.</u> Assume $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is a short exact sequence in $\mathcal{L}$. If the induced map $\varphi^*: \mathrm{Ext}(B,G) \longrightarrow \mathrm{Ext}(A,G)$ is an epimorphism for all discrete G, then it is an epimorphism for all G in $\mathcal{L}$.

Proof. From Theorem 2.7 [9] it is easily seen that G contains a compactly generated subgroup K such that G/K is discrete. Since compactly generated groups have injective resolutions, we may apply the technique of Theorem 2.9 to obtain a proper exact sequence

$$G \rightarrowtail I \oplus G/K \twoheadrightarrow J$$

where I and J are injective in $\mathcal{L}$. We now construct the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(B,J) \arrow[r] \arrow[d, two heads, "\lambda"'] & \mathrm{Ext}(B,G) \arrow[r] \arrow[d, two heads, "\varphi^*"] & \mathrm{Ext}(G, I \oplus G/K) \arrow[r] \arrow[d, two heads, "\sigma"'] & \mathrm{Ext}(B,J) = 0 \arrow[d, equal] \\
\mathrm{Hom}(A,J) \arrow[r] & \mathrm{Ext}(A,G) \arrow[r] & \mathrm{Ext}(A, I \oplus G/K) \arrow[r] & \mathrm{Ext}(A,J) = 0
\end{tikzcd}$$

where the rows are exact. Since I is injective and G/K is discrete, it follows that σ is an epimorphism. The morphism λ is an epimorphism since $\mathrm{Ext}(C,J) = 0$. The Five Lemma implies that φ^* is an epimorphism.

Lemma 4.2. Assume $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is a proper exact sequence in $\mathcal{L}$ with A, B, and C totally disconnected. For each group G in $\mathcal{L}$, the induced map $\varphi^*: \mathrm{Ext}(B,G) \longrightarrow \mathrm{Ext}(A,G)$ is an epimorphism.

Proof. By Theorem 4.1, we may assume G is discrete. There is an exact sequence $G \rightarrowtail D \twoheadrightarrow D_1$ where D and D_1 are discrete divisible groups. By Theorem 3.4, $\mathrm{Ext}(A,D) = 0$, $\mathrm{Ext}(B,D) = 0$, and $\mathrm{Ext}(C,D) = 0$. Hence we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(B,D_1) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Ext}(B,G) \arrow[d, "\varphi^*"] \\
\mathrm{Hom}(A,D_1) \arrow[r, two heads] & \mathrm{Ext}(A,G).
\end{tikzcd}$$

It follows immediately that φ^* is an epimorphism.

<u>Theorem 4.3.</u> Assume $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is a proper short exact sequence in $\mathcal{L}$ and that G is a group in $\mathcal{L}$. The induced map

$$\varphi^*: \ \mathrm{Ext}(B,G) \longrightarrow \mathrm{Ext}(A,G)$$

is an epimorphism provided any one of the following conditions hold:

(1) G has an injective resolution,

(2) the sequence $A_o \xrightarrow{\varphi_o} B_o \xrightarrow{\theta_o} C$ is short exact where $\varphi_o = \varphi|A_o$ and $\theta_o = \theta|B_o$,

(3) A is connected, or

(4) C is totally disconnected.

<u>Proof.</u> First assume (1) holds. By Theorem 3.6 [9], G has a short injective resolution. The remainder of the proof proceeds as in the case of discrete group theory (see MacLane [6]).

We now assume (3) as we plan to use case (3) to prove the theorem in case (2) holds. By Theorem 4.1 we may also assume that G is discrete. Let $G \overset{\delta}{\rightarrowtail} X \overset{\gamma}{\twoheadrightarrow} A$ represent an element of $\mathrm{Ext}(A,G)$. By Theorem 3.5, $X = X_o \oplus H$ where $\gamma(X_o) = A$ and H is discrete. Hence

$K \overset{\delta'}{\rightarrowtail} X_o \overset{\gamma'}{\twoheadrightarrow} A$ represents an element of $\mathrm{Ext}(A,K)$ where $K = \delta^{-1}(\delta(G) \cap X_o)$ and where δ' and γ' are restrictions of δ and γ, respectively. By Theorem 3.6 [9], X_o has an injective resolution in $\mathcal{L}$ since necessarily X_o is compactly generated. It follows that K also has an injective resolution in $\mathcal{L}$. By case (1) the natural homomorphism from $\mathrm{Ext}(B,K) \longrightarrow \mathrm{Ext}(A,K)$ is an epimorphism. Hence we obtain the proper diagram

$$\begin{tikzcd}
K \arrow[r, tail, "\delta'"] \arrow[d, equal] & X \arrow[r, two heads, "\gamma'"] \arrow[d, tail, "\psi'"] & A \arrow[d, tail] \\
K \arrow[r, tail, "\lambda'"] & Y \arrow[r, two heads, "\sigma'"] & B.
\end{tikzcd}$$

Let $W = Y \oplus H$ and define $\sigma\colon W \longrightarrow B$ and $\lambda\colon G \longrightarrow W$ as follows: $\sigma\colon (y,h) \longrightarrow \sigma'(y) + \gamma(h)$ and for $\delta(g) = (x,h) \in X_o \oplus H$ define $\lambda\colon g \longrightarrow (\psi'(x), h)$. Straightforward computation shows that $\ker \sigma = \mathrm{Im}\,\lambda$ and that the diagram

$$\begin{tikzcd}
E_1\colon\ G \arrow[r, tail, "\delta"] \arrow[d, equal] & X_o \oplus H \arrow[r, two heads, "\gamma"] \arrow[d, tail, "\psi"] & A \arrow[d, tail, "\varphi"] \\
E_2\colon\ G \arrow[r, tail, "\lambda"] & Y \oplus H \arrow[r, two heads, "\sigma"] & B
\end{tikzcd}$$

is commutative (as a diagram of discrete groups) where $\psi\colon (x,h) \longrightarrow (\psi'(x),h)$. Since λ and σ are clearly continuous, it suffices to show that λ and σ

are proper. We have that ψ is proper since ψ' is proper and H is discrete. Thus λ is the composition of two proper monomorphisms and hence is itself proper. Since H is discrete and since σ' is proper, then $\sigma(U \times \{0\}) = \sigma'(U)$ is open in B for each open subset U of Y. It follows that σ is proper. Thus $\varphi^*([E_2]) = [E_1]$, and the theorem follows if (3) holds.

Now assume the sequence $A_o \xrightarrow{\varphi_o} B_o \xrightarrow{\theta_o} C_o$ is short exact. We have a proper commutative diagram:

$$\begin{tikzcd}
A_o \arrow[r, tail, "\varphi_o"] \arrow[d, tail, "\alpha"'] & B_o \arrow[r, two heads] \arrow[d, tail, "\beta"] & C_o \arrow[d, tail] \\
A \arrow[r, tail, "\varphi"] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\
A_1 \arrow[r, tail, "\varphi_1"] & B_1 \arrow[r, two heads] & C_1
\end{tikzcd}$$

where A_1, B_1, and C_1 are totally disconnected. The above diagram yields the following commutative diagram of discrete groups with exact rows

$$\begin{tikzcd}
\mathrm{Hom}(B_o,G) \arrow[r] \arrow[d] & \mathrm{Ext}(B_1,G) \arrow[r] \arrow[d, two heads, "\varphi_1^*"] & \mathrm{Ext}(B,G) \arrow[r] \arrow[d, "\varphi^*"] & \mathrm{Ext}(B_o,G) \arrow[r] \arrow[d, two heads, "\varphi_o^*"] & 0 \arrow[d, equal] \\
\mathrm{Hom}(A_o,G) \arrow[r] & \mathrm{Ext}(A_1,G) \arrow[r] & \mathrm{Ext}(A,G) \arrow[r] & \mathrm{Ext}(A_o,G) \arrow[r] & 0 .
\end{tikzcd}$$

By the fact that we have shown that (3) implies the theorem, we have

that α^*, β^*, and φ_o^* are epimorphisms. By Lemma 4.2, φ_1^* is an epimorphism. Thus by the Five Lemma, it follows that φ^* is an epimorphism.

Finally assume (4) holds. One has a proper commutative diagram

$$\begin{tikzcd}
A_o \arrow[r, tail, two heads, "\bar{\varphi}"] \arrow[d, tail, "i"'] & L \arrow[d, tail, "j"] & \\
A \arrow[r, tail, "\varphi"] \arrow[d, two heads, "\eta"'] & B \arrow[r, two heads, "\theta"] \arrow[d, two heads, "\beta'"] & C \arrow[d, equal] \\
A_1 \arrow[r, tail, "\varphi'"] & P \arrow[r, two heads, "\theta'"] & C
\end{tikzcd}$$

where $L = \operatorname{Im}\bar{\varphi}$, $\bar{\varphi}$ is the map induced from φ, i and j are inclusion maps, η is the natural map of A onto $A_1 = A/A_o$, and the last row is obtained from a pushout of η and φ. The diagram has exact rows and columns. Thus we obtain the following commutative diagram of discrete groups with exact rows:

$$\begin{tikzcd}
\operatorname{Ext}(P,G) \arrow[r] \arrow[d, two heads, "\varphi'^*"'] & \operatorname{Ext}(B,G) \arrow[r, "j^*"] \arrow[d, two heads, "\varphi^*"] & \operatorname{Ext}(L,G) \arrow[r] \arrow[d, two heads, "\bar{\varphi}^*"] & 0 \arrow[d, equal] \\
\operatorname{Ext}(A_1, G) \arrow[r] & \operatorname{Ext}(A,G) \arrow[r, "i^*"] & \operatorname{Ext}(A_o, G) \arrow[r] & 0
\end{tikzcd}$$

In the latter diagram, φ'^* is an epimorphism since A_1, P, and C are totally disconnected, and i^*, j^*, and $\bar{\varphi}^*$ are epimorphisms since

A_o and L are connected. By the Five Lemma, φ^* is an epimorphism.

By use of a duality argument one may show the following.

<u>Theorem 4.4.</u> Assume $A \overset{\varphi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is a proper short exact sequence in $\mathcal{L}$ and that G is a group in $\mathcal{L}$. The induced map

$$\theta^*: \ \mathrm{Ext}(G,B) \longrightarrow \mathrm{Ext}(G,C)$$

is an epimorphism provided any one of the following conditions hold:

(1) G has a projective resolution,

(2) $C = R^n \oplus H$ for some discrete torsion free group H,

(3) A is an extension of a compact group by a discrete torsion group; in particular any compact group or any discrete torsion group satisfies this condition.

<u>Proposition 4.5.</u> Let $\mathrm{Ext}_Z(\ ,\ )$ denote the extension functor on the category of discrete abelian groups and let $\mathcal{F}$ be the forgetful functor from $\mathcal{L}$ into the category of discrete abelian groups which forgets topological structure.

(a) If A and B are discrete, $\mathrm{Ext}(A,B) \cong \mathrm{Ext}_Z(A,B)$.

(b) If A and B are compact, $\mathrm{Ext}(A,B) \cong \mathrm{Ext}_Z(B^\wedge, A^\wedge)$.

(c) If A is discrete and B in $\mathcal{L}$, $\mathrm{Ext}(A,B) \cong \mathrm{Ext}_Z(A,\mathcal{F}(B))$.

(d) If A is discrete and B is divisible, $\mathrm{Ext}(A,B) = 0$.

(e) Suppose that A is a totally disconnected group which contains a compact, bounded torsion subgroup K such that A/K is discrete. If B is connected, then $\mathrm{Ext}(A,B) = 0$.

(f) If G is discrete, then $\mathrm{Ext}(T,G) \cong G$. Thus the range of $\mathrm{Ext}(\ ,\ )$ is all of the discrete abelian groups.

Proof: Both (a) and (b) are clear. If A is discrete and if B is in $\mathcal{L}$, then it is easily seen that there is a unique way of putting a topology on any extension group X of $\mathcal{F}(B)$ by A so that the sequence $B \rightarrowtail X \twoheadrightarrow A$ is proper exact. This establishes (c). Clearly (d) follows from (c). Let A satisfy the hypothesis of (e). By (d) we have that $\mathrm{Ext}(A/K, B) = 0$ since a connected group is divisible. Now $B = C \oplus R^n$ where C is a compact connected group. Since $K^\wedge$ is a discrete, bounded torsion group and since $C^\wedge$ is torsion free and discrete, it follows that $0 = \mathrm{Ext}(C^\wedge, K^\wedge) = \mathrm{Ext}(K,C)$. Hence $\mathrm{Ext}(K,B) = 0$ since R^n is injective in $\mathcal{L}$. Thus, by exactness, $\mathrm{Ext}(A,B) = 0$. We remark that (e) generalizes a result of Braconnier [1]. Let G be discrete. Then the exact sequence $Z \rightarrowtail R \twoheadrightarrow T$ yields the exact sequence

$$0 = \mathrm{Hom}(R,G) \longrightarrow \mathrm{Hom}(Z,G) \xrightarrow{\partial} \mathrm{Ext}(T,G) \longrightarrow \mathrm{Ext}(R,G) = 0.$$

hence $\mathrm{Hom}(Z,G) \cong \mathrm{Ext}(T,G)$. Since G is discrete, $G \cong \mathrm{Hom}(Z,G) = \mathrm{Ext}(T,G)$.

5. Projectives and Injectives in the Category of Totally Disconnected, Locally Compact Abelian Groups.

Let $\mathcal{D}$ denote the category of totally disconnected, locally compact abelian groups. We first remark that Moskowitz [9] has described the projectives and injectives in $\mathcal{L}$. In the next section we shall make use of the injectives for $\mathcal{L}$ and $\mathcal{D}$.

Theorem 5.1. P is a projective in $\mathcal{D}$ if and only if P is a discrete free group.

Proof. By Theorem 3.3 [9] the sufficiency is clear. Now suppose that K is a compact projective in $\mathcal{D}$.

Since $0 = \mathrm{Ext}(K, (Q/Z)^\wedge) \cong \mathrm{Ext}_Z(Q/Z, K^\wedge)$ where Q denotes the discrete group of rational numbers, it follows that $K^\wedge$ is a discrete, divisible torsion group. If $K \neq 0$, then K is a direct product of p-adic groups for various primes p. Viewed as an object of $\mathcal{L}$, K is projective relative to the compact groups in $\mathcal{L}$. If K were also projective in $\mathcal{D}$, it would follow that K is projective in $\mathcal{L}$. However, by Theorem 3.3 [9], K is not projective in $\mathcal{L}$ and hence not in $\mathcal{D}$. Thus $K = 0$. It follows that a projective P in $\mathcal{D}$ has no non-trivial compact subgroups and hence P must be discrete. Therefore P has a free resolution in $\mathcal{D}$ and thus P is necessarily free.

<u>Theorem 5.2.</u> E is injective in $\mathcal{D}$ if and only if E is a discrete divisible group.

<u>Proof.</u> The sufficiency follows from Theorem 3.4. Hence let E be an injective in $\mathcal{D}$. Now E contains a compact subgroup K such that E/K is discrete. We therefore obtain the exact sequence

$$\mathrm{Hom}(K, E/K) \xrightarrow{\partial} \mathrm{Ext}(K,K) \longrightarrow \mathrm{Ext}(K,E) = 0.$$

Hence ∂ is an epimorphism. Since E/K is discrete and since K is compact, then for each $f \in \mathrm{Hom}(K, E/K)$ we have that $\mathrm{Im} f$ is finite. It follows that $\mathrm{Ext}(K,K) = \mathrm{Ext}_Z(K^\wedge, K^\wedge)$ is a torsion group. Since $K^\wedge$ is torsion, we also have $\mathrm{Ext}_Z(K^\wedge, K^\wedge)$ is a reduced cotorsion group. Therefore $\mathrm{Ext}_Z(K^\wedge, K^\wedge)$ is bounded and hence $K^\wedge$ is bounded. Thus K must be a bounded torsion group. Since E/K is discrete it follows that all compact subgroups of E are bounded torsion groups. Now suppose that K is infinite.

Then there exists a prime p and a positive integer n such that $K^\wedge = \sum_m Z(p^n) \oplus L$ for some infinite cardinal m and some subgroup L of $K^\wedge$. Let $S = \sum_m Z(p^n) \oplus \sum_{k=1}^{\infty} \sum_m Z(p^{nk})$. Then $p^n S = \sum_{k=2}^{\infty} \sum_m Z(p^{nk})$ and $S/p^n S \cong \sum_m Z(p^n)$. Therefore, there is a morphism $\theta: K^\wedge \twoheadrightarrow S/p^n S$. Since $E^\wedge$ is necessarily projective relative to discrete torsion groups, there is a morphism $f: E^\wedge \twoheadrightarrow S$ such that the diagram

$$\begin{tikzcd}
 & E^\wedge \arrow[dl, two heads, "f"'] \arrow[d, two heads, "\theta"] \\
p^n S \arrow[r, tail] & S \arrow[r, two heads] & S/p^n S
\end{tikzcd}$$

is commutative. Clearly f is proper since S is discrete and also $S = p^n S + H$ where $H = \mathrm{Im} f$. It is clear that H is an unbounded direct sum of cyclic groups. By Pontryagin Duality, E contains an unbounded compact subgroup which is a contradiction. Hence K is finite and E is discrete. Since $\mathrm{Ext}(Q/Z, E) = 0$ where Q denotes the discrete group of rational numbers, it follows that E must be divisible.

<u>6. The Component of Zero as a Direct Summand.</u> We first remark that Proposition 4.5 (a) provides a sufficient condition on G/G_o in order that G_o be a direct summand of G. In this section we wish to find necessary and sufficient conditions on a connected group C such that C be a direct summand of every group containing C as its component of zero. More generally we describe the structure of a group E which

has the property that $\mathrm{Ext}(X,E) = 0$ for all totally disconnected groups X in $\mathcal{L}$.

<u>Theorem 6.1.</u> Let C be a connected group in $\mathcal{L}$. Then $\mathrm{Ext}(X,C) = 0$ for all totally disconnected groups X in $\mathcal{L}$ if and only if C is an injective in $\mathcal{L}$.

<u>Proof.</u> The sufficiency is clear. Hence suppose $\mathrm{Ext}(X,C) = 0$ for all totally disconnected groups X in $\mathcal{L}$. Since C is connected, $C = K \oplus R^n$ where K is a compact connected group. It follows that $K^\wedge$ is torsion free and that $\mathrm{Ext}_Z(K^\wedge, G) = 0$ for all discrete torsion groups G. Hence $K^\wedge$ is a Baer-group which implies by Corollary 3.2 [2] that $K^\wedge$ is a free discrete group. Therefore $K = T^\sigma$ and $C = T^\sigma \oplus R^n$. By Theorem 3.2 [9], C is injective in $\mathcal{L}$.

<u>Lemma 6.2.</u> Assume X is a totally disconnected locally compact abelian group. If E is a group in $\mathcal{L}$, then the map

$$\mathrm{Ext}(X,E) \longrightarrow \mathrm{Ext}(X,E/E_o)$$

induced by the natural map from E onto E/E_o is an epimorphism.

<u>Proof.</u> Choose a subgroup M of E such that M contains some compact open subgroup K and $E = R^n \oplus M$. Clearly $E_o = R^n \oplus M_o$ and $E/E_o = (R^n \oplus M)/(R^n \oplus M_o) = M/M_o$. Thus $\mathrm{Ext}(X,E) \longrightarrow \mathrm{Ext}(X, E/E_o)$ is an epimorphism whenever $\mathrm{Ext}(X,M) \longrightarrow \mathrm{Ext}(X, M/M_o)$ is an epimorphism.

Since $M_o \rightarrowtail M \twoheadrightarrow M/M_o$ is a proper exact sequence, so is $(M/M_o)^\wedge \rightarrowtail M^\wedge \twoheadrightarrow M_o^\wedge$. Note that $M_o \subseteq K$ and thus $M_o^\wedge$ is discrete. By Theorem 4.3 (4), $\mathrm{Ext}(M^\wedge, X^\wedge) \longrightarrow \mathrm{Ext}((M/M_o)^\wedge, X^\wedge)$

is an epimorphism for each X in $\mathcal{L}$. It follows that $\mathrm{Ext}(X, M) \longrightarrow \mathrm{Ext}(X, M/M_o)$ is an epimorphism for each X in $\mathcal{L}$. The lemma follows.

<u>Theorem 6.3.</u> Let E be in $\mathcal{L}$. Then $\mathrm{Ext}(X, E) = 0$ for all totally disconnected groups X in $\mathcal{L}$ if and only if $E = I \oplus D$ where I is injective in $\mathcal{L}$ and D is a discrete divisible group.

<u>Proof.</u> The sufficiency follows from Theorem 3.4. Since $\mathrm{Ext}(X, E) \twoheadrightarrow \mathrm{Ext}(X, E/E_o)$, it follows that E/E_o is an injective in $\mathcal{D}$. By Theorem 5.2, E/E_o is a discrete divisible group. It follows from Proposition 4.5 (d), that $E = E_o \oplus D$ where D is a discrete divisible group. Theorem 6.1 yields that E_o is injective in $\mathcal{L}$ which completes the proof.

REFERENCES

1. Jean Braconnier, Sur les groupes topologiques localement compacts, J. Math. Pures Appl., N.S. 27, 1-85 (1948).
2. Phillip A. Griffith, A solution to the splitting mixed group problem of Baer, to appear.
3. Edwin Hewitt and Kenneth Ross, Abstract Harmonic Analysis I, Springer-Verlag, Berlin, 1963.
4. G. Hochschild, Group extensions of Lie groups. I, Annals of Math., Vol. 54, No. 1, 96-109 (1951).
5. _____, Group extensions of Lie groups. II, Annals. of Math., Vol. 54, No. 1, 537-551 (1951).
6. Saunders MacLane, Homology, Springer-Verlag, New York, 1967.
7. Calvin C. Moore, Extensions and lower dimensional cohomology theory of locally compact groups. I, Transactions of the Amer. Math. Soc., Vol. 113, No. 1, 40-63 (1964).
8. _____, Extensions and lower dimensional cohomology theory of locally compact groups. II, Transactions of the Amer. Math. Soc., Vol. 113, No. 1, 64-86 (1964).
9. Martin Moskowitz, Homological algebra in locally compact Abelian groups, Transactions of the Amer. Math. Soc., Vol. 127, No. 3, 361-404 (1967).
10. L. S. Pontryagin, Topological Groups, Gordon and Breach, Science Publishers, Inc., New York, 1966.

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