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CONCERNING A CONJECTURE OF MARSHALL HALL

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by

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Introduction.

An $n \times n$ matrix A is said to be doubly stochastic if $a_{ij} \geq 0$ and if $\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} = 1$ for all i and j. The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n.

The permanent of an $n \times n$ matrix A is defined by

$$\text{per } A = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the sum is taken over all permutations σ of $1, \ldots, n$.

An unresolved conjecture of B. L. Van der Waerden states that the minimal value of the permanent of the $n \times n$ doubly stochastic matrices is $n!/n^n$, and is uniquely achieved at the matrix J_n in which every element is $1/n$. Marshall Hall has made the following observation. Let Λ_n denote the collection of $n \times n$ (0,1)-matrices (i.e. those for which every element is either 0 or 1) which have exactly three ones in each row and each column. If $A \in \Lambda_n$, then $\frac{1}{3} A \in \Omega_n$, and thus if the Van der Waerden conjecture is correct, per $\frac{1}{3} A \geq n!/n^n$, i.e. per $A \geq 3^n n!/n^n$. Thus, in particular

$$\operatorname*{Inf}_{A \in \Lambda_n} \text{per } A \geq 3^n n!/n^n,$$

and therefore

$$\lim_{n \to \infty} \left(\operatorname*{Inf}_{A \in \Lambda_n} \text{per } A \right) = +\infty.$$

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Marshall Hall conjectures therefore that (1) holds. It is the purpose of this paper to show that the Hall conjecture is correct independent of the Van der Waerden conjecture.

The following notations and definitions will be used. An $n \times n$ matrix A is said to be partly decomposable if there exist permutation matrices P and Q such that PAQ has the form $\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$ where B and D are square. If no such P and Q exist, the matrix is said to be fully indecomposable. If A is fully indecomposable, but is such that if any single $a_{ij} \neq 0$ in A is replaced by 0 the resulting matrix becomes partly decomposable, then A is said to be nearly decomposable. An important theorem concerning nearly decomposable matrices follows. A proof may be found in [2].

THEOREM 1. <u>Let</u> A <u>be a nonnegative</u> $n \times n$ <u>nearly decomposable</u> $(0,1)$-<u>matrix where</u> $n > 1$. <u>Then there exist permutation matrices</u> P <u>and</u> Q <u>and an integer</u> $s > 1$ <u>such that</u>

$$PAQ = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 & 0 & E_1 \\ E_2 & A_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & E_3 & A_3 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & E_{s-1} & A_{s-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & E_s & A_s \end{pmatrix} \tag{2}$$

<u>where each</u> E_i <u>has exactly one entry equal to</u> 1 <u>and each</u> A_i <u>is nearly decomposable.</u>

A set of elements $a_{1\sigma(1)}, \dots, a_{n\sigma(n)}$ in a matrix A, where σ is a permutation of $1, \dots, n$, is called a diagonal of A.

Results and Consequences.

LEMMA 1. <u>Let</u> u <u>be a positive integer and suppose that</u> $v_1 \geq v_2 \geq \cdots \geq v_u \geq 3$. <u>Then</u>. $\prod_{k=1}^{u} v_k - u - \sum_{k=1}^{u} v_k + 1 \geq 0$.

<u>Proof:</u> The result holds for $u = 1$. If it holds for u, then

$$\prod_{k=1}^{u+1} v_k - (u+1) - \sum_{k=1}^{u+1} v_k + 1$$

$$= v_{u+1} \prod_{k=1}^{u} v_k - u - \sum_{k=1}^{u+1} v_k$$

$$\geq v_{u+1} \left(u + \sum_{k=1}^{u} v_k - 1\right) - u - \sum_{k=1}^{u+1} v_k$$

$$= (v_{u+1} - 1)\, u + (v_{u+1} - 1) \sum_{k=1}^{u} v_k - 2v_{u+1}$$

$$\geq 2u + 2v_u - 2v_{u+1}$$

$$\geq 2u \geq 2 > 0,$$

completing the induction.

LEMMA 2. <u>Let</u> u, p, <u>and</u> q <u>be integers where</u> $u > 1$ <u>and</u> $0 \leq p \leq q$, <u>and suppose that</u> $v_1 \geq v_2 \geq \cdots \geq v_u \geq 3$. <u>Then</u>

$$1 + 2^p\, 3^{q-p} \prod_{k=1}^{u} v_k \geq u + 3q + \sum_{k=1}^{u} v_k.$$

<u>Proof:</u> Put $f(x) = 2^x \prod_{k=1}^{u} v_k - u - 3x - \sum_{k=1}^{u} v_k + 1$. By lemma 1,

$$f(0) = \prod_{k=1}^{u} v_k - u - \sum_{k=1}^{u} v_k + 1 \geq 0$$

and

$$f(1) = 2 \prod_{k=1}^{u} v_k - u - 3 - \sum_{k=1}^{u} v_k + 1 = f(0) + (\prod_{k=1}^{u} v_k - 3) \geq 0.$$

If $x \geq 1$,

$$f'(x) = 2^x(\prod_{k=1}^{u} v_k) \ln 2 - 3 \geq 2 \cdot 3 \ln 2 - 3 = 3(\ln 4 - 1) > 0.$$

Thus $f(x) \geq f(1) \geq 0$ if $x \geq 1$. Then since $2^p\, 3^{q-p} \geq 2^q$,

$$2^p\, 3^{q-p} \prod_{k=1}^{u} v_k - u - 3q - \sum_{k=1}^{u} v_k + 1 \geq f(q) \geq 0.$$

LEMMA 3. <u>Let</u> A <u>be an</u> $n \times n$ <u>nearly decomposable</u> (0,1)-<u>matrix with at least two and not more than three ones in each row. If at least one row of</u> A <u>contains three ones, then</u> per $A \geq 3$.

<u>Proof:</u> It follows from the Frobenius-König theorem [1, p. 97-98] that every positive element in a nonnegative fully indecomposable matrix lies on a positive diagonal. Assume A to be in the form (2). The some A_i is at least 2×2 and therefore has at least two positive diagonals. Whence per $A_i \geq 2$. Allowing for at least one positive diagonal of A to pass through the E_i, we see that per $A \geq 1 + \prod_{i=1}^{s}$ per $A_i \geq 1 + 2 = 3$.

LEMMA 4. <u>If in a fully indecomposable</u> (0,1)-matrix a zero is replaced <u>by</u> 1, <u>the permanent is increased.</u>

<u>Proof:</u> If A is a fully indecomposable (0,1)-matrix and if B is obtained from A by replacing a zero by 1, then B is still fully indecomposable. By the Frobenius-König theorem every 1 in B, including the 1 not in A, lies on a positive diagonal. Thus B has at least one more positive diagonal than does A. Hence per $B >$ per A.

THEOREM 2. *Let* A *be a nearly decomposable* $n \times n$ *(0,1)-matrix with* m *rows containing exactly three ones and* $n - m$ *rows containing exactly two ones. Then* per $A \geq m$.

Proof: The proof is by induction on n. We must have $n \geq 2$, and if $n = 2$, then $m = 0$ and the result holds. In fact, if $m = 0$, the result holds for all $n \geq 2$; whence if $n > 2$ we suppose that $m > 0$. We may assume that A is in the form (2). Since $m > 0$, at least one of the A_i's must be larger than 1×1. Suppose that $A_{i_1}, \ldots, A_{i_r}$ are at least 2×2 and that each of $A_{i_1}, \ldots, A_{i_p}$, $p \leq r$, has exactly two ones in each row (and thus in each column) while $A_{i_{p+1}}, \ldots, A_{i_r}$ have $m_{p+1}, \ldots, m_r$ rows with three ones, respectively. Note that $m = r + \sum_{k=p+1}^{r} m_k$. Let $m_k \leq 2$ for $k = p+1, \ldots, t$, and $m_k \geq 3$ for $k = t+1, \ldots, r$. Then by lemma 3 and the induction hypothesis, per $A_{i_k} \geq 3$ for $k = p+1, \ldots, t$ and per $A_{i_k} \geq m_k$ for $k = t+1, \ldots, r$. Since per $A_{i_k} = 2$ for $k = 1, \ldots, p$, we have, allowing for at least one positive diagonal through the E_i,

$$\text{per } A \geq 1 + \prod_{k=1}^{r} \text{per } A_{i_k} \geq 1 + 2^p\, 3^{t-p} \prod_{k=t+1}^{r} m_k .$$

The proof will be complete if we show that $1 + 2^p\, 3^{t-p} \prod_{k=t+1}^{r} m_k \geq r + \sum_{k=p+1}^{r} m_k$. If $t = 0$, then $p = 0$, and the result is an immediate consequence of lemma 1. Hence suppose $t > 0$. If $t = r$, the inequality becomes $1 + 2^p\, 3^{t-p} \geq t + \sum_{k=p+1}^{t} m_k$. We prove this by showing that $1 + 2^p \cdot 3^{t-p} \geq 3t - 2p$. This is clear for $p = 0$. If $0 < p \leq t$, we have

$$1 + 2^p\, 3^{t-p} \geq 1 + 2^t > 3t - 2 \geq 3t - 2p.$$

Then since $m_k \leq 2$ if $k = p+1, \ldots, t$, $\sum_{k=p+1}^{t} m_k \leq 2(t-p)$. Thus

$$1 + 2^p\, 3^{t-p} \geq t + 2(t-p) \geq t + \sum_{k=p+1}^{t} m_k.$$

Whence we can assume that $r > t$. Then by lemma 2,

$$1 + 2^p\, 3^{t-p} \prod_{k=t+1}^{r} m_k \geq (r-t) + 3t + \sum_{k=t+1}^{r} m_k = r + 2t + \sum_{k=t+1}^{r} m_k.$$

But $\sum_{k=p+1}^{t} m_k \leq 2(t-p) \leq 2t$, if $t > p$, so $r + 2t + \sum_{k=t+1}^{r} m_k \geq$ $r + \sum_{k=p+1}^{r} m_k$, and the result follows.

THEOREM 3. <u>If</u> A <u>is a fully indecomposable member of</u> A_n <u>then</u> per $A \geq n$.

<u>Proof:</u> If a certain set of ones in A, k in number, are replaced by zeros, there results a nearly decomposable matrix A'. The matrix A', being fully indecomposable, has at least two ones in every row (and column), and, of course, no more than three. Thus the k ones were removed from k different rows (and columns). A' has k rows with exactly two ones and $n-k$ rows with exactly three ones. By theorem 2, per $A' \geq n-k$. But by lemma 4, A has at least k more positive diagonals than did A' (one at least for each of the k ones removed). Thus per $A \geq k + (n-k) = n$.

THEOREM 4. $\operatorname{Inf}_{A \in A_n}$ per $A \geq n$.

<u>Proof:</u> Suppose $A_0 \in A_n$ is partly decomposable. By permuting rows and

columns we may suppose that

$$A_o = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1 is $r \times r$ and A_2 is $(n - r) \times (n - r)$. Then since the row sums of A_o are all 3, the sum of the elements in A_1 is $3r$. This is precisely the sum of the elements in A_1 and B. Thus $B = 0$ and in fact $A_o = A_1 \oplus A_2$, where $A_1 \in \Lambda_r$ and $A_2 \in \Lambda_{n-r}$. Clearly the process can be continued so that

$$PA_oQ = A_1 \oplus \ldots \oplus A_k ,$$

for some permutation matrices P and Q, where the A_j are fully indecomposable matrices in Λ_{n_j}, $j = 1, \ldots, k$. Hence, by theorem 2,

$$\text{per } A_j \geq n_j .$$

Now each row of A_j has three ones and therefore $n_j \geq 3$, $j = 1, \ldots, k$, and we have

$$\begin{aligned} \text{per } A_o &= \prod_{j=1}^{k} \text{per } A_j \\ &\geq \prod_{j=1}^{k} n_j \\ &\geq \sum_{j=1}^{k} n_j \\ &= n , \end{aligned}$$

where equality can occur only if $k = 1$.

The Marshall Hall conjecture stated in (1) is now immediate.

REFERENCES

1. Marvin Marcus and Henryk Minc, "A Survey of Matrix Theory and Matrix Inequalities", Allyn and Bacon, Boston, 1964.

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