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by A. Khalanay

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PERIODIC AND NEAR-PERIODIC SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A LAGGING ARGUMENT

by A. Khalanay

1. In [1] we have established the following simple theorem: if the system $\dot{x}(t) = f[t, x(t), x(t-\tau)]$, $f(t, u, v) \equiv f(t+\omega, u, v)$ has a limited solution $x_0(t)$, such that $\lim_{t\to\infty} [x_0(t+\omega) - x_0(t)] = 0$, then it has a periodic solution of the ω period.

Hence the following corollary:

If the system $\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t-\tau) + f(t)$, where

$$A(t+\omega) \equiv A(t), \quad B(t+\omega) \equiv B(t), \quad f(t+\omega) \equiv f(t)$$

has a limited solution, and if the trivial solution corresponding to a homogeneous system is uniformly and asymptotically stable, then the system has a periodic solution.

We will show that if a trivial solution of a homogeneous system is uniformly asymptotically stable, the existence of a limited solution of the inhomogeneous system is assured.

<u>Lemma</u>. Under consideration is the following system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t-\tau) + f(t),$$

where A(t), B(t), f(t) are limited when $t \geqslant 0$. Let us assume that the correct solution of a corresponding homogeneous system is uniformly asymptotically stable. Then all the solutions of the system are limited when $t \geqslant 0$.

<u>Proof</u>. As the trivial solution of the system $\dot{y}(t) = A(t)\,y(t) + B(t)\,y(t-\tau)$ is uniformly asymptotically stable, the following evaluation takes place

$$|y(t; t_0, \varphi)| \leqslant Ke^{-\alpha(t-t_0)} \|\varphi\|.$$

Let

$$V[t,\varphi]=\int_0^\infty \|y(t+u+s;\ t,\ \varphi)\|^2\ du+\sup_{\sigma\geqslant 0}\|y(t+\sigma+s;\ t,\ \varphi)\|^2,$$

$$-\tau\leqslant s\leqslant 0,\ \|\varphi\|=\sup_{-\tau\leqslant s\leqslant 0}|\varphi(s)|.$$

It is obvious that $\|\varphi\|^2\leqslant V[t,\ \varphi]\leqslant K_1\|\varphi\|^2$.

It is easy to show that $|V[t,\varphi_1]-V[t,\varphi_2]|\leqslant L_0\ (\|\varphi_1\|+\|\varphi_2\|)\|\varphi_1-\varphi_2\|$.

Further, we have

$$V[t,\ y\ (t+s;t_0,\ \varphi)]=\int_0^\infty \|y\ (t+u+s;\ t,\ y\ (t+s;\ t_0,\ \varphi))\|^2\ du+$$

$$+\sup_{\sigma\geqslant 0}\|y\ (t+\sigma+s;\ t,\ y\ (t+s;t_0,\ \varphi))\|^2=\int_0^\infty \|y\ (t+u+s;\ t_0,\ \varphi)\|^2\ du+$$

$$+\sup_{\sigma\geqslant 0}\|y(t+\sigma+s;\ t_0,\ \varphi)\|^2=\int_t^\infty \|y\ (u+s;t_0,\ \varphi)\|^2du+\sup_{\sigma\geqslant t}\|y\ (\sigma+s;\ t_0,\ \varphi)\|^2.$$

It follows from this that

$$\limsup_{h\to 0+}\frac{V\ [t+h,\ y\ (t+h+s;\ t_0,\ \varphi)]-V\ [t,\ y\ (t+s;\ t_0,\ \varphi)]}{h}\leqslant -\|y\ (t+s;\ t_0,\ \varphi)\|^2.$$

For any solution of $x(v;t,\psi)$ we have $t\leqslant v\leqslant t+h\leqslant t+\tau$

$$x\ (v;\ t,\ \psi)=\psi\ (t)+\int_t^v\ [A(u)\ x\ (u;t,\psi)+B(u)\psi\ (u-\tau)+f(u)]\ du,$$

consequently,

$$|x(v;\ t,\ \psi)|\leqslant\|\psi\|\ (1+\int_t^{t+h}|B(u)|\ du)+\int_t^{t+h}|f(u)|\ du+$$

$$+\int_t^v|A\ (u)|\cdot|x\ (u;t,\ \psi)|\ du.$$

Hence, on the basis of the known lemma

$$|x(v;t,\ \psi)|\leqslant\left\{\|\psi\|(1+\int_t^{t+h}|\ B\ (u)|du)+\int_t^{t+h}|f\ (u)|\ du\right\}e^{\int_t^{t+h}|\ A(u)|\ du}.$$

We have also

$$x\ (v;t,\ \psi)-y\ (v;t,\ \psi)=\int_t^v\{A\ (u)\ [x\ (u;\ t,\ \psi)-y\ (u;\ t,\ \psi)]+f\ (u)\}\ du.$$

consequently,

$$|x(v;t,\ \psi)-y(v;t,\ \psi)|\leqslant\int_t^{t+h}|f(u)|\ du+\int_t^v|A(u)|\ |x(u;t,\ \psi)-y(u;t,\ \psi)|du$$

$$|x\ (v;\ t,\ \psi)-y\ (v;\ t,\ \psi)|\leqslant\int_t^{t+h}|f\ (u)|\ .e^{\int_t^{t+h}|A\ (u)|\ du},\ t\leqslant v\leqslant t+h.$$

It follows from the resulting evaluations that

$$\limsup_{h\to 0+} \frac{V[t+h, x(t+h+s; t, \varphi)] - V[t, \varphi]}{h} \leqslant \limsup_{h\to 0+} \frac{V[t+h, y(t+h+s; t, \varphi)] - V[t, \varphi]}{h} +$$

$$+ \limsup_{h\to 0+} \frac{V[t+h, x(t+h+s; t, \varphi)] - V[t+h, y(t+h+s; t, \varphi)]}{h} \leqslant -\|\varphi\|^2 +$$

$$+ \limsup_{h\to 0+} \frac{V[t+h, x(t+h+s; t, \varphi)] - V[t+h, y(t+h+s; t, \varphi)]}{h};$$

$$|V[t+h, x(t+h+s; t, \varphi)] - V[t+h, y(t+h+s; t, \varphi)]| \leqslant L_0 [\|x(t+$$
$$+h+s; t, \varphi)\| + \|y(t+h+s; t, \varphi)\|] \|x(t+h+s; t, \varphi) - y(t+h+s; t, \varphi)\| \leqslant$$

$$\leqslant L_0 \left\{ 2\|\varphi\| (1 + \int_t^{t+h} |B(u)| du) + \int_t^{t+h} |f(u)| du \right\} \int_t^{t+h} |f(u)| du \; e^{2\int_t^{t+h} |A(u)| du};$$

$$\limsup_{h\to 0+} \frac{V[t+h; x(t+h+s; t, \varphi)] - V[t+h; y(t+h+s; t, \varphi)]}{h} \leqslant 2 L_0 \|\varphi\| . |f(t)|.$$

If we designate $M = \sup_{t\geqslant 0} |f(t)|$, we will finally get

$$\limsup_{h\to 0+} \frac{V[t+h, x(t+h+s; t, \varphi)] - V[t, \varphi]}{h} \leqslant -\|\varphi\|^2 + 2 L_0 M \|\varphi\|.$$

We will now consider the arbitrary solution $x(t, \varphi)$ such that $V[0, \varphi] < 4 K_1 L_0^2 M^2$. We will prove that whenever $t \geqslant 0$, we have $V[t, x(t+s; \varphi)] \leqslant 4 K_1 L_0^2 M^2$. Let us assume it is incorrect; then there exists $t_1 > 0$, such that $V[t_1, x(t_1+s; \varphi)] > 4 K_1 L_0^2 M^2$ and, consequently, there exists $t_2 < t_1$ such that $V[t_2, x(t_2+s; \varphi)] = 4 K_1 L_0^2 M^2$ and $V[t, x(t+s; \varphi)] > 4 K_1 L_0^2 M^2$ with $t_2 < t \leqslant t_1$. From this it follows that

$$V[t_2 + h; x(t_2 + h + s; t_2, x(t_2 + s; \varphi))] > V[t_2, x(t_2 + s; \varphi)],$$

consequently,

$$\limsup_{h\to 0+} \frac{V[t_2 + h; x(t_2 + h + s; t_2, x(t_2 + s; \varphi))] - V[t_2, x(t_2 + s; \varphi)]}{h} \geqslant 0.$$

But

$$\limsup_{h\to 0+} \frac{V[t_2 + h; x(t_2 + h + s; t_2, x(t_2 + s; \varphi))] - V[t_2, x(t_2 + s; \varphi)]}{h} \leqslant$$
$$\leqslant -\|x(t_2 + s; \varphi)\|^2 + 2L_0 M \|x(t_2 + s; \varphi)\| =$$
$$= \|x(t_2 + s; \varphi)\| \{-\|x(t_2 + s; \varphi)\| + 2 L_0 M\}.$$

It follows from $V[t_2, x(t_2 + s; \varphi)] = 4 K_1 L_0^2 M^2$ and $V[t, \varphi] \leqslant K_1 \|\varphi\|^2$ that $4 K_1 L_0^2 M^2 < K_1 \|x(t_2 + s; \varphi)\|^2$, consequently, $\|x(t_2 + s; \varphi)\|^2 > 4 L_0^2 M^2$, that is $\|x(t_2 + s; \varphi)\| > 2 L_0 M$.

From this it follows that

$$\limsup_{h \to 0+} \frac{V[t_2 + h; x(t_2 + h + s; (t_2, x(t_2 + s; \varphi))] - V[t_2, x(t_2 + s; \varphi)]}{h} < 0.$$

The result is a contradiction, consequently

$V[t, x(t+s; \varphi)] \leqslant 4 K_1 L_0^2 M^2$ whenever $t \geqslant 0$; but it follows from $V[t, x(t+s; \varphi)] \leqslant 4 K_1 L_0^2 M^2$ and $V[t, \varphi] \geqslant \|\varphi\|^2$ that $\|x(t+s; \varphi)\|^2 \leqslant 4 K_1 L_0^2 M^2$, that is $\|x(t+s; \varphi)\| \leqslant 2\sqrt{K_1} L_0 M = LM$.

The lemma has been proved.

The direct corollary of the proved lemma is

Theorem 1. The system $\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + f(t)$ where

$$A(t+\omega) \equiv A(t),\ B(t+\omega) \equiv B(t),\ f(t+\omega) \equiv f(t)$$

has a periodic solution of the period if the trivial solution of a corresponding homogeneous system is uniformly asymptotically stable. This periodic solution is obviously the only one and it satisfies the $/x_0(t)/ \leqslant LM$ type of evaluation where $M = \sup/f(t)/$.

With this theorem it is possible to provide a new proof of the following theorem:

Theorem 2. Under consideration is the following system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + f(t) + \mu X(t, x(t), x(t-\tau), \mu),$$

where $A(t)$, $B(t)$, $f(t)$, $X(t,u,v,\mu)$ are periodic of the ω period,

$$|X(t, u, v, \mu) - X(t, u', v', \mu)| \leqslant K\{|u - u'| + |v - v'|\}.$$

If the trivial solution of the system $\dot{y}(t) = A(t)y(t) + B(t)y(t-\tau)$ is uniformly asymptotically stable, then the system has a periodic solution for a sufficiently small $|\mu|$.

Proof. Let $x_0(t)$ be a periodic solution of the system $\dot{x}_0(t) = A(t)x(t) + B(t)x(t-\tau) + f(t)$, which exists on the basis of theorem 1. Let us assume that $Z(t) = x(t) - x_0(t)$.

We get

$$\dot{z}(t) = \dot{x}(t) - \dot{x}_0(t) = A(t)\,[z(t) + x_0(t)] + B(t)\,[z(t-\tau) + x_0(t-\tau)] + f(t) +$$
$$+ \mu X\,[t, z(t) + x_0(t), z(t-\tau) + x_0(t-\tau), \mu] - A(t)\,x_0(t) - B(t)\,x_0(t-\tau) - f(t),$$

consequently

$$\dot{z}(t) = A(t)\,z(t) + B(t)\,z(t-\tau) + \mu Z(t, z(t), z(t-\tau), \mu).$$

Let $z_1(t)$ be the periodic solution of the following system

$$\dot{z}(t) = A(t)\,z(t) + B(t)\,z(t-\tau) + \mu Z(t, 0, 0, \mu),$$

and $z_k(t)$ the periodic solution of the following system

$$\dot{z}(t) = A(t)\,z(t) + B(t)\,z(t-\tau) + \mu Z(t, z_{K-1}(t), z_{K-1}(t-\tau), \mu).$$

Resulting from Theorem 1 is the following evaluation $|z_{K+1} - z_K| < |\mu| KL|z_K - z_{K-1}|$. The corollary is a uniform convergence of successive approximations; the end result is a periodic solution of the system.

2. It is possible to establish similar facts in connection with near-periodic solutions.

<u>Theorem 3</u>. The system $\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t-\theta) + f(t)$, where A, B, f are continuous near-periodic functions, has a near-periodic solution, if the trivial solution of a corresponding homogeneous system is uniformly asymptotically stable. This near-periodic solution is obviously the only one, and it satisfies the $|x_0(t)| \leqslant LM$, type of evaluation where $M = \sup|f(t)|$.

<u>Proof</u>. The existence of a limited solution of u (t) is deduced from the lemma. We have $\dot{u}(t) = A(t)\,u(t) + B(t)\,u(t-\theta) + f(t)$,

$$\dot{u}(t+\tau) = A(t+\tau)\,u(t+\tau) + B(t+\tau)\,u(t+\tau-\theta) + f(t+\tau)$$
$$\dot{u}(t+\tau) - \dot{u}(t) = A(t)\,[u(t+\tau) - u(t)] + B(t)[u(t+\tau-\theta) - u(t-\theta)] +$$
$$+ [A(t+\tau) - A(t)]\,u(t+\tau) + [B(t+\tau) - B(t)]\,u(t+\tau-\theta) + f(t+\tau) - f(t).$$

Let $M_1 = \sup\,[\,|u(t)| + 1]$, and τ the common $\frac{\varepsilon}{4M_1L}$ near-period for A, B, f; L a constant, depending on the homogeneous system, resulting from the proof of the lemma.

Let y(t) be the solution of the homogeneous system which coincides with $u(t+\tau) - u(t)$ when $-\theta \leqslant t \leqslant 0$. We will designate

$$v(t) = u(t+\tau) - u(t) - y(t).$$

We have
$$\dot{v}(t) = A(t)\,v(t) + B(t)\,v(t-\theta) + [A(t+\tau) - A(t)]\,u(t+\tau) + [B(t+\tau) - B(t)]\,u(t+\tau-\theta) + f(t+\tau) - f(t),$$

$v(t) \equiv 0$ when $t \in [-\theta, 0]$.

It follows from the lemma that

$$|v(t)| \leqslant L\{|A(t+\tau) - A(t)|\cdot|u(t+\tau)| + |B(t+\tau) - B(t)|\cdot|u(t+\tau-\theta)| + |f(t+\tau) - f(t)|\} \leqslant L\left\{\frac{\varepsilon}{4M_1L}\cdot M_1 + \frac{\varepsilon}{4M_1L}\cdot M_1 + \frac{\varepsilon}{4M_1L}\right\} < \frac{3\varepsilon}{4}.$$

Consequently, $|u(t+\tau) - u(t) - y(t)| < \frac{3\varepsilon}{4}$ when $t \geqslant -\theta$.

Since the stereotyped solution of a homogeneous system is uniformly asymptotically stable, and the major term of the initial function for the solution of y(t) is the constant $2M_1$, it follows that there exists $T(\varepsilon)$ such that, when $t > T$, we have $|y(t)| < \frac{\varepsilon}{4}$. Hence, when $t > T$, we get

$$|u(t+\tau) - u(t)| \leqslant |u(t+\tau) - u(t) - y(t)| + |y(t)| < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

We have thus found such $l(\varepsilon)$ and $T(\varepsilon)$ for the arbitrary $\varepsilon > 0$ that in any interval of the length l there is a number τ such that $|u(t+\tau) - u(t)| < \varepsilon$ when $t > T$; but this means that the solution of u(t) is asymptotically near-periodic.

Then $u(t) = x_0(t) + \omega(t)$, where $u_0(t)$ is a near-periodic function and $\lim_{t\to\infty} \omega(t) = 0$. We have

$$\dot{x}_0(t) + \dot{\omega}(t) = A(t)\,x_0(t) + A(t)\,\omega(t) + B(t)\,x_0(t-\theta) + B(t)\,\omega(t-\theta) + f(t),$$

consequently, $\dot{x}_0(t) = A(t)\,x_0(t) + B(t)\,x_0(t-\theta) + f(t)$. The existence of a near-periodic solution has thus been proved.

It follows from $|u(t)| \leqslant ML$ that $|x_0(t)| \leqslant ML + |\omega(t)|$. Let $\varepsilon > 0$; it follows from $\lim_{t\to\infty} \omega(t) = 0$ that there exists $T > 0$ such that with $t > T$ we have $|\omega(t)| < \frac{\varepsilon}{2}$.

We will write down that $|x_0(t)| \leqslant ML + \frac{\varepsilon}{2}$, with $t > T$. Let t be an arbitrary value; there exists an $\frac{\varepsilon}{2}$ - near-period τ, such that $t + \tau > T$. Then $|x_0(t+\tau)| \leqslant ML + \frac{\varepsilon}{2}$ and $|x_0(t+\tau) - x_0(t)| < \frac{\varepsilon}{2}$.
We get $|x_0(t)| \leqslant ML + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = ML + \varepsilon$. But since ε was arbitrary, it follows that $|x_0(t)| \leqslant ML$.

The theorem has been proved.

We should point out that the proved theorem generalizes S. N. Shimanov's result [2] to a great extent.

Theorem 4. Under consideration is the following system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\theta) + f(t) + \mu X[t, x(t), x(t-\theta), \mu]$$

where $A, B, f, X[t, u, v, \mu]$ are near periodic,

$$|X(t, u, v, \mu) - X(t, u', v', \mu)| \leqslant K\{|u - u'| + |v - v'|\}.$$

If the trivial solution of the system $y(t) = A(t)y(t) + B(t)y(t-\theta)$ is uniformly asymptotically stable, the system will have a near periodic solution when $|\mu|$ is sufficiently small.

The proof is based on the same scheme as that of Theorem 2 and it, in effect, coincides with the corresponding proof for the systems of ordinary differential equations [3].

We should point out that a more general assumption is proved in the same way.

Under consideration is the following system

$$\dot{x}(t) = X_0[t, x(t), x(t-\theta)] + \mu X_1[t, x(t), x(t-\theta), \mu],$$

where X_0 and X_1 are near periodic functions of t. Let us assume that the initial system $x_0(t) = X_0[t, x_0(t), x_0(t-\theta)]$ has a near-periodic solution, such that a trivial solution of different variations of a corresponding system is uniformly asymptotically stable. Then, if $|\mu|$ is sufficiently small, the system will have a near-periodic solution.

By using a slight generalization of the lemma and repeating the scheme of the proof of Theorem 3 it is possible to construct the following theorem.

Theorem 5. Under consideration is the system $\dot{x}(t) = X[t, x(t), x(t-\theta)]$, where X[t,u,v] is a near-periodic function of t which is uniform in relation to u,v. Let us assume that the system has a limited and, on the whole, an asymptotically stable solution u(t) such that the correct solution of different variations of a corresponding system is uniformly asymptotically stable. Then u(t) is an asymptotically near-periodic function, and the system has a near-periodic solution.

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