N-83830

VIBRATION ANALYSIS OF TOROIDAL SHELLS OF CIRCULAR CROSS SECTION

By

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

March 1960

N71-72243
(ACCESSION NUMBER) (THRU)
98 None
(PAGES) (CODE)
TMX 67022
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)
FACILITY FORM 602

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ABSTRACT

The problem treated is the free-free vibration of a thin toroidal shell of circular cross section and uniform thickness. The governing equations of equilibrium are derived on the basis of Love's first approximation in terms of orthogonal curvilinear coordinates on the middle surface. The strain displacement and curvature displacement relations are used to show the apparent strains produced by small rigid body displacements and rotations. The apparent strains result from the basic assumptions of Love's first approximation and are not particular to the toroidal shell. The equilibrium equations are shown to reduce to Love's first approximation equations for circular cylinders of constant thickness. This reduction is accomplished by allowing the major generating radius of the middle surface of the toroidal shell to approach infinity.

The equilibrium equations for the free-free vibration of the complete toroidal shell vibrating in vacuo with no internal pressure are presented for simple harmonic motion with all three components of the inertia forces. The equilibrium equations are shown to reduce to a set of ordinary differential equations if the motion is assumed to be composed of complete waves in the form of trigonometric functions in the direction of principal curvature, and arbitrary functions in the direction

of minimum curvature. The resulting equations are linear differential equations with variable coefficients. Although the general closed-form solutions to these equations were not found, it is evident that the displacement functions must be continuous periodic functions with continuous periodic derivatives of the same period. Thus the solutions of the equilibrium equations are expressed in terms of complete Fourier series. The series coefficients that satisfy the governing equations were determined by the well-known Galerkin Procedure. The resulting solutions showed that the vibrations of the toroidal shell are characterized by symmetrical and antisymmetrical parts, and that the modes and frequencies of vibration are dependent on two shell parameters and the number of complete waves in the direction of principal curveture. One of the shell parameters is the ratio of the radius of the cross section to the radius from the axis of revolution to the center of the cross section, and the other parameter is the ratio of the shell thickness to the radius of the cross section.

In order to show graphically the nature of the natural modes of vibration of the toroidal shell, numerical calculations were made for the lowest three modes of symmetrical vibration for several values of the shell parameters. These calculations were made using ten terms of each displacement series. The results of the calculations were used to show the effects of the shell parameters on the modes and frequency, and on the rate of convergence of the series representing the three components of displacement. It was found that ten terms of each series are not sufficient to insure adequate convergence of the mode shapes for

all values of the shell parameters. Comparison of the results for the lowest three modes of vibration indicates that the nature of the vibration is not as sensitive to variations in the thickness-to-radius ratio for the range of values considered, as it is to the radius-ratio parameter. As the radius-ratio parameter approaches 1, the components of displacement in the lowest mode approach each other in magnitude.

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Blacksburg, Virginia

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IV. SYMBOLS

A_1, A_2	parameters in linear element defined by equation (7)
B_k, C_k, D_k	coefficients defined in equation (28)
D	flexural rigidity of shell $\left(D = \frac{Eh^3}{12(1 - \nu^2)}\right)$
E	Young's modulus
G	shear modulus
K	stiffness parameter $\left(K = \frac{Eh}{1 - \nu^2}\right)$
M_1, M_2, M_{12}	resultant bending and twisting moments in shell
N_1, N_2, N_{12}	resultant forces in middle surface of shell
R_1, R_2	principal radii of curvature
R	radius of the centroid of cross section of shell
$\vec{U}$	displacement vector $(\vec{U} = U_1\vec{t_1} + U_2\vec{t_2} + W\vec{n})$
U_1, U_2, W	components of displacement vector
$a_i, b_i, c_i, d_i, f_i, g_i$	Fourier coefficients defined in equation (29)
h	thickness of shell
$\vec{i}, \vec{j}, \vec{k}$	unit vectors in direction of x, y, z coordinates
i, j, k, n	integers
m	mass per unit area of the shell $(m = \mu h)$
$\vec{p}$	effective external force per unit area; applied to middle surface $(\vec{p} = p_1\vec{t_1} + p_2\vec{t_2} + q\vec{n})$
p_1, p_2, q	components of force vector
$\vec{r}$	position vector to point on middle surface $(\vec{r} = x\vec{i} + y\vec{j} + z\vec{k})$
r	radius of cross section of shell

ds	distance between adjacent points in the shell
$\vec{t}_1, \vec{t}_2, \vec{n}$	unit vectors in direction of ξ_1, ξ_2, ζ coordinates
u, v, w	dimensionless displacement functions defined in equation (16)
u_1, u_2, v_1, v_2, w	displacement functions defined in equation (5)
$\bar{u}, \bar{v}, \bar{w}$	dimensionless displacement functions defined in equation (27)
x, y, z	rectangular coordinates
a_1, a_2	parameters in linear element defined by equation (8)
α, β, γ	parameters defined by equation (17)
γ_{12}	component of shearing strain
δ_{ij}	Kronecker delta
ϵ_1, ϵ_2	components of direct strain
ζ	coordinate normal to middle surface
κ_1, κ_2, τ	strain functions defined in equation (11)
$\bar{\lambda}$	frequency parameter $\left(\bar{\lambda}_i = r\omega_i \sqrt{\frac{m}{K}}\right)$
μ	mass density of the shell material
ν	Poisson's ratio
ξ_1, ξ_2	curvilinear coordinates of middle surface of shell
$\vec{\rho}$	position vector to a point in the shell $(\vec{\rho} = \vec{r} + \zeta\vec{n})$
$\sigma_1, \sigma_2, \sigma_\zeta$	components of direct stress
$\tau_{12}, \tau_{1\zeta}, \tau_{2\zeta}$	components of shearing stress
ω_i	natural circular frequency of ith mode of vibration

V. INTRODUCTION

In the design of thin elastic shell structures, it is important to know their dynamic and vibrational characteristics as well as their load carrying ability. The dynamic behavior is particularly important for missiles and space vehicles since they are subject to a variety of dynamic loadings. The structural configuration of these vehicles depends on their intended mission but in general thin shells are utilized since the lightest possible structural weight is required. For manned satellites the toroidal shell has many attractive features. The design of such a satellite requires theoretical methods for predicting the vibrational characteristics as well as the stresses and deflections under static loadings.

A number of authors have studied the static behavior of toroidal shells; several of the recent investigations are described in references 1 to 6. Reports of other investigations on the stresses and deflections of toroidal shells are listed in these references and in references 7 and 8.

Although there is considerable information on the static behavior there is no information on the dynamic behavior of toroidal shells. It is the purpose of this dissertation to present the results of a theoretical investigation of the vibrational characteristics of toroidal shells. The analysis of the shell is based on Love's first approximation theory for thin shells[9] and the equilibrium equations for the displacements are derived in terms of orthogonal curvilinear

coordinates of the middle surface. Solutions to these equations are presented for the natural mode shapes in the form of Fourier series for the three components of displacement. The natural modes for the toroidal shell with circular cross section and uniform thickness are shown to depend on two shell parameters and the number of complete waves around the toroid. One of the shell parameters is the ratio of the shell thickness to the radius of the cross section and the other parameter is the ratio of the cross-section radius to the radius from the axis of symmetry to the center of the cross section.

Numerical calculations are used to illustrate the nature of the lowest modes of vibration and the effect of the shell parameters on the vibrational characteristics of the toroidal shell.

VI. GOVERNING EQUATIONS

1. Formulation of the Problem

In this section the equations governing the vibration of a thin toroidal shell of circular cross section are discussed. The middle surface of the shell is generated by the revolution of a circle of radius r with its center a distance R from the axis of revolution. A section of the toroid with the coordinate system used to define a point in the shell is shown in figure 1. Thus the position of any point in the shell may be defined by three coordinates ξ_1, ξ_2, and ζ, where ξ_1 and ξ_2 specify position on the middle surface and ζ specifies the distance from the middle surface along the outward normal. The position vector to a point in the shell is given by

$$\vec{\rho}(\xi_1,\xi_2,\zeta) = \vec{r}(\xi_1,\xi_2) + \zeta\vec{n}(\xi_1,\xi_2) \tag{1}$$

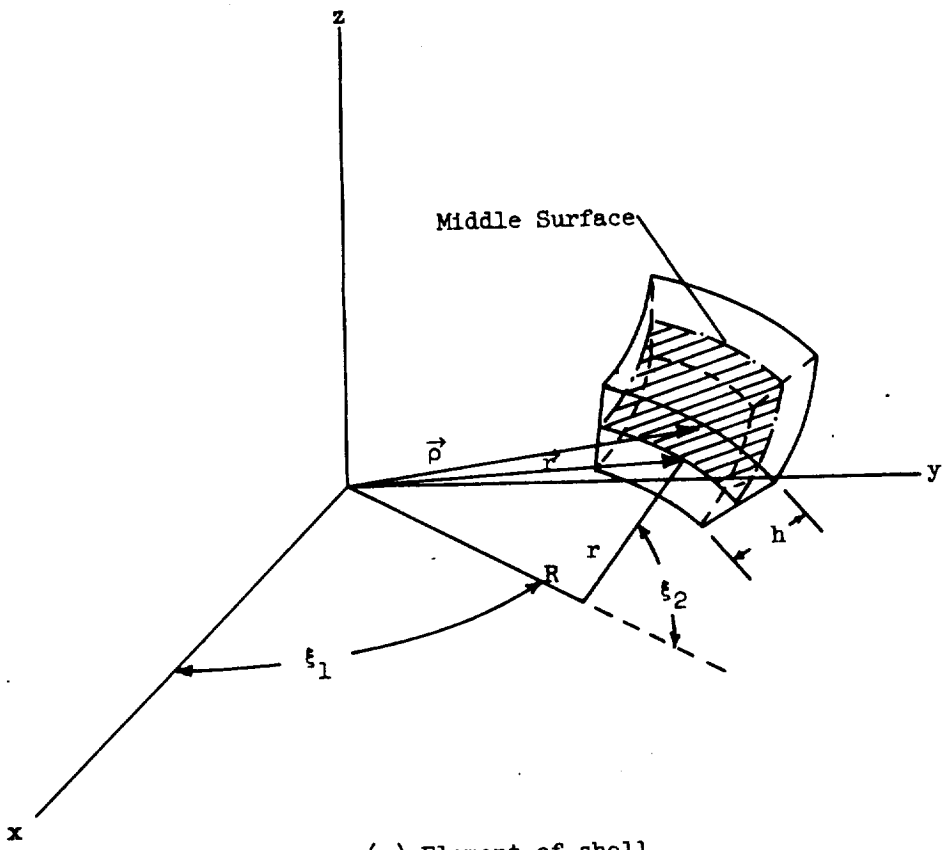
where $\vec{r}$ is the position vector to a point on the middle surface and $\vec{n}$ is the unit normal vector to the middle surface. From figure 1 it is seen that

$$\vec{r} = (R + r\cos\xi_2)\cos\xi_1\,\vec{i} + (R + r\cos\xi_2)\sin\xi_1\,\vec{j} + r\sin\xi_2\,\vec{k} \tag{2}$$

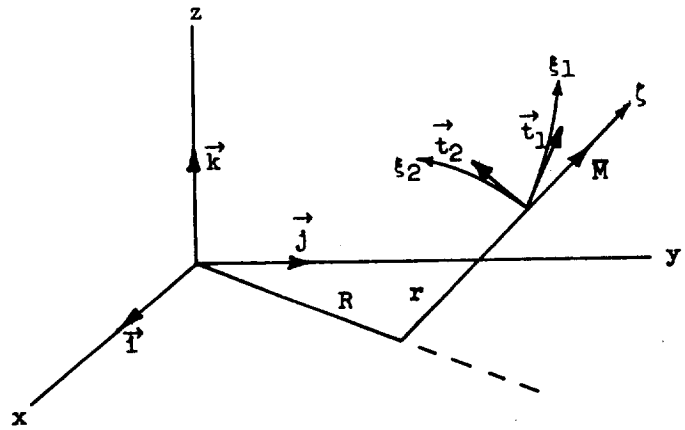
where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit base vectors. The square of the line element, ds^2, can now be expressed in terms of the coordinates ξ_1, ξ_2, and ζ by the use of equations (1) and (2)

$$ds^2 = d\vec{\rho}\cdot d\vec{\rho} = \left[R + (r+\zeta)\cos\xi_2\right]^2 d\xi_1^2 + (r+\zeta)^2\, d\xi_2^2 + d\zeta^2 \tag{3}$$

from which we see that the coordinate curves form an orthogonal system.



(a) Element of shell.



(b) Unit tangent vectors

Figure 1.- Shell element and coordinate system.

If the displacement vector is written in the form

$$\vec{U} = U_1\vec{t_1} + U_2\vec{t_2} + W\vec{n} \tag{4}$$

where $\vec{t_1}$ and $\vec{t_2}$ denote the unit tangent vectors to the ξ_1- and the ξ_2-curves, respectively, then the strains can be expressed in terms of the components of $\vec{U}$.

The analysis that follows is based on thin-shell theory according to Love's first approximation[9]; hence, before writing the strain-displacement relations the basic assumptions of this theory will be discussed.

The conventional assumptions[10] for thin-shell theory based on an order-of-magnitude consideration of the equilibrium equations for stress show that, in the absence of concentrated surface loads, the transverse normal stress, σ_ζ, and the transverse shear stresses, $\tau_{1\zeta}$ and $\tau_{2\zeta}$, are of smaller magnitude than the normal stresses, σ_1 and σ_2, and the shear stress, τ_{12}. It is also assumed that the displacement components in equation (4) are defined as

$$\left.\begin{aligned} U_1 &= u_1(\xi_1, \xi_2) + \zeta\, u_2(\xi_1, \xi_2) \\ U_2 &= v_1(\xi_1, \xi_2) + \zeta\, v_2(\xi_1, \xi_2) \\ W &= w_1(\xi_1, \xi_2) \end{aligned}\right\} \tag{5}$$

that is, the displacements in the direction of ξ_1 and ξ_2 vary linearly through the thickness of the shell while the normal component of displacement is constant through the thickness and is defined as the normal displacement of the middle surface. In addition to the above assumptions, Love's first approximation states that the shell is assumed to be very thin in comparison with the minimum radius of

curvature, R_2, and hence, ζ/R_2 can be neglected in comparison with unity in the equations relating the displacements and strains, and in the equations relating the stress resultants and couples to the components of strain. It is also assumed that normals to the undeformed middle surface remain normal to the deformed middle surface and suffer no extension.

2. The Strain-Displacement Relations

In order to write the strain-displacement relations based on the above assumptions it is convenient to make use of the relations given by Hildebrand, Reissner, and Thomas[11] for general orthogonal curvilinear coordinates. These general expressions are based on the following definitions

$$ds^2 = d\vec{\rho} \cdot d\vec{\rho} = A_1^2\, d\xi_1^2 + A_2^2\, d\xi_2^2 + d\zeta^2 \tag{6}$$

with

$$\left.\begin{aligned} A_1 &= \alpha_1(1 + \zeta/R_1) \\ A_2 &= \alpha_2(1 + \zeta/R_2) \end{aligned}\right\} \tag{7}$$

where R_1 and R_2 are the principle radii of curvature of the middle surface and

$$\left.\begin{aligned} \alpha_1^2 &= \frac{\partial \vec{r}}{\partial \xi_1} \cdot \frac{\partial \vec{r}}{\partial \xi_1} & \alpha_2^2 &= \frac{\partial \vec{r}}{\partial \xi_2} \cdot \frac{\partial \vec{r}}{\partial \xi_2} \\ \frac{1}{R_1} &= \frac{1}{\alpha_1^2} \frac{\partial \vec{n}}{\partial \xi_1} \cdot \frac{\partial \vec{r}}{\partial \xi_1} & \frac{1}{R_2} &= \frac{1}{\alpha_2^2} \frac{\partial \vec{n}}{\partial \xi_2} \cdot \frac{\partial \vec{r}}{\partial \xi_2} \end{aligned}\right\} \tag{8}$$

The quantity $\vec{n}$ is defined by

$$\vec{n} = \vec{t}_1 \times \vec{t}_2 = \frac{1}{\alpha_1\alpha_2} \frac{\partial \vec{r}}{\partial \xi_1} \times \frac{\partial \vec{r}}{\partial \xi_2} \tag{9}$$

The quantities in equations (8) and (9) can be obtained in the notation of the present problem by using the expression for $\vec{r}$ in equation (2) and performing the indicated operations.

$$\left.\begin{aligned}
\vec{n} &= \cos \xi_2 \cos \xi_1 \, \vec{i} + \cos \xi_2 \sin \xi_1 \, \vec{j} + \sin \xi_2 \, \vec{k} \\
\alpha_1 &= R + r \cos \xi_2 \\
\alpha_2 &= r \\
R_1 &= \frac{R + r \cos \xi_2}{\cos \xi_2} = \frac{\alpha_1}{\cos \xi_2} \\
R_2 &= r
\end{aligned}\right\} \tag{10}$$

The stress-strain relations for Love's first approximation of the toroidal shell can now be written using equations (5) and (10) in the general stress-strain relations given by equations (43), (45), and (46)[11]. These substitutions give the following strain-displacement and curvature-displacement relations

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{\alpha_1} \frac{\partial u_1}{\partial \xi_1} + \frac{v_1}{\alpha_1\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} + \frac{w_1}{R_1} = \frac{\partial}{\partial \xi_1}\left(\frac{u_1}{\alpha_1}\right) - \frac{v_1 \sin \xi_2}{\alpha_1} + \frac{w \cos \xi_2}{\alpha_1} \\
\epsilon_2 &= \frac{1}{\alpha_2} \frac{\partial v_1}{\partial \xi_2} + \frac{u_1}{\alpha_1\alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} + \frac{w_1}{R_2} = \frac{\partial}{\partial \xi_2}\left(\frac{v_1}{r}\right) + \frac{w_1}{r} \\
\gamma_{12} &= \frac{\alpha_2}{\alpha_1} \frac{\partial}{\partial \xi_1}\left(\frac{v_1}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2} \frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right) = \frac{r}{\alpha_1} \frac{\partial}{\partial \xi_1}\left(\frac{v_1}{r}\right) + \frac{\alpha_1}{r} \frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right)
\end{aligned}\right\} \tag{11}$$

$$\kappa_1 = \frac{1}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(-\frac{1}{\alpha_1}\frac{\partial w_1}{\partial \xi_1} + \frac{u_1}{R_1}\right) + \frac{1}{\alpha_1\alpha_2}\left(-\frac{1}{\alpha_2}\frac{\partial w_1}{\partial \xi_2} + \frac{v_1}{R_2}\right)\frac{\partial \alpha_1}{\partial \xi_2} = -\frac{r}{\alpha_1^2}\frac{\partial^2}{\partial \xi_1^2}\left(\frac{w_1}{r}\right) + \frac{\cos \xi_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_1}{\alpha_1}\right) + \frac{\sin \xi_2}{\alpha_1}\frac{\partial}{\partial \xi_2}\left(\frac{w_1}{r}\right) - \frac{\sin \xi_2}{\alpha_1}\left(\frac{v_1}{r}\right)$$

$$\kappa_2 = \frac{1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(-\frac{1}{\alpha_2}\frac{\partial w_1}{\partial \xi_2} + \frac{v_1}{R_2}\right) + \frac{1}{\alpha_1\alpha_2}\left(-\frac{1}{\alpha_1}\frac{\partial w_1}{\partial \xi_1} + \frac{u_1}{R_1}\right)\frac{\partial \alpha_2}{\partial \xi_1} = -\frac{1}{r}\frac{\partial^2}{\partial \xi_2^2}\left(\frac{w_1}{r}\right) + \frac{1}{r}\frac{\partial}{\partial \xi_2}\left(\frac{v_1}{r}\right)$$

$$\tau = \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(-\frac{1}{\alpha_2^2}\frac{\partial w_1}{\partial \xi_2} + \frac{v_1}{\alpha_2 R_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(-\frac{1}{\alpha_1^2}\frac{\partial w_1}{\partial \xi_1} + \frac{u_1}{\alpha_1 R_1}\right)$$

$$= -\frac{2}{\alpha_1}\frac{\partial^2}{\partial \xi_1 \partial \xi_2}\left(\frac{w_1}{r}\right) + \frac{1}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{v_1}{r}\right) + \frac{\cos \xi_2}{r}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right) - \frac{2r \sin \xi_2}{\alpha_1^2}\frac{\partial}{\partial \xi_1}\left(\frac{w_1}{r}\right) - \frac{R \sin \xi_2}{r\alpha_1}\left(\frac{u_1}{\alpha_1}\right) \tag{11}$$

and the expressions for the stress resultants and couples in terms of the displacement components for a shell of constant thickness, h

$$N_{11} = \frac{Eh}{1 - \nu^2}(\epsilon_1 + \nu\epsilon_2) = \frac{Eh}{1 - \nu^2}\left\{\frac{\partial}{\partial \xi_1}\left(\frac{u_1}{\alpha_1}\right) - \frac{r \sin \xi_2}{\alpha_1}\left(\frac{v_1}{r}\right) + \frac{r \cos \xi_2}{\alpha_1}\left(\frac{w_1}{r}\right) + \nu\left[\frac{\partial}{\partial \xi_2}\left(\frac{v_1}{r}\right) + \left(\frac{w_1}{r}\right)\right]\right\}$$

$$N_{22} = \frac{Eh}{1 - \nu^2}(\epsilon_2 + \nu\epsilon_1) = \frac{Eh}{1 - \nu^2}\left\{\frac{\partial}{\partial \xi_2}\left(\frac{v_1}{r}\right) + \frac{w_1}{r} + \nu\frac{\partial}{\partial \xi_1}\left(\frac{u_1}{\alpha_1}\right) + \nu\left[\frac{r \cos \xi_2}{\alpha_1}\left(\frac{w_1}{r}\right) - \frac{r \sin \xi_2}{\alpha_1}\left(\frac{v_1}{r}\right)\right]\right\} \tag{12}$$

$$N_{12} = N_{21} = Gh\gamma_{12} \tag{12}$$

$$= \frac{1 - \nu}{2} \frac{Eh}{1 - \nu^2}\left[\frac{r}{\alpha_1} \frac{\partial}{\partial \xi_1}\left(\frac{v_1}{r}\right) + \frac{\alpha_1}{r} \frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right)\right]$$

$$M_{11} = \frac{Eh^3}{12(1 - \nu^2)} (\kappa_1 + \nu\kappa_2)$$

$$= \frac{Eh^3}{12(1 - \nu^2)}\left\{\frac{\cos \xi_2}{\alpha_1} \frac{\partial}{\partial \xi_1}\left(\frac{u_1}{\alpha_1}\right) - \frac{r}{\alpha_1^2} \frac{\partial^2}{\partial \xi_1^2}\left(\frac{w_1}{r}\right) + \frac{\sin \xi_2}{\alpha_1} \frac{\partial}{\partial \xi_2}\left(\frac{w_1}{r}\right) - \right.$$

$$\left. \frac{\sin \xi_2}{\alpha_1}\left(\frac{v_1}{r}\right) + \nu\left[\frac{1}{r} \frac{\partial}{\partial \xi_2}\left(\frac{v_1}{r}\right) - \frac{1}{r} \frac{\partial^2}{\partial \xi_2^2}\left(\frac{w_1}{r}\right)\right]\right\}$$

$$M_{22} = \frac{Eh^3}{12(1 - \nu^2)} (\kappa_2 + \nu\kappa_1)$$

$$= \frac{Eh^3}{12(1 - \nu^2)}\left\{\frac{1}{r} \frac{\partial}{\partial \xi_2}\left(\frac{v_1}{r}\right) - \frac{1}{r} \frac{\partial^2}{\partial \xi_2^2}\left(\frac{w_1}{r}\right) + \nu\left[\frac{\cos \xi_2}{\alpha_1} \frac{\partial}{\partial \xi_1}\left(\frac{u_1}{\alpha_1}\right) - \right.\right.$$

$$\left.\left. \frac{r}{\alpha_1^2} \frac{\partial^2}{\partial \xi_1^2}\left(\frac{w_1}{r}\right) + \frac{\sin \xi_2}{\alpha_1} \frac{\partial}{\partial \xi_2}\left(\frac{w_1}{r}\right) - \frac{\sin \xi_2}{\alpha_1}\left(\frac{v_1}{r}\right)\right]\right\}$$

$$M_{12} = M_{21} = G \frac{h^3}{12} \tau$$

$$= \frac{1 - \nu}{2} \frac{Eh^3}{12(1 - \nu^2)}\left\{\frac{1}{\alpha_1} \frac{\partial}{\partial \xi_1}\left(\frac{v_1}{r}\right) - \frac{2}{\alpha_1} \frac{\partial^2}{\partial \xi_1 \partial \xi_2}\left(\frac{w_1}{r}\right) + \frac{\cos \xi_2}{r} \frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right) - \right.$$

$$\left. \frac{2r \sin \xi_2}{\alpha_1^2} \frac{\partial}{\partial \xi_1}\left(\frac{w_1}{r}\right) - \frac{R \sin \xi_2}{r\alpha_1}\left(\frac{u_1}{\alpha_1}\right)\right\}$$

In equations (11) and (12) E is Young's modulus of elasticity, G is the shear modulus of the material, and ν is Poisson's ratio.

[illegible]

[illegible]

[illegible]

$$[illegible] \qquad (13)$$

[illegible]

$$\left.\begin{aligned}
&\frac{\partial c_1 M_{11}}{\partial \xi_1} + \frac{\partial c_1 M_{21}}{\partial \xi_2} - [illegible] \frac{\partial c_1}{\partial \xi_1} - [illegible] \frac{\partial c_2}{\partial \xi_1} + [illegible] + \frac{1}{[illegible]}\left[\frac{\partial c_1 M_{11}}{\partial \xi_1} + \frac{\partial c_1 M_{21}}{\partial \xi_2} + \right. \\
&\qquad\qquad \left. [illegible] \frac{\partial c_1}{\partial \xi_1} - [illegible] \frac{\partial c_2}{\partial \xi_1}\right] = \\
&\frac{\partial c_2 M_{12}}{\partial \xi_1} + \frac{\partial c_1 M_{22}}{\partial \xi_2} + [illegible] \frac{\partial c_2}{\partial \xi_1} - [illegible] \frac{\partial c_2}{\partial \xi_2} + [illegible] + \frac{1}{[illegible]}\left[\frac{\partial c_2 M_{12}}{\partial \xi_1} + \frac{\partial c_1 M_{22}}{\partial \xi_2} + \right. \\
&\qquad\qquad \left. [illegible] \frac{\partial c_2}{\partial \xi_1} - [illegible] \frac{\partial c_1}{\partial \xi_1}\right] = \\
&-\left(\frac{[illegible]}{[illegible]} + \frac{[illegible]}{[illegible]}\right) [illegible] + [illegible] + \frac{\partial}{\partial \xi_1}\left[\frac{1}{c_1}\frac{\partial c_1 M_{11}}{\partial \xi_1} + \frac{1}{c_1}\frac{\partial c_2 M_{21}}{\partial \xi_2} + \frac{M_{12}}{c_1}\frac{\partial c_1}{\partial \xi_2} - \right. \\
&\left. \frac{[illegible]}{c_1}\frac{\partial c_2}{\partial \xi_1}\right] + \frac{\partial}{\partial \xi_2}\left[\frac{1}{c_2}\frac{\partial c_2 M_{12}}{\partial \xi_1} + \frac{1}{c_2}\frac{\partial c_1 M_{22}}{\partial \xi_2} + \frac{M_{21}}{c_2}\frac{\partial c_2}{\partial \xi_1} - \frac{M_{11}}{c_2}\frac{\partial c_1}{\partial \xi_2}\right] = 0
\end{aligned}\right\} \qquad (14)$$

or in terms of the present notation

$$r\frac{\partial N_{11}}{\partial \xi_1} - 2N_{21}r\sin\xi_2 + a_1\frac{\partial N_{12}}{\partial \xi_2} + ra_1p_1 + \frac{r\cos\xi_2}{a_1}\frac{\partial M_{11}}{\partial \xi_1} -$$

$$2M_{12}\frac{r\sin\xi_2\cos\xi_2}{a_1} + \frac{\partial M_{21}}{\partial \xi_2}\cos\xi_2 = 0 \qquad (15a)$$

$$r\frac{\partial N_{12}}{\partial \xi_1} - N_{22}r\sin\xi_2 + a_1\frac{\partial N_{22}}{\partial \xi_2} + N_{11}r\sin\xi_2 + ra_1p_2 + \frac{\partial M_{12}}{\partial \xi_1} -$$

$$M_{22}\sin\xi_2 + \frac{a_1}{r}\frac{\partial M_{22}}{\partial \xi_2} + M_{11}\sin\xi_2 = 0 \qquad (15b)$$

$$-N_{11}r\cos\xi_2 - a_1N_{22} + ra_1q + \frac{r}{a_1}\frac{\partial^2 M_{11}}{\partial \xi_1^2} - \frac{2r\sin\xi_2}{a_1}\frac{\partial M_{12}}{\partial \xi_1} +$$

$$2\frac{\partial^2 M_{12}}{\partial \xi_1\partial \xi_2} - 2\frac{\partial M_{22}}{\partial \xi_2}\sin\xi_2 - M_{22}\cos\xi_2 + \frac{a_1}{r}\frac{\partial^2 M_{22}}{\partial \xi_2^2} + \frac{\partial M_{11}}{\partial \xi_2}\sin\xi_2 +$$

$$M_{11}\cos\xi_2 = 0 \qquad (15c)$$

The substitution of equation (12) into equation (15) results in the following three equilibrium equations in terms of the three components of displacement.

$$r^2a_1^3\ddot{u} - \frac{1-\nu}{2}\left(3a_1^4 r \sin\xi_2 u' - a_1^3 u''\right) + \frac{1+\nu}{2} r^2 a_1^2 \dot{v}' -$$

$$\frac{3-\nu}{2} r^3 a_1^2 \sin\xi_2 \dot{v} + r^2 a_1^2 (r\cos\xi_2 + \nu a_1)\dot{w} + \frac{D}{K}\left\{\frac{r^2 a_1}{2}(1 +\right.$$

$$\cos 2\xi_2)\ddot{u} + \frac{1-\nu}{2}\left[\frac{a_1^3}{2}(1+\cos 2\xi_2)u'' + r a_1^2 \cos\xi_2 u - \frac{r a_1^2}{4}(\sin\xi_2 +\right.$$

$$\left.\sin 3\xi_2)u' - \frac{r^2 a_1}{8}(1 - \cos 4\xi_2)u - a_1^3 \sin 2\xi_2 u' - \frac{a_1^3}{2}(1+\cos 2\xi_2)u\right] -$$

$$\frac{3-\nu}{4} r^2 a_1 \sin 2\xi_2 \dot{v} + \frac{1+\nu}{2} r a_1^2 \cos\xi_2 \dot{v}' - r^3 \cos\xi_2 \ddot{w} + \frac{r^2 a_1}{2}\sin 2\xi_2 \dot{w}' -$$

$$\left.r a_1^2 \cos\xi_2 \dot{w}'' - \frac{1-\nu}{2}\left[r^2 a_1 (1+\cos 2\xi_2)\dot{w}\right]\right\} + \frac{r^2 a_1^4}{K} p_1 = 0 \qquad (16a)$$

$$\frac{1+\nu}{2} a_1^3 r^2 \ddot{u}' + (1-\nu) r^3 a_1^2 \sin\xi_2 \ddot{u} - r^3 a_1^2 \sin\xi_2 v' + a_1^3 r^2 v'' -$$

$$\frac{r^4 a_1}{2}(1 - \cos 2\xi_2)v + \frac{1-\nu}{2} r^4 a_1 \ddot{v} - \nu r^3 a_1^2 \cos\xi_2 v + a_1^3 r^2 w' -$$

$$r^3 a_1^2 \sin\xi_2 w + \frac{r^4 a_1}{2}\sin 2\xi_2 w + \nu r^3 a_1^2 \cos\xi_2 w' + \frac{D}{K}\left\{\frac{r^2 a_1}{2}\sin 2\xi \ddot{u} +\right.$$

$$\nu(r a_1^2 \cos\xi_2 \ddot{u}' - r a_1^2 \sin\xi_2 \ddot{u}) + \frac{1-\nu}{2}(r a_1^2 \cos\xi_2 \ddot{u}' - r a_1^2 \sin\xi_2 \ddot{u} +$$

$$\frac{r^2 a_1}{2}\sin 2\xi_2 \ddot{u}) + a_1^3 v'' - \frac{r^2 a_1}{2}(1 - \cos 2\xi_2)v - r a_1^2 \sin\xi_2 v' -$$

$$\nu r a_1^2 \cos\xi_2 v + \frac{1-\nu}{2}(a_1 r^2 \ddot{v}) + r a_1^2 \sin\xi_2 w'' - r^3 \sin\xi_2 \ddot{w} +$$

$$\frac{r^2 a_1}{2}(1 - \cos 2\xi_2)w' - a_1^3 w''' + \nu(r a_1^2 \cos\xi_2 w' - r^3 \sin\xi_2 \ddot{w} - r^2 a_1 \ddot{w}') -$$

$$\frac{1-\nu}{2}\left(2r^2a_1\ddot{w}' + 2r^3 \sin \xi_2 \ddot{w}\right)\Bigg\} + \frac{r^3a_1^3}{K} p_2 = 0 \tag{16b}$$

$$-(r \cos \xi_2 + \nu a_1)\dot{u} + \left(\frac{r \cos \xi_2}{a_1} + \nu\right) r \sin \xi_2 v - \left(\frac{r \cos \xi_2}{a_1} + 2\nu\right) r \cos \xi_2 w -$$

$$a_1 w - (a + \nu r \cos \xi_2)v' + \frac{D}{K}\Bigg\{\frac{r \cos \xi_2}{a_1^2}\ddot{u} - \frac{r^2}{a_1^3}\ddot{w} - \frac{r \sin \xi_2}{a_1^2}\ddot{v} +$$

$$\frac{2 \sin \xi_2}{r} w'''' - \frac{a_1}{r^2} w'''' + \frac{a_1}{r^2} v''' - \frac{2 \sin \xi_2}{r} v'' - \left(\frac{\cos \xi_2}{r} + \frac{1}{2a_1} -\right.$$

$$\left.\frac{\cos 2\xi_2}{2a_1} + \frac{\nu \cos \xi_2}{r}\right)(v' - w'') + \left(\frac{\sin 2\xi_2}{2a_1} - \frac{2 \sin \xi_2}{r}\right)\dot{u}' +$$

$$\left[\frac{3-\nu}{2a_1} \cos 2\xi_2 - \frac{\cos \xi_2}{r} + \frac{1-\nu}{2a_1} + \frac{r}{4a_1^2}(\cos \xi_2 - \cos 3\xi_2)\right]\dot{u} +$$

$$\left[-\frac{2r^2}{a_1^3} + \frac{2r^2 \cos 2\xi_2}{a_1^5} - (3+\nu)\frac{r \cos \xi_2}{a_1^2}\right]\ddot{w} + \left[\frac{3r \sin \xi_2}{4a_1^2} - \frac{r \sin 3\xi_2}{4a_1^2} +\right.$$

$$\left.\frac{\sin 2\xi_2}{a_1} - \frac{\nu \sin \xi_2}{r}\right](v' - v) + \frac{1}{a_1}\ddot{v}' - \frac{2}{a_1}\ddot{w}'' + \frac{\cos \xi_2}{r}\dot{u}'' -$$

$$\frac{2r \sin \xi_2}{a_1^2}\ddot{w}'\Bigg\} + \frac{ra_1}{K} q = 0 \tag{16c}$$

where $u = \frac{u_1}{a_1}$, $v = \frac{v_1}{r}$, $w = \frac{w_1}{r}$, $D = \frac{Eh^3}{12(1-\nu^2)}$, $K = \frac{Eh}{1-\nu^2}$, and h is the thickness of the shell. Also in equation (16) a prime denotes differentiation with respect to ξ_2 and a dot denotes differentiation with respect to ξ_1.

The equilibrium equations (16) can be written in terms of dimensionless quantities upon introduction of the following parameters:

$$\left.\begin{aligned} \alpha &= \frac{a_1}{R} = 1 + \beta \cos \xi_2 \\ \beta &= r/R \\ \gamma &= h/r \\ \frac{D}{K} &= \frac{\gamma^2}{12} \beta^2 R^2 \end{aligned}\right\} \qquad (17)$$

The introduction of equation (17) into equation (16) and the combining of like terms of u, v, and w gives

$$\frac{1-\nu}{2}\frac{\gamma^2}{12}\left\{\beta^3\alpha^2 \cos \xi_2 - \frac{\beta^4\alpha}{8}(1 - \cos 4\xi_2) - \frac{\beta^2\alpha^3}{2}(1 + \cos 2\xi_2)\right\} u +$$

$$\left\{ - \frac{1-\nu}{2}(3\alpha^4\beta \sin \xi_2) - \frac{1-\nu}{2}\frac{\gamma^2}{12}\left[\frac{\beta^3\alpha^2}{4}(\sin \xi_2 + \sin 3\xi_2) + \right.\right.$$

$$\left.\left.\beta^2\alpha^3 \sin 2\xi_2\right]\right\} u' + \left\{\beta^2\alpha^3 + \frac{\gamma^2}{12}\left[\frac{\beta^4\alpha}{2}(1 + \cos 2\xi_2)\right]\right\}\ddot{u} + \frac{1-\nu}{2}\left\{\alpha^3 + \right.$$

$$\left.\frac{1-\nu}{2}\frac{\gamma^2}{12}\left[\frac{\beta^2\alpha^3}{2}(1 + \cos 2\xi_2)\right]\right\} u'' - \frac{3-\nu}{2}\left\{\beta^3\alpha^2 \sin \xi_2 + \right.$$

$$\left.\frac{\gamma^2}{12}\left[\frac{\beta^4\alpha}{2} \sin 2\xi_2\right]\right\}\dot{v} + \frac{1+\nu}{2}\left\{\beta^2\alpha^3 + \frac{\gamma^2}{12}\left[\beta^3\alpha^2 \cos \xi_2\right]\right\}\dot{v}' +$$

$$\left\{\beta^2\alpha^2(\beta \cos \xi_2 + \nu\alpha) - \frac{\gamma^2}{12}\frac{1-\nu}{2}\left[\beta^4\alpha(1 + \cos 2\xi_2)\right]\right\}\dot{w} + \frac{\gamma^2}{12}\left\{\frac{\beta^4\alpha}{2} \sin 2\xi_2\right\}\dot{w}' -$$

$$\frac{\gamma^2}{12}\left\{\beta^3\alpha^2 \cos \xi_2\right\}\dot{w}'' - \frac{\gamma^2}{12}\left\{\beta^5 \cos \xi_2\right\}\dddot{w} + \frac{R\beta^2\alpha^4}{K} p_1 = 0 \qquad (18a)$$

$$\left\{(1-\nu)\beta^3\alpha^2 \sin \xi_2 + \frac{\gamma^2}{12}\left[-\frac{1+\nu}{2}(\beta^3\alpha^2 \sin \xi_2) + \frac{3-\nu}{2}\left(\frac{\beta^4\alpha}{2}\sin 2\xi_2\right)\right]\right\}\dot{u} +$$

$$\frac{1+\nu}{2}\left\{\beta^2\alpha^3 + \frac{\gamma^2}{12}\left[\beta^3\alpha^2 \cos \xi_2\right]\right\}\dot{u}' - \left\{\frac{\beta^4\alpha}{2}(1-\cos 2\xi_2) + \nu\beta^3\alpha^2 \cos \xi_2\right\}v -$$

$$\left\{\beta^3\alpha^2 \sin \xi_2\right\}v' + \left\{\alpha^3\beta^2\right\}v'' + \left\{\frac{1-\nu}{2}\beta^4\alpha\right\}\ddot{v} + \left\{\frac{\beta^4\alpha}{2}\sin 2\xi_2 -\right.$$

$$\left.\beta^3\alpha^2 \sin \xi_2\right\}w + \left\{\beta^2\alpha^3 + \nu\beta^3\alpha^2 \cos \xi_2 + \frac{\gamma^2}{12}\left[\frac{\beta^4\alpha}{2}(1-\cos 2\xi_2)\right]\right\}w' +$$

$$\left\{\frac{\gamma^2}{12}\beta^3\alpha^2 \sin \xi_2\right\}w'' - \left\{\frac{\gamma^2}{12}\beta^2\alpha^3\right\}w''' - \left\{\frac{\gamma^2}{12}\left[2\beta^3 \sin \xi_2\right]\right\}\ddot{w}' - \left\{\frac{\gamma^2}{12}\beta^4\alpha\right\}\ddot{w} +$$

$$\frac{R\beta^3\alpha^3}{K}p_2 = 0 \qquad (18b)$$

$$\left\{-(\beta\alpha^3 \cos \xi_2 + \nu\alpha^4) + \frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\alpha^2\beta^2 \cos 2\xi_2 + \frac{1-\nu}{2}\alpha^2\beta^2 + \frac{\beta^3\alpha}{4}(\cos \xi_2 -\right.\right.$$

$$\left.\left.\cos 3\xi_2)\right]\right\}\dot{u} + \left\{\frac{\gamma^2}{12}\left(\frac{\beta^2\alpha^2}{2}\sin 2\xi_2 - 2\beta\alpha^3 \sin \xi_2\right)\right\}\dot{u}' + \left\{\frac{\gamma^2}{12}\beta\alpha^3 \cos \xi_2\right\}\dot{u}'' +$$

$$\left\{\frac{\gamma^2}{12}\beta^3\alpha \cos \xi_2\right\}\ddot{u}' + \left\{(\beta\alpha^4 \cos \xi + \nu\alpha^3)\beta \sin \xi_2 - \frac{\gamma^2}{12}\left[\frac{\beta^3\alpha}{4}(3 \sin \xi_2 -\right.\right.$$

$$\left.\left.\sin 3\xi_2)\right]\right\}v + \left\{-(\alpha^4 + \nu\alpha^3\beta \cos \xi_2) + \frac{\gamma^2}{12}\left[-\beta\alpha^3 \cos \xi_2 - \frac{\beta^2\alpha^2}{2} +\right.\right.$$

$$\left.\left.\frac{\beta^2\alpha^2}{2}\cos 2\xi_2\right]\right\}v' - \left\{\frac{\gamma^2}{12}\left[2\beta\alpha^3 \sin \xi_2\right]\right\}v'' + \left\{\frac{\gamma^2}{12}\alpha^4\right\}v''' - \left\{\frac{\gamma^2}{12}\beta^3\alpha \sin \xi_2\right\}\ddot{v} +$$

$$\left\{\frac{\gamma^2}{12}\beta^2\alpha^2\right\}\ddot{v}' + \left\{-(\beta\alpha^2 \cos \xi_2 + 2\nu\alpha^3)\beta \cos \xi_2 - \alpha^4\right\}w + \left\{\frac{\gamma^2}{12}\left[\frac{\beta^3\alpha}{4}(3 \sin \xi_2 -\right.\right.$$

$$\sin 3\xi_2) + \beta^2\alpha^2 \sin 2\xi_2 - \nu\beta\alpha^3 \sin \xi_2\Big]\Big\} w' + \Big\{\frac{\gamma^2}{12}\Big[\beta\alpha^3 \cos \xi_2 + \frac{\beta^2\alpha^2}{2}(1 -$$

$$\cos 2\xi_2) + \nu\beta\alpha^3 \cos \xi_2\Big]\Big\} w'' + \Big\{\frac{\gamma^2}{12}(2\beta\alpha^3 \sin \xi_2)\Big\} w''' - \Big\{\frac{\gamma^2}{12}\alpha^4\Big\} w'''' +$$

$$\Big\{\frac{\gamma^2}{12}\Big[-2\beta^4(1 - \cos 2\xi_2) - (3 + \nu)\beta^3\alpha \cos \xi_2\Big]\Big\} \ddot{w} - \Big\{\frac{\gamma^2}{12}(2\beta\alpha^3)\Big\} \ddot{w}'' -$$

$$\Big\{\frac{\gamma^2}{12}\Big[2\beta^3\alpha \sin \xi_2\Big]\Big\} \ddot{w}' - \Big\{\frac{\gamma^2}{12}\beta^4\Big\} \ddot{\ddot{w}} + \frac{\beta\alpha^4 R}{K} q = 0 \qquad (18c)$$

In order to obtain equations (18) in the present form from equations (16) use was made of the assumption $1 + \gamma^2 \approx 1$ which is basic in Love's first approximation.

4. The Equilibrium Equations for the Vibration of the Toroidal Shell

For the vibrations of the shell, equations (18) can be used directly if the external force term given by equation (13) is written in terms of the inertia forces. The three components of the inertia force are

$$\left.\begin{aligned} p_1 &= -\mu h \frac{\partial^2 u_1}{\partial t^2} = -\mu h\alpha \frac{\partial^2 u}{\partial t^2} \\ p_2 &= -\mu h \frac{\partial^2 v_1}{\partial t^2} = -\mu h r \frac{\partial^2 v}{\partial t^2} \\ q &= -\mu h \frac{\partial^2 w_1}{\partial t^2} = -\mu h r \frac{\partial^2 w}{\partial t^2} \end{aligned}\right\} \qquad (19b)$$

where μ is the mass density of the shell material.

In order to obtain the natural modes of vibration, it is sufficient to assume that the structure is vibrating in simple harmonic motion in each of the natural modes. Then accelerations can be written in terms of the displacements and the square of the natural circular frequency, ω. The inertia force components in equations (19a) become

$$\left.\begin{aligned} p_1 &= u\omega^2\mu hr \\ p_2 &= v\omega^2\mu hr \\ q &= w\omega^2\mu hr \end{aligned}\right\} \qquad (19b)$$

The equilibrium equations for the vibrations of the thin toroidal shell of constant thickness are given by equations (18) with the three force components expressed in terms of the inertias by equations (19b).

VII. ANALYTICAL SOLUTIONS

1. The Problem Defined

In this section the particular problem to be solved is defined. The toroidal shell is considered to be a complete torus that is unsupported (i.e., is in the "free-free" condition) and is vibrating in vacuum with no pressure differential across the shell throughout the torus. The dimensions of the shell structure are taken to be consistent with the assumptions of thin-shell theory and Love's first approximation as expressed in chapter VI. The ratio of the minimum radius of curvature to the thickness of the shell, $\frac{r}{h} = \frac{1}{\gamma}$, is considered to be greater than 100 and the ratio $\frac{r}{R} = \beta$ is considered to lie in the range $0 < \beta < 1$ in such a way that r always remains finite and compatible with the restriction on γ.

The problem of determining the analytical expressions for the modes and frequencies for the free-free vibration of the toroidal shell as defined above is the subject of this chapter. For these solutions the only restrictions on the displacements are that they be continuous functions of ξ_1 and ξ_2, since the shell is assumed to be continuous, closed, and unsupported.

2. The Limiting Case of an Infinitely Long Circular Cylinder

The torus shell defined in chapter VI can be shown to limit to an infinitely long circular cylinder as the radius R becomes very large. Hence it is reasonable to expect that the equilibrium equations (18) should limit to the well known equations for a circular cylinder according to Love's first approximation[9]. In the limiting process the independent variable, ξ_2 remains unchanged but

the variable ξ_1 tends toward the longitudinal coordinate x and the following relations hold

$$R\alpha d\xi_1 = R(1 + \beta \cos \xi_2)d\xi_1 = dx \tag{20}$$

where it is seen that x depends on both ξ_1 and ξ_2. The behavior of the various terms in equations (18) as $R \to \infty$ can be expressed as follows:

$$\lim_{R \to \infty} \dot{u}\beta = \lim_{\beta \to 0} \frac{\partial^2}{\partial \xi_1^2}\left(\frac{u_1}{\alpha}\right)\beta = r\,\frac{\partial u_1}{\partial x}, \quad \lim_{R \to \infty} \frac{\ddot{u}}{\alpha}\,\beta^2 = \lim_{\beta \to 0} \beta^2 \frac{\partial^2}{\partial \xi_1^2}\left(\frac{u_1}{\alpha^2}\right) = r^2 \frac{\partial u_1^2}{\partial x^2}$$

$$\lim_{R \to \infty} \alpha = \lim_{\beta \to 0} (1 + \beta \cos \xi) = 1$$

$$\lim_{R \to \infty} u' = \lim_{\beta \to 0} \frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha}\right) = \frac{\partial u_1}{\partial \varphi} \qquad \xi_2 \to \varphi \tag{21}$$

with similar expressions for the other terms in equations (18). Thus, if the terms in equations (18) that contain derivatives with respect to ξ_1 are multiplied and divided by α in accordance with equations (21) and the limit taken as $\beta \to 0$, then the following equilibrium equations for the circular cylinder are obtained:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{1 - \nu}{2r^2}\frac{\partial^2 u_1}{\partial \varphi^2} + \frac{1 + \nu}{2r}\frac{\partial^2 v_1}{\partial x\,\partial \varphi} - \frac{\nu}{r}\frac{\partial w_1}{\partial x} + \frac{r}{K} p_1 = 0$$

$$\frac{1 + \nu}{2}\frac{\partial^2 u_1}{\partial x\,\partial \varphi} + r\,\frac{1 - \nu}{2}\frac{\partial^2 v_1}{\partial x^2} + \frac{1}{r}\frac{\partial^2 v_1}{\partial \varphi^2} - \frac{1}{r}\frac{\partial w_1}{\partial \varphi} + \frac{h^2}{12r}\left(\frac{\partial^3 w_1}{\partial x^2\,\partial \varphi} + \frac{\partial^3 w_1}{r^2\,\partial \varphi^3}\right) +$$

$$\frac{r}{K} p_2 = 0 \tag{22}$$

$$\nu \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{r\partial\varphi} - \frac{w_1}{r} - \frac{h^2}{12}\left(r \frac{\partial^4 w_1}{\partial x^4} + \frac{2}{r}\frac{\partial^4 w_1}{\partial x^2 \partial\varphi^2} + \frac{\partial^4 w_1}{r^3 \partial\varphi^4}\right) + \frac{h^2}{12}\left(\frac{1}{r}\frac{\partial^3 v_1}{\partial x^2 \partial\varphi} + \right. \tag{22}$$

$$\left. \frac{\partial^3 v_1}{r^3 \partial\varphi^3}\right) + \frac{r}{K} q = 0$$

These equations can be obtained directly for the circular cylinder from the relations given for Love's first approximation by Hildebrand, Reissner, and Thomas[11]. It may be noted that the next to the last bending term involving v_1 in the third equation of equation (22) is not the same as in the expressions usually given for this set of equations. (See Timoshenko[12], page 440.) A similar situation was found to exist for circular cylinders by Fung[13] and it was shown that the effect on numerical results is negligible.

3. Small Rigid-Body Motion

It is well known that Love's first approximation theory contains an inconsistency in that, except for the special case of axisymmetric loading of shells of revolution, the strains do not vanish for small rigid-body motion. This inconsistency of Love's theory is discussed in a recent paper by Sanders[14]. For the present problem the apparent strains that result from rigid-body motions of the toroidal shell are illustrated by the following examples:

(a) Unit displacement parallel to axis of symmetry.- The displacement vector of a point on the middle surface of the shell is given by

$$\begin{aligned} \vec{U} &= \vec{k} \\ &= \cos \xi_2 \vec{t_2} + \sin \xi_2 \vec{n} \end{aligned} \tag{23a}$$

For this displacement the strains given by equations (11) become

$$\epsilon_1 = \epsilon_2 = \gamma_{12} = K_1 = K_2 = \tau = 0 \tag{23b}$$

(b) <u>Unit displacement normal to axis of symmetry</u>.- The displacement vector of a point on the middle surface of the shell due to a unit displacement in the x-direction is given by

$$\begin{aligned} \vec{U} &= \vec{i} \\ &= -\sin \xi_1 \bar{t}_1 - \cos \xi_1 \sin \xi_2 \bar{t}_2 + \cos \xi_1 \cos \xi_2 \bar{n} \end{aligned} \tag{24a}$$

The apparent strains due to their displacement are

$$\begin{aligned} \epsilon_1 &= \epsilon_2 = \gamma_{12} = K_1 = K_2 = 0 \\ \tau &= \frac{R}{r}\left(\frac{\alpha_1 - 1}{\alpha_1}\right) \sin \xi_1 \sin \xi_2 \end{aligned} \tag{24b}$$

(c) <u>Unit rotation about axis of symmetry</u>.- For this motion the displacement vector for a point on the middle surface is

$$\begin{aligned} \vec{U} &= -\alpha_1 \sin \xi_1 \vec{i} + \alpha_1 \cos \xi_1 \vec{j} \\ &= \alpha_1 \vec{t}_1 \end{aligned} \tag{25a}$$

and the strains given by equations (11) for this motion are

$$\begin{aligned} \epsilon_1 &= \epsilon_2 = \gamma_{12} = K_1 = K_2 = 0 \\ \tau &= -\frac{R \sin \xi_2}{r\alpha_1} \end{aligned} \tag{25b}$$

(d) <u>Unit rotation about an axis normal to axis of symmetry</u>.- If the rigid body rotation is taken as a rotation about the x-axis, then

the vector displacement of a point on the middle surface of the shell is given by

$$\vec{U} = -r \sin \xi_2 \vec{j} + a_1 \sin \xi_1 \vec{k}$$

$$= -r \sin \xi_2 \cos \xi_1 \vec{t}_1 + (r + R \cos \xi_2) \sin \xi_1 \vec{t}_2 + R \sin \xi_1 \sin \xi_2 \vec{n} \tag{26a}$$

The apparent strains resulting from this displacement are given by

$$\epsilon_1 = \epsilon_2 = \gamma_{12} = K_1 = K_2 = 0$$

$$\tau = -\frac{R}{ra_1} \cos \xi_1 \cos \xi_2 \tag{26b}$$

4. <u>Development of the solution</u>

The equilibrium equations with the inertia force terms from equations (19b) can be uncoupled and reduced to ordinary differential equations in the single variable ξ_2 and a wave number parameter along the length of the torus. This uncoupling can be accomplished by writing

$$\begin{aligned} u &= \bar{u} \cos n \xi_1 \\ v &= \bar{v} \sin n \xi_1 \\ w &= \bar{w} \sin n \xi_1 \end{aligned} \qquad n = 1,2,3,\ldots \tag{27}$$

where $\bar{u}$, $\bar{v}$, and $\bar{w}$ are functions of ξ_2 only and n represents the number of complete waves in the ξ_1 direction. If equations (27) are substituted into equation (18) with equations (19b) taken into account, the equilibrium equations take the form

let $\qquad \lambda^2 = \frac{r^2 \omega p^2}{K}, \qquad 1 + \gamma^2 = 1$, and $\xi_2 = \xi$

$$B_1\bar{u} + B_2\bar{u}' + B_3\bar{u}'' + B_4\bar{v} + B_5\bar{v}' + B_6\bar{w} + B_7\bar{w}' + B_8\bar{w}'' = 0 \qquad (28a)$$

where

$$B_1 = \bar{\lambda}^2\alpha^5 - n^2\beta^2\alpha^3 + \frac{1-\nu}{2}\frac{\gamma^2}{12}\left[\beta^3\alpha^2\cos\xi - \left(\frac{\alpha^3}{2} + \frac{n^2\beta^2\alpha}{1-\nu}\right)\beta^2(1+\cos 2\xi) - \frac{1}{8}\beta^4\alpha(1-\cos 4\xi)\right]$$

$$B_2 = -\frac{1-\nu}{2}\left\{3\beta\alpha^4\sin\xi + \frac{\gamma^2}{12}\left[\frac{1}{4}\beta^3\alpha^2(\sin\xi + \sin 3\xi) + \beta^2\alpha^3\sin 2\xi\right]\right\}$$

$$B_3 = \frac{1-\nu}{2}\left\{\alpha^3 + \frac{\gamma^2}{12}\left[\frac{1}{2}\beta^2\alpha^3(1+\cos 2\xi)\right]\right\}$$

$$B_4 = -\frac{1-\nu}{2}\left\{n\beta^3\alpha^2\sin\xi + \frac{\gamma^2 n}{12}\left[\frac{1}{2}\beta^4\alpha\sin 2\xi\right]\right\}$$

$$B_5 = \frac{1+\nu}{2}\left\{\beta^2\alpha^3 n + \frac{\gamma^2 n}{12}\left[\beta^3\alpha^2\cos\xi\right]\right\}$$

$$B_6 = n(\beta^3\alpha^2\cos\xi + \nu\beta^2\alpha^3) + \frac{\gamma^2 n}{12}\left[n^2\beta^3\cos\xi - \frac{1-\nu}{2}\beta^4\alpha(1+\cos 2\xi)\right]$$

$$B_7 = \frac{\gamma^2 n}{12}\frac{1}{2}\beta^4\alpha\sin 2\xi$$

$$B_8 = -\frac{\gamma^2 n}{12}\beta^3\alpha^2\cos\xi$$

$$C_1\bar{u} + C_2\bar{u}' + C_3\bar{v} + C_4\bar{v}' + C_5\bar{v}'' + C_6\bar{w} + C_7\bar{w}' + C_8\bar{w}'' + C_9\bar{w}''' = 0 \qquad (28b)$$

where

$$C_1 = -(1 - \nu)n\beta\, a^2 \sin \xi + \frac{\gamma^2 n}{12}\left[\frac{1 + \nu}{2}\, \beta a^2 \sin \xi - \frac{3 - \nu}{4}\, \beta^2 a \sin 2\xi\right]$$

$$C_2 = -\frac{1 + \nu}{2}\left\{n a^3 + \frac{\gamma^2 n}{12}\left[\beta a^2 \cos \xi\right]\right\}$$

$$C_3 = \bar{\lambda}^2 a^3 - \frac{1}{2}\beta^2 a(1 - \cos 2\xi) - \frac{1 + \nu}{2}\, n^2\beta^2 a - \nu\beta a^2 \cos \xi$$

$$C_4 = -\beta a^2 \sin \xi$$

$$C_5 = a^3$$

$$C_6 = -\beta a^2 \sin \xi + \frac{1}{2}\beta^2 a \sin 2\xi + \frac{\gamma^2}{12}\left(2\beta^3 n^2 \sin \xi\right)$$

$$C_7 = a^3 + \nu\beta a^2 \cos \xi + \frac{\gamma^2}{12}\left[\frac{1}{2}\beta^2 a(1 - \cos 2\xi) + \beta^2 n^2 a\right]$$

$$C_8 = \frac{\gamma^2}{12}\beta a^2 \sin \xi$$

$$C_9 = -\frac{\gamma^2}{12} a^3$$

and

$$D_1\bar{u} + D_2\bar{u}' + D_3\bar{u}'' + D_4\bar{v} + D_5\bar{v}' + D_6\bar{v}'' + D_7\bar{v}''' + D_8\bar{w} + D_9\bar{w}' + D_{10}\bar{w}'' +$$

$$D_{11}\bar{w}''' + D_{12}\bar{w}'''' = 0 \qquad (28c)$$

where

$$D_1 = n(\beta a^3 \cos \xi + \nu a^4) + \frac{\gamma^2 n}{12}\left[n^2\beta^3 a \cos \xi - \frac{3 - \nu}{2}(\beta^2 a^2 \cos 2\xi) - \frac{1}{4}\beta^3 a(\cos \xi - \cos 3\xi) - \frac{1 - \nu}{2}\beta^2 a^2\right]$$

$$D_2 = \frac{-\gamma^2 n}{12}\left[\frac{1}{2}\beta^2 a^2 \sin 2\xi - 2\beta a^3 \sin \xi\right]$$

$$D_3 = \frac{-\gamma^2 n}{12}\left[\beta a^3 \cos \xi\right]$$

$$D_4 = \frac{1}{2}\beta^2 a^2 \sin 2\xi + \nu\beta a^3 \sin \xi + \frac{\gamma^2}{12}\left[n^2\beta^3 a \sin \xi - \frac{1}{4}\beta^3 a(3\sin \xi - \sin 3\xi)\right]$$

$$D_5 = -(a^4 + \nu\beta a^3 \cos \xi) - \frac{\gamma^2}{12}\left[\beta a^3 \cos \xi + \frac{1}{2}\beta^2 a^2(1 - \cos 2\xi) + n^2\beta^2 a^2\right]$$

$$D_6 = -\frac{\gamma^2}{12} 2\beta a^3 \sin \xi$$

$$D_7 = \frac{\gamma^2}{12} a^4$$

$$D_8 = \bar{\lambda}^2 a^4 - \frac{1}{2}\beta^2 a^2(1 + \cos 2\xi) - 2\nu\beta a^3 \cos \xi - a^4 - \frac{\gamma^2}{12}\left[n^4\beta^4 - 2\beta^4 n^2(1 - \cos 2\xi) - (3 + \nu)\beta^3 n^2 a \cos \xi\right]$$

$$D_9 = \frac{\gamma^2}{12}\left[\frac{1}{4}\beta^3 a(3\sin \xi - \sin 3\xi) + \beta^2 a^2 \sin 2\xi - \nu\beta a^3 \sin \xi + 2\beta^3 n^2 a \sin \xi\right]$$

$$D_{10} = \frac{\gamma^2}{12}\left[\beta a^3 \cos \xi + \frac{1}{2}\beta^2 a^2(1 - \cos 2\xi) + \nu\beta a^3 \cos \xi + 2\beta^2 n^2 a^2\right]$$

$$D_{11} = \frac{\gamma^2}{12}\left[2\beta a^3 \sin \xi\right]$$

$$D_{12} = -\frac{\gamma^2}{12} a^4$$

The equations (28) are seen to be linear differential equations with variable coefficients. In these equations the displacement functions $\bar{u}$, $\bar{v}$, and $\bar{w}$ must be continuous functions of ξ_2 and the slope of the middle surface of the shell must also be continuous. Therefore the displacement functions and their derivatives must be continuous, periodic functions of ξ_2 with period 2π in order to satisfy the conditions on the problem under consideration. Even with this information it seems highly improbable that general solutions to equations (28) can be obtained. However, the displacement functions satisfying equations (28) could be expected to be expressable in terms of Fourier series. Let

$$\bar{u} = \sum_{i=1}^{\infty} (a_i \cos i\xi_2 + b_i \sin i\xi_2) + \frac{a_o}{2}$$

$$\bar{v} = \sum_{i=1}^{\infty} (c_i \cos i\xi_2 + d_i \sin i\xi_2) + \frac{c_o}{2} \qquad (29)$$

$$\bar{w} = \sum_{i=1}^{\infty} (f_i \cos i\xi_2 + g_i \sin i\xi_2) + \frac{f_o}{2}$$

where the coefficients a_i, b_i, c_i, d_i, f_i, g_i, a_o, c_o, and f_o are to be determined to satisfy the equilibrium equations and the conditions on the displacement functions mentioned above. The substitution of equations (29) into equations (28) gives

$$B_1 \frac{a_o}{2} + \sum_i (B_1 a_i + B_2 i b_i - B_3 i^2 a_i) \cos i\xi_2 + \sum_i (B_1 b_i - B_2 i a_i - B_3 i^2 b_i) \sin i\xi_2 + B_4 \frac{c_o}{2} + \sum_i (B_4 c_i + B_5 i d_i) \cos i\xi_2 + \sum_i (B_4 d_i - B_5 i c_i) \sin i\xi_2 + B_6 \frac{f_o}{2} + \sum_i (B_6 f_i + B_7 i g_i - B_8 i^2 f_i) \cos i\xi_2 + \sum_i (B_6 g_i - B_7 i f_i - B_8 i^2 g_i) \sin i\xi_2 = 0 \tag{30a}$$

$$C_1 \frac{a_o}{2} + \sum_i (C_1 a_i + C_2 i b_i) \cos i\xi_2 + \sum_i (C_1 b_i - C_2 i a_i) \sin i\xi_2 + C_3 \frac{c_o}{2} + \sum_i (C_3 c_i + C_4 i d_i - C_5 i^2 c_i) \cos i\xi_2 + \sum_i (C_3 d_i - C_4 i c_i - C_5 i^2 d_i) \sin i\xi_2 + C_6 \frac{f_o}{2} + \sum_i (C_6 f_i + C_7 i g_i - C_8 i^2 f_i - C_9 i^3 g_i) \cos i\xi_2 + \sum_i (C_6 g_i - C_7 i f_i - C_8 i^2 g_i + C_9 i^3 f_i) \sin i\xi_2 = 0 \tag{30b}$$

$$D_1 \frac{a_0}{2} + \sum_i (D_1 a_i + D_2 i b_i - D_3 i^2 a_i) \cos i\xi_2 + \sum_i (D_1 b_i - D_2 i a_i -$$

$$D_3 i^2 b_i) \sin i\xi_2 + D_4 \frac{c_0}{2} + \sum_i (D_4 c_i + D_5 i d_i - D_6 i^2 c_i - D_7 i^3 d_i) \cos i\xi_2 +$$

$$\sum_i (D_4 d_i - D_5 i c_i - D_6 i^2 d_i + D_7 i^3 c_i) \sin i\xi_2 + D_8 \frac{f_0}{2} + \sum_i (D_8 f_i + D_9 i e_i -$$

$$D_{10} i^2 f_i - D_{11} i^3 e_i + D_{12} i^4 f_i) \cos i\xi_2 + \sum_i (D_8 e_i - D_9 i f_i - D_{10} i^2 e_i +$$

$$D_{11} i^3 f_i + D_{12} i^4 e_i) \sin i\xi_2 = 0 \qquad (30c)$$

Thus the differential equations (28) have been reduced to a set of equations in terms of the variables $\sin i\xi_2$ and $\cos i\xi_2$. These equations are seen to be nonlinear in these variables since the coefficients defined in equations (28), that is, the B_k's, C_k's, and the D_k's, contain powers of the trigonometric functions. The form of equations (30) makes it all but impossible to solve for the coefficients a_i, b_i, c_i, d_i, f_i, and e_i directly; however, it would be expected that the nature of the solution would change but very little around the torus, that is, as ξ_2 varies from 0 to 2π. Thus the equations (30) can be satisfied on the average and the solution be sufficiently accurate to define the basic character of the vibrating torus. The solution that follows is based on the well-known Galerkin method.

If the equations (30) are multiplied in turn by $\sin j\xi_2$ and by $\cos j\xi_2$ and integrated over the interval $0 \leq \xi_2 \leq 2\pi$ then six equations will be obtained which are written as

$$\int_0^{2\pi} \sum_i (B_1 a_i - B_3 i^2 a_i + B_5 i d_i + B_6 f_i - B_9 i^2 f_i) \cos i\xi_2 \cos j\xi_2 \, d\xi_2 +$$

$$\int_0^{2\pi} \sum_i (-B_2 i a_i + B_4 d_i - B_7 i f_i) \sin i\xi_2 \cos j\xi_2 \, d\xi_2 + \frac{1}{2} \int_0^{2\pi} (B_1 a_0 +$$

$$B_6 f_1) \cos j\xi_2 \, d\xi_2 = 0 \qquad (31a)$$

$$\int_0^{2\pi} \sum_i (B_2 i b_i + B_4 c_i + B_7 i e_i) \cos i\xi_2 \sin j\xi_2 \, d\xi_2 + \frac{1}{2} \int_0^{2\pi} B_4 c_0 \sin j\xi_2 \, d\xi_2 +$$

$$\int_0^{2\pi} \sum_i (B_1 b_i - B_3 i^2 b_i - B_5 i c_i + B_6 e_i - B_8 i^2 e_i) \sin i\xi_2 \sin j\xi_2 \, d\xi_2 = 0$$

$$(31b)$$

$$\int_0^{2\pi} \sum_i (C_2 i b_i + C_3 c_i - C_5 i^2 c_i + C_7 i e_i - C_9 i^2 e_i) \cos i\xi_2 \cos j\xi_2 \, d\xi_2 +$$

$$\frac{1}{2} \int_0^{2\pi} (C_3 c_0 \cos j\xi_2) d\xi_2 + \int_0^{2\pi} \sum_i (C_1 b_i - C_4 i c_i + C_6 e_i -$$

$$C_8 i^2 e_i) \sin i\xi_2 \cos j\xi_2 \, d\xi_2 = 0 \qquad (31c)$$

$$\int_0^{2\pi} \sum_i^{\infty} (C_1 a_i + C_4 i d_i + C_6 f_i - C_8 i^2 f_i) \cos i\xi_2 \sin j\xi_2 \, d\xi_2 +$$

$$\frac{1}{2} \int_0^{2\pi} (C_1 a_0 + C_6 f_0) \sin j\xi_2 \, d\xi_2 + \int_0^{2\pi} \sum_i^{\infty} (-C_2 i a_i + C_3 d_i - C_5 i^2 d_i -$$

$$C_7 i f_i + C_9 i^3 f_i) \sin i\xi_2 \sin j\xi_2 \, d\xi_2 = 0 \qquad (31d)$$

$$\int_0^{2\pi} \sum_i^{\infty} (D_1 a_i - D_3 i^2 a_i + D_5 i d_i - D_7 i^3 d_i + D_8 f_i - D_{10} i^2 f_i +$$

$$D_{12} i^4 f_i) \cos i\xi_2 \cos j\xi_2 \, d\xi_2 + \int_0^{2\pi} \sum_i^{\infty} (-D_2 i a_i + D_4 d_i - D_6 i^2 d_i -$$

$$D_9 i f_i + D_{11} i^3 f_i) \sin i\xi_2 \cos j\xi_2 \, d\xi_2 + \frac{1}{2} \int_0^{2\pi} (D_1 a_0 + D_8 f_0) \cos j\xi_2 \, d\xi_2 = 0 \qquad (31e)$$

$$\int_0^{2\pi} \sum_i^{\infty} (D_2 i b_i + D_4 c_i - D_6 i^2 c_i + D_9 i e_i - D_{11} i^3 e_i) \cos i\xi_2 \sin j\xi_2 \, d\xi_2 +$$

$$\int_0^{2\pi} \sum_i^{\infty} (D_1 b_i + D_3 i^2 b_i - D_5 i c_i + D_7 i^3 c_i + D_8 e_i - D_{10} i^2 e_i +$$

$$D_{12} i^4 e_i) \sin i\xi_2 \sin j\xi_2 \, d\xi_2 + \frac{1}{2} \int_0^{2\pi} D_4 c_0 \sin j\xi_2 \, d\xi_2 = 0 \qquad (31f)$$

In order to obtain equations (31) from equations (30), the powers and products of the trigonometric terms in the coefficients B_k, C_k, and D_k were first removed by the use of the relations for the integral powers of sine and cosine such as given in section 3.180 of Adams[7], and then use was made of the expressions

$$\left.\begin{aligned} &\int_0^{2\pi} \sin k\xi \cos i\xi \cos j\xi \, d\xi = 0 \\ &\int_0^{2\pi} \cos k\xi \cos i\xi \sin j\xi \, d\xi = 0 \\ &\int_0^{2\pi} \sin k\xi \sin i\xi \sin j\xi \, d\xi = 0 \end{aligned}\right\} \quad \left| i \pm j \right| \neq k \qquad (32)$$

for all integral values of i, j, and k.

By using the relation

$$\left.\begin{aligned} \int_0^{2\pi} \cos k\xi \cos i\xi \cos j\xi \, d\xi &= 0, \; i + j \neq k, \; i - j \neq k \text{ and } j - i \neq k \\ &= \frac{\pi}{2}, \; i + j = k, \text{ or } i - j = k, \text{ or } j - i = k \end{aligned}\right\} \qquad (33a)$$

$$\left.\begin{aligned} \int_0^{2\pi} \sin k\xi \cos i\xi \sin j\xi \, d\xi &= 0, \; i + j \neq k, \; i - j \neq k \text{ and } j - i \neq k \\ &= \frac{\pi}{2}, \; i + j = k \text{ or } j - i = k \\ &= -\frac{\pi}{2}, \; i - j = k \end{aligned}\right\} \qquad (33b)$$

$$\int_0^{2\pi} \cos k\xi \sin i\xi \sin j\xi \, d\xi = 0, \; i + j \neq k, \; i - j \neq k, \; j - i \neq k$$
$$= -\frac{\pi}{2}, \; i + j = k$$
$$= \frac{\pi}{2}, \; i - j = k, \text{ or } \; j - i = k \qquad (33c)$$

$$\int_0^{2\pi} \sin k\xi \sin \xi \cos j\xi \, d\xi = 0, \; i + j \neq k, \; i - j \neq k, \text{ and } \; j - i \neq k$$
$$= \frac{\pi}{2}, \; i + j, \text{ or } \; i - j = k$$
$$= -\frac{\pi}{2}, \; j - i = k \qquad (33d)$$

the integrals and the summations in equation (31) can be removed and two sets of algebraic equations obtained. One set will contain five equations in the five unknowns a_1, d_1, f_1, a_0, and f_0 and the other set in the four unknowns b_1, c_1, e_1, and c_0. Thus, it is seen by considering equations (29) that the vibratory motion is characterized by two distinct parts which can be referred to as "symmetrical" and "antisymmetrical" parts with respect to ξ_2, using $\bar{u}$ displacement as reference. That is, the motion in the ξ_1 direction has a symmetrical or antisymmetrical distribution with respect to ξ_2.

The integrations of equations (31) using the results of equations (33) lead to the following equations for the coefficients a_1, d_1, and f_1 in the symmetrical case;

with $\frac{D}{R^4K} = \frac{1}{12}\gamma^2\beta^2$

$$\left\{\bar{\lambda}^2\left(2 + 10\beta^2 + \frac{15}{4}\beta^4\right) - n^2(2\beta^2 + 3\beta^4) - (1 - \nu)\left(1 + 5\beta^2 + \frac{15}{8}\beta^4\right)j^2 - \right.$$

$$\left.\frac{\gamma^2}{12}\left[\frac{1 - \nu}{2}\left(\beta^2 + \frac{1}{2}\beta^4\right) + n^2\beta^4 + \frac{1 - \nu}{2}\left(\beta^2 + \frac{2}{4}\beta^4\right)j^2\right]\right\}a_j + \left\{\bar{\lambda}^2\left(5\beta + \frac{15}{2}\beta^3 + \right.\right.$$

$$\left.\frac{5}{8}\beta^5\right) - n^2\left(3\beta^3 + \frac{3}{4}\beta^5\right) + \frac{3(1 - \nu)}{2}\left(\beta + \frac{3}{2}\beta^3 + \frac{1}{8}\beta^5\right)(j + 1) - \frac{1 - \nu}{2}\left(5\beta + \right.$$

$$\left.\frac{15}{2}\beta^3 + \frac{5}{8}\beta^5\right)(j + 1)^2 - \frac{\gamma^2}{12}\left[\frac{1 - \nu}{2}\left(\frac{3}{4}\beta^3\right) + \frac{3}{4}n^2\beta^5 - \frac{1 - \nu}{2}\left(\frac{7}{4}\beta^3 + \frac{3}{8}\beta^5\right)(j + 1) + \right.$$

$$\left.\left.\frac{1 - \nu}{2}\left(\frac{3}{4}\beta^3 + \frac{5}{8}\beta^5\right)(j + 1)^2\right]\right\}a_{j+1} + \left\{\bar{\lambda}^2\left(5\beta + \frac{15}{2}\beta^3 + \frac{5}{8}\beta^5\right) - n^2\left(3\beta^3 + \right.\right.$$

$$\left.\frac{3}{4}\beta^5\right) - \frac{3(1 - \nu)}{2}\left(\beta + \frac{3}{2}\beta^3 + \frac{1}{8}\beta^5\right)(j - 1) - \frac{1 - \nu}{2}\left(5\beta + \frac{15}{2}\beta^3 + \frac{5}{8}\beta^5\right)(j - 1)^2 - $$

$$\frac{\gamma^2}{12}\left[\frac{1 - \nu}{2}\left(\frac{3}{4}\beta^3\right) + \frac{3}{4}n^2\beta^5 + \frac{1 - \nu}{2}\left(\frac{7}{4}\beta^3 + \frac{3}{8}\beta^5\right)(j - 1) + \frac{1 - \nu}{2}\left(\frac{3}{4}\beta^3 + \right.\right.$$

$$\left.\left.\frac{5}{8}\beta^5\right)(j - 1)^2\right]\right\}a_{j-1} + \left\{\bar{\lambda}^2\left(5\beta^2 + \frac{5}{2}\beta^4\right) - \frac{3}{2}n^2\beta^4 + \frac{3(1 - \nu)}{2}(2\beta^2 + \beta^4)(j + 2) - \right.$$

$$\frac{1 - \nu}{2}\left(5\beta^2 + \frac{5}{2}\beta^4\right)(j + 2)^2 - \frac{\gamma^2}{12}\left[\frac{1 - \nu}{2}\left(\frac{1}{2}\beta^2 + \frac{1}{2}\beta^4\right) + \frac{1}{2}n^2\beta^4 - \frac{1 - \nu}{2}(\beta^2 + \right.$$

$$\left.\left.2\beta^4)(j + 2) + \frac{1 - \nu}{2}\left(\frac{1}{2}\beta^2 + \frac{3}{2}\beta^4\right)(j + 2)^2\right]\right\}a_{j+2} + \left\{\bar{\lambda}^2\left(5\beta^2 + \frac{5}{2}\beta^4\right) - \right.$$

$$\frac{3}{2}n^2\beta^4 - \frac{3(1 - \nu)}{2}(2\beta^2 + \beta^4)(j - 2) - \frac{1 - \nu}{2}\left(5\beta^2 + \frac{5}{2}\beta^4\right)(j - 2)^2 - $$

$$\frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{1}{2}\beta^2+\frac{1}{2}\beta^4\right)+\frac{1}{2}n^2\beta^4+\frac{1-\nu}{2}(\beta^2+2\beta^4)(j-2)+\frac{1-\nu}{2}\left(\frac{1}{2}\beta^2+\right.\right.$$

$$\left.\left.\frac{3}{2}\beta^4\right)(j-2)^2\right]\Bigg\}a_{j-2}+\Bigg\{\bar{\lambda}^2\left(5\beta^2+\frac{5}{2}\beta^4\right)-\frac{3}{2}n^2\beta^4+\frac{3(1-\nu)}{2}(2\beta^2+$$

$$\beta^4)(2-j)-\frac{1-\nu}{2}\left(5\beta^2+\frac{5}{2}\beta^4\right)(2-j)^2-\frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{1}{2}\beta^2+\frac{1}{2}\beta^4\right)+\right.$$

$$\left.\frac{1}{2}n^2\beta^4-\frac{1-\nu}{2}(\beta^2+2\beta^4)(2-j)+\frac{1-\nu}{2}\left(\frac{1}{2}\beta^2+\frac{3}{2}\beta^4\right)(2-j)^2\right]\Bigg\}a_{2-j}+$$

$$\Bigg\{\bar{\lambda}^2\left(\frac{5}{2}\beta^3+\frac{5}{16}\beta^5\right)-\frac{1}{4}n^2\beta^5+\frac{3(1-\nu)}{2}\left(\frac{3}{2}\beta^3+\frac{3}{16}\beta^5\right)(j+3)-\frac{1-\nu}{2}\left(\frac{5}{2}\beta^3+\right.$$

$$\left.\frac{5}{16}\beta^5\right)(j+3)^2-\frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{3}{4}\beta^3\right)+\frac{1}{4}n^2\beta^5-\frac{1-\nu}{2}\left(\frac{7}{4}\beta^3+\frac{2}{16}\beta^5\right)(j+3)+\right.$$

$$\left.\frac{1-\nu}{2}\left(\frac{3}{4}\beta^3+\frac{5}{16}\beta^5\right)(j+3)^2\right]\Bigg\}a_{j+3}+\Bigg\{\bar{\lambda}^2\left(\frac{5}{2}\beta^3+\frac{5}{16}\beta^5\right)+\frac{1}{4}n^2\beta^5-$$

$$\frac{3(1-\nu)}{2}\left(\frac{3}{2}\beta^3+\frac{3}{16}\beta^5\right)(j-3)-\frac{1-\nu}{2}\left(\frac{5}{2}\beta^3+\frac{2}{16}\beta^5\right)(j-3)^2-\frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{3}{4}\beta^3\right)+\right.$$

$$\left.\frac{1}{4}n^2\beta^5+\frac{1-\nu}{2}\left(\frac{7}{4}\beta^3+\frac{2}{16}\beta^5\right)(j-3)+\frac{1-\nu}{2}\left(\frac{3}{4}\beta^3+\frac{5}{16}\beta^5\right)(j-3)^2\right]\Bigg\}a_{j-3}+$$

$$\Bigg\{\bar{\lambda}^2\left(\frac{5}{2}\beta^3+\frac{5}{16}\beta^5\right)-\frac{1}{4}n^2\beta^5+\frac{3(1-\nu)}{2}\left(\frac{3}{2}\beta^3+\frac{3}{16}\beta^5\right)(3-j)-\frac{1-\nu}{2}\left(\frac{5}{2}\beta^3+\right.$$

$$\left.\frac{5}{16}\beta^5\right)(3-j)^2-\frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{3}{4}\beta^3\right)+\frac{1}{4}n^2\beta^5-\frac{1-\nu}{2}\left(\frac{7}{4}\beta^3+\frac{2}{16}\beta^5\right)(3-j)+\right.$$

$$\left.\frac{1-\nu}{2}\left(\frac{3}{4}\beta^3+\frac{5}{16}\beta^5\right)(3-j)^2\right]\Bigg\}a_{3-j}+\Bigg\{\bar{\lambda}^2\left(\frac{5}{8}\beta^4\right)+\frac{3(1-\nu)}{2}\left(\frac{1}{2}\beta^4\right)(j+4)-$$

$$\frac{1-\nu}{2}\left(\frac{5}{8}\beta^4\right)(j+4)^2 - \frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{1}{4}\beta^4\right) - \frac{1-\nu}{2}(\beta^4)(j+4) + \frac{1-\nu}{2}\left(\frac{3}{8}\beta^4\right)(j+4)^2\right]\Bigg\}a_{j+4} + \Bigg\{\bar{\lambda}^2\left(\frac{5}{8}\beta^4\right) - \frac{3(1-\nu)}{2}\left(\frac{1}{2}\beta^4\right)(j-4) - \frac{1-\nu}{2}\left(\frac{5}{8}\beta^4\right)(j-4)^2 -$$

$$\frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{1}{4}\beta^4\right) + \frac{1-\nu}{2}(\beta^4)(j-4) + \frac{1-\nu}{2}\left(\frac{3}{8}\beta^4\right)(j-4)^2\right]\Bigg\}a_{j-4} +$$

$$\Bigg\{\bar{\lambda}^2\left(\frac{5}{8}\beta^4\right) + \frac{3(1-\nu)}{2}\left(\frac{1}{2}\beta^4\right)(4-j) - \frac{1-\nu}{2}\left(\frac{5}{8}\beta^4\right)(4-j)^2 -$$

$$\frac{\gamma^2}{12}\left[\frac{1-\nu}{2}\left(\frac{1}{4}\beta^4\right) - \frac{1-\nu}{2}(\beta^4)(4-j) + \frac{1-\nu}{2}\left(\frac{3}{8}\beta^4\right)(4-j)^2\right]\Bigg\}a_{4-j} +$$

$$\Bigg\{\bar{\lambda}^2\left(\frac{1}{16}\beta^5\right) + \frac{3(1-\nu)}{2}\left(\frac{1}{16}\beta^5\right)(j+5) - \frac{1-\nu}{2}\left(\frac{1}{16}\beta^5\right)(j+5)^2 +$$

$$\frac{\gamma^2}{12}\,\frac{1-\nu}{2}\left[\frac{3}{16}\beta^5(j+5) - \frac{1}{16}\beta^5(j+5)^2\right]\Bigg\}a_{j+5} + \Bigg\{\bar{\lambda}^2\left(\frac{1}{16}\beta^5\right) -$$

$$\frac{1-\nu}{2}\left(\frac{3}{16}\beta^5\right)(j-5) - \frac{1-\nu}{2}\left(\frac{1}{16}\beta^5\right)(j-5)^2 - \frac{\gamma^2}{12}\,\frac{1-\nu}{2}\left[\frac{3}{16}\beta^5(j-5) +\right.$$

$$\left.\frac{1}{16}\beta^5(j-5)^2\right]\Bigg\}a_{j-5} + \Bigg\{\bar{\lambda}^2\left(\frac{1}{16}\beta^5\right) + \frac{1-\nu}{2}\left(\frac{3}{16}\beta^5\right)(5-j) - \frac{1-\nu}{2}\left(\frac{1}{16}\beta^5\right)(5-j)^2 +$$

$$\frac{\gamma^2}{12}\,\frac{1-\nu}{2}\left[\frac{3}{16}\beta^5(5-j) - \frac{1}{16}\beta^5(5-j)^2\right]\Bigg\}a_{5-j} + \Bigg\{(1+\nu)\left(\beta^2 + \frac{3}{2}\beta^4 +\right.$$

$$\left.\frac{\gamma^2}{12}\beta^4\right)n\Bigg\}a_j + \Bigg\{-\left(\frac{3-\nu}{2}\right)\left(\beta^3 + \frac{1}{4}\beta^5\right)n + \frac{1+\nu}{2}\left(3\beta^3 + \frac{3}{4}\beta^5\right)n(j+1) -$$

$$\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{4}\beta^5\right)n - \frac{1+\nu}{2}\left(\beta^3 + \frac{3}{4}\beta^5\right)(j+1)n\right]\Bigg\}a_{j+1} + \Bigg\{\frac{3-\nu}{2}\left(\beta^3 + \frac{1}{4}\beta^5\right)n +$$

$$\frac{1+\nu}{2}\left(3\beta^3+\frac{3}{4}\beta^5\right)n(j-1)+\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{4}\beta^5\right)n+\frac{1+\nu}{2}\left(\beta^3+\frac{3}{4}\beta^5\right)(j-1)n\right]\Bigg\}d_{j-1}+$$

$$\Bigg\{-\frac{3-\nu}{2}\beta^4 n+\frac{1+\nu}{2}\left(\frac{3}{2}\beta^4\right)(j+2)n-\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{2}\beta^4\right)n-\frac{1+\nu}{2}(\beta^4)n(j+$$

$$2)\right]\Bigg\}d_{j+2}+\Bigg\{\frac{3-\nu}{2}(\beta^4)n+\frac{1+\nu}{2}\left(\frac{3}{2}\beta^4\right)(j-2)n+\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{2}\beta^4\right)n+$$

$$\frac{1+\nu}{2}(\beta^4)(j-2)n\right]\Bigg\}d_{j-2}+\Bigg\{-\left(\frac{3-\nu}{2}\right)\beta^4 n+\frac{1+\nu}{2}\left(\frac{3}{2}\beta^4\right)(2-j)n-$$

$$\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{2}\beta^4\right)n-\frac{1+\nu}{2}(\beta^4)(2-j)n\right]\Bigg\}d_{2-j}+\Bigg\{-\left(\frac{3-\nu}{2}\right)\left(\frac{1}{4}\beta^5\right)n+$$

$$\frac{1+\nu}{2}\left(\frac{1}{4}\beta^5\right)(j+3)n-\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{4}\beta^5\right)n-\frac{1+\nu}{2}\left(\frac{1}{4}\beta^5\right)(j+3)n\right]\Bigg\}d_{j+3}+$$

$$\Bigg\{\frac{3-\nu}{2}\left(\frac{1}{4}\beta^5\right)n+\frac{1+\nu}{2}\left(\frac{1}{4}\beta^5\right)(j-3)n+\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{4}\beta^5\right)n+$$

$$\frac{1+\nu}{2}\left(\frac{1}{4}\beta^5\right)(j-3)n\right]\Bigg\}d_{j-3}+\Bigg\{-\left(\frac{3-\nu}{2}\right)\left(\frac{1}{4}\beta^5\right)n+\frac{1+\nu}{2}\left(\frac{1}{4}\beta^5\right)(3-j)n-$$

$$\frac{\gamma^2}{12}\left[\frac{3-\nu}{2}\left(\frac{1}{4}\beta^5\right)n-\frac{1+\nu}{2}\left(\frac{1}{4}\beta^5\right)(3-j)n\right]\Bigg\}d_{3-j}+\Bigg\{(2\beta^4+2\nu\beta^2+3\nu\beta^4)n-$$

$$\frac{\gamma^2}{12}\left[(1-\nu)\beta^4 n-2\beta^4 n j^2\right]\Bigg\}f_j+\Bigg\{\left[\beta^3+\frac{3}{4}\beta^5+\nu\left(3\beta^3+\frac{3}{4}\beta^5\right)\right]n+$$

$$\frac{\gamma^2 n}{12}\left[n^2\beta^5-\frac{1-\nu}{2}\left(\frac{3}{2}\beta^5\right)-\frac{1}{4}\beta^5(j+1)+\left(\beta^3+\frac{3}{4}\beta^5\right)(j+1)^2\right]\Bigg\}f_{j+1}+$$

$$\Bigg\{\left[\beta^3+\frac{3}{4}\beta^5+\nu\left(3\beta^3+\frac{3}{4}\beta^5\right)\right]n+\frac{\gamma^2 n}{12}\left[n^2\beta^5-\frac{1-\nu}{2}\left(\frac{3}{2}\beta^5\right)+\frac{1}{4}\beta^5(j-1)+\right.$$

$$\left(\beta^3 + \frac{3}{4}\beta^5\right)(j-1)^2\Bigg]\Bigg\} f_{j-1} + \Bigg\{\left(1 + \frac{3}{2}\nu\right)n\beta^4 - \frac{\chi^2 n}{12}\Bigg[\frac{1-\nu}{2}\beta^4 + \frac{1}{2}\beta^4(j+2) -$$

$$\beta^4(j+2)^2\Bigg]\Bigg\} f_{j+2} + \Bigg\{\left(1 + \frac{3}{2}\nu\right)n\beta^4 - \frac{\chi^2 n}{12}\Bigg[\frac{1-\nu}{2}\beta^4 - \frac{1}{2}\beta^4(j-2) -$$

$$\beta^4(j-2)^2\Bigg]\Bigg\} f_{j-2} + \Bigg\{\left(1 + \frac{3}{2}\nu\right)n\beta^4 - \frac{\chi^2 n}{12}\Bigg[\frac{1-\nu}{2}\beta^4 + \frac{1}{2}\beta^4(2-j) -$$

$$\beta^4(2-j)^2\Bigg]\Bigg\} f_{2-j} + \Bigg\{\frac{1}{4}(1+\nu)n\beta^5 - \frac{\chi^2 n}{12}\Bigg[\frac{1-\nu}{2}\left(\frac{1}{2}\beta^5\right) + \frac{1}{4}\beta^5(j+3) -$$

$$\frac{1}{4}\beta^5(j+3)^2\Bigg]\Bigg\} f_{j+3} + \Bigg\{\frac{1}{4}(1+\nu)n\beta^5 - \frac{\chi^2 n}{12}\Bigg[\frac{1-\nu}{2}\left(\frac{1}{2}\beta^5\right) - \frac{1}{4}\beta^5(j-3) -$$

$$\frac{1}{4}\beta^5(j-3)^2\Bigg]\Bigg\} f_{j-3} + \Bigg\{\frac{1}{4}(1+\nu)n\beta^5 - \frac{\chi^2 n}{12}\Bigg[\frac{1-\nu}{2}\left(\frac{1}{2}\beta^5\right) + \frac{1}{4}\beta^5(3-j) -$$

$$\frac{1}{4}\beta^5(3-j)^2\Bigg]\Bigg\} f_{3-j} + \Bigg[\Bigg\{5\bar{\lambda}^2\left(\beta + \frac{3}{2}\beta^3 + \frac{1}{8}\beta^5\right) - 3n^2\beta^3\left(1 + \frac{1}{4}\beta^2\right) -$$

$$\frac{1-\nu}{2}\frac{\chi^2}{12}\left[\frac{5}{4}\beta^3 + \frac{n^2\beta^5}{1-\nu}\left(\frac{3}{2}\right)\right]\Bigg\}\delta_{j1} + \Bigg\{5\bar{\lambda}^2\beta^2\left(1 + \frac{1}{2}\beta^2\right) - \frac{3}{2}n^2\beta^4 -$$

$$\frac{1-\nu}{2}\frac{\chi^2}{12}\left[\frac{1}{2}\beta^2 + \frac{1}{2}\beta^4 + \frac{n^2\beta^4}{1-\nu}\right]\Bigg\}\delta_{j2} + \Bigg\{5\bar{\lambda}^2\beta^3\left(\frac{1}{2} + \frac{1}{16}\beta^2\right) - \frac{1}{4}n^2\beta^5 -$$

$$\frac{1-\nu}{2}\frac{\chi^2}{12}\left[\frac{3}{4}\beta^3 + \frac{n^2\beta^5}{2(1-\nu)}\right]\Bigg\}\delta_{j3} + \Bigg\{\frac{5}{8}\bar{\lambda}^2\beta^4 - \frac{1-\nu}{2}\frac{\chi^2}{12}\left(\frac{\beta^4}{4}\right)\Bigg\}\delta_{j4} +$$

$$\Bigg\{\frac{1}{16}\bar{\lambda}^2\beta^5\Bigg\}\delta_{j5}\Bigg]a_0 + \Bigg[\Bigg\{n\beta^3(1+3\nu) + \frac{3}{4}n\beta^5(1+\nu) + \frac{\chi^2}{12}\Bigg[n^3\beta^5 -$$

$$\frac{1-\nu}{2}\left(\frac{3}{2}n\beta^5\right)\Bigg]\Bigg\}\delta_{j1} + \Bigg\{n\beta^4\left(1 + \frac{3}{2}\nu\right) - \frac{1-\nu}{2}\frac{\chi^2}{12}(n\beta^4)\Bigg\}\delta_{j2} + \Bigg\{\frac{1}{4}n\beta^5(1+\nu) -$$

$$\frac{1-\nu}{2}\frac{\chi^2}{12}\left(\frac{1}{2}n\beta^5\right)\Bigg\}\delta_{j3}\Bigg] f_0 = 0 \qquad (34a)$$

$$\left\{\bar{\lambda}^2\left(1 + 5\beta^2 + \frac{15}{8}\beta^4\right) - n^2\beta^2\left(1 + \frac{3}{2}\beta^2\right) - \frac{1-\nu}{2}\frac{\chi^2}{12}\left[\frac{1}{2}\beta^2 + \frac{1}{4}\beta^4 + \frac{n\beta^4}{1-\nu}\right]\right\}a_0 + \left\{\bar{\lambda}^2 5\beta\left(1 + \frac{3}{2}\beta^2 + \frac{1}{8}\beta^4\right) - 3n^2\beta^3\left(1 + \frac{1}{4}\beta^2\right) - \frac{1-\nu}{2}\left(2\beta + 3\beta^3 + \frac{1}{4}\beta^5\right) - \frac{\chi^2}{12}\left[\frac{1-\nu}{2}\left(\frac{7}{4}\beta^3 + \frac{1}{4}\beta^5\right) + \frac{3}{4}\beta^5 n^2\right]\right\}a_1 + \left\{\bar{\lambda}^2 5\beta^2\left(1 + \frac{1}{2}\beta^2\right) - \frac{3}{2}n^2\beta^4 - \frac{1-\nu}{2}(8\beta^2 + 4\beta^4) - \frac{\chi^2}{12}\left[\frac{1}{2}n^2\beta^4 + \frac{1-\nu}{2}\left(\frac{1}{2}\beta^2 + \frac{5}{2}\beta^4\right)\right]\right\}a_2 + \left\{\bar{\lambda}^2\frac{5}{2}\beta^3\left(1 + \frac{1}{8}\beta^2\right) - \frac{1}{4}n^2\beta^5 - \frac{1-\nu}{2}\left(9\beta^3 + \frac{9}{8}\beta^5\right) - \frac{\chi^2}{12}\left[\frac{1}{4}n^2\beta^5 + \frac{1-\nu}{2}\left(\frac{9}{4}\beta^3 + \frac{9}{8}\beta^5\right)\right]\right\}a_3 + \left\{\bar{\lambda}^2\left(\frac{5}{8}\beta^4\right) - \frac{1-\nu}{2}\left(4\beta^4\right) - \frac{1-\nu}{2}\frac{\chi^2}{12}\left[\frac{9}{4}\beta^4\right]\right\}a_4 + \left\{\bar{\lambda}^2\left(\frac{1}{16}\beta^5\right) - \frac{1-\nu}{2}\left(\frac{5}{8}\beta^5\right) - \frac{1-\nu}{2}\frac{\chi^2}{12}\left[\frac{5}{8}\beta^5\right]\right\}a_5 - \left\{\frac{3-\nu}{2}n\beta^3\left(1 + \frac{1}{4}\beta^2\right) - \frac{1+\nu}{2}3n\beta^3\left(1 + \frac{1}{4}\beta^2\right) + \frac{\chi^2}{12}\left[\frac{3-\nu}{2}\left(\frac{n\beta^5}{4}\right) - \frac{1-\nu}{2}n\beta^3\left(1 + \frac{3}{4}\beta^2\right)\right]\right\}d_1 - \left\{\frac{3-\nu}{2}n\beta^4 - \frac{1+\nu}{2}3n\beta^4 + \frac{\chi^2}{12}\left[\frac{3-\nu}{2}\frac{n\beta^4}{2} - \frac{1+\nu}{2}2n\beta^4\right]\right\}d_2 - \left\{\frac{3-\nu}{2}\frac{n\beta^5}{4} - \frac{1+\nu}{2}\frac{3n\beta^5}{4} + \frac{\chi^2}{12}\left[\frac{3-\nu}{2}\frac{n\beta^5}{4} - \frac{1+\nu}{2}\frac{3n\beta^5}{4}\right]\right\}d_3 + \left\{n\beta^4 + n\nu\beta^2\left(1 + \frac{3}{2}\beta^2\right) - \frac{1-\nu}{2}\frac{\chi^2}{12}n\beta^4\right\}f_0 + \left\{n\beta^3\left(1 + \frac{3}{4}\beta^2\right) + 3n\nu\beta^3\left(1 + \frac{1}{4}\beta^2\right) + \frac{\chi^2}{12}\left[n^3\beta^5 + \frac{3}{4}n\beta^3(1 + \beta^2) - \frac{1-\nu}{2}\frac{3n\beta^5}{2}\right]\right\}f_1 + \left\{n\beta^4 + \frac{3}{2}n\nu\beta^4 + \frac{\chi^2}{12}\left[3n\beta^4 - \frac{1-\nu}{2}n\beta^4\right]\right\}f_2 + \left\{\frac{1}{4}n\beta^5(1 + \nu) + \frac{\chi^2}{12}\left[\frac{3}{2}n\beta^5 - \frac{1-\nu}{2}\frac{n\beta^5}{2}\right]\right\}f_3 = 0 \qquad (34b)$$

$$\left\{(1+\nu)\left(1+\frac{3}{2}\beta^2\right)nj+\frac{\gamma^2 n}{12}(1+\nu)\beta^2 j\right\}a_j+\left\{(1-\nu)\left(\beta+\frac{1}{4}\beta^3\right)n+\right.$$

$$\frac{1+\nu}{2}\left(3\beta+\frac{3}{4}\beta^3\right)(j+1)n-\frac{\gamma^2 n}{12}\left[\frac{1+\nu}{2}\beta-\frac{1-\nu}{4}\beta^3-\frac{1+\nu}{2}\left(\beta+\frac{3}{4}\beta^3\right)(j+\right.$$

$$\left.\left.1)\right]\right\}a_{j+1}+\ -(1-\nu)\left(\beta+\frac{1}{4}\beta^3\right)n+\frac{1+\nu}{2}\left(3\beta+\frac{3}{4}\beta^3\right)(j-1)n+\frac{\gamma^2 n}{12}\left[\frac{1+\nu}{2}\beta-\right.$$

$$\left.\left.\frac{1-\nu}{4}\beta^3+\frac{1+\nu}{2}\left(\beta+\frac{3}{4}\beta^3\right)(j-1)\right]\right\}a_{j-1}+\left\{(1-\nu)n\beta^2+\frac{1+\nu}{2}\frac{3}{2}\beta^2(j+2)n+\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{1-3\nu}{4}\beta^2+\frac{1+\nu}{2}\beta^2(j+2)\right]\right\}a_{j+2}+\left\{-(1-\nu)n\beta^2+\frac{1+\nu}{2}\frac{3}{2}\beta^2(j-2)n-\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{1-3\nu}{4}\beta^2-\frac{1+\nu}{2}\beta^2(j-2)\right]\right\}a_{j-2}+\left\{-(1-\nu)n\beta^2-\frac{1+\nu}{2}\frac{3}{2}\beta^2(2-j)n-\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{1-3\nu}{4}\beta^2+\frac{1+\nu}{2}\beta^2(2-j)\right]\right\}a_{2-j}+\left\{\frac{1-\nu}{4}n\beta^3+\frac{1+\nu}{2}\frac{1}{4}\beta^3(j+3)n+\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{1-\nu}{4}\beta^3+\frac{1+\nu}{2}\frac{1}{4}\beta^3(j+3)\right]\right\}a_{j+3}+\left\{-\frac{1-\nu}{4}n\beta^3+\frac{1+\nu}{2}\frac{1}{4}\beta^3(j-3)n-\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{1-\nu}{4}\beta^3-\frac{1+\nu}{2}\frac{1}{4}\beta^3(j-3)\right]\right\}a_{j-3}+\left\{-\frac{1-\nu}{4}n\beta^3-\frac{1+\nu}{2}\frac{1}{4}\beta^3(3-j)n-\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{1-\nu}{4}\beta^3+\frac{1+\nu}{2}\frac{1}{4}\beta^3(3-j)\right]\right\}a_{3-j}+\left\{\bar{\lambda}^2(2+3\beta^2)-\left[1+n^2(1-\nu)+\right.\right.$$

$$\left.\left.2\nu\right]\beta^2-2\left(1+\frac{3}{2}\beta^2\right)j^2\right\}d_j+\left\{\bar{\lambda}^2\left(3\beta+\frac{3}{4}\beta^3\right)-\left[\frac{1}{4}(1+3\nu)+\frac{1-\nu}{2}n^2\right]\beta^3-\right.$$

$$\left.\nu\beta+\left(\beta+\frac{1}{4}\beta^3\right)(j+1)-\left(3\beta+\frac{3}{4}\beta^3\right)(j+1)^2\right\}d_{j+1}+\left\{\bar{\lambda}^2\left(3\beta+\frac{3}{4}\beta^3\right)-\right.$$

$$\left[\frac{1}{4}(1 + 3\nu) + \frac{1 - \nu}{2} n^2\right]\beta^3 - \nu\beta - \left(\beta + \frac{1}{4}\beta^3\right)(j - 1) - \left(3\beta + \frac{3}{4}\beta^3\right)(j - 1)^2\Bigg\}d_{j-1} +$$

$$\left\{\frac{3}{2}\bar{\lambda}^2\beta^2 + \left(\frac{1}{2} - \nu\right)\beta^2 + \beta^2(j + 2) - \frac{3}{2}\beta^2(j + 2)^2\right\}d_{j+2} + \left\{\frac{3}{2}\bar{\lambda}^2\beta^2 + \left(\frac{1}{2} - \nu\right)\beta^2 -$$

$$\beta^2(j - 2) - \frac{3}{2}\beta^2(j - 2)^2\Bigg\}d_{j-2} + \left\{-\frac{3}{2}\bar{\lambda}^2\beta^2 - \frac{1}{2} - \nu\,\beta^2 - \beta^2(2 - j) +\right.$$

$$\left.\frac{3}{2}\beta^2(2 - j)^2\right\}d_{2-j} + \left\{\frac{1}{4}\bar{\lambda}^2\beta^3 + \frac{1}{4}(1 - \nu)\beta^3 + \frac{1}{4}\beta^3(j + 3) - \frac{1}{4}\beta^3(j + 3)^2\right\}d_{j+3} +$$

$$\left\{\frac{1}{4}\bar{\lambda}^2\beta^3 + \frac{1}{4}(1 - \nu)\beta^3 - \frac{1}{4}\beta^3(j - 3) - \frac{1}{4}\beta^3(j - 3)^2\right\}d_{j-3} - \left\{\frac{1}{4}\bar{\lambda}^2\beta^3 +\right.$$

$$\left.\frac{1}{4}\beta^3(1 - \nu) + \frac{1}{4}\beta^3(3 - j) - \frac{1}{4}\beta^3(3 - j)^2\right\}d_{3-j} - \left\{\left[2 + (3 + 2\nu)\beta^2\right]j +\right.$$

$$\left.\frac{\gamma^2}{12}\left[(1 + 2n^2)\beta^2 j + (2 + 3\beta^2)j^3\right]\right\}f_j + \left\{\beta - \left[(3 + \nu)\beta + \frac{3}{4}(1 + \nu)\beta^3\right](j + 1) -\right.$$

$$\frac{\gamma^2}{12}\left[2n^2\beta^3 + \frac{1}{4}(1 + 4n^2)\beta^3(j + 1) - \left(\beta + \frac{1}{4}\beta^3\right)(j + 1)^2 + 3\left(\beta + \frac{1}{4}\beta^3\right)(j +\right.$$

$$\left.\left.1)^3\right]\right\}f_{j+1} - \left\{\beta + \left[(3 + \nu)\beta + \frac{3}{4}(1 + \nu)\beta^3\right](j - 1) + \frac{\gamma^2}{12}\left[-2n^2\beta^3 + \frac{1}{4}(1 +\right.\right.$$

$$\left.\left.4n^2)\beta^3(j - 1) + \left(\beta + \frac{1}{4}\beta^3\right)(j - 1)^2 + \left(3\beta + \frac{3}{4}\beta^3\right)(j - 1)^3\right]\right\}f_{j-1} +$$

$$\left\{\frac{1}{2}\beta^2 - \left(\frac{3}{2} + \nu\right)\beta^2(j + 2) + \frac{\gamma^2}{12}\left[\frac{1}{2}\beta^2(j + 2) + \beta^2(j + 2)^2 - \frac{3}{2}\beta^2(j + 2)^3\right]\right\}f_{j+2} -$$

$$\left\{\frac{1}{2}\beta^2 + \left(\frac{3}{2} + \nu\right)\beta^2(j - 2) - \frac{\gamma^2}{12}\left[\frac{1}{2}\beta^2(j - 2) - \beta^2(j - 2)^2 - \frac{3}{2}\beta^2(j - 2)^3\right]\right\}f_{j-2} -$$

$$\left\{\frac{1}{2}\beta^2 - \left(\frac{3}{2} + \nu\right)\beta^2(2 - j) + \frac{\chi^2}{12}\left[\frac{1}{2}\beta^2(2 - j) + \beta^2(2 - j)^2 - \frac{3}{2}\beta^2(2 - j)^3\right]\right\} f_{2-j} -$$

$$\left\{\frac{1}{4}(1 + \nu)\beta^3(j + 3) - \frac{\chi^2}{12}\left[\frac{1}{4}\beta^3(j + 3) + \frac{1}{4}\beta^3(j + 3)^2 - \frac{1}{4}\beta^3(j + 3)^3\right]\right\} f_{j+3} -$$

$$\left\{\frac{1}{4}(1 + \nu)\beta^3(j - 3) - \frac{\chi^2}{12}\left[\frac{1}{4}\beta^3(j - 3) - \frac{1}{4}\beta^3(j - 3)^2 - \frac{1}{4}\beta^3(j - 3)^3\right]\right\} f_{j-3} +$$

$$\left\{\frac{1}{4}(1 + \nu)\beta^3(3 - j) - \frac{\chi^2}{12}\left[\frac{1}{4}\beta^3(3 - j) + \frac{1}{4}\beta^3(3 - j)^2 - \frac{1}{4}\beta^3(3 - j)^3\right]\right\} f_{3-j} -$$

$$\left[\left\{(1 - \nu)n\beta\left(1 + \frac{1}{4}\beta^2\right) + \frac{\chi^2}{12}\left[\frac{3 - \nu}{2}\frac{1}{4}n\beta^3 - \frac{1 + \nu}{2}n\beta\left(1 + \frac{1}{4}\beta^2\right)\right]\right\}\delta_{j1} +\right.$$

$$\left\{(1 - \nu)n\beta^2 + \frac{\chi^2}{12}\left[\frac{3 - \nu}{2}\frac{n\beta^2}{2} - \frac{1 + \nu}{2}(n\beta^2)\right]\right\}\delta_{j2} + \left\{(1 - \nu)\frac{1}{4}\beta^3 n +\right.$$

$$\left.\left.\frac{\chi^2}{12}\left[\frac{3 - \nu}{2}\frac{1}{4}n\beta^3 - \frac{1 + \nu}{2}\frac{1}{4}n\beta^3\right]\right\}\delta_{j3}\right] a_0 + \left[\left\{-\beta + \frac{\chi^2}{12}(2n^2\beta^3)\right\}\delta_{j1} +\right.$$

$$\left.\left\{-\frac{1}{2}\beta^2\right\}\delta_{j2}\right] f_0 = 0 \qquad (34c)$$

$$\left\{n\left[2\nu + 3(1 + 2\nu)\beta^2 + \frac{3}{4}(1 + \nu)\beta^4\right] + \frac{\chi^2 n}{12}\left[(\nu - 1)\beta^2 + \left(n^2 - \frac{3}{2} + \frac{3}{4}\nu\right)\beta^4 +\right.\right.$$

$$\left.\left.\left(3\beta^2 + \frac{3}{4}\beta^4\right)j^2\right]\right\} a_j + \left\{n\left[(1 + 4\nu)\beta + 3\left(\frac{3}{4} + \nu\right)\beta^3\right] + \frac{\chi^2 n}{12}\left[\left(n^2 - \frac{11}{4} + \frac{3\nu}{2}\right)\beta^3 -\right.\right.$$

$$\left.\left.(2\beta + \beta^3)(j + 1) + \left(\beta + \frac{3}{4}\beta^3\right)(j + 1)^2\right]\right\} a_{j+1} + \left\{n\left[(1 + 4\nu)\beta + 3\left(\frac{3}{4} + \nu\right)\beta^3\right] +\right.$$

$$\frac{\chi^2 n}{12}\left[\left(n^2 - \frac{11}{4} + \frac{3\nu}{2}\right)\beta^3 + (2\beta + \beta^3)(j - 1) + \left(\beta + \frac{2}{4}\beta^3\right)(j - 1)^2\right]\Bigg\} a_{j-1} +$$

$$\left\{n\left[3\left(\frac{1}{2} + \nu\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right] + \frac{\chi^2 n}{12}\left[\frac{\nu - 3}{2}\beta^2 + \left(\frac{n^2}{2} - 1 + \frac{\nu}{2}\right)\beta^4 - \left(\frac{5}{2}\beta^2 + \right.\right.\right.$$

$$\left.\left.\frac{1}{4}\beta^4\right)(j + 2) + \left(\frac{3}{2}\beta^2 + \frac{1}{2}\beta^4\right)(j + 2)^2\right]\Bigg\} a_{j+2} + \left\{n\left[3\left(\frac{1}{2} + \nu\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right] + \right.$$

$$\frac{\chi^2 n}{12}\left[\frac{\nu - 3}{2}\beta^2 + \left(\frac{n^2}{2} - 1 + \frac{\nu}{2}\right)\beta^4 + \left(\frac{5}{2}\beta^2 + \frac{1}{4}\beta^4\right)(j - 2) + \left(\frac{3}{2}\beta^2 + \right.\right.$$

$$\left.\left.\frac{1}{2}\beta^4\right)(j - 2)^2\right]\Bigg\} a_{j-2} + \left\{n\left[3\left(\frac{1}{2} + \nu\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right] + \frac{\chi^2 n}{12}\left[\frac{\nu - 3}{2}\beta^2 + \left(\frac{n^2}{2} - \right.\right.\right.$$

$$\left.\left.1 + \frac{\nu}{2}\right)\beta^4 - \left(\frac{5}{2}\beta^2 + \frac{1}{4}\beta^4\right)(2 - j) + \left(\frac{3}{2}\beta^2 + \frac{1}{2}\beta^4\right)(2 - j)^2\right]\Bigg\} a_{2-j} +$$

$$\left\{n\left(\frac{3}{4} + \nu\right) + \frac{\chi^2 n}{12}\left[-\frac{1}{2}\left(\frac{5}{2} - \nu\right) - (j + 3) + \frac{3}{4}(j + 3)^2\right]\right\}\beta^3 a_{j+3} + \left\{n\left(\frac{3}{4} + \nu\right) + \right.$$

$$\left.\frac{\chi^2 n}{12}\left[-\frac{1}{2}\left(\frac{5}{2} - \nu\right) + (j - 3) + \frac{3}{4}(j - 3)^2\right]\right\}\beta^3 a_{j-3} + \left\{n\left(\frac{3}{4} + \nu\right) + \frac{\chi^2 n}{12}\left[-\frac{1}{2}\left(\frac{5}{2} - \nu\right) - \right.\right.$$

$$\left.\left.(3 - j) + \frac{3}{4}(3 - j)^2\right]\right\}\beta^3 a_{j-3} + \left\{\frac{n}{8}(1 + \nu) + \frac{\chi^2 n}{12}\left[-\frac{2 - \nu}{8} - \frac{1}{8}(j + 4) + \right.\right.$$

$$\left.\left.\frac{1}{8}(j + 4)^2\right]\right\}\beta^4 a_{j+4} + \left\{\frac{n}{8}(1 + \nu) + \frac{\chi^2 n}{12}\left[-\frac{2 - \nu}{8} + \frac{1}{8}(j - 4) + \frac{1}{8}(j - 4)^2\right]\right\}\beta^4 a_{j-4} +$$

$$\left\{\frac{n}{8}(1 + \nu) + \frac{\chi^2 n}{12}\left[-\frac{2 - \nu}{8} - \frac{1}{8}(4 - j) + \frac{1}{8}(4 - j)^2\right]\right\}\beta^4 a_{4-j} - \left\{\left[2 + 3(2 + \nu)\beta^2 + \right.\right.$$

$$\left.\left.\frac{3}{4}(1 + \nu)\beta^4\right]j + \frac{\chi^2}{12}\left\{\left[(4 + 2n^2)\beta^2 + (1 + n^2)\beta^4\right]j + \left[2 + 6\beta^2 + \frac{3}{4}\beta^4\right]j^3\right\}\right\} d_j +$$

$$\left\{\nu\beta + \left(1 + \frac{3}{2}\nu\right)\frac{\beta^3}{2} - \left[(4 + \nu)\beta + 3\left(1 + \frac{3\nu}{4}\right)\beta^3\right](j + 1) + \frac{\gamma^2}{12}\left\{\left(n^2 - \frac{3}{4}\right)\beta^3 - \right.\right.$$

$$\left.\left.\left[\beta + \left(\frac{11}{4} + 2n^2\right)\beta^3\right](j + 1) + \left[2\beta + \frac{3}{2}\beta^3\right](j + 1)^2 - \left[4\beta + 3\beta^3\right](j + 1)^3\right\}\right\} d_{j+1} -$$

$$\left\{\nu\beta + \frac{1}{2}\left(1 + \frac{3\nu}{2}\right)\beta^3 + \left[(4 + \nu)\beta + 3\left(1 + \frac{3\nu}{4}\right)\beta^3\right](j - 1) + \frac{\gamma^2}{12}\left\{\left(n^2 - \frac{3}{4}\right)\beta^3 + \right.\right.$$

$$\left.\left.\left[\beta + \left(\frac{11}{4} + 2n^2\right)\beta^3\right](j - 1) + \left[2\beta + \frac{3}{2}\beta^3\right](j - 1)^2 + \left[4\beta + 3\beta^3\right](j - 1)^3\right\}\right\} d_{j-1} +$$

$$\left\{\frac{1 + 3\nu}{2}\beta^2 + \frac{1 + \nu}{4}\beta^4 - \left[3\left(1 + \frac{\nu}{2}\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right](j + 2) + \frac{\gamma^2}{12}\left\{\left(\frac{n^2}{2} - \frac{1}{4}\right)\beta^4 - \right.\right.$$

$$\left.\left.\beta^2 - \left[\beta^2 + \frac{1}{2}(1 + n^2)\beta^4\right](j + 2) + \left[3\beta^2 + \frac{1}{2}\beta^4\right]\left[(j + 2)^2 - (j + 2)^3\right]\right\}\right\} d_{j+2} -$$

$$\left\{\frac{1 + 3\nu}{2}\beta^2 + \frac{1 + \nu}{4}\beta^4 + \left[3\left(1 + \frac{\nu}{2}\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right](j - 2) + \frac{\gamma^2}{12}\left\{\left(\frac{n^2}{2} - \frac{1}{4}\right)\beta^4 + \right.\right.$$

$$\left.\left.\left[\beta^2 + \frac{1}{2}(1 + n^2)\beta^4\right](j - 2) + \left[3\beta^2 + \frac{1}{2}\beta^4\right]\left[(j - 2)^2 + (j - 2)^3\right]\right\}\right\} d_{j-2} +$$

$$\left\{\frac{1 + 3\nu}{2}\beta^2 + \frac{1 + \nu}{4}\beta^4 - \left[3\left(1 + \frac{\nu}{2}\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right](2 - j) + \frac{\gamma^2}{12}\left\{\left(\frac{n^2}{2} - \frac{1}{4}\right)\beta^4 - \right.\right.$$

$$\left.\left.\left[\beta^2 + \frac{1}{2}(1 + n^2)\beta^4\right](2 - j) + \left[3\beta^2 + \frac{1}{2}\beta^4\right]\left[(2 - j)^2 - (2 - j)^3\right]\right\}\right\} d_{2-j} +$$

$$\left\{\frac{1}{2}\left(1 + \frac{3\nu}{2}\right) - \left(1 + \frac{3\nu}{4}\right)(j + 3) + \frac{\gamma^2}{12}\left[\frac{1}{4} - \frac{1}{4}(j + 3) + \frac{3}{2}(j + 3)^2 - \right.\right.$$

$$\left.\left.(j + 3)^3\right]\right\}\beta^3 d_{j+3} - \left\{\frac{1}{2}\left(1 + \frac{3\nu}{2}\right) + \left(1 + \frac{3\nu}{4}\right)(j - 3) + \frac{\gamma^2}{12}\left[\frac{1}{4} + \frac{1}{4}(j - 3) + \right.\right.$$

$$\frac{3}{2}(j-3)^2 + (j-3)^3\Big]\Big\}\beta^3 d_{j-3} + \Big\{\frac{1}{2}\Big(1+\frac{3\nu}{2}\Big) - \Big(1+\frac{3\nu}{4}\Big)(3-j) + \frac{\gamma^2}{12}\Big[\frac{1}{4} -$$

$$\frac{1}{4}(3-j) + \frac{3}{2}(3-j)^2 - (3-j)^3\Big]\Big\}\beta^3 d_{3-j} + \Big\{\frac{1+\nu}{8} - \frac{1+\nu}{8}(j+4) +$$

$$\frac{\gamma^2}{12}\Big[\frac{1}{8} + \frac{1}{4}(j+4)^2 - \frac{1}{8}(j+4)^3\Big]\Big\}\beta^4 d_{j+4} - \Big\{\frac{1+\nu}{8} + \frac{1+\nu}{8}(j-4) + \frac{\gamma^2}{12}\Big[\frac{1}{8} +$$

$$\frac{1}{4}(j-4)^2 + \frac{1}{8}(j-4)^3\Big]\Big\}\beta^4 d_{j-4} + \Big\{\frac{1+\nu}{8} - \frac{1+\nu}{8}(4-j) + \frac{\gamma^2}{12}\Big[\frac{1}{8} +$$

$$\frac{1}{4}(4-j)^2 - \frac{1}{8}(4-j)^3\Big]\Big\}\beta^4 d_{4-j} - \Big\{2 + (7+6\nu)\beta^2 + \frac{3}{2}(1+\nu)\beta^4 -$$

$$\bar{\lambda}^2\Big(2 + 6\beta^2 + \frac{3}{4}\beta^4\Big) + \frac{\gamma^2}{12}\Big\{n^2(2n^2 - 7 - \nu)\beta^4 + \Big[(4 + 4n^2 - 3\nu)\beta^2 +$$

$$\Big(1 + 2n^2 + \frac{3\nu}{4}\Big)\beta^4\Big]j^2 + \Big[2 + 6\beta^2 + \frac{3}{4}\beta^4\Big]j^4\Big\}\Big\}f_j - \Big\{\frac{9}{2}(1+\nu)\beta^3 + 2(2+\nu)\beta -$$

$$\bar{\lambda}^2(4\beta + 3\beta^3) + \frac{\gamma^2}{12}\Big\{-n^2(3+\nu)\beta^3 + \Big[\Big(\frac{7}{4} + 2n^2 - \frac{3\nu}{4}\Big)\beta^3 - \nu\beta\Big](j+1) +$$

$$\Big[(1+\nu)\beta + \Big(\frac{11}{4} + 4n^2 + \frac{9\nu}{4}\Big)\beta^3\Big](j+1)^2 - \Big[2\beta + \frac{3}{2}\beta^3\Big](j+1)^3 + \Big[4\beta +$$

$$3\beta^3\Big](j+1)^4\Big\}\Big\}f_{j+1} - \Big\{\frac{9}{2}(1+\nu)\beta^3 + 2(2+\nu)\beta - \bar{\lambda}^2(4\beta + 3\beta^3) +$$

$$\frac{\gamma^2}{12}\Big\{-n^2(3+\nu)\beta^3 - \Big[\Big(\frac{7}{4} + 2n^2 - \frac{3\nu}{4}\Big)\beta^3 - \nu\beta\Big](j-1) + \Big[(1+\nu)\beta + \Big(\frac{11}{4} + 4n^2 +$$

$$\frac{9\nu}{4}\Big)\beta^3\Big](j-1)^2 + \Big[2\beta + \frac{3}{2}\beta^3\Big](j-1)^3 + \Big[4\beta + 3\beta^3\Big](j-1)^4\Big\}\Big\}f_{j-1} -$$

$$\left\{\frac{7+6\nu}{2}\beta^2 + (1+\nu)\beta^4 - \bar{\lambda}^2\left(3\beta^2 + \frac{1}{2}\beta^4\right) + \frac{\gamma^2}{12}\left\{\frac{1-\nu}{2}n^2\beta^4 + \left[\left(1 - \frac{3\nu}{2}\right)\beta^2 + \right.\right.\right.$$

$$\left.\left(\frac{3}{4} - \frac{\nu}{4} + n^2\right)\beta^4\right](j+2) + \left[\left(1 + \frac{3\nu}{2}\right)\beta^2 + \left(\frac{1}{2} + n^2 + \frac{\nu}{2}\right)\beta^4\right](j+2)^2 -$$

$$\left.\left.\left[3\beta^2 + \frac{1}{2}\beta^4\right]\left[(j+2)^3 - (j+2)^4\right]\right\}\right\} f_{j+2} - \left\{\frac{7+6\nu}{2}\beta^2 + (1+\nu)\beta^4 - \right.$$

$$\bar{\lambda}^2\left(3\beta^2 + \frac{1}{2}\beta^4\right) + \frac{\gamma^2}{12}\left\{\frac{1-\nu}{2}n^2\beta^4 - \left[\left(1 - \frac{3\nu}{2}\right)\beta^2 + \left(\frac{3-\nu}{4} + n^2\right)\beta^4\right](j-2) + \right.$$

$$\left[\left(1 + \frac{3\nu}{2}\right)\beta^2 + \left(\frac{1+\nu}{2} + n^2\right)\beta^4\right](j-2)^2 + \left[3\beta^2 + \frac{1}{2}\beta^4\right]\left[(j-2)^3 + \right.$$

$$\left.\left.\left.(j-2)^4\right]\right\}\right\} f_{j-2} - \left\{\frac{7+6\nu}{2}\beta^2 + (1+\nu)\beta^4 - \bar{\lambda}^2\left(3\beta^2 + \frac{1}{2}\beta^4\right) + \right.$$

$$\frac{\gamma^2}{12}\left\{\frac{1-\nu}{2}n^2\beta^4 + \left[\left(1 - \frac{3\nu}{2}\right)\beta^2 + \left(\frac{3-\nu}{4} + n^2\right)\beta^4\right](2-j) + \left[\left(1 + \frac{3\nu}{2}\right)\beta^2 + \right.\right.$$

$$\left.\left.\left(\frac{1+\nu}{2} + n^2\right)\beta^4\right](2-j)^2 - \left[3\beta^2 + \frac{1}{2}\beta^4\right]\left[(2-j)^3 - (2-j)^4\right]\right\}\right\} f_{2-j} -$$

$$\left\{\frac{3}{2}(1-\nu) - \bar{\lambda}^2 + \frac{\gamma^2}{12}\left[\frac{3}{4}(1-\nu)(j+3) + \frac{1+3\nu}{4}(j+3)^2 - \frac{3}{2}(j+3)^3 + \right.\right.$$

$$\left.\left.(j+3)^4\right]\right\}\beta^3 f_{j+3} - \left\{\frac{3}{2}(1+\nu) - \bar{\lambda}^2 + \frac{\gamma^2}{12}\left[-\frac{3}{4}(1-\nu)(j-3) + \frac{1+3\nu}{4}(j-3)^2 + \right.\right.$$

$$\left.\left.\frac{3}{2}(j-3)^3 + (j-3)^4\right]\right\}\beta^3 f_{j-3} - \left\{\frac{3}{2}(1+\nu) - \bar{\lambda}^2 + \frac{\gamma^2}{12}\left[\frac{3}{4}(1-\nu)(3-j) + \right.\right.$$

$$\left.\left.\frac{1+3\nu}{4}(3-j)^2 - \frac{3}{2}(3-j)^3 + (3-j)^4\right]\right\}\beta^3 f_{3-j} - \left\{\frac{1+\nu}{4} - \frac{1}{8}\bar{\lambda}^2 + \right.$$

$$\frac{\gamma^2}{12}\left[\frac{1-\nu}{8}(j+4)+\frac{\nu}{8}(j+4)^2-\frac{1}{4}(j+4)^3+\frac{1}{8}(j+4)^4\right]\Bigg\}\beta^4 f_{j+4}-$$

$$\left\{\frac{1+\nu}{4}-\frac{1}{8}\bar{\lambda}^2+\frac{\gamma^2}{12}\left[\frac{1-\nu}{8}(j-4)+\frac{\nu}{8}(j-4)^2+\frac{1}{4}(j-4)^3+\right.\right.$$

$$\left.\left.\frac{1}{8}(j-4)^4\right]\right\}\beta^4 f_{j-4}-\left\{\frac{1+\nu}{4}-\frac{1}{8}\bar{\lambda}^2+\frac{\gamma^2}{12}\left[\frac{1-\nu}{8}(4-j)+\frac{\nu}{8}(4-j)^2-\right.\right.$$

$$\left.\left.\frac{1}{4}(4-j)^3+\frac{1}{8}(4-j)^4\right]\right\}\beta^4 f_{4-j}+\left\{\left\{n\beta\left(1+\frac{9}{4}\beta^2\right)+n\nu(4\beta+3\beta^3)+\right.\right.$$

$$\left.\frac{\gamma^2}{12}n\beta^3\left[n^2-\frac{3-\nu}{2}-\frac{1}{4}-(1-\nu)\right]\right\}\delta_{j1}+\left\{n\beta\left(\frac{3}{2}\beta+\frac{1}{2}\beta^3\right)+n\nu\left(3\beta^2+\right.\right.$$

$$\left.\left.\frac{1}{2}\beta^4\right)+\frac{\gamma^2}{12}n\beta^2\left[\frac{1}{2}n^2\beta^2-\frac{3-\nu}{2}\left(1+\frac{1}{2}\beta^2\right)-\frac{1-\nu}{2}\frac{1}{2}\beta^2\right]\right\}\delta_{j2}+$$

$$\left\{\frac{3}{4}n\beta^3\left(1+\frac{4}{3}\nu\right)+\frac{\gamma^2}{12}n\beta^3\left[\frac{1}{4}-\frac{3-\nu}{2}\right]\right\}\delta_{j3}+\left\{\frac{1}{8}n\beta^4(1+\nu)-\right.$$

$$\left.\left.\frac{\gamma^2}{12}n\beta^4\frac{2-\nu}{8}\right\}\delta_{j4}\right\}a_o+\left[\left\{(\bar{\lambda}^2-1)(4\beta+3\beta^3)-\frac{3}{2}\beta^3-2\nu\beta\left(1+\frac{9}{4}\beta^2\right)+\right.\right.$$

$$\left.\frac{\gamma^2}{12}(3+\nu)n^2\beta^3\right\}\delta_{j1}+\left\{(\bar{\lambda}^2-1)\left(3\beta^2+\frac{1}{2}\beta^4\right)-\frac{1}{2}\beta^2(1+\beta^2)-2\nu\beta\left(\frac{3}{2}\beta+\right.\right.$$

$$\left.\left.\frac{1}{2}\beta^3\right)+\frac{\gamma^2}{12}n^2\beta^4\left[\frac{3+\nu}{2}-2\right]\right\}\delta_{j2}+\left\{(\bar{\lambda}^2-1)\beta^3-\frac{1}{2}\beta^3-\frac{3}{2}\nu\beta^3\right\}\delta_{j3}+$$

$$\left.\left\{(\bar{\lambda}^2-1)\frac{1}{8}\beta^4-\frac{1}{8}\beta^4-\frac{1}{2}\nu\beta^4\right\}\delta_{j4}\right]f_o=0 \qquad (34d)$$

where

$$\delta_{ij}=1,\quad i=j$$

$$\delta_{ij}=0,\quad i\neq j$$

$$\left\{\frac{3n}{2}\beta^2\left(1+\frac{1}{4}\beta^2\right)+n\nu\left(1+3\beta^2+\frac{3}{8}\beta^4\right)+\frac{\gamma^2 n}{12}\left[\frac{1}{2}\beta^4\left(n^2-\frac{1}{4}\right)-\frac{3-\nu}{2}\frac{\beta^4}{4}-\right.\right.$$

$$\left.\left.\frac{1-\nu}{2}\beta^2\left(1+\frac{1}{2}\beta^2\right)\right]\right\}a_0+\left\{n\beta\left(1+\frac{9}{4}\beta^2\right)+\nu n(4\beta+3\beta^3)+\frac{\gamma^2 n}{12}\left[n^2\beta^3-\right.\right.$$

$$\left.\left.\beta+\beta^3-\frac{3-\nu}{2}\beta^3-(1-\nu)\beta^3\right]\right\}a_1+\left\{\frac{n\beta^2}{2}(3+\beta^2)+\nu n\beta^2\left(3+\frac{1}{2}\beta^2\right)+\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{n^2\beta^4}{2}+\beta^2+\frac{3}{2}\beta^4-\frac{1-\nu}{2}\frac{\beta^4}{2}-\frac{3-\nu}{2}\beta^2\left(1+\frac{1}{2}\beta^2\right)\right]\right\}a_2+\left\{\frac{3}{4}n\beta^3+\right.$$

$$\left.\nu n\beta^3+\frac{\gamma^2 n}{12}\left[4\beta^3-\frac{3-\nu}{2}\beta^3\right]\right\}a_3+\left\{\frac{1}{8}n\beta^4(1+\nu)+\frac{\gamma^2 n}{12}\left[\frac{13}{8}\beta^4-\right.\right.$$

$$\left.\left.\frac{1-\nu}{2}\frac{1}{4}\beta^4\right]\right\}a_4-\left\{+4\beta+\frac{2}{2}\beta^3+\frac{3}{2}\nu\beta^3+\frac{\gamma^2}{12}\left[+n^2\beta^3+5\beta+5\beta^3\right]\right\}d_1-$$

$$\left\{+\frac{11}{2}\beta^2+\frac{3}{4}\beta^4+\nu\beta^2\left(\frac{3}{2}+\frac{3}{4}\beta^2\right)+\frac{\gamma^2}{12}\left[\frac{1}{2}n^2\beta^4+14\beta^2+\frac{13}{4}\beta^4\right]\right\}d_2-$$

$$\left\{\frac{3}{2}\nu\beta^3+\frac{2}{2}\beta^3+\frac{\gamma^2}{12}\left[14\beta^3\right]\right\}d_3-\left\{\frac{3}{8}\beta^4+\frac{3}{8}\nu\beta^4+\frac{\gamma^2}{12}\left[\frac{31}{8}\beta^4\right]\right\}d_4+$$

$$\left\{\bar{\lambda}^2\left(1+3\beta^2+\frac{3}{8}\beta^4\right)-1-\frac{7}{2}\beta^2-\frac{3}{4}\beta^4-3\nu\beta^2\left(1+\frac{1}{4}\beta^2\right)-\frac{\gamma^2}{12}\left[n^4\beta^4-\right.\right.$$

$$\left.\left.2n^2\beta^4-(3+\nu)\frac{n^2\beta^4}{2}\right]\right\}f_0+\left\{(\bar{\lambda}^2-1)(4\beta+3\beta^3)-\frac{3}{2}\beta^3-2\nu\beta\left(1+\frac{9}{4}\beta^2\right)-\right.$$

$$\left.\frac{\gamma^2}{12}\left[2n^2\beta^3+3\beta-(3+\nu)n^2\beta^3+6\beta^3+\frac{3}{2}\nu\beta^3\right]\right\}f_1+(\bar{\lambda}^2-1)\left(3\beta^2+\frac{1}{2}\beta^4\right)-$$

$$\pm\beta(1+\beta)-\nu\beta(3\beta+\beta^3)-\frac{\gamma^2}{12}\left[8n^2\beta^4-(3+\nu)\frac{n^2\beta^4}{2}+30\beta^2+3\nu\beta^2+\right.$$

$$\frac{3}{2}\nu\beta^4 + \frac{15}{2}\beta^4\Bigg]\Bigg\} f_2 + \Bigg\{(\bar{\lambda}^2 - 1)\beta^3 - \frac{1}{2}\beta^3(1 + 3\nu) - \frac{\gamma^2}{12}\Bigg[\frac{9}{2}\nu\beta^3 + 45\beta^3\Bigg]\Bigg\} f_3 +$$

$$\Bigg\{(\bar{\lambda}^2 - 1)\frac{\beta^4}{8} - \frac{1}{8}\beta^4(1 + 2\nu) - \frac{\gamma^2}{12}\Bigg[\frac{3}{2}\nu\beta^4 + \frac{33}{2}\beta^4\Bigg]\Bigg\} f_4 = 0 \qquad (34e)$$

The notation used in equations (34) requires that the following restrictions be placed on the coefficients a_i, d_i, and f_i in order to be consistent with equation (29).

$$\left.\begin{aligned} a_{j-k} &= 0 \\ d_{j-k} &= 0 \\ f_{j-k} &= 0 \end{aligned}\right\} \quad j - k \leq 0 \quad \text{or} \quad k \geq j \qquad (35a)$$

$$\left.\begin{aligned} a_{k-j} &= 0 \\ d_{k-j} &= 0 \\ f_{k-j} &= 0 \end{aligned}\right\} \quad k - j \leq 0 \quad \text{or} \quad j \geq k$$

That is, the series assumed in equations (29) does not include coefficients with negative subscripts.

The integration of equations (31b), (31c), and (31f) using equations (33) leads to the following equations for the coefficients b_i, c_i, and e_i of the antisymmetrical case:

$$\left\{-n^2(2\beta^2 + 3\beta^4) + \bar{\lambda}^2\left(2 + 10\beta^2 + \frac{15}{4}\beta^4\right) - \frac{1-\nu}{2}\left(2 + 10\beta^2 + \frac{15}{4}\beta^4\right)j^2 - \right.$$

$$\left.\frac{\gamma^2}{12}\left\{n^2\beta^4 + \frac{1-\nu}{2}\left[\beta^2 + \frac{1}{2}\beta^4 + \left(\beta^2 + \frac{9}{4}\beta^4\right)j^2\right]\right\}\right\} b_j + \left\{-n^2\left(3\beta^3 + \frac{3}{4}\beta^5\right) + \right.$$

$$\bar{\lambda}^2\left(5\beta + \frac{15}{2}\beta^3 + \frac{5}{8}\beta^5\right) - \frac{1-\nu}{2}\left(5\beta + \frac{15}{2}\beta^3 + \frac{5}{8}\beta^5\right)(j+1)^2 +$$

$$\frac{3(1-\nu)}{2}\left(\beta + \frac{3}{2}\beta^3 + \frac{1}{8}\beta^5\right)(j+1) - \frac{\gamma^2}{12}\left\{\frac{3}{4}n^2\beta^5 + \frac{1-\nu}{2}\left[\frac{5}{4}\beta^3 - \left(\frac{7}{4}\beta^3 + \right.\right.\right.$$

$$\left.\left.\left.\frac{3}{8}\beta^5\right)(j+1) + \left(\frac{9}{4}\beta^3 + \frac{5}{8}\beta^5\right)(j+1)^2\right]\right\}\right\} b_{j+1} + \left\{-n^2\left(3\beta^3 + \frac{3}{4}\beta^5\right) + \right.$$

$$\bar{\lambda}^2\left(5\beta + \frac{15}{2}\beta^3 + \frac{5}{8}\beta^5\right) - \frac{3(1-\nu)}{2}\left(\beta + \frac{3}{2}\beta^3 + \frac{1}{8}\beta^5\right)(j-1) -$$

$$\frac{1-\nu}{2}\left(5\beta + \frac{15}{2}\beta^3 + \frac{5}{8}\beta^5\right)(j-1)^2 - \frac{\gamma^2}{12}\left\{\frac{3}{4}n^2\beta^5 + \frac{1-\nu}{2}\left[\frac{5}{8}\beta^3 + \left(\frac{7}{4}\beta^3 + \right.\right.\right.$$

$$\left.\left.\left.\frac{3}{8}\beta^5\right)(j-1) + \left(\frac{9}{4}\beta^3 + \frac{5}{8}\beta^5\right)(j-1)^2\right]\right\}\right\} b_{j-1} + \left\{\bar{\lambda}^2\left(5\beta^2 + \frac{5}{2}\beta^4\right) - \right.$$

$$\frac{3}{2}n^2\beta^4 + \frac{3(1-\nu)}{2}(2\beta^2 + \beta^4)(j+2) - \frac{1-\nu}{2}\left(5\beta^2 + \frac{5}{2}\beta^4\right)(j+2)^2 -$$

$$\frac{\gamma^2}{12}\left\{\frac{1}{2}n^2\beta^4 + \frac{1-\nu}{2}\left[\frac{1}{2}\beta^2 + \frac{1}{2}\beta^4 - (\beta^2 + 2\beta^4)(j+2) + \left(\frac{1}{2}\beta^2 + \frac{3}{2}\beta^4\right)(j + \right.\right.$$

$$\left.\left.\left.2)^2\right]\right\}\right\} b_{j+2} + \left\{\bar{\lambda}^2\left(5\beta^2 + \frac{5}{2}\beta^4\right) - \frac{3}{2}n^2\beta^4 - \frac{3(1-\nu)}{2}(2\beta^2 + \beta^4)(j-2) - \right.$$

$$\frac{1-\nu}{2}\left(5\beta^2 + \frac{5}{2}\beta^4\right)(j-2)^2 - \frac{\gamma^2}{12}\left\{\frac{1}{2}n^2\beta^4 + \frac{1-\nu}{2}\left[\frac{1}{2}\beta^2 + \frac{1}{2}\beta^4 + (\beta^2 + \right.\right.$$

$$2\beta^4)(j - 2) + \left(\frac{1}{2}\beta^2 + \frac{3}{2}\beta^4\right)(j - 2)^2\Big]\Big\}\Big] b_{j-2} - \Big\{\bar{\lambda}^2\left(5\beta^2 + \frac{5}{2}\beta^4\right) - \frac{3}{2}n^2\beta^4 +$$

$$\frac{3(1 - \nu)}{2}(2\beta^2 + \beta^4)(2 - j) - \frac{1 - \nu}{2}\left(5\beta^2 + \frac{5}{2}\beta^4\right)(2 - j)^2 - \frac{\gamma^2}{12}\Big\{\frac{1}{2}n^2\beta^4 +$$

$$\frac{1 - \nu}{2}\left(\frac{1}{2}\beta^2 + \frac{1}{2}\beta^3\right) - (\beta^2 + 2\beta^4)(2 - j) + \left(\frac{1}{2}\beta^2 + \frac{3}{2}\beta^4\right)(2 - j)^2\Big]\Big\}\Big\} b_{2-j} +$$

$$\Big\{\bar{\lambda}^2\left(\frac{5}{2}\beta^3 + \frac{5}{16}\beta^5\right) - \frac{1}{4}n^2\beta^5 + \frac{3(1 - \nu)}{2}\left(\frac{3}{2}\beta^3 + \frac{3}{16}\beta^5\right)(j + 3) - \frac{1 - \nu}{2}\left(\frac{5}{2}\beta^3 +$$

$$\frac{5}{16}\beta^5\right)(j + 3)^2 - \frac{\gamma^2}{12}\Big\{\frac{1}{4}n^2\beta^5 + \frac{1 - \nu}{2}\Big[\frac{3}{4}\beta^3 - \left(\frac{7}{4}\beta^3 + \frac{9}{16}\beta^5\right)(j + 3) +$$

$$\left(\frac{3}{4}\beta^3 + \frac{5}{16}\beta^5\right)(j + 3)^2\Big]\Big\}\Big\} b_{j+3} + \Big\{\bar{\lambda}^2\left(\frac{5}{2}\beta^3 + \frac{5}{16}\beta^5\right) - \frac{1}{4}n^2\beta^5 -$$

$$\frac{3(1 - \nu)}{2}\left(\frac{3}{2}\beta^3 + \frac{3}{16}\beta^5\right)(j - 3) - \frac{1 - \nu}{2}\left(\frac{5}{2}\beta^3 + \frac{5}{16}\beta^5\right)(j - 3)^2 -$$

$$\frac{\gamma^2}{12}\Big\{\frac{1}{4}n^2\beta^5 + \frac{1 - \nu}{2}\Big[\frac{3}{4}\beta^3 + \left(\frac{7}{4}\beta^3 + \frac{9}{16}\beta^5\right)(j - 3) + \left(\frac{3}{4}\beta^3 + \frac{5}{16}\beta^5\right)(j -$$

$$3)^2\Big]\Big\}\Big\} b_{j-3} - \Big\{\bar{\lambda}^2\left(\frac{5}{2}\beta^3 + \frac{5}{16}\beta^5\right) - \frac{1}{4}n^2\beta^5 + \frac{3(1 - \nu)}{2}\left(\frac{3}{2}\beta^3 + \frac{3}{16}\beta^5\right)(3 - j) -$$

$$\frac{1 - \nu}{2}\left(\frac{5}{2}\beta^3 + \frac{5}{16}\beta^5\right)(3 - j)^2 - \frac{\gamma^2}{12}\Big\{\frac{1}{4}n^2\beta^5 + \frac{1 - \nu}{2}\Big[\frac{3}{4}\beta^3 - \left(\frac{7}{4}\beta^3 +$$

$$\frac{9}{16}\beta^5\right)(3 - j) + \left(\frac{3}{4}\beta^3 + \frac{5}{16}\beta^5\right)(3 - j)^2\Big]\Big\}\Big\} b_{3-j} + \Big\{\bar{\lambda}^2\,\frac{5}{8}\beta^4 +$$

$$\frac{3(1 - \nu)}{2}\,\frac{1}{2}\beta^4(j + 4) - \frac{1 - \nu}{2}\,\frac{5}{8}\beta^4(j + 4)^2 - \frac{\gamma^2}{12}\Big\{\frac{1 - \nu}{2}\Big[\frac{1}{4}\beta^4 - \beta^4(j + 4) +$$

$$\frac{3}{8}\beta^4(j+4)^2\Big]\Big\}\Big\} b_{j+4} + \Big\{\bar{\lambda}^2 \frac{3}{8}\beta^4 - \frac{3(1-\nu)}{2}\frac{1}{2}\beta^4(j-4) -$$

$$\frac{1-\nu}{2}\frac{3}{8}\beta^4(j-4)^2 - \frac{1-\nu}{2}\frac{\chi^2}{12}\Big[\frac{1}{4}\beta^4 + \beta^4(j-4) + \frac{3}{8}\beta^4(j-4)^2\Big]\Big\} b_{j-4} -$$

$$\Big\{\bar{\lambda}^2 \frac{3}{8}\beta^4 + \frac{3(1-\nu)}{2}\frac{1}{2}\beta^4(4-j) - \frac{1-\nu}{2}\frac{3}{8}\beta^4(4-j)^2 - \frac{1-\nu}{2}\frac{\chi^2}{12}\Big[\frac{1}{4}\beta^4 -$$

$$\beta^4(4-j) + \frac{3}{8}\beta^4(4-j)^2\Big]\Big\} b_{4-j} + \Big\{\bar{\lambda}^2 \frac{1}{16}\beta^5 + \frac{3(1-\nu)}{2}\frac{1}{16}\beta^5(j+5) -$$

$$\frac{1-\nu}{2}\frac{1}{16}\beta^5(j+5)^2 + \frac{1-\nu}{2}\frac{\chi^2}{12}\Big[\frac{3}{16}\beta^5(j+5) - \frac{1}{16}\beta^5(j+5)^2\Big]\Big\} b_{j+5} +$$

$$\Big\{\bar{\lambda}^2 \frac{1}{16}\beta^5 - \frac{3(1-\nu)}{2}\frac{1}{16}\beta^5(j-5) - \frac{1-\nu}{2}\frac{1}{16}\beta^5(j-5)^2 -$$

$$\frac{1-\nu}{2}\frac{\chi^2}{12}\Big[\frac{3}{16}\beta^5(j-5) + \frac{1}{16}\beta^5(j-5)^2\Big]\Big\} b_{j-5} - \Big\{\bar{\lambda}^2 \frac{1}{16}\beta^5 +$$

$$\frac{3(1-\nu)}{2}\frac{1}{16}\beta^5(5-j) - \frac{1-\nu}{2}\frac{1}{16}\beta^5(5-j)^2 + \frac{1-\nu}{2}\frac{\chi^2}{12}\Big[\frac{3}{16}\beta^5(5-j) -$$

$$\frac{1}{16}\beta^5(5-j)^2\Big]\Big\} b_{5-j} - \Big\{\frac{1+\nu}{2}(2\beta^2 + 3\beta^4)nj + \frac{1+\nu}{2}\frac{\chi^2 n}{12}(2\beta^4 j)\Big\} c_j +$$

$$\Big\{\frac{3-\nu}{2}\Big(\beta^3 + \frac{1}{4}\beta^5\Big)n - \frac{1+\nu}{2}\Big(3\beta^3 + \frac{3}{4}\beta^5\Big)n(j+1) + \frac{\chi^2 n}{12}\Big[\frac{3-\nu}{2}\frac{1}{4}\beta^5 -$$

$$\frac{1+\nu}{2}\Big(\beta^3 + \frac{3}{4}\beta^5\Big)(j+1)\Big]\Big\} c_{j+1} - \Big\{\frac{3-\nu}{2}\Big(\beta^3 + \frac{1}{4}\beta^5\Big)n + \frac{1+\nu}{2}\Big(3\beta^3 +$$

$$\frac{3}{4}\beta^5\Big)n(j-1) + \frac{\chi^2 n}{12}\Big[\frac{3-\nu}{2}\frac{1}{4}\beta^5 + \frac{1+\nu}{2}\Big(\beta^3 + \frac{3}{4}\beta^5\Big)(j-1)\Big]\Big\} c_{j-1} +$$

$$\left\{\frac{3-\nu}{2}(n\beta^4) - \frac{1+\nu}{2}\frac{3}{2}n\beta^4(j+2) + \frac{\gamma^2 n}{12}\left[\frac{3-\nu}{2}\frac{\beta^4}{2} - \frac{1+\nu}{2}\beta^4(j+2)\right]\right\}c_{j+2} -$$

$$\left\{\frac{3-\nu}{2}(n\beta^4) + \frac{1+\nu}{2}\frac{3}{2}n\beta^4(j-2) + \frac{\gamma^2 n}{12}\left[\frac{3-\nu}{2}\frac{\beta^4}{2} + \frac{1+\nu}{2}\beta^4(j-2)\right]\right\}c_{j-2} -$$

$$\left\{\frac{3-\nu}{2}(n\beta^4) - \frac{1+\nu}{2}\frac{3}{2}n\beta^4(2-j) + \frac{\gamma^2 n}{12}\left[\frac{3-\nu}{2}\frac{\beta^4}{2} - \frac{1+\nu}{2}\beta^4(2-j)\right]\right\}c_{2-j} +$$

$$\left\{\frac{3-\nu}{2}\frac{1}{4}n\beta^5 - \frac{1+\nu}{2}\frac{1}{4}n\beta^5(j+3) + \frac{\gamma^2 n}{12}\left[\frac{3-\nu}{2}\frac{1}{4}\beta^5 -\right.\right.$$

$$\left.\left.\frac{1+\nu}{2}\frac{1}{4}\beta^5(j+3)\right]\right\}c_{j+3} - \left\{\frac{3-\nu}{2}\frac{1}{4}n\beta^5 + \frac{1+\nu}{2}\frac{1}{4}n\beta^5(j-3) +\right.$$

$$\left.\frac{\gamma^2 n}{12}\left[\frac{3-\nu}{2}\frac{1}{4}\beta^5 + \frac{1+\nu}{2}\frac{1}{4}\beta^5(j-3)\right]\right\}c_{j-3} - \left\{\frac{3-\nu}{2}\frac{1}{4}n\beta^5 -\right.$$

$$\left.\frac{1+\nu}{2}\frac{1}{4}n\beta^5(3-j) + \frac{\gamma^2 n}{12}\left[\frac{3-\nu}{2}\frac{1}{4}\beta^5 - \frac{1+\nu}{2}\frac{1}{4}\beta^5(3-j)\right]\right\}c_{3-j} +$$

$$\left\{2\nu n\beta^2 + n(2+3\nu)\beta^4 - \frac{\gamma^2 n}{12}\left[\frac{1-\nu}{2}(2\beta^4) - (2\beta^4)j^2\right]\right\}e_j + \left\{n(1+3\nu)\beta^3 +\right.$$

$$\frac{3}{4}(1+\nu)\beta^5 n + \frac{\gamma^2 n}{12}\left[\left(n^2 - \frac{3}{4} + \frac{3}{4}\nu\right)\beta^5 - \frac{1}{4}\beta^5(j+1) + \left(\beta^3 +\right.\right.$$

$$\left.\left.\left.\frac{3}{4}\beta^5\right)(j+1)^2\right]\right\}e_{j+1} + \left\{n(1+3\nu)\beta^3 + \frac{3}{4}(1+\nu)\beta^5 n + \frac{\gamma^2 n}{12}\left[\left(n^2 - \frac{3}{4} + \frac{3}{4}\nu\right)\beta^5 +\right.\right.$$

$$\left.\left.\frac{1}{4}\beta^5(j-1) + \left(\beta^3 + \frac{3}{4}\beta^5\right)(j-1)^2\right]\right\}e_{j-1} + \left\{n\left(1 + \frac{3}{2}\nu\right)\beta^4 - \frac{\gamma^2 n}{12}\left[\frac{1-\nu}{2}\beta^4 +\right.\right.$$

$$\left.\left.\frac{1}{2}\beta^4(j+2) - \beta^4(j+2)^2\right]\right\}e_{j+2} + \left\{n\left(1 + \frac{3}{2}\nu\right)\beta^4 - \frac{\gamma^2 n}{12}\left[\frac{1-\nu}{2}\beta^4 -\right.\right.$$

$$\frac{1}{2}\beta^4(j-2) - \beta^4(j-2)^2\Big]\Big\}a_{j-2} - \Big\{n\Big(1+\frac{3}{2}\nu\Big)\beta^4 - \frac{\gamma^2 n}{12}\Big[\frac{1-\nu}{2}\beta^4 +$$

$$\frac{1}{2}\beta^4(2-j) - \beta^4(2-j)^2\Big]\Big\}a_{2-j} + \Big\{\frac{n}{4}(1+\nu)\beta^5 - \frac{\gamma^2 n}{12}\Big[\frac{1-\nu}{2}\,\frac{1}{2}\beta^5 +$$

$$\frac{1}{4}\beta^5(j+3) - \frac{1}{4}\beta^5(j+3)^2\Big]\Big\}a_{j+3} + \Big\{\frac{n}{4}(1+\nu)\beta^5 - \frac{\gamma^2 n}{12}\Big[\frac{1-\nu}{2}\,\frac{1}{2}\beta^5 -$$

$$\frac{1}{4}\beta^5(j-3) - \frac{1}{4}\beta^5(j-3)^2\Big]\Big\}a_{j-3} - \Big\{\frac{n}{4}(1+\nu)\beta^5 - \frac{\gamma^2 n}{12}\Big[\frac{1-\nu}{2}\,\frac{1}{2}\beta^5 +$$

$$\frac{1}{4}\beta^5(3-j) - \frac{1}{4}\beta^5(3-j)^2\Big]\Big\}a_{3-j} - \frac{3-\nu}{2}\Big\{n\beta^3\Big[\Big(1+\frac{1}{4}\beta^2\Big)\delta_{j1} + \beta\delta_{j2} +$$

$$\frac{1}{4}\beta^2\delta_{j3}\Big] + \frac{\gamma^2 n}{12}\,\frac{\beta^4}{2}\Big[\frac{1}{2}\beta\delta_{j1} + \delta_{j2} + \frac{1}{2}\beta\delta_{j3}\Big]\Big\}c_0 = 0 \qquad (36a)$$

$$-\Big\{(1+\nu)\Big(1+\frac{3}{2}\beta^2\Big)nj + \frac{\gamma^2 n}{12}(1+\nu)\beta^2 j\Big\}b_j - \Big\{(1-\nu)\Big(\beta+\frac{1}{4}\beta^3\Big)n +$$

$$\frac{1+\nu}{2}\Big(3\beta+\frac{3}{4}\beta^3\Big)(j+1)n - \frac{\gamma^2 n}{12}\Big[\frac{1+\nu}{2}\beta - \frac{1-\nu}{4}\beta^3 -$$

$$\frac{1+\nu}{2}\Big(\beta+\frac{3}{4}\beta^3\Big)(j+1)\Big]\Big\}b_{j+1} - \Big\{-(1-\nu)\Big(\beta+\frac{1}{4}\beta^3\Big)n +$$

$$\frac{1+\nu}{2}\Big(3\beta+\frac{3}{4}\beta^3\Big)(j-1)n + \frac{\gamma^2 n}{12}\Big[\frac{1+\nu}{2}\beta - \frac{1-\nu}{4}\beta^3 + \frac{1+\nu}{2}\Big(\beta +$$

$$\frac{3}{4}\beta^3\Big)(j-1)\Big]\Big\}b_{j-1} - \Big\{(1-\nu)n\beta^2 + \frac{1+\nu}{2}\,\frac{3}{2}\beta^2(j+2)n + \frac{\gamma^2 n}{12}\Big[\frac{1-3\nu}{4}\beta^2 +$$

$$\frac{1+\nu}{2}\beta^2(j+2)\Big]\Big\}b_{j+2} - \Big\{-(1-\nu)n\beta^2 + \frac{1+\nu}{2}\frac{3}{2}\beta^2(j-2)n - \frac{\gamma^2 n}{12}\Big[\frac{1-3\nu}{4}\beta^2 -$$

$$\frac{1+\nu}{2}\beta^2(j-2)\Big]\Big\}b_{j-2} - \Big\{(1-\nu)n\beta^2 + \frac{1+\nu}{2}\frac{3}{2}\beta^2(2-j)n + \frac{\gamma^2 n}{12}\Big[\frac{1-3\nu}{4}\beta^2 +$$

$$\frac{1+\nu}{2}\beta^2(2-j)\Big]\Big\}b_{2-j} - \Big\{\frac{1-\nu}{4}n\beta^3 + \frac{1+\nu}{2}\frac{1}{4}\beta^3(j+3)n + \frac{\gamma^2 n}{12}\Big[\frac{1-\nu}{4}\beta^3 +$$

$$\frac{1+\nu}{2}\frac{1}{4}\beta^3(j+3)\Big]\Big\}b_{j+3} - \Big\{-\frac{1-\nu}{4}n\beta^3 + \frac{1+\nu}{2}\frac{1}{4}\beta^3(3-j)n -$$

$$\frac{\gamma^2 n}{12}\Big[\frac{1-\nu}{4}\beta^3 - \frac{1+\nu}{2}\frac{1}{4}\beta^3(j-3)\Big]\Big\}b_{j-3} - \Big\{\frac{1-\nu}{4}n\beta^3 + \frac{1+\nu}{2}\frac{1}{4}\beta^3(3-j)n +$$

$$\frac{\gamma^2 n}{12}\Big[\frac{1-\nu}{4}\beta^3 + \frac{1+\nu}{2}\frac{1}{4}\beta^3(3-j)\Big]\Big\}b_{3-j} + \Big\{\bar{\lambda}^2(2+3\beta^2) - \Big[n^2(1-\nu) + 1 +$$

$$2\nu\Big]\beta^2 - 2\Big(1+\frac{3}{2}\beta^2\Big)j^2\Big\}c_j + \Big\{\bar{\lambda}^2\Big(3\beta + \frac{3}{4}\beta^3\Big) - \Big[\frac{1}{4}(1+3\nu) + \frac{1-\nu}{2}n^2\Big]\beta^3 -$$

$$\nu\beta + \Big(\beta + \frac{1}{4}\beta^3\Big)(j+1) - \Big(3\beta + \frac{3}{4}\beta^3\Big)(j+1)^2\Big\}c_{j+1} + \Big\{\bar{\lambda}^2\Big(3\beta + \frac{3}{4}\beta^3\Big) -$$

$$\Big[\frac{1}{4}(1+3\nu) + \frac{1-\nu}{2}n^2\Big]\beta^3 - \nu\beta - \Big(\beta + \frac{1}{4}\beta^3\Big)(j-1) - \Big(3\beta + \frac{3}{4}\beta^3\Big)(j-1)^2\Big\}c_{j-1} +$$

$$\Big\{\bar{\lambda}^2\frac{3}{2}\beta^2 + \Big(\frac{1}{2}-\nu\Big)\beta^2 + \beta^2(j+2) - \frac{3}{2}\beta^2(j+2)^2\Big\}c_{j+2} + \Big\{\bar{\lambda}^2\frac{3}{2}\beta^2 +$$

$$\Big(\frac{1}{2}-\nu\Big)\beta^2 - \beta^2(j-2) - \frac{3}{2}\beta^2(j-2)^2\Big\}c_{j-2} + \Big\{\bar{\lambda}^2\frac{3}{2}\beta^2 + \Big(\frac{1}{2}-\nu\Big)\beta^2 -$$

$$\beta^2(2-j) - \frac{3}{2}\beta^2(2-j)^2\Big\}c_{2-j} + \Big\{\bar{\lambda}^2\frac{1}{4}\beta^3 + \frac{1}{4}(1-\nu)\beta^3 + \frac{1}{4}\beta^3(j+3) -$$

$$\frac{1}{4}\beta^3(j+3)^2\Big\}c_{j+3} + \Big\{\bar{\lambda}^2\frac{1}{4} + \frac{1}{4}(1-\nu) - \frac{1}{4}(j-3) - \frac{1}{4}(j-3)^2\Big\}\beta^3 c_{j-3} +$$

$$\Big\{\bar{\lambda}^2\frac{1}{4} + \frac{1}{4}(1-\nu) - \frac{1}{4}(3-j) - \frac{1}{4}(3-j)^2\Big\}\beta^3 c_{3-j} + \Big\{\Big[2 + (3+2\nu)\beta^2\Big]j +$$

$$\frac{\gamma^2}{12}\Big[(1+2n^2)\beta^2 j + (2+3\beta^2)j^3\Big\}\varepsilon_j - \Big\{\beta - \Big[(3+\nu)\beta + \frac{3}{4}(1+\nu)\beta^3\Big](j+1) -$$

$$\frac{\gamma^2}{12}\Big[2n^2\beta^3 + \frac{1}{4}(1+4n^2)\beta^3(j+1) - \Big(\beta + \frac{1}{4}\beta^3\Big)(j+1)^2 + \Big(3\beta + \frac{3}{4}\beta^3\Big)(j+$$

$$1)^3\Big]\Big\}\varepsilon_{j+1} + \Big\{\beta + \Big[(3+\nu)\beta + \frac{3}{4}(1+\nu)\beta^2\Big](j-1) + \frac{\gamma^2}{12}\Big[-2n^2\beta^3 + \frac{1}{4}(1+4n^2)(j-1) +$$

$$\Big(\beta + \frac{1}{4}\beta^3\Big)(j-1)^2 + \Big(3\beta + \frac{3}{4}\beta^3\Big)(j-1)^3\Big]\Big\}\varepsilon_{j-1} + \Big\{-\frac{1}{2}\beta^2 + \Big(\frac{3}{2}+\nu\Big)\beta^2(j+2) +$$

$$\frac{\gamma^2}{12}\Big[-\frac{1}{2}\beta^2(j+2) - \beta^2(j+2)^2 + \frac{3}{2}\beta^2(j+2)^3\Big]\Big\}\varepsilon_{j+2} + \Big\{\frac{1}{2}\beta^2 + \Big(\frac{3}{2}+\nu\Big)\beta^2(j-2) +$$

$$\frac{\gamma^2}{12}\Big[-\frac{1}{2}\beta^2(j-2) + \beta^2(j-2)^2 + \frac{3}{2}\beta^2(j-2)^3\Big]\Big\}\varepsilon_{j-2} + \Big\{-\frac{1}{2}\beta^2 +$$

$$\Big(\frac{3}{2}+\nu\Big)\beta^2(2-j) + \frac{\gamma^2}{12}\Big[-\frac{1}{2}\beta^2(2-j) - \beta^2(2-j)^2 + \frac{3}{2}\beta^2(2-j)^3\Big]\Big\}\varepsilon_{2-j} +$$

$$\Big\{\frac{1+\nu}{4}\beta^3(j+3) + \frac{\gamma^2}{12}\Big[-\frac{1}{4}\beta^3(j+3) - \frac{1}{4}\beta^3(j+3)^2 + \frac{1}{4}\beta^3(j+3)^3\Big]\Big\}\varepsilon_{j+3} +$$

$$\Big\{\frac{1+\nu}{4}\beta^3(j-3) + \frac{\gamma^2}{12}\Big[-\frac{1}{4}\beta^3(j-3) + \frac{1}{4}\beta^3(j+3)^2 + \frac{1}{4}\beta^3(j-3)^3\Big]\Big\}\varepsilon_{j-3} +$$

$$\Big\{\frac{1+\nu}{4}\beta^3(3-j) + \frac{\gamma^2}{12}\Big[-\frac{1}{4}\beta^3(3-j) - \frac{1}{4}\beta^3(3-j)^2 + \frac{1}{4}\beta^3(3-j)^3\Big]\Big\}\varepsilon_{3-j} +$$

$$\left\{\left[\overline{\lambda}^2\left(3\beta + \frac{3}{4}\beta^3\right) - \frac{1-\nu}{2}n^2\beta^3 - \frac{1}{4}\beta^3 - \nu\beta\left(1 + \frac{3}{4}\beta^2\right)\right]\delta_{j1} + \left[\frac{3}{2}\beta^2\overline{\lambda}^2 + \frac{1}{2}\beta^2 - \nu\beta^2\right]\delta_{j2} + \left[\frac{1}{4}\beta^3\overline{\lambda}^2 + \frac{1}{4}\beta^3 - \frac{1}{4}\nu\beta^3\right]\delta_{j3}\right\}c_o = 0 \tag{36b}$$

$$-\left\{(1-\nu)n\beta\left(1 + \frac{1}{4}\beta^2\right) + \frac{1+\nu}{2}(3n\beta)\left(1 + \frac{1}{4}\beta^2\right) + \frac{\chi^2 n}{12}\left[\frac{1+\nu}{2}\frac{1}{2}\beta^3 + \frac{3-\nu}{2}\frac{1}{4}\beta^3\right]\right\}b_1 - \left\{(1-\nu)n\beta^2 + \frac{1+\nu}{2}(3n\beta^2) + \frac{\chi^2 n}{12}\left[\frac{1+\nu}{2}\beta^2 + \frac{3-\nu}{2}\frac{1}{2}\beta^2\right]\right\}b_2 - \left\{(1-\nu)\frac{1}{4}n\beta^3 + \frac{1+\nu}{2}\frac{3}{4}n\beta^2 + \frac{\chi^2 n}{12}\left[\frac{1+\nu}{2}\frac{1}{2}\beta^3 + \frac{3-\nu}{2}\frac{1}{4}\beta^3\right]\right\}b_3 + \left\{\overline{\lambda}^2\left(1 + \frac{3}{2}\beta^2\right) - \frac{1+2\nu}{2}\beta^2 - \frac{1-\nu}{2}n^2\beta^2\right\}c_o + \left\{\overline{\lambda}^2(3\beta)\left(1 + \frac{1}{4}\beta^2\right) - (2+\nu)\beta - \frac{1-\nu}{2}n^2\beta^3 - \frac{3}{4}(1+\nu)\beta^3\right\}c_1 + \left\{\overline{\lambda}^2\frac{3}{2}\beta^2 - \frac{7}{2}\beta^2 - \nu\beta^2\right\}c_2 + \left\{\overline{\lambda}^2\frac{1}{4}\beta^3 - \frac{5+\nu}{4}\beta^3\right\}c_3 + \left\{(2+\nu)\beta + \frac{3}{4}(1+\nu)\beta^3 + \frac{\chi^2}{12}\left[2\beta + 3\left(n^2 + \frac{1}{4}\right)\beta^3\right]\right\}e_1 + \left\{\frac{5}{2}\beta^2 + 2\nu\beta^2 + \frac{\chi^2}{12}7\beta^2\right\}e_2 + \left\{\frac{3}{4}(1+\nu)\beta^3 + \frac{\chi^2}{12}\frac{15}{4}\beta^3\right\}e_3 = 0 \tag{36c}$$

$$\left\{n\left[2\nu + 3(1 + 2\nu)\beta^2 + \frac{3}{4}(1 + \nu)\beta^4\right] + \frac{\chi^2 n}{12}\left[(\nu - 1)\beta^2 + \left(n^2 - \frac{3}{2} + \frac{3}{4}\nu\right)\beta^4 + \right.\right.$$

$$\left.\left.\left(3\beta^2 + \frac{3}{4}\beta^4\right)j^2\right]\right\}b_j + \left\{n\left[(1 + 4\nu)\beta + 3\left(\frac{3}{4} + \nu\right)\beta^3\right] + \frac{\chi^2 n}{12}\left[\left(n^2 - \frac{11}{4} + \frac{3\nu}{2}\right)\beta^3 + \right.\right.$$

$$\left.\left.(2\beta + \beta^3)(j + 1) - \left(\beta + \frac{9}{4}\beta^3\right)(j + 1)^2\right]\right\}b_{j+1} + \left\{n\left[(1 + 4\nu)\beta + 3\left(\frac{3}{4} + \nu\right)\beta^3\right] + \right.$$

$$\left.\frac{\chi^2 n}{12}\left[\left(n^2 - \frac{11}{4} + \frac{3\nu}{2}\right)\beta^3 - (2\beta + \beta^3)(j - 1) - \left(\beta + \frac{9}{4}\beta^3\right)(j - 1)^2\right]\right\}b_{j-1} +$$

$$\left\{n\left[3\left(\frac{1}{2} + \nu\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right] + \frac{\chi^2 n}{12}\left[\frac{\nu - 3}{2}\beta^2 + \left(\frac{n^2}{2} - 1 + \frac{\nu}{2}\right)\beta^4 + \left(\frac{9}{2}\beta^2 + \right.\right.\right.$$

$$\left.\left.\left.\frac{1}{4}\beta^4\right)(j + 2) - \left(\frac{3}{2}\beta^2 + \frac{1}{2}\beta^4\right)(j + 2)^2\right]\right\}b_{j+2} + \left\{n\left[3\left(\frac{1}{2} + \nu\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right] + \right.$$

$$\left.\frac{\chi^2 n}{12}\left[\frac{\nu - 3}{2}\beta^2 + \left(\frac{n^2}{2} - 1 + \frac{\nu}{2}\right)\beta^4 - \left(\frac{9}{2}\beta^2 + \frac{1}{4}\beta^4\right)(j - 2) - \left(\frac{3}{2}\beta^2 + \right.\right.\right.$$

$$\left.\left.\left.\frac{1}{2}\beta^4\right)(j - 2)^2\right]\right\}b_{j-2} - \left\{n\left[3\left(\frac{1}{2} + \nu\right)\beta^2 + \frac{1 + \nu}{2}\beta^4\right] + \frac{\chi^2 n}{12}\left[\frac{\nu - 3}{2}\beta^2 + \right.\right.$$

$$\left.\left.\left(\frac{n^2}{2} - 1 + \frac{\nu}{2}\right)\beta^4 + \left(\frac{9}{2}\beta^2 + \frac{1}{4}\beta^4\right)(2 - j) - \left(\frac{3}{2}\beta^2 + \frac{1}{2}\beta^4\right)(2 - j)^2\right]\right\}b_{2-j} +$$

$$\left\{n\left(\frac{3}{4} + \nu\right) + \frac{\chi^2 n}{12}\left[-\frac{1}{2}\left(\frac{9}{2} - \nu\right) + (j + 3) - \frac{3}{4}(j + 3)^2\right]\right\}\beta^3 b_{j+3} + \left\{n\left(\frac{3}{4} + \nu\right) + \right.$$

$$\left.\frac{\chi^2 n}{12}\left[-\frac{1}{2}\left(\frac{9}{2} - \nu\right) - (j - 3) - \frac{3}{4}(j - 3)^2\right]\right\}\beta^3 b_{j-3} - \left\{n\left(\frac{3}{4} + \nu\right) + \frac{\chi^2 n}{12}\left[-\frac{1}{2}\left(\frac{9}{2} - \nu\right) + \right.\right.$$

$$\left.\left.(3 - j) - \frac{3}{4}(3 - j)^2\right]\right\}\beta^3 b_{3-j} + \left\{\frac{n}{8}(1 + \nu) + \frac{\chi^2 n}{12}\left[-\frac{2 - \nu}{8} + \frac{1}{8}(j + 4) - \right.\right.$$

$$\frac{1}{8}(j + 4)^2\Big]\Big\}\beta^4 b_{j+4} + \Big\{\frac{n}{8}(1 + \nu) + \frac{\gamma^2 n}{12}\Big[-\frac{2 - \nu}{8} - \frac{1}{8}(j - 4) - \frac{1}{8}(j - 4)^2\Big]\Big\}\beta^4 b_{j-4} -$$

$$\Big\{\frac{n}{8}(1 + \nu) + \frac{\gamma^2 n}{12}\Big[-\frac{2 - \nu}{8} + \frac{1}{8}(4 - j) - \frac{1}{8}(4 - j)^2\Big]\Big\}\beta^4 b_{4-j} + \Big\{\Big[2 + 3(2 + \nu)\beta^2 +$$

$$\frac{3}{4}(1 + \nu)\beta^4\Big]j + \frac{\gamma^2}{12}\Big\{\Big[(4 + 2n^2)\beta^2 + (1 + n^2)\beta^4\Big]j + \Big[2 + 6\beta^2 +$$

$$\frac{3}{4}\beta^4 j^3\Big]\Big\}\Big\}c_j + \Big\{\nu\beta + \Big(1 + \frac{3}{2}\nu\Big)\frac{\beta^2}{2} + \Big[(4 + \nu)\beta + 3\Big(1 + \frac{3\nu}{4}\Big)\beta^3\Big](j + 1) +$$

$$\frac{\gamma^2}{12}\Big\{\Big(n^2 - \frac{3}{4}\Big)\beta^3 + \Big[\beta + \Big(\frac{11}{4} + 2n^2\Big)\beta^3\Big](j + 1) + \Big[2\beta + \frac{3}{2}\beta^3\Big](j + 1)^2 +$$

$$(4\beta + 3\beta^3)(j + 1)^3\Big\}\Big\}c_{j+1} - \Big\{\nu\beta + \frac{1}{2}\Big(1 + \frac{3\nu}{2}\Big)\beta^3 - \Big[(4 + \nu)\beta +$$

$$3\Big(1 + \frac{3\nu}{4}\Big)\beta^3\Big](j - 1) + \frac{\gamma^2}{12}\Big\{\Big(n^2 - \frac{3}{4}\Big)\beta^3 - \Big[\beta + \Big(\frac{11}{4} + 2n^2\Big)\beta^3\Big](j - 1) +$$

$$\Big(2\beta + \frac{3}{2}\beta^3\Big)(j - 1)^2 - (4\beta + 3\beta^3)(j - 1)^3\Big\}\Big\}c_{j-1} + \Big\{\frac{1 + 3\nu}{2}\beta^2 +$$

$$\frac{1 + \nu}{4}\beta^4 + \Big[3\Big(1 + \frac{\nu}{2}\Big)\beta^2 + \frac{1 + \nu}{2}\beta^4\Big](j + 2) + \frac{\gamma^2}{12}\Big\{\Big(\frac{n^2}{2} - \frac{1}{4}\Big)\beta^4 - \beta^2 +$$

$$\Big[\beta^2 + \frac{1}{2}(1 + n^2)\beta^4\Big](j + 2) + \Big(3\beta^2 + \frac{1}{2}\beta^4\Big)\Big[(j + 2)^2 + (j + 2)^3\Big]\Big\}\Big\}c_{j+2} -$$

$$\Big\{\frac{1 + 3\nu}{2}\beta^2 + \frac{1 + \nu}{4}\beta^4 - \Big[3\Big(1 + \frac{\nu}{2}\Big)\beta^2 + \frac{1 + \nu}{2}\beta^4\Big](j - 2) + \frac{\gamma^2}{12}\Big\{\Big(\frac{n^2}{2} - \frac{1}{4}\Big)\beta^4 -$$

$$\Big[\beta^2 + \frac{1}{2}(1 + n^2)\beta^4\Big](j - 2) + \Big(3\beta^2 + \frac{1}{2}\beta^4\Big)\Big[(j - 2)^2 - (j - 2)^3\Big]\Big\}\Big\}c_{j-2} +$$

$$\left\{-\frac{1+3\nu}{2}\beta^2-\frac{1+\nu}{4}\beta^4-\left[3\left(1+\frac{\nu}{2}\right)\beta^2+\frac{1+\nu}{2}\beta^4\right](2-j)+\frac{\chi^2}{12}\left\{-\left(\frac{n^2}{2}-\right.\right.\right.$$

$$\left.\left.\frac{1}{4}\right)\beta^4-\left[\beta^2+\frac{1}{2}(1+n^2)\beta^4\right](2-j)-\left(3\beta^2+\frac{1}{2}\beta^4\right)\left[(2-j)^2-(2-j)^3\right]\right\}\right\}c_{2-j}+$$

$$\left\{+\frac{1}{2}\left(1+\frac{3\nu}{2}\right)+\left(1+\frac{3\nu}{4}\right)(j+3)+\frac{\chi^2}{12}\left[\frac{1}{4}+\frac{1}{4}(j+3)+\frac{3}{2}(j+3)^2+\right.\right.$$

$$\left.\left.(j+3)^3\right]\right\}\beta^3c_{j+3}-\left\{\frac{1}{2}\left(1+\frac{3\nu}{2}\right)+\left(1+\frac{3\nu}{4}\right)(j-3)+\frac{\chi^2}{12}\left[\frac{1}{4}-\frac{1}{4}(j-3)+\right.\right.$$

$$\left.\left.\frac{3}{2}(j-3)^2-(j-3)^3\right]\right\}\beta^3c_{j-3}+\left\{-\frac{1}{2}\left(1+\frac{3\nu}{2}\right)-\left(1+\frac{3\nu}{4}\right)(3-j)+\right.$$

$$\left.\frac{\chi^2}{12}\left[-\frac{1}{4}-\frac{1}{4}(3-j)-\frac{3}{2}(3-j)^2-(3-j)^3\right]\right\}\beta^3c_{3-j}+\left\{\frac{1+\nu}{8}+\frac{1+\nu}{8}(j+4)+\right.$$

$$\left.\frac{\chi^2}{12}\left[\frac{1}{8}+\frac{1}{4}(j+4)^2+\frac{1}{8}(j+4)^3\right]\right\}\beta^4c_{j+4}-\left\{\frac{1+\nu}{8}-\frac{1+\nu}{8}(j-4)+\frac{\chi^2}{12}\left[\frac{1}{8}+\right.\right.$$

$$\left.\left.\frac{1}{4}(j-4)^2-\frac{1}{8}(j-4)^3\right]\right\}\beta^4c_{j-4}+\left\{-\frac{1+\nu}{8}-\frac{1+\nu}{8}(4-j)+\frac{\chi^2}{12}\left[-\frac{1}{8}-\right.\right.$$

$$\left.\left.\frac{1}{4}(4-j)^2-\frac{1}{8}(4-j)^3\right]\right\}\beta^4c_{4-j}-\left\{2+(7+6\nu)\beta^2+\frac{3}{2}(1+\nu)\beta^4-\right.$$

$$\bar{\lambda}^2\left(2+6\beta^2+\frac{3}{4}\beta^4\right)+\frac{\chi^2}{12}\left\{n^2(2n^2-7-\nu)\beta^4+\left[(4+4n^2-3\nu)\beta^2+\right.\right.$$

$$\left.\left.\left(1+2n^2+\frac{3\nu}{4}\right)\beta^4\right]j^2+\left[2+6\beta^2+\frac{3}{4}\beta^4\right]j^4\right\}\Bigg\}c_j-\left\{\frac{3}{2}(1+\nu)\beta^3+2(2+\nu)\beta-\right.$$

$$\bar{\lambda}^2(4\beta+3\beta^3)+\frac{\chi^2}{12}\left\{-n^2(3+\nu)\beta^3-\left[\left(\frac{7}{4}+2n^2-\frac{3\nu}{4}\right)\beta^3-\nu\beta\right](j+1)+\right.$$

$$\left[(1+\nu)\beta+\left(\frac{11}{4}+4n^2+\frac{9\nu}{4}\right)\beta^3\right](j+1)^2+\left(2\beta+\frac{3}{2}\beta^3\right)(j+1)^3+\left(4\beta+\right.$$

$$\left.3\beta^3\right)(j+1)^4\Bigg\}\Bigg]\varepsilon_{j+1}-\Bigg\{\frac{9}{2}(1+\nu)\beta^3+2(2+\nu)\beta-\bar{\lambda}^2(4\beta+3\beta^3)+$$

$$\frac{\gamma^2}{12}\Bigg\{-n^2(3+\nu)\beta^3+\left[\left(\frac{7}{4}+2n^2-\frac{3\nu}{4}\right)\beta^3-\nu\beta\right](j-1)+\left[(1+\nu)\beta+\right.$$

$$\left.\left(\frac{11}{4}+4n^2+\frac{9\nu}{4}\right)\beta^3\right](j-1)^2-\left(2\beta+\frac{3}{2}\beta^3\right)(j-1)^3+\left(4\beta+3\beta^3\right)(j-4)^4\Bigg\}\Bigg\}\varepsilon_{j-1}-$$

$$\Bigg\{\frac{7+6\nu}{2}\beta^2+(1+\nu)\beta^4-\bar{\lambda}^2\left(3\beta^2+\frac{1}{2}\beta^4\right)+\frac{\gamma^2}{12}\Bigg\{\frac{1-\nu}{2}n^2\beta^4-\Bigg[\left(1-\frac{3\nu}{2}\right)\beta^2+$$

$$\left(\frac{3}{4}-\frac{\nu}{4}+n^2\right)\beta^4\Bigg](j+2)+\Bigg[\left(1+\frac{3\nu}{2}\right)\beta^2+\left(\frac{1}{2}+n^2+\frac{\nu}{2}\right)\beta^4\Bigg](j+2)^2+$$

$$\left(3\beta^2+\frac{1}{2}\beta^4\right)\Bigg[(j+2)^3+(j+2)^4\Bigg]\Bigg\}\Bigg\}\varepsilon_{j+2}-\Bigg\{\frac{7+6\nu}{2}\beta^2+(1+\nu)\beta^4-$$

$$\bar{\lambda}^2\left(3\beta^2+\frac{1}{2}\beta^4\right)+\frac{\gamma^2}{12}\Bigg\{\frac{1-\nu}{2}n^2\beta^4+\Bigg[\left(1-\frac{3\nu}{2}\right)\beta^2+\left(\frac{3-\nu}{4}+n^2\right)\beta^4\Bigg](j-2)+$$

$$\Bigg[\left(1+\frac{3\nu}{2}\right)\beta^2+\left(\frac{1+\nu}{2}+n^2\right)\beta^4\Bigg](j-2)^2-\Bigg[3\beta^2+\frac{1}{2}\beta^4\Bigg]\Bigg[(j-2)^3-$$

$$(j-2)^4\Bigg]\Bigg\}\Bigg\}\varepsilon_{j-2}+\Bigg\{\frac{7+6\nu}{2}\beta^2+(1+\nu)\beta^4-\bar{\lambda}^2\left(3\beta^2+\frac{1}{2}\beta^4\right)+$$

$$\frac{\gamma^2}{12}\Bigg\{\frac{1-\nu}{2}n^2\beta^4-\Bigg[\left(1-\frac{3\nu}{2}\right)\beta^2+\left(\frac{3-\nu}{4}+n^2\right)\beta^4\Bigg](2-j)+\Bigg[\left(1+\frac{3\nu}{2}\right)\beta^2+$$

$$\left(\frac{1+\nu}{2}+n^2\right)\beta^4\Bigg](2-j)^2+\left(3\beta^2+\frac{1}{2}\beta^4\right)\Bigg[(2-j)^3+(2-j)^4\Bigg]\Bigg\}\Bigg\}\varepsilon_{2-j}-$$

$$\left\{\frac{3}{2}(1+\nu) - \bar{\lambda}^2 + \frac{\gamma^2}{12}\left[-\frac{3}{4}(1-\nu)(j+3) + \frac{1+3\nu}{4}(j+3)^2 + \frac{3}{2}(j+3)^3 + (j+3)^4\right]\right\}\beta^3 c_{j+3} - \left\{\frac{3}{2}(1+\nu) - \bar{\lambda}^2 + \frac{\gamma^2}{12}\left[\frac{3}{4}(1-\nu)(j-3) + \frac{1+3\nu}{4}(j-3)^2 - \frac{3}{2}(j-3)^3 + (j-3)^4\right]\right\}\beta^3 c_{j-3} - \left\{-\frac{3}{2}(1+\nu) + \bar{\lambda}^2 + \frac{\gamma^2}{12}\left[\frac{3}{4}(1-\nu)(3-j) - \frac{1+3\nu}{4}(3-j)^2 - \frac{3}{2}(3-j)^3 - (3-j)^4\right]\right\}\beta^3 c_{3-j} - \left\{\frac{1+\nu}{4} - \frac{1}{8}\bar{\lambda}^2 + \frac{\gamma^2}{12}\left[-\frac{1-\nu}{8}(j+4) + \frac{\nu}{8}(j+4)^2 + \frac{1}{4}(j+4)^3 + \frac{1}{8}(j+4)^4\right]\right\}\beta^4 c_{j+4} - \left\{\frac{1+\nu}{4} - \frac{1}{8}\bar{\lambda}^2 + \frac{\gamma^2}{12}\left[\frac{1-\nu}{8}(j-4) + \frac{\nu}{8}(j-4)^2 - \frac{1}{4}(j-4)^3 + \frac{1}{8}(j-4)^4\right]\right\}\beta^4 c_{j-4} - \left\{-\frac{1+\nu}{4} + \frac{1}{8}\bar{\lambda}^2 + \frac{\gamma^2}{12}\left[\frac{1-\nu}{8}(4-j) - \frac{\nu}{8}(4-j)^2 - \frac{1}{4}(4-j)^3 - \frac{1}{8}(4-j)^4\right]\right\}\beta^4 c_{4-j} + \left\{\left[\frac{1}{2}\beta^3 + \nu\beta\left(1 + \frac{3}{4}\beta^2\right) + \frac{\gamma^2}{12}\left(n^2\beta^3 - \frac{3}{4}\beta^3\right)\right]\delta_{j1} + \left[\frac{1}{2}\beta^2\left(1 + \frac{1}{2}\beta^2\right) + \nu\beta\left(\frac{3}{2}\beta + \frac{1}{4}\beta^3\right) + \frac{\gamma^2}{12}\left(\frac{1}{2}n^2\beta^4 - \frac{1}{16}\beta^4\right)\right]\delta_{j2} + \left[\frac{1}{2}\beta^3 + \frac{3}{4}\nu\beta^3 + \frac{\gamma^2}{12}\frac{1}{4}\beta^3\right]\delta_{j3} + \left[\frac{1}{8}\beta^4 + \frac{1}{8}\nu\beta^4 + \frac{\gamma^2}{12}\frac{1}{8}\beta^4\right]\delta_{j4}\right\}c_o = 0 \qquad (36d)$$

The vibration of the toroidal shell is completely defined by equations (34) and (36) within the limits imposed by thin shell theory and Love's first approximation. It is seen that these equations are dependent on three parameters; two of the parameters, β and γ, depend

on the geometry of the shell and the third parameter, n, depends on the number of complete waves in the ξ_1 direction.

VIII. NUMERICAL EXAMPLES

1. Scope of Calculations

Although the natural mode shapes and frequencies are defined by equations (27), (29), (34), and (36), very little insight as to the nature of the motion in each mode can be gained by direct examination of these equations. In this section the results of numerical calculations are presented for several values of the parameters β and γ with $n = 1$ and these results are used to show graphically the nature of the lowest mode of vibration of the toroidal shell. These calculated results are intended to illustrate the nature of the vibrations and do not include all permissible values of the shell parameters, nor are the results intended to cover all possible shell configurations or modes of vibration.

The equations (34) for the symmetrical vibrations were used for the calculations of the numerical examples. Solutions to these equations were obtained by use of the IBM 704 computer at the Langley Research Center of the National Aeronautics and Space Administration. At the time the computer was available it was somewhat limited in storage capacity due, in part, to a change over in equipment and relocation of the facilities. The nature of equations (34) with the frequency parameter appearing in off-diagonal elements does not permit the use of many standard techniques for determining the roots of the frequency equation by digital computers so that a major factor to be considered in setting up the problem was the largest determinant that

would be expected to give satisfactory results using single precision of eight digits plus sign and decimal point. In view of the considerations mentioned above, equations (34) were limited to 30 simultaneous algebraic equations for the unknown coefficients of the first 10 terms of each of the series for $\bar{u}$, $\bar{v}$, and $\bar{w}$ in equations (13).

2. Calculated Results

Solutions were obtained for the lowest three modes of vibration of the toroidal shell for $\beta = 0.1$, 0.3, and 0.5 and for $\gamma = 0.010$, 0.005, and 0.001. Calculated values of the 30 series coefficients for the first two modes of vibrations are presented in table 1 through 5 along with the corresponding values of the frequency parameters. The frequency parameters for the lowest three modes are shown in table 6 where the effect of the shell parameters β and γ on the frequencies is illustrated. The series coefficients in part (a) of tables 1 through 5 have been used to calculate the displacements u_1, v_1, and w_1 corresponding to the lowest mode of vibration and the results are shown in figures 2 through 6. In these figures the displacements are given in the form of $u_1/u_{1_{max}}$, $v_1/v_{1_{max}}$, and $w_1/w_{1_{max}}$; the ratio of the maximum value of v_1 and w_1 to the maximum value of u_1 is shown also.

3. Discussion of Calculated Results

Examination of the numerical values of the series coefficients for tables 1 through 5 shows that 10 terms of the series for each displacement are not sufficient to insure adequate convergence of the

TABLE 1. SERIES COEFFICIENTS FOR $\beta = 0.1$ AND $\gamma = 0.01$

(All coefficients normalized to $a_0 = 1$)

(a) First mode, $\lambda_1 = 0.00653488$

j	a_j	d_j	f_j
0	1.000000	0	0.8301
1	-0.270404	1.07506	1.1578
2	.348133	31.68810	-105.5288
3	-.174878	-31.79138	99.0859
4	.096334	32.54057	-130.7001
5	.005260	7.41819	-36.0064
6	-.010266	-5.21954	31.6756
7	.001074	-.31669	1.7880
8	.000264	.26357	-2.1050
9	-.000056	-.00138	.0427

(b) Second mode, $\lambda_2 = 0.0097105$

j	a_j	d_j	f_j
0	1.000000	0	0.8355
1	-0.191467	2.08309	-.7462
2	.301456	35.27024	-73.8707
3	-.261256	-51.23002	158.5717
4	.185844	48.24114	-199.4461
5	-.108875	-44.04753	222.2525
6	-.000171	-8.08355	45.9840
7	.007667	4.59862	-32.6224
8	-.000938	.19411	-1.0801
9	-.000093	-.16350	1.4319

TABLE 2. SERIES COEFFICIENTS FOR $\beta = 0.3$ AND $\gamma = 0.01$

(All coefficients normalized to $a_0 = 1$)

(a) First mode, $\lambda_1 = 0.0359960$

j	a_j	d_j	f_j
0	1.000000	0	0.73466
1	-0.932288	-.393176	.65495
2	.606546	5.883020	-12.17018
3	-.229336	-2.463398	7.81600
4	.032587	.190637	-.73485
5	.033576	1.240300	-6.46717
6	-.025038	-.795916	4.97514
7	.002702	-.287753	1.88073
8	.003408	.259115	-2.11294
9	-.001154	.050653	-.03556

(b) Second mode, $\lambda_2 = 0.0409868$

j	a_j	d_j	f_j
0	1.000000	0	0.77930
1	-0.949982	-0.420533	.53198
2	.658085	5.326667	-11.60001
3	-.294148	-3.501887	11.04427
4	.066846	.581751	-2.50953
5	.024884	.964837	-5.10986
6	-.031813	-1.160867	7.22543
7	.008678	-.051118	.15954
8	.002533	.364787	-2.88737
9	-.001466	.013078	-.02273

TABLE 3. SERIES COEFFICIENTS FOR $\beta = 0.5$ AND $\gamma = 0.01$

(All coefficients normalized to $a_0 = 1$)

(a) First mode, $\lambda_1 = 0.0572017$

j	a_j	d_j	f_j
0	1.000000	0	0.457151
1	-1.095887	-.485019	.245274
2	.720392	1.436450	-2.878356
3	-.325757	-.734418	2.295355
4	.078239	-.016673	.080680
5	.014909	.343747	-1.804583
6	-.018347	-.118993	.795210
7	.002670	-.114372	.761707
8	.002536	.053065	-.453162
9	-.001071	.028748	-.214107

(b) Second mode, $\lambda_2 = 0.0817285$

j	a_j	d_j	f_j
0	1.000000	0	0.528608
1	-1.185486	-.596696	.147259
2	0.873386	1.211924	-2.428027
3	-.484396	-.121348	3.786317
4	.169714	.199542	-.919375
5	-.009629	.267578	-1.426770
6	-.028756	-.302492	1.952498
7	.014712	-.003982	-.059009
8	-.000973	.144045	-1.112153
9	-.000902	.017886	-.100245

TABLE 4. SERIES COEFFICIENTS FOR $\beta = 0.1$ AND $\gamma = 0.005$

(All coefficients normalized to $a_0 = 1$)

(a) First mode, $\lambda_1 = 0.00500112$

j	a_j	d_j	f_j
0	1.000000	0	0.8819
1	-0.334829	.95723	2.0712
2	.395362	66.51034	-134.2790
3	-.109320	-17.00030	53.4088
4	.040960	14.51474	-37.9888
5	-.018788	8.18315	-41.7277
6	-.024574	-13.09621	79.0431
7	.001167	-1.20906	7.7178
8	.003291	2.57021	-20.6188
9	-.000508	.06904	-.3182

(b) Second mode, $\lambda_2 = 0.00598213$

j	a_j	d_j	f_j
0	1.000000	0	0.9295
1	-0.322718	1.12109	1.3475
2	.433435	59.05885	-121.9444
3	-.359880	-56.87731	173.8010
4	.096429	22.30288	-92.8755
5	-.031252	-15.36049	75.9704
6	-.032485	-19.21464	115.6501
7	.018388	9.74161	-69.3420
8	.001818	3.35312	-25.9762
9	-.001914	-1.30183	12.0105

TABLE 5. SERIES COEFFICIENTS FOR $\beta = 0.1$ AND $\gamma = 0.001$

(All coefficients normalized to $a_0 = 1$)

(a) First mode, $\lambda_1 = 0.00352047$

j	a_j	d_j	f_j
0	1.000000	0	0.8671
1	-0.370761	.73426	3.0203
2	.388086	77.84006	-154.9957
3	.036973	13.85026	-42.3842
4	-.067592	-25.02042	100.2760
5	.006438	-3.86080	16.9099
6	-.011888	-9.48191	55.7327
7	-.014144	-7.80273	55.9138
8	.017168	13.33750	-106.5309
9	.004370	6.81181	-59.6593

(b) Second mode, $\lambda_2 = 0.00379588$

j	a_j	d_j	f_j
0	1.000000	0	0.9638
1	-0.390034	.68014	2.5619
2	.482307	73.43231	-150.3405
3	-.178414	-49.03286	146.2042
4	-.062473	-28.76626	114.6836
5	.040996	11.14373	-58.7432
6	-.031579	-13.26121	81.8414
7	.011335	10.16534	-69.4289
8	.025644	20.04016	-160.3992
9	-.011438	-6.42446	60.0424

TABLE 6. FREQUENCY PARAMETER

(a) Effect of β on frequency parameter, $\gamma = 0.01$

	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$
λ_1	0.00654	0.0560	0.0572
λ_2	.00978	.0410	.0817
λ_3	.01552	.0514	.0923

(b) Effect of γ on frequency parameter, $\beta = 0.1$

	$\gamma = 0.001$	$\gamma = 0.005$	$\gamma = 0.01$
λ_1	0.00352	0.00500	0.00654
λ_2	.00580	.00598	.00978
λ_3	.00505	.00770	.01552

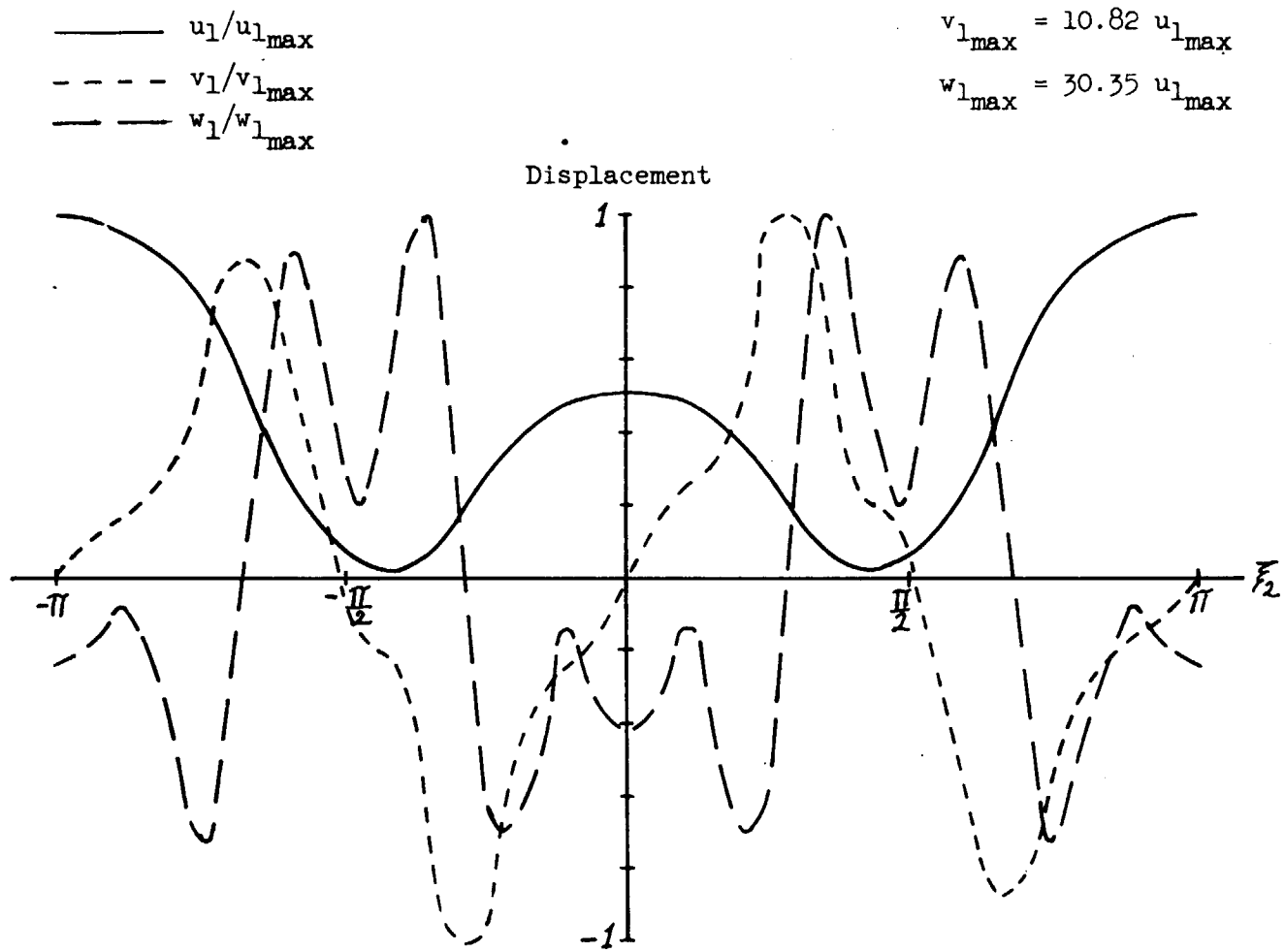


Figure 2.- Displacements of first mode of toroidal shell for $\beta = 0.1$, $\gamma = 0.001$ and $n = 1$.

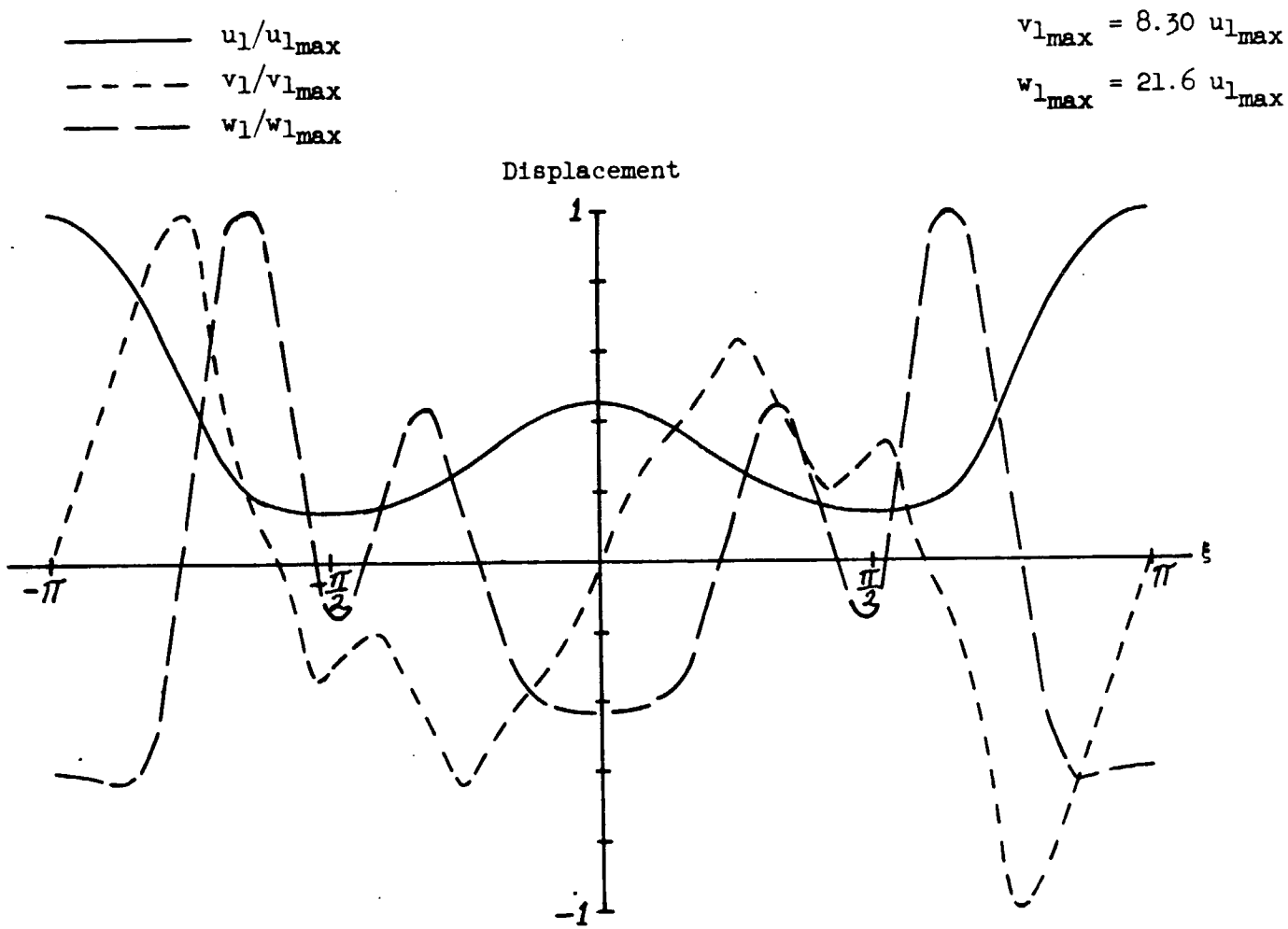


Figure 3.- Displacements of first mode of toroidal shell for $\beta = 0.1$, $\gamma = 0.005$, and $n = 1$.

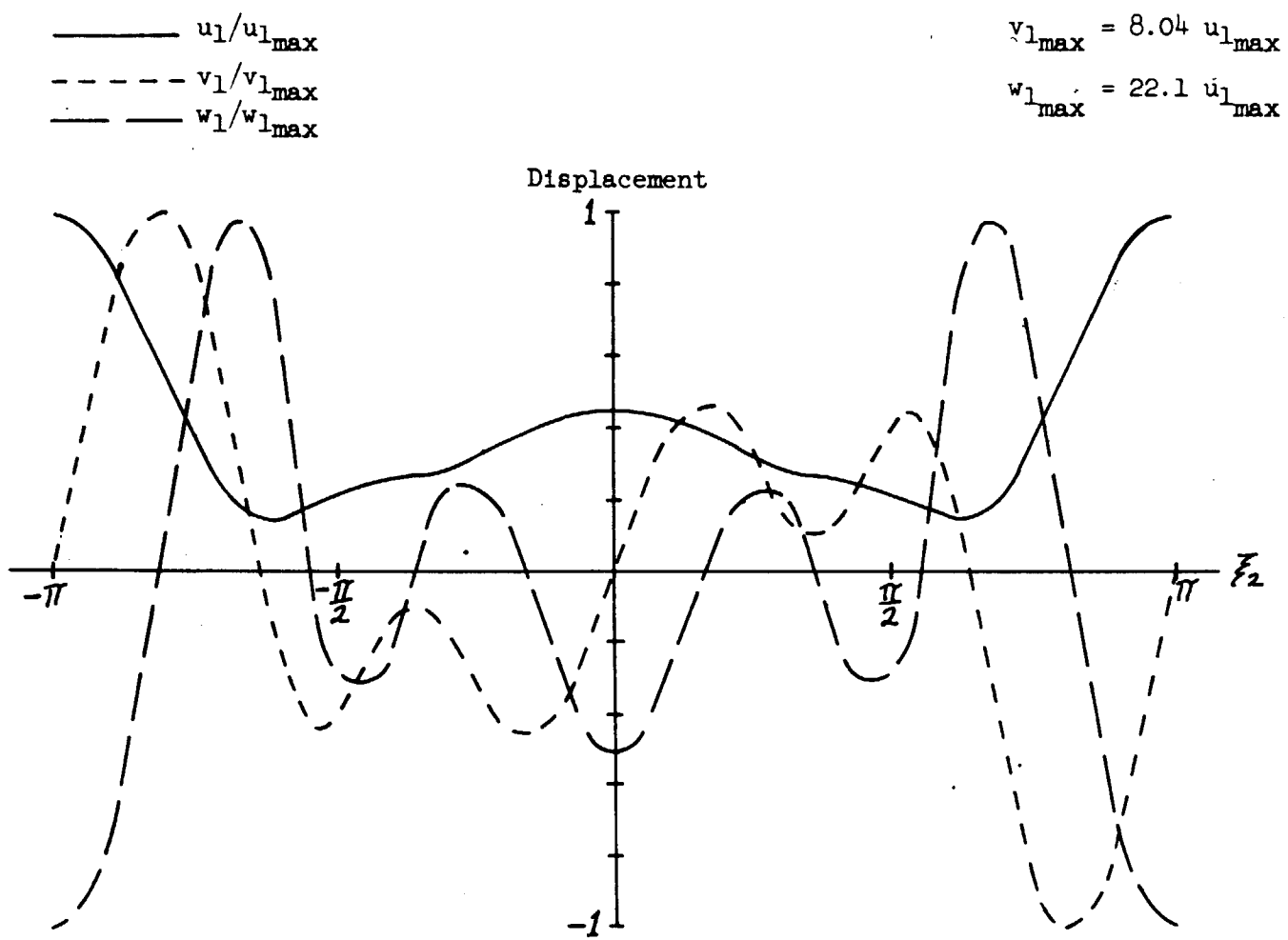


Figure 4.- Displacements of first mode of toroidal shell for $\beta = 0.1$, $\gamma = 0.01$, and $n = 1$.

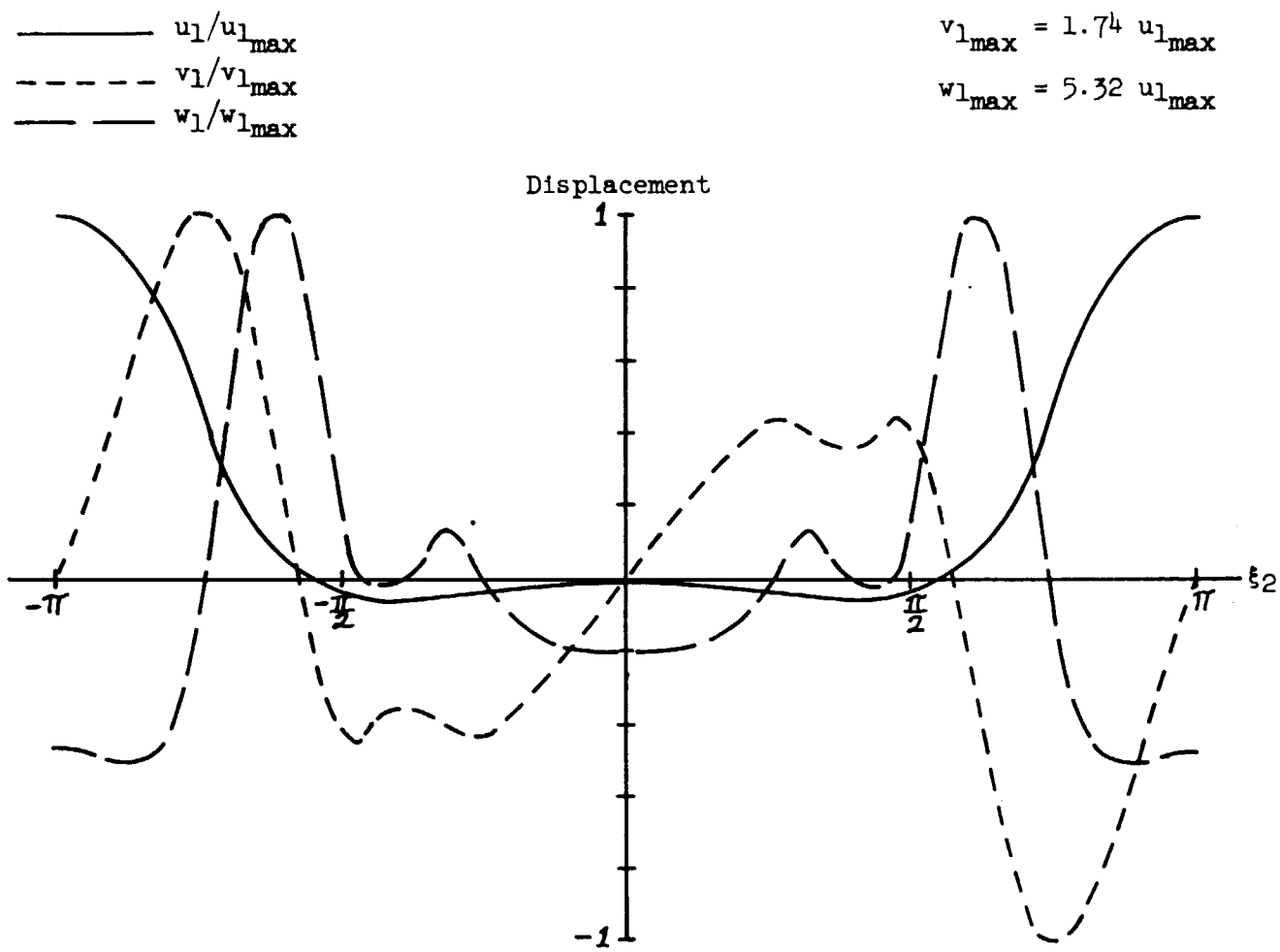


Figure 5.- Displacements of first mode of toroidal shell for $\beta = 0.3$, $\gamma = 0.01$, and $n = 1$.

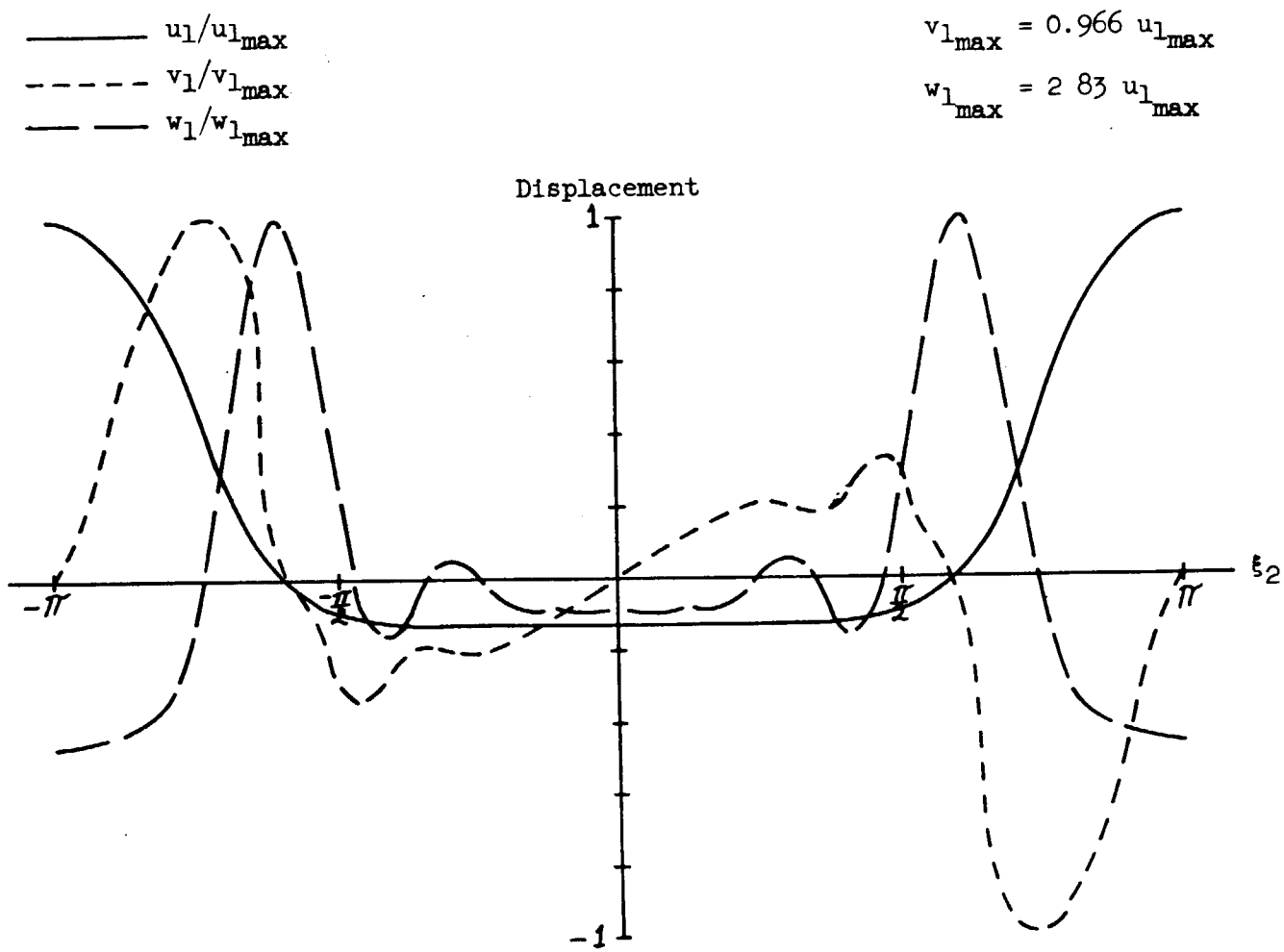


Figure 6.- Displacements of first mode of toroidal shell for $\beta = 0.5$, $\gamma = 0.01$, and $n = 1$.

displacements. The displacement values shown in figures 2 through 6 also indicate that convergence is not as complete as desired for a true description of the mode shape; however, the modes in their present form provide a considerable amount of information on the behavior of the vibrating shell. Figures 2 through 4 show the effect of increasing the thickness of the shell through increasing values of γ with the radius ratio, β, held constant. The results also indicate that, for reasonable convergence, more terms of each series (especially for v_1 and w_1) are required for very thin shells (small values of γ) than for values of γ near 0.01.

The effect of increasing the value of the radius ratio, β, is shown in figures 4 through 6. Examination of these figures as well as the series coefficients also shows that convergence is not as good for larger values of β as for $\beta = 0.1$.

It is of interest to note that the relative magnitudes of the displacements for each case presented is changed very little with relatively large changes in the values of γ. The effect of β on the relative magnitudes of displacements is seen to be quite different. As β increases, the displacements become more nearly the same in magnitude with $u_{1_{max}}$ and $v_{1_{max}}$ having nearly equal values for $\beta = 0.5$ whereas the value of $v_{1_{max}}$ is eight times as large as $u_{1_{max}}$ for $\beta = 0.1$.

As a check on the convergence of the frequency parameter, the calculations were repeated using eight terms of each series and using nine terms of each series. The results of these calculations are

compared with the results using ten terms and this comparison is presented in table 7. From this result it is seen that the maximum difference between the frequency parameters for a given mode is less than 10.5 percent. This difference occurs for the third mode with $\beta = 0.3$ and $\gamma = 0.01$ when the results for eight terms are compared with the results for ten terms. For this same mode the difference is only 4 percent when the results for nine terms are compared with the results for ten terms. Increasing the number of terms in each series from nine to ten affects the values of the frequency parameter by less than 0.5 percent for the nine cases presented in table 7. Convergence of the frequency parameter for the case with $\beta = 0.1$ and $\gamma = 0.01$ shown in table 7(a) is very good since the difference shown in the last column is 1 percent or less for all three modes.

The numerical results presented herein have provided information on the effects of the shell parameters β and γ on the frequency parameters for the lowest three modes and the mode shapes for the lowest mode of vibration for the toroidal shell. The results also show that a large number of terms of the series for each displacement is required before the mode's shapes are known accurately. As would be expected, the convergence for frequency parameters was found to be much better than convergence for mode shape.

TABLE 7. CONVERGENCE OF FREQUENCY PARAMETER

(a) $\beta = 0.1$, $\gamma = 0.01$

	8 terms	9 terms	10 terms	Percent difference between	
				8 and 10 terms	9 and 10 terms
λ_1	0.006529	0.006535	0.006535	1.5	0
λ_2	.009768	.009777	.009781	1.3	.4
λ_3	.015165	.015506	.015522	2.3	1.0

(b) $\beta = 0.3$, $\gamma = 0.01$

	8 terms	9 terms	10 terms	Percent difference between	
				8 and 10 terms	9 and 10 terms
λ_1	0.03561	0.03480	0.03596	6.5	3.2
λ_2	.04151	.04116	.04099	1.3	0.4
λ_3	.06671	.04933	.05145	10.3	4.0

(c) $\beta = 0.5$, $\gamma = 0.01$

	8 terms	9 terms	10 terms	Percent difference between	
				8 and 10 terms	9 and 10 terms
λ_1	0.06140	0.06071	0.05720	7.3	6.1
λ_2	.08500	.08425	.08173	4.0	3.1
λ_3	.15935	.09458	.09234	7.3	2.4

IX. CONCLUDING REMARKS

In this dissertation a general solution has been developed for the determination of the modes and frequencies of vibration for complete toroidal shells of uniform thickness and circular cross section. The solution is based on Love's first approximation theory for thin shells. The solution is obtained in the form of Fourier series for the three displacements u_1, v_1, and w_1 and it was found that the vibrations are dependent on three shell parameters β, γ, and n where β is the ratio of the radius of the cross section to the radius from the axis of revolution to the center of the cross section, γ is the ratio of the shell thickness to the radius of the cross section, and n is the number of complete waves around the torus. Although the solution has been obtained for the shell with constant thickness, the method is sufficiently general that it may be applied to toroidal shells with variable thickness.

The equilibrium equations for the toroidal shell obtained in this study are reducible to the equilibrium equations for the circular cylinder given by Love's first approximation. This reduction is accomplished by letting β approach zero in such a way that the distance from the axis of revolution to the center of the cross section goes to infinity.

The general solution has been used to obtain numerical results for the frequency parameter and the corresponding mode in terms of the displacements at resonance for free vibrations. These calculated results are used to show the effects of the parameters β and γ on the natural vibrations of the toroidal shell.

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