N72-21932

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THEORY OF FIBER REINFORCED MATERIALS

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Prepared by
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Philadelphia, Pa. 19104
for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1972

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-1974		

4. Title and Subtitle	5. Report Date
THEORY OF FIBER REINFORCED MATERIALS	March 1972
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
ZVI HASHIN	
9. Performing Organization Name and Address	10. Work Unit No.
UNIVERSITY OF PENNSYLVANIA PHILADELPHIA, PA.	126-14-17-02
	11. Contract or Grant No.*
	NAS1-8818
12. Sponsoring Agency Name and Address	13. Type of Report and Period Covered
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION WASHINGTON, D.C. 20546	CONTRACTOR REPORT
	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

A unified and rational treatment of the theory of fiber reinforced composite materials is presented. Fundamental geometric and elasticity considerations are thoroughly covered, and detailed derivations of the effective elastic moduli for these materials are presented. Biaxially reinforced materials which take the form of laminates are then discussed. Based on the fundamentals presented in the first portion of this volume, the theory of fiber-reinforced composite materials is extended to include viscoelastic and thermoelastic properties. Thermal and electrical conduction, electrostatics and magnetostatics behavior of these materials are discussed. Finally, a brief statement of the very difficult subject of physical strength is included.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Composite materials, fiber reinforced materials, elastic properties, laminates	Unclassified

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	704	$9.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

TABLE OF CONTENTS

Part 1. INTRODUCTION 1

Part 2. GEOMETRY 6

Chapter 2.1 FIBROUS AND FIBER REINFORCED MATERIAL 7

Chapter 2.2 STATISTICAL GEOMETRY 11

Figures 19

Part 3. ELASTICITY 25

Chapter 3.1 TENSORIAL AVERAGE THEOREMS 26

3.1.1 Average Strain and Strain Rate Theorems 26

3.1.2 Average Stress and Stress Rate Theorems 30

3.1.3 Average Virtual Work Theorems 34

Appendix - Divergence Theorem. Theorem of Virtual Work 38

Chapter 3.2 THE ELASTICITY PROBLEM FOR HOMOGENEOUS AND HETEROGENEOUS BODIES 40

3.2.1 Homogeneous and Continuously Nonhomogeneous Bodies 40

3.2.2 Exact Solutions for Homogeneous Bodies of Arbitrary Shape 44

3.2.3 Heterogeneous Bodies 48

Chapter 3.3 EFFECTIVE STRESS-STRAIN RELATIONS OF GENERAL COMPOSITES 50

3.3.1 Heterogeneous Bodies with Homogeneous Boundary Conditions 50

3.3.2 Statistically Homogeneous Bodies 56

3.3.3 Effective Elastic Properties in Terms of Phase Averages 63

Chapter 3.4 EFFECTIVE STRESS-STRAIN RELATIONS OF FIBER REINFORCED MATERIALS 67

3.4.1 Elastic Symmetry 67

3.4.2 Orthotropic Materials 69

3.4.3 Square Symmetric and Transversely Isotropic Materials 80

3.4.4 Isotropic Materials 96

3.4.5 Structural Relations for Effective Elastic Moduli 98

3.4.6 Effective Elastic Moduli in Terms of Phase Averages 107

Chapter 3.5 EXACT SOLUTIONS 110

3.5.1 Formulation of Boundary Value Problems 110

3.5.2 Elementary Solutions for Fibrous Materials 130

3.5.3 Composite Cylinder Assemblages 136

3.5.4 Dilute Reinforcement 163

3.5.5 Numerical Analysis 171

Appendix 1 - Isotropy - Transverse Isotropy Analogy for Effective Elastic Moduli 181

Appendix 2 - Internal Stresses 183

Chapter 3.6 BOUNDING METHODS FOR EFFECTIVE ELASTIC MODULI: CLASSICAL EXTREMUM PRINCIPLES 195

3.6.1 Motivation 195

3.6.2 Principles of Minimum Potential Energy and Minimum Complementary Energy 196

3.6.3 Elementary Bounds for Fibrous Materials 209

3.6.4 Bounds for Circular-Fiber Reinforcement 224

Chapter 3.7 BOUNDING METHODS FOR EFFECTIVE ELASTIC MODULI: POLARIZATION EXTREMUM PRINCIPLES 243

3.7.1 Elastic Polarization 243

3.7.2 Polarization Extremum Principles 250

3.7.3 Bounds for Fibrous Materials 262

3.7.4 Bounds for Circular-Fiber Reinforcement 294

3.7.5 Bounds for Isotropic Composites; Randomly Oriented Fibers 296

Appendix-Computation of $\tilde{U}'$ 299

Chapter 3.8 FIBER EMBEDDING APPROXIMATION 313

Chapter 3.9 COMMENTS ON MACRO-MECHANICS OF COMPOSITES 324

Chapter 3.10 BIAXIAL AND MULTIAXIAL FIBER REINFORCEMENT: LAMINATES 331

3.10.1 Introduction 331

3.10.2 Laminae Stress-Strain Relations 333

3.10.3 Laminated Plates 339

Figures 351

Tables 379

Part 4. VISCOELASTICITY 384

Chapter 4.1 QUASI-STATIC LINEAR VISCOELASTICITY 385

4.1.1 Stress-Strain Relations 385

4.1.2 Boundary Value Problems 401

Chapter 4.2 GENERAL QUASI-STATIC THEORY OF VISCO-ELASTIC COMPOSITES 407

4.2.1 Definition of Effective Viscoelastic Properties of Composites 407

4.2.2 Static Correspondence Principle for Visco-Elastic Composites 413

Chapter 4.3 VISCOELASTIC BEHAVIOR OF FIBER REINFORCED MATERIALS 423

4.3.1 Effective Transform Domain (TD) Moduli 423

4.3.2 Effective Relaxation Moduli and Creep Compliances 432

4.3.2.1 Methods of investigation 432

4.3.2.2 Axial and isotropic-plane stressing and straining 434

4.3.2.3 Axial and transverse shearing 447

4.3.2.4 Transverse stressing and straining 459

Appendix - Internal Stresses 462

Chapter 4.4 DYNAMIC LINEAR VISCOELASTICITY 470

4.4.1 General Dynamic Problem 470

4.4.2 Viscoelastic Vibrations: Complex Moduli 471

4.4.3 Viscoelastic Vibrations: Boundary Value Problems and Correspondence Principle 479

Chapter 4.5 EFFECTIVE COMPLEX MODULI 482

4.5.1 Definition and Correspondence Principle 482

4.5.2 Effective Complex Moduli of Fiber Reinforced Materials 487

Chapter 4.6 STRUCTURAL APPLICATIONS 504

4.6.1 Quasi-Static Examples 504

4.6.2 Torsional Forced Vibrations of Fiber Reinforced Cylinder 514

Figures 520

Part 5. CONDUCTION, DIELECTRICS AND MAGNETICS 531

Chapter 5.1 VECTORIAL AVERAGE THEOREMS 533

5.1.1 Average Gradient or Intensity Theorem 533

5.1.2 Average Flux Theorem 536

5.1.3 Average Virtual Work Theorems 538

Chapter 5.2 STEADY STATE BOUNDARY VALUE PROBLEMS 542

5.2.1 Formulation 542

5.2.2 Elementary Solutions for Homogeneous Bodies of Arbitrary Shape. Homogeneous Boundary Conditions 547

Chapter 5.3 EFFECTIVE CONDUCTIVITIES OF GENERAL COMPOSITES 549

5.3.1 Heterogeneous Bodies with Homogeneous Boundary Conditions 549

5.3.2 Statistically Homogeneous Composites 553

5.3.3 Effective Physical Constants in Terms of Phase Averages 555

Chapter 5.4 EFFECTIVE CONSTITUTIVE RELATIONS OF FIBER REINFORCED MATERIALS 557

5.4.1 Symmetry Reductions 557

5.4.2 Axial Conductivity 561

Chapter 5.5 TRANSVERSE CONDUCTIVITY 565

5.5.1 Formulation and Axial Shearing-Transverse Conduction Analogy 565

5.5.2 Expressions and Bounds 569

Appendix - Isotropic Composites; Randomly Oriented Fibers 572

Table 573

Part 6. THERMO-ELASTICITY 574

Chapter 6.1 UNCOUPLED THERMO-ELASTICITY 575

6.1.1 Formulation 575

6.1.2 Elementary Solutions 581

Chapter 6.2 EFFECTIVE THERMO-ELASTIC STRESS-STRAIN RELATIONS 583

6.2.1 Establishment of Stress-Strain Relations 583

6.2.2 Relations Between Effective Elastic Properties and Effective Thermal Expansion Coefficients 589

Chapter 6.3 THERMO-ELASTIC FIBER REINFORCED MATERIALS 600

6.3.1 Effective Thermal Expansion Coefficients 600

6.3.2 Internal Stresses 607

Appendix - Specific Heat 621

Figures 623

Table 625

Part 7. STRENGTH 626

Chapter 7.1 INTRODUCTION 627

Chapter 7.2 AXIAL STRENGTH 630

7.2.1 Tensile Strength 630

7.2.2 Compressive Strength 639

Chapter 7.3 TRANSVERSE STRENGTH 640

Chapter 7.4 AXIAL SHEAR STRENGTH 650

Chapter 7.5 FAILURE CRITERIA FOR COMBINED LOADING 653

Figures 663

Table 667

REFERENCES 668

PRINCIPAL NOTATION 680

LIST OF ABBREVIATIONS 690

LIST OF FIGURES

Part 2. - GEOMETRY

Fig. 2.1.1 - Fibrous Cylinder 19

Fig. 2.1.2 - Fiber Reinforced Material. Transverse Cut 20

Fig. 2.1.3 - Composite Cylinder Assemblage 21

Fig. 2.1.4 - Rectangular Array 22

Fig. 2.1.5 - Square Array 22

Fig. 2.1.6 - Hexagonal Array 23

Fig. 2.2.1 - Two Point Probability Function 24

Part 3. - ELASTICITY

Fig. 3.2.1 - Variation of Average Stress with Size of Volume Element 351

Fig. 3.2.2 - Variation of Statistically Homogeneous Stress in a Composite 351

Fig. 3.2.3 - Variation of Statistically Nonhomogeneous Stress in a Composite 352

Fig. 3.4.1 - Uniaxial Stressing of Fiber Reinforced Cylinder 353

Fig. 3.4.2 - Transversely Isotropic Stressing of Circular Fiber Reinforced Cylinder 354

Fig. 3.4.3 - Shearing of Fiber Reinforced Cylinder 354

Fig. 3.4.4 - Statistically Isotropic Fiber Reinforced Material 355

Fig. 3.4.5 - Verification of Linear Relation between ℓ^* and k* for Numerical Results; Hexagonal Circular Fiber Array 356

Fig. 3.5.1 - Composite Cylinder 357

Fig. 3.5.2 - Local Coordinate System of Composite Cylinder 357

Fig. 3.5.3 - Axial Young's Modulus, E^*_A 358

Fig. 3.5.4 - Axial Poisson's Ratio, ν^*_A, CCA Model 359

Fig. 3.5.5 - Axial Shear Modulus, G^*_A, CCA Model and Experimental Results 360

Fig. 3.5.6 - Transverse Young's Modulus, E^*_T, of Glass Reinforced Epoxy - Comparison of Theory and Experiment 361

Fig. 3.5.7 - Transverse Young's Modulus, E^*_T, of Boron Reinforced Epoxy - Comparison of Theory and Experiment 362

Fig. 3.5.8 - Bounds for ν^*_T, CCA Model 363

Fig. 3.5.9 - Repeating Elements for Periodic Fiber Arrays 364

Fig. 3.5.10 - Comparison of Square Array, CCA Model and Experimental Results: Transverse Young's Modulus, E^*_T 365

Fig. 3.5.11 - Comparison of Square Array and CCA Model Results: Axial Shear Modulus G^*_A 366

Fig. 3.6.1 - Composite Cylinder Construction 367

Fig. 3.6.2 - Composite Cylinder Construction for Periodic Array Bounding 368

Fig. 3.7.1 - Typical Composite Cylinders of Different Assemblages with Same Phase Volume Fractions 369

Fig. 3.7.2 - Bounds for G^*_T, Boron-Aluminum Fibrous Material 370

Fig. 3.7.3 - Bounds for E^*_T, Boron-Aluminum Fibrous Material 371

Fig. 3.7.4 - Bounds for G^*_A, Boron-Aluminum Fibrous Material 372

Fig. 3.7.5 - Bounds for Axial Shear Modulus G^*_A of Square Array of Circular Fibers 373

Fig. 3.7.6 - Bounds for Axial Shear Modulus G^*_A of Hexagonal Array of Circular Fibers 374

Fig. 3.8.1 - Geometry of Fiber Embedding Approximation 375

Fig. 3.8.2 - Relation between Fibrous Material Bounds and FEA Approximation (Schematic) 375

Fig. 3.10.1 - Biaxially Reinforced Material 376

Fig. 3.10.2 - Coordinate Systems of Laminae 377

Fig. 3.10.3 - Lamina Convention 378

Fig. 3.10.4 - Symmetric Laminae Pair 378

Part 4. - VISCOELASTICITY

Fig. 4.3.1 - Initial and Final Relaxation Modulus, ℓ^* 520

Fig. 4.3.2 - Initial and Final Transverse Bulk Relaxation Modulus, k^* 521

Fig. 4.3.3 - Initial and Final Axial Young's Modulus, E^*_A 522

Fig. 4.3.4 - Initial and Final Axial Young's Creep Compliance, e^*_A 523

Fig. 4.3.5 - Initial and Final Transverse Bulk Creep Compliance, κ^* 524

Fig. 4.3.6 - Initial and Final Creep Compliance, s^* 525

Fig. 4.3.7 - Dependence of Axial Shear Relaxation Modulus $G^*_A(t)$ on Fiber/Matrix Stiffness Ratio. Maxwell Model Matrix 526

Fig. 4.5.1 - Real Part of Complex Axial Shear Modulus, $G^{*R}_A(w)$, Polyisobutylene Matrix, Rigid Fibers 527

Fig. 4.5.2 - Imaginary Part of Complex Axial Shear Modulus, $G^{*I}_A(w)$, Polyisobutylene Matrix, Rigid Fibers 528

Fig. 4.6.1 - Bending of Fiber Reinforced Beam 529

Fig. 4.6.2 - Torsion of Fiber Reinforced Cylinder 529

Fig. 4.6.3 - Amplitude of Angle of Twist in Torsional Vibrations 530

Part 6. - THERMO-ELASTICITY

Fig. 6.3.1 - Effective Thermal Expansion Coefficients 623

Fig. 6.3.2 - Best Possible Bounds for Effective Thermal Expansion Coefficients 624

Part 7. - STRENGTH

Fig. 7.2.1 - Stresses near Fiber Breaks - Schematic 663

Fig. 7.2.2 - Effect of Fiber Coefficient of Variation and Reference Fiber Strength upon the Ratio of Composite Tensile Strength (Cumulative Damage Mode) to Mean Fiber Strength 664

Fig. 7.3.1 - Plastic Transverse Strength for Some Geometries 665

Fig. 7.3.2 - Plastic Transverse Strength of Some Square Arrays 666

LIST OF TABLES

Table 3.5.1 -	Elastic Properties of Fiber and Matrix Materials	379
Table 3.5.2 -	Effective Elastic Properties, E-Glass Fibers - Epoxy Matrix. CCA Model	380
Table 3.5.3 -	Effective Elastic Properties, Boron Fibers - Epoxy Matrix. CCA Model	381
Table 3.5.4 -	Effective Elastic Properties, Boron Fibers - Aluminum Matrix. CCA Model	382
Table 3.5.5 -	Comparison of Effective Elastic Properties Predicted by CCA Analysis and Hexagonal Array Numerical Analysis	383
Table 3.7.1 -	Comparison of Elementary and Improved Bounds for Boron - Epoxy Fibrous Materials	291
Table 5.5.1 -	Thermal Conductivities of Fiber and Matrix Materials	573
Table 6.3.1 -	Thermal Expansion Coefficients of Fiber and Matrix Materials	625
Table 7.1.1 -	Specific Weights and Tensile Strengths of Fiber and Matrix Materials	667

1. INTRODUCTION

The concept of composite materials and structures appears to be as old as mankind; but it is only in recent years that the true technological potential of such materials has begun to be recognized and exploited. The following lines are an admirable summary of the meaning and significance of the modern composite material concept:

"More important than any one new material or any one new application is the new "materials" concept itself. It marks a shift from concern with substances to concern with structures, a shift from artisan to scientist as man's artificer, a shift from chemistry to physics as the basic discipline, and a shift, above all, from the concrete experience of the workshop to abstract mathematics, a shift from starting with what nature provides to what man wants to accomplish." - Peter F. Drucker - The Age of Discontinuity(*)

From the engineering point of view the most important type of composites at the present time are Fiber Reinforced Materials (from now on abbreviated FRM). The idea behind this kind of material is simple and the results are startling. The very high strength of various kinds of fibers of minute cross sections is exploited by embedding them in a relatively soft matrix. It thus becomes possible to manufacture materials whose strength and stiffness is comparable to that of the strongest metals and whose specific weight is as low as one third of that of steel.

(*) I am indebted to Harper and Row, Publishers, for permission to quote this material.

This combination of high stiffness and strength with low weight makes these materials natural candidates for aerospace applications. Such applications have been the chief motivation for the intense research and development activity concerning FRM in the last decade.

If the engineer is to use these new materials with confidence, he must have detailed and reliable knowledge of their physical properties. While for conventional engineering materials, such as metals and plastics, physical properties are almost exclusively determined by experiment, such an approach is impractical for FRM because of their great structural and physical variety. Typical aspects of internal structure of FRM are: volumes occupied by fibers and matrix, directions of fiber reinforcement, shapes of fiber cross sections and the relative positions of fibers. Variation of these geometrical parameters alone leads to an enormous number of possibilities.

Additional variety is introduced by choice of constituents. At the present time fibers are chiefly made of Glass, Carbon, and Boron while matrices in use are plastics such as epoxy and lightweight metals such as Aluminum and Magnesium.

More variety is introduced by the scope of physical properties which must be studied. Of primary interest are: elasticity, time dependence (e.g. viscoelasticity), thermal and electrical conduction, dielectric and magnetic properties, thermomechanical behavior, yielding and strength, fatigue and dynamical characteristics.

Finally, it should be noted that FRM are anisotropic, which requires in each case the determination of a whole set of physical constants to account for properties in different directions.

It is seen that a purely experimental program to study physical properties would call for a stupendous number of experiments. Even if such experiments were carried out it is hardly likely that the resulting multitude of experimental data could lead to guide lines for the engineer.

A more hopeful avenue of approach is to construct a theory of FRM whose predictions shall have to be verified by experiment. Not only is such a theory necessary to determine the properties of materials in use. More important, it is indispensable as a guide to design materials with required properties, which is the ultimate goal of materials engineering.

Theory of composite materials, in general, and of FRM, in particular, has been the subject of a very large number of papers and reports, most of which have been written in recent years. Many different approaches to the very difficult problems involved have been proposed, some of which are rigorous while others are based on assumptions and approximations whose validity is hard to assess. It is not the purpose of the present report to review the voluminous existing literature(*); it is an attempt to present a theory of FRM which is reasonably rigorous and is at the same time oriented towards the engineering uses of such materials.

(*) References [1.1-1.12] which are, except for [1.4], collections of papers and chapters by different authors cover a great variety of important aspects of modern composite materials research.

The most basic FRM is uniaxially reinforced (or may be so idealized), that is to say, the fibers are all in one and the same direction. This report is primarily.concerned with such materials. Engineering applications frequently make it necessary to lay the fibers in two or more different directions, thus producing biaxially or multiaxially reinforced materials. Such materials mostly consist of laminated layers of uniaxially reinforced material and are thus called laminates . Once the properties of the uniaxial laminae are known, laminate theory serves to analyze laminated structures. Examples of this approach are discussed in this report.

The presentation of theory of FRM in this report begins with description of geometry of FRM, Part. 2. Part 3 is devoted to elastic analysis of FRM; Part 4 deals with viscoelastic static and dynamic properties; Part 5 is concerned with thermal and electrical conduction, dielectrics and magnetics. In Part 6 there is given a theory of thermoelastic behavior. Part 7 discusses the very difficult problem of strength prediction. This material is to be regarded primarily as qualitative, as work to date in this area has not reached the quantitative analytical level of the material given in Parts 3-6.

An important and satisfying aspect of the theories presented is the close mathematical interrelationship between elastic theory and theories of other physical behavior. Thus it will be seen that viscoelastic properties of FRM are directly related to elastic properties by simple mathematical analogies. Thermal and electrical conduction, dielectrics and magnetics present one and the same mathematical problem which is found to be analogous to a certain elastic

problem arising in shearing of FRM. Finally, the thermal expansion coefficients of composites can be expressed in terms of their elastic properties. It is thus seen that theory of elastic properties of FRM is of pivotal importance in the whole development.

2.1 FIBROUS AND FIBER REINFORCED MATERIAL

Consider a two phase material whose phases are of cylindrical shape, with all generators oriented in one and the same direction. For convenience and without loss of generality we shall mostly be concerned with material specimens of cylindrical shape with generators parallel to phase region generators, figure 2.1.1. It shall be also assumed that phase cylindrical regions continue without interruption from base to base of the cylindrical specimen. The material thus described is called a <u>fibrous material</u> (henceforth abbreviated FM). Its geometry is completely described by the plane geometry of any transverse cross section.

The cylindrical specimen is referred to a cartesian system of axes x_1, x_2, x_3 where x_1 is in generator direction and x_2, x_3 are in the plane of the transverse cross section. The two phases are arbitrarily assigned the numbers 1 and 2. The cross section area is denoted A and is bounded by a plane curve C. The phase partial areas are A_1, A_2, respectively, and the aggregate of arcs common to both phase regions is denoted C_{12}.

In the three dimensional description of such a specimen of height H, the specimen volume is denoted by V with phase volumes V_1 and V_2, occupying regions R_1 and R_2. The specimen is bounded by the surface S which is composed of the curved cylindrical surface S_c and the lower and upper faces A_o and A_H. The aggregate of phase cylindrical interfaces is denoted S_{12}.

Fig. 2.1.1.

The <u>phase volume fractions</u> v_1, v_2 are given by

$$v_1 = \frac{V_1}{V} = \frac{A_1}{A} \tag{2.1.1}$$

$$v_2 = \frac{V_2}{V} = \frac{A_2}{A} \tag{2.1.2}$$

Obviously

$$v_1 + v_2 = 1 \tag{2.1.3}$$

In the most general kind of fibrous material there is no specific geometrical distinction between the two phases. If we impose the topological restriction that one of the phases is in the form of cylinders which are completely embedded in the other phase, we shall call the embedded cylinders the <u>fibers</u> and the embedding phase the <u>matrix</u>. The two phase material in that case is called a <u>fiber reinforced material</u> henceforward abbreviated FRM. See figure 2.1.2 for an example of such a material.

As a further restriction it may now be assumed that the fibers are of specified shapes, e.g. circular, elliptical or diamond shaped, however, their locations within the cross section are random. The resulting geometry has deterministic as well as random features and is therefore called semi-random.

We shall now consider a certain semi-random FRM which is of fundamental importance in theory of FRM. We construct composite circular cylinders each of which is made of a circular cylindrical fiber and a concentric matrix shell. In the nth composite cylinder the fiber radius is a_n and the

composite cylinder radius is b_n. In all composite cylinders the ratios a_n/b_n are identical and the cylinders are all of equal height H. Now a cylindrical specimen of height H and cross section A is progressively filled out with non-overlapping composite cylinders. This is done by first placing a number of such composite cylinders into the specimen and then filling out the remaining spaces by smaller and smaller composite cylinders. At each stage of filling, the volume V consists of V_c, the volume filled out by composite cylinders, and the remaining volume V'. At the same time, a portion A_c of the section is filled out by composite circles and there remains the area A', fig. 2.1.3. Since composite cylinders are assumed to be available in all sizes, the remaining volume V' can be made indefinitely small by progressive filling and in the limit the volume V consists entirely of non-overlapping composite cylinders. The resulting material is semi-random and is called <u>composite cylinder assemblage</u>. It has been introduced as a FRM model in [3.10] and its detailed analysis for various kinds of physical properties will be given in subsequent chapters.

Finally we may consider a FRM whose geometry is specified in all details. While in principle any of the previous kinds of geometries may be completely specified, such minute specifications would be completely useless from any practical point of view. Complete specification of geometry is practically feasible only when this can be done in terms of a small number of geometrical parameters. The most important cases of deterministic geometry are, therefore regular arrays of identical fibers. Figures 2.1.4,

2.1.5, 2.1.6 show rectangular, square and hexagonal arrays, respectively of identical circular fibers.

The arrays listed above are important examples of materials which exhibit various geometrical symmetries. It is to be noted in this respect that by definition the transverse x_2 x_3 plane is a plane of geometrical symmetry in any fibrous material or uniaxially FRM. The rectangular array has two additional planes of symmetry, namely the x_1 x_2 and x_2 x_3 coordinate planes, thus in all three mutually perpendicular planes of symmetry. Such a material is called geometrically orthotropic.

The square array is a special case of the rectangular array. The additional symmetry is in that the x_1 axis is an axis of two fold symmetry, by which is meant that the x_2 x_3 plane geometry has the two perpendicular axes of symmetry x_2 and x_3. A material having this kind of symmetry is here called geometrically square symmetric.

The hexagonal array has an x_1 axis of three fold symmetry as it is seen that there are now three angularly equally spaced axes of symmetry in the x_2 x_3 plane.

The various symmetries of FRM are a most important aspect of the subject and we shall return to them later on in relation to the concept of material symmetry.

Frequently, technological applications require biaxially or multiaxially FRM. Such materials are usually produced as laminates of uniaxially reinforced parallel layers, the reinforcement direction being different in adjacent layers. A short discussion of such FRM is given later in chap. 3.10. For geometrical examples the reader is referred to fig. 3.10.1.

2.2 STATISTICAL GEOMETRY(*)

As will be seen later on, the physical properties of a FRM depend in general upon all the details of the phase geometry. Specification of random or semi-random geometry in minute detail is, however, a hopeless task. Even if this were done for one specimen not much would be achieved, for another specimen would have different phase geometry details . Unless the geometry is simply deterministic as in a regular array, the details cannot be controlled by the FRM manufacturer. It is therefore necessary to consider global geometrical information rather than detailed one, and this is best done by means of statistics and theory of random processes.

Consider a collection of N fiber reinforced (or fibrous) cylindrical specimens. The specimens have the same external geometry, however, their phase geometries, i.e. internal geometries may be quite different. Each specimen is referred to the cartesian system of axes, x_1, x_2, x_3 described in 2.1.1 and moreover, the origin of the system of axes is at the same point in each specimen. In the language of theory of random processes, such a collection of specimens is called an <u>ensemble</u> and each specimen is a member of the ensemble.

We now consider the same point B with position vector $\underline{x}$ in all members of the ensemble and we pose the question: what is the probability that the point is in either one of the phases? To answer this question we

(*) This chapter is not absolutely required for most of subsequent development.

perform in our minds a counting experiment. We denote by N_1 the number of specimens in which B is in phase 1 and by N_2 the number of specimens in which B is in phase 2. Then the probabilities P_1, P_2 that B is in R_1 or R_2 respectively are defined by

$$P(B \supset R_1) = P_1(\underline{x}) = \lim_{N \to \infty} \frac{N_1}{N}$$

$$P(B \supset R_2) = P_2(\underline{x}) = \lim_{N \to \infty} \frac{N_2}{N} \qquad (2.2.1)$$

The existence of the limits in the extreme right sides of (2.2.1) is a question of fundamental importance in probability theory and in order to avoid this difficulty modern probability theory has been based on set theory. Such discussions are not within the scope of the present treatment and the interested reader is referred, for example, to [2.1] .

The probabilities (2.2.1) are known as <u>one point probabilities</u>. Similarly two point, three point and multipoint probabilities may be defined. Consider two points, B_1, B_2 in each specimen member of the ensemble, which have the same position vectors $\underline{x}^1$ and $\underline{x}^2$ in each specimen. There are now four <u>two point probabilities</u> which are defined as follows:-

$$P(B_1 \supset R_1, B_2 \supset R_1) = P_{11}(\underline{x}^1, \underline{x}^2) = \lim \frac{N_{11}}{N}$$

$$P(B_1 \supset R_1, B_2 \supset R_2) = P_{12}(\underline{x}^1, \underline{x}^2) = \lim \frac{N_{12}}{N}$$

$$P(B_1 \supset R_2, B_2 \supset R_1) = P_{21}(\underline{x}^1, \underline{x}^2) = \lim_{N \to \infty} \frac{N_{21}}{N}$$

$$P(B_1 \supset R_2, B_2 \supset R_2) = P_{22}(\underline{x}^1, \underline{x}^2) = \lim_{N \to \infty} \frac{N_{22}}{N} \qquad (2.2.2)$$

Here N_{11} is the number of times that both points fall simultaneously into phase 1, with analogous interpretations for N_{12}, N_{21}, N_{22}.

It is to be carefully noted that while it is theoretically possible for a point to be on the phase interface such a possibility can be ignored since the number of points inside the phases is infinitely larger than the number of points on the interfaces. In the language of probability theory, the probability of a point falling on the interface is assigned zero value.

There are similarly eight three point probabilities and in general 2^n, n point probabilities. Such n-point probabilities may be written in the form

$$P_{i_1, i_2, \ldots i_n}(\underline{x}^1, \underline{x}^2 \ldots \underline{x}^n) \qquad (2.2.3)$$

where each of the subscripts $i_1, i_2, \ldots i_n$ assumes either one of the phase numbers 1, 2 and its position in the subscript sequence is attached to that of the corresponding position vector within the parenthesis.

There is, of course, no difficulty to define the same kind of multi-point probabilities when there are more than two phases.

The probability functions defined above obey the obvious relations

$$P_1(x) + P_2(\underline{x}) = 1 \qquad \text{(a)}$$

$$P_{11}(\underline{x}^1, \underline{x}^2) + P_{12}(\underline{x}^1, \underline{x}^2) = P_1(\underline{x}^1) \qquad \text{(b)}$$

$$P_{21}(\underline{x}^1, \underline{x}^2) + P_{22}(\underline{x}^1, \underline{x}^2) = P_2(\underline{x}^1) \qquad \text{(c)}$$

$$\Sigma\, P_{ij}(\underline{x}^1, \underline{x}^2) = 1 \qquad \text{(d)} \qquad (2.2.4)$$

Again similar relations are easily established for any number of points and any number of phases.

The reader will have noticed that the preceding discussion has not really been concerned with the specific case of fibrous materials but applied to any kind of heterogeneous material. The distinctive feature of the fibrous material is that its phase geometry is independent of the longitudinal x_1 coordinate. Hence for such a material all the multipoint probabilities are independent of the x_1 coordinate and it is therefore sufficient to consider points which lie in the same cross section. Therefore, the ensemble of cylindrical specimens may be replaced by an ensemble of two phase cross sections, each of which is referred to a plane cartesian system x_2, x_3 and all position vectors are in the x_2, x_3 plane. Thus all vector variables in (2.2.4) lie in the $x_2 x_3$ plane.

It may in general be safely assumed that all multipoint probability functions are continuous functions of position, because of the smoothening out effect of the, in theory, infinite number of ensemble members.

We now proceed to define the very important concept of <u>statistical homegeneity</u>, henceforward abbreviated as SH. For this purpose, the system of n points entering into (2.2.3) may be considered as a rigid body which is described by the vector differences

$$\underline{r}^1 = \underline{x}^1 - \underline{x}^n \,, \underline{r}^2 = \underline{x}^2 - \underline{x}^n, \ldots \underline{r}^{n-1} = \underline{x}^{n-1} - \underline{x}^n \tag{2.2.5}$$

Suppose that any multipoint probability such as (2.2.3) depends only on the relative configuration of the points and not on their absolute position with respect to the coordinate system; then the ensemble is called statistically homogeneous. Mathematically this means

$$P_{i_1}, P_{i_2} \ldots_{i_n} (\underline{x}^1, \underline{x}^2 \ldots \underline{x}^n) = P_{i_1}, {}_{i_2} \ldots_{i_n} (\underline{r}^1, \underline{r}^2, \ldots, \underline{r}^{n-1}) \tag{2.2.6}$$

The meaning of this statement is that in the counting experiments to determine probabilities which were described above, it is not important where the n point system is located within the ensemble members.

Statistical homogeneity is a theoretical assumption which is fundamental in theory of heterogeneous media. It plays the same role as the assumption of homogeneity (constant properties) in classical continuum theories. Neither one of these assumptions is ever literally satisfied but in their absence theories become hopelessly difficult.

A most important aspect of SH is expressed by an <u>ergodic type hypothesis</u>. It is assumed that in a SH ensemble the counting experiment for the determination of any n point probabilities may be performed by moving the rigid n point system through a large number of positions within any <u>one</u> member of the ensemble. This hypothesis is fundamental for the actual experimental determination of a probability function. It is hardly possible for the experimenter to make a large enough ensemble. Instead he simply considers one typical case.

We now consider the consequence of the SH assumption for the probability functions (2.2.1). The one point probabilities (2.2.1) cannot be functions of position and are therefore constant. Furthermore, in view of the ergodic hypothesis, the one point probabilities can be obtained by randomly throwing a point into <u>one</u> specimen a very large number of times and counting N_1 and N_2, as defined in (2.2.1). It is, therefore, clear that the one point probabilities are just the volume fractions (2.2.1-2), so

$$P_1 = v_1$$

$$P_2 = v_2 \tag{2.2.7}$$

In view of (2.2.6) the two point probabilities (2.2.2) now assume the form

$$P_{ij}(\underline{x}^1, \underline{x}^2) = P_{ij}(\underline{r}) \tag{2.2.8}$$

where

$$\underline{r} = \underline{r}^2 - \underline{r}^1 \tag{2.2.9}$$

The two point probabilities do not have as simple an interpretation as the one point probabilities. They have in general to be found by a scanning experiment or on the basis of plausible theoretical assumptions. However, some important general aspects of these quantities are readily deduced. We first consider the case when the two points coincide, i.e. $\underline{r}$ vanishes. Then

$$P_{11}(o) = P_1 = v_1 \qquad \text{(a)}$$

$$P_{22}(o) = P_2 = v_2 \qquad \text{(b)}$$

$$P_{12}(o) = P_{21}(o) = 0 \qquad \text{(c)} \tag{2.2.10}$$

The first two results stem from the fact that since the points coincide the two point probability becomes a one point probability. The result (2.2.10c) is due to the neglect of the probability of a point falling on an interface.

Next, we consider the case when the two points are infinitely apart, i.e. $r = |\underline{r}| \to \infty$. In this event we make the assumption that whatever happens at one point is independent of what happens at the other point. This situation is expressed in statistical language as <u>statistical independence</u>. If two events are statistically independent the probability of their simultaneous occurrence is simply the product of the probabilities of their individual occurrences, see e.g. [2.1]. The events are in the present case the falling of points into a certain phase. We thus conclude that

$$P_{ij}(\underline{r}) = P_i(\underline{x}^1)\, P_j\,(\underline{x}^2) = v_i v_j \qquad (2.2.11)$$
$$r \to \infty$$

The last equality in (2.2.11) follows from (2.2.7). In detail (2.2.11) are:

$$P_{11} = v_1^{\,2}\ ,\ P_{22} = v_2^{\,2}\ ,\ P_{12} = P_{21} = v_1 v_2 \qquad (2.2.12)$$

It is seen that the sum of these probabilities (2.2.12) obeys (2.2.4d), as it should, because of (2.1.3).

A typical plot of a two point probability function is shown in fig. 2.2.1.

Another fundamentally important feature is statistical symmetry. In chapter 2.1 we have discussed certain geometrical symmetries and all of

these have statistical counterparts. From the practical point of view only statistical transverse isotropy is of importance since this kind of symmetry is frequently obtained when the fibers are randomly placed. Orthotropy and square symmetry appear for the deterministic periodic geometries described in chapter 2.1. It is hard to imagine a random geometry with this kind of statistical symmetry.

It will be recalled that in a fibrous material the position vectors appearing as arguments in the probability functions can without loss of generality be taken as plane vectors in a transverse plane. Statistical transverse isotropy (abbreviated STI) requires that all multipoint probability functions be not affected by rigid body rotations of the plane point system with which they are associated. The simplest example is a one point probability function $P(\underline{x})$. In the case of STI

$$P(\underline{x}) = P(x) \tag{2.2.13}$$

where x is the magnitude of the vector $\underline{x}$.

It should be carefully noted that SH and STI are independent properties. We shall, however, not be concerned with STI geometry which is not SH. If the geometry fulfills both requirements we conclude that the values of the one point probabilities remain as given by (2.2.7).

The only difference in the two point probabilities is that now

$$P_{ij}(\underline{r}) = P_{ij}(r) \tag{2.2.14}$$

where r is the magnitude of $\underline{r}$. The results (2.2.10) and (2.2.11-12) remain, of course, the same. For further discussion of statistical geometry the reader is referred to e.g. [2.2-3].

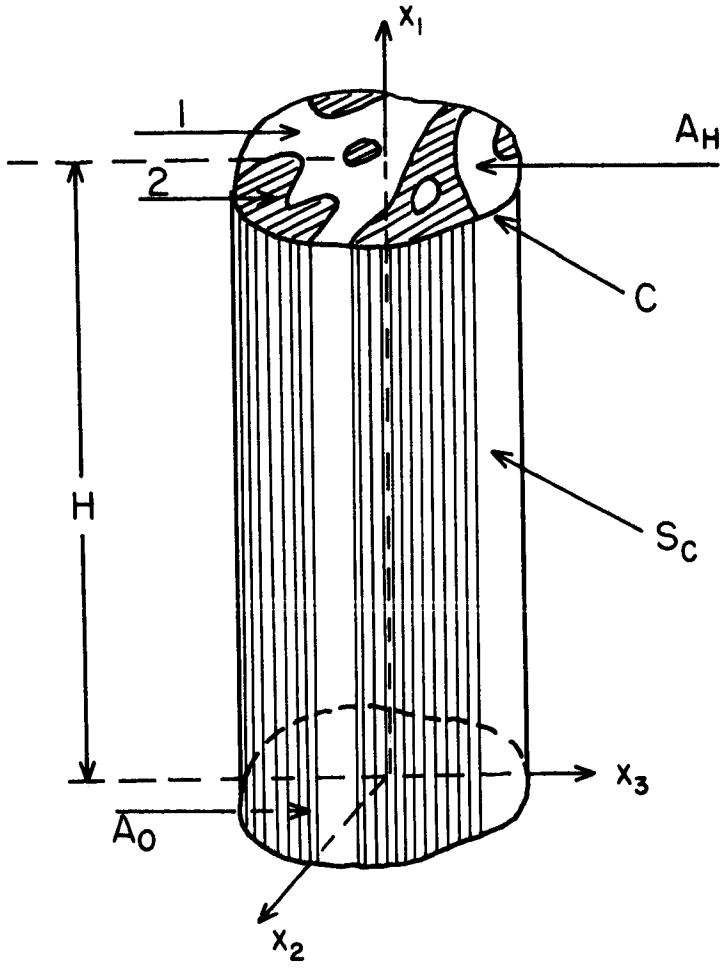


FIG. 2.1.1 - FIBROUS CYLINDER

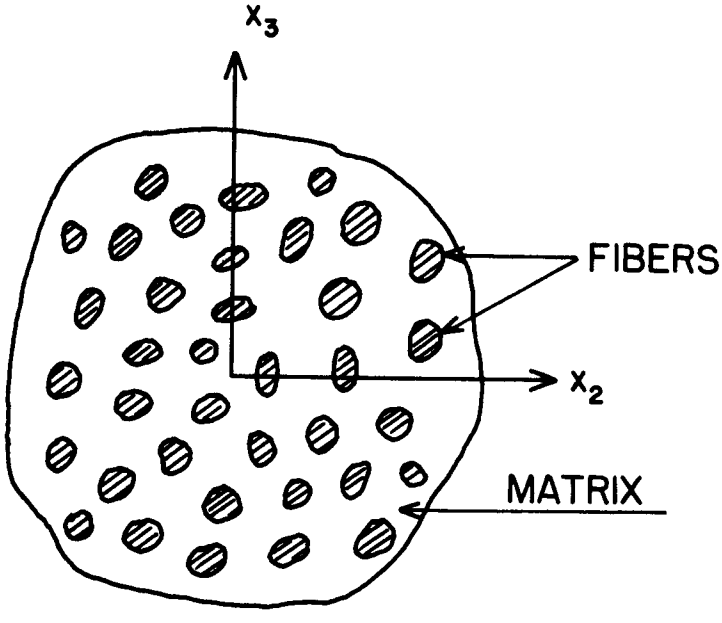


FIG. 2.1.2 - FIBER REINFORCED MATERIAL. TRANSVERSE CUT

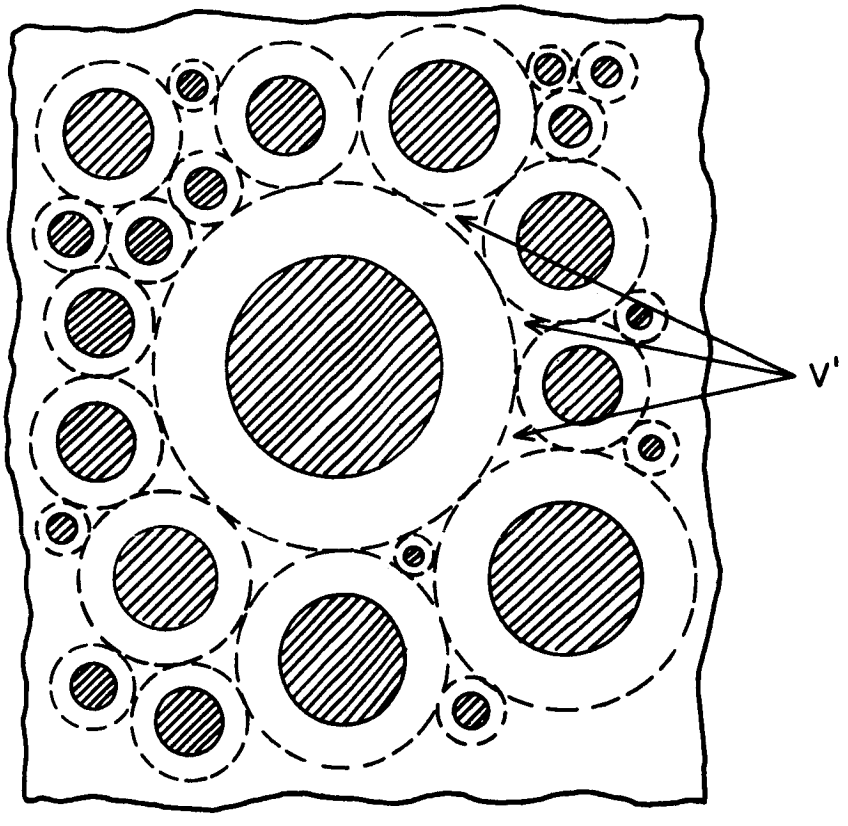


FIG. 2.1.3 - COMPOSITE CYLINDER ASSEMBLAGE

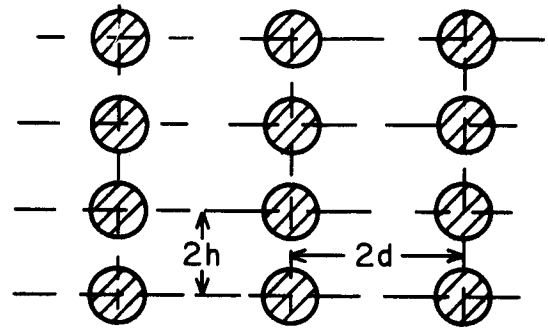


FIG. 2.1.4 - RECTANGULAR ARRAY

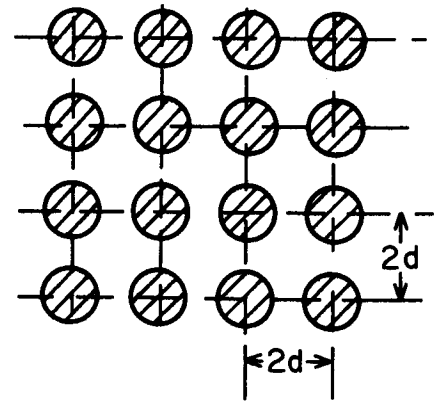


FIG. 2.1.5 - SQUARE ARRAY

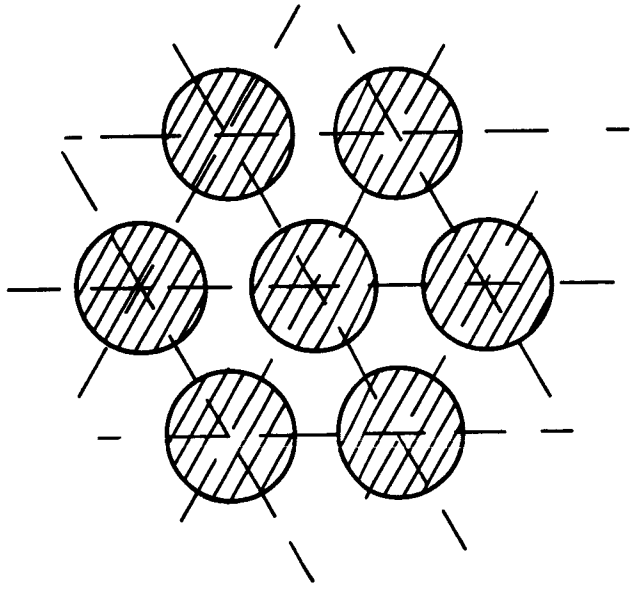


FIG. 2.1.6 - HEXAGONAL ARRAY

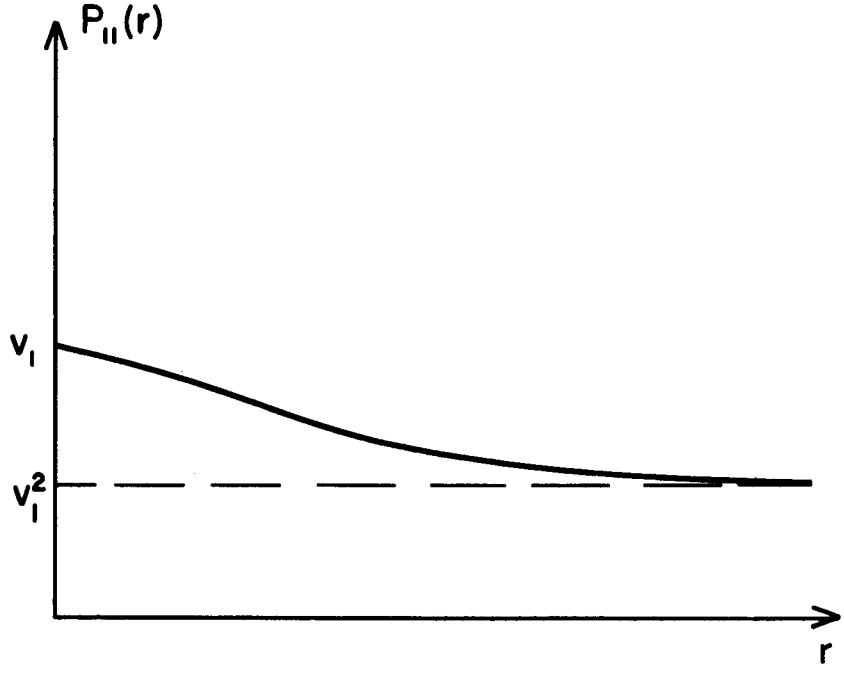


FIG. 2.2.1 - TWO POINT PROBABILITY FUNCTION

3. ELASTICITY

3.1 TENSORIAL AVERAGE THEOREMS

We shall here be concerned with the derivation of some theorems for second rank tensor averages, which are fundamental in the theory of heterogeneous materials. It is emphasized in advance that these are general continuum mechanics theorems and do not presuppose any kind of specific mechanical behavior of the material. The theorems hold for homogeneous as well as multiphase bodies. For the sake of simplicity they will be derived for the specific case of a two phase body.

It is assumed that the reader has some knowledge of continuum mechanics and cartesian tensors. Subscripts used range over 1,2,3, repeated subscripts denote summation over their range and a comma before a subscript denotes partial differentiation with respect to the space coordinate associated with the subscript. A position vector with cartesian components x_1, x_2, x_3 is denoted $\underline{x}$.

3.1.1 Average Strain and Strain Rate Theorems

Consider a two phase body with phases occupying regions R_1 and R_2. The displacement fields in the phases are $u_i^{(1)}(\underline{x},t)$ and $u_i^{(2)}(\underline{x},t)$ where t is the time. Associated with these are velocity fields $v_i^{(1)}(\underline{x},t)$ and $v_i^{(2)}(\underline{x},t)$ where

$$v_i = \frac{\partial u_i}{\partial t} = \dot{u}_i \tag{3.1.1}$$

The volume V of the two phase body, the phase volumes V_1 and V_2, the bounding surface S, and the interface S_{12} may all be time dependent. The velocities are prescribed on S, i.e.

$$v_i\,(S) = v_i^o \tag{3.1.2}$$

Also the velocities are continuous within the phases and at the interface, the last condition being expressed by

$$v_i^{(1)} = v_i^{(2)} \quad \text{on } S_{12}(t)\ . \tag{3.1.3}$$

Define the strain rate tensor γ_{ij} by

$$\gamma_{ij} = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \tag{3.1.4}$$

The volume average $\bar{\gamma}_{ij}$ of γ_{ij} is given by

$$\bar{\gamma}_{ij}(t) = \frac{1}{V} \int_V \gamma_{ij}\,(\underline{x},t)\,dV \tag{3.1.5}$$

The average strain rate theorem is expressed by the statement

$$\bar{\gamma}_{ij}\,(t) = \frac{1}{2V} \int_S (v_i^o\,n_j + v_j^o\,n_i)\,dS \tag{3.1.6}$$

It should be noted that the components of the normal, n_i, in (3.1.6) are time dependent since S is time dependent

<u>Proof:</u> Substitute (3.1.4) into (3.1.5) and use the extended divergence theorem (see appendix of chap. 3.1). Then

$$2\bar{\gamma}_{ij}(t) = \frac{1}{V}\left[\int_{S_1}(v_i^{(1)}\,n_j + v_j^{(1)}\,n_i)\,dS + \int_{S_2}(v_i^{(2)}\,n_j + v_j^{(2)}\,n_i)\,dS\right] \tag{3.1.7}$$

where $S_1(t)$ and $S_2(t)$ are the bounding surfaces of the phase regions R_1 and R_2. Now each of S_1 and S_2 is at most composed of part of the external surface S and the interface S_{12}. Therefore (3.1.7) may be rewritten

$$2\bar{\gamma}_{ij}(t) = \frac{1}{V}\left[\int_S (v_i\, n_j + v_j\, n_i)\, dS + \int_{S_{12}} (v_i^{(1)}\, n_j + v_j^{(1)}\, n_i)\, dS\right.$$

$$\left. + \int_{S_{12}} (v_i^{(2)}\, n_j + v_j^{(2)}\, n_i)\, dS\right] \tag{3.1.8}$$

It is to be carefully noted that in the divergence theorem n_i is always the outward normal. At the interface S_{12} the outward normal reverses sign according to whether it is taken from the inside of R_1 or R_2. Therefore at each point on S_{12} the n_i in the second and third integrals in (3.1.8) are of opposite signs. This in conjunction with (3.1.3) makes S_{12} surface integrals in (3.1.8) cancel one another and consequently the result 3.1.6 follows.

We now proceed to prove an important corollary of (3.1.6): If

$$v_i\,(S) = \gamma^{o}_{ij}\,(t)\, x_j \tag{3.1.9}$$

then

$$\bar{\gamma}_{ij}\,(t) = \gamma^{o}_{ij}\,(t) \tag{3.1.10}$$

<u>Proof:</u> Substitute (3.1.9) into 3.1.6). Then

$$\bar{\gamma}_{ij}\,(t) = \frac{1}{2V}\left[\gamma^{o}_{ik}(t)\int_S x_k\, n_j\, dS + \gamma^{o}_{jk}\,(t)\int_S x_k\, n_i\, dS\right] \tag{3.1.11}$$

By the divergence theorem

$$\int_S x_k\, n_j\, dS = \int_V x_{k,j}\, dV = V\delta_{kj}$$

$$\int_S x_k\, n_i\, dS = \int_V x_{k,i}\, dV = V\delta_{ki} \tag{3.1.12}$$

where δ_{kj}, δ_{ki} are Kronecker delta.

Substitution of (3.1.12) into (3.1.11) immediately leads to (3.1.10)

Average small strain theorems may be derived in a completely analogous fashion. The strains ϵ_{ij} are defined by

$$\epsilon_{ij}(\underline{x}, t) = \frac{1}{2}(u_{i,j}(\underline{x},t) + u_{j,i}(\underline{x},t)) \tag{3.1.13}$$

If the displacements are prescribed on S, i.e.

$$u_i(S, t) = u_i^o \tag{3.1.14}$$

and

$$u_i^{(1)} = u_i^{(2)} \quad \text{on} \quad S_{12}(t) \tag{3.1.15}$$

then

$$\bar{\epsilon}_{ij}(t) = \frac{1}{2V}\int_S (u_i^o n_j + u_j^o n_i)\, dS \tag{3.1.16}$$

Also if

$$u_i(S, t) = \epsilon_{ij}^o(t)\, x_j \tag{3.1.17}$$

then

$$\bar{\epsilon}_{ij}(t) = \epsilon_{ij}^o(t) \tag{3.1.18}$$

It should, however, be carefully noted that while

$$\gamma_{ij}(\underline{x}, t) = \dot{\epsilon}_{ij}(\underline{x}, t) \tag{3.1.19}$$

in general

$$\bar{\gamma}_{ij}(t) \neq \dot{\bar{\epsilon}}_{ij}(t) \tag{3.1.20}$$

The reason for the inequality is clearly seen when (3.1.16) is differentiated with respect to time. Because of the time dependence of V, S and n_i in (3.1.16) the time derivative of the right hand side is not equal to (3.1.6). Equality occurs in (3.1.20) if, and only if, the geometry of the body is time independent.

3.1.2 Average Stress and Stress Rate Theorems

Next we consider average stress and stress rate theorems. Let the stress field inside the body be $\sigma_{ij}(\underline{x}, t)$ and the body force field per unit volume be $F_i(\underline{x}, t)$. The body is assumed to be in quasi-static equilibrium, so that at every point

$$\sigma_{ij,j} + F_i = 0 \tag{3.1.21}$$

and also

$$\dot{\sigma}_{ij,j} + \dot{F}_i = 0 \tag{3.1.22}$$

On the external surface S the tractions are prescribed

$$T_i(S, t) = \sigma_{ij} n_j = T_i^o \tag{3.1.23}$$

At the interface the tractions are continuous

$$T_i^{(1)} = \sigma_{ij}^{(1)} n_j = T_i^{(2)} = \sigma_{ij}^{(2)} n_j \quad \text{on } S_{12} \tag{3.1.24}$$

Equation (3.1.24) involves a tacit sign convention with respect to the normal at the interface, which should be clarified. It is customary to take a normal at a surface in the outward direction. While "outward" is clearly defined at the external surface S, this is not the case at the interface for what is outward with respect to R_1 is inward with respect to R_2. It is

understood in (3.1.24) that the normal components on both sides refer to the same vector, outward into R_1, say, and consequently inward into R_2 (or vice versa). If it is desired to preserve the outward sense of the normal with respect to both phases at the same time, then one of the sides of (3.1.24) must be given a minus sign.

The average stress is defined by

$$\bar{\sigma}_{ij}(t) = \frac{1}{V} \int_V \sigma_{ij}(\underline{x}, t)\, dV \tag{3.1.25}$$

Then the <u>average stress theorem</u> asserts that

$$\boxed{\bar{\sigma}_{ij}(t) = \frac{1}{V} \left[\int_S x_j T_i \, dS + \int_V x_j F_i \, dV \right]} \tag{3.1.26}$$

<u>Proof:</u> We first prove the identity

$$\sigma_{ij} = (\sigma_{ik} x_j)_{,k} + F_i x_j \tag{3.1.27}$$

Evidently

$$(\sigma_{ik} x_j)_{,k} = \sigma_{ik,k} x_j + \sigma_{ik} \delta_{jk} = - F_i x_j + \sigma_{ij}$$

where the last step follows from (3.1.21). This establishes (3.1.27). Now (3.1.27) is introduced into (3.1.25) and the divergence theorem is applied. Then

$$\bar{\sigma}_{ij}(t) = \frac{1}{V} \left[\int_{S_1} x_j \sigma_{ik}^{(1)} n_k^{(1)} \, dS + \int_{S_2} x_j \sigma_{ik}^{(2)} n_k^{(2)} \, dS + \int_V F_i x_j \, dV \right] \tag{3.1.28}$$

which may be rewritten in the form

$$\bar{\sigma}_{ij}(t) = \frac{1}{V}\left[\int_S x_j\,\sigma_{ik}\,n_k\,dS + \int_{S_{12}} x_j\,\sigma_{ik}^{(1)}\,n_k^{(1)}\,dS\right.$$

$$\left. + \int_{S_{12}} x_j\,\sigma_{ik}^{(2)}\,n_k^{(2)}\,dS + \int_V F_i\,x_j\,dV\right] \qquad (3.1.29)$$

The superscripts on the normals indicate component of <u>outward</u> normals with respect to surfaces enclosing R_1 and R_2, because of the application of the divergence theorem. Because of (3.1.24) and the normal sign convention, as explained above, the integrands of the interface surface integrals in (3.1.29) cancel one another at each interface point and thus the two interface surface integrals cancel. Then in view of the traction definition (3.1.23), (3.1.29) immediately reduces to (3.1.26).

Also, since

$$\sigma_{ij}(\underline{x}, t) = \sigma_{ji}(\underline{x}, t)$$

it follows that

$$\bar{\sigma}_{ij}(t) = \bar{\sigma}_{ji}(t) \qquad (3.1.30)$$

Therefore (3.1.26) can be symmetrized in the form

$$\boxed{\bar{\sigma}_{ij}(t) = \frac{1}{2V}\left[\int_S (x_j\,T_i + x_i\,T_j)\,dS + \int_V (x_j\,F_i + x_i\,F_j)\,dV\right]} \qquad (3.1.31)$$

Precisely the same result holds for stress rates when (3.1.22) is fulfilled, (3.1.24) is replaced by traction rate continuity and (3.1.23) - by prescribed traction rates. Then,

$$\dot{\bar{\sigma}}_{ij}(t) = \frac{1}{V}\left[\int_S x_j \dot{T}_i \, dS + \int_V x_j \dot{F}_i \, dV\right]$$

$$= \frac{1}{2V}\left[\int_S (x_j \dot{T}_i + x_i \dot{T}_j) dS + \int_V (x_j \dot{F}_i + x_i \dot{F}_j) \, dV\right] \tag{3.1.32}$$

Again it should be noted that because of the time dependence of V, S and S_{12}, (3.1.32) is not the time derivative of (3.1.31). This becomes true if, and only if, the geometry is time independent.

We now prove an important corollary which is similar to (3.1.17 - 18).

If

$$T_i(S,t) = \sigma^o_{ij}(t) \; n_j \tag{3.1.33}$$

$$F_i(\underline{x}, t) = 0 \tag{3.1.34}$$

then

$$\bar{\sigma}_{ij}(t) = \sigma^o_{ij}(t) \tag{3.1.35}$$

<u>Proof:</u> Substitute (3.1.33-34) into (3.1.29). Then

$$\bar{\sigma}_{ij}(t) = \frac{1}{V} \sigma^o_{ik}(t) \int_S x_j n_k \, dS$$

and (3.1.35) follows immediately by use of (3.1.12).

By precisely the same proof we also have: If

$$\dot{T}_i(S, t) = \dot{\sigma}^o_{ij}(t) \; n_j \tag{3.1.36}$$

$$\dot{F}_i(\underline{x}, t) = 0 \tag{3.1.37}$$

then

$$\dot{\bar{\sigma}}_{ij}(t) = \dot{\sigma}^{o}_{ij}(t) \tag{3.1.38}$$

3.1.3 Average Virtual Work Theorems

The third class of theorems to be proved are virtual work type theorems. We shall prove such theorems in detail for stresses and strains and then we shall write down corresponding theorems for stress and strain rates by analogy.

Suppose that within a two phase body the displacement field $u_i(\underline{x}, t)$ is continuous and obeys (3.1.15). The associated strain field is given by (3.1.13). Also consider a stress field $\sigma_{ij}(\underline{x}, t)$ within the same two phase body which may be unrelated to the strain field. The stress field obeys (3.1.21) with vanishing body forces and also the continuity condition (3.1.24).

The strains and stresses at each point are split into averages and deviations from the average, thus

$$\epsilon_{ij}(\underline{x}, t) = \bar{\epsilon}_{ij}(t) + \epsilon'_{ij}(\underline{x}, t) \tag{3.1.39}$$

$$\sigma_{ij}(\underline{x}, t) = \bar{\sigma}_{ij}(t) + \sigma'_{ij}(\underline{x}, t) \tag{3.1.40}$$

It follows from the definition of the volume average and (3.1.39-40) that

$$\int_V \epsilon'_{ij}(\underline{x}, t)\, dV = \int_V \sigma'_{ij}(\underline{x}, t)\, dV = 0 \tag{3.1.41}$$

It then follows from (3.1.39-41) that

$$J = \int_V \sigma_{ij}\, \epsilon_{ij}\, dV = \bar{\sigma}_{ij}\, \bar{\epsilon}_{ij}\, V + \int_V \sigma'_{ij}\, \epsilon'_{ij}\, dV \tag{3.1.42}$$

The strain field $\bar{\epsilon}_{ij}$ (t) is derivable from a displacement field

$$u^{o}_{i} = \bar{\epsilon}_{ij} \ (t) \ x_j \tag{3.1.43}$$

Therefore the displacement at each point in the body may be written as

$$u_i \ (\underline{x}, t) = u^{o}_{i} + u'_i \ (\underline{x}, t) \tag{3.1.44}$$

where

$$\epsilon'_{ij} = \frac{1}{2} (u'_{i,j} + u'_{j,i}) \tag{3.1.45}$$

Now $\bar{\sigma}_{ij}$ (t) is not space dependent and it therefore trivially satisfies equilibrium without body forces. Since σ_{ij} also satisfies equilibrium it follows from (3.1.40) that

$$\sigma'_{ij,j} = 0 \tag{3.1.46}$$

In view of (3.1.45-46) the theorem of virtual work (see appendix to chap. 3.1) is applicable to the second integral in the right side of (3.1.42). Thus

$$J' = \int_V \sigma'_{ij} \ \epsilon'_{ij} \ dV = \int_{S_1} \sigma'^{(1)}_{ij} n^{(1)}_j u'^{(1)}_i \ dS + \int_{S_2} \sigma'^{(2)}_{ij} \ n^{(2)}_j \ u'^{(2)}_i \ dS$$

which may be rewritten as

$$J' = \int_S \sigma'_{ij} \ n_j \ u'_i \ dS + \int_{S_{12}} \sigma'^{(1)}_{ij} \ n^{(1)}_j \ u'^{(1)}_i \ dS$$

$$+ \int_{S_{12}} \sigma'^{(2)}_{ij} \ n^{(2)}_j \ u'^{(2)}_i \ dS \tag{3.1.47}$$

In view of (3.1.43-44) and (3.1.15), the displacements u_i' are continuous across S_{12}. Also in view of (3.1.40) and (3.1.24) the tractions $\sigma_{ij}' \; n_j$ are continuous across the interface S_{12}. Thus the integrands in the S_{12} surface integrals in (3.1.47) cancel at each point because of difference in normal sign convections involved in (3.1.24) and the divergence theorem, respectively. Therefore, we have the result,

$$J = \bar{\sigma}_{ij} \; \bar{\epsilon}_{ij} \; V + \int_S \sigma_{ij}' \; n_j \; u_i' \; dS \tag{3.1.48}$$

The result (3.1.48) leads to the following two theorems

(a) If on the bounding surface of the heterogeneous body the displacements are

$$u_i(S, t) = \epsilon_{ij}^{o}(t) \; x_j \tag{3.1.17}$$

then

$$J = \bar{\sigma}_{ij}(t) \; \bar{\epsilon}_{ij}(t) \; V = \bar{\sigma}_{ij}(t) \; \epsilon_{ij}^{o}(t) \; V \tag{3.1.49}$$

(b) If on the bounding surface of the heterogeneous body

$$T_i(S, t) = \sigma_{ij}^{o}(t) \; n_j \tag{3.1.33}$$

then

$$J = \bar{\sigma}_{ij}(t) \; \bar{\epsilon}_{ij}(t) V = \sigma_{ij}^{o}(t) \; \bar{\epsilon}_{ij}(t) \; V \tag{3.1.50}$$

These theorems will be called the <u>average theorems of virtual work.</u>

Proof: Consider first (3.1.17). In view of (3.1.43-44), (3.1.17-18), $u_i'(S) = 0$, and consequently the second integral on the right side of (3.1.48) vanishes. This establishes (3.1.49).

Next consider (3.1.33). Because of (3.1.40) and (3.1.35) we have $\sigma'_{ij}\ n_j = 0$ on S and therefore again the second integral in (3.1.48) vanishes, establishing (3.1.50). By precisely the same method <u>average rate theorems of virtual work</u> may be established. If $\overset{o}{\epsilon}_{ij}$ and/or $\overset{o}{\sigma}_{ij}$ are replaced by strain rates and/or stress rates, (3.1.49) and/or (3.1.50) hold with corresponding rate replacement.

For example: if

$$v_i(S, t) = \dot{\overset{o}{\epsilon}}_{ij}(t)\ x_j \tag{3.1.51}$$

then

$$\int_V \sigma_{ij}\ \dot{\epsilon}_{ij}\, dV = \bar{\sigma}_{ij}(t)\ \dot{\bar{\epsilon}}_{ij}(t)\, V = \bar{\sigma}_{ij}(t)\ \dot{\overset{o}{\epsilon}}_{ij}(t)\, V \tag{3.1.52}$$

APPENDIX

Divergence Theorem

The usual divergence theorem for a vector F_i asserts that

$$\int_V F_{i,i}\, dV = \int_S F_i\, n_i\, dS \tag{1}$$

The extended divergence theorem asserts that

$$\int F_{i,j}\, dV = \int_S F_i\, n_j\, dS \tag{2}$$

Note that (2) is more general than (1), since (1) follows from (2) by contraction (summation over equal values of i and j).

A more general result than (2) is

$$\int_V a_{ij,k}\, dV = \int_S a_{ij}\, n_k\, dS \tag{3}$$

with obvious further generalizations.

Theorem of Virtual Work

This extremely important theorem may be summarized as follows. Let $\sigma_{ij}(\underline{x})$ be a stress field in a body of volume V and bounding surface S, which satisfies static equilibrium everywhere, i.e.

$$\sigma_{ij,j} + F_i = 0$$

Let ϵ_{ij} $(\underline{x})$ be a compatible strain system defined throughout the same body , but completely unrelated to the previously considered stresses. The theorem of virtual work asserts that

$$\int_V \sigma_{ij}\, \epsilon_{ij}\, dV = \int_S T_i\, u_i\, dS + \int_V F_i\, u_i\, dV$$

where

$$T_i = \sigma_{ij}\, n_j$$

and u_i are the displacements from which the ϵ_{ij} are derived.

It cannot be too strongly emphasized that the theorem of virtual work has no physical meaning in its present form. It becomes physically meaningful when it is considered in the special case of stresses which are related to the strains such as in elasticity. In that latter case it becomes Clapeyron's theorem which states that in an elastic body the internal work (strain energy) equals the external work.

The theorem of virtual work in its above form remains valid for a multiphase body if the tractions $\sigma_{ij}\, n_j$ and the displacements u_i are continuous across phase interfaces.

3.2 THE ELASTICITY PROBLEM FOR HOMOGENEOUS AND HETEROGENEOUS BODIES

3.2.1 Homogeneous and Continuously Nonhomogeneous Bodies

We consider in this chapter the formulation of classical elasticity problems for homogeneous and heterogeneous bodies.

Let $u_i(\underline{x})$ be the small displacement field in an elastic body. The small strain tensor $\epsilon_{ij}(\underline{x})$ is defined by

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.2.1}$$

The symmetric stress tensor $\sigma_{ij}(\underline{x})$ is related to the strain tensor by Hooke's law which for the general anisotropic case has the form

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \tag{3.2.2}$$

The components of the fourth rank tensor C_{ijkl} are known as the <u>elastic moduli</u>. They obey the symmetry relations

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{3.2.3}$$

As a consequence of (3.2.3) there remain at most 21 different C_{ijkl}.

The inversion of (3.2.2) is written in the form

$$\epsilon_{ij} = S_{ijkl}\,\sigma_{kl} \tag{3.2.4}$$

where S_{ijkl} are known as the <u>elastic compliances</u>. The S_{ijkl} also obey symmetry relations of type (3.2.3), i.e.

$$S_{ijkl} = S_{jikl} = S_{ijlk} = S_{klij} \tag{3.2.5}$$

The relationship between C_{ijkl} and S_{ijkl} may be written compactly in the form

$$S_{ijrs}\ C_{rskl} = I_{ijkl} \tag{3.2.6}$$

$$I_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{3.2.7}$$

where δ_{ij} is the Kronecker delta and I_{ijkl} is a symmetric fourth rank unit tensor.

It should be noted that for an isotropic elastic body the C_{ijkl} tensor assumes the form

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + 2G\,I_{ijkl}$$

where λ is the usual Lamé modulus and G is the shear modulus.

The elastic moduli C_{ijkl} and the compliances S_{ijkl} are further restricted by positive definiteness requirements of the elastic energy density. This energy density is defined by

$$W = \frac{1}{2}\,\sigma_{ij}\,\epsilon_{ij} \tag{3.2.8}$$

Substitution of (3.2.2) into (3.2.8) yields the strain energy density

$$W^{\epsilon} = \frac{1}{2}\,C_{ijkl}\,\epsilon_{ij}\,\epsilon_{kl} \geq 0 \tag{3.2.9}$$

Substitution of (3.2.4) into (3.2.6) yields the stress energy density

$$W^{\sigma} = \frac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl} \geq 0 \quad (*) \tag{3.2.10}$$

Equality to zero in (3.2.9-10) occurs if, and only if, $\varepsilon_{ij} = 0$ or $\sigma_{ij} = 0$, respectively.

In the static case the stresses obey the equilibrium equations

$$\sigma_{ij,j} + F_i = 0 \tag{3.2.11}$$

where F_i are the body forces per unit volume. Substitution of (3.2.1) into (3.2.2) and the result into (3.2.4) yields a set of differential equations for displacements

$$(C_{ijkl} u_{k,l})_{,j} + F_i = 0 \tag{3.2.12}$$

where the symmetry relations (3.1.3) have been exploited. It has been assumed in (3.2.12) that the C_{ijkl} are <u>continuously</u> space variable. In the important special case when C_{ijkl} are constant, the elastic body is called <u>homogeneous</u>. In that event (3.2.12) simplifies to

$$C_{ijkl} u_{k,lj} + F_i = 0 \tag{3.2.13}$$

For an isotropic elastic body (3.2.13) reduces to the well known Navier equations

$$(\lambda + G) u_{j,ji} + G u_{i,jj} + F_i = 0$$

(*) Strictly speaking (3.2.9-10) define positive semi-definite quadratic forms. For reasons of compactness the expression positive definite will be used throughout this work.

The traction vector components T_i at a surface point where the normal has the components n_i with respect to some cartesian coordinate system, are given by

$$T_i = \sigma_{ij} \; n_j \tag{3.2.14}$$

The tractions (3.2.14) are easily expressed in terms of displacement gradients by substitution of (3.2.1 - 2) into (3.2.14). This yields

$$T_i = C_{ijkl} \, u_{k,l} \, n_j \tag{3.2.15}$$

where the symmetry (3.2.3) has been exploited.

In a typical boundary value problem the boundary conditions are

$$u_i = u^o_i \qquad \text{on } S_u \tag{3.2.16}$$

$$T_i = C_{ijkl} \, u_{k,l} \, n_j = T^o_i \qquad \text{on } S_T \tag{3.2.17}$$

That is to say, displacements are prescribed on the part S_u of the boundary and tractions on the part S_T of the boundary.

If the system of differential equations (3.2.12) has a solution which satisfies (3.2.16-17) then this solution is unique, apart from a possible rigid body motion (in the event that tractions are prescribed over the entire boundary). For proof see e.g. [3.1] . In any event, the strains and consequently the stresses are always unique since a rigid body motion produces no strains.

3.2.2 Exact Solutions for Homogeneous Bodies of Arbitrary Shape

We shall now consider two cases in which the elasticity problem can be solved for bodies of arbitrary shape. Let the body be homogeneous and let the body forces vanish. The differential equations (3.2.13) then assume the form

$$C_{ijkl}\, u_{k,lj} = 0 \tag{3.2.18}$$

In the first case linear boundary displacements are applied to the entire surface S, i.e.

$$u_i\,(S) = \epsilon^{o}_{ij}\, x_j \tag{3.2.19}$$

where ϵ^{o}_{ij} are symmetric constants and x_j are the surface coordinates. In that event the displacement field inside the body is given by

$$u_i\,(\underline{x}) = \epsilon^{o}_{ij}\, x_j \tag{3.2.20}$$

The proof of this statement is immediate: The displacements (3.2.20) obviously satisfy (3.2.19) . Since they are linear in x_j and all displacement derivatives in (3.2.18) are of second order they also satisfy (3.2.18), trivially. By the previously mentioned uniqueness theorem it follows that (3.2.20) is the unique solution.

Insertion of (3.2.20) into (3.2.1) shows that the strains in the body are given by

$$\epsilon_{ij}\,(\underline{x}) = \epsilon^{o}_{ij} \tag{3.2.21}$$

and they are thus constant. Insertion of (3.2.21) into (3.2.2) shows that the stresses are also constant and are given by

$$\sigma_{ij} = C_{ijkl} \; \epsilon^{o}_{ij} \tag{3.2.22}$$

Next we consider a body which has tractions prescribed over its entire surface, in the form

$$T_i \, (S) = \sigma^{o}_{ij} \; n_j \tag{3.2.23}$$

where σ^{o}_{ij} are symmetric constants and n_j are the components of the outward normal to the surface. To find a solution first define the constants α_{ij} by

$$\alpha_{ij} = S_{ijkl} \; \sigma^{o}_{kl} \tag{3.2.24}$$

The solution is then

$$u_i \, (\underline{x}) = \alpha_{ij} \, x_j \tag{3.2.25}$$

To prove this statement it is again noted that, by its structure, (3.2.25) trivially satisfies (3.2.18). Insertion of (3.2.25) into (3.2.15) shows in view of (3.2.24), (3.1.6) and (3.2.7), that the boundary conditions (3.2.23) are satisfied. By the uniqueness theorem (3.2.25) is the displacement, apart from rigid body motion. It follows that the strains are constant and are given by (3.2.24) and the stresses are also constant and are given by

$$\sigma_{ij} \, (\underline{x}) = \sigma^{o}_{ij} \tag{3.2.26}$$

Since either one of the boundary conditions (3.2.19) or (3.2.23) leads to homogeneous (constant) fields of strain and stress in homogeneous elastic

bodies of arbitrary shape, such boundary conditions will from now on be called homogeneous boundary conditions. (*)

It is to be carefully noted that the elementary solutions derived do not hold for bodies with variable elastic moduli.

The preceding solutions illuminate the significance of homogeneous boundary conditions. It was seen that when such boundary conditions are applied to homogeneous elastic bodies the fields of strain and stress throughout are uniform. Thus important cases such as isotropic (hydrostatic) stress and strain, uniaxial stress and pure shearing and straining are all covered as special cases.

When such boundary conditions are applied to the surface of a heterogeneous body the fields inside are no longer uniform. However, the surface is deformed or loaded as if the body were homogeneous with homogeneous strain and stress inside. Thus these boundary conditions express mathematically the fundamental tests of material behavior such as simple extension, biaxial stressing, pure shearing and hydrostatic stress which are performed in the laboratory on heterogeneous specimens. The forms of the ϵ^{o}_{ij} and σ^{o}_{ij} matrices for such cases are given below

Isotropic Deformation

$$\left[\epsilon^{o}_{ij}\right] = \begin{bmatrix} \epsilon^{o} & 0 & 0 \\ 0 & \epsilon^{o} & 0 \\ 0 & 0 & \epsilon^{o} \end{bmatrix}$$

(*) This should not be confused with the meaning of homogeneous boundary conditions in theory of differential equations.

Isotropic (Hydrostatic) Stress

$$\left[\sigma^{o}_{ij}\right] = \begin{bmatrix} \sigma^{o} & 0 & 0 \\ 0 & \sigma^{o} & 0 \\ 0 & 0 & \sigma^{o} \end{bmatrix}$$

Pure Shear Deformation in x_1 x_2 Plane

$$\left[\epsilon^{o}_{ij}\right] = \begin{bmatrix} 0 & \epsilon^{o}_{12} & 0 \\ \epsilon^{o}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Pure Shear Stress in x_1 x_2 Plane

$$\left[\sigma^{o}_{ij}\right] = \begin{bmatrix} 0 & \sigma^{o}_{12} & 0 \\ \sigma^{o}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Uniaxial Stress in x_1 Direction

$$\left[\sigma^{o}_{ij}\right] = \begin{bmatrix} \sigma^{o}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.2.3 Heterogeneous Bodies

We shall now discuss typical elasticity problems for heterogeneous bodies. For simplicity only two phase bodies shall be considered, extension to multiphase bodies being a straightforward matter.

The body of volume V and surface S is now composed of two phases, one occupying the domain R_1 with volume V_1, the other occupying the domain R_2 with volume V_2. The phases are assumed to be homogeneous with elastic moduli $C^{(1)}_{ijkl}$ and $C^{(2)}_{ijkl}$, respectively. The interface between the two phases is denoted S_{12}. The elasticity problem is then formulated as follows

$$\begin{aligned} C^{(1)}_{ijkl}\, u^{(1)}_{k,lj} + F^{(1)}_{i} &= 0 \qquad \text{in } R_1 \\ C^{(2)}_{ijkl}\, u^{(2)}_{k,lj} + F^{(2)}_{i} &= 0 \qquad \text{in } R_2 \end{aligned} \tag{3.2.27}$$

where superscripts define fields in the respective phases.

The boundary conditions (3.2.16 - 17) remain unchanged with the added provision that they apply to both displacement fields $u^{(1)}_i$ and $u^{(2)}_i$ on boundary parts composed of respective phase materials. There are added the interface conditions

$$\left.\begin{aligned} u^{(1)}_i &= u^{(2)}_i \qquad \text{(a)} \\ T^{(1)}_i &= T^{(2)}_i \qquad \text{(b)} \end{aligned}\right\} \quad \text{on } S_{12} \tag{3.2.28}$$

These interface conditions assume perfect bonding. The first assures that there is no separation between the phase materials and the second assures equilibrium at a bonded interface.

In the event of interface separation, voids appear which may be regarded as a third phase. The continuity conditions are then replaced by zero traction conditions on that part of S_{12} which is separated.

It is of interest to consider two extreme cases of two phase bodies. In the first case one phase, say 2, is considered as perfectly rigid. In that event (3.2.28) is replaced by

$$u^{(1)}_i = 0 \qquad \text{on} \qquad S_{12} \tag{3.2.29}$$

In the second case phase 2 is composed of voids. In that event

$$T^{(1)}_i = 0 \qquad \text{on } S_{12} \tag{3.2.30}$$

Problems for two phase bodies are by an order of magnitude more difficult than problems for homogeneous bodies. In particular, the simple results given above for homogeneous bodies under homogeneous boundary conditions are not valid any more because of the appearance of the interface conditions (3.2.28).

It is important to note that the uniqueness theorem is easily extended to two phase and multiphase bodies. It now asserts that if fields can be found which satisfy (3.2.27), (3.2.28) and (3.2.16-17) there is no other solution which satisfies the same equations (apart from arbitrary rigid body motions for $S_u = 0$).

3.3 EFFECTIVE STRESS-STRAIN RELATIONS OF GENERAL COMPOSITES

3.3.1 Heterogeneous Bodies with Homogeneous Boundary Conditions

In the present chapter the concept of effective elastic moduli (EEM) will be defined and discussed for general SH composite elastic materials without particular reference to the more special case of fibrous or fiber reinforced materials. The present paragraph is concerned with some general theorems which hold for any elastic body, homogeneous or nonhomogeneous.

Let a composite body with no body forces be subjected to the homogeneous boundary condition

$$u_i(S) = \epsilon^o_{ij} x_j \tag{3.3.1}$$

The formulation of the mathematical problem for determination of the displacements u_i at every point has been given in chapter 3.2. It is here desired to establish a relationship between field averages and the ϵ^o_{ij}

The ϵ^o_{ij} matrix can be separated into 6 matrices in each of which there occurs only one non-vanishing ϵ^o_{ij} strain. Thus

$$\left[\epsilon^o_{ij}\right] = \begin{bmatrix} \epsilon^o_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon^o_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon^o_{33} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \epsilon^o_{12} & 0 \\ \epsilon^o_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon^o_{23} \\ 0 & \epsilon^o_{23} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \epsilon^o_{13} \\ 0 & 0 & 0 \\ \epsilon^o_{13} & 0 & 0 \end{bmatrix}$$

To these correspond six boundary displacement vectors which for convenience are also written in matrix form

$$\left|u^{o}_{i}\right| = \begin{bmatrix} \epsilon^{o}_{11}\, x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon^{o}_{22}\, x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \epsilon^{o}_{33} x_3 \end{bmatrix} + \begin{bmatrix} \epsilon^{o}_{12}\, x_2 \\ \epsilon^{o}_{12}\, x_1 \\ 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ \epsilon^{o}_{23}\, x_3 \\ \epsilon^{o}_{23}\, x_2 \end{bmatrix} + \begin{bmatrix} \epsilon^{o}_{13}\, x_3 \\ 0 \\ \epsilon^{o}_{13}\, x_1 \end{bmatrix} \qquad (3.3.2)$$

Because of the superposition principle of the linear theory of elasticity the elastic field which is produced by (3.3.1) is equal to the sum of the six fields which are produced by the application of each of (3.3.2), separately, on the boundary. Consider for example the application of the first displacement on the right side of (3.3.2). Suppose that $\epsilon^{o}_{11} = 1$ and let the resulting displacement field be denoted $u^{(11)}_{i}(\underline{x})$. Then when $\epsilon^{o}_{11} \neq 1$ the field is by linearity $\epsilon^{o}_{11}\, u^{(11)}_{i}(\underline{x})$. Similar considerations for each of the displacements on the right side of (3.3.2) and superposition show that the displacement field in the body due to (3.3.1) on the boundary can be written in the form.

$$u_i(\underline{x}) = \epsilon^{o}_{kl}\; u^{(kl)}_{i}(\underline{x}) \qquad (3.3.3)$$

Here kl is summed. By (3.2.1) and (3.3.3) the strain at any point is given by

$$\epsilon_{ij}(\underline{x}) = \epsilon^{o}_{kl}\ \epsilon^{(kl)}_{ij}(\underline{x}) \tag{3.3.4}$$

where

$$\epsilon^{(kl)}_{ij}(\underline{x}) = \frac{1}{2}\left(u^{(kl)}_{ij} + u^{(kl)}_{j,i}\right) \tag{3.3.5}$$

Finally, the stress at any point is given in view of (3.3.4) and (3.2.2) by

$$\sigma_{ij}(\underline{x}) = \epsilon^{o}_{kl}\ C_{ijmn}(\underline{x})\ \epsilon^{(kl)}_{mn}(\underline{x}) \tag{3.3.6}$$

where the C_{ijmn} are the space dependent elastic moduli of the heterogeneous body. In the case of a two phase body they assume only the two values $C^{(1)}_{ijmn}$ and $C^{(2)}_{ijmn}$

Now let (3.3.6) be volume averaged. The result is written in the form

$$\bar{\sigma}_{ij} = C^{*}_{ijkl}\ \epsilon^{o}_{kl} \tag{3.3.7}$$

where

$$C^{*}_{ijkl} = \frac{1}{V}\int_{V} C_{ijmn}(\underline{x})\ \epsilon^{(kl)}_{mn}(\underline{x})\ dV \tag{3.3.8}$$

It is thus seen that a field quantity at any point and its average are linearly related to ϵ^{o}_{ij}. It should be borne in mind that this property is solely a consequence of the linearity of the governing equations.

In view of the average strain theorem (3.1.17 - 18), eqn. (3.3.7) can be written in the important form.

$$\boxed{\bar{\sigma}_{ij} = C^*_{ijkl}\,\bar{\epsilon}_{kl}} \tag{3.3.9}$$

Since $\bar{\sigma}_{ij}$ and $\bar{\epsilon}_{ij}$ are tensors of second rank it follows by the quotient law of tensor analysis (see e.g. [3.2]) that C^*_{ijkl} is a tensor of fourth rank.

If a composite elastic body is subjected to the homogeneous traction boundary conditions

$$T_i\,(S) = \sigma^o_{ij}\;n_j \tag{3.3.10}$$

then it can be shown by arguments completely analogous to the ones given above that all field quantities now are linearly related to σ^o_{ij} . In particular the average strains can be written as

$$\bar{\epsilon}_{ij} = S^*_{ijkl}\,\sigma^o_{kl} \tag{3.3.11}$$

In view of the average stress theorem (3.1.33 - 35), equation (3.3.11) can be rewritten in the important form

$$\boxed{\bar{\epsilon}_{ij} = S^*_{ijkl}\,\bar{\sigma}_{kl}} \tag{3.3.12}$$

It follows again by the quotient law that S^*_{ijkl} is a tensor of fourth rank.

The average theorems of virtual work which have been derived in chapter 3.1 are now recalled. Obviously these theorems apply as a special case to the elastic energy of a composite body subjected to the homogeneous boundary conditions (3.3.1) or (3.3.10). The elastic energy U is defined by

$$U = \frac{1}{2} \int_V \sigma_{ij}(\underline{x})\, \epsilon_{ij}(\underline{x})\, dV \tag{3.3.13}$$

When (3.3.1) is applied to the boundary we have

$$U = \frac{1}{2} \bar{\sigma}_{ij}\, \epsilon^{o}_{ij}\, V = \frac{1}{2} C^{*}_{ijkl}\, \epsilon^{o}_{ij}\, \epsilon^{o}_{kl}\, V \tag{3.3.14}$$

where the last equality follows from (3.3.7).

When (3.3.10) is applied to the boundary we have

$$U = \frac{1}{2} \sigma^{o}_{ij}\, \bar{\epsilon}_{ij}\, V = \frac{1}{2} S^{*}_{ijkl}\, \sigma^{o}_{ij}\, \sigma^{o}_{kl}\, V \tag{3.3.15}$$

where the last equality follows from (3.3.11). Expressions (3.3.14) and (3.3.15) may be called the <u>strain and stress energies</u>, respectively. The notation used will be

$$\begin{aligned} U^{\epsilon} &= \frac{1}{2} C^{*}_{ijkl}\, \epsilon^{o}_{ij}\, \epsilon^{o}_{kl}\, V = \frac{1}{2} C^{*}_{ijkl}\, \bar{\epsilon}_{ij}\, \bar{\epsilon}_{kl}\, V \qquad \text{(a)} \\ U^{\sigma} &= \frac{1}{2} S^{*}_{ijkl}\, \sigma^{o}_{ij}\, \sigma^{o}_{kl}\, V = \frac{1}{2} S^{*}_{ijkl}\, \bar{\sigma}_{ij}\, \bar{\sigma}_{kl}\, V \qquad \text{(b)} \end{aligned} \tag{3.3.16}$$

It is easily shown that both C^{*}_{ijkl} and S^{*}_{ijkl} obey the symmetry relations of elastic moduli, (3.2.3). Symmetry with respect to i,j and k,l interchange follows from (3.3.7) and (3.3.11) by stress and strain symmetry. To show symmetry with respect to ij, kl interchange we proceed by writing (3.3.9) in the form

$$\bar{\sigma}_{ij} = \overline{C_{ijkl}(\underline{x})\, \epsilon_{kl}(\underline{x})} = C^{*}_{ijkl}\, \bar{\epsilon}_{kl} \tag{3.3.17}$$

where $C_{ijkl}(\underline{x})$ denote the space variable (piecewise constant) elastic moduli of the heterogeneous body. It follows from (3.3.17) that

$$(C^*_{ijkl} - C^*_{klij})\,\bar{\epsilon}_{kl} = \overline{(C_{ijkl} - C_{klij})\,\epsilon_{kl}} = 0 \tag{3.3.18}$$

where the right side of (3.3.18) vanishes because of the last equality in (3.2.3). Therefore the left side of (3.3.18) also vanishes and since ϵ_{kl} is an arbitrary tensor it follows that each coefficient of $\bar{\epsilon}_{kl}$ must vanish. Therefore $C^*_{ijkl} = C^*_{klij}$.

In summary, then

$$\boxed{C^*_{ijkl} = C^*_{jikl} = C^*_{ijlk} = C^*_{klij}} \tag{3.3.19}$$

$$\boxed{S^*_{ijkl} = S^*_{jikl} = S^*_{ijlk} = S^*_{klij}} \tag{3.3.20}$$

which leads to at most 21 independent C^*_{ijkl} or S^*_{ijkl}. Note that (3.3.19 - 20) are entirely analogous to the classical relations (3.2.3) and (3.2.5).

It should be carefully noted that all of the results established so far in this chapter are rigorous results for any heterogeneous elastic body which is subjected to homogeneous boundary conditions. They apply to bodies containing one foreign inclusion as well as to bodies containing an immense number of inclusions.

3.3.2 Statistically Homogeneous Bodies

We now consider the special class of heterogeneous bodies which are statistically homogeneous. The concept of the statistically homogeneous body has been discussed in chapter 2.2. We shall now need another concept which is that of the statistically homogeneous field. The precise definition of the SH field must be given in probabilistic terms as was done with the SH body. We shall here discuss statistical field homogeneity in elementary fashion. For detailed statistical discussion the reader is referred to [2.2] .

Consider a volume element ΔV which is a small part of an heterogeneous body yet large enough to represent its structure. In a composite which consists of particles and matrix such an element must contain many particles. In a FRM we may choose a cylindrical element whose generators are parallel to the cylindrical composite specimen and whose cross section contains many fibers. We shall call such a volume element a representative volume element, henceforward abbreviated RVE. If the field is statistically homogeneous then the volume average taken over RVE approaches the whole body average, wherever the RVE may be located. Considering for example a SH stress field $\sigma_{ij}(\underline{x})$ we have

$$\bar{\sigma}_{ij} = \frac{1}{V} \int_V \sigma_{ij}(\underline{x})\, dV \cong \frac{1}{\Delta V} \int_{\Delta V} \sigma_{ij}(\underline{x})\, dV \qquad (3.3.21)$$

Suppose for illustrative purposes that the volume element is of circular cylindrical shape and its centroid is located at the point $\underline{x}$. The cylinder is now

expanded radially and the volume average is continuously taken. The average starts out with the value $\sigma_{ij}(\underline{x})$ at the point $\underline{x}$ and converges asymptotically to the body average $\bar{\sigma}_{ij}$. A schematic plot of average stress as function of ΔV is shown in fig. 3.2.1. Also fig. 3.2.2 shows a schematic variation of SH $\sigma_{ij}(\underline{x})$ along a line taken through the composite. In contrast fig. 3.2.3 shows the variation of statistically <u>non-homogeneous</u> σ_{ij} along a line taken through the composite.

Similar considerations apply to surface averages over large surface elements ΔA.

The statistical definition of SH involves <u>ensemble</u> averages. Such averages have been defined in chap. 2.2 in a geometrical context and there is no difficulty to give similar definitions in the present case. Thus $\langle \sigma_{ij} \rangle$, the ensemble average of σ_{ij}, is defined as the average of σ_{ij} at the same identical point taken over all the infinity of members of an ensemble of heterogeneous specimens. In general $\langle \sigma_{ij} \rangle$ is a function of position, but if σ_{ij} is SH this average is space independent and by the ergodic hypothesis equal to the body average or RVE average.

SH in the strict sense requires that the infinite set of n point averages be independent of the position of the point system within the ensemble members. For example

$$\langle \sigma_{ij}(\underline{x}^1)\, \sigma_{kl}(\underline{x}^2) \rangle = R_{ijkl}(\underline{x}^2 - \underline{x}^1)$$

$$\langle \sigma_{ij}(\underline{x}^1)\, \sigma_{kl}(\underline{x}^2)\, \sigma_{mn}(\underline{x}^3) \rangle = R_{ijklmn}(\underline{x}^2 - \underline{x}^1, \underline{x}^3 - \underline{x}^1)$$

For detailed discussion the reader is referred to [2.2]

It is to be noted that in a composite of periodic geometry the RVE is simply the repeating element. Thus in the FRM shown in fig. 2.1.5 the RVE is a composite cylinder of square section containing a symmetrically located circular fiber.

The question which now arises is under what circumstances are SH stress and strain fields produced in SH bodies? The answer to this question is contained in what shall be called the fundamental postulate of the theory of elasticity of heterogeneous media:

The stress and strain fields in a very large SH heterogeneous body, subjected to homogeneous boundary conditions, are SH, except in a narrow boundary layer near the external surface.

While the validity of this postulate is not in doubt, a general proof does not seem to be available. By way of some explanation it should be noted that homogeneity is certainly a special, albeit trivial, case of statistical homogeneity. Now it has been shown in par. 3.2.2 that the fields of stress and strain in homogeneous elastic bodies are homogeneous if the boundary conditions are homogeneous. Thus the present postulate extrapolates this rather simple state of affairs to the much more complicated case of heterogeneous SH bodies. It is clear that near the surface there must be a boundary layer where SH is not fulfilled. For a SH field looks roughly as the one shown in fig. 3.2.2 and the homogeneous boundary conditions (3.3.1) and (3.3.10) impose a constraint which does not permit the random fluctuation

to fully develop near the surface. The surface effect, however, diminishes very rapidly at points removed from the surface.

It is instructive to note that if the strain field is SH the displacement field is statistically nonhomogeneous. This is certainly to be expected, for if the strain field in a homogeneous body is uniform, ϵ^{o}_{ij}, say, then the displacement field is linear and has the form

$$u_i(\underline{x}) = \epsilon^{o}_{ij} x_j$$

It may be shown that for a SH strain field with average $\bar{\epsilon}_{ij}$ the ensemble average $\langle u_i \rangle$ of the displacement field is given by

$$\langle u_i \rangle (\underline{x}) = \bar{\epsilon}_{ij} x_j$$

which is thus space variable. Therefore $u_i(\underline{x})$ is not SH.

We now return to the results (3.3.9) and (3.3.12) which give the relations between stress and strain averages in heterogeneous bodies which are subjected to homogeneous boundary conditions. Evidently, the results remain valid for the present case of large SH bodies. The new significance of the results is in that, the coefficients C^{*}_{ijkl} and S^{*}_{ijkl} in (3.3.9) and (3.3.12) are now related. One matrix is the inverse of the other, which is mathematically expressed by

$$\boxed{\begin{aligned} & C^{*}_{ijmn} S^{*}_{mnkl} = I_{ijkl} && \text{(a)} \\ & I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) && \text{(b)} \end{aligned}} \qquad (3.3.22)$$

A general proof of this assertion does not seem to be available but its validity can hardly be in doubt. For in any large SH body in which the fields of stress and strain are statistically homogeneous, the imposition of average strains $\bar{\epsilon}_{ij}$ leads to average stresses $\bar{\sigma}_{ij}$ via (3.3.9). It is quite clear that imposition of the same $\bar{\sigma}_{ij}$ first, will lead to the same $\bar{\epsilon}_{ij}$ which were imposed initially. This implies that (3.3.9) can be substituted into the right side of (3.3.12) to yield

$$\bar{\epsilon}_{ij} = S^*_{ijmn} C^*_{mnkl} \bar{\epsilon}_{kl} \tag{3.3.23}$$

where $\bar{\epsilon}_{ij}$ and $\bar{\epsilon}_{kl}$ are by hypothesis the same average strain system. Recalling I_{ijkl} as defined by (3.3.22b) we can write

$$\bar{\epsilon}_{ij} = I_{ijkl} \bar{\epsilon}_{kl} \tag{3.3.24}$$

Subtraction of (3.3.24) from (3.3.23) gives

$$(S^*_{ijmn} C^*_{mnkl} - I_{ijkl}) \bar{\epsilon}_{kl} = 0 \tag{3.3.25}$$

Since $\bar{\epsilon}_{kl}$ is an arbitrary tensor the parenthesis in (3.3.25) must vanish which leads to the desired reciprocity relation (3.3.22a).

Note that such reciprocity is by no means valid for an arbitrary heterogeneous body such as one containing only two or three inclusions or fibers.

On the basis of the preceding discussion the C^*_{ijkl} are defined as the <u>effective elastic moduli</u> (EEM) of the SH heterogeneous body while the S^*_{ijkl} are defined as the <u>effective elastic compliances</u> (EEC). Equations (3.3.9)

and (3.3.12) are called the <u>effective stress-strain relations</u> of the heterogeneous body. Because of the relations (3.3.20) a SH elastic body has at most 21 independent EEM or EEC.

The reciprocity of the C^*_{ijkl} and S^*_{ijkl} tensors means that it does not matter whether homogeneous displacement or traction boundary conditions are used in effective elastic properties determination. So the choice can be made on the basis of convenience alone.

Moreover, it is sometimes necessary to use mixed homogeneous boundary conditions, i.e. (3.3.1) on a part S_u of S and (3.3.10) on a part S_T of S. This is permissible if σ^o_{ij} in (3.3.10) on S_T is given by (3.3.7) or dually ϵ^o_{ij} in (3.3.1) on S_u is given by (3.3.11).

Furthermore, it is sometimes necessary to prescribe one or two traction components of form (3.3.10) in one or two directions and two or one displacement components of form (3.3.1) in the remaining direction(s), over the entire surface. This is again permissible under similar conditions, i.e. if the σ^o_{ij} and ϵ^o_{ij} in the boundary conditions are related by (3.3.9) or (3.3.11).

It is intuitively clear (again there is no proof available) that the elastic energy stored within a RVE of a SH heterogeneous body, subjected to homogeneous boundary conditions, does not depend upon the RVE location within the body. If the RVE is arbitrarily defined as unit volume then it follows from (3.3.16) and the preceding that the strain energy $\overline{W}$ stored in a RVE is given by

$$\boxed{\bar{W}^{\epsilon} = \frac{1}{2} C^{*}_{ijkl} \bar{\epsilon}_{ij} \bar{\epsilon}_{kl} = \frac{1}{2} \bar{\sigma}_{ij} \bar{\epsilon}_{ij}} \tag{3.3.26}$$

and it follows also that the stress energy within a RVE is given by

$$\boxed{\bar{W}^{\sigma} = \frac{1}{2} S^{*}_{ijkl} \bar{\sigma}_{ij} \bar{\sigma}_{kl} = \frac{1}{2} \bar{\sigma}_{ij} \bar{\epsilon}_{ij}} \tag{3.3.27}$$

The expressions (3.3.26 - 27) may thus be called the <u>strain and stress energy densities</u>, respectively.

The procedure of computation of EEM is now outlined. A large heterogeneous body of some convenient shape is subjected to homogeneous boundary conditions of type (3.3.1). The average strains are then known and the average stresses have to be computed. To do this the elasticity problem of the heterogeneous body, as described in par. 3.2.3, has to be solved in detail, the stresses have to be found throughout the body and have then to be averaged. To isolate different EEM, boundary displacements of type (3.3.2) have to be applied separately. Thus, the first of these is

$$u^{o}_{i}(S) = \epsilon^{o}_{11} x_1 \ , \ u^{o}_{2}(S) = u^{o}_{3}(S) = 0 \tag{3.3.28}$$

In that case the only nonvanishing average strain is $\bar{\epsilon}_{11} = \epsilon^{o}_{11}$ according to (3.1.18). If the average stresses in the body subjected to (3.3.28) are known, then we have because of (3.3.7)

$$\bar{\sigma}_{ij} = C^{*}_{ij11} \epsilon^{o}_{11}$$

which defines the EEM C^{*}_{1111}, C^{*}_{2211}, C^{*}_{3311}, C^{*}_{1211}, C^{*}_{2311}, C^{*}_{31111}

By application of each of the displacements in the right side of (3.3.2), separately, all EEM are given similarly as ratios between an average stress and an average strain.

The procedure for computation of EEC is precisely the same, via homogeneous traction boundary conditions of type (3.3.10).

It should now be clear that the actual computation of EEM or EEC is an extremely difficult problem, since it is necessary to solve in detail an elasticity problem for a heterogeneous body. Because of the interface conditions (3.2.28) the solution depends upon all the details of the phase geometry and therefore also the EEM and EEC depend upon the entire phase geometry and of course also upon the phase elastic moduli. The dependence upon the entire phase geometry must be strongly emphasized for it has been frequently assumed in the literature that such simple geometrical information as volume fractions is sufficient for computation of EEM. This is, of course, in general incorrect and is true only in some very limited and special cases.

It should also be realized that if the phase geometry is not known in all detail there is in general not sufficient information available to solve the boundary value problem and thus there is also insufficient information for computation of the EEM. This immediately leads to the conclusion that actual computation of EEM must be limited to simple geometries.

3.3.3 Effective Elastic Properties in Terms of Phase Averages

Consider the effective stress-strain relation (3.3.9) for a two phase SH body, subjected to (3.3.1). Then

$$\bar{\epsilon}_{ij} = \epsilon^{o}_{ij} = \bar{\epsilon}^{(1)}_{ij} \; v_1 + \bar{\epsilon}^{(2)}_{ij} \; v_2 \qquad \text{(a)}$$

$$\bar{\sigma}_{ij} = \bar{\sigma}^{(1)}_{ij} \; v_1 + \bar{\sigma}^{(2)}_{ij} \; v_2 \qquad \text{(b)} \qquad (3.3.29)$$

where v_1 and v_2 are the phase volume fractions and quantities as $\bar{\epsilon}^{(1)}_{ij}$ etc. are averages over phase volumes, thus

$$\bar{\epsilon}^{(1)}_{ij} = \frac{1}{V_1} \int_{V_1} \epsilon^{(1)}_{ij} \; (\underline{x}) \, dV \qquad \text{etc}$$

The phases are elastic homogeneous and anisotropic and their stress-strain relations are

$$\sigma^{(1)}_{ij} = C^{(1)}_{ijkl} \; \epsilon^{(1)}_{kl} \qquad \text{(a)}$$

$$\sigma^{(2)}_{ij} = C^{(2)}_{ijkl} \; \epsilon^{(2)}_{kl} \qquad \text{(b)} \qquad (3.3.30)$$

Equations (3.3.30) may be averaged over the respective phase volumes and then become

$$\bar{\sigma}^{(1)}_{ij} = C^{(1)}_{ijkl} \; \bar{\epsilon}^{(1)}_{kl} \qquad \text{(a)}$$

$$\bar{\sigma}^{(2)}_{ij} = C^{(2)}_{ijkl} \; \bar{\epsilon}^{(2)}_{kl} \qquad \text{(b)} \qquad (3.3.31)$$

Now (3.3.9), (3.3.29) and (3.3.31) are five equations, from which we choose to eliminate the four quantities $\bar{\sigma}_{ij}$, $\bar{\sigma}^{(1)}_{ij}$, $\bar{\sigma}^{(2)}_{ij}$ and $\bar{\epsilon}^{(1)}_{ij}$

The result is

$$C^{*}_{ijkl}\,\epsilon^{o}_{kl} = C^{(1)}_{ijkl}\,\epsilon^{o}_{kl} + (C^{(2)}_{ijkl} - C^{(1)}_{ijkl})\,\bar{\epsilon}^{(2)}_{kl}\,v_2 \qquad (3.3.32)$$

Evidently $\bar{\epsilon}^{(2)}_{ij}$ could have been eliminated instead of $\bar{\epsilon}^{(1)}_{ij}$ The result for this case can be written down at sight by interchange of 1 with 2 and 2 with 1 in (3.3.32).

An explicit result for C^{*}_{ijkl} can be obtained in the following fashion: Because of the linearity of the problem the average $\bar{\epsilon}^{(2)}_{ij}$ must be linearly related to the ϵ^{o}_{ij} in the boundary conditions. (A similar argument has been given in detail in par. 3.3.1). Write

$$\bar{\epsilon}^{(2)}_{ij} = A^{(2)}_{ijkl}\,\epsilon^{o}_{kl} \qquad (3.3.33)$$

where $A^{(2)}_{ijkl}$ is a strain average influence tensor for phase 2. Introducing (3.3.33) into (3.3.32) we obtain

$$\left\{C^{*}_{ijkl} - \left[C^{(1)}_{ijkl} + (C^{(2)}_{ijmn} - C^{(1)}_{ijmn})\,A^{(2)}_{mnkl}\,v_2\right]\right\}\epsilon^{o}_{kl} = 0 \qquad (3.3.34)$$

Since ϵ^{o}_{kl} are arbitrary strains, the parenthesis in (3.3.34) must vanish. Thus

$$\boxed{C^{*}_{ijkl} = C^{(1)}_{ijkl} + (C^{(2)}_{ijmn} - C^{(1)}_{ijmn})\,A^{(2)}_{mnkl}\,v_2} \qquad (3.3.35)$$

The whole preceding development may be easily repeated for homogeneous traction boundary conditions (3.3.10). This leads to

$$S^{*}_{ijkl}\,\sigma^{o}_{kl} = S^{(1)}_{ijkl}\,\sigma^{o}_{kl} + (S^{(2)}_{ijkl} - S^{(1)}_{ijkl})\,\bar{\sigma}^{(2)}_{kl}\,v_2 \tag{3.3.36}$$

or explicitly to

$$\boxed{S^{*}_{ijkl} = S^{(1)}_{ijkl} + (S^{(2)}_{ijmn} - S^{(1)}_{ijmn})\,B^{(2)}_{mnkl}\,v_2} \tag{3.3.37}$$

In (3.3.37), $B^{(2)}_{mnkl}$ is the stress average influence tensor for phase 2, which enters into the linear relationship

$$\bar{\sigma}^{(2)}_{ij} = B^{(2)}_{ijkl}\,\sigma^{o}_{kl} \tag{3.3.38}$$

The proof of (3.3.37-38) is left as an exercise to the reader. Note that insertion of (3.3.35) and (3.3.39) into (3.3.22) provides a relation between $A^{(2)}_{ijkl}$ and $B^{(2)}_{ijkl}$.

3.4 EFFECTIVE STRESS-STRAIN RELATIONS OF FIBER REINFORCED MATERIALS

3.4.1 Elastic Symmetry

It has been seen in the previous chapter that a SH heterogeneous body has at most 21 independent EEM in the most general anisotropic case. This situation is entirely analogous to the one encountered in the case of the general anisotropic homogeneous elastic body, par. 3.2.1. Fortunately, various symmetry considerations reduce the number of independent EEM and thus the effective stress-strain relations (3.3.9) or (3.3.12) can be greatly simplified.

Various kinds of geometrical symmetry have been briefly discussed in chapter 2.1. Here, we shall be concerned with elastic symmetry, i.e. symmetry considerations which are specifically tied to linear elastic stress - strain relations. It is to be noted that other kinds of materials constitutive relations would give rise to different symmetry properties. Thus elastic symmetry is a certain aspect of the more general subject of material symmetry.

The elastic moduli C_{ijkl} in the microscopic stress-strain law (3.2.2), as well as the effective elastic moduli C^*_{ijkl} in the macroscopic stress - strain law (3.3.9) are the components of fourth rank tensors. Their values are therefore defined in reference to a coordinate system. If the coordinate system is rotated the elastic moduli assume different values with respect to the new coordinate system, the new values being connected to the old values by the laws of tensor transformation. Elastic symmetry is expressed by the

property that under certain coordinate changes the elastic moduli remain unchanged. Basic coordinate changes are (a) Reflection in a plane, (b) Rotation about an axis (c) Rotation about an axis combined with reflection in a plane perpendicular to the axis. ([3.3] ., p. 152). If reflection in a plane leaves the stress-strain law unchanged then the plane is called a plane of elastic symmetry. If rotation of a coordinate direction about an axis leaves elastic properties unchanged then the initial and final coordinate directions, which are perpendicular to the axis, are equivalent elastic directions. If all rotations about a fixed axis do not change elastic properties, the axis is one of rotational elastic symmetry, in short an axis of symmetry.

In a heterogeneous body the symmetry may be microscopic, i.e. for the elastic properties at a point, or the symmetry may be macroscopic in terms of effective elastic moduli when the point is replaced by a RVE. For example, in a fibrous material any transverse x_2, x_3 plane is a plane of microscopic as well as of macroscopic elastic symmetry (if it is also a plane of elastic symmetry for the phase materials). If the fibers are randomly placed in the cross section, then the x_1 axis may in many cases be assumed to be an axis of macroscopic rotational symmetry but it is not in general an axis of microscopic rotational symmetry.

Reductions of the stress-strain law (3.2.2) for various kinds of elastic symmetry may be found in the literature in books such as Love [3.3] , Sokolnikoff [3.2] and Lekhnitski [3.4] . Since the macroscopic stress-strain law (3.3.9) is the complete mathematical analogue of (3.2.2) it is evident

that symmetry reductions for the latter are mathematically identical to reductions for the former. Therefore there is no need to perform such reductions here in detail.

Before proceeding further a comment about the concept of independent elastic moduli is in order: The reduction of the number of independent moduli from 21 to any lower number is based upon information which is available to us about the elastic material. At present the only information to be used is that of elastic symmetry. We shall see later on that information about the structure of the material can also be used to reduce the number of independent moduli.

3.4.2 Orthotropic Materials

The most complicated case to be considered in the present work is the macroscopically orthotropic body. The orthotropic elastic body is one which has three mutually perpendicular planes of elastic symmetry. An example for such a material is provided by the rectangular array of identical circular fibers shown in fig. 2.1.4. It is clear that the coordinate planes are planes of geometrical symmetry, however, this in itself is not sufficient for we are concerned here with elastic symmetry.

In order to ensure macroscopic elastic orthotropy the phases elastic behaviour must obey certain conditions, which are of two kinds. The first kind involves elastic symmetry of the phase materials and the second involves the direction of the phase elastic axes with respect to fixed composite body axes x_1, x_2, x_3. It would be tedious and superfluous to list all the

conditions under which macroscopic orthotropy is fulfilled for a geometrically orthotropic body. Suffice it here to give some important examples: The geometrically orthotropic FM or uniaxially FRM is elastically orthotropic if the phase materials elastic symmetry is not less than orthotropic, with fixed elastic axes parallel to the x_1, x_2, x_3 axes. On the other hand macroscopic elastic orthotropy is also fulfilled if in the fibers the x_1 axes remain fixed, while the other two axes are randomly oriented in the x_2, x_3 plane.

The most important case is isotropic fibers and matrix which is of course included as a special case in both examples. In that case there is no directional effect of phase elastic axes.

The orthotropic form of the effective stress-strain laws (3.3.9) and (3.3.12) involve only nine independent constants. Eqns. (3.3.9) assume the form

$$\bar{\sigma}_{11} = C^*_{11}\,\bar{\epsilon}_{11} + C^*_{12}\,\bar{\epsilon}_{22} + C^*_{13}\,\bar{\epsilon}_{13} \qquad \text{(a)}$$

$$\bar{\sigma}_{22} = C^*_{12}\,\bar{\epsilon}_{11} + C^*_{22}\,\bar{\epsilon}_{22} + C^*_{23}\,\bar{\epsilon}_{33} \qquad \text{(b)} \qquad (3.4.1)$$

$$\bar{\sigma}_{33} = C^*_{13}\,\bar{\epsilon}_{11} + C^*_{23}\,\bar{\epsilon}_{22} + C^*_{33}\,\bar{\epsilon}_{33} \qquad \text{(c)}$$

$$\bar{\sigma}_{12} = 2\,C^*_{44}\,\bar{\epsilon}_{12} \qquad \text{(a)}$$

$$\bar{\sigma}_{23} = 2\,C^*_{55}\,\bar{\epsilon}_{23} \qquad \text{(b)} \qquad (3.4.2)$$

$$\bar{\sigma}_{31} = 2\,C^*_{66}\,\bar{\epsilon}_{31} \qquad \text{(c)}$$

The two subscript notations for the effective elastic moduli in (3.4.1 - 2) is connected to the four subscript notation of (3.3.9) by

$$C^*_{11} = C^*_{1111},\ C^*_{22} = C^*_{2222},\ C^*_{33} = C^*_{3333} \qquad \text{(a)}$$

$$C^*_{12} = C^*_{1122},\ C^*_{23} = C^*_{2233},\ C^*_{31} = C^*_{3311} \qquad \text{(b)} \qquad (3.4.3)$$

$$C^*_{44} = C^*_{1212},\ C^*_{55} = C^*_{2323},\ C^*_{66} = C^*_{3131} \qquad \text{(c)}$$

It is seen that (3.4.2) are shear moduli for shears taking place in the coordinate planes. At times the following notation will be used for these

$$C^*_{44} = G^*_{12},\ C^*_{55} = G^*_{23},\ C^*_{66} = G^*_{31} \qquad (3.4.4)$$

The inverse of (3.4.1 - 2), i.e. the form which corresponds to (3.3.12) is written as

$$\boxed{\begin{aligned} \bar{\epsilon}_{11} &= S^*_{11}\,\bar{\sigma}_{11} + S^*_{12}\,\bar{\sigma}_{22} + S^*_{13}\,\bar{\sigma}_{33} \\ \bar{\epsilon}_{22} &= S^*_{12}\,\bar{\sigma}_{11} + S^*_{22}\,\bar{\sigma}_{22} + S^*_{23}\,\bar{\sigma}_{33} \\ \bar{\epsilon}_{33} &= S^*_{13}\,\bar{\sigma}_{11} + S^*_{23}\,\bar{\sigma}_{22} + S^*_{33}\,\bar{\sigma}_{33} \end{aligned}} \qquad \begin{aligned} &\text{(a)} \\ &\text{(b)} \\ &\text{(c)} \end{aligned} \qquad (3.4.5)$$

$$\bar{\epsilon}_{12} = 2\, S^*_{44}\, \bar{\sigma}_{12} \qquad \text{(a)}$$

$$\bar{\epsilon}_{23} = 2\, S^*_{55}\, \bar{\sigma}_{23} \qquad \text{(b)} \qquad (3.4.6)$$

$$\bar{\epsilon}_{31} = 2\, S^*_{66}\, \bar{\sigma}_{31} \qquad \text{(c)}$$

where S^*_{ij} in (3.4.5 - 6) are connected to S^*_{ijkl} in (3.3.12) by the same kind of relations as (3.4.3).

The S^*_{ij} in (3.4.5) are given in terms of C^*_{ij} in (3.4.1) by complicated algebraic expressions which may be found in [3.4] The relation between the shear moduli in (3.4.2) and the shear compliances in (3.4.6) is, however, rather simple and is obviously

$$S^*_{44} = \frac{1}{4C^*_{44}} \qquad S^*_{55} = \frac{1}{4C^*_{55}} \qquad S^*_{66} = \frac{1}{4C^*_{66}} \qquad (3.4.7)$$

It is furthermore customary to write the S^*_{ij} in (3.4.5) in the form

$$S^*_{11} = \frac{1}{E^*_1} \qquad S^*_{22} = \frac{1}{E^*_2} \qquad S^*_{33} = \frac{1}{E^*_3} \qquad (3.4.8)$$

$$S^*_{12} = -\frac{\nu^*_{12}}{E^*_1} = -\frac{\nu^*_{21}}{E^*_2} \qquad \text{(a)}$$

$$S^*_{23} = -\frac{\nu^*_{23}}{E^*_2} = -\frac{\nu^*_{32}}{E^*_3} \qquad \text{(b)} \qquad (3.4.9)$$

$$S^*_{31} = -\frac{\nu^*_{31}}{E^*_3} = -\frac{\nu^*_{13}}{E^*_1} \qquad \text{(c)}$$

Here, E^*_1, E^*_2, E^*_3 are effective Young's moduli associated with uniaxial stresses in coordinate directions. A ν^*_{ij} appearing in (3.4.9) indicates an effective Poisson's ratio in which i is the direction of uniaxial stress, producing transverse Poisson's strain in the j direction.

We shall now discuss appropriate homogeneous boundary conditions which are to be applied to an orthotropic specimen in order to determine the EEM and Poisson's ratios in (3.4.4) and (3.4.8 - 9). The development is based on the strain-stress relations (3.4.5 - 6) and the general theory of par. 3.2.2. We shall establish for each EEM two dual sets of appropriate homogeneous boundary conditions. It will be later seen that this is of crucial importance for bounding methods of EEM.

From the technical point of view the important elastic properties are the ones appearing in (3.4.4) and (3.4.8 - 9, for these enter into governing equations of structures of orthotropic materials and are also the ones which are measured in the laboratory.

We start by listing below homogeneous displacement boundary conditions

and corresponding average strain and stress matrices, for determination of the three effective shear moduli. It is convenient, though not necessary, to assume that the boundary conditions are applied to a cubical specimen whose sides are parallel to coordinate planes.

$\underline{G^*_{12}}$

$$u_1(S) = \epsilon^o_{12}\, x_2 \quad (a) \qquad u_2(S) = \epsilon^o_{12}\, x_1 \quad (b) \qquad u_3(S) = 0 \quad (c) \tag{3.4.10}$$

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & \epsilon^o_{12} & 0 \\ \epsilon^o_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (a) \qquad \left[\bar{\sigma}_{ij}\right] = \begin{bmatrix} 0 & \bar{\sigma}_{12} & 0 \\ \bar{\sigma}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (b) \tag{3.4.11}$$

$$G^*_{12} = \frac{\bar{\sigma}_{12}}{2\,\epsilon^o_{12}} \tag{3.4.12}$$

$\underline{G^*_{23}}$

$$u_1(S) = 0 \quad (a) \qquad u_2(S) = \epsilon^o_{23}\, x_3 \quad (b) \qquad u_3(S) = \epsilon^o_{23}\, x_2 \quad (c) \tag{3.4.13}$$

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon^o_{23} \\ 0 & \epsilon^o_{23} & 0 \end{bmatrix} \quad (a) \qquad \left[\bar{\sigma}_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\sigma}_{23} \\ 0 & \bar{\sigma}_{23} & 0 \end{bmatrix} \quad (b) \tag{3.4.14}$$

$$G^*_{23} = \frac{\bar{\sigma}_{23}}{2\,\epsilon^o_{23}} \tag{3.4.15}$$

$\underline{G^*_{13}}$

$$u_1(S) = \epsilon^o_{13} x_3 \quad (a) \qquad u_2(S) = 0 \quad (b) \qquad u_3(S) = \epsilon^o_{13} x_1 \quad (c) \tag{3.4.16}$$

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & 0 & \epsilon^o_{13} \\ 0 & 0 & 0 \\ \epsilon^o_{13} & 0 & 0 \end{bmatrix} \quad (a) \qquad \left[\bar{\sigma}_{ij}\right] = \begin{bmatrix} 0 & 0 & \bar{\sigma}_{13} \\ 0 & 0 & 0 \\ \bar{\sigma}_{13} & 0 & 0 \end{bmatrix} \quad (b) \tag{3.4.17}$$

$$G^*_{13} = \frac{\bar{\sigma}_{13}}{2\epsilon^o_{13}} \tag{3.4.18}$$

It is seen that in each case it is necessary to compute a single average shear stress to determine the effective shear modulus.

Dually, we may subject the cubical specimen to homogeneous traction boundary conditions. We list below the traction boundary conditions, the corresponding average shear stresses and the resulting average shear strains.

$\underline{G^*_{12}}$

$$T_1(S) = \sigma^o_{12} n_2 \quad (a) \qquad T_2(S) = \sigma^o_{12} n_1 \quad (b) \qquad T_3(S) = 0 \quad (c) \tag{3.4.19}$$

$$\left[\bar{\sigma}_{ij}\right] = \begin{bmatrix} 0 & \sigma^o_{12} & 0 \\ \sigma^o_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (a) \qquad \left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & \bar{\epsilon}_{12} & 0 \\ \bar{\epsilon}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (b) \tag{3.4.20}$$

$$G^*_{12} = \frac{\sigma^o_{12}}{2\bar{\epsilon}_{12}} \tag{3.4.21}$$

$\underline{G^*_{23}}$

$$T_1(S) = 0 \quad (a) \qquad T_2(S) = \sigma^o_{23}\, n_3 \quad (b) \qquad T_3(S) = \sigma^o_{23}\, n_2 \quad (c) \tag{3.4.22}$$

$$\bar{\sigma}_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma^o_{23} \\ 0 & \sigma^o_{23} & 0 \end{bmatrix} \quad (a) \qquad \bar{\epsilon}_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\epsilon}_{23} \\ 0 & \bar{\epsilon}_{23} & 0 \end{bmatrix} \quad (b) \tag{3.4.23}$$

$$G^*_{23} = \frac{\sigma^o_{23}}{2\bar{\epsilon}_{23}} \tag{3.4.24}$$

$\underline{G^*_{13}}$

$$T_1(S) = \sigma^o_{13}\, n_3 \quad (a) \qquad T_2(S) = 0 \quad (b) \qquad T_3(S) = \sigma^o_{13}\, n_1 \quad (c) \tag{3.4.25}$$

$$\bar{\sigma}_{ij} = \begin{bmatrix} 0 & 0 & \sigma^o_{13} \\ 0 & 0 & 0 \\ \sigma^o_{13} & 0 & 0 \end{bmatrix} \quad (a) \qquad \bar{\epsilon}_{ij} = \begin{bmatrix} 0 & 0 & \bar{\epsilon}_{13} \\ 0 & 0 & 0 \\ \bar{\epsilon}_{13} & 0 & 0 \end{bmatrix} \quad (b) \tag{3.4.26}$$

$$G^*_{13} = \frac{\sigma^o_{13}}{2\bar{\epsilon}_{13}} \tag{3.4.27}$$

It is seen that now a single average shear strain has to be computed in each case to determine an effective shear modulus. It is to be noted that dual determination of the effective shear moduli by (3.4.12), (3.4.15), (3.4.18) and (3.4.21), (3.4.24) , (3.4.27) is a direct consequence of the general assumption of reciprocity of C^*_{ijkl} and S^*_{ijkl} tensors, which was discussed in par. (3.2.2).

For the determination of effective Young's moduli and Poisson's ratios it is convenient to use cylindrical specimens each of whose axes is in the direction associated with the Young's modulus to be determined. Fig. 3.4.1 shows such a specimen with axis in x_1 direction. We list below homogeneous traction boundary conditions with associated average stress matrices and resulting average strain matrices which define the effective properties.

$\underline{E_1^*,\ \nu_{12}^*,\ \nu_{13}^*}$

$$T_1(S) = \sigma_{11}^o n_1 \quad (a) \qquad T_2(S) = 0 \quad (b) \qquad T_3(S) = 0 \quad (c) \qquad (3.4.28)$$

$$\left[\bar{\sigma}_{ij}\right] = \begin{bmatrix} \sigma_{11}^o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (a) \qquad \left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} \bar{\epsilon}_{11} & 0 & 0 \\ 0 & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix} \quad (b) \qquad (3.4.29)$$

$$E_1^* = \frac{\sigma_{11}^o}{\bar{\epsilon}_{11}} \quad (a) \qquad \nu_{12}^* = -\frac{\bar{\epsilon}_{22}}{\bar{\epsilon}_{11}} \quad (b) \qquad \nu_{13}^* = -\frac{\bar{\epsilon}_{33}}{\bar{\epsilon}_{11}} \quad (c) \qquad (3.4.30)$$

$\underline{E_2^*,\ \nu_{21}^*,\ \nu_{23}^*}$

$$T_1(S) = 0 \quad (a) \qquad T_2(S) = \sigma_{22}^o n_2 \quad (b) \qquad T_3(S) = 0 \quad (c) \qquad (3.4.31)$$

$$\left[\bar{\sigma}_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22}^o & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (a) \qquad \left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} \bar{\epsilon}_{11} & 0 & 0 \\ 0 & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix} \quad (b) \qquad (3.4.32)$$

$$E_2^* = \frac{\sigma_{22}^o}{\bar{\epsilon}_{22}} \quad (a) \qquad \nu_{21}^* = -\frac{\bar{\epsilon}_{11}}{\bar{\epsilon}_{22}} \quad (b) \qquad \nu_{23}^* = -\frac{\bar{\epsilon}_{33}}{\bar{\epsilon}_{22}} \quad (c) \qquad (3.4.33)$$

$\underline{E_3^*, \nu_{31}^*, \nu_{32}^*}$

$$T_1(S) = 0 \quad (a) \qquad T_2(S) = 0 \quad (b) \qquad T_3(S) = \sigma_{33}^o n_3 \quad (c) \tag{3.4.34}$$

$$\left[\bar{\sigma}_{ij}\right] \doteq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33}^o \end{bmatrix} \quad (a) \qquad \left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} \bar{\epsilon}_{11} & 0 & 0 \\ 0 & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix} (b) \tag{3.4.35}$$

$$E_3^* = \frac{\sigma_{33}^o}{\bar{\epsilon}_{33}} \quad (a) \qquad \nu_{31}^* = -\frac{\bar{\epsilon}_{11}}{\bar{\epsilon}_{33}} \quad (b) \qquad \nu_{32}^* = -\frac{\bar{\epsilon}_{22}}{\bar{\epsilon}_{33}} \quad (c) \tag{3.4.36}$$

A dual determination in terms of homogeneous displacement boundary conditions is not useful. Since such boundary conditions define average strains, it is seen that (3.4.1) will have to be used and only C_{ij}^* can be thus determined. The complicated relations between the C_{ij}^* and the S_{ij}^* make such an undertaking impractical. We can, however, obtain a simple dual formulation for the determination of effective Young's moduli and Poisson's ratios by use of mixed homogeneous boundary conditions. Consider for example (3.4.28). We replace the traction boundary conditions by the mixed boundary conditions

$$u_1(S) = \epsilon_{11}^o x_1 \quad (a) \qquad T_2(S) = T_3(S) = 0 \quad (b) \tag{3.4.37}$$

where ϵ_{11}^o may be interpreted as the $\bar{\epsilon}_{11}$ appearing in (3.4.29b). Then the states of strain and stress in the specimen are still statistically homogeneous.

The components of the outward normal on the surface of the cylindrical specimen shown in fig. 3.4.1 are subject to the following restrictions

$$\begin{array}{llll} n_1 = 1 & n_2 = n_3 = 0 & \text{on } A_H & (a) \\ n_1 = -1 & n_2 = n_3 = 0 & \text{on } A_o & (b) \\ n_1 = 0 & & \text{on } S_C & (c) \end{array} \tag{3.4.38}$$

Application of the average strain theorem (3.1.16) and of the average stress theorem

(3.1.26), with zero body forces, for (3.4.37) with (3.4.38), easily gives the results

$$\bar{\epsilon}_{11} = \epsilon^{o}_{11} \qquad \bar{\sigma}_{22} = \bar{\sigma}_{33} = \bar{\sigma}_{12} = \bar{\sigma}_{23} = \bar{\sigma}_{31} = 0 \tag{3.4.39}$$

It follows from (3.4.39), (3.4.5) and (3.4.8 - 9) that

$$E^{*}_{1} = \frac{\bar{\sigma}_{11}}{\epsilon^{o}_{11}} \quad \text{(a)} \qquad \nu^{*}_{12} = -\frac{\bar{\epsilon}_{22}}{\epsilon^{o}_{11}} \quad \text{(b)} \qquad \nu^{*}_{13} = -\frac{\bar{\epsilon}_{33}}{\epsilon^{o}_{11}} \quad \text{(c)} \tag{3.4.40}$$

We have thus obtained a dual formulation for E^{*}_{1}, ν^{*}_{12} and ν^{*}_{13} which requires the computation of $\bar{\sigma}_{11}$, $\bar{\epsilon}_{22}$ and $\bar{\epsilon}_{33}$ under boundary conditions (3.4.37).

In a completely analogous fashion we can obtain similar results for the other effective Young's moduli and Poisson's ratios, by using cylindrical specimens with axes in x_2 and x_3 directions, respectively. These results are now listed

$$\begin{aligned} &T_1(S) = 0 \qquad u_2(S) = \epsilon^{o}_{22}\, x_2 \qquad T_3(S) = 0 \qquad \text{(a)} \\ &\bar{\epsilon}_{22} = \epsilon^{o}_{22} \qquad \bar{\sigma}_{11} = \bar{\sigma}_{33} = \bar{\sigma}_{12} = \bar{\sigma}_{23} = \bar{\sigma}_{31} = 0 \qquad \text{(b)} \end{aligned} \tag{3.4.41}$$

$$E^{*}_{2} = \frac{\bar{\sigma}_{22}}{\epsilon^{o}_{22}} \qquad \nu^{*}_{21} = -\frac{\bar{\epsilon}_{11}}{\epsilon^{o}_{22}} \qquad \nu^{*}_{23} = -\frac{\bar{\epsilon}_{33}}{\epsilon^{o}_{22}} \tag{3.4.42}$$

$$\begin{aligned} &T_1(S) = 0 \qquad T_2(S) = 0 \qquad u_3(S) = \epsilon^{o}_{33}\, x_3 \qquad \text{(a)} \\ &\bar{\epsilon}_{33} = \epsilon^{o}_{33} \qquad \bar{\sigma}_{11} = \bar{\sigma}_{22} = \bar{\sigma}_{12} = \bar{\sigma}_{23} = \bar{\sigma}_{31} = 0 \qquad \text{(b)} \end{aligned} \tag{3.4.43}$$

$$E_3^* = \frac{\bar{\sigma}_{33}}{\overset{o}{\epsilon}_{33}} \qquad \nu_{31}^* = -\frac{\bar{\epsilon}_{11}}{\overset{o}{\epsilon}_{33}} \qquad \nu_{32}^* = -\frac{\bar{\epsilon}_{22}}{\overset{o}{\epsilon}_{33}} \tag{3.4.44}$$

It is emphasized again that all the problems listed are very difficult to solve since the elastic fields in the specimen, which must satisfy phase differential equations and phase interface continuity, have to be found in order to compute the required stress and strain averages.

3.4.3 Square Symmetric and Transversely Isotropic Materials

A square symmetric material is an orthotropic material in which two axes x_2 and x_3 say, are elastically equivalent. This means that the stress-strain law is insensitive to a 90° rotation of the x_2, x_3 axes around the x_1 axis. As will be seen later the square symmetric material is in a certain sense the two dimensional analogue of a cubic material.

A transversely isotropic material is one in which the x_1 axis is an axis of rotational elastic symmetry. This means that the elastic stress-strain law is insensitive to any rotation of the x_2, x_3 axes around the x_1 axis. This material is thus a special case of the square symmetric material.

An example for a geometrically square symmetric material is the square array of circular fibers, shown in fig. 2.1.5. The most important case in which such a material is also elastically square symmetric is when the phases elastic symmetry is not less than square, with fixed axes parallel to x_1, x_2, x_3. This, of course, includes transversely isotropic and

completely isotropic phases as special cases.

The most important case of symmetry in a FM or FRM is transverse isotropy, since it is applicable to random and semi-random geometries. Suppose that the transverse plane geometry is of such nature and that the phases are transversely isotropic, with fixed x_1 axis, or completely isotropic. Another case of some interest is transversely isotropic or isotropic matrix and orthotropic fibers with fixed x_1 axes and the other two axes randomly oriented in the x_2, x_3 plane. If the geometry has no directional bias in the statistical sense, in the x_2, x_3 plane (statistically transversely isotropic geometry, see chapter 2.2), then the material described cannot be expected to have any directional bias in the x_2, x_3 plane for its effective elastic behavior. It may thus be assumed that the material is elastically transversely isotropic. This is the most important kind of FRM since it is of such frequent occurrence.

Another example is the hexagonal array of identical fibers, fig. 2.1.6 with same restriction on phase elastic symmetry. The transverse isotropy of this material is a consequence of the theorem that a material which has an n fold axis of symmetry where n = 3, 5, 6,.... is transversely isotropic , see e.g. Love [3.3]. In the present case n = 3.

It is easily shown that for square symmetry there are the following relations among the EEM of (3.4.1 - 2).

$$C^*_{12} = C^*_{13} \qquad C^*_{22} = C^*_{33} \qquad C^*_{44} = C^*_{66} \tag{3.4.45}$$

If the material is transversely isotropic there is added to (3.4.45) the relation

$$C^*_{55} = \frac{1}{2}\left(C^*_{22} - C^*_{33}\right) \tag{3.4.46}$$

The effective stress strain relations of both materials are thus

$$\bar{\sigma}_{11} = C^*_{11}\,\bar{\epsilon}_{11} + C^*_{12}\,\bar{\epsilon}_{22} + C^*_{12}\,\bar{\epsilon}_{33} \quad \text{(a)}$$

$$\bar{\sigma}_{22} = C^*_{12}\,\bar{\epsilon}_{11} + C^*_{22}\,\bar{\epsilon}_{22} + C^*_{23}\,\bar{\epsilon}_{33} \quad \text{(b)} \qquad (3.4.47)$$

$$\bar{\sigma}_{33} = C^*_{12}\,\bar{\epsilon}_{11} + C^*_{23}\,\bar{\epsilon}_{22} + C^*_{22}\,\bar{\epsilon}_{33} \quad \text{(c)}$$

$$\bar{\sigma}_{12} = 2C^*_{44}\,\bar{\epsilon}_{12} \quad \text{(a)}$$

square symmetry $\quad \bar{\sigma}_{23} = 2C^*_{55}\,\bar{\epsilon}_{12} \quad$ (b)

(3.4.48)

transverse isotropy $\quad \bar{\sigma}_{23} = (C^*_{22} - C^*_{23})\,\bar{\epsilon}_{23} \quad$ (c)

$$\bar{\sigma}_{13} = 2C^*_{44}\,\bar{\epsilon}_{13} \quad \text{(d)}$$

It is seen that there are six independent EEM for square symmetry and five for transverse isotropy.

Inversion of (3.4.47) shows that the normal strain-normal stress relations have the form

$$\bar{\epsilon}_{11} = S^*_{11}\,\bar{\sigma}_{11} + S^*_{12}\,\bar{\sigma}_{22} + S^*_{12}\,\bar{\sigma}_{33} \quad \text{(a)}$$

$$\bar{\epsilon}_{22} = S^*_{12}\,\bar{\sigma}_{11} + S^*_{22}\,\bar{\sigma}_{22} + S^*_{23}\,\bar{\sigma}_{33} \quad \text{(b)} \qquad (3.4.49)$$

$$\bar{\epsilon}_{33} = S^*_{12}\,\bar{\sigma}_{11} + S^*_{23}\,\bar{\sigma}_{22} + S^*_{22}\,\bar{\sigma}_{33} \quad \text{(c)}$$

where

$$S^*_{11} = \frac{C^*_{22} + C^*_{23}}{C^*_{11}\,(C^*_{22} + C^*_{23}) - 2\,C^{*\,2}_{12}} \qquad \text{(a)}$$

$$S^*_{12} = \frac{C^*_{12}}{C^*_{11}\,(C^*_{22} + C^*_{23}) - 2\,C^{*\,2}_{12}} \qquad \text{(b)}$$

$$S^*_{22} = \frac{C^*_{11}\,C^*_{22} - C^{*\,2}_{12}}{\left[C^*_{11}\,(C^*_{22} + C^*_{23}) - 2\,C^{*\,2}_{12}\right]\,(C^*_{22} - C^*_{23})} \qquad \text{(c)} \qquad (3.4.50)$$

$$S^*_{23} = -\frac{C^*_{11}\,C^*_{23} - C^{*\,2}_{12}}{\left[C^*_{11}\,(C^*_{22} + C^*_{23}) - 2\,C^{*\,2}_{12}\right]\,(C^*_{22} - C^*_{23})} \qquad \text{(d)}$$

The inversion of (3.4.48) is of course immediate and need not be written down. It is seen that the effective strain-stress relations of the square symmetric and transversely isotropic materials have the same kind of symmetry as their effective stress-strain relations, as they should.

The C^*_{ij} set of EEM while notationally convenient for writing stress-strain relations is inconvenient for computation of EEM. We shall therefore introduce a different set which is both analytically convenient and physically significant. We first write down the new set and we shall then explain its physical significance and list the boundary conditions which have to be

applied to fiber reinforced specimens in order to compute them. The new set is for transverse isotropy.

$$n^* = C^*_{11} \tag{3.4.51}$$

$$\ell^* = C^*_{12} \tag{3.4.52}$$

$$k^* = \frac{1}{2}(C^*_{22} + C^*_{23}) \tag{3.4.53}$$

$$G^*_T = \frac{1}{2}(C^*_{22} - C^*_{23}) \tag{3.4.54}$$

$$G^*_A = C^*_{44} \tag{3.4.55}$$

For square symmetry there is added

$$G'^*_T = C^*_{55} \tag{3.4.56}$$

In order to define the boundary value problems which have to be solved to compute these EEM it is convenient to consider a cylindrical specimen in which the fibers are parallel to the axis. For n^* and ℓ^* determination we apply the homogeneous displacement boundary condition.

$$u_1\ (S) = \epsilon^o_{11}\, x_1 \ \text{(a)} \qquad u_2\ (S) = u_3\ (S) = 0 \ \text{(b)} \tag{3.4.57}$$

By the average strain theorem (3.1.18) the average strains are

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} \epsilon^o_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then from (3.4.47) and (3.4.51 - 52)

$$\bar{\sigma}_{11} = n^* \epsilon^o_{11} \qquad \text{(a)}$$

$$\bar{\sigma}_{22} = \bar{\sigma}_{33} = \ell^* \epsilon^o_{11} \qquad \text{(b)} \qquad (3.4.58)$$

It is seen that (3.4.57) corresponds to uniaxial straining with transverse deformation prevented (by a smooth rigid enclosure, for example). Determination of n^* and ℓ^* requires computation of axial and transverse average stresses.

The modulus k^* is called <u>effective transverse bulk modulus</u>. To obtain it we impose on the cylindrical specimen the homogeneous displacement boundary conditions

$$u_1 (S) = 0 \quad \text{(a)} \qquad u_2 (S) = \epsilon^o x_2 \quad \text{(b)} \qquad u_3 (S) = \epsilon^o x_3 \quad \text{(c)} \qquad (3.4.59)$$

By the average strain theorem the average strains are then

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon^o & 0 \\ 0 & 0 & \epsilon^o \end{bmatrix} \qquad (3.4.60)$$

which is an isotropic plane strain. Inserting (3.4.60) into (3.4.47) and using (3.4.51), (3.4.53), the surviving average stresses are

$$\bar{\sigma}_{11} = 2\, \ell^* \epsilon^o \qquad \text{(a)}$$

$$\bar{\sigma}_{22} = \bar{\sigma}_{33} = 2k^* \epsilon^o \qquad \text{(b)} \qquad (3.4.61)$$

We thus have isotropic average stress in the transverse plane and an axial average stress which is due to the plane strain restraint (3.4.59a).

A dual formulation is obtained by use of mixed boundary conditions in the form

$$u_1 (S) = 0 \quad (a) \qquad T_2 (S) = \sigma^o n_2 \quad (b) \quad T_3 (S) = \sigma^o n_3 \quad (c) \qquad (3.4.62)$$

Fig. 3.4.2 illustrates such a situation. Recalling (3.4.38), the average strain theorem (3.1.16) and the average stress theorem (3.1.26) (with no body forces) yield for (3.4.62)

$$\bar{\epsilon}_{11} = 0 \qquad \bar{\sigma}_{22} = \bar{\sigma}_{33} = \sigma^o \qquad \bar{\sigma}_{12} = \bar{\sigma}_{23} = \bar{\sigma}_{31} = 0 \qquad (3.4.63)$$

Introducing (3.4.63) into (3.4.47) and using (3.4.53) we find

$$\bar{\epsilon}_{22} = \bar{\epsilon}_{33} = \frac{\sigma^o}{2k^*} \qquad (3.4.64)$$

The EEM G_T^* and G_A^* are evidently effective shear moduli as is seen from (3.4.54 - 55) and (3.4.48). It is seen that G_T^* is involved in shearing in the transverse x_2, x_3 plane; hence it is called <u>effective transverse shear modulus</u>. On the other hand G_A^* is involved in shearing in x_1 x_2 and x_1 x_3 planes, which contain the axis i.e. fiber direction; hence it is called <u>effective axial shear modulus</u>. The formulation of boundary value problems to determine these EEM is just as for the effective shear moduli in orthotropic FRM. G_T^* is the analogue of G_{23}^*, eqns. (3.4.13 - 15) and (3.4.22 - 24);

G_A^* is the analogue of either one of G_{12}^* or G_{13}^*, eqns. (3.4.10 - 12), (3.4.16 - 18), (3.4.19 - 21), (3.4.25 - 27). All these boundary conditions may now be thought of as being applied to the present cylindrical specimen. See fig. 3.4.3.

Discussion of the square symmetric material is almost identical. The EEM ℓ^*, n^*, k^* and G_A^* are to be determined just as for transverse isotropy and G'^*_T is now the analogue of the orthotropic G_{23}^*. For G_T^* determination, eqn. (3.4.54), the homogeneous displacement

$$u_1 (S) = 0 \qquad u_2 (S) = \epsilon^o x_2 \qquad u_3 (S) = - \epsilon^o x_3 \tag{3.4.65}$$

may be applied to the specimen. This yields

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon^o & 0 \\ 0 & 0 & -\epsilon^o \end{bmatrix} \tag{3.4.66}$$

Then from (3.4.47) and (3.4.54)

$$\begin{aligned} \bar{\sigma}_{22} &= - \bar{\sigma}_{33} = 2\, G_T^* \epsilon^o \\ \bar{\sigma}_{11} &= \bar{\sigma}_{12} = \bar{\sigma}_{23} = \bar{\sigma}_{13} = 0 \end{aligned} \tag{3.4.67}$$

It follows from (3.4.66 - 67) that G_T^* relates average shear stress to average shear strain in the system of axes x_1, x_2', x_3' where $(x_2', x_2) = (x_3', x_3) = 45^o$.

It is of interest to note that both the transversely isotropic and square symmetric materials are isotropic in axial shear. To see this suppose that the specimen is subjected to average pure shear

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & \bar{\epsilon}_{12} & 0 \\ \bar{\epsilon}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$\left[\bar{\sigma}_{ij}\right] = \begin{bmatrix} 0 & 2G_A^* \bar{\epsilon}_{12} & 0 \\ 2G_A^* \bar{\epsilon}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now if the x_2, x_3 axes are rotated to new positions$(x_2', x_2) = (x_3', x_3) = \theta$ then by tensor transformation

$$\bar{\epsilon}'_{12} = \bar{\epsilon}_{12} \cos\theta \qquad \bar{\sigma}'_{12} = 2\, G_A^* \, \bar{\epsilon}_{12} \cos\theta$$

and so

$$\bar{\sigma}'_{12} = 2\, G_A^* \, \bar{\epsilon}'_{12}$$

which proves our assertion.

The EEM k^* and G_T^* are sufficient to describe transversely plane states in transversely isotropic materials. Suppose that through application of appropriate homogeneous boundary conditions the average state of strain imposed is

$$\left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{\epsilon}_{22} & \bar{\epsilon}_{23} \\ 0 & \bar{\epsilon}_{23} & \bar{\epsilon}_{23} \end{bmatrix} \tag{3.4.68}$$

It is convenient to split the strain system (3.4.68) into so-called isotropic and deviatoric parts. The separation is given by

$$\bar{\epsilon}_{\alpha\beta} = \bar{\epsilon}\,\delta_{\alpha\beta} + \bar{e}_{\alpha\beta} \qquad \text{(a)}$$

$$\bar{\epsilon} = \frac{1}{2}\,\bar{\epsilon}_{\gamma\gamma} = \frac{1}{2}\,(\bar{\epsilon}_{22} + \bar{\epsilon}_{33}) \qquad \text{(b)} \tag{3.4.69}$$

where $\bar{\epsilon}\,\delta_{\alpha\beta}$ is the isotropic part and $\bar{e}_{\alpha\beta}$ the deviatoric part. Here, and from now on indices which have the range 2, 3 are denoted by Greek letters α, β, etc. while i, j and other latin indices continue to range over 1, 2, 3. If (3.4.68) is inserted into (3.4.47 b,c) and (3.4.48 c) these three expressions can be written compactly as

$$\boxed{\begin{aligned} \bar{\sigma} &= 2k^{*}\,\bar{\epsilon} \qquad \text{(a)} \\ \bar{s}_{\alpha\beta} &= 2G_T^{*}\,\bar{e}_{\alpha\beta} \qquad \text{(b)} \end{aligned}} \tag{3.4.70}$$

where

$$\bar{\sigma}_{\alpha\beta} = \bar{\sigma}\,\delta_{\alpha\beta} + \bar{s}_{\alpha\beta} \qquad \text{(a)}$$

$$\bar{\sigma} = \frac{1}{2}\,\bar{\sigma}_{\gamma\gamma} = \frac{1}{2}\,(\bar{\sigma}_{22} + \bar{\sigma}_{33}) \qquad \text{(b)} \tag{3.4.71}$$

It is seen that (3.4.70) is a two dimensional isotropic stress-strain relation where k^* plays the role of a two dimensional bulk modulus. Hence the name transverse bulk modulus.

Note that the representation (3.4.70) fails for the square symmetric material. In this case (3.4.70b) is valid only for $\alpha\beta = 11, 22$ while for the case $\alpha\beta = 12$ (3.4.70b) must be replaced by (3.4.48b). The resulting stress-strain relation which involves k^*, G_T^* and $G_T'^*$ is the two dimensional analogue of a cubic stress-strain relation.

We now consider other EEM which are primarily of engineering importance, namely effective Young's moduli and effective Poisson's ratios. Comparison of the strain-stress relation (3.4.49) with its general orthotropic counterpart (3.4.5) shows that in the present case

$$S_{12}^* = S_{13}^* \qquad S_{22}^* = S_{23}^* \tag{3.4.72}$$

It follows from (3.4.72) and (3.4.8 - 9) that

$$\nu_{12}^* = \nu_{13}^* \qquad E_2^* = E_3^*$$

$$\nu_{23}^* = \nu_{32}^* \qquad \nu_{31}^* = \nu_{21}^*$$

We now introduce the following notation

$$E_1^* = E_A^* \tag{3.4.73}$$

$$E_2^* = E_3^* = E_T^* \tag{3.4.74}$$

$$\nu^*_{12} = \nu^*_{13} = \nu^*_A \tag{3.4.75}$$

$$\nu^*_{23} = \nu^*_{32} = \nu^*_T \tag{3.4.76}$$

where E^*_A is the <u>effective axial Young's modulus</u>, E^*_A - the <u>effective transverse Young's modulus</u>, ν^*_A - the <u>effective axial Poisson's ratio</u>, and ν^*_T - the <u>effective transverse Poisson's ratio.</u>

The problems involved in the computation of (3.4.73 - 76) have been previously discussed for the orthotropic material. For E^*_A and ν^*_A the formulations (3.4.28 - 30) and (3.4.37) (3.4.39 - 40) are appropriate. For E^*_T and ν^*_T we can use (3.4.31 - 33) or (3.4.41 - 42) or the other remaining set.

Note that it follows from (3.4.9c), and (3.4.73 - 75) that

$$\nu^*_{21} = \nu^*_{31} = \frac{E^*_T}{E^*_A} \nu^*_A \tag{3.4.77}$$

By use of (3.4.8 - 9) and (3.4.73 - 77) we can now rewrite (3.4.49) in the form

$$\boxed{\begin{aligned}
\bar{\epsilon}_{11} &= \frac{1}{E^*_A} \bar{\sigma}_{11} - \frac{\nu^*_A}{E^*_A} \bar{\sigma}_{22} - \frac{\nu^*_A}{E^*_A} \bar{\sigma}_{33} && \text{(a)} \\
\bar{\epsilon}_{22} &= - \frac{\nu^*_A}{E^*_A} \bar{\sigma}_{11} + \frac{1}{E^*_T} \bar{\sigma}_{22} - \frac{\nu^*_T}{E^*_T} \bar{\sigma}_{33} && \text{(b)} \\
\bar{\epsilon}_{33} &= - \frac{\nu^*_A}{E^*_A} \bar{\sigma}_{11} - \frac{\nu^*_T}{E^*_T} \bar{\sigma}_{22} + \frac{1}{E^*_T} \bar{\sigma}_{33} && \text{(c)}
\end{aligned}} \tag{3.4.78}$$

It is important for computational purposes to relate the EEM appearing in (3.4.78) to previously defined EEM. Note that identification of the S^*_{ij} in (3.4.49) with the coefficients in (3.4.78) , and use of (3.4.50) immediately gives E^*_A, E^*_T, ν^*_T and ν^*_A , in terms of C^*_{ij} . We have

$$E^*_A = C^*_{11} - \frac{2\,C^{*\,2}_{12}}{C^*_{22} + C^*_{23}} \qquad \text{(a)}$$

$$\nu^*_A = \frac{C^*_{12}}{C^*_{22} + C^*_{23}} \qquad \text{(b)}$$

$$E^*_T = \frac{\left[C^*_{11}(C^*_{22} + C^*_{23}) - 2\,C^{*\,2}_{12}\right](C^*_{22} - C^*_{23})}{C^*_{11}\,C^*_{22} - C^{*\,2}_{12}} \qquad \text{(c)} \qquad (3.4.79)$$

$$\nu^*_T = \frac{C^*_{11}\,C^*_{23} - C^{*\,2}_{12}}{C^*_{11}\,C^*_{22} - C^{*\,2}_{12}} \qquad \text{(d)}$$

Introduction of (3.4.51 - 54) into (3.4.79) gives the results

$$\boxed{\begin{aligned} E^*_A &= n^* - 4\,k^*\,\nu^{*\,2}_A \qquad (3.4.80) \\ \ell^* &= 2\,k^*\,\nu^*_A \qquad (3.4.81) \end{aligned}}$$

$$E_T^* = \frac{4\,k^*\,G_T^*}{k^* + m\,G_T^*} \tag{3.4.82}$$

$$\nu_T^* = \frac{k^* - m\,G_T^*}{k^* + m\,G_T^*} = 1 - \frac{2}{1 + k^*/m\,G_T^*} \tag{3.4.83}$$

where

$$m = 1 + \frac{4\,k^*\,\nu_A^{*2}}{E_A^*} \tag{3.4.84}$$

From (3.4.82) - 83) there follows the important relation

$$G_T^* = \frac{E_T^*}{2\,(1+\nu_T^*)} \tag{3.4.85}$$

which is identical to the well known relation in isotropic elasticity.

For the square symmetric material (3.4.78 - 85) are still valid in the x_1, x_2, x_3 coordinate system. The difference between the square symmetric and transversely isotropic materials appears when the x_2, x_3 axes are rotated around the x_1 axis to new positions x'_2, x'_3. We may then define transverse Young's modulus, Poisson's ratio and shear modulus with respect to the new system of axes. In the transversely isotropic material these will have the same values as in the x_1, x_2, x_3 system but in the square symmetric material they will be different and will have to be found by tensor transformation.

To describe the effective elastic properties of a macroscopically transversely isotropic or square symmetric material any convenient set of five or six independent EEM, respectively, may be used. Care should be taken that the EEM be independent. For example E_T^*, G_T^*, and ν_T^* are related by (3.4.85) and thus count as two EEM.

Finally we list stress-strain relations of homogeneous transversely isotropic and of completely isotropic phases, since we shall have frequent occasion to refer to these. On the basis of the treatment given above we can write transversely isotropic stress-strain relations as

$$\bar{\sigma}_{11} = n\ \epsilon_{11} + \ell\ (\epsilon_{22} + \epsilon_{23}) \qquad \text{(a)}$$

$$\sigma_{22} = \ell\ \epsilon_{11} + (k + G_T)\ \epsilon_{22} + (k - G_T)\ \epsilon_{33} \qquad \text{(b)}$$

$$\sigma_{33} = \ell\ \epsilon_{11} + (k - G_T)\ \epsilon_{22} + (k + G_T)\ \epsilon_{33} \qquad \text{(c)}$$

$$\sigma_{12} = 2\ G_A\ \epsilon_{12} \qquad \text{(d)}$$

$$\sigma_{23} = 2\ G_T\ \epsilon_{23} \qquad \text{(e)}$$

$$\sigma_{31} = 2\ G_A\ \epsilon_{31} \qquad \text{(f)}$$

(3.4.86)

The strain-stress relations are

$$\epsilon_{11} = \frac{1}{E_A}\ \sigma_{11} - \frac{\nu_A}{E_A}\ (\sigma_{22} + \sigma_{33}) \qquad \text{(a)}$$

$$\epsilon_{22} = -\frac{\nu_A}{E_A}\sigma_{11} + \frac{1}{E_T}\sigma_{22} - \frac{\nu_T}{E_T}\sigma_{33} \qquad \text{(b)}$$

$$\epsilon_{33} = -\frac{\nu_A}{E_A}\sigma_{11} - \frac{\nu_T}{E_T}\sigma_{22} + \frac{1}{E_T}\sigma_{33} \qquad \text{(c)}$$

$$\epsilon_{12} = \frac{\sigma_{12}}{2G_A} \qquad \text{(d)} \qquad (3.4.87)$$

$$\epsilon_{23} = \frac{\sigma_{23}}{2G_T} \qquad \text{(e)}$$

$$\epsilon_{31} = \frac{\sigma_{31}}{2G_A} \qquad \text{(f)}$$

All of the relations (3.4.80-85) are obviously valid for the elastic constants in (3.4.86-87).

The isotropic stress-strain relations are written in the well known compact form

$$\sigma_{ij} = \lambda\,\epsilon_{kk}\,\delta_{ij} + 2G\,\epsilon_{ij} \qquad (3.4.88)$$

Comparison of (3.4.86) with (3.4.88) shows that for an isotropic material

$$n = \lambda + 2G \qquad \text{(a)}$$

$$\ell = \lambda \qquad \text{(b)} \qquad (3.4.89)$$

$$k = \lambda + G \qquad \text{(c)}$$

$$G_T = G_A = G \qquad \text{(d)}$$

The modulus k is called <u>plane strain bulk modulus</u> for the isotropic material.

The moduli λ and G are related to the Young's modulus E and the Poisson's ratio ν by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \text{(a)}$$

$$G = \frac{E}{2(1+\nu)} \qquad \text{(b)} \qquad (3.4.90)$$

For an isotropic material we thus have

$$E_A = E_T = E \qquad \text{(a)}$$

$$\nu_A = \nu_T = \nu \qquad \text{(b)} \qquad (3.4.91)$$

$$k = \frac{E}{2(1+\nu)(1-2\nu)} = \frac{G}{1-2\nu} \qquad \text{(c)}$$

3.4.4 Isotropic Materials

Finally, we consider the case of a statistically isotropic heterogeneous material. Obviously a uniaxial fibrous or fiber reinforced material cannot be isotropic. However, if the fibers are oriented in many different directions in a completely random manner, fig. 3.4.4 , the material may assumed to be statistically isotropic. In that case the well known reduction of the general anisotropic Hooke's law to the isotropic case (see e.g. [3.1]) is valid. Eqns. (3.3.9) reduce to

$$\bar{\sigma}_{ij} = \lambda^* \bar{\epsilon}_{kk} \delta_{ij} + 2G^* \bar{\epsilon}_{ij} \qquad (3.4.92)$$

where λ^* is the effective Lamé modulus and G^* is the effective shear modulus.

The three dimensional separation of $\bar{\sigma}_{ij}$ and $\bar{\epsilon}_{ij}$ into isotropic and deviatoric parts is

$$\bar{\sigma}_{ij} = \bar{\sigma}\,\delta_{ij} + \bar{s}_{ij} \qquad \text{(a)}$$

$$\bar{\sigma} = \frac{1}{3}\,\bar{\sigma}_{kk} \qquad \text{(b)} \qquad (3.4.93)$$

$$\bar{\epsilon}_{ij} = \bar{\epsilon}\,\delta_{ij} + \bar{e}_{ij} \qquad \text{(a)}$$

$$\bar{\epsilon} = \frac{1}{3}\,\bar{\epsilon}_{kk} \qquad \text{(b)} \qquad (3.4.94)$$

Introduction of (3.4.93 - 94) into (3.4.92) leads to

$$\bar{\sigma} = 3\,K^*\,\bar{\epsilon} \qquad \text{(a)}$$

$$\bar{s}_{ij} = 2\,G^*\,\bar{e}_{ij} \qquad \text{(b)} \qquad (3.4.95)$$

where K^* is the effective bulk modulus given by

$$K^* = \lambda^* + \frac{2}{3}\,G^* \qquad (3.4.96)$$

An effective Young's modulus E^* is defined by application of average uniaxial stress in any direction, x_1 say. Then the average stress is (3.4.29a) and E^* is defined by

$$E^* = \frac{\bar{\sigma}_{11}}{\bar{\epsilon}_{11}} \qquad (3.4.97)$$

The effective Poisson's ratio ν^* is defined by

$$\nu^* = - \frac{\bar{\epsilon}_{22}}{\bar{\epsilon}_{11}} = - \frac{\bar{\epsilon}_{33}}{\bar{\epsilon}_{11}} \tag{3.4.98}$$

Introduction of (3.4.29a) into (3.4.92) gives

$$E^* = \frac{9 K^* G^*}{3 K^* + G^*} \tag{3.4.99}$$

$$\nu^* = \frac{3 K^* - 2 G^*}{2 (3 K^* + G^*} \tag{3.4.100}$$

These results are, of course, the same as in classical elasticity theory for homogeneous isotropic bodies.

3.4.5 Structural Relations for Effective Elastic Moduli

Classical procedures to find the number of independent effective elastic moduli for various FRM have been exploited in the preceding paragraphs of this chapter. It is necessary at the present time to re-examine the concept of independence of elastic moduli.

Independence or non-independence of elastic moduli is based on available information. Thus the generalized Hooke's law (3.2.2) contains

at first sight 81 elastic moduli. The information that the stress and strain tensors are symmetric reduces the number to 36. The existence of an elastic energy function further reduces the number to 21. So far nothing has been said about internal structure of the material.

The existence of planes of elastic symmetry, axes of rotational elastic symmetry and equivalent elastic directions is information which is based on some knowledge of the internal structure of the material. Examples of this kind have been given in paragraphs 3.4.3 - 4. However, the information is expressed in a macroscopic sense. From a formal theoretical point of view one can very well consider elastic symmetry without knowledge of its origin.

In the present paragraph we consider additional structural information which is of purely microscopic nature and is peculiar to the heterogeneity of the material. To give some conceptual examples consider the periodic square and hexagonal arrays of equal circular fibers shown in figs. 2.1.5 - 6. On the basis of the results given in par. 3.4.3 only, we know that in the first case there are six independent EEM while in the second case there are five. It is clear that in both cases the EEM are functions of the phase moduli and the ratio a/d where a is fiber radius and d is the spacing. This may be symbolically expressed in the form

$$C^*_{ij} = C^*_{ij}\,(\underline{C}\,,\ a/d)$$

where C stands for phase moduli. In principle, a/d could be eliminated between any two moduli which would result in a relation between these moduli, involving only phase properties. We have thus reached the conclusion that there is a relation between any two EEM in the cases considered and thus each material has really only one independent EEM. The reduction from six to one or five to one was based on detailed knowledge of the internal geometry of the material. But it should be noted that the fact that there is only one independent modulus is here of no usefulness since the relations are not known for the materials under consideration. Surprisingly enough, however, it is possible to establish some general relations between some of the EEM of two phase fibrous or fiber reinforced material of arbitrary phase geometry. These remarkable relations have been established by Hill [3.5] and shall now be derived.

Suppose that a cylindrical specimen of the composite is subjected to homogeneous displacement boundary conditions (3.3.1) with average strain matrix

$$\left[\epsilon^{o}_{ij}\right] = \left[\bar{\epsilon}_{ij}\right] = \begin{bmatrix} \epsilon^{o}_{11} & 0 & 0 \\ 0 & \epsilon^{o} & 0 \\ 0 & 0 & \epsilon^{o} \end{bmatrix} \tag{3.4.101}$$

The boundary displacements are then

$$u_1\,(S) = \epsilon^{o}_{11}\, x_1 \qquad \text{(a)}$$

$$u_2\,(S) = \epsilon^{o}\, x_2 \qquad \text{(b)} \qquad (3.4.102)$$

$$u_3\,(S) = \epsilon^{o}\, x_3 \qquad \text{(c)}$$

This corresponds to average uniform straining in x_1 direction and average isotropic straining in the x_2, x_3 plane. We insert the strains (3.4.101) into the stress-strain relations (3.4.47) which are valid for square symmetric and transversely isotropic materials. Adding (3.4.47 b,c) and using the notation (3.4.51 - 53) we obtain

$$\bar{\sigma}_{11} = n^{*}\, \epsilon^{o}_{11} + 2\, \ell^{*} \epsilon^{o} \qquad \text{(a)}$$

$$(3.4.103)$$

$$\bar{\sigma} = \ell^{*}\, \epsilon^{o}_{11} + 2\, k^{*}\, \epsilon^{o} \qquad \text{(b)}$$

where $\bar{\sigma}$ is defined by (3.4.71b).

The average stresses may be written in terms of phase averages as was done in par. 3.3.3. Thus

$$\bar{\sigma}_{11} = \bar{\sigma}^{(1)}_{11}\, v_1 + \bar{\sigma}^{(2)}_{11}\, v_2 \qquad \text{(a)}$$

$$(3.4.104)$$

$$\bar{\sigma} = \bar{\sigma}^{(1)}_{11}\, v_1 + \bar{\sigma}^{(2)}\, v_2 \qquad \text{(b)}$$

It is notationally convenient to assume that the phase materials are square symmetric or transversely isotropic with elastic axes parallel to the composite's x_1, x_2, x_3 system. Then the phase stress - strain relations are of the form (3.4.86) and it follows very simply that phase averages obey relations of the form (3.4.104). For example

$$\bar{\sigma}_{11}^{(1)} = n_1 \bar{\epsilon}_{11}^{(1)} + 2 \ell_1 \bar{\epsilon}^{(1)}$$

To continue we need a result which is of crucial importance here. It will be shown later in par. 3.5.1 that for boundary conditions of type (3.4.102) <u>the strain ϵ_{11} is uniform in both phases</u>. (See eqs. (3.5.10a)) We thus conclude that in the present case

$$\bar{\epsilon}_{11}^{(1)} = \epsilon_{11}^{(2)} = \bar{\epsilon}_{11}^{(2)} = \epsilon_{11}^{o} \tag{3.4.105}$$

Consequently the phase stress averages assume the following forms

$$\bar{\sigma}_{11}^{(1)} = n_1 \epsilon_{11}^{o} + 2 \ell_1 \bar{\epsilon}^{(1)} \quad \text{(a)}$$

$$\bar{\sigma}^{(1)} = \ell_1 \epsilon_{11}^{o} + 2 k_1 \bar{\epsilon}^{(1)} \quad \text{(b)} \tag{3.4.106}$$

$$\bar{\sigma}_{11}^{(2)} = n_2 \epsilon_{11}^{o} + 2 \ell_2 \bar{\epsilon}^{(2)} \quad \text{(a)}$$

$$\bar{\sigma}^{(2)} = \ell_2 \epsilon_{11}^{o} + 2 k_2 \bar{\epsilon}^{(2)} \quad \text{(b)} \tag{3.4.107}$$

Now substitute (3.4.106-107) into (3.4.104) and the resulting expression into (3.4.103). After rearrangement we find

$$\bar{\epsilon}^{(1)} (k^* - k_1) v_1 + \bar{\epsilon}^{(2)} (k^* - k_2) v_2 + \frac{1}{2} \epsilon^o_{11} (\ell^* - \ell_2 v_2 - \ell_1 v_1) = 0 \quad \text{(a)}$$

(3.4.108)

$$\bar{\epsilon}^{(1)} (\ell^* - \ell_1) v_1 + \bar{\epsilon}^{(2)} (\ell^* - \ell_2) v_2 + \frac{1}{2} \epsilon^o_{11} (n^* - n_2 v_2 - n_1 v_1) = 0 \quad \text{(b)}$$

Eqns. (3.4.108) determine $\bar{\epsilon}^{(1)}$, $\bar{\epsilon}^{(2)}$ in terms of effective moduli, phase moduli and volume fractions. There is, however, a third equation

$$\bar{\epsilon}^{(1)} v_1 + \bar{\epsilon}^{(2)} v_2 = \epsilon^o \qquad (3.4.109)$$

from the average strain theorem. Consequently eqns (3.4.108-109) are not independent. If any two linear equations in two unknown are the same, their coefficients must be proportional. Now it should be borne in mind that ϵ^o_{11} and ϵ^o are independent quantities. So assumption that (3.4.109) is identical to any of (3.4.108) would lead to specification of the ratio $\epsilon^o_{11} / \epsilon^o$ which is a contradiction. We thus conclude that (3.4.108a,b) are the same. Therefore

$$\frac{k^* - k_1}{\ell^* - \ell_1} = \frac{k^* - k_2}{\ell^* - \ell_2} = \frac{\ell^* - (\ell_1 v_1 + \ell_2 v_2)}{n^* - (n_1 v_1 + n_2 v_2)} \qquad (3.4.110)$$

It thus follows that if any EEM of the group k^*, ℓ^*, n^* is known, the other two follow from (3.4.110).

It is easily shown that the first pair in (3.4.10) leads to

$$\frac{k^* - k_1}{\ell^* - \ell_1} = \frac{k^* - k_2}{\ell^* - \ell_2} = \frac{k_2 - k_1}{\ell_2 - \ell_1} \tag{3.4.111}$$

We see that each modulus is linearly related to another one. Thus

$$\boxed{\begin{aligned} & k^* (\ell_2 - \ell_1) - \ell^* (k_2 - k_1) = \ell_2 k_1 - \ell_1 k_2 && \text{(a)} \\ & \ell^* (\ell_2 - \ell_1) - n^* (k_2 - k_1) = \bar{\ell} (\ell_2 - \ell_1) - \bar{n} (k_2 - k_1) && \text{(b)} \\ & k^* \frac{\ell_2 - \ell_1}{k_2 - k_1} - n^* \frac{k_2 - k_1}{\ell_2 - \ell_1} = \bar{k} \frac{\ell_2 - \ell_1}{k_2 - k_1} - \bar{n} \frac{k_2 - k_1}{\ell_2 - \ell_1} && \text{(c)} \end{aligned}} \tag{3.4.112}$$

where

$$\begin{aligned} \bar{n} &= n_1 v_1 + n_2 v_2 && \text{(a)} \\ \bar{\ell} &= \ell_1 v_1 + \ell_2 v_2 && \text{(b)} \end{aligned} \tag{3.4.113}$$

It is to be noted that for isotropic phases n, ℓ, and k are given by (3.4.89).

It is now seen that because of the relations (3.4.80-81) and (3.4.110), E_A^* and ν_A^* can also be related to ℓ^*, n^* and k^*. This can of course be done by simple algebra but the method given by Hill [3.5] is preferable. For this purpose, eqns. (3.4.103) are rewritten in different form. If the strains (3.4.101) are substituted into (3.4.70) it is easily seen that

$$\bar{\sigma}_{22} = \bar{\sigma}_{33} = \bar{\sigma}$$

Therefore (3.4.78a) can be written as

$$\bar{\sigma}_{11} = 2\nu_A^* \bar{\sigma} + E_A^* \epsilon_{11}^o \tag{3.4.114}$$

which is the equivalent of (3.4.103a). Also use of (3.4.81) in (3.4.103b) transforms this equation into

$$\epsilon^o = \frac{\bar{\sigma}}{2k^*} - \nu_A^* \epsilon_{11}^o \tag{3.4.115}$$

The same procedure previously used is now employed in (3.4.114-115). This is left as an exercise to the reader. The results are

$$\frac{\frac{1}{k^*} - \frac{1}{k_1}}{\nu_A^* - \nu_1} = \frac{\frac{1}{k^*} - \frac{1}{k_2}}{\nu_A^* - \nu_2} = \frac{\nu^* - \nu_1 v_1 - \nu_2 v_2}{-\frac{1}{4}(E_A^* - E_1 v_1 - E_2 v_2)} = \frac{\frac{1}{k_1} - \frac{1}{k_2}}{\nu_1 - \nu_2} \tag{3.4.116}$$

where the last equality follows directly from the first pair. If the phase materials are square symmetric or transversely isotropic, then E_1, E_2, ν_1 and ν_2 are the axial Young's moduli and Poisson's ratios respectively of the phases. For isotropic phases they are the usual Young's moduli and Poisson's ratios. Eqns. (3.4.116) give the following linear relationships between the quantities

$\frac{1}{k^*}$, ν_A^* and E_A^*.

$$\boxed{\begin{aligned} E_A^* &= \bar{E} + \frac{4(\nu_2 - \nu_1)^2}{\left(\frac{1}{k_2} - \frac{1}{k_1}\right)^2} \left[\left(\frac{\bar{1}}{k}\right) - \frac{1}{k^*}\right] && (3.4.117) \\ \nu_A^* &= \bar{\nu} + \frac{\nu_2 - \nu_1}{\frac{1}{k_2} - \frac{1}{k_1}} \left[\frac{1}{k^*} - \left(\frac{\bar{1}}{k}\right)\right] && (3.4.118) \end{aligned}}$$

where

$$\bar{E} = E_1 v_1 + E_2 v_2 \qquad \text{(a)}$$

$$\bar{\nu} = \nu_1 v_1 + \nu_2 v_2 \qquad \text{(b)} \qquad (3.4.119)$$

$$\left(\frac{\bar{1}}{k}\right) = \frac{v_1}{k_1} + \frac{v_2}{k_2} \qquad \text{(c)}$$

A numerical verification of one of the general relations between the various EEM is shown in fig. 3.4.5. Numerical values of EEM of a regular hexagonal array of identical circular fibers (fig. 2.1.6) have been obtained with the aid of electronic computers in ref. [3.6] . The numerical values of ℓ^* and k^* have been plotted for various fiber volume fractions while the straight line represents (3.4.112a) for the phase elastic properties used in [3.6]. It is seen that there is excellent agreement with (3.4.112a).

3.4.6 Effective Elastic Moduli in Terms of Phase Averages

We shall here reduce the general results obtained in par. 3.3.3 to uniaxially FRM. This reduction must be performed with four subscript notation for elastic moduli. We note in this respect the relations (3.4.3) for a macroscopically orthotropic FRM and we set down here similar relations for macroscopically square symmetric and transversely isotropic FRM. In view of (3.4.3) and (3.4.45) we have

$$C^*_{1111} = C^*_{11} \qquad C^*_{2222} = C^*_{3333} = C^*_{22} \qquad \text{(a)}$$

$$C^*_{1122} = C^*_{1133} = C^*_{12} \qquad C^*_{2233} = C^*_{23} \qquad \text{(b)} \qquad (3.4.120)$$

$$C^*_{1212} = C^*_{1313} = C^*_{44} \qquad C^*_{2323} = C^*_{55} \qquad \text{(c)}$$

for the square symmetric FRM, and all others vanish.

For the transversely isotropic FRM the last of (3.4.120c) is replaced by

$$C^*_{2323} = \frac{1}{2}\,(C^*_{22} - C^*_{23}) \qquad (3.4.121)$$

because of (3.4.46).

We also note that for isotropic phases

$$C_{1111} = C_{2222} = C_{3333} = \lambda + 2G \qquad \text{(a)}$$

$$C_{1122} = C_{2233} = C_{3311} = \lambda \qquad \text{(b)} \qquad (3.4.122)$$

$$C_{1212} = C_{2323} = C_{3131} = G \qquad \text{(c)}$$

We start with the transverse bulk modulus k^* of a square symmetric or transversely isotropic FRM, whose constituents have at least the same symmetry. Appropriate homogeneous boundary displacements and associated average strains are given by (3.4.59 - 60). Now consider (3.3.32) for the present specific case. The relations are written out for ij = 22, 33, taking into account (3.4.60), (3.4.120) and the constituent symmetry, and are then added together. The result is easily found to be

$$k^* = k_1 + (k_2 - k_1) \frac{\bar{\epsilon}^{(2)}}{\epsilon^o} v_2 \tag{3.4.123}$$

where $\bar{\epsilon}^{(2)}$ is given by

$$\bar{\epsilon}^{(2)} = \frac{1}{V_2} \int_{V_2} \frac{1}{2} (\epsilon_{22}^{(2)} + \epsilon_{33}^{(2)}) \, dV$$

and k_1, k_2 are the transverse bulk moduli of the phases.

Next we consider the axial shear modulus G_A^*. We apply the average strain system (3.4.11) and write out (3.3.32) for ij = 12. Taking into account (3.4.120) it easily follows that

$$G_A^* = G_1 + (G_2 - G_1) \frac{\bar{\epsilon}_{12}^{(2)}}{\epsilon_{12}^o} v_2 \tag{3.4.124}$$

A similar procedure for transverse shear, with the homogeneous displacement system (3.4.13), yields

$$G_T^* = G_1 + (G_2 - G_1) \frac{\bar{\epsilon}_{23}^{(2)}}{\epsilon_{23}^{o}} v_2 \tag{3.4.125}$$

There is no difficulty whatsoever in finding similar expressions for other EEM such as n^*, ℓ^*, E_A^* etc. However, the expressions are not as simple as (3.4.55 - 57), since they involve averages of several strain components over phase 2. In this respect it should be recalled that if k^* can actually be computed by use of (3.4.123), then n^*, ℓ^*, E_A^* and ν_A^* can be found by use of (3.4.112), (3.4.117) and (3.4.118), respectively.

It is also possible to establish expressions for EEM in terms of phase stress averages by use of (3.3.36). For k^* we apply (3.4.62) with associated averages (3.4.63). We then obtain by a similar method

$$\frac{1}{k^*} = \frac{1}{k_1} + \left(\frac{1}{k_2} - \frac{1}{k_1} \right) \frac{\bar{\sigma}^{(2)}}{\sigma^o} v_2 \tag{3.4.126}$$

where $\bar{\sigma}^{(2)}$ is the average of $\frac{1}{2}(\sigma_{22} + \sigma_{33})$ over phase 2. Similarly, we find by use of (3.4.19 - 20) and (3.4.22 - 23)

$$\frac{1}{G_A^*} = \frac{1}{G_1} + \left(\frac{1}{G_2} - \frac{1}{G_1} \right) \frac{\bar{\sigma}_{12}^{(2)}}{\sigma_{12}^{o}} v_2 \tag{3.4.127}$$

$$\frac{1}{G_T^*} = \frac{1}{G_1} + \left(\frac{1}{G_2} - \frac{1}{G_1} \right) \frac{\bar{\sigma}_{23}^{(2)}}{\sigma_{23}^{o}} v_2 \tag{3.4.128}$$

Establishment of expressions for EEM in terms of averages over one phase is also a straightforward affair for orthotropic FRM.

3.5 EXACT SOLUTIONS

3.5.1 Formulation of Boundary Value Problems

As has been mentioned in chapter 3.3 the computation of EEM or EEC calls in general for a detailed solution for the stress or strain fields in a heterogeneous body, under homogeneous boundary conditions. Such elasticity problems have been formulated in chapter 3.2 for the general two phase body. In a uniaxially fibrous or fiber reinforced specimen the problem is much simplified, though still formidable, because of the cylindrical geometry. Therefore, such problems will now be discussed for the specific case under consideration.

It is assumed that in any cylindrical fiber reinforced specimen the fibers are continuous from base to base in the cylindrical specimen. In practice there are always broken fibers and so the present analysis cannot account for the state of stress near fiber breaks. The local stresses which occur there are of great importance for failure considerations; they are, however, of negligible importance for effective moduli calculations since, as has been seen,such calculations involve averaging over a representative volume. (It should be borne in mind that the problem of the state of stress near a fiber break is of such complexity that it defies exact analysis at the present time.)

It will also be assumed that the cylindrical specimen's height is much larger than a typical cross section dimension, so that Saint Venants principle applies with respect to stress distributions on the end faces. It should be noted that the boundary value problems to be formulated here apply for any long two phase cylinder whose cross section geometry is invariant with height.

Let the specimen be subjected to the homogeneous displacement boundary condition

$$u_i(S) = \epsilon^o_{ij}\, x_j \tag{3.5.1}$$

For present purposes it is convenient to split the ϵ^o_{ij} matrix in (3.5.1), thus

$$\left[\epsilon^o_{ij}\right] = \begin{bmatrix} \epsilon^o_{11} & 0 & 0 \\ 0 & \epsilon^o_{22} & \epsilon^o_{23} \\ 0 & \epsilon^o_{23} & \epsilon^o_{33} \end{bmatrix} + \begin{bmatrix} 0 & \epsilon^o_{12} & \epsilon^o_{13} \\ \epsilon^o_{12} & 0 & 0 \\ \epsilon^o_{13} & 0 & 0 \end{bmatrix} \tag{3.5.2}$$

The boundary displacements associated with the first strain matrix in (3.5.2) are

$$u_i(S) = \epsilon^o_{11} x_1 \qquad \text{(a)}$$

$$u_2(S) = \epsilon^o_{22}\, x_2 + \epsilon^o_{23}\, x_3 \qquad \text{(b)} \qquad (3.5.3)$$

$$u_3(S) = \epsilon^o_{23}\, x_2 + \epsilon^o_{33}\, x_3 \qquad \text{(c)}$$

while the boundary displacements associated with the second strain matrix are

$$u_1\,(S) = \epsilon^{o}_{12}\,x_2 + \epsilon^{o}_{13}\,x_3 \qquad \text{(a)}$$

$$u_2\,(S) = \epsilon^{o}_{12}\,x_1 \qquad \text{(b)} \qquad (3.5.4)$$

$$u_3\,(S) = \epsilon^{o}_{13}\,x_1 \qquad \text{(c)}$$

If the specimen is subjected to homogeneous traction boundary conditions

$$T_1\,(S) = \sigma^{o}_{ij}\,n_j \qquad (3.5.5)$$

we proceed analogously. Thus

$$\left[\sigma^{o}_{ij}\right] = \begin{bmatrix} \sigma^{o}_{11} & 0 & 0 \\ 0 & \sigma^{o}_{22} & \sigma^{o}_{23} \\ 0 & \sigma^{o}_{23} & \sigma^{o}_{33} \end{bmatrix} + \begin{bmatrix} 0 & \sigma^{o}_{12} & \sigma^{o}_{13} \\ \sigma^{o}_{12} & 0 & 0 \\ \sigma^{o}_{13} & 0 & 0 \end{bmatrix} \qquad (3.5.6)$$

To these stress matrices correspond the boundary tractions

$$T_1\,(S) = \sigma^{o}_{11}\,n_1 \qquad \text{(a)}$$

$$T_2\,(S) = \sigma^{o}_{22}\,n_2 + \sigma^{o}_{23}\,n_3 \qquad \text{(b)} \qquad (3.5.7)$$

$$T_3\,(S) = \sigma^{o}_{23}\,n_2 + \sigma^{o}_{33}\,n_3 \qquad \text{(c)}$$

and

$$T_1 (S) = \sigma^o_{12} n_2 + \sigma^o_{13} n_3 \qquad \text{(a)}$$

$$T_2 (S) = \sigma^o_{12} n_1 \qquad \text{(b)} \qquad (3.5.8)$$

$$T_3 (S) = \sigma^o_{13} n_1 \qquad \text{(c)}$$

The elasticity problems of the fibrous or fiber reinforced cylinder with boundary conditions of type (3.5.3) or (3.5.7) are fundamentally different from those of the cylinder subjected to (3.5.4) or (3.5.6). We shall begin with a discussion of the first type of problem and we shall afterwards consider the second one. The cylinder geometry notation to be used is the one employed in chapter 2.2 and fig. 2.11.

We shall assume for simplicity that there are two phases only. However, the present formulation applies just as well for any number of phases. The phases themselves are assumed transversely isotropic with axes of elastic symmetry in x_1 direction. It will be seen that the formulations developed are mathematically identical in the case of completely isotropic phases.

The first problem may be conveniently formulated as one of plane strain with uniform axial strain (see e.g. Love [3.3]). According to such a formulation

$$u_1^{(1)}(\underline{x}) = u_1^{(2)}(\underline{x}) = \epsilon_{11}^{o} x_1 \qquad \text{(a)}$$

$$u_2^{(1)}(\underline{x}) = u_2^{(1)}(x_2, x_3) \quad u_3^{(1)}(\underline{x}) = u_3^{(1)}(x_2, x_3) \qquad \text{(b)} \qquad (3.5.9)$$

$$u_2^{(2)}(\underline{x}) = u_2^{(2)}(x_2, x_3) \quad u_3^{(2)}(\underline{x}) = u_3^{(2)}(x_2, x_3) \qquad \text{(c)}$$

Such a method of solution where some features of the solution are assumed in advance, is known as semi-inverse in the theory of elasticity. The justification of the assumptions is obtained if the problem which is formulated on the basis of the assumptions is well defined and has a unique solution.

The strains associated with (3.5.9) are

$$\epsilon_{11}^{(1)} = \epsilon_{11}^{(2)} = \epsilon_{11}^{o} \qquad \text{(a)}$$

$$\epsilon_{12}^{(1)} = \epsilon_{13}^{(1)} = \epsilon_{12}^{(2)} = \epsilon_{13}^{(2)} = 0 \qquad \text{(b)} \qquad (3.5.10)$$

$$\epsilon_{\alpha\beta}^{(1)} = \epsilon_{\alpha\beta}^{(1)}(x_2, x_3) \quad \epsilon_{\alpha\beta}^{(2)} = \epsilon_{\alpha\beta}^{(2)}(x_2, x_3) \qquad \text{(c)}$$

where it is recalled that greek indices such as α, β are confined to the range 2, 3. Using the transversely isotropic Hooke's law (3.4.86) the stresses are found to be

$$\sigma_{11}^{(1)} = n_1 \epsilon_{11}^{o} + \ell_1 (\epsilon_{22}^{(1)} + \epsilon_{33}^{(1)}) \qquad \text{(a)}$$

$$(3.5.11)$$

$$\sigma_{11}^{(2)} = n_2 \epsilon_{11}^{o} + \ell_2 (\epsilon_{22}^{(2)} + \epsilon_{33}^{(2)}) \qquad \text{(b)}$$

$$\sigma^{(1)}_{\alpha\beta} = \left[\ell_1 \, \epsilon^{o}_{11} + (k_1 - G_1) \, \epsilon^{(1)}_{\gamma\gamma}\right] \delta_{\alpha\beta} + 2 \, G_1 \, \epsilon^{(1)}_{\alpha\beta} \quad \text{(a)}$$

$$\sigma^{(2)}_{\alpha\beta} = \left[\ell_2 \, \epsilon^{o}_{11} + (k_2 - G_2) \, \epsilon^{(2)}_{\gamma\gamma}\right] \delta_{\alpha\beta} + 2 \, G_2 \, \epsilon^{(2)}_{\alpha\beta} \quad \text{(b)} \qquad (3.5.12)$$

$$\sigma^{(1)}_{12} = \sigma^{(1)}_{13} = \sigma^{(2)}_{12} = \sigma^{(2)}_{13} = 0 \quad \text{(c)}$$

where G_T has been denoted G for simplicity. It is seen that the stresses (3.5.11-12) are not functions of x_1. In view of this and (3.5.12c) the stress equilibrium equations without body forces reduce to

$$\sigma_{\alpha\beta,\beta} = 0 \qquad (3.5.13)$$

Inserting (3.5.12a,b) in terms of displacement derivatives into (3.5.13) and using (3.4.89c) we obtain

$$k_1 \, u^{(1)}_{\beta,\beta\alpha} + G_1 \, u^{(1)}_{\alpha,\beta\beta} = 0 \quad \text{in } R_1 \quad \text{(a)}$$

$$k_2 \, u^{(2)}_{\beta,\beta\alpha} + G_2 \, u^{(2)}_{\alpha,\beta\beta} = 0 \quad \text{in } R_2 \quad \text{(b)} \qquad (3.5.14)$$

where R_1 and R_2 are the plane phase regions of a cylinder section, as the governing differential equations for the displacements u_2 and u_3. The boundary conditions for these displacements are (3.5.3b,c) on the contour C of the section. Note that (3.5.3a) is automatically satisfied by the choice (3.5.9a) for u_1.

At the phase interface the continuity conditions (3.2.28a) must be satisfied. Continuity for u_1 is already satisfied by the choice (3.5.9a).

Hence, displacement continuity requires

$$u^{(1)}_{\alpha} = u^{(2)}_{\alpha} \quad \text{on} \quad C_{12} \tag{3.5.15}$$

To establish the traction continuity conditions (3.2.28b) we note that the interface S_{12} is cylindrical; hence the interface normal is in the x_2, x_3 plane and thus

$$n_1 = 0 \quad \text{on} \quad S_{12} \tag{3.5.16}$$

In view of (3.2.14), (3.5.13) and (3.5.16), the interface tractions have the form

$$T_1 = 0 \qquad \text{(a)}$$

$$T_2 = \sigma_{22}\, n_2 + \sigma_{23}\, n_3 \qquad \text{(b)} \qquad (3.5.17)$$

$$T_3 = \sigma_{23}\, n_2 + \sigma_{33}\, n_3 \qquad \text{(c)}$$

Therefore the traction continuity requirement is

$$\sigma^{(1)}_{\alpha\beta}\, n_{\beta} = \sigma^{(2)}_{\alpha\beta}\, n_{\beta} \quad \text{on } C_{12} \tag{3.5.18}$$

In view of (3.5.12) this condition can be written in the form

$$\left[(k_2 - G_2)\, u^{(2)}_{\beta,\beta} - (k_1 - G_1)\, u^{(1)}_{\beta,\beta}\right] n_{\alpha} + \left[G_2\, (u^{(2)}_{\alpha,\beta} + u^{(2)}_{\beta,\alpha}) - \right.$$

$$\left. - G_1\, (u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha})\right] n_{\beta} = (\ell_1 - \ell_2)\, \epsilon^{o}_{11}\, n_{\alpha} \text{ on } C_{12} \tag{3.5.19}$$

This completes the mathematical formulation of the first problem. It is seen that the initial three dimensional problem has been reduced to a two dimensional problem for the displacements $u_2(x_2, x_3)$ and $u_3(x_2, x_3)$. Evidently the three dimensional uniqueness proof also applies to the present two dimensional formulation.

We now turn to the case when the tractions (3.5.7) are applied. We note that n_1 vanishes on the cylindrical boundary S_c and therefore T_1 also vanishes there. On the other hand we have on the terminal sections A_o, A_H

$$n_1 = 1,\ n_2 = n_3 = 0 \quad \text{on } A_H$$
$$n_1 = -1,\ n_2 = n_3 = 0 \quad \text{on } A_o \tag{3.5.20}$$

Thus the traction boundary conditions may be written as

$$\sigma_{11} = \sigma^o_{11} \qquad x_1 = o, H \qquad \text{(a)}$$
$$T_\alpha(C) = \sigma^o_{\alpha\beta} n_\beta \qquad \text{(b)} \tag{3.5.21}$$

In spite of the traction boundary conditions it is analytically convenient to retain the displacement formulation of the problem. Thus the tractions have to be expressed in terms of displacement gradients, as was done in (3.5.19) and the results must be equated to the right side of (3.5.21b). The problem for u_2 and u_3 determination then again consists of (3.5.14) with (3.5.15) and (3.5.19), and (3.5.3) is replaced by (3.5.21b). Again,

the problem is unique for the determination of u_2 and u_3 in the specimen, but here there arises a problem in the satisfaction of the boundary condition (3.5.21a). The σ_{11} stress in the phases is given by (3.5.11) where $\overset{o}{\epsilon}_{11}$ is now unknown, while the ($\overset{o}{\epsilon}_{11}$ dependent) strains ϵ_{22} and ϵ_{23} are determined by the unique solution for u_2 and u_3. In order to comply with (3.5.22) rigorously, the expressions (3.5.11) must be equal to one and the same constant $\overset{o}{\sigma}_{11}$. This can in general not be expected. Therefore it is necessary to appeal to Saint Venants principle for long cylinders and to make the much less stringent requirement that the stress resultants on the end faces A_o, A_H be the same as the stress resultants produced by the applied tractions.

The stresses (3.5.21) produce only an axial force $\overset{o}{\sigma}_{11} A$ on the end faces. There are no shear forces, torques or bending moments. In view of (3.5.13) the solution produces no shear force or torque on any section. The expression for axial force on the section is in view of (3.5.11)

$$\int_A \sigma_{11}\, dA = \overset{o}{\sigma}_{11} A = (n_1 A_1 + n_2 A_2)\, \overset{o}{\epsilon}_{11}$$
$$+ \ell_1 \int_{A_1} (\epsilon_{22}^{(1)} + \epsilon_{33}^{(1)})\, dA + \ell_2 \int_{A_2} (\epsilon_{22}^{(2)} + \epsilon_{33}^{(2)})\, dA \tag{3.5.22}$$

Equation (3.5.22) serves to determine the unknown strain $\overset{o}{\epsilon}_{11}$.

To make the bending moments on the sections vanish we must have

$$\int_A \sigma_{11} x_2\, dA = \int_A \sigma_{11} x_3\, dA = 0 \tag{3.5.23}$$

These requirements may or may not be satisfied by the solution and must be checked for each individual case. There are, however, two important cases when (3.5.3) are known to be satisfied. Firstly, if the problem is such that σ_{11} is a symmetric function of x_2 and x_3, i.e.

$$\sigma_{11}(x_2, x_3) = \sigma_{11}(-x_2, x_3) = \sigma_{11}(x_2, -x_3) = \sigma_{11}(-x_2, -x_3)$$

then since x_2 and x_3 are odd functions, the integrands in (3.5.23) are also odd functions. If also the section of the cylindrical specimen is chosen, without loss of generality, as x_2 and x_3 symmetric then (3.5.23) is evidently satisfied.

Secondly, if the stress σ_{11} is statistically homogeneous, then its variation over the section is of the kind shown in fig. 3.2.2 and it is evident that the associated bending moment is negligible.

Finally, it should be noted that plane strain analysis of a fiber reinforced cylinder is included in the preceding development as a special case. The plane strain situation is obtained by setting

$$\epsilon^{o}_{11} = 0 \tag{3.5.24}$$

in which event there is no axial displacement of the cylinder.

It is seen that in the plane strain case the only phase elastic moduli which enter into the boundary value problem for the displacements and into the expressions for $\sigma_{\alpha\beta}$ are k and G. If the phases are completely isotropic, nothing is changed. The moduli k and G have now merely to be interpreted as plane strain bulk modulus and as usual shear modulus, respectively.

There is thus complete mathematical analogy between the plane strain problems for transversely isotropic and for isotropic phases and solutions for the latter case can be rewritten at once as solutions to the former case. Care must be taken to first rewrite any elastic constant in the plane strain isotropic phase solution in terms of k and G.

Corresponding elastic constants for the analogy are given below

Isotropic Phases	Transversely Isotropic Phases	
k	k	
G	G_T	(3.5.25)
λ	$k - G_T$	
ν	$\frac{1}{2}(1 - G_T/k)$	

The analogy expressed in (3.5.25) is here called the <u>first isotropy-transverse isotropy analogy.</u> It should be carefully noted that it applies only to the case of plane strain and not directly to the stress σ_{11} as given in (3.5.11).

In the more general case of plane strain and axial strain, as previously considered, there is no such direct analogy as (3.5.25) between the isotropic and transversely isotropic phase solution since the modulus ℓ now also enters into the problem through (3.5.19) and also through (3.5.7) in the event that these are prescribed. Still, the transversely isotropic phase problem is mathematically completely similar to the isotropic phase problem, the difference between the two being merely in reinterpretation of constants.

Some consequences of the analogy described for EEM will be discussed in the appendix to this paragraph.

The present formulation of the problem is easily extended to the case when both phases are orthotropic, provided that the elastic axes of the phases are parallel to the cylinder axes. Let the orthotropic stress-strain laws of the phases be given by

$$\sigma_{11}^{(m)} = C_{11}^{(m)} \epsilon_{11}^{(m)} + C_{12}^{(m)} \epsilon_{22}^{(m)} + C_{13}^{(m)} \epsilon_{33}^{(m)} \qquad \text{(a)}$$

$$\sigma_{22}^{(m)} = C_{12}^{(m)} \epsilon_{11}^{(m)} + C_{22}^{(m)} \epsilon_{22}^{(m)} + C_{23}^{(m)} \epsilon_{33}^{(m)} \qquad \text{(b)}$$

$$\sigma_{33}^{(m)} = C_{13}^{(m)} \epsilon_{11}^{(m)} + C_{23}^{(m)} \epsilon_{22}^{(m)} + C_{33}^{(m)} \epsilon_{33}^{(m)} \qquad \text{(c)}$$

$$\sigma_{12}^{(m)} = 2\, C_{44}^{(m)} \epsilon_{12}^{(m)} \qquad \text{(d)} \qquad (3.5.26)$$

$$\sigma_{23}^{(m)} = 2\, C_{55}^{(m)} \epsilon_{23}^{(m)} \qquad \text{(e)}$$

$$\sigma_{31}^{(m)} = 2\, C_{66}^{(m)} \epsilon_{31}^{(m)} \qquad \text{(f)}$$

where m denotes the phase number, 1 or 2. Assuming again displacements of form (3.5.9), the stresses in the phases become

$$\sigma_{11}^{(m)} = C_{11}^{(m)} \epsilon_{11}^{o} + C_{12}^{(m)} \epsilon_{22}^{(m)} + C_{13}^{(m)} \epsilon_{33}^{(m)} \qquad \text{(a)}$$

$$\sigma_{22}^{(m)} = C_{12}^{(m)} \epsilon_{11}^{o} + C_{22}^{(m)} \epsilon_{22}^{(m)} + C_{23}^{(m)} \epsilon_{33}^{(m)} \qquad \text{(b)}$$

$$\sigma_{33}^{(m)} = C_{13}^{(m)} \epsilon_{11}^{o} + C_{23}^{(m)} \epsilon_{22}^{(m)} + C_{33}^{(m)} \epsilon_{33}^{(m)} \qquad \text{(c)} \qquad (3.5.27)$$

$$\sigma_{23}^{(m)} = 2\, C_{55}^{(m)}\ \epsilon_{23}^{(m)} \qquad \text{(d)}$$

$$\sigma_{12}^{(m)} = \sigma_{31}^{(m)} = 0 \qquad \text{(e)}$$

The equilibrium equations (3.5.13) remain unchanged. Insertion of (3.5.27) into (3.5.13) and expression of the strains in terms of displacement gradients, gives the differential equations

$$C_{22}^{(m)}\ u_{2,22}^{(m)} + C_{55}^{(m)}\ u_{2,33}^{(m)} + (C_{23}^{(m)} + C_{55}^{(m)})\ u_{3,23}^{(m)} = 0 \qquad \text{(a)}$$

$$C_{55}^{(m)}\ u_{3,22}^{(m)} + C_{33}^{(m)}\ u_{3,33}^{(m)} + (C_{23}^{(m)} + C_{55}^{(m)})\ u_{2,23}^{(m)} = 0 \qquad \text{(b)} \qquad (3.5.28)$$

The continuity condition (3.5.15) remains unchanged while (3.5.19) is replaced by

$$\left[C_{22}^{(2)}\ u_{2,2}^{(2)} + C_{23}^{(2)}\ u_{3,3}^{(2)} - (C_{22}^{(1)}\ u_{2,2}^{(1)} + C_{23}^{(1)}\ u_{3,3}^{(1)})\right]\ n_2$$

$$+\ \left[C_{55}^{(2)}\ (u_{2,3}^{(2)} + u_{3,2}^{(2)}) - C_{55}^{(1)}\ (u_{2,3}^{(1)} + u_{3,2}^{(1)})\right]\ n_3 =$$

$$= (C_{12}^{(1)} - C_{12}^{(2)})\ \epsilon_{11}^{o}\ n_2 \qquad \text{(a)}$$

$$\left[C_{55}^{(2)}\ (u_{2,3}^{(2)} + u_{3,2}^{(2)}) - C_{55}^{(1)}(u_{2,3}^{(1)} + u_{3,2}^{(1)})\right] n_2$$

$$+\ \left[C_{23}^{(2)}\ u_{2,2}^{(2)} + C_{33}^{(2)}\ u_{3,3}^{(2)} - (C_{23}^{(1)}\ u_{2,2}^{(1)} + C_{33}^{(1)}\ u_{3,3}^{(1)})\right]\ n_3 =$$

$$= (C_{13}^{(1)} - C_{13}^{(2)})\ \epsilon_{11}^{o}\ n_3 \qquad \text{(b)} \qquad (3.5.29)$$

It is to be noted that the problem of the fiber reinforced cylinder under boundary conditions (3.5.3) or (3.5.7) can also be formulated in terms of Airy stress functions. However, this is analytically inconvenient for satisfaction of displacement boundary or continuity conditions since representation of displacements by Airy stress functions is only possible in terms of indefinite integrals of derivatives of stress functions (see e.g. [3.7]).

An Airy stress function formulation has, however, been used by Pickett [3.6] for numerical analysis of periodic square and hexagonal arrays of circular fibers.

We now consider the second kind of problem in which the fiber reinforced cylinder is subjected to boundary conditions of type (3.5.4) or (3.5.8). The present problem is also treated by a semi-inverse method. The displacements in the cylinder are assumed to be of the form

$$u_1^{(1)}(\underline{x}) = u_1^{(1)}(x_2, x_3) \qquad \text{(a)}$$

$$u_1^{(2)}(\underline{x}) = u_1^{(2)}(x_2, x_3) \qquad \text{(b)} \qquad (3.5.30)$$

$$u_2^{(1)}(\underline{x}) = u_2^{(2)}(\underline{x}) = \overset{o}{e}_{12}\, x_1 \qquad \text{(a)}$$

$$u_3^{(1)}(\underline{x}) = u_3^{(2)}(\underline{x}) = \overset{o}{e}_{13}\, x_1 \qquad \text{(b)} \qquad (3.5.31)$$

A displacement field as the one given above is called anti-plane in the theory of elasticity (see e.g. Milne-Thomson [3.8]).

For reasons of convenience we define the functions $\varphi^{(1)}$ and $\varphi^{(2)}$ by

$$u_1^{(1)} = \varphi^{(1)} - \epsilon_{12}^{o} x_2 - \epsilon_{13}^{o} x_3 \qquad \text{(a)}$$

$$u_1^{(2)} = \varphi^{(2)} - \epsilon_{12}^{o} x_2 - \epsilon_{13}^{o} x_3 \qquad \text{(b)} \qquad (3.5.32)$$

The strains associated with (3.5.30 - 32) are then

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_{23} = 0 \qquad \text{(a)}$$

$$\epsilon_{12}^{(1)} = \frac{1}{2} \frac{\partial \varphi^{(1)}}{\partial x_2} \qquad \epsilon_{13}^{(1)} = \frac{1}{2} \frac{\partial \varphi^{(1)}}{\partial x_3} \qquad \text{(b)} \qquad (3.5.33)$$

$$\epsilon_{12}^{(2)} = \frac{1}{2} \frac{\partial \varphi^{(2)}}{\partial x_2} \qquad \epsilon_{13}^{(2)} = \frac{1}{2} \frac{\partial \varphi^{(2)}}{\partial x_3} \qquad \text{(c)}$$

The associated stresses are from (3.4.86)

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0 \qquad \text{(a)}$$

$$\sigma_{12}^{(1)} = G_1 \frac{\partial \varphi^{(1)}}{\partial x_2} \qquad \sigma_{13}^{(1)} = G_1 \frac{\partial \varphi^{(1)}}{\partial x_3} \qquad \text{(b)} \qquad (3.5.34)$$

$$\sigma_{12}^{(2)} = G_2 \frac{\partial \varphi^{(2)}}{\partial x_2} \qquad \sigma_{13}^{(2)} = G_2 \frac{\partial \varphi^{(2)}}{\partial x_3} \qquad \text{(c)}$$

where G now denotes the axial shear modulus G_A.

In view of (3.5.34) and the fact that the non-vanishing stresses are not functions of x_1, the only surviving equilibrium equation is

$$\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \tag{3.5.35}$$

Inserting (3.5.34) into (3.5.35) we find

$$\frac{\partial^2 \varphi^{(1)}}{\partial x_2^2} + \frac{\partial^2 \varphi^{(1)}}{\partial x_3^2} = 0 \qquad \text{in } R_1 \qquad \text{(a)}$$

$$\frac{\partial^2 \varphi^{(2)}}{\partial x_2^2} + \frac{\partial^2 \varphi^{(2)}}{\partial x_3^2} = 0 \qquad \text{in } R_2 \qquad \text{(b)} \tag{3.5.36}$$

Thus the functions $\varphi^{(1)}$ and $\varphi^{(2)}$ are plane harmonic (satisfy Laplace equations) within the phase areas.

The boundary conditions (3.5.4b,c) are identically satisfied by the assumed solution. Boundary condition (3.5.4a) with (3.5.28) leads to

$$\varphi(C) = 2\epsilon^{o}_{12} x_2 + 2\epsilon^{o}_{13} x_3 \tag{3.5.37}$$

But note that it is also required that

$$\varphi(x_2, x_3) = 2\epsilon^{o}_{12} x_2 + 2\epsilon^{o}_{13} x_3 \qquad \text{on } A_o, A_H \tag{3.5.38}$$

The last condition can obviously not be satisfied by the present two dimensional formulation. For a long cylinder, however, nonsatisfaction of (3.5.38) produces only insignificant end effects.

Because of the form of (3.5.31), continuity of u_2 and u_3 at phase interfaces is satisfied. Continuity of u_1 requires in view of (3.5.28) that

$$\varphi^{(1)} = \varphi^{(2)} \qquad \text{on } C_{12} \tag{3.5.39}$$

To set up the traction continuity conditions we recall that on the cylindrical interface $n_1 = 0$. Furthermore, four stresses vanish according to (3.5.34a). Thus the only nonvanishing traction component on the interface is

$$T_1 = \sigma_{12}\, n_2 + \sigma_{13}\, n_3 \tag{3.5.40}$$

Inserting (3.5.30b,c) into (3.5.40) we find

$$G_1 \frac{\partial \varphi^{(1)}}{\partial n} = G_2 \frac{\partial \varphi^{(2)}}{\partial n} \qquad \text{on } C_{12} \tag{3.5.41}$$

where the normal derivative $\frac{\partial \varphi}{\partial n}$ is given by

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial x_2}\, n_2 + \frac{\partial \varphi}{\partial x_3}\, n_3 \tag{3.5.42}$$

It is seen that the problem of axial shear is now defined in terms of the functions $\varphi^{(1)}$, $\varphi^{(2)}$ which must satisfy the Laplace equations (3.5.36), the boundary condition (3.5.37) and the continuity conditions (3.5.39) and (3.5.41). By a simple extension of the uniqueness proof for harmonic functions where the function is prescribed on the boundary (Dirichlet problem; for uniqueness proof see e.g. [3.9]) it follows that the functions $\varphi^{(1)}$, $\varphi^{(2)}$ are uniquely defined by the present formulation.

Next we consider the fiber reinforced cylinder with the traction boundary conditions (3.5.8). On the cylindrical surface S_c (3.5.16) must be satisfied and also (3.5.20 - 21) hold on the terminal sections. Therefore (3.5.8) assumes the form

$$T_1(S_c) = \sigma^o_{12}\, n_2 + \sigma^o_{13}\, n_3 \qquad \text{(a)}$$

$$T_2\,(S_c) = T_3\,(S_c) = 0 \qquad \text{(b)} \qquad (3.5.43)$$

$$\sigma_{11} = 0, \quad \sigma_{12} = \sigma^o_{12}, \quad \sigma_{13} = \sigma^o_{13} \text{ on } A_o, A_H \qquad \text{(c)}$$

The formulation (3.5.30 - 34) and its consequences also apply to the solution of the present problem. The differential equations (3.5.36) and the continuity conditions (3.5.39) and (3.5.41) remain unchanged. The boundary condition (3.5.36) is now replaced by (3.5.43a) which, in view of (3.5.34) and (3.5.42), can be written in the convenient form

$$G\,\frac{\partial\varphi}{\partial n} = \sigma^o_{12}\, n_2 + \sigma^o_{13}\, n_3 \quad \text{on } C \qquad (3.5.44)$$

In (3.5.44) G and φ assume 1 and 2 values for parts of the boundary C which are in the 1 and 2 phases respectively. Note that (3.5.43b) is identically satisfied since the stress system (3.5.34) produces no T_2 and T_3 tractions on any cylindrical surface.

The problem (3.5.36), (3.5.39), (3.5.41) and (3.5.44) is a plane harmonic problem in which the normal derivative is prescribed on the boundary.

This is known in the literature as the Neumann problem. The uniqueness proof for homogeneous domains is well known (see eg. [3.9]) and is easily extended to the present two phase case. Note that now φ is uniquely defined by the problem apart from an arbitrary additive constant. However, for strain and stress calculation the arbitrary constant is immaterial since only derivatives of φ are involved. In view of (3.5.32) the arbitrary constant is merely a rigid body displacement of the whole cylinder in the x_1 direction.

It is seen that the first of (3.5.43c) is identically satisfied while the other two are generally not satisfied. For a long cylinder this non-satisfaction produces only insignificant end effects.

Evidently, nothing is changed in the formulation if the phases are isotropic. In that case G_A merely becomes the isotropic G. This implies that any axial shearing displacement, strain or stress field for isotropic phases is also one for transversely isotropic phases with $G \longleftrightarrow G_A$ equivalence. This analogy is called the <u>second isotropy-transverse isotropy analogy</u>, complementing the first one which was given above.

The preceding formulation of the axial shear problem is readily extended to the case of orthotropic phases. Assuming again displacements of form (3.5.30 - 31) and defining the functions $\varphi^{(1)}$ and $\varphi^{(2)}$ by (3.5.32), it follows from (3.5.33) and (3.5.26) that the only surviving stresses are

$$\sigma_{12}^{(m)} = C_{44}^{(m)} \frac{\partial \varphi^{(m)}}{\partial x_2} \qquad \text{(a)}$$

$$\sigma_{13}^{(m)} = C_{66}^{(m)} \frac{\partial \varphi^{(m)}}{\partial x_3} \qquad \text{(b)} \qquad (3.5.45)$$

The equilibrium equation (3.5.35) now leads to

$$C_{44}^{(m)} \frac{\partial^2 \varphi^{(m)}}{\partial x_2^2} + C_{66}^{(m)} \frac{\partial^2 \varphi^{(m)}}{\partial x_3^2} = 0 \qquad (3.5.46)$$

as the governing differential equations for $\varphi^{(m)}$. The boundary condition (3.5.37) and the displacement continuity condition (3.5.39) remain unchanged. The traction continuity condition (3.5.41) now becomes

$$C_{44}^{(1)} \frac{\partial \varphi^{(1)}}{\partial x_2} n_2 + C_{66}^{(1)} \frac{\partial \varphi^{(1)}}{\partial x_3} n_3 = C_{44}^{(2)} \frac{\partial \varphi^{(2)}}{\partial x_2} + C_{66}^{(2)} \frac{\partial \varphi^{(2)}}{\partial x_3} \qquad (3.5.47)$$

3.5.2 Elementary Solutions for Fibrous Materials

We consider some simple cases in which the problem of the fibrous cylinder of arbitrary transverse geometry, subjected to homogeneous boundary conditions, can be solved in elementary fashion.

Suppose that the two phases are isotropic and have equal Poisson's ratios

$$\nu_1 = \nu_2 = \nu \tag{3.5.48}$$

while

$$E_1 \neq E_2 \tag{3.5.49}$$

If the cylinder is macroscopically transversely isotropic or square symmetric it follows at once from (3.5.118) and the general relations (3.4.117) and (3.4.119) that

$$\boxed{E_A^* = E_1 v_1 + E_2 v_2} \tag{3.5.50}$$

$$\nu_A^* = \nu \tag{3.5.51}$$

It is also easily realized that since the relations (3.4.117), (3.4.118) are valid for transversely isotropic phases with parallel material axes of symmetry in cylinder axis direction, the results remain valid if the Poisson's ratios in (3.5.48) are replaced by axial phase Poisson's ratios while all other phase elastic properties remain distinct. The result (3.5.50) is of utmost importance as it will be shown later that it is an excellent approximation for E_A^* of any FM or FRM.

We shall now show that under homogeneous boundary conditions appropriate for effective axial Young's modulus determination, the results (3.5.50 - 51) remain valid for more general fibrous cylinders and the internal stress and strain fields can also easily be obtained. Let a fibrous two phase cylinder be subjected to the boundary conditions (3.4.28) which when written out in detail for the cylinder are

$$\sigma_{11} = \sigma^{o}_{11}, \ \sigma_{12} = \sigma_{13} = 0 \qquad x_1 = o, H \tag{3.5.52}$$

$$\left.\begin{aligned} T_1 &= 0 \\ T_2 &= \sigma_{22}\, n_2 + \sigma_{23}\, n_3 = 0 \\ T_3 &= \sigma_{23}\, n_2 + \sigma_{33}\, n_3 = 0 \end{aligned}\right\} \text{ on } S_c \tag{3.5.53}$$

It is seen that (3.5.53) are a special case of (3.5.21). Therefore, if the phases are isotropic or transversely isotropic the par. 3.5.1 formulation of plane strain with uniform axial strain applies. Consequently

$$u_1\,(\underline{x}) = \epsilon^{o}_{11}\, x_1 \tag{3.5.54}$$

in both phases, at sufficient distance from the terminal sections of the cylinder. The uniform axial strain ϵ^{o}_{11} is at present unknown.

The displacements u_2 and u_3 are guessed to be

$$\begin{aligned} u_2\,(\underline{x}) &= -\,\nu\,\epsilon^{o}_{11}\, x_2 \\ u_3\,(\underline{x}) &= -\,\nu\,\epsilon^{o}_{11}\, x_3 \end{aligned} \tag{3.5.55}$$

in both phases. Then the strains associated with (3.5.54 - 55) are

$$\epsilon_{11} = \epsilon^{o}_{11}$$

$$\epsilon_{22} = \epsilon_{33} = -\nu\,\epsilon^{o}_{11} \qquad (3.5.56)$$

$$\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0$$

It follows by Hooke's law that the stresses are

$$\sigma^{(1)}_{11} = E_1\,\epsilon^{o}_{11} \qquad \sigma^{(2)}_{11} = E_2\,\epsilon^{o}_{11} \qquad \text{(a)}$$

$$(3.5.57)$$

$$\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0 \qquad \text{(b)}$$

where E_1 and E_2 are axial Young's moduli for transversely isotropic phases and are the usual Young's moduli for isotropic phases.

To verify the solution it is noted that (3.5.55) trivially satisfy the phase differential equations (3.5.14) and interface continuity (3.5.15). It is also seen in view of (3.5.17) and and (3.5.57) that the interface tractions vanish and thus interface traction continuity (3.5.18) is also trivially satisfied.

The unknown strain ϵ^{o}_{11} is determined by the condition (3.5.22) with (3.5.57). This leads to

$$\epsilon^{o}_{11} = \frac{\sigma^{o}_{11}}{E_1\,v_1 + E_2\,v_2} \qquad (3.5.58)$$

and so the denominator of (3.5.58) is recognized as the effective axial Young's modulus, in accordance with (3.5.50). Furthermore, since it has been verified that (3.5.55) are the actual uniform strains in both phases it follows that ν is the effective axial Poisson's ratio, in accordance with (3.5.51).

It should be noted that the solution is subject to the further Saint Venant restriction (3.5.23) which in view of (3.5.57a) assumes the form

$$E_1 \int_{A_1} x_2 \, dA + E_2 \int_{A_2} x_2 \, dA = 0$$

$$E_1 \int_{A_1} x_3 \, dA + E_2 \int_{A_2} x_3 \, dA = 0 \qquad (3.5.59)$$

Since E_1 and E_2 are arbitrary the integrals in (3.5.59) must vanish separately. This is the case if the geometry is symmetric with respect to the x_2, x_3 axes and also, in the limit, if the geometry is statistically homogeneous.

Note that it was not necessary to assume any specific macroscopic symmetry of the fibrous cylinder. Consequently (3.5.50 - 51) are valid for any statistically homogeneous fibrous cylinder, if (3.5.59) is satisfied.

The solution is easily generalized to the case of any number N of cylindrical phases which all have equal Poisson's ratios in which case

$$E_A^* = \sum_{m=1}^{m=N} E_m v_m = \bar{E} \qquad \text{(a)}$$

$$\nu_A^* = \nu \qquad \text{(b)} \qquad (3.5.60)$$

$$\sigma_{11}^{(m)} = \frac{\sigma_{11}^{o} E_m}{E_A^*} \qquad \text{(c)}$$

The solution may also be generalized in another way. Suppose that the phases are orthotropic with a common orthotropy axis x_1. Let the phase Young's moduli in x_1 direction be $E_1^{(1)}$, $E_1^{(2)}$ and the associated Poisson's ratios $\nu_{12}^{(1)}$, $\nu_{12}^{(2)}$ and $\nu_{13}^{(1)}$, $\nu_{13}^{(2)}$ where superscripts now denote the phases. If

$$\nu_{12}^{(1)} = \nu_{12}^{(2)} = \nu_{12}$$
$$\nu_{13}^{(1)} = \nu_{13}^{(2)} = \nu_{13} \tag{3.5.61}$$

Then it follows easily that

$$E_1^* = E_1^{(1)} v_1 + E_1^{(2)} v_2$$
$$\nu_{12}^* = \nu_{12} \tag{3.5.62}$$
$$\nu_{13}^* = \nu_{13}$$

It is also of some interest to note that if one phase, 2 say, is empty that is consists of parallel cylindrical voids, then it is rigorously true that

$$\nu_{12}^*, \nu_{13}^*, \nu_A^* = \nu_1$$
$$E_1^*, E_A^* = E_1 v_1 \tag{3.5.63}$$

This is easily proved by retracement of previous analysis.

Another special situation which can be analyzed is the case of transversely isotropic phases in which

$$\ell_1 = \ell_2 = \ell \tag{3.5.64}$$

which for isotropic phases assumes the form

$$\lambda_1 = \lambda_2 = \lambda \tag{3.5.65}$$

It follows from the general relations (3.4.112) and (3.5.64) that

$$\ell^* = \ell \tag{3.5.66}$$

$$n^* = n_1 v_1 + n_2 v_2 = \bar{n} \tag{3.5.67}$$

It is easily shown by direct analysis through application of the boundary conditions (3.4.57) that the stresses in the phases are

$$\sigma_{11}^{(1)} = n_1 \epsilon_{11}^{o} \qquad \sigma_{11}^{(2)} = n_2 \epsilon_{11}^{o}$$
$$\sigma_{22} = \sigma_{33} = \ell \epsilon_{11}^{o} \tag{3.5.68}$$

and that (3.5.66 - 68) are valid for any statistically homogeneous fibrous cylinder as long as (3.5.64) is fulfilled. The results are also easily generalized to any number of phases in obvious fashion.

If the phases are orthotropic and

$$C_{12}^{(1)} = C_{12}^{(2)} = C_{12}$$
$$C_{13}^{(1)} = C_{13}^{(2)} = C_{13} \tag{3.5.69}$$

all other phase moduli being distinct, then

$$C_{11}^{*} = C_{11}^{(1)} v_1 + C_{11}^{(2)} v_2$$
$$C_{12}^{*} = C_{12} \tag{3.5.70}$$
$$C_{13}^{*} = C_{13}$$

3.5.3 Composite Cylinder Assemblages

The composite cylinder assemblage model for uniaxially FRM has been described in chapter 2.1, fig. 2.1.3. We shall here be concerned with the computation of EEM for this model.

The inner cylinders are assigned the role of fibers and their material is labeled 2. The remaining material is the matrix which is labeled 1. If the phase materials are transversely isotropic about an axis in fiber direction, the assemblage is macroscopically transversely isotropic as will be explained later on. There is thus a basic set of five different EEM. However, only three are independent because of the general relations (3.4.112) and (3.4.117 - 118).

The fundamental importance of the present model is in that four EEM can be exactly calculated in terms of simple closed form expressions. A fifth EEM is bounded from below and above; however, recent work indicates that the upper bound may actually be the correct result.

Most of the results which will be here obtained were first given by Hashin and Rosen [3.10] where this model was introduced. A related model is that of a composite sphere assemblage introduced previously by Hashin [3.11] for isotropic composites. The method of analysis to be here employed is different and much simpler than the one used in [3.10].

We consider first the calculation of the transverse bulk modulus k^*. The homogeneous displacement boundary conditions to be applied to the surface of a fiber reinforced cylinder are in this case given by (3.4.59).

Consider first a single composite cylinder subjected to (3.4.59). The inner circular cylindrical fiber is of radius a and the outer cylindrical matrix shell has radius b, fig. 3.5.1. Using cylindrical coordinates the boundary displacements (3.4.59) transform into

$$u_1 (S) = u_z (b,\theta,z) = 0 \qquad \text{(a)}$$

$$u_\theta^{(1)} (b,\theta,z) = 0 \qquad \text{(b)} \qquad (3.5.71)$$

$$u_r^{(1)} (b,\theta,z) = \varepsilon^o b \qquad \text{(c)}$$

In view of (3.5.71a) the cylinder is in plane strain and moreover the cylinder is in an axially symmetric state because of (3.5.71b). Consequently, we have

$$u_z^{(1)} = u_z^{(2)} = u_\theta^{(1)} = u_\theta^{(2)} = 0 \qquad \text{(a)}$$

$$(3.5.72)$$

$$u_r^{(1)} = u^{(1)} (r) \qquad u_r^{(2)} = u^{(2)} (r) \qquad \text{(b)}$$

throughout the cylinder. As is well known

$$u = Br + C/r \qquad \text{(a)}$$

$$\sigma_{rr} = 2kB - 2GC/r^2 \qquad \text{(b)}$$

$$\sigma_{\theta\theta} = 2kB + 2GC/r^2 \qquad \text{(c)} \qquad (3.5.73)$$

$$\sigma_{zz} = 2\ell B \qquad \text{(d)}$$

$$\sigma_{rz} = \sigma_{r\theta} = \sigma_{\theta z} = 0 \qquad \text{(e)}$$

for the axially symmetric cylinder in plane strain (see e.g. [3.6]). There are here two different solutions, for fiber and matrix. In view of the first transverse isotropy - isotropy analogy of par. 3.5.1, k and G in (3.5.73) may be interpreted as transverse bulk and shear modulus respectively, of a transversely isotropic material, or as their isotropic counterparts for an isotropic material.

It is seen that in the fiber C must vanish in order to avoid infinite displacement and stress at the center. Thus we may write

$$u^{(1)} = B_1 r + C_1/r \qquad \text{(a)}$$

$$\sigma_{rr}^{(1)} = 2k_1 B_1 - 2G_1 C_1/r^2 \qquad \text{(b)} \qquad (3.5.74)$$

$$\sigma_{zz}^{(1)} = 2\ell_1 B_1 \qquad \text{(c)}$$

$$u^{(2)} = B_2 r \qquad \text{(a)}$$

$$\sigma_{rr}^{(2)} = 2k_2 B_2 \qquad \text{(b)} \qquad (3.5.75)$$

$$\sigma_{zz}^{(2)} = 2\ell_2 B_2 \qquad \text{(c)}$$

In view of (3.5.72) displacement continuity at fiber-matrix interface, $r = a$, requires only

$$u^{(1)}(a) = u^{(2)}(a) \qquad (3.5.76)$$

Because of the axial symmetry $\sigma_{r\theta} = \sigma_{rz} = 0$ throughout the cylinder and thus traction continuity at $r = a$ reduces to

$$\sigma_{rr}^{(1)}(a) = \sigma_{rr}^{(2)}(a) \tag{3.5.77}$$

Insertion of (3.5.74 - 75) into (3.5.71c) and (3.5.76 - 77) yields three linear equations for the constants B_1, C_1 and B_2. For future use we record the value of B_2 which is given by

$$B_2 = \frac{k_1 + G_1}{k_2 + G_1 - (k_2 - k_1)(a/b)^2}\,\epsilon^o \tag{3.5.78}$$

For reasons which will become apparent we are at present interested only in the tractions on the composite cylinder surface. Because of (3.5.73e) the only surviving traction components are $T_r = \sigma_{rr}$ on $r = b$ and $T_z = \sigma_{zz}$ on $z = o, H$, the terminal sections.

It follows from (3.5.74b) and the computed values of B_1, C_1 that

$$\sigma_{rr}^{(1)}(b) = 2k_c\,\epsilon^o \tag{3.5.79}$$

where

$$k_c = \frac{k_1(k_2 + G_1)[1-(a/b)^2] + k_2(k_1+G_1)(a/b)^2}{(k_2 + G_1)[1 - (a/b)^2] + (k_1 + G_1)(a/b)^2} \tag{3.5.80}$$

In view of (3.5.74c) and (3.5.75c), σ_{zz} assumes different constant values in the fiber and matrix shell. We compute the average of σ_{zz} over the cross section which can be written in the form

$$\bar{\sigma}_{zz} = 2\ell_c\,\epsilon^o \tag{3.5.81}$$

where

$$\ell_c = \frac{\ell_1(k_2+G_1)[1-(a/b)^2] + \ell_2(k_1+G_1)(a/b)^2}{(k_2+G_1)[1-(a/b)^2] + (k_1+G_1)(a/b)^2} \tag{3.5.82}$$

If in the composite cylinder $H >> b$, then replacement of the actual piecewise constant σ_{zz} stresses on the end sections by the uniform stress (3.5.81) will merely produce end effects because of Saint Venant's principle, so it is henceforth assumed that (3.5.81) is the normal stress on the end sections.

It is instructive to transform the tractions on the composite cylinder surface to the cartesian x_1, x_2, x_3 system. We have

$$\left.\begin{aligned} T_1 &= T_r \cos(r, x_1) = T_r n_1 = 0 \\ T_2 &= \qquad\qquad = T_r n_2 = 2k_c \epsilon^o n_2 \\ T_3 &= \qquad\qquad = T_r n_3 = 2k_c \epsilon^o n_3 \end{aligned}\right\} r = b \tag{3.5.83}$$

$$\left.\begin{aligned} T_1 &= 2\ell_c \epsilon^o \\ T_2 &= T_3 = 0 \end{aligned}\right\} z = o, H \tag{3.5.84}$$

It is seen that (3.5.83 - 84) is a homogeneous traction system, $\sigma^o_{ij} n_j$, on the cylinder surface, where

$$[\sigma^o_{ij}] = \begin{bmatrix} 2\ell_c\epsilon^o & 0 & 0 \\ 0 & 2k_c\epsilon^o & 0 \\ 0 & 0 & 2k_c\epsilon^o \end{bmatrix} \tag{3.5.85}$$

Suppose now that (3.4.59) is applied to the surface of a homogeneous transversely isotropic circular cylinder of radius b. The solution is most elementary, having the form (3.5.75). We obtain in particular that

$$\sigma_{rr} = 2k_o \epsilon^o \qquad \text{(a)}$$

$$\sigma_{zz} = 2\ell_o \epsilon^o \qquad \text{(b)} \qquad (3.5.86)$$

where k_o, ℓ_o are two of the five moduli of the homogeneous material (see (3.4.86)). Evidently the tractions on the homogeneous cylinder surface will be of the same form as (3.5.83 - 84), with k_c, ℓ_c replaced by k_o, ℓ_o. It is thus seen that to an external observer the composite cylinder is indistinguishable from a homogeneous cylinder with transverse bulk modulus k_c and modulus ℓ_c. This is also apparent from (3.5.85). Accordingly k_c and ℓ_c as given by (3.5.80) and (3.5.82) may be called apparent moduli of the composite cylinder.

Dually, the composite cylinder may be subjected to the mixed boundary conditions (3.4.62). Transformation to cylindrical coordinates again produces an axially symmetric plane strain problem. An analysis similar to the previous one again shows that the composite cylinder has apparent moduli k_c and ℓ_c, given by (3.5.80) and (3.5.82).

Consider now a homogeneous transversely isotropic cylindrical specimen of arbitrary cross section A, with moduli k_c and ℓ_c. If the specimen is subjected to the homogeneous boundary condition (3.4.59) we know by the theorem on

homogeneous elastic bodies of arbitrary shape, under homogeneous boundary conditions, par. 3.2.2, that the displacements in the cylindrical specimen are just of the form (3.4.59), that is

$$u_1(\underline{x}) = 0 \qquad u_2(\underline{x}) = \epsilon^o x_2 \qquad u_3(\underline{x}) = \epsilon^o x_3 \qquad (3.5.87)$$

Consequently the strains are given by

$$[\epsilon_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon^o & 0 \\ 0 & 0 & \epsilon^o \end{bmatrix}$$

and the stresses are found from Hookes' law for homogeneous transversely isotropic bodies (3.4.86) to be precisely (3.5.85).

Now consider any circular cylinder, of radius b, within the cylindrical specimen extending from base to base and with axis parallel to the specimen axis. Let the center of the circular cylinder be at the point $x_2^{(m)}$, $x_3^{(m)}$ in the transverse plane. Introduce a local coordinate system y_α defined by

$$\begin{aligned} x_2 &= x_2^{(m)} + y_2 \\ x_3 &= x_3^{(m)} + y_3 \end{aligned} \qquad (3.5.88)$$

(fig. 3.5.2). The displacements on the curved surface r = b are in view of (3.5.87) and (3.5.88)

$$u_1\ (b) = 0 \qquad \text{(a)}$$

$$u_2\ (b) = \epsilon^o x_2^{(m)} + \epsilon^o y_2 \qquad \text{(b)} \qquad (3.5.89)$$

$$u_3\ (b) = \epsilon^o x_3^{(m)} + \epsilon^o y_3 \qquad \text{(c)}$$

It is seen that the first parts of the right sides of (3.5.89b,c) are rigid body motions of the circular cylinder, thus producing no stresses and strains. The remaining parts of (3.5.89) are referred to the circular cylinders local coordinate system and thus are equivalent to (3.5.71). Furthermore, since the stresses throughout the cylindrical specimen are (3.5.85) the tractions on the circular cylinder's surface are given by (3.5.83 - 84). The conclusion is that if the circular cylinder is replaced by the composite cylinder whose apparent moduli are k_c and ℓ_c the cylindrical specimen will not know the difference since the displacements and tractions on the surface enclosing the replaced cylinder have been preserved. Thus the states of strain and stress in the remainder of the cylindrical specimen are not affected by the replacement.

Consequently, such replacements can be performed again and again, starting with composite cylinders of relatively large radii and filling the remaining volume with smaller and smaller composite cylinders. In order to preserve the same k_c and ℓ_c in all cylinders we require that all quantities in (3.5.80) and (3.5.82) be the same in all composite cylinders. Thus, in particular, the ratio a/b is the same and the composite cylinders are

all geometrically similar. In the limit the whole volume of the cylindrical specimen is filled out with composite cylinders and thus becomes a composite cylinder assemblage, Fig. 2.1.3. Evidently $(a/b)^2$ is now the phase volume fraction of the fibers, thus

$$(a/b)^2 = v_2 \qquad \text{(a)}$$

$$1-(a/b)^2 = v_1 \qquad \text{(b)} \qquad (3.5.90)$$

and k_c as given by (3.5.80) becomes the effective transverse bulk modulus k^* of the composite cylinder assemblage. In view of (3.5.90) it may be written in the equivalent forms

$$\boxed{k^* = \frac{k_1(k_2 + G_1)\, v_1 + k_2\,(k_1 + G_1)\, v_2}{(k_2 + G_1)v_1 + (k_1 + G_1)\, v_2} \qquad \text{(a)}}$$

$$\boxed{k^* = k_1 + \frac{v_2}{\dfrac{1}{k_2 - k_1} + \dfrac{v_1}{k_1 + G_1}} \qquad \text{(b)}} \qquad (3.5.91)$$

Similarly, (3.5.82) becomes ℓ^* of the composite cylinder assemblage. By use of (3.5.90) it may be written in the form

$$\boxed{\ell^* = \frac{\ell_1\,(k_2 + G_1)\, v_1 + \ell_2\,(k_1 + G_1)\, v_2}{(k_2 + G_1)v_1 + (k_1 + G_1)\, v_2}} \qquad (3.5.92)$$

It is recalled that k^* and ℓ^* of a FRM must be connected by the general relation (3.4.112a). It may indeed be verified that (3.5.91 - 92) satisfy this relation.

It is recalled that (3.5.91 - 92) are results for transversely isotropic phases, with G interpreted as G_T. For isotropic phases ℓ becomes λ while k and G are plane strain bulk modulus and isotropic shear modulus, respectively.

The analysis of k^* and ℓ^* may even be carried out for hollow or composite fibers. Suppose that all fibers have concentric cylindrical circular voids. If 'a' is a typical fiber radius let a_o be the radius of the void. Suppose that the ratio a_o/a is the same in all fibers. Denote

$$(a_o/a)^2 = v_o \tag{3.5.93}$$

where v_o is the volume fraction of voids relative to fiber volume. We know from previous analysis that also the hollow fiber behaves to an external observer as a homogeneous fiber with some apparent transverse bulk modulus k'_2. To find k'_2 it is merely necessary to reinterpret (3.5.80) for the hollow fiber in the following way: k_2 becomes an elastic modulus of the void and it therefore vanishes, k_1, G_1 become the fiber elastic moduli and $(a/b)^2$ is replaced by (3.5.93). Thus

$$k'_2 = \frac{k_2 G_2 (1 - v_o)}{G_2 + v_o k_2} \tag{3.5.94}$$

Then the transverse bulk modulus for the composite hollow cylinder assemblage becomes

$$k^* = \frac{k_1 (k_2' + G_1) v_1 + k_2' (k_1 + G_1) v_2}{(k_2' + G_1) v_1 + (k_1 + G_1) v_2} \tag{3.5.95}$$

where v_2 is the gross volume fraction of hollow fibers relative to the composite.

Next we consider the axial Young's modulus E_A^* and the axial Poisson's ratio ν_A^*. These elastic properties can be obtained in a similar way by subjecting a single composite cylinder to axial extension without load on its lateral boundary, obtaining the apparent axial Young's modulus and Poisson's ratio of the composite cylinder, and then filling out an arbitrary cylindrical specimen with composite cylinders to obtain an assemblage. Such a procedure has indeed been originally adopted in [3,10] but fortunately this is no longer necessary for the general relations (3.4.117) and (3.4.118) now permit calculation of E_A^* and ν_A^* directly in terms of k^*. The results are

$$E_A^* = E_1 v_1 + E_2 v_2 + \frac{4(\nu_2 - \nu_1)^2 v_1 v_2}{v_1/k_2 + v_2/k_1 + 1/G_1} \tag{3.5.96}$$

$$\nu_A^* = \nu_1 v_1 + \nu_2 v_2 + \frac{(\nu_2 - \nu_1)(1/k_1 - 1/k_2) v_1 v_2}{v_1/k_2 + v_2/k_1 + 1/G_1} \tag{3.5.97}$$

The effective moduli E_A^* and ν_A^* were first obtained in [3.10] in very complicated form. The much simpler forms (3.5.96 - 97) were later given by

Hill [3.5] as "effective moduli" of a single composite cylinder.

In view of the equs. (1), (2) in the appendix to par. 3.5.1 it is seen that in the case of transversely isotropic phases E, ν, k, G in (3.5.96 - 97) are to be interpreted as axial Young's modulus E_A, axial Poisson's ratio ν_A, transverse bulk modulus k and transverse shear modulus G_T, respectively. For isotropic phases they are the usual elastic moduli.

For hollow fibers E_2, ν_2 and k_2 in (3.5.95 - 96) are replaced by the corresponding apparent elastic constants of the hollow fibers. These apparent constants may be directly found from (3.5.96 - 97) by letting the fiber moduli vanish in these expressions and replacing the matrix moduli by fiber (2) moduli. We then find

$$E_2' = E_2\,(1-v_o) \qquad \text{(a)}$$

$$\nu_2' = \nu_2 \qquad \text{(b)} \qquad (3.5.98)$$

Introduction of (3.5.98) into (3.5.96 - 97) instead of E_2, ν_2 and (3.5.94) instead of k_2 yields the hollow fiber expressions for E_A^* and ν_A^*.

It should be noted that instead of hollow fibers it is just as simple to consider fibers which are themselves composite concentric cylinders. This may be of some interest for boron fibers which consist of a tungsten core on which the boron is vapor deposited. In this case the primed moduli (3.5.94) and (3.5.98) merely have to be replaced by the equivalent moduli of the composite fiber, which are again easily obtained from previous expressions of equivalent moduli of a composite cylinder.

Evidently, n^* of a composite cylinder assemblage may now also be computed by use of the relation (3.4.112b). For transversely isotropic phases this modulus is given by

$$\boxed{n^* = n_1 v_1 + n_2 v_2 - \frac{(\ell_1 - \ell_2)^2 v_1 v_2}{k_1 v_2 + k_2 v_1 + G_{T1}}} \tag{3.5.99}$$

where the phase moduli are defined by (3.4.86). For isotropic phases n, ℓ, k and G_T in (3.5.99) are given by (3.4.89).

We now consider the axial shear modulus G_A^* of a composite cylinder assemblage. The method is in principle the same as for k^* calculation. The boundary conditions (3.4.10) may be applied to the surface of the cylindrical fiber reinforced specimen and consequently we consider the problem of a single composite cylinder subjected to (3.4.10). The general axial shearing formulation given in par. 3.5.1 is immediately applicable to the present case, with $\epsilon^o_{13} = 0$. We set for convenience $\epsilon^o_{12} = \alpha$. The problem then becomes

$$\nabla^2 \varphi^{(1)} = 0 \qquad a \le r \le b \qquad \text{(a)}$$

$$\nabla^2 \varphi^{(2)} = 0 \qquad 0 \le r \le a \qquad \text{(b)} \tag{3.5.100}$$

$$\varphi^{(1)} = 2\alpha x_2 \qquad r = b \qquad \text{(a)}$$

$$\left.\begin{aligned} &\varphi^{(1)} = \varphi^{(2)} \\ &G_1 \frac{\partial \varphi^{(1)}}{\partial n} = G_2 \frac{\partial \varphi^{(2)}}{\partial n} \end{aligned}\right\} r = a \qquad \begin{aligned} &\text{(b)} \\ &\text{(c)} \end{aligned} \tag{3.5.101}$$

Evidently the problem (3.5.100 - 101) should be transformed to polar coordinates r, θ given by

$$x_2 = r\cos\theta$$

$$x_3 = r\sin\theta$$

In terms of these (compare e.g. [3.9]) the harmonic problem (3.5.99 - 101) becomes

$$\frac{\partial^2 \varphi^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi^{(1)}}{\partial \theta^2} = 0 \qquad a \le r \le b \quad \text{(a)}$$

$$\frac{\partial^2 \varphi^{(2)}}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi^{(2)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi^{(2)}}{\partial \theta^2} = 0 \qquad 0 \le r \le a \quad \text{(b)} \tag{3.5.102}$$

$$\varphi^{(1)}(b, \theta) = 2\alpha b\cos\theta \tag{3.5.103}$$

$$\varphi^{(1)}(a, \theta) = \varphi^{(2)}(a, \theta) \qquad \text{(a)}$$

$$G_1 \frac{\partial \varphi^{(1)}(a,\theta)}{\partial r} = G_2 \frac{\partial \varphi^{(2)}(a,\theta)}{\partial r} \qquad \text{(b)} \tag{3.5.104}$$

Solution of this problem is readily obtained by separation of variables or adaption of the general harmonic solution in polar coordinates in circular annular domains (compare [3.9]) to the present problem. The results are

$$\varphi^{(1)}(r,\theta) = (B_1 r + C_1/r)\cos\theta \qquad \text{(a)}$$

$$\varphi^{(2)}(r,\theta) = B_2 r\cos\theta \qquad \text{(b)} \tag{3.5.105}$$

where B_1, C_1 and B_2 are arbitrary constants. Insertion of (3.5.105) into (3.5.103 - 104) yields three linear equations for these constants, thus completely specifying (3.5.105). We record for future use the value of B_2 which is given by

$$B_2 = \frac{4\alpha G_1}{G_2 + G_1 - (G_2 - G_1)(a/b)^2} \tag{3.5.106}$$

Again we are interested only in the tractions on the boundary r = b. We know from the general analysis in par. 3.5.1 that the only surviving traction there is T_1 which in view of (3.5.40 - 41) is $G_1 \frac{\partial \varphi^{(1)}}{\partial n}$ i.e. $G_1 \frac{\partial \varphi^{(1)}}{\partial r}$ in polar coordinates. We then obtain on the basis of the present solution

$$T_1 (b,\theta) = 2\, G_{Ac}\, \alpha \cos \theta \tag{3.5.107}$$

where

$$G_{Ac} = G_1 \left[1 + \frac{(a/b)^2}{\frac{G_1}{G_2 - G_1} + \frac{1}{2}\left[1-(a/b)^2\right]} \right] \tag{3.5.108}$$

Now suppose that (3.4.10) with $\epsilon^{o}_{12} = \alpha$ is applied to the surface r = b of a homogeneous transversely isotropic circular cylinder with axial shear modulus G_{Ao}. The simplest way to find the solution is to appeal again to the theorem on homogeneous elastic bodies of arbitrary shape under homogeneous boundary conditions. It follows that the displacements in the cylinder are just of form (3.4.10) throughout. Consequently, the strains are everywhere

$$[\epsilon_{ij}] = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the stresses are everywhere

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 2G_{Ao}\alpha & 0 \\ 2G_{Ao}\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.5.109}$$

In view of (3.5.109) the only surviving traction on $r = b$ is

$$T_1 = \sigma_{12}\, n_2 = 2G_{Ao}\, \alpha \cos\theta \tag{3.5.110}$$

since $n_2 = \cos\theta$. Comparison of (3.5.110) with (3.5.107) shows that to an external observer the composite cylinder is indistinguishable from a homogeneous transversely isotropic cylinder with axial shear modulus as given by (3.5.108).

There remains, however, a problem of end conditions on the terminal sections of the composite cylinder. In view of (3.4.10) and (3.5.38) it is seen that we should have $\varphi = 2\alpha x_2$ on the terminal sections which condition is certainly not satisfied by (3.5.105). To assess the magnitude of the effect involved it is best to consider the boundary tractions. It is easily shown that the section resultant associated with (3.5.105) is only a shear force in x_2 direction. Thus if in the composite $H >> b$ the actual variation of φ on the end section may be replaced by (3.4.19a) and the end effect produced is macroscopically insignificant.

We now follow the same reasoning that led to the construction of the composite cylinder assemblage for k^* calculation. A homogeneous transversely isotropic cylindrical specimen of arbitrary cross section is assigned the axial shear modulus G_{Ac}. Parallel circular cylinders within the specimen can be replaced by composite cylinders with constant a/b ratios without affecting the stresses and strains in the remaining volume. In the limit of filling out the specimen becomes a composite cylinder assemblage with axial shear modulus G_{Ac}. This modulus is now denoted G_A^* and becomes in terms of fiber and matrix volume fractions

$$G_A^* = G_1 + \frac{v_2}{\frac{1}{G_2 - G_1} + \frac{v_1}{2G_1}} \qquad \text{(a)}$$

$$G_A^* = G_1 \frac{G_1 v_1 + G_2(1 + v_2)}{G_1(1 + v_2) + G_2 v_1} \qquad \text{(b)} \qquad (3.5.111)$$

If the phases are transversely isotropic or even square symmetric, G_2, G_1 in (3.5.111) are the phase axial shear moduli. If the fibers are hollow with void volume fractions defined by (3.5.93), then G_2 in (3.5.111) has to be replaced by

$$G_2' = G_2 \frac{1 - v_o}{1 + v_o} \qquad (3.5.112)$$

where v_2' is given by (3.5.93).

The axial effective shear modulus is unique in that it depends only on the phase shear moduli and not on other phase elastic properties. This is so even for the axial shear modulus of any fibrous material since by par. 3.5.1 only G_1 and G_2 enter into the field analysis for this case.

It is again very easy to generalize to the case of composite fibers in the manner described above, after equ. (3.5.98).

There remains the problem of the calculation of a fifth EEM in order to complete the analysis of the composite cylinder assemblage. In view of (3.4.82) and (3.4.85) and the results obtained above for the composite cylinder assemblage, it is seen that any one of the group G_T^*, E_T^* and ν_T^* can be taken as a fifth elastic constant. It turns out that of these the easiest to consider is G_T^*. But, unfortunately, the replacement method which has been previously employed fails for this case since a composite cylinder has no apparent transverse shear modulus.

It is, however, possible to find upper and lower bounds for G_T^* on the basis of variational methods. The discussion of such methods and the detailed derivation of bounds is deferred to chapters 3.6, 3.7. The bounds are here presented for the sake of completeness.

Upper and lower bounds are denoted $G_{T(+)}^*$ and $G_{T(-)}^*$ respectively. Then for fibers (2) which are stiffer than the matrix (1) i.e. $k_2 > k_1, G_2 > G_1$ (*)

(*) Bounds for the case of matrix stiffer than fibers are given in par. 3.7.4.

$$G^*_{T(+)} = G_1 \frac{(1+\alpha v_2^3)(\rho + \beta_1 v_2) - 3v_2 v_1^2 \beta_1^2}{(1+\alpha v_2^3)(\rho - v_2) - 3v_2 v_1^2 \beta_1^2} \qquad \text{(a)}$$

where

$$\alpha = \frac{\beta_1 - \gamma\beta_2}{1+\gamma\beta_2} \qquad \text{(b)}$$

$$\rho = \frac{\gamma + \beta_1}{\gamma - 1} \qquad \text{(c)} \qquad (3.5.113)$$

$$\gamma = \frac{G_2}{G_1} \qquad \text{(d)}$$

$$\beta_1 = \frac{1}{3-4\nu_1} \qquad \beta_2 = \frac{1}{3-4\nu_2} \qquad \text{(e)}$$

$$G^*_{T(-)} = G_1 + \frac{v_2}{\dfrac{1}{G_2 - G_1} + \dfrac{k_1 + 2G_1}{2G_1(k_1+G_1)} v_1} \qquad \text{(a)}$$

or equivalently (3.5.114

$$G^*_{T(-)} = G_1 \left[1 + \frac{v_2}{\dfrac{1}{\gamma - 1} + \dfrac{v_1}{1+\beta_1}} \right] \qquad \text{(b)}$$

The upper bound (3.5.113) has been originally derived in [3.10] in terms of a parameter which had to be obtained by solution of a six by six system of linear equations. (See par. 3.6.2) The present equivalent and much simplified result (3.5.115) has been obtained recently by algebraic solution of the six by six system, [3.12].

A lower bound was also derived in [3.10]. Later, bounds for statistically transversely isotropic fibrous materials of otherwise arbitrary phase geometry were derived by Hashin [3.13]. Surprisingly enough it was found that the general lower bound of [3.13] is always higher than the lower bound of [3.10], in the case of fibers which are stiffer than the matrix. Since the composite cylinder assemblage is a special case of the general fibrous geometry of [3.13], the general bounds are certainly also valid for the composite cylinder assemblage model. Therefore the lower bound of [3.13] supersedes the lower bound of [3.10]. Thus the bound (3.5.114) is the general fibrous geometry lower bound.

Recent work by Hashin and Rosen [3.12] indicates that the upper bound (3.5.113) may actually be the expression for G_T^* of the composite cylinder assemblage model in the event that the fibers are stiffer than the matrix.

If the phases are transversely isotropic the results (3.5.113 - 114) are still valid with the following interpretation of phase properties: G_2, G_1 are the phase transverse shear moduli and the Poisson's ratios are replaced by

$$\nu = \frac{1}{2}\,(1 - G_T/k)$$

It is of some interest to specialize the results obtained so far for the composite cylinder assemblage model to extreme cases. In the first case it is assumed that the fibers are perfectly rigid, which means physically that they are very much stiffer than the matrix. We thus assume that all fiber moduli are infinitely larger than matrix moduli. We then obtain from the previous results (3.5.91), (3.5.96), (3.5.111) and (3.5.113)

Rigid fibers

$$k^* = k_1 + (k_1 + G_1)\frac{v_2}{1 - v_2} \qquad \text{(a)}$$

$$E_A^* \approx E_2\, v_2 \qquad \text{(b)} \qquad (3.5.115)$$

$$G_A^* = G_1 \frac{1 + v_2}{1 - v_2} \qquad \text{(c)}$$

$$G_1 \left[1 + (1 + \beta_1)\frac{v_2}{v_1}\right] \le G_T^* \le G_1 \frac{(1-v_2^3)(1+\beta_1 v_2) - 3v_2 v_1^2 \beta_1^2}{(1-v_2^3)(1-v_2) - 3v_2 v_1^2 \beta_1^2} \qquad \text{(d)}$$

Note that (3.5.115b) becomes infinite for rigid fibers. This result implies that for very stiff fibers (3.5.115b) is a very accurate approximation of (3.5.96). The modulus ℓ^* and ν_A^* are not meaningful for rigid fibers since ν_2 remains in the expressions and for a rigid material ν_2 is indeterminate.

The second extreme case is that of cylindrical parallel voids. This is obtained from previous results by letting fiber moduli be infinitely smaller than matrix moduli. We then have:

Cylindrical voids

$$k^* = \frac{k_1 G_1 (1-v_2)}{G_1 + k_1 v_2} \qquad \text{(a)}$$

$$\ell^* = \frac{\ell_1 G_1 (1 - v_2)}{G_1 + k_1 v_2} \qquad \text{(b)}$$

$$E_A^* = E_1 (1-v_2) \qquad \text{(c)} \qquad (3.5.116)$$

$$\nu_A^* = \nu_1 \qquad \text{(d)}$$

$$G_A^* = G_1 \frac{1 - v_2}{1 + v_2} \qquad \text{(e)}$$

Note that (3.5.116c,d) are merely special cases of the general results (3.5.61).

The bounds (3.5.113 - 114) are not valid in the present case. Bounds for matrix which is stiffer than the fibers, which situation includes cylindrical voids as a special case will be given later, par. 3.7.4.

The third extreme case to be considered is that of incompressible isotropic matrix in which case

$$\nu_1 = \frac{1}{2} \qquad \text{(a)}$$

$$(3.5.117)$$

$$\ell_1, k_1 \to \infty \qquad \text{(b)}$$

where (3.5.117b) follows from (3.4.89 - 91). It should be noted that great care has to be exercised in the use of (3.5.117b) in (3.5.91 - 92) and (3.5.96 - 97) for the meaning of (3.5.117b) is really that ℓ_1, k_1 become very large. It may be safely assumed that they are much larger than G_1 but it is questionable whether they become much larger than fiber moduli in the event that the fibers are much stiffer than the matrix. Since the last case is of most practical interest it is preferable not to simplify (3.5.91 - 92) and (3.5.96 - 97) by use of (3.5.117).

It is noted that no Poisson's ratios appear in the G_A^* expression (3.5.111). This, of course, is in accordance with the general axial shearing boundary value problem formulation of par. 3.5.1, which does not involve phase Poisson's ratios. Therefore (3.5.111) remains the same whether or not the matrix is incompressible.

The G_T^* bounds (3.5.113 - 114) assume the following form for incompressible matrix

$$G_1 \left[1 + \frac{v_2}{\frac{1}{\gamma-1} + \frac{v_1}{2}} \right] \leq G_T^* \leq G_1 \frac{(1+\alpha v_2)^3 (\rho+v_2) - 3v_2 v_1^2}{(1+\alpha v_2)^3 (\rho-v_2) - 3v_2 v_1^2} \qquad (3.5.118)$$

It is of interest to note that the left side of (3.5.118) is the G_A^* result (3.5.111) in different form. The significance of this, if any, is not known to the writer.

For rigid fibers <u>and</u> incompressible matrix, (3.5.118) reduces to

$$G_1 \left(1 + 2\frac{v_2}{v_1}\right) \le G_T^* \le G_1 \frac{1 - v_2 + 5v_2^2 + v_2^3}{(1 - v_2)^3} \tag{3.5.119}$$

To obtain results for E_T^* and ν_T^* we exploit the relations (3.4.82 - 84). It is seen that for the composite cylinder assemblage all EEM entering into the right sides of these expressions are known in closed form, except for G_T^* which is bounded. It is easily shown that in terms of G_T^* bounds we have

$$\boxed{\begin{aligned} &\frac{4k^* G_{T(-)}^*}{k^* + mG_{T(-)}^*} \le E_T^* \le \frac{4k^* G_{T(+)}^*}{k^* + mG_{T(+)}^*} && \text{(a)} \\ &\frac{k^* - mG_{T(+)}^*}{k^* + mG_{T(+)}^*} \le \nu_T^* \le \frac{k^* - mG_{T(-)}^*}{k^* + mG_{T(-)}^*} && \text{(b)} \end{aligned}} \tag{3.5.120}$$

If it is accepted that $G_{T(+)}^*$ is the actual expression for G_T^* then

$$E_T^* = E_{T(+)}^* = \frac{4k^* G_{T(+)}^*}{k^* + mG_{T(+)}^*}$$

$$\nu_T^* = \nu_{T(-)}^* = \frac{k^* - mG_{T(-)}^*}{k^* + mG_T^*(+)}$$

We now re-examine the statement made at the beginning of this paragraph that a composite cylinder assemblage with transversely isotropic phases is macroscopically transversely isotropic. Indeed it has been seen that four EEM of a composite cylinder assemblage are just the apparent moduli of a single composite cylinder. A single composite cylinder is obviously transversely isotropic because of its axial symmetry. This argument cannot be applied to G_T^*. However, it is clear that because of the geometry involved the value of G_T^* cannot depend upon the orientation of x_2, x_3 axes with respect to which the analysis is performed. Consequently the composite cylinder assemblage is transversely isotropic.

The reader will recall that it has been emphasized in chap. 3.3 that EEM depend in general on all the details of the phase geometry and not just on the volume fractions. It is seen, however, that in spite of the complex geometry of the composite cylinder assemblage, the volume fractions are its only geometrical parameters which enter into the EEM expressions. This is a rather special and fortunate situation which is a result of premeditated construction of the model in such manner that its EEM are the equivalent moduli of one single composite cylinder. Consequently, the results are so surprisingly simple.

The present results for EEM of the composite cylinder assemblage model will be referred to very frequently in the remainder of this work. They shall then be identified by an additional subscript c, thus k_c^*, G_{Ac}^*, E_{Ac}^*, etc.

Finally, it should be pointed out that the stress and strain fields in the composite cylinder assemblage are just the stresses and strains in the composite cylinders, which are known by previous analysis for those boundary conditions which lead to closed expressions for EEM, i.e. k^*, n^*, ℓ^* and E_A^*. For those boundary conditions which lead to G_T^* and E_T^* the stresses and strains are not known, unless it is accepted that (3.5.113) is the actual result for G_T^* in which case the stresses and strains are also known in those cases. Results for internal stresses are given in appendix 2.

Some numerical examples will now be given. Table 3.5.1 lists some elastic properties of various fiber and matrix materials. These should not be taken too literally. In particular, different epoxies have significantly different Young's moduli which range between about $0.3 - 0.6 \times 10^6$ psi.

Tables 3.5.2-4 contain effective elastic properties of various fiber reinforced materials which have been computed on the basis of composite cylinder assemblage (CCA) theory, using the properties listed in table 3.5.1. The effective properties $G_{T(+)}^*$, $E_{T(+)}^*$ and $\nu_{T(-)}^*$ have been underlined as these may be the actual results for the composite cylinder assemblage model.

Figs. 3.5.3 - 3.5.8 show various plots of effective elastic properties and also comparisons with experimental results. It is seen that E_A^* shown in fig. 3.5.3 varies to all practical considerations linearly with v_2. This is due to the fact that the third term in (3.5.96) is numerically insignificant in comparison to the first two terms. Accordingly E_A^* is excellently approximated by (3.5.50). It will be shown in chaps. 3.6.-7 that this is true for any fibrous geometry.

Fig. 3.5.4 shows ν_A^*, (3.5.97) which is represented by a very flat curve which can be approximated with fair accuracy by a straight line. It will be seen in chap. 3.7 that this feature of ν_A^* is also common to general fibrous materials.

Fig. 3.5.5 shows G_A^* variation as given by (3.5.111), together with experimental results from [3.14]. It is seen that the experimental results tend to be above the predicted values. It will be shown in chap. 3.7 that (3.5.111) is also a lower bound for G_A^* of a material of any fibrous geometry. This, at least, is certainly verified by fig. 3.5.5.

Figs. 3.5.6-7 show bounds for E_T^* which were computed by use of (3.5.120), and also experimental results. It is seen that there is very good agreement between theory and experiment. The experimental results in fig. 3.5.7 definitely tend to favor the upper bound which may lend additional credence to the previously stated conjecture that $E_{T(+)}^*$ is the actual result for E_T^* of the CCA model.

Fig. 3.5.8 shows ν_T^* bounds. The lower bound has been emphasized by a full curve since it is the actual result if $G_{T(+)}^*$ (thus $E_{T(+)}^*$ is the actual result.

For additional numerical results for effective elastic properties the reader is referred to [3.17 - 19].

3.5.4 Dilute Reinforcement

The problem to be considered here is that of uniaxially FRM when the volume fraction of fibers is very small. We denote the fiber volume fraction v_2 by c and assume that

$$c^2 << c << 1 \tag{3.5.121}$$

It should be realized at the outset that the case is not one of practical interest since in FRM fiber volume fractions are generally of the order of 40 - 70%. However, the case is of importance as a check on expressions for any fiber volume fractions and also the method used has its applications in other cases as will be seen later, chap. 3.8.

The crucial assumption which makes analysis possible is that the fibers do not interact. Mathematically this assumption can be expressed in the following manner: Suppose that a dilutely reinforced cylindrical specimen is subjected to homogeneous displacement or traction boundary conditions. Then the state of stress and strain in any fiber can be computed with sufficient accuracy from the problem of a homogeneous cylindrical specimen containing one single fiber, when the cross section dimensions are infinitely larger than the fiber cross section dimensions.

Suppose that the fibers are circular, with equal or unequal radii. We start with analysis of the effective bulk modulus k^* when the composite is transversely isotropic or square symmetric. Without loss of generality we may consider a cylindrical specimen of circular section with radius R,

in which a circular fiber of radius 'a' is embedded concentrically. By hypothesis

$$a << R \tag{3.5.122}$$

The boundary conditions are (3.5.59) at r = R and we realize that such a problem has already been solved in par. 3.5.3 for the more general case when the outer radius is not large with respect to a.

We now recall the result (3.4.123) and we realize that for k^* calculation it is only necessary to know $\bar{\epsilon}^{(2)}$ in the fibers. In cylindrical coordinates and plane strain axial symmetry

$$\epsilon^{(2)} = \frac{1}{2}\left(\epsilon^{(2)}_{rr} + \epsilon^{(2)}_{\theta\theta}\right) = \frac{1}{2}\left(\frac{du^{(2)}_r}{dr} + \frac{u^{(2)}_r}{r}\right)$$

Introducing (3.5.75a) into the last expression we have

$$\epsilon^{(2)} = B_2 \tag{3.5.123}$$

Now B_2 is given by (3.5.78) for any ratio between inner and outer radii. In the present case, because of (3.5.121), the quantity $(a/R)^2$ can be neglected in the denominator of (3.5.78). We then have in view of (3.5.123)

$$\epsilon^{(2)} = \frac{k_1 + G_1}{k_2 + G_1}\,\epsilon^{o} \tag{3.5.124}$$

Since by (3.5.124) $\epsilon^{(2)}$ is a constant in any fiber, its average $\bar{\epsilon}^{(2)}$ over all fibers is also given by (3.5.120). Inserting the result into (3.4.123) we find

$$k^* = k_1 + (k_2 - k_1)\,\frac{k_1+G_1}{k_2+G_1}\,c \tag{3.5.125}$$

The result (3.5.125) can also be directly obtained from the composite cylinder assemblage result for k^* given by (3.5.91). If in the latter we set

$$v_2 = c$$
$$v_1 = 1-c \tag{3.5.126}$$

and expand the resulting expression as a power series in c, it is seen that

$$k^* = k_1 + (k_2 - k_1)\frac{k_1+G_1}{k_2+G_1}c + \ldots. \tag{3.5.127}$$

Now if c is as small as required by (3.5.116), all terms with c powers higher than 1 can be neglected in (3.5.127) and k^* reduces to (3.5.125). It should however be noted that (3.5.125) is more general than a composite cylinder assemblage result since in order to derive it the special geometrical construction by which a composite cylinder assemblage is obtained was not necessary. It was only necessary to assume that there is no fiber interaction and therefore (3.5.125) is valid for any sufficiently dilute fiber arrangement (as long as it can be assumed to have the required macroscopic symmetry). Thus (3.5.125) is also valid for dilute reinforcement by circular fibers of equal cross sections and would in particular also apply to the cases of periodic hexagonal and square arrays.

Next we consider E_A^* and ν_A^*. These can most easily be obtained in the following manner: Since the composite cylinder assemblage results (3.5.96 - 97) were obtained only from k^* as given by (3.5.91), and since we

now know that for dilute reinforcement k^* can be obtained by truncation of the c power expansion of k^* after two terms, it follows that E_A^* and ν_A^* for dilute reinforcement can be similarly obtained by retention of the first two terms in their c power expansions. The results are

$$E_A^* = E_1 + [E_2 - E_1 + \frac{4k_2 G_1 (\nu_2 - \nu_1)^2}{k_2 + G_1}] c \tag{3.5.128}$$

$$\nu_A^* = \nu_1 + (\nu_2 - \nu_1) \cdot \frac{k_2}{k_1} \cdot \frac{k_1 + G_1}{k_2 + G_1} c \tag{3.5.129}$$

The method used to obtain k^* for dilute reinforcement is directly applicable to obtain G_A^* for the same case. We use the general result (3.4.124) and the axial shear composite cylinder solution obtained in par. 3.5.4 for the case of very small inner radius. It is seen that $\epsilon_{12}^{(2)}$ in the fiber, is given by the first of (3.5.33c). Now $\varphi^{(2)}$ in the fiber as given by (3.5.105b) can be written

$$\varphi^{(2)} = B_2 x_2$$

Therefore

$$\epsilon_{12}^{(2)} = \frac{1}{2} B_2 \tag{3.5.130}$$

For very small a/b (or a/R), B_2 as given by (3.5.106) reduces to a constant and thus the average $\epsilon_{12}^{(2)}$ is also given by (3.5.130. Thus we have from (3.4.124)

$$G_A^* = G_1 \ (1 + 2 \frac{G_2 - G_1}{G_2 + G_1} c) \tag{3.5.131}$$

Again this result is directly obtainable from G_A^* as given by (3.5.111) if (3.5.126) and (3.5.120) are used in these expressions.

An expression for the transverse shear modulus G_T^* in the case of dilute circular reinforcement can be similarly derived. We can introduce (3.5.120), (3.5.126) into (3.5.111 - 114) and it is not difficult to show that they coincide to give

$$G_T^* = G_1 + (G_2 - G_1) \frac{2(k_1+G_1)}{k_1+(k_1+2G_1)G_2/G_1} c \qquad (3.5.132)$$

Alternatively, we may consider a circular cylinder of radius R in which there is embedded concentrically a fiber of radius a, when R is much larger than a. The cylinder's external surface is subjected to the transverse shearing boundary conditions (3.4.13) and it is seen that this elasticity problem falls into the general category of the first kind of boundary value problem discussed in par. 3.5.1.

The problem may be solved in closed form, and it is found that the only nonvanishing strain in the fiber is a uniform $\epsilon_{23}^{(2)}$ which is given by(*)

$$\epsilon_{23}^{(2)} = \frac{2(k_1+G_1)}{k_1+(k_1+2G_1)G_2/G_1} \epsilon_{23}^{o} \qquad (3.5.133)$$

(*) The result (3.5.133) can be obtained on the basis of a solution for a composite cylinder under transverse shearing displacement boundary conditions which is given in par. 3.6.4, by specializing the solution to the case b >>a. Alternatively, (3.5.133) may be obtained by plane strain version of methods given in [3.20]. See also [3.21].

Since $\epsilon_{23}^{(2)}$ is uniform in any fiber it is also equal to $\bar{\epsilon}_{23}^{(2)}$. Thus (3.5.133) may be used for $\bar{\epsilon}_{23}^{(2)}$ in the general result (3.4.125). When this is done, (3.5.132) is G_T^* for any sufficiently dilute reinforcement of circular fibers.

By the transverse isotropy -- isotropy analogies (par. 3.5.1), all the dilute reinforcement results obtained are also valid for transversely isotropic phases, the interpretation of phase moduli being just as in the case of composite cylinder assemblages. To recapitulate, k becomes the phase transverse bulk modulus in all expressions, G becomes the phase axial shear modulus in (3.5.131) and the transverse shear modulus in all others, and E and ν become axial phase Young's modulus and Poisson's ratio, respectively.

This completes the analysis of effective elastic properties of transversely isotropic materials with dilute reinforcement of circular fibers. It should be noted that for a square symmetric material (3.5.132) represents the transverse shear modulus G'^*_T, (3.4.56), and an additional related calculation is necessary to find the other transverse shear modulus. It should also be mentioned that all of the dilute reinforcement results can be obtained by use of proper homogeneous traction boundary conditions instead of the more convenient homogeneous displacement boundary conditions which have been employed.

Dilute reinforcement results can also be obtained for elliptical fibers since the necessary boundary value problem for one elliptical fiber in an infinite matrix can be solved. It is known that under homogeneous boundary conditions at infinity the strains in the elliptical fiber are uniform. Thus

all the results would have the same general form as for circular fibers, i.e.

$$M^* = M_1(1 + A_M c) \tag{3.5.134}$$

where M^* is some effective modulus, M_1 its matrix counterpart and A_M is some number depending upon phase properties and ellipse geometry (ratio between minor and major axes).

It should be noted that the orientation of the ellipses would specify the macroscopic symmetry of the FRM. Thus if all fiber sections are identical, equal orientation of all ellipses would produce an orthotropic material, while random orientation would produce a transversely isotropic material. Although the analysis for elliptical fibers is a relatively straightforward affair, the writer is not aware of such results in the literature.

As is seen from the results obtained, all dilute reinforcement results depend only upon phase properties and phase volume fractions. The reason for this is the basic assumption of non-interaction between the fibers which implies that for any fiber placement the states of stress and strain in the fibers do not depend upon their position.

An important interpretation of the dilute reinforcement results is as follows. Suppose that in a FRM the fibers are all circular with volume fraction c which is not small. Let M^* be any effective modulus. Presumably such a modulus could be expanded in a power series in c, thus

$$M^* = M_1 \ (1 + A_1 c + A_2 c^2 + \ldots \ ) \tag{3.5.135}$$

The value of A_1 is now known from the previous results and it is seen that it is geometry independent. However, the other coefficients in the expansion are in general geometry dependent. (An exception are the composite cylinder assemblage results of par. 3.5.4). Yet, it is seen that for all different fiber placements the results for dilute reinforcement reduce to one and the same expression.

It is also seen that the coefficient A_1 in (3.5.131) can be interpreted as

$$A_1 = \frac{1}{M_1} \left. \frac{dM^*}{dc} \right|_{c=0}$$

thus giving the slope of the M^* versus c curve at the origin.

Recently Chow and Hermans [3.46] have attempted to compute the second coefficient, A_2 in the expansion (3.5.135) for EEM of unixial FRM with circular fibers, by a method of "reflection."

It should be emphasized that, unlike A_1, the coefficient A_2 will in general depend on the detailed arrangement of circular fibers, while in the analysis given in [3.46] it depends only on the volume fractions. If the A_2 computed in [3.46] were indeed rigorous universal results they would have to be the same as the A_2 found by expansion of the rigorous composite cylinder assemblage EEM, which were given in par. 3.5.3, as power series in $v_2 = c$.

Comparison of the two sets of A_2 shows no agreement whatsoever, the A_2 predicted in [3.46] being very much smaller than the ones obtained from the composite cylinder assemblage expressions. It is therefore concluded that the A_2 obtained in [3.46] are not of universal validity and it would appear that the results should be considered as approximations.

3.5.5 Numerical Analysis

Numerical analysis of internal fields in FRM is carried out by numerical solution of the elasticity equations, subject to appropriate boundary conditions, with the aid of electronic computers. Such analyses have mostly been limited to the cases of periodic rectangular, square and hexagonal arrays, examples of which have been shown in figs. (2.1.4-6). Because of the periodic geometry it is possible to define repeating elements in which, by symmetry, the stress and strain fields are identical. Such elements are in the present cases the RVE of the composite.

Appropriate repeating elements for rectangular , square and hexagonal arrays, respectively, of identical circular fibers are shown in fig. 3.5.9 . Similar repeating elements can, of course, be constructed when the (identical) fibers have other shapes which must, however, be properly symmetric. For rectangular and square arrays the fiber section must have x_2 and x_3 axes of symmetry. For hexagonal arrays the fiber section must have three axes of symmetry, equally spaced at 120°.

The boundary conditions on a repeating element are found by symmetry considerations, as will be explained for a subsequent example. It should be borne in mind that the analysis for a repeating element does not apply for elements which are situated close to the bounding surface of the composite. It is thus seen that also in the present cases there is a boundary layer (compare: fundamental postulate of theory of elasticity of heterogeneous media, par. 3.2.2) which can be disregarded for a body which contains a very large number of fibers.

We consider as an illustrative example the computation of $E_2^* = E_T^*$ and the associated internal fields of a square array of identical circular fibers, fig. 2.1.4. For convenience we choose a cylindrical fiber reinforced specimen of square section with side 2D, which contains a square array of many fibers. Appropriate cylinder boundary conditions are given by (3.4.31). In the present case these assume the form: no shear stresses on the entire cylinder surface, zero σ_{11} stress on cylinders terminal sections and

$$\sigma_{22}(\pm D, x_3) = \sigma^{o}_{22} = \sigma^{o}$$
$$\sigma_{33}(x_2, \pm D) = 0 \qquad (3.5.136)$$

Consider now the repeating element, fig. (3.5.9b). It is clear from the loading and from the geometry that the repeating element is bounded by four axes of symmetry. On such an axis of symmetry the shear stresses must vanish and after deformation the axis remains a parallel straight line. Consequently, the following boundary conditions are valid for any repeating element, apart from immaterial rigid body displacements

$$u_{2,3} + u_{3,2} = 0 \qquad x_2 \text{ or } x_3 = 0, d \qquad \text{(a)}$$

$$u_2(0, x_3) = 0, \qquad u_2(d, x_3) = \delta_2 \qquad \text{(b)} \qquad (3.5.137)$$

$$u_3(x_2, 0) = 0, \qquad u_3(x_2, d) = \delta_3 \qquad \text{(c)}$$

Condition (a) comes from

$$\sigma^{(1)}_{23} = G_1 (u^{(1)}_{2,3} + u^{(1)}_{3,2})$$

$$\sigma^{(2)}_{23} = G_2 (u^{(2)}_{2,3} + u^{(2)}_{3,2})$$

which apply according to whether the boundary is composed of phase 1 (matrix) or phase 2 (fiber). Conditions (b), (c) involve the presently unknown constants δ_2, δ_3.

In addition, the terminal sections of the repeating element are free of load. The boundary value problem is thus of the nature of the first kind of boundary value problem discussed in par. 3.5.1, i.e. plane strain with axial strain. The governing differential equations are given by (3.5.14) and the interface conditions at fiber-matrix interface by (3.5.15) and (3.5.19). It is to be carefully noted that stresses have to be determined from (3.5.11 - 12), which requires determination of the unknown axial strain ϵ^{o}_{11} by means of (3.5.22), with the left side of this equation vanishing in the present case. It is not permissible to assume plane strain conditions, which would make $\epsilon^{o}_{11} = 0$.

It is convenient to consider instead of (3.5.137) the two separate sets of boundary conditions for displacement fields $\overset{2}{u}_{\alpha}$ and $\overset{3}{u}_{\alpha}$

$$\overset{2}{u}_{2,3} + \overset{2}{u}_{3,2} = 0 \qquad x_2 \text{ or } x_3 = 0, d$$

$$\overset{2}{u}_2 (0, x_3) = 0, \quad \overset{2}{u}_2 (d, x_3) = 1 \tag{3.5.138}$$

$$\overset{2}{u}_3 (x_2, 0) = 0, \quad \overset{2}{u}_3 (x_2, d) = 0$$

$$\overset{3}{u}_{2,3} + \overset{3}{u}_{3,2} = 0 \qquad x_2 \text{ or } x_3 = 0, d$$

$$\overset{3}{u}_2 (0, x_3) = 0, \quad \overset{3}{u}_2 (d, x_3) = 0 \tag{3.5.139}$$

$$\overset{3}{u}_3 (x_2, 0) = 0, \quad \overset{3}{u}_3 (x_2, d) = 1$$

Otherwise, each of $\overset{2}{u}_\alpha$, $\overset{3}{u}_\alpha$ must satisfy the same differential equations interface continuity and terminal section conditions as the previously considered u_α. Then by superposition the solution for boundary conditions (3.5.137) is given by

$$u_\alpha (x_2, x_3) = \delta_2 \overset{2}{u}_\alpha (x_2, x_3) + \delta_3 \overset{3}{u}_\alpha (x_2, x_3) \qquad (3.5.140)$$

It is seen that the boundary conditions (3.5.138 - 139) actually define identical problems, one being obtained from the other by rotation. Thus it follows that

$$\overset{2}{u}_\alpha (x_2, x_3) = \overset{3}{u}_\alpha (x_3, - x_3)$$

The problem of the repeating element under boundary conditions (3.5.138 - 139) is now solved numerically. In this respect it is mentioned that Pickett [3.6] employed a stress formulation of such problems since this makes it possible to use Airy stress functions and to make use of general two dimensional solutions of plane elasticity problems. Displacements are then expressed in form of integrals of stress function derivatives and continuity conditions at fiber-matrix interface and boundary conditions are satisfied at a finite set of points. Adams, Doner and Thomas [3.22] and others used a displacement formulation and replaced the elasticity differential equations by finite difference equations. Continuity and boundary conditions were again pointwise satisfied.

Once the displacements $\overset{2}{u}_\alpha$ and $\overset{3}{u}_\alpha$ have been found, numerically, the associated strains $\overset{2}{\epsilon}_{\alpha\beta}$ and $\overset{3}{\epsilon}_{\alpha\beta}$ and stresses $\overset{2}{\sigma}_{\alpha\beta}$ and $\overset{3}{\sigma}_{\alpha\beta}$ are obtained by numerical differentiation. It follows from (3.5.139) that the strains and stresses in the repeating element which is subjected to (3.5.137) are given by

$$\epsilon_{\alpha\beta} = \delta_2 \overset{2}{\epsilon}_{\alpha\beta} + \delta_3 \overset{3}{\epsilon}_{\alpha\beta} \qquad \text{(a)}$$

$$\sigma_{\alpha\beta} = \delta_2 \overset{2}{\sigma}_{\alpha\beta} + \delta_3 \overset{3}{\sigma}_{\alpha\beta} \qquad \text{(b)} \qquad (3.5.141)$$

Since the strains and stresses in all repeating elements are the same (except in those near the boundary) the average strains and stresses in any repeating element are the same as in the whole fiber reinforced cylinder. By the average stress theorem $\bar{\sigma}_{22}$ and $\bar{\sigma}_{33}$ are given by the right side of (3.5.136). It thus follows from (3.5.141b) that

$$\delta_2 \overset{2}{\bar{\sigma}}_{22} + \delta_3 \overset{3}{\bar{\sigma}}_{22} = \bar{\sigma}_{22} = \sigma^o$$

$$\delta_2 \overset{2}{\bar{\sigma}}_{33} + \delta_3 \overset{3}{\bar{\sigma}}_{33} = 0 \qquad (3.5.142)$$

Equs. (3.5.142) serve to determine the unknown constants δ_2, δ_3 in terms of the averages of the numerically known stresses in the repeating element. The strains are now determined by (3.5.141a). The effective Young's modulus E_T^* and the effective transverse Poisson's ratio ν_T^* are then given by

$$E_T^* = \frac{\sigma^o}{\bar{\epsilon}_{22}}$$

$$\nu_T^* = -\frac{\bar{\epsilon}_{33}}{\bar{\epsilon}_{22}} \tag{3.5.143}$$

A somewhat simplified computation of E_T^* may be carried out by subjecting the square section fiber reinforced specimen to the following boundary conditions

$$u_1 (S) = 0$$

$$T_2 (S) = \sigma^o_{22} n_2 \tag{3.5.144}$$

$$T_3 (S) = 0$$

These boundary conditions imply that the specimen is in plane strain, there is no shear on the boundary and conditions (3.5.136) remain valid. The preceding description of numerical analysis remains the same in the present case but E_T^* and ν_T^* are no longer given by (3.5.143). To see this we consider the states of average stress and strain associated with (3.5.144). Since the specimen is in a state of plane strain it follows from the analysis in par. 3.5.1 that $\epsilon_{11} = 0$ throughout the specimen and so its average also vanishes. Furthermore, from (3.5.144) and the average stress theorem, $\bar{\sigma}_{22} = \sigma^o$, $\bar{\sigma}_{33} = 0$. Inserting these results into the effective strain-stress relations (3.4.78) we obtain

$$0 = \bar{\sigma}_{11} - \nu_A^* \bar{\sigma}_{22}$$

$$\bar{\epsilon}_{22} = - \frac{\nu_A^*}{E_A^*} \bar{\sigma}_{11} + \frac{1}{E_T^*} \bar{\sigma}_{22}$$

$$\bar{\epsilon}_{33} = - \frac{\nu_A^*}{E_A^*} \bar{\sigma}_{11} - \frac{\nu_T^*}{E_T^*} \bar{\sigma}_{22}$$

from which it follows that

$$\bar{\epsilon}_{22} = \left(- \frac{\nu_A^{*2}}{E_A^*} + \frac{1}{E_T^*} \right) \sigma_{22}^o$$

$$\bar{\epsilon}_{33} = - \left(\frac{\nu_A^{*2}}{E_A^*} + \frac{\nu_T^*}{E_T^*} \right) \sigma_{22}^o \qquad (3.5.145)$$

It is seen that once $\bar{\epsilon}_{22}$ and $\bar{\epsilon}_{33}$ have been determined numerically it is necessary to know ν_A^* and E_A^* in order to find E_T^* and ν_T^*. Now it has been mentioned in par. 3.5.2, and it will be later shown, that to a high degree of accuracy

$$E_A^* \cong E_1 v_1 + E_2 v_2$$

for any fiber reinforced material. Also, with lesser accuracy

$$\nu_A^* \sim \nu_1 v_1 + \nu_2 v_2$$

Since in the parenthesis in the right sides of (3.5.145) the first terms are much smaller than the second terms the above approximations may be used with high accuracy.

Other EEM may be computed by related methods. For numerical results and description of methods the reader is referred to refs. [3.6], [3.22 - 24].

It is of great interest to compare numerical values of EEM as predicted by the composite cylinder assemblage analysis and by numerical analysis of square and hexagonal arrays of circular fibers. Table 3.5 lists numerical results given in [3.6] for effective elastic properties of an hexagonal array of identical circular fibers, fig. 2.1.6. The phase properties represent E-Glass fibers and epoxy matrix. The hexagonal array results are denoted HA. Also listed are composite cylinder assemblage results which are denoted CCA. It is seen that in most cases the numerical values are so close that they are indistinguishable for all practical purposes.

As a further comparison numerical results for square arrays of identical circular fibers, fig. 2.1.5, which were obtained in [3.19] have been plotted together with CCA results for same phase properties. Fig. 3.5.10 shows such plots for transverse Young's modulus E_T^*, where $E_{T(+)}^*$ has been used as CCA result. Also shown are experimental results from [3.19]. It is seen that the two curves are quite close up to $v_2 = 0.7$ and both agree reasonably well with the experimental results.

It should be noted that the square array plot terminates at $v_2 = \pi/4 = 0.785$, this being the maximum possible fiber packing for a square array. It is also to be noted that in FRM the fiber volume fraction can usually not exceed 0.7 because of manufacturing difficulties.

Fig. 3.5.11 shows a similar comparison for G_A^*. Again, it is seen that for a similar range of fiber volume fraction the two curves are quite close. Experimental results for this case have been given in [3.19]. As they are extremely scattered they are considered unreliable and are therefore not reproduced.

The previous comparisons between numerical and analytical results lead to the following conclusions: both kinds of results are numerically extremely close for hexagonal arrays and quite close for square arrays. Comparison of all results with experimental data shows about the same kind of agreement. However, the composite cylinder assemblage results have an overwhelming advantage in that they are simple closed form expressions which can be evaluated in very short time with a slide rule. In contrast numerical regular array analyses must be performed by computers with tedious programming and at great expense.

APPENDIX 1

Isotropy - Transverse Isotropy Analogies for Effective Elastic Moduli

We consider two fibrous or fiber reinforced specimens of entirely identical phase geometry. In the first specimen the phases are isotropic, while in the second the phases are transversely isotropic about an axis in cylinder generator direction.

Let the specimens be macroscopically transversely isotropic. We choose as basic EEM the set k^*, G_T^*, G_A^*, E_A^* and ν_A^*. For isotropic phases these are denoted ${}_ik^*$, ${}_iG_T^*$ etc. while for transversely isotropic phase, they are denoted ${}_tk^*$, ${}_tG_T^*$ etc.

Consider first the pair k^* and G_T^*. It has been seen in paragraph 3.4.3, equs. (3.4.68 - 70) that the computation of these EEM is based on a plane strain problem. Hence the first isotropy-transverse isotropy analogy as expressed by (3.5.25) applies at once to these EEM. Thus

(a) If expressions for ${}_ik^*$ and ${}_iG_T^*$ are known for isotropic phases, expressions for ${}_tk^*$ and ${}_tG_T^*$ are obtained by use of the replacement scheme (3.5.25) in the expressions for ${}_ik^*$ and ${}_iG_T^*$.

Next consider G_A^*. Its computation is defined by the anti-plane problem discussed in paragraph 3.5.1 and hence the second isotropy-transverse isotropy analogy applies. Thus

(b) If ${}_iG_A^*$ is known for isotropic phases ${}_tG_A^*$ is obtained by replacement of phase shear moduli in ${}_iG_A^*$ by corresponding axial phase shear moduli.

Finally, consider E_A^* and ν_A^*. Here the relations (3.4.117 - 118) can be used to advantage since these were explicitly derived for transversely isotropic phases.

These relations are here rewritten as follows

$$\boxed{\begin{aligned} {}_tE_A^* &= \bar{E}_A + \frac{4(\nu_{A2}-\nu_{A1})^2}{\left(\frac{1}{k_2}-\frac{1}{k_1}\right)^2}\left[\frac{\bar{1}}{k} - \frac{1}{{}_tk^*}\right] && (1) \\ {}_t\nu_A^* &= \bar{\nu}_A + \frac{\nu_{A2}-\nu_{A1}}{\frac{1}{k_2}-\frac{1}{k_1}}\left[\frac{1}{{}_tk^*} - \frac{\bar{1}}{k}\right] && (2) \end{aligned}}$$

Now since ${}_tk^*$ can be simply obtained from analogy (a), ${}_tE_A^*$ and ${}_t\nu_A^*$ are given for transversely isotropic phases by (1), (2).

It should be noted that everything is simply generalized to the case of macroscopic square symmetry with transversely isotropic phases. In that case there is an additional effective shear modulus $G_T'^*$, (3.4.56), which is also defined by a plane strain problem and thus obeys the analogy (a). Everything else remains the same.

APPENDIX 2

Internal Stresses

The general problem of the determination of internal stresses in a composite requires the complete solution of the elasticity problem for the composite material, as described in par. 3.2.3. It has been shown in par. 3.5.1 that for a fiber reinforced material with continuous fibers the problem can be considerably simplified by the establishment of two dimensional formulation. Only in some very special cases, par. 3.5.2, is it possible to determine stresses for fibers of arbitrary cross sections. In general, the internal stresses depend upon the detailed internal geometry i.e. fiber cross section shapes, fiber positions, and therefore even the two dimensional formulation becomes extremely difficult. Consequently, stress analysis must be limited to simple geometries such as regular arrays of identical circular or elliptical fibers and the composite cylinder assemblage model.

It should, however, be borne in mind that the reliability of internal stress computation for such simple models of fiber reinforced materials is quite problematic from a practical point of view. There are inevitably geometrical discrepancies between a real fiber reinforced material and the simplified model which is being analyzed, e.g. broken fibers, matrix-fiber interface separation. Such local imperfections may lead to important local stress concentrations whose determination is not only extremely difficult but in a sense impossible, since no precise description of the imperfections is generally available.

It may of course be argued that the role of such imperfections should also be considered in EEM analysis but there they are fortunately of minor importance, for EEM computation is based on determination of field averages. While local stress or strain discrepancies may be severe, their contributions to their averages appear to be insignificant.

A (perhaps exaggerated) analogy which comes to mind is the kinetic theory of gases. While relations between macroscopic variables such as pressure, temperature, entropy etc. can be accurately predicted, it is impossible to determine the detailed motion of a single molecule.

Numerical analyses of internal stress fields for hexagonal, square and rectangular arrays of identical circular or elliptical fibers have been given in e.g. [3.6], [3.22-24] for isotropic fibers and matrix. Recently a number of papers have been concerned with isolated aspects of extension of stress analysis to the case of transversely isotropic phases. The problem of such phase anisotropy has been resolved in general fashion in par. 3.5.1, where a complete formulation for transversely isotropic phases has been given. It is recalled that such formulation is in no sense more difficult than the isotropic phase formulation.

Here we shall be concerned with internal stresses in composite cylinder assemblages, with transversely isotropic fibers and matrix, which will be given by simple analytical expressions. The reason for the simplicity of the results is that for boundary conditions which lead to exact closed expressions for EEM, i.e. k^*, E_A^*, ν_A^*, G_A^*, ℓ^* and n^*, the internal fields

in the composite cylinder assemblage are exactly known and are merely the internal fields in any composite cylinder under the same boundary conditions. The situation is different and complicated for boundary conditions associated with determination of G_T^* and E_T^*.

We commence the composite cylinder assemblage stress analysis with plane isotropic straining as given by (3.4.59). In that event the stress and displacement fields in any composite cylinder, fig. 3.5.1, are given by (3.5.74-75). The boundary conditions (3.5.71c), (3.5.76-77) easily lead to the determination of the constants B_1, C_1 and B_2 and thus all stresses become known.

Resulting important matrix and fiber stresses are

$$\sigma^{(1)}_{\theta\theta(max)} = \sigma^{(1)}_{\theta\theta}(a) = 2\epsilon^o k_2 \frac{k_1 + G_1}{(k_2+G_1)v_1 + (k_1+G_1)v_2} \tag{1}$$

$$\sigma^{(1)}_{rr(max)} = \sigma^{(1)}_{rr}(b) = 2\epsilon^o k^* \tag{2}$$

$$\sigma^{(1)}_{zz} = 2\epsilon^o \ell_1 \frac{k_2 + G_1}{(k_2+G_1)v_1 + (k_1+G_1)v_2} \tag{3}$$

$$\sigma^{(2)}_{zz} = 2\epsilon^o \ell_2 \frac{k_1 + G_1}{(k_2+G_1)v_1 + (k_1+G_1)v_2} \tag{4}$$

Here the usual notation 1 - matrix, 2 - fibers has been adopted, k^* is given by (3.5.91), G_1 is transverse shear modulus G_{T1} and $\sigma^{(1)}_{zz}$ and $\sigma^{(2)}_{zz}$ are constant throughout their phase regions. In the case of isotropic phases the phase moduli in (1-4) are given by (3.4.89).

Stress concentration factors in the matrix may be defined by division of stresses by their corresponding values in the absence of fibers. Since in the absence of fibers the stresses in any cylinder, and also in the whole specimen, are given by

$$\sigma_{rr} = \sigma_{\theta\theta} = 2k_1 \varepsilon^o \qquad \sigma_{zz} = 2\ell_1 \varepsilon^o \tag{5}$$

We have from (1-3) and (5)

$$S^{(1)}_{\theta\theta}(a) = \frac{k_2}{k_1} \frac{k_1 + G_1}{(k_2+G_1)\, v_1 + (k_1+G_1)\, v_2} \tag{6}$$

$$S^{(1)}_{rr}(b) = \frac{k^*}{k_1} \tag{7}$$

$$S^{(1)}_{zz} = \frac{\ell_2}{\ell_1} \frac{k_1 + G_1}{(k_2+G_1)\, v_1 + (k_1+G_1)\, v_2} \tag{8}$$

where S denotes stress concentration factor.

If the fiber reinforced specimen is subjected to the plane isotropic stressing boundary conditions (3.4.62) then the stresses can be immediately obtained from (1-4) by the replacement

$$\epsilon^{o} \rightarrow \frac{\sigma^{o}}{2k^{*}} \tag{9}$$

where k^{*} is given (3.5.91). It follows that all stress concentration factors (6-8) <u>remain the same</u>.

Next we consider the case of uniaxial straining of a cylindrical specimen in fiber direction with no load on the lateral boundary of the specimen, as expressed by (3.4.37). In that event a typical composite cylinder is in axially symmetric state and is subjected to the boundary conditions

$$\begin{aligned} &u_1 = u_z = \epsilon^{o} x_1 = \epsilon^{o} z && \text{(a)} \\ &\sigma_{rr}\,(b) = 0 && \text{(b)} \end{aligned} \tag{10}$$

It is seen that the problem is one of plane strain with uniform axial strain, whose general formulation was given in par. 3.5.1. Elementary considerations lead to the results

$$u_z^{(1)} = \epsilon^o z \qquad \text{(a)}$$

$$u_r^{(1)} = B_1 r + C_1/r \qquad \text{(b)}$$

$$\sigma_{rr}^{(1)} = \ell_1 \epsilon^o + 2k_1 B_1 - 2G_1 B_1/r^2 \qquad \text{(c)} \qquad (11)$$

$$\sigma_{\theta\theta}^{(1)} = \ell_1 \epsilon^o + 2k_1 B_1 + 2G_1 B_1/r^2 \qquad \text{(d)}$$

$$\sigma_{zz}^{(1)} = n_1 \epsilon^o + 2\ell_1 B_1 \qquad \text{(e)}$$

$$u_z^{(2)} = \epsilon^o z \qquad \text{(a)}$$

$$u_r^{(2)} = B_2 r \qquad \text{(b)}$$

$$\sigma_{rr}^{(2)} = \sigma_{\theta\theta}^{(2)} = \ell_2 \epsilon^o + 2k_2 B_2 \qquad \text{(c)} \qquad (12)$$

$$\sigma_{zz}^{(2)} = n_2 \epsilon^o + 2\ell_2 B_2 \qquad \text{(d)}$$

while all other displacements and stresses vanish.

The constants B_1, C_1 and B_2 are easily found from the conditions

$$\sigma_{rr}^{(1)}(b) = 0 \qquad \text{(a)}$$

$$u_r^{(1)}(a) = u_r^{(2)}(a) \qquad \text{(b)} \qquad (13)$$

$$u_{rr}^{(1)}(a) = u_{rr}^{(2)}(a) \qquad \text{(c)}$$

Important resulting stresses are

$$\sigma_{zz}^{(1)} = \epsilon^{o} E_1 \left[1 + \frac{4G_1}{E_1} \frac{\nu_1(\nu_1-\nu_2)\, v_2}{v_2\, G_1/k_1 + v_1\, G_1/k_2 + 1} \right] \tag{14}$$

$$\sigma_{rr}^{(2)}(a) = \sigma_{rr}^{(1)}(a) = 2\epsilon^{o} G_1 \frac{(\nu_2 - \nu_1)\, v_1}{v_2\, G_1/k_1 + v_1\, G_1/k_2 + 1} \tag{15}$$

$$\sigma_{\theta\theta}^{(1)}(a) = 2\epsilon^{o} G_1 \frac{(\nu_1 - \nu_2)(1 + v_2)}{v_2\, G_1/k_1 + v_1\, G_1/k_2 + 1} \tag{16}$$

$$\sigma_{zz}^{(2)} = \epsilon^{o} E_2 \left[1 + \frac{4G_1}{E_2} \frac{\nu_2(\nu_2 - \nu_1)\, v_1}{v_2\, G_1/k_1 + v_1\, G_1/k_2 + 1} \right] \tag{17}$$

Here E is axial Young's modulus, ν is axial Poisson's ratio and G is transverse shear modulus. The stresses $\sigma_{zz}^{(1)}$ and $\sigma_{zz}^{(2)}$ are constant throughout their phase regions. It is easily seen that computation of $\bar{\sigma}_{zz}$ from (14), (17) and division by ϵ^{o} leads to E_A^* of the composite cylinder assemblage as given by (3.5.96).

It is recalled that for the case $\nu_2 = \nu_1$ a general internal stress solution for arbitrary cylindrical geometry was given in par. 3.5.2, equs. (3.5.57). Indeed, it is seen that for $\nu_2 = \nu_1$ the stresses (14-17) reduce to the elementary results (3.5.57). It is of interest to compare the numerical values of (14-17)

for $\nu_2 = \nu_1$ with those predicted by (3.5.57). Taking as an example an E-Glass-Epoxy FRM with elastic properties as listed in table 3.1, for the case $v_1 = v_2 = 0.5$, we have from (14-17)

$$\sigma_{zz}^{(1)} = 1.0334\ \epsilon^{o}\ E_1$$

$$\sigma_{zz}^{(2)} = 0.9991\ \epsilon^{o}\ E_2$$

$$\frac{\sigma_{rr}^{(1)}(a)}{\sigma_{zz}^{(1)}} = -\ 0.046$$

$$\frac{\sigma_{\theta\theta}^{(1)}(a)}{\sigma_{zz}^{(1)}} = 0.138$$

It is seen that the axial stresses $\sigma_{zz}^{(1)}$, $\sigma_{zz}^{(2)}$ are very close to the results (3.5.57a) and it may be therefore surmised <u>that (3.5.57a) should closely approximate the axial stresses in any FRM with continuous fibers.</u>

Finally, it is noted that if the terminal sections of the fiber reinforced specimen are loaded by uniform stress σ_{11}^{o} then the stresses corresponding to (14-17) are found by the replacement

$$\epsilon^{o} \rightarrow \frac{\sigma_{11}^{o}}{E_A^{*}} \tag{18}$$

where E_A^* is given by (3.5.96).

Next we consider the case of axial shearing with boundary conditions

$$u_1(S) = \epsilon_{12}^{o} x_2 \qquad u_2(S) = \epsilon_{12}^{o} x_1 \qquad u_3(S) = 0 \tag{19}$$

The problem of a typical composite cylinder in the assemblage in this case is described by (3.5.102-105) where in view of (3.5.34)

$$\sigma_{rz}^{(m)} = G_m \frac{\partial \varphi^{(m)}}{\partial r} \qquad \sigma_{\theta z}^{(m)} = G_m \frac{1}{r} \frac{\partial \varphi^{(m)}}{\partial \theta} \tag{20}$$

$$m = 1,2$$

A simple calculation yields

$$\sigma_{rz}^{(1)} = 2\epsilon_{12}^{o} G_1 \frac{\gamma + 1 + (\gamma-1) a^2/r^2}{\gamma v_1 + 1+v_2} \cos\theta \tag{a}$$

$$\sigma_{\theta z}^{(1)} = -2\epsilon_{12}^{o} G_1 \frac{\gamma + 1 - (\gamma-1) a^2/r^2}{\gamma v_1 + 1+v_2} \sin\theta \tag{b}$$

$$\sigma_{rz}^{(2)} = \frac{4\epsilon_{12}^{o} G_2 \cos\theta}{\gamma v_1 + 1+v_2} \tag{c}$$

(21)

$$\sigma_{\theta z}^{(2)} = -\frac{4\epsilon_{12}^{o} G_2 \sin\theta}{\gamma v_1 + 1+v_2} \tag{d}$$

$$\gamma = G_2/G_1 \tag{e}$$

and all other stresses vanish. The resultant shear stress τ at any point is given by

$$\tau = \sqrt{\sigma_{rz}^2 + \sigma_{\theta z}^2}$$

Its maximum in the matrix is located at $r = a$, $\theta = 0$, with value

$$\tau_{max} = 2\epsilon_{12}^{o} G_1 \frac{2\gamma}{\gamma v_1 + 1 + v_2} \tag{22}$$

while throughout the fiber

$$\tau = \frac{4\epsilon_{12}^{o} G_1}{\gamma v_1 + 1 + v_2} \tag{23}$$

In the absence of fibers the only surviving stress is

$$\sigma_{12} = \tau = 2\epsilon_{12}^{o} G_1 \tag{24}$$

Therefore the maximum matrix shear stress concentration is from (22) and (24)

$$S_\tau = \frac{2\gamma}{\gamma v_1 + 1 + v_2} \tag{25}$$

It should be recalled that G_1, G_2 are axial shear moduli G_{A1}, G_{A2} for transversely isotropic phases.

If instead of (19) the specimen is subjected to axial shearing

$$T_1(S) = \sigma^o_{12}\, n_2 \qquad T_2(S) = \sigma^o_{12}\, n_1 \qquad T_3(S) = 0 \tag{26}$$

then the stresses are given by (21) with the replacement

$$\epsilon^o_{12} \rightarrow \frac{\sigma^o_{12}}{2G^*_A} \tag{27}$$

where G^*_A is given by (3.5.111).

There remains the difficult problem of transverse shearing. An exact solution for internal stresses in the composite cylinder assemblage is not available in this case. On the other hand internal fields which lead to lower and upper bounds on G^*_T have been constructed. As has been indicated in par. 3.5.3 the upper bound on G^*_T, (3.5.113), may actually be the exact result for this EEM in the event that fibers are stiffer than matrix. If this conjecture is correct, the admissible displacements and associated stress fields on which the upper bound construction is based are the actual displacement and stress fields in the composite cylinder assemblage. Consequently, it is not unreasonable to regard these fields as approximations, at least.

The stress fields are the ones produced in any composite cylinder by the boundary conditions (3.4.13), with traction and displacement continuity satisfied at fiber-matrix interface. A solution for the displacement fields is given in par. (3.6.4), equs. (3.6.85-88). Computation of the associated stresses is an easy matter but the resulting expressions are complicated. Suffice it here to give the fiber-matrix interface stresses σ_{rr} and $\sigma_{r\theta}$

$$\sigma_{rr}(a,\theta) = 2G_2 A_5^{\epsilon} \sin 2\theta$$

$$\sigma_{r\theta}(a,\theta) = 2G_2 \left(A_5^{\epsilon} + \frac{3}{3-2\nu_2} A_6^{\epsilon}\right) \cos 2\theta$$

where A_5^{ϵ}, A_6^{ϵ} are constants to be found by solution of the six-by-six system of equations (3.6.87) .

3.6 BOUNDING METHODS FOR EFFECTIVE ELASTIC MODULI: CLASSICAL EXTREMUM PRINCIPLES

3.6.1 Motivation

In the preceding chapter we have been concerned with computation of EEM of FM and FRM by direct rigorous methods. In view of the extreme difficulty of the problem it is not surprising that results could be obtained only in special circumstances. Solutions were obtained either for special relations between phase moduli, par. 3.5.2, or for special geometries, pars. 3.5.3-5.

In the present and next chapter we shall develop variational bounding methods for EEM of FM and FRM. Such methods are of crucial importance since they enable us to estimate EEM by bracketing them between lower and upper bounds in cases where direct computation is impossible or extremely laborious and difficult.

Bounding methods will be applied in two different classes of problems. In the first class the phase geometry is only partially known. Direct computation of EEM is then impossible, not just because of mathematical difficulty but because the problem is then indeterminate. It is, however, possible to obtain bounds on EEM in terms of the available information. The most important example for this is a FM in which only phase moduli and phase volume fractions are known.

In the second class of problems the phase geometry is completely specified but direct computation is difficult. It has been seen that even such simple geometries as square and hexagonal arrays of identical circular fibers could only be treated by numerical analysis with the aid of computers. In such cases it is also possible to construct simple closed form bounds which provide valuable information about the magnitude of the EEM.

It happens at times that bounds obtained are so far apart that they are useless from a practical point of view. It happens at other times that bounds are so close together that they determine EEM to all practical purposes.

3.6.2 Principles of Minimum Potential Energy and Minimum Complementary Energy

Derivations of elasticity extremum principles may be found in many textbooks (see e.g. Fung [3.25]). Usually the principles are derived by methods of the calculus of variations, for homogeneous or continuously non-homogeneous bodies. The present derivation of the principles departs from conventional derivations in that (a) the principles are derived for heterogeneous bodies; (b) finite differences are used instead of variations; (c) the principles are constructed in a straightforward manner, starting out with the governing elasticity equations.

We consider a multi-phase elastic body whose phases are homogeneous and anisotropic. For present purposes we need only consider the case of vanishing body forces and we note in passing that all theorems which will be derived may be easily generalized to the case of non-vanishing body forces.

Suppose that the boundary conditions on the external surface are

$$\begin{aligned} &u_i(S) = u_i^o && \text{on } S_u && \text{(a)} \\ &T_i(S) = C_{ijkl}u_{k,l}n_j = T_i^o && \text{on } S_T && \text{(b)} \end{aligned} \qquad (3.6.1)$$

Other types of boundary conditions will be considered further below. We choose to formulate the problem in terms of displacements, so the governing equations in the mth phase are of form (3.2.27),

$$C^{(m)}_{ijkl}\, u^{(m)}_{k,lj} = 0 \qquad \text{in } R_m \qquad (3.6.2)$$

Furthermore, continuity conditions of type (3.2.28) must be satisfied at all interfaces. These are written here in the form

$$\left.\begin{aligned} &u_i \\ &T_i = \sigma_{ij}n_j = C_{ijkl}\, u_{k,l}\, n_j \end{aligned}\right\} \text{ continuous on } S_{int.} \qquad \begin{aligned} &\text{(a)} \\ &\text{(b)} \end{aligned} \qquad (3.6.3)$$

where $S_{int.}$ denotes all interface surfaces. The unique solution of this problem is denoted

$$u_i = u_i^{(m)} \quad \text{in } R_m \qquad (3.6.4)$$

We now define a so-called <u>admissible displacement field</u> $\tilde{u}_i$ which is continuous throughout the phases and also satisfies the following requirements

$$\begin{aligned} &\tilde{u}_i = u_i^o && \text{on } S_u && \text{(a)} \\ &\tilde{u}_i \quad \text{continuous} && \text{on } S_{int.} && \text{(b)} \end{aligned} \qquad (3.6.5)$$

where u_i^o is the same as in (3.6.1). We define the difference field Δu_i by

$$\Delta u_i = \tilde{u}_i - u_i \qquad (3.6.6)$$

It follows from (3.6.1a), (3.6.5a) and from (3.6.3a), (3.6.5b), respectively that

$$\begin{aligned} &\Delta u_i = 0 && \text{on } S_u && \text{(a)} \\ &\Delta u_i \quad \text{continuous on } S_{int.} && && \text{(b)} \end{aligned} \qquad (3.6.7)$$

We also denote Δu_i in a form similar to (3.6.4)

$$\Delta u_i = \Delta u_i^{(m)} \qquad \text{in } R_m \qquad (3.6.8)$$

Now multiply (3.6.2) by $\Delta u_i^{(m)}$ and integrate over the phase region

$$\int_{V_m} C_{ijkl}^{(m)} u_{k,lj}^{(m)} \Delta u_i^{(m)} dV = 0$$

$$= \int_{V_m} [(C_{ijkl}^{(m)} u_{k,l}^{(m)} \Delta u_i^{(m)})_{,j} - C_{ijkl}^{(m)} u_{k,l}^{(m)} \Delta u_{i,j}^{(m)}] dV$$

$$= \int_{S_m} C_{ijkl}^{(m)} u_{k,l}^{(m)} \Delta u_i^{(m)} n_j dS - \int_{V_m} C_{ijkl}^{(m)} u_{k,l}^{(m)} \Delta u_{i,j}^{(m)} dV$$

where S_m is the surface enclosing V_m and the divergence theorem has been used. Now

$$\int_V C_{ijkl}\, u_{k,lj}\, \Delta u_i\, dV = \sum_m \int_{V_m} C^{(m)}_{ijkl}\, u^{(m)}_{k,lj}\, \Delta u^{(m)}_i dV = 0$$

$$= \sum_m \int_{S_m} C^{(m)}_{ijkl}\, u^{(m)}_{k,l}\, \Delta u^{(m)}_i\, n_j\, dS - \sum_m \int_{V_m} C^{(m)}_{ijkl}\, u^{(m)}_{k,l}\, \Delta u^{(m)}_{i,j}\, dV$$

$$= \sum_m \int_{S_m} T^{(m)}_i \Delta u^{(m)}_i dS - \sum_m \int_{V_m} C^{(m)}_{ijkl}\, u^{(m)}_i\, \Delta u^{(m)}_{i,j}\, dV \qquad (3.6.9)$$

The surface integrals in (3.6.9) are taken twice over the interfaces and once over the external surface S. The interface integrals all cancel because of $T^{(m)}_i$ and $\Delta u^{(m)}_i$ continuity at the interfaces (see discussion in par. 3.1.3 after (3.1.47)). Therefore (3.6.9) assumes the form

$$\int_S T_i\, \Delta u_i\, dS - \sum_m \int_{V_m} C^{(m)}_{ijkl}\, u^{(m)}_{k,l}\, \Delta u^{(m)}_{i,j}\, dV = 0$$

which in view of (3.6.1b) and (3.6.7a) can be written as

$$\int_{S_T} T^o_i\, \Delta u_i\, dS - \sum_m \int_{V_m} C^{(m)}_{ijkl}\, u^{(m)}_{k,l}\, \Delta u^{(m)}_{i,j}\, dV = 0 \qquad (3.6.10)$$

Now consider the integral

$$U_p = \frac{1}{2} \int_V C_{ijkl}\, u_{i,j}\, u_{k,l}\, dV - \int_{S_T} T^o_i\, u_i\, dS$$

$$= \sum_m \int_{V_m} W^{\epsilon}_m\, dV - \int_{S_T} T^o_i\, u_i\, dS \qquad \text{(a)}$$

$$(3.6.11)$$

$$W_m^{\epsilon} = \frac{1}{2} C_{ijkl}^{(m)} u_{i,j}^{(m)} u_{k,l}^{(m)} = \frac{1}{2} C_{ijkl}^{(m)} \epsilon_{ij}^{(m)} \epsilon_{kl}^{(m)} \qquad \text{(b)}$$

where u_i is the elasticity solution of the problem and W_m^{ϵ} is the strain energy density of the mth phase. The expression (3.6.11) is traditionally known as the <u>potential energy</u>.

If the actual u_i in (3.6.11) is replaced by the admissible $\tilde{u}_i$ we have

$$\tilde{U}_P = \sum_m \int_{V_m} \tilde{W}_m^{\epsilon} \, dV - \int_{S_T} T_i^o \tilde{u}_i \, dS \qquad \text{(a)}$$

$$\tilde{W}_m^{\epsilon} = \frac{1}{2} C_{ijkl}^{(m)} \tilde{u}_{i,j}^{(m)} \tilde{u}_{k,l}^{(m)} = \frac{1}{2} C_{ijkl}^{(m)} \tilde{\epsilon}_{ij}^{(m)} \tilde{\epsilon}_{kl}^{(m)} \qquad \text{(b)} \qquad (3.6.12)$$

where

$$\tilde{\epsilon}_{ij}^{(m)} = \frac{1}{2} (\tilde{u}_{i,j}^{(m)} + \tilde{u}_{j,i}^{(m)}) \qquad (3.6.13)$$

The expression (3.6.12) is here called the <u>potential energy functional</u>.

In view of (3.6.6) the difference between (3.6.12) and (3.6.11) can be written in the form

$$\tilde{U}_P - U_P = \Delta U_P = \Delta^{(1)} U_P + \Delta^{(2)} U_P \qquad \text{(a)}$$

$$\Delta^{(1)} U_P = \sum_m \int_{V_m} C_{ijkl}^{(m)} u_{k,l}^{(m)} \Delta u_{i,j}^{(m)} \, dV - \int_{S_T} T_i^o \Delta u_i \, dS = 0 \qquad \text{(b)} \qquad (3.6.14)$$

$$\Delta^{(2)} U_P = \frac{1}{2} \sum_m \int_{V_m} C_{ijkl}^{(m)} \Delta u_{i,j}^{(m)} \Delta u_{k,l}^{(m)} \, dV \qquad \text{(c)}$$

Here (3.6.14b) vanishes because of (3.6.10). Note that in the (3.6.14b) volume integrals the symmetry $C^{(m)}_{ijkl} = C^{(m)}_{klij}$ has been used to cancel the 1/2.

Now if we define

$$\Delta \epsilon^{(m)}_{ij} = \frac{1}{2}(\Delta u^{(m)}_{i,j} + \Delta u^{(m)}_{j,i})$$

then (3.6.14c) can be written as

$$\Delta^{(2)} U_P = \frac{1}{2} \sum_m \int_{V_m} C^{(m)}_{ijkl} \Delta \epsilon^{(m)}_{kl} \Delta \epsilon^{(m)}_{ij} dV \tag{3.6.15}$$

It follows from the positive definiteness property of elastic moduli, (3.2.9), that the integrands in (3.6.15) are always positive and, therefore, all integrals are positive. Therefore,

$$\Delta^{(2)} U_P \geq 0 \tag{3.6.16}$$

with equality occurring if and only if $\Delta \epsilon^{(m)}_{ij} = 0$. But in that event $\Delta u^{(m)}_i$ is at most a rigid body motion; moreover, one and the same rigid body motion for all phases, and thus of the whole body because of Δu_i interface continuity, (3.6.7b). But because of (3.6.7a) this rigid body motion must vanish. Consequently, (3.6.16) can vanish only when $\Delta u^{(m)}_i = 0$, in which event $\tilde{u}_i$ is the elasticity solution u_i, because of (3.6.6).

We now have from (3.6.14a,b) and (3.6.16)

$$\tilde{U}_P \geq U_P \tag{3.6.17}$$

where equality occurs if and only if the admissible displacement happens to be the true elasticity displacement.

The inequality (3.6.17) expresses the principle of minimum potential energy which may be stated as: <u>The potential energy is the absolute minimum of the potential energy functional</u>.

For purpose of future applications it is important to consider mixed boundary conditions of different form than (3.6.1). Suppose that there are prescribed on the entire boundary traction components in one or more directions and displacement components in the remaining direction or directions. (But never both a traction component and displacement component in the same direction). Examples are

$$u_1(S) = u_1^o \qquad T_2(S) = T_2^o \qquad T_3(S) = T_3^o \qquad \text{(a)}$$

$$T_1(S) = T_1^o \qquad u_2(S) = u_2^o \qquad u_3(S) = u_3^o \qquad \text{(b)} \tag{3.6.18}$$

It is recalled that such mixed boundary conditions occur in formulations of problems of computation of EEM, pars. 3.4.2-3.

The principle of minimum potential energy is easily adapted to this kind of boundary conditions. The admissible displacement now has to satisfy only those displacement component boundary conditions which are prescribed and the surface integrals in (3.6.11-12) have to be considered only for those traction components which are prescribed. Thus, for example, for (3.6.18a) we have

$$\tilde{u}_1(S) = u_1^o \qquad \text{(a)}$$

$$\tilde{u}_2(S)\ ,\ \tilde{u}_3(S) \quad \text{arbitrary} \qquad \text{(b)} \tag{3.6.19}$$

$$U_P = \sum_m \int_{V_m} W_m^\epsilon \, dV \quad - \int_S (T_2^o u_2 + T_3^o u_3) dS \qquad \text{(a)}$$

$$\tilde{U}_P = \sum_m \int_{V_m} \tilde{W}_m^\epsilon \, dV \quad - \int_S (T_2^o \tilde{u}_2 + T_3^o \tilde{u}_3) dS \qquad \text{(b)} \tag{3.6.20}$$

Next we consider the important special case of displacements prescribed over the entire surface S ($S_T = 0$ in (3.6.1)). In that event the surface integrals in (3.6.11) and (3.6.12) vanish. Therefore, (3.6.11 - 12) can be written in the forms

$$U^\epsilon = \sum_m \int_{V_m} W_m^\epsilon \, dV \qquad \text{(a)}$$

$$\tilde{U}^\epsilon = \sum_m \int_{V_m} \tilde{W}_m^\epsilon \, dV \qquad \text{(b)} \tag{3.6.21}$$

Now U^ϵ is the strain energy and $\tilde{U}^\epsilon$ is called the <u>strain energy functional</u> and the theorem becomes:

$$\tilde{U}^\epsilon \le U^\epsilon \tag{3.6.22}$$

where again equality occurs if and only if the admissible displacement is the true elasticity displacement.

The statement (3.6.22) may be called the <u>principle of minimum strain energy</u>. In words: <u>For a body with displacements prescribed over its entire surface the strain energy is the absolute minimum of the strain energy functional</u>.

We now proceed to derive another class of extremum principles which involve admissible stress fields. We consider again a heterogeneous body consisting of any number of anisotropic phases, which is subjected to the boundary conditions (3.6.1). The continuity conditions are still (3.6.3). It is desired to formulate the boundary value problem in terms of stresses. We have in the mth phase

$$\sigma^{(m)}_{ij,j} = 0 \qquad \text{(a)}$$

$$S^{(m)}_{ijkl}\,\sigma^{(m)}_{kl} = \frac{1}{2}\left(u^{(m)}_{i,j} + u^{(m)}_{j,i}\right) = \epsilon^{(m)}_{ij} \qquad \text{(b)} \qquad (3.6.23)$$

The displacements may be eliminated from (3.6.23b) in the following manner: since (3.6.23b) are strains they must satisfy the strain compatibility equations which may thus be expressed in terms of stresses. Thus is obtained a set of Beltrami-Michell equations for the anisotropic case. However, u_i are still present in (3.6.1a) and (3.6.3b). If $S_u = 0$ in (3.6.1) the displacements

disappear from the boundary conditions but not from the interface continuity conditions. We thus conclude that a stress formulation is not useful for actual solution of problems in heterogeneous bodies. (Recall a similar situation in par. 3.5.1 for Airy stress function representation).

For present purposes we retain the formulation (3.6.23). We define an admissible stress field

$$\tilde{\sigma}_{ij} = \tilde{\sigma}^{(m)}_{ij} \quad \text{in } R_m \tag{3.6.24}$$

which satisfies the following requirements

$$\begin{aligned} &\tilde{\sigma}_{ij}\, n_j = T_i^{o} && \text{on } S_T && \text{(a)} \\ &\tilde{\sigma}^{(m)}_{ij,j} = 0 && \text{in } R_m && \text{(b)} \\ &\tilde{T}_i = \tilde{\sigma}_{ij}\, n_j && \text{continuous on } S_{int} && \text{(c)} \end{aligned} \tag{3.6.25}$$

We further define the difference

$$\Delta\sigma^{(m)}_{ij} = \tilde{\sigma}^{(m)}_{ij} - \sigma^{(m)}_{ij} \tag{3.6.26}$$

It follows from (3.6.23), (3.6.25) and (3.6.23a) that

$$\begin{aligned} &\Delta T_i = \Delta\sigma_{ij}\, n_j = 0 && \text{on } S_T && \text{(a)} \\ &\Delta\sigma^{(m)}_{ij,j} = 0 && \text{in } R_m && \text{(b)} \\ &\Delta T_i = \Delta\sigma_{ij}\, n_j && \text{continuous on } S_{int.} && \text{(c)} \end{aligned} \tag{3.6.27}$$

Now multiply (3.6.23) by $\Delta\sigma_{ij}^{(m)}$ and integrate over V_m. Then

$$\int_{V_m} (S_{ijkl}^{(m)} \sigma_{kl}^{(m)} \Delta\sigma_{ij}^{(m)} - \epsilon_{ij}^{(m)} \Delta\sigma_{ij}^{(m)}) \, dV = 0$$

$$= \int_{V_m} S_{ijkl}^{(m)} \sigma_{kl}^{(m)} \Delta\sigma_{ij}^{(m)} - \int_{S_m} u_i^{(m)} \Delta T_i^{(m)} \, dS \tag{3.6.28}$$

where the conversion to the surface integral is done by aid of the theorem of virtual work. We now sum (3.6.28) over all phases. Then by a previous argument the surface integrals over the interface cancel because of $u_i^{(m)}$ and $\Delta T_i^{(m)}$ continuity. Thus

$$\sum_m \int_{V_m} S_{ijkl}^{(m)} \sigma_{kl}^{(m)} \Delta\sigma_{ij}^{(m)} \, dV - \int_S u_i \, \Delta T_i \, dS = 0$$

$$= \sum_m \int_{V_m} S_{ijkl}^{(m)} \sigma_{kl}^{(m)} \Delta\sigma_{kl}^{(m)} \, dV - \int_{S_u} u_i^o \, \Delta T_i \, dS$$

where the last step is due to (3.6.1a) and (3.6.27a).

Now consider the expressions

$$U_C = \sum_m \int_{V_m} W_m^\sigma \, dV - \int_{S_u} T_i \, u_i^o \, dV \tag{a}$$

$$W_m^\sigma = \frac{1}{2} S_{ijkl}^{(m)} \sigma_{ij}^{(m)} \sigma_{kl}^{(m)} \tag{b}$$

(3.6.29)

$$\tilde{U}_C = \Sigma \int_{V_m} \tilde{W}_m^{\sigma}\, dV - \int_{S_u} \tilde{T}_i\, u_i^o\, dV \qquad \text{(c)}$$

$$\tilde{W}_m^{\sigma} = \frac{1}{2} S_{ijkl}^{(m)} \tilde{\sigma}_{ij}^{(m)} \tilde{\sigma}_{kl}^{(m)} \qquad \text{(d)}$$

The first integral is traditionally called the <u>complementary energy</u> and the second is here called the <u>complementary energy functional</u>. The difference between (3.6.29c,a) may be written in view of (3.6.27) in the form

$$\tilde{U}_C - U_C = \Delta U_C = \Delta^{(1)} U_C + \Delta^{(2)} U_C \qquad \text{(a)}$$

$$\Delta^{(1)} U_C = \sum_m \int_{V_m} S_{ijkl}^{(m)}\, \sigma_{kl}^{(m)}\, \Delta\sigma_{ij}^{(m)}\, dV - \int_{S_u} \Delta T_i u_i^o\, dS = 0 \qquad \text{(b)} \qquad (3.6.30)$$

$$\Delta^{(2)} U_C = \frac{1}{2} \sum_m \int_{V_m} S_{ijkl}^{(m)}\, \Delta\sigma_{kl}^{(m)}\, \Delta\sigma_{ij}^{(m)}\, dV \qquad \text{(c)}$$

Here (3.6.30b) vanishes because of (3.6.28). Note that in (3.6.30b) the symmetry $S_{ijkl}^{(m)} = S_{klij}^{(m)}$ has been used to cancel the $\frac{1}{2}$.

Now it follows from compliance positive definiteness, (3.2.10), that

$$\Delta^{(2)} U_C \geq 0 \qquad (3.6.31)$$

equality occurring if and only if $\Delta\sigma_{ij}^{(m)} = 0$. But in that event $\tilde{\sigma}_{ij}^{(m)} = \sigma_{ij}^{(m)}$ which is the true elasticity stress field.

We now have from (3.6.30a) and (3.6.31) that

$$\tilde{U}_C \geq U_C \tag{3.6.32}$$

equality occurring if, and only if, the admissible stress field happens to be the true elasticity stress field. Accordingly, the <u>principle of minimum complementary energy</u> is stated as: <u>The complementary energy is the absolute minimum of the complementary energy functional</u>.

Again we consider mixed boundary conditions of type (3.6.18). Now we require that the admissible stress field satisfy only those traction component conditions which are prescribed and in the surface integrals in (3.6.29) we retain only the contributions of the displacement components which are prescribed. Thus, for (3.6.18a) we have

$$\left.\begin{array}{ll} \tilde{\sigma}_{ij}\, n_j & \text{arbitrary} \\ \tilde{\sigma}_{2j}\, n_j = T^o_2 & \\ \tilde{\sigma}_{3j}\, n_j = T^o_3 & \end{array}\right\} \quad \text{on } S \tag{3.6.33}$$

$$U_C = \sum_m \int_{V_m} W^\sigma_m \, dV - \int_S T_1 \, u^o_1 \, dS \tag{a}$$

$$\tilde{U}_C = \sum_m \int_{V_m} \tilde{W}^\sigma \, dV - \int_S \tilde{T}_1 \, u^o_1 \, dS \tag{b}$$

(3.6.34)

Next we consider the important special case when $S_u = 0$ in (3.6.1), i.e. tractions are prescribed over the entire surface S. In that event the surface integrals in (3.6.29) vanish, and thus (3.6.29a,c) reduce to

$$U^{\sigma} = \sum_m \int_{V_m} W_m^{\sigma} \, dV \qquad \text{(a)}$$

$$\tilde{U}^{\sigma} = \sum_m \int_{V_m} \tilde{W}_m^{\sigma} \, dV \qquad \text{(b)} \qquad (3.6.35)$$

The first integral is the <u>stress energy</u> (this expression is here used instead of the cumbersome expression: the strain energy in terms of stresses) and (3.6.35b) is called the <u>stress energy functional</u>. The theorem becomes

$$\tilde{U}^{\sigma} \geq U^{\sigma} \qquad (3.6.36)$$

equality occurring if, and only if, the admissible stress field is the true elasticity stress field.

The result (3.6.36) may be called the <u>principle of minimum stress energy</u>. It states that: <u>The stress energy is the absolute minimum of the stress energy functional.</u>

3.6.3 Elementary Bounds for Fibrous Materials

In the present paragraph we shall exploit the previously derived extremum principles to obtain lower and upper bounds for EEM of fibrous materials.

The phase geometry of such materials is, by definition, irregular. The information which is generally available to us about such a material is phase elastic moduli, phase volume fractions, and the macroscopic elastic symmetry of the material. The EEM bounds which will be derived are based on this information alone. Thus, the present analysis is an example of the utilization of partial information.

It is our first task to express elastic energies in terms of EEM or EEC. This has already been done for a general macroscopically anisotropic body in par. 3.3.2, equations (3.3.26 - 27). It is a very easy matter to specialize these energy densities for various cases of macroscopic symmetry. It should, however, be noted that it is generally more convenient to directly evaluate $\frac{1}{2}\bar{\sigma}_{ij}\bar{\epsilon}_{ij}$ for a specific case instead of simplifying C^*_{ijkl} and S^*_{ijkl} in (3.3.26 - 27).

For the macroscopically orthotropic material we have in view of (3.4.1 - 2) and (3.4.5 - 9)

$$2\bar{W}^{\epsilon} = C^*_{11}\bar{\epsilon}^2_{11} + C^*_{22}\epsilon^2_{22} + C^*_{33}\bar{\epsilon}^2_{33}$$

$$+ 2C^*_{12}\bar{\epsilon}_{11}\bar{\epsilon}_{22} + 2C^*_{23}\bar{\epsilon}_{22}\bar{\epsilon}_{33} + 2C^*_{13}\bar{\epsilon}_{11}\bar{\epsilon}_{33}$$

$$+ 4C^*_{44}\bar{\epsilon}^2_{12} + 4C^*_{55}\bar{\epsilon}^2_{23} + 4C^*_{66}\bar{\epsilon}^2_{13} \qquad (3.6.37)$$

$$2\bar{W}^{\sigma} = \frac{\bar{\sigma}_{11}^2}{E_1^*} + \frac{\bar{\sigma}_{22}^2}{E_2^*} + \frac{\bar{\sigma}_{33}^2}{E_3^*} - \frac{2\nu_{12}^*}{E_1^*}\bar{\sigma}_{11}\bar{\sigma}_{22}$$

$$- \frac{2\nu_{23}^*}{E_2^*}\bar{\sigma}_{22}\bar{\sigma}_{33} - \frac{2\nu_{13}^*}{E_3^*}\bar{\sigma}_{11}\bar{\sigma}_{33} + \frac{\bar{\sigma}_{12}^2}{C_{44}^*} + \frac{\bar{\sigma}_{23}^2}{C_{55}^*} + \frac{\bar{\sigma}_{13}^2}{C_{66}^*} \qquad (3.6.38)$$

For the transversely isotropic material it is most convenient to use the effective or macroscopic counterparts of (3.4.86 - 87). We then have

$$2\bar{W}^{\epsilon} = n^*\bar{\epsilon}_{11}^2 + 2\ell^*\bar{\epsilon}_{11}(\bar{\epsilon}_{22} + \bar{\epsilon}_{33}) + k^*(\bar{\epsilon}_{22} + \bar{\epsilon}_{33})^2$$

$$+ G_T^*(\bar{\epsilon}_{22} - \bar{\epsilon}_{33})^2 + 4G_T^*\bar{\epsilon}_{23}^2 + 4G_A^*(\bar{\epsilon}_{12}^2 + \bar{\epsilon}_{13}^2) \qquad (3.6.39)$$

$$2\bar{W}^{\sigma} = \frac{\bar{\sigma}_{11}^2}{E_A^*} + \frac{1}{E_T^*}(\bar{\sigma}_{22}^2 + \bar{\sigma}_{33}^2)$$

$$- \frac{2\nu_A^*}{E_A^*}\bar{\sigma}_{11}\bar{\sigma}_{22} - \frac{2\nu_T^*}{E_T^*}\bar{\sigma}_{22}\bar{\sigma}_{33} - \frac{2\nu_A^*}{E_T^*}\bar{\sigma}_{11}\bar{\sigma}_{33}$$

$$+ \frac{\bar{\sigma}_{23}^2}{G_T^*} + \frac{1}{G_A^*}(\bar{\sigma}_{12}^2 + \bar{\sigma}_{13}^2) \qquad (3.6.40)$$

For square symmetric materials, G_T^* in the second terms from the end in (3.6.3.3-4) must be replaced by G'^*_T, equation (3.4.56).

For a statistically isotropic material the most convenient energy densities are obtained by use of (3.4.95). Substituting these into

$$\bar{\sigma}_{ij}\,\bar{\epsilon}_{ij} = 3\bar{\sigma}\,\bar{\epsilon} + \bar{s}_{ij}\,\bar{e}_{ij} \tag{3.6.41}$$

we obtain

$$2\bar{W}^{\epsilon} = 9K^*\,\bar{\epsilon}^2 + 2G^*\,\bar{e}_{ij}\,\bar{e}_{ij} \tag{3.6.42}$$

$$2\bar{W}^{\sigma} = \frac{\bar{\sigma}^2}{K^*} + \frac{\bar{s}_{ij}\,\bar{s}_{ij}}{2G^*} \tag{3.6.43}$$

Finally, we record for future use energy densities of homogeneous materials

Transverse isotropy

$$2W^{\epsilon} = n\,\epsilon_{11}^2 + 2\ell\,\epsilon_{11}\,(\epsilon_{22} + \epsilon_{33}) + k\,(\epsilon_{22} + \epsilon_{33})^2$$

$$+ G_T\,(\epsilon_{22} - \epsilon_{33})^2 + 4G_T\,\epsilon_{23}^2 + 4G_A\,(\epsilon_{12}^2 + \epsilon_{13}^2) \qquad \text{(a)}$$

(3.6.44)

$$2W^{\sigma} = \frac{\sigma_{11}^2}{E_A} + \frac{1}{E_T}\,(\sigma_{22}^2 + \sigma_{33}^2) - \frac{2\nu_A}{E_A}\,\sigma_{11}(\sigma_{22} + \sigma_{33})$$

$$- \frac{2\nu_T}{E_T}\,\sigma_{22}\,\sigma_{33} - \frac{\sigma_{23}^2}{G_T} + \frac{1}{G_A}\,(\sigma_{12}^2 + \sigma_{13}^2) \qquad \text{(b)}$$

Isotropy

$$2W^{\epsilon} = 9K\,\epsilon^2 + 2\,G\,e_{ij}\,e_{ij} \qquad \text{(a)}$$

$$2W^{\sigma} = \frac{\sigma^2}{K} + \frac{s_{ij}\,s_{ij}}{2G} \qquad \text{(b)} \qquad (3.6.45)$$

where K is the three dimensional bulk modulus.

It is recalled that in the problem of computation of EEM or EEC of a heterogeneous body the boundary conditions are homogeneous. In the present method of establishment of bounds the admissible displacement and stress fields will be chosen in homogeneous form throughout the body. This implies that the admissible displacement fields are all linear and the admissible stress fields are all constant i.e. fields as occur in homogeneous bodies under homogeneous boundary conditions.

We begin with macroscopically orthotropic bodies and we start with the shear moduli: As a first example we consider G^*_{12}. Dual formulations for this modulus are given by (3.4.10-12) and (3.4.19-21). In the first formulation displacements are prescribed over the entire surface and, therefore, the principle of minimum strain energy (3.6.22) is appropriate. An obvious admissible displacement is

$$\tilde{u}_1 = \epsilon^{o}_{12}\,x_2 \qquad \tilde{u}_2 = \epsilon^{o}_{12}\,x_1 \qquad \tilde{u}_3 = 0 \qquad (3.6.46)$$

It is seen that (3.6.46) satisfies the boundary conditions and is continuous everywhere and, therefore, also at phase interfaces.

From (3.4.11a), (3.6.37), (3.6.21) and (3.4.4)

$$U^{\epsilon} = 2G^{*}_{12}\, \epsilon^{o\,2}_{12}\, V \tag{3.6.47}$$

For simplicity we first assume that the phases are isotropic. It follows from (3.6.21b), (3.6.11b), (3.6.46) and (3.6.45a) that

$$\tilde{U}_{\epsilon} = 2(G_1\, v_1 + G_2\, v_2)\, \epsilon^{o\,2}_{12}\, V \tag{3.6.48}$$

Inserting (3.6.47-48) into (3.6.22) we have

$$G^{*}_{12} \leq G_1\, v_1 + G_2\, v_2$$

To obtain a lower bound we use the formulation (3.4.19-21). Since tractions are prescribed over the entire boundary the principle of minimum stress energy (3.6.36) is now to be used. An obvious admissible stress field is

$$[\tilde{\sigma}_{ij}] = \begin{bmatrix} 0 & \sigma^{o}_{12} & 0 \\ \sigma^{o}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.6.49}$$

This stress field satisfies (3.4.19). It satisfies equilibrium since it is constant. Its associated tractions are continuous everywhere and, therefore, also at phase interfaces.

The stress energy is in view of (3.4.20), (3.6.38), (3.4.4) and (3.6.35a)

$$U^{\sigma} = \frac{\sigma^{o\,2}_{12}}{2G^{*}_{12}}\, V \tag{3.6.50}$$

Now from (3.6.49), (3.6.35), (3.6.29d), and (3.6.45b) we have for isotropic phases

$$\tilde{U}^{\sigma} = \frac{\sigma_{12}^{o\,2}}{2} \left(\frac{v_1}{G_1} + \frac{v_2}{G_2} \right) V \tag{3.6.51}$$

Inserting (3.6.50-51) into (3.6.36) we find

$$\left(\frac{v_1}{G_1} + \frac{v_2}{G_2} \right)^{-1} \leq G^*_{12}$$

Obviously nothing is changed in the bounding procedure for the EEM G^*_{23} and G^*_{13}. We can, therefore, summarize that for a macroscopically orthotropic body with isotropic phases

$$\left(\frac{\bar{1}}{G} \right)^{-1} \leq G^*_{12}, G^*_{23}, G^*_{13} \leq \bar{G} \tag{3.6.52}$$

where

$$\bar{G} = G_1 v_1 + G_2 v_2 \qquad \text{(a)}$$

$$\frac{\bar{1}}{G} = \frac{v_1}{G_1} + \frac{v_2}{G_2} \qquad \text{(b)} \tag{3.6.53}$$

Suppose now that the phases are orthotropic with elastic axes parallel to the x_1, x_2, x_3 system and shear moduli $G^{(1)}_{12}$, $G^{(2)}_{12}$ etc. The preceding method easily generalizes (3.6.52) to read

$$\left(\frac{\bar{1}}{G_{12}}\right)^{-1} \le G^*_{12} \le \bar{G}_{12} \qquad \text{(a)}$$

$$\left(\frac{\bar{1}}{G_{23}}\right)^{-1} \le G^*_{23} \le \bar{G}_{23} \qquad \text{(b)} \qquad (3.6.54)$$

$$\left(\frac{\bar{1}}{G_{13}}\right)^{-1} \le G^*_{13} \le \bar{G}_{13} \qquad \text{(c)}$$

Next we consider the Young's moduli E^*_1, E^*_2 and E^*_3. We start with the boundary traction formulation (3.4.28-36). It is seen that the principle of minimum stress energy should be used. In each case the admissible stress fields are chosen as the constant fields (3.4.29a), (3.4.32a) and (3.4.35a). The analysis is in all aspects similar to that employed in the previous establishment of lower bounds for shear moduli. If the phases are isotropic we find that

$$\left(\frac{\bar{1}}{E}\right)^{-1} \le E^*_1, E^*_2, E^*_3 \qquad (3.6.55)$$

where $\frac{\bar{1}}{E}$ is the analogue of (3.6.53b) in terms of isotropic phase Young's moduli:

If the phases are orthotropic with elastic axes parallel to the composite's axes, (3.6. 55) assumes the form

$$\left(\frac{\bar{1}}{E_i}\right)^{-1} \le E^*_i \qquad (3.6.56)$$

where i = 1,2,3. An example for the left side is

$$\frac{\bar{1}}{E_1} = \frac{v_1}{E_1^{(1)}} + \frac{v_2}{E_1^{(2)}}$$

where the superscripts on the Young's moduli indicate the phases.

The construction of upper bounds on the Young's moduli is not so straightforward. For simplicity we shall limit the treatment to transversely isotropic phases with axis of symmetry parallel to the composite's x_1 axis. Treatment for orthotropic phases is analogous, but the results are cumbersome.

We now make use of the formulations (3.4.37), (3.4.39-40), (3.4.41-42) and (3.4.43-44). We see that the boundary conditions are now of the mixed type (3.6.18) and we use the principle of minimum potential energy for such cases. Starting with (3.4.37) we see that these are a special case of (3.6.18a). Inspection of (3.6.20) shows that the surface integrals vanish since the entering traction components vanish. Therefore, the principle of minimum potential energy reduces to that of minimum strain energy.

We choose an admissible linear displacement system of the form

$$\tilde{u}_1 = \epsilon^{o}_{11} x_1 \qquad \tilde{u}_2 = \beta x_2 \qquad \tilde{u}_3 = \gamma x_3 \qquad (3.6.57)$$

These $\tilde{u}_i$ satisfy the boundary condition (3.4.37a) and continuity everywhere. Because of (3.6.19b) it is seen that β and γ are now arbitrary constants. The strains $\tilde{\epsilon}_{ij}$ associated with (3.6.57) are

$$[\tilde{\epsilon}_{ij}] = \begin{bmatrix} \epsilon^{o}_{11} & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \tag{3.6.58}$$

We insert (3.6.58) into (3.6.44) for the two phases and compute $\tilde{U}^{\epsilon}$ from (3.6.21b). The result is

$$2\tilde{U}^{\epsilon} = [\bar{n}\,\epsilon^{o\,2}_{11} + 2\bar{\ell}\,\epsilon^{o}_{11}\,(\beta+\gamma) + \bar{k}\,(\beta+\gamma)^2 + \bar{G}_T\,(\beta-\gamma)^2]V \tag{3.6.59}$$

where the overbars denote averages as in (3.6.53a).

Now because of (3.4.39), (3.4.5), and (3.4.8) the actual strain energy is given by

$$2U^{\epsilon} = E^{*}_{1}\,\epsilon^{o\,2}_{11}\,V \tag{3.6.60}$$

Since $\tilde{U}^{\epsilon}$ is larger than or equal to U^{ϵ}, (3.6.59-60) yield an upper bound on E^{*}_{1} in terms of the unknown parameters β, γ. To find the best upper bound (3.6.59) is minimized with respect to β, γ. This procedure yields

$$2\tilde{U}^{\epsilon}_{min} = \epsilon^{o\,2}_{11}\,(\bar{n} - \frac{\bar{\ell}^2}{\bar{k}})\,V$$

Hence we have

$$(\frac{\bar{1}}{E_A})^{-1} \le E^{*}_{1} \le \bar{n} - \frac{\bar{\ell}^2}{\bar{k}} \tag{3.6.61}$$

where the left side inequality follows from the adaption of (3.6.56) to transversely isotropic phases. Recalling the relation (3.4.80-81) it is seen that it follows from these that

$$E_A^* = n^* - \frac{\ell^{*2}}{k^*}$$

and the upper bound in (3.6.61) is of the same form in terms of averages of moduli. It is not difficult to prove that this form of the upper bound is an inherent feature of the present bounding method. Consequently bounds for E_2^*, E_3^* can be written down at once from (3.4.82). We have

$$\left(\frac{1}{\bar{E}_T}\right)^{-1} \le E_2^*, E_3^* \le \frac{4\bar{k}\bar{G}_T}{\bar{k} + \bar{\bar{m}}\bar{G}_T} \tag{3.6.62}$$

$$\bar{\bar{m}} = 1 + \frac{4\bar{k}\bar{\nu}_A^2}{\bar{E}_A}$$

where the left side of (3.6.62) follows from (3.6.56).

If one or both phases are isotropic we can use (3.4.89) and (3.4.91) to obtain the bounds (3.6.61-62) in that case.

We have so far constructed bounds for the three effective shear moduli and three effective Young's moduli. Bounds for the three remaining EEM or EEC are difficult to obtain for the present general geometry. Since the bounds which are obtained by the present elementary method do not, in general, give good estimates we shall not concern ourselves at the present time with the remaining EEM or EEC.

We now consider the transversely isotropic material with transversely isotropic or isotropic phases. The bounds for the two effective shear moduli can immediately be written down by adaption of (3.6.54). We have

$$\left(\frac{\bar{1}}{G_A}\right)^{-1} \le G_A^* \le \bar{G}_A \qquad \text{(a)}$$

$$\left(\frac{\bar{1}}{G_T}\right)^{-1} \le G_T^* \le \bar{G}_T \qquad \text{(b)} \qquad (3.6.63)$$

If the phases are isotropic (3.6.63) reduce to

$$\left(\frac{\bar{1}}{G}\right)^{-1} \le G_A^*, G_T^* \le \bar{G} \qquad (3.6.64)$$

Next we establish bounds for k^*. Dual formulations for this EEM are given by (3.4.59-61) and (3.4.62-64). In the first formulation we use the principle of strain energy and in the second formulation the principle of minimum complementary energy with mixed boundary conditions of type (3.6.18a). Since the analysis is completely analogous to previous ones we dispense with the details. The bounds are found to be

$$\left(\frac{v_1}{k_1} + \frac{v_2}{k_2}\right)^{-1} = \left(\frac{\bar{1}}{k}\right)^{-1} \le k^* \le \bar{k} = k_1 v_1 + k_2 v_2 \qquad (3.6.65)$$

We now recall the general relations between k^*, n^*, ℓ^*, E_A^* and ν_A^* given by (3.4.112), (3.4.117) and (3.4.118). These can be used conveniently to construct bounds on the EEM involved, in terms of k^* bounds, Hill [3.5].

We consider first E_A^* as expressed by (3.4.117). It is seen that because of the left inequality in (3.6.65) the parenthesis in (3.4.117) cannot be negative. It is also seen that the factor before the parenthesis cannot be negative. It is, therefore, concluded that E_A^* is a monotonically increasing function of k^*. Consequently, replacement of k^* by any upper or lower bound produces corresponding upper and lower bounds for E_A^*. We express this symbolically by

$$\boxed{\begin{aligned} E^*_{A(+)} &= E^*_A\,(k^*_{(+)}) \qquad \text{(a)} \\ E^*_{A(-)} &= E^*_A\,(k^*_{(-)}) \qquad \text{(b)} \end{aligned}} \qquad (3.6.66)$$

Insertion of (3.6.65) into (3.4.117), therefore, yields the following bounds.

$$\boxed{\bar{E} \le E^* \le \bar{E} + 4(\nu_2-\nu_1)^2\, v_1 v_2\, \frac{k_1\, k_2}{\bar{k}}} \qquad (3.6.67)$$

It should be noted that the phase Young's moduli and Poisson's ratios in (3.6.67) are all axial. The lower bound (3.6.67) has been first obtained by Hill [3.5].

Now it should be realized that the bounds (3.6.61) also apply for the transversely isotropic material and the question is which are the better bounds? It turns out that the upper bounds in (3.6.61) and (3.6.67) are exactly the same. This follows by simple algebra from the relations (3.4.80-81) for the phase moduli. The lower bounds are, however, different and it is easily shown that (3.6.67) gives the better one.

Next we consider ν_A^* as expressed by (3.4.118). It is seen that the sign of $(\nu_1-\nu_2)/(k_2-k_1)$ specifies whether ν_A^* is a monotonically increasing or decreasing function of k^*. It is easily seen that by the same reasoning as employed in (3.6.66) we have

$$\boxed{\begin{array}{ll} \nu^*_{A(+)} = \nu^*_A(k^*_{(-)}) & \dfrac{\nu_1 - \nu_2}{k_2 - k_1} > 0 \quad \text{(a)} \\ \nu^*_{A(-)} = \nu^*_A(k^*_{(+)}) & \\ & \\ \nu^*_{A(+)} = \nu^*_A(k^*_{(+)}) & \dfrac{\nu_1 - \nu_2}{k_2 - k_1} < 0 \quad \text{(b)} \\ \nu^*_{A(-)} = \nu^*_A(k^*_{(-)}) & \end{array}} \qquad (3.6.68)$$

We then have from (3.6.65), (3.4.118) and (3.6.68) that

$$\boxed{\frac{\nu_1 k_1 v_1 + \nu_2 k_2 v_2}{k_1 v_1 + k_2 v_2} = \frac{\overline{\nu_A k}}{\bar{k}} \underset{>}{\overset{<}{=}} \nu_A^* \underset{>}{\overset{<}{=}} \bar{\nu}_A = \nu_1 v_1 + \nu_2 v_2} \qquad (3.6.69)$$

where the upper inequality signs are valid for (a) and the lower inequality signs are valid for (b).

Bounds for n^* and ℓ^* are similarly easily established by use of (3.6.65) and the relations (3.4.112).

For statistically isotropic two phase materials methods analogous to the ones used above yield the results

$$\left(\frac{\bar{1}}{K}\right)^{-1} \le K^* \le \bar{K} \qquad \text{(a)}$$

$$\left(\frac{\bar{1}}{G}\right)^{-1} \le G^* \le \bar{G} \qquad \text{(b)} \qquad (3.6.70)$$

$$\left(\frac{\bar{1}}{E}\right)^{-1} \le E^* \le \frac{9\bar{K}\bar{G}}{3\bar{K}+\bar{G}} \qquad \text{(c)}$$

The results (3.6.70a,c) were first given by Paul [3.26] .

Finally it should be noted that all results obtained in this paragraph are easily generalized to any number of phases. It is merely necessary for this purpose to rewrite the averages in the bounds for that case. Thus, for example

$$\bar{G} = \sum_m G_m v_m$$

To obtain some idea about the numerical values of the bounds we consider a typical case of a fiber reinforced material composed of isotropic glass fibers and isotropic epoxy matrix. Phase moduli values are given by

Table 3.6.1

	Epoxy (1)	Glass fiber (2)
E	0.4×10^6 psi	10.5×10^6 psi
ν	0.35	0.20
G	0.148×10^6 psi	4.38×10^6 psi
k	0.494×10^6 psi	7.3×10^6 psi

Suppose that the material is transversely isotropic and that the volume fractions are equal $v_1 = v_2 = 0.5$. Then from (3.6.63), (3.6.65), (3.6.67) and (3.6.69)

$$0.29 \times 10^6 \text{ psi} \le G_A^*, G_T^* \le 2.26 \times 10^6 \text{ psi}$$

$$0.92 \times 10^6 \text{ psi} \le k^* \le 3.90 \times 10^6 \text{ psi}$$

$$5.45 \times 10^6 \text{ psi} \le E_A^* \le 5.47 \times 10^6 \text{ psi}$$

$$0.210 \le \nu_A^* \le 0.275$$

It is seen that the bounds for G_A^*, G_T^* and k^* do not provide good estimates. On the other hand, the E_A^* bounds are excellent and the ν_A^* bounds are fair.

The present E_A^* bounds already show that <u>$\bar{E}$ is an excellent approximation for E_A^* of a typical FM or FRM.</u>

It will be seen in chapter 3.7 that it is possible to obtain substantially better bounds for EEM of fibrous materials, in terms of phase elastic moduli and phase volume fractions.

3.6.4 Bounds for Circular Fiber Reinforcement

In contrast to the general fibrous materials considered in paragraph 3.6.3 we shall here be concerned with FRM whose geometry is completely specified.

We consider the important case of a FRM in which the fibers have circular cross sections. The geometry is completely defined if the radii of

the fibers and the locations of their centers are known. We shall here derive lower and upper bounds for the EEM of such a FRM by use of the principles of minimum potential and minimum complementary energies and the composite cylinder results which were developed in par. 3.5.3.

It should be noted that the elementary bounds which were established in par. 3.6.3 are certainly also valid for the present material. However, those bounds are valid for all cylindrical geometries. It is our present purpose to obtain improved bounds by explicit use of the present geometry.

To construct admissible fields which are suitable for the present specific geometry we surround each fiber by an imaginary concentric circular cylindrical surface which is entirely within the matrix. It is, moreover, required that none of the imaginary cylindrical surfaces overlap. Fig. 3.6.1 shows an example of such a geometrical construction. It is seen that the FRM has now been divided into a part of volume V_c which is the sum of the volumes of all composite cylinders and into a remaining matrix volume V'.

Suppose now that a fiber reinforced cylindrical specimen is subjected to homogeneous displacement boundary conditions (3.3.1). We choose as an admissible displacement in V' the linear field $\epsilon^o_{ij} x_j$. In order to satisfy continuity the admissible composite cylinder displacement fields must now assume the values $\epsilon^o_{ij} x_j$ on the outer surfaces of the composite cylinders and they must also be continuous at fiber matrix interfaces. We may thus choose as admissible fields in the composite cylinders their elastic displacement

fields under the homogeneous boundary conditions (3.3.1). Thus, the admissible field may be written as

$$\tilde{u}_i = \begin{cases} \epsilon^o_{ij} x_j & \text{in } V' \quad \text{(a)} \\ u^{(c)}_i & \text{in } V_c \quad \text{(b)} \end{cases} \tag{3.6.71}$$

where $u^{(c)}_i$ is the aggregate of all composite cylinder displacements under homogeneous boundary conditions. Thus, the field (3.6.71) satisfies continuity and the boundary conditions.(*)

Next we suppose that the fiber reinforced cylinder is subjected to the homogeneous traction boundary conditions (3.3.10). We use the same geometrical construction and choose the following admissible field

$$\tilde{\sigma}_{ij} = \begin{cases} \sigma^o_{ij} & \text{in } V' \quad \text{(a)} \\ \sigma^{(c)}_{ij} & \text{in } V_c \quad \text{(b)} \end{cases} \tag{3.6.72}$$

where σ^o_{ij} is the homogeneous stress field in (3.3.10) and $\sigma^{(c)}_{ij}$ is the aggregate of all actual elasticity stress fields in the composite cylinders under the boundary conditions (3.3.10).

(*) For $u^{(c)}_i$ computation we may consider any composite cylinder under boundary conditions $u^{(c)}_i(S_c) = \epsilon^o_{ij} y_j$, where y_j is the local coordinate system with origin at cylinder axis. To comply with $u^{(c)}_i(S_c) = \epsilon^o_{ij} x_j$ there is added the rigid body displacement $\epsilon^o_{ij} x^{(n)}_j$ where $x^{(n)}_j$ are the coordinates of the cylinder center. See similar reasoning, par. 3.5.3, eqns. (3.5.88-89) and fig. 3.5.2. The rigid body displacements do not contribute to the strain energy functional, they only serve to assure displacement continuity.

It is seen that (3.6.72) satisfies equilibrium in V' because it is constant there and also in V_c because $\sigma_{ij}^{(c)}$ is an actual elastic field in a composite cylinder. The traction boundary conditions (3.3.10) are satisfied because of (3.6.72a). Traction continuity at composite cylinder-matrix imaginary interfaces is satisfied by definition of $\sigma_{ij}^{(c)}$, and traction continuity at fiber-matrix interfaces is certainly satisfied as a requirement of a composite cylinder solution.

As has been seen before it is sometimes necessary to use mixed boundary conditions for EEM calculation. In that event $\tilde{u}_i$ in V' is chosen as the linear displacement which satisfies such boundary conditions and similarly $\tilde{\sigma}_{ij}$ in V' is taken as the constant stress field which satisfies the boundary conditions. The fields in V_c are taken as elasticity solutions of composite cylinders which are subjected to such boundary conditions.

In order to exploit the extremum principles we need the strain and stress energy functionals (3.6.21b) and (3.6.35b) which are associated with the admissible fields. The strain energy functional associated with (3.6.71) can be written in the form

$$\tilde{U}^{\epsilon} = W_1^{\epsilon} V' + U_c^{\epsilon} \qquad \text{(a)}$$

$$U_c^{\epsilon} = \sum_n U_{cn}^{\epsilon} \qquad \text{(b)} \qquad (3.6.73)$$

where W_1^{ϵ} is the strain energy density of the matrix material with strains ϵ_{ij}^{o}, and U_{cn}^{ϵ} is the actual strain energy of the nth composite cylinder under the homogeneous displacement boundary conditions.

Similarly, the stress energy functional associated with (3.6.72) is

$$\tilde{U}^{\sigma} = W_1^{\sigma} V' + U_c^{\sigma} \qquad \text{(a)}$$

$$U_c^{\sigma} = \sum_n U_{cn}^{\sigma} \qquad \text{(b)} \qquad (3.6.74)$$

where W_1^{σ} is the stress energy density of the matrix material with stresses σ_{ij}^{o} and U_{cn}^{σ} is the actual stress energy of the nth composite cylinder under the homogeneous traction boundary conditions.

Expressions similar to (3.6.73-74) are obtained for mixed boundary conditions in the event that the boundary conditions are such that the surface integrals in (3.6.12a) and (3.6.29c) vanish.

For the purpose of computation of strain or stress energy in a composite cylinder under homogeneous boundary conditions we go back to par. 3.5.3. It has been shown there that such a composite cylinder has a set of apparent elastic moduli which are the composite cylinder assemblage moduli. These are here denoted by subscripts c, thus; k_c^*, E_{Ac}^*, ν_{Ac}^*, n_c^* and ℓ_c^* which implies that under homogeneous boundary conditions associated with these moduli the composite cylinder behaves precisely as a homogeneous cylinder with such elastic moduli. Since the strain or stress energy of an elastic body

is given by the surface integral $\frac{1}{2}\int T_i u_i \, dS$ and since the pertinent displacements and tractions on the composite cylinder surface for the cases considered are just as in a homogeneous cylinder having the apparent moduli, it is concluded that also for purpose of computation of strain or stress energy the composite cylinder can be replaced by a homogeneous cylinder with apparent moduli.

Suppose that in the geometrical construction shown in fig. 3.6.1 the ratio a_n/b_n assumes M different values. For each of these values we have a group of composite cylinders which by the results of par. 3.5.3 have the same apparent moduli. Thus, such a group may be replaced by homogeneous cylinders with appropriate apparent moduli. Then we obtain instead of the original FRM another M+1 phase equivalent FRM consisting of matrix of volume V' and M different kinds of homogeneous circular cylinders of total volume V_o. Then, equivalent FRM bounds can be written down at once by use of the bounds of par. 3.6.3 for fibrous materials, with M+1 phases.

The procedure will be illustrated by construction of bounds for square and hexagonal arrays of identical circular fibers. Composite cylinder construction for these cases is shown in fig. 3.6.2 and it is seen that all composite cylinders are identical. Let the volume fraction occupied by V_c be denoted v_c. Clearly v_c is the fractional volume of a composite circle with respect to its circumscribing square or hexagon, respectively. By elementary geometry

$$v_c = \begin{cases} \dfrac{\pi}{4} = 0.785 & \text{square array} \quad \text{(a)} \\ \\ \dfrac{\pi}{2\sqrt{3}} = 0.907 & \text{hexagonal array} \quad \text{(b)} \end{cases} \tag{3.6.75}$$

Let the volume fraction of fibers relative to the whole composite be v_2. Since $(a/b)^2$ in each composite circle is the fiber volume fraction relative to the composite cylinder it follows that

$$(a/b)^2 = \frac{v_2}{v_c} = v_{2c} \tag{3.6.76}$$

The apparent moduli of a composite cylinder are then found by substitution of (3.6.76) into their expressions given in par. 3.5.3. This is equivalent to the replacement

$$\begin{aligned} v_2 &\longrightarrow v_{2c} && \text{(a)} \\ v_1 &\longrightarrow 1 - v_{2c} && \text{(b)} \end{aligned} \tag{3.6.77}$$

in the expressions for the EEM of the composite cylinder assemblage.

Resulting expressions of apparent moduli of composite cylinders are denoted $k_c^*(v_{2c})$ etc. It is seen that in the present cases the equivalent FRM has only two phases: matrix and composite cylinders of one kind. Hence, we can use the two phase bounds of par. 3.6.3. We then have

$$\left[\frac{1-v_c}{k_1} + \frac{v_c}{k_c^*(v_{2c})}\right]^{-1} \le k^* \le k_1(1-v_c) + k_c^*(v_{2c})\, v_c \tag{3.6.78}$$

$$E_1(1-v_c) + E_{Ac}^*(v_{2c})\, v_c \le E_A^* \le E_1(1-v_c) + E_{Ac}^*(v_{2c})\, v_c$$

$$+ 4\,[\nu_{Ac}^*(v_{2c}) - \nu_1]^2\, v_c\,(1-v_c)\,\frac{k_1\, k_c^*(v_{2c})}{k_1(1-v_c)+k_c^*(v_{2c})\, v_c} \tag{3.6.79}$$

$$\left[\frac{1-v_c}{G_1} + \frac{v_c}{G_{Ac}^*(v_{2c})}\right]^{-1} \le G_A^* \le G_1(1-v_c) + G_{Ac}^*(v_{2c})\, v_c \tag{3.6.80}$$

To obtain bounds on ν_A^* the preceding method is not applicable since bounds on ν_A^* cannot be found by extremum principles. We can, however, use the results (3.6.68) with (3.6.78) and (3.4.118) to find bounds for ν_A^*. The results are

$$\frac{k_1 k_2(\nu_1-\nu_2)}{k_1-k_2}\left[\frac{1-v_c}{k_1} + \frac{v_c}{k_c^*(v_{2c})}\right] \lesseqgtr \nu_A^* - \frac{\nu_2 k_2 - \nu_1 k_1}{k_2-k_1} \lesseqgtr$$

$$\frac{k_1 k_2(\nu_1-\nu_2)}{k_2-k_1}\;\frac{1}{k_1(1-v_c)+k_c^*(v_{2c})v_c} \tag{3.6.81}$$

where the upper inequality signs are valid for (3.6.68a) and the lower ones for (3.6.68b).

It is to be noted that all the bounds apply for transversely isotropic phases. In this case the shear moduli in (3.6.78) (introduced through (3.5.91)) are transverse, the Young's moduli and Poisson's ratios in (3.6.79) are axial, the shear moduli in (3.6.80) are axial and the Poisson's ratios in (3.6.81) are axial.

The EEM of the composite cylinder assemblage can also be obtained by the same method. Indeed the initial derivation in [3.10] was based on the present variational treatment. To see this we recall that in the composite cylinders used to construct the assemblage the radii a_n/b_n are the same in all cylinders. The remaining matrix volume at any filling stage becomes V', with volume fraction $v'=1-v_c$, according to the present construction. In the limit of complete filling by composite cylinders

$$v' \cong 0 \qquad v_c \cong 1 \qquad \text{(a)}$$

$$(a_n/b_n)^2 = v_{2c} \cong v_2 \qquad \text{(c)} \qquad (3.6.82)$$

Insertion of (3.6.82) into (3.6.78-81) shows that <u>all bounds coincide and reduce to the composite cylinder assemblage results</u> (3.5.91), (3.5.96-97), and (3.5.111).

It is recalled that G_T^* could not be obtained by the replacement method given in par. 3.5.3, but with the present method it is possible to obtain bounds for

G_T^*, both for composite cylinder assemblages and any other circular fiber arrangement.

Let a cylindrical fiber reinforced specimen be subjected to (3.4.13). Then (3.6.71a) becomes

$$\tilde{u}_1 = 0 \qquad \tilde{u}_2 = \epsilon_{23}^{o} x_3 \qquad \tilde{u}_3 = \epsilon_{23}^{o} x_2 \tag{3.6.83}$$

The composite cylinder solution is denoted $u_i^{(1)}$ in the matrix shell and $u_i^{(2)}$ in the fiber. The boundary conditions are

$$u_i^{(1)} = 0 \qquad u_2^{(1)} = \epsilon_{23}^{o} x_3 \qquad \tilde{u}_3 = \epsilon_{23}^{o} x_2 \ , \ r=b \tag{3.6.84}$$

It is seen that we have a plain strain problem which falls into the general category of the first kind of boundary value problem discussed in par. 3.5.1. Hence, $u_\alpha^{(1)}$ and $u_\alpha^{(2)}$ must satisfy (3.5.14) in matrix shell and fiber, respectively, and also the interface conditions (3.5.15) and (3.5.19) on r=a.

A solution of this problem has been constructed in [3.10]. The general form of the solution is in vectorial notation

$$\underline{u}^{(1)}/\epsilon_{23}^{o} = A_1^{\epsilon} \nabla (x_2 x_3) + A_2^{\epsilon}/a^2 \cdot [r^2 \nabla (x_2 x_3) + \alpha^{(1)} x_2 x_3 \underline{r}]$$

$$+ A_3^{\epsilon} a^4 \nabla \left(\frac{x_2 x_3}{r^4} \right) + A_4^{\epsilon} a^2 \left[r^2 \nabla \left(\frac{x_2 x_3}{r^4} \right) + \beta^{(1)} \frac{x_2 x_3}{r^4} \underline{r} \right] \quad \text{(a)}$$

$$\underline{u}^{(2)}/\epsilon_{23}^{o} = A_5^{\epsilon} \nabla (x_2 x_3) + A_6^{\epsilon}/a^2 \cdot [r^2 \nabla (x_2 x_3) + \alpha^{(2)} x_2 x_3 \underline{r}] \quad \text{(b)} \tag{3.6.85}$$

where $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ are the displacement vector fields in matrix and fiber, respectively. Here A_k^{ϵ} $k = 1,2,\ldots,6$ are non-dimensional arbitrary

constants, ∇ stands for the gradient operator, $\underline{r}$ is the plane radial vector with components x_2, x_3 and magnitude r, and the constants $\alpha^{(1)}$, $\beta^{(1)}$, $\alpha^{(2)}$ are given by

$$\alpha^{(1)} = -\frac{2\,(3-4\nu_1)}{3-2\nu_1} \qquad \text{(a)}$$

$$\beta^{(1)} = \frac{2\,(3-4\nu_1)}{1-2\nu_1} \qquad \text{(b)} \qquad (3.6.86)$$

$$\alpha^{(2)} = -\frac{2\,(3-4\nu_2)}{3-2\nu_2} \qquad \text{(c)}$$

This solution was prompted by the material contained in Love [3.3], chapter XI, and a related composite sphere solution which was constructed in [3.11].

So far the present solution satisfies the differential equations (3.5.14). If inserted into (3.6.84) and into (3.5.15) and (3.5.19) at r = a, there are obtained precisely 6 linear equations for determination of the constants A_k^{ϵ} $k = 1,2,\ldots,6$. These equations may be written

$$c_{k\ell}^{\epsilon} A_{\ell}^{\epsilon} = d_k^{\epsilon} \qquad k,\ell = 1,2,\ldots,6 \qquad \text{(a)} \qquad (3.6.87)$$

where $c_{k\ell}^{\epsilon}$ and d_k^{ϵ} are given in the table which follows.

$k \backslash \ell$	$c^{\epsilon}_{k\ell}$ 1	2	3	4	5	6	d^{ϵ}_{k}
1	1	$1/v$	v^2	v	0	0	1
2	0	$-\frac{3-4\nu_1}{3-2\nu_1}\frac{1}{v}$	$-2v^2$	$\frac{1}{1-2\nu_1}v$	0	0	0
3	1	1	1	1	-1	-1	0
4	0	$-\frac{3-4\nu_1}{3-2\nu_1}$	-2	$\frac{1}{1-2\nu_1}$	0	$\frac{3-4\nu_2}{3-2\nu_2}$	0
5	1	$\frac{3}{3-2\nu_1}$	-3	$\frac{1}{1-2\nu_1}$	$-\gamma$	$-\frac{3}{3-2\nu_2}\gamma$	0
6	0	$-\frac{1}{3-2\nu_1}$	2	$-\frac{1}{1-2\nu_1}$	0	$\frac{1}{3-2\nu_2}\gamma$	0

(b) (3.6.87)

where

$$v = (a/b)^2 \qquad \text{(a)}$$

$$\gamma = \frac{G_2}{G_1} \qquad \text{(b)} \qquad (3.6.88)$$

By the first transverse isotropy-isotropy analogy, par. 3.5.1, the present solution is valid also for transversely isotropic phases. In this case G_2, G_1 are to be interpreted as the phases transverse shear moduli and the Poisson's ratios ν_2, ν_1 are replaced by the last of (3.5.25).

Computation of the strain energy in the composite cylinder is best carried out by computation of the boundary tractions and evaluation of the surface integral $\frac{1}{2}\int T_i\, u_i\, dS$. We find that

$$U^{\epsilon}_{cn} = 2G_1 \left[1 - \frac{2(1-\nu_1)}{1-2\nu_1} A^{\epsilon}_4 (v_{2n})\, v_{2n} \right] \epsilon^{o\,2}_{23}\, V_{cn} \tag{3.6.89}$$

where V_{cn} is the volume of the composite cylinder,

$$v_{2n} = (a_n/b_n)^2 \tag{3.6.90}$$

and $A^{\epsilon}_4 (v_{2n})$ denotes the coefficient A^{ϵ}_4 in (3.6.87) computed for (3.6.88a) of value (3.6.90).

In view of (3.6.83) and (3.6.89) , (3.6.73) now assumes the form

$$\tilde{U}^{\epsilon} = 2\epsilon^{o\,2}_{23}\, G_1 \left\{ V' + \sum_n \left[1 - \frac{2(1-\nu_1)}{1-2\nu_1} A^{\epsilon}_4 (v_{2n})\, v_{2n} \right] V_{cn} \right\} \tag{3.6.91}$$

If the FRM is transversely isotropic the actual strain energy is given in view of (3.6.39) by

$$U^{\epsilon} = 2\epsilon^{o\,2}_{23}\, G^*_T\, V \tag{3.6.92}$$

and it is recalled that G^*_T has to be interpreted as (3.4.56) for square symmetry. Because of (3.6.22), (3.6.91-92) now give the upper bound

$$G^*_T \le G_1 \left\{ v' + \frac{1}{V} \sum_n \left[1 - \frac{2(1-\nu_1)}{1-2\nu_1} A^{\epsilon}_4 (v_{2n})\, v_{2n} \right] v_{cn} \right\} \tag{3.6.93}$$

where $v_{cn} = V_{cn}/V$

To construct a lower bound the cylindrical fiber reinforced specimen is subjected to (3.4.22). The stress field $\tilde{\sigma}_{ij}$ in (3.6.72a) is then given by (3.4.23a) and W_1^{σ} in (3.6.74a) becomes

$$W_1^{\sigma} = \frac{\sigma_{23}^{o\,2}}{2G_1} \tag{3.6.94}$$

Now the composite cylinders are subjected to the traction system (3.4.22). On the surface r = b we have

$$\left.\begin{aligned} T_1 &= 0 && \text{(a)} \\ T_2 &= \sigma_{23}^{o}\,\frac{x_3}{b} && r = b \quad \text{(b)} \\ T_3 &= \sigma_{23}^{o}\,\frac{x_2}{b} && \text{(c)} \end{aligned}\right\} \tag{3.6.95}$$

The solution of the composite cylinder under (3.6.95) is again of the form (3.6.85). It is now written as

$$\frac{2G_1}{\sigma_{23}^{o}}\,\underline{u}^{(1)} = A_1^{\sigma}\,\nabla(x_2\,x_3) + A_2^{\sigma}/a^2 \cdot [r^2\nabla(x_2\,x_3) + \alpha^{(1)}x_2\,x_3\underline{r}]$$

$$+ A_3^{\sigma}\,a^4\nabla\left(\frac{x_2\,x_3}{r^4}\right) + A_4^{\sigma}\,a^2\left[r^2\nabla\left(\frac{x_2x_3}{r^4}\right) + \beta^{(1)}\frac{x_2x_3}{r^4}\,\underline{r}\right] \quad \text{(a)}$$

$$\frac{2G_1}{\sigma_{23}^{o}}\,\underline{u}^{(2)} = A_5^{\sigma}\,\nabla(x_2\,x_3) + A_6^{\sigma}/a^2 \cdot [r^2\nabla(x_2\,x_3) + \alpha^{(2)}x_2\,x_3\,\underline{r}] \quad \text{(b)} \tag{3.6.96}$$

The boundary conditions (3.6.95) and the interface conditions (3.5.15) and (3.5.19) at $r = a$ now lead to the system of equations

$$c^{\sigma}_{k\ell} A^{\sigma}_{\ell} = d^{\sigma}_{k} \qquad k,\ell = 1,2,\ldots,6 \qquad \text{(a)} \qquad (3.6.97)$$

where $c^{\sigma}_{k\ell}$ and d^{σ}_{k} are given in the table below.

	$c^{\sigma}_{k\ell}$						d^{σ}_{k}
$k \backslash \ell$	1	2	3	4	5	6	
1	1	$\frac{3}{3-2\nu_1}\frac{1}{v}$	$-3v^2$	$\frac{1}{1-2\nu_1}v$	0	0	1
2	0	$-\frac{1}{3-2\nu_1}\frac{1}{v}$	$2v^2$	$-\frac{1}{1-2\nu_1}v$	0	0	0
3	1	1	1	1	-1	-1	0
4	0	$-\frac{3-4\nu_1}{3-2\nu_1}$	-2	$\frac{1}{1-2\nu_1}$	0	$\frac{3-4\nu_2}{3-2\nu_2}$	0
5	1	$\frac{3}{3-2\nu_1}$	-3	$\frac{1}{1-2\nu_1}$	$-\gamma$	$-\frac{3}{3-2\nu_2}\gamma$	0
6	0	$-\frac{1}{3-2\nu_1}$	2	$-\frac{1}{1-2\nu_1}$	0	$\frac{1}{3-2\nu_2}\gamma$	0

(b) (3.6.97)

where v and γ are given by (3.6.88).

It is to be noted that (3.6.95a) is only satisfied in the Saint Venant sense by the solution, which is of no consequence for a long cylinder. See par. 3.5.1 for discussion.

Now the stress energy in any composite cylinder is found to be

$$U^{\sigma}_{cn} = \frac{\sigma^{o\,2}_{23}}{2G_1} \left[1 + \frac{2(1-\nu_1)}{1-2\nu_1} A^{\sigma}_4 (v_{2n})\, v_{2n} \right] V_{cn} \tag{3.6.98}$$

where v_{2n} is given by (3.6.90) and $A^{\sigma}_4 (v_{2n})$ denotes the constant A^{σ}_4 in (3.6.97) computed for $v = v_{2n}$.

Insertion of (3.6.98) and (3.6.94) into (3.6.74) gives $\tilde{U}^{\sigma}$. If the FRM is transversely isotropic or square symmetric it follows from (3.4.23a) and (3.6.40) that the stress energy is

$$U^{\sigma} = \frac{\sigma^{o\,2}_{23}}{2G^*_T} V \tag{3.6.99}$$

where G^*_T is to be interpreted as (3.4.56) for square symmetry. Insertion of (3.6.99) and the previously computed $\tilde{U}^{\sigma}$ into the extremum principle (3.6.36) yields the following lower bound for G^*_T

$$G_1 \left\{ v' + \frac{1}{V} \sum_n \left[1 + \frac{2(1-\nu_1)}{1-2\nu_1} A^{\sigma}_4 (v_{2n})\, v_{2n} \right] v_{cn} \right\}^{-1} \le G^*_T \tag{3.6.100}$$

We now apply the bounds (3.6.93) and (3.6.100) for special cases. For square and hexagonal arrays as shown in fig. 3.6.2, v_{2n} and v_{cn} are all the same. It is seen from (3.6.76) and (3.6.90) that now

$$v_{2n} = v_{2c} = v_2/v_c \tag{3.6.101}$$

where it is recalled that v_2 is the fibers volume fraction relative to the composite and v_c is given by (3.6.75). Since V_{cn} is the volume of a composite cylinder, we have

$$Nv_{cn} = \frac{NV_{cn}}{V} = v_c$$

$$v' = 1 - v_c \tag{3.6.102}$$

where N is the number of fibers. Inserting (3.6.101-102) into (3.6.93) and (3.6.100) we obtain

$$G_1 \left[1 + \frac{2(1-\nu_1)}{1-2\nu_1} A_4^{\sigma} (v_{2c}) v_2 \right]^{-1} \leq G_T^* \leq G_1 \left[1 - \frac{2(1-\nu_1)}{1-2\nu_1} A_4^{\epsilon} (v_{2c}) v_2 \right] \tag{3.6.103}$$

For the composite cylinder assemblage v_{2c} in (3.6.103) becomes v_2, (3.6.82b). We then have

$$G_{Tc(-)}^* = G_1 \left[1 + \frac{2(1-\nu_1)}{1-2\nu_1} A_4^{\sigma} (v_2) v_2 \right]^{-1} \leq G_{Tc}^* \leq G_1 \left[1 - \frac{2(1-\nu_1)}{1-2\nu_1} A_4^{\epsilon} (v_2) v_2 \right] = G_{Tc(+)}^* \tag{3.6.104}$$

These are the bounds which were given in [3.10].

It may be noted that the bound expressions (3.6.104) can be obtained in the following fashion. Let a single composite cylinder of the assemblage be subjected to the displacement boundary conditions (3.6.84). Then

$$G^*_{Tc(+)} = \frac{\bar{\sigma}_{23}}{2\epsilon^o_{23}} \tag{3.6.105}$$

where the overbar denotes average over the composite cylinder. If the composite cylinder is subjected to the traction boundary conditions (3.4.22), then

$$G^*_{Tc(-)} = \frac{\sigma^o_{23}}{2\bar{\epsilon}_{23}} \tag{3.6.106}$$

It is of some interest to note that the bounds (3.6.103) can now be expressed in a different form. If the bounds (3.6.104) are denoted $G^*_{(+)}(v_2)$ and $G^*_{(-)}(v_2)$, respectively then (3.6.103) assumes the form

$$\left[\frac{1-v_c}{G_1} + \frac{v_c}{G^*_{(-)}(v_{2c})}\right]^{-1} \le G^*_T \le G_1(1-v_c) + G^*_{(+)}(v_{2c})\, v_c \tag{3.6.107}$$

which resembles the results (3.6.78-80).

The system of equations (3.6.87) has recently been solved algebraically for A^ϵ_4. Insertion of this result into the upper bound (3.6.104) yields the explicit expression (3.5.113) in par. 3.5.3 for $G^*_{T(+)}$. No doubt the lower bound in (3.6.104) can be similarly simplified but this has not been done since for fibers stiffer than matrix this bound is superseded by a better lower bound which will be derived in par. 3.7.3. This better lower bound is (3.5.114), par. 3.5.3.

The results (3.6.78-80), (3.6.81), and (3.6.103) provide bounds for five effective elastic properties. Bounds on other EEM can be derived in terms of these. To find bounds on E_T^* , for example, it is easily shown, in general, by use of (3.4.82) and (3.4.84) that

$$E_{T(+)}^* = \frac{4k_{(+)}^* G_{T(+)}^*}{k_{(+)}^* + m_{(+)} G_{T(+)}^*} \qquad \text{(a)}$$

$$E_{T(-)}^* = \frac{4k_{(-)}^* G_{T(-)}^*}{k_{(-)}^* + m_{(-)} G_{T(-)}^*} \qquad \text{(b)} \qquad (3.6.108)$$

where

$$m_{(+)} = 1 + \frac{4k_{(+)}^* \nu_{A(-)}^{*\,2}}{E_{A(+)}^*} \qquad \text{(a)}$$

$$m_{(-)} = 1 + \frac{4k_{(-)}^* \nu_{A(+)}^{*\,2}}{E_{A(-)}^*} \qquad \text{(b)} \qquad (3.6.109)$$

and the bounds in right sides of (3.6.108-109) are any bounds.

For the composite cylinder assemblage k^*, E_A^* and ν_A^* are known and therefore the bounds (3.6.105-106) reduce to (3.5.120).

3.7 BOUNDING METHODS FOR EFFECTIVE ELASTIC MODULI: POLARIZATION EXTREMUM PRINCIPLES

3.7.1 Elastic Polarization

In par. 3.2.3 we have formulated the elasticity problem of heterogeneous bodies in terms of displacements. The problem may also, though inconveniently, be formulated in terms of stresses as has been shown in par. 3.6.2. It is our present purpose to formulate the elasticity problem of a heterogeneous body in terms of two new variables: the elastic polarization tensor and a displacement deviation. These will be defined further below.

Consider two elastic bodies of identical external geometries which are subjected to identical surface displacements, u_i^S. The first body is homogeneous anisotropic with elastic moduli C^o_{ijkl}. The second body is heterogeneous anisotropic, composed of any number of homogeneous anisotropic phases with elastic moduli $C^{(m)}_{ijkl}$ in the mth phase. For convenience the first body is called the C^o body and the second is called the C body.

The elastic fields in the C^o body are denoted u^o_i, ϵ^o_{ij}, σ^o_{ij} while the fields in the C body are denoted u_i, ϵ_{ij}, σ_{ij} assuming the values $u^{(m)}_i$, $\epsilon^{(m)}_{ij}$, $\sigma^{(m)}_{ij}$ in the mth phase. We summarize the elasticity problems of the two bodies for the case of zero body forces:

<u>C^o body</u>

$$\sigma^o_{ij,j} = 0 \qquad \text{(a)}$$

$$\sigma^o_{ij} = C^o_{ijkl}\,\epsilon^o_{kl} \qquad \text{(b)}$$

$$\epsilon^o_{ij} = \frac{1}{2}\,(u^o_{i,j} + u^o_{j,i}) \qquad \text{(c)} \qquad (3.7.1)$$

$$u^o_i\,(S) = u^S_i \qquad \text{(d)}$$

<u>C body</u>

$$\sigma_{ij,j} = 0 \qquad \text{(a)}$$

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \qquad \text{(b)}$$

$$\epsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i} \qquad \text{(c)} \qquad (3.7.2)$$

$$\left.\begin{array}{ll} u_i & \text{(d)} \\ T_i = \sigma_{ij}\,n_j = C_{ijkl}\,\epsilon_{kl}\,n_i & \text{(e)} \end{array}\right\} \text{continuous on } S_{int}$$

$$u_i\,(S) = u_i^S \qquad \text{(f)}$$

Equs. (3.7.2a,b,c) apply for each phase separately and all quantities may then be given a superscript m.

We now define the <u>stress polarization tensor</u> p_{ij} by

$$\sigma_{ij} = C^{o}_{ijkl}\,\epsilon_{kl} + p_{ij} \qquad \text{(a)}$$

$$\sigma^{(m)}_{ij} = C^{o}_{ijkl}\,\epsilon^{(m)}_{kl} + p^{(m)}_{ij} \qquad \text{(b)} \qquad (3.7.3)$$

where (a) is a general definition and (b) is an explicit expression for the mth phase. The polarization may be interpreted as the necessary correction if it is erroneously assumed that stress and strain in the C body are related by the stress strain law of the C^{o} body.

Next we define the deviating fields u'_i, ϵ'_{ij}, σ'_{ij} as

$$u'_i = u_i - u^{o}_i \qquad \text{(a)}$$

$$\epsilon'_{ij} = \epsilon_{ij} - \epsilon^{o}_{ij} \qquad \text{(b)} \qquad (3.7.4)$$

$$\sigma'_{ij} = \sigma_{ij} - \sigma^{o}_{ij} \qquad \text{(c)}$$

It follows from (3.7.1c-2c) and (3.7.4b) that

$$\epsilon'_{ij} = \frac{1}{2}(u'_{i,j} + u'_{j,i}) \qquad (3.7.5)$$

from (3.7.1a-2a),

$$\sigma'_{ij,j} = 0 \qquad (3.7.6)$$

from (3.7.1b), (3.7.3a) and (3.7.4c),

$$\sigma'_{ij} = C^{o}_{ijkl}\,\epsilon'_{kl} + p_{ij} = C^{o}_{ijkl}\,u'_{k,l} + p_{ij} \tag{3.7.7}$$

Insertion of (3.7.7) into (3.7.6) yields

$$(C^{o}_{ijkl}\,\epsilon'_{kl} + p_{ij})_{,j} = C^{o}_{ijkl}\,u'_{k,lj} + p_{ij,j} = 0 \tag{3.7.8}$$

We replace in (3.7.3) the left side by (3.7.2b); we then have

$$p_{ij} = (C_{ijkl} - C^{o}_{ijkl})\epsilon_{kl} \qquad \text{(a)}$$

$$\epsilon_{ij} = \epsilon^{o}_{ij} + \epsilon'_{ij} = H_{ijkl}\,p_{kl} \qquad \text{(b)} \tag{3.7.9}$$

where H_{ijkl} is the inverse of $C_{ijkl} - C^{o}_{ijkl}$. The relation between these two tensors is written both in matrix and subscript notation

$$\underline{H} \cdot \underline{C}' = \underline{I} \qquad H_{ijrs}\,C'_{rskl} = I_{ijkl} \qquad \text{(a)}$$

$$\underline{C}' = \underline{C} - \underline{C}^{o} \qquad C'_{rskl} = C_{rskl} - C^{o}_{rskl} \qquad \text{(b)} \tag{3.7.10}$$

where $\underline{I}$ or I_{ijkl} is the fourth rank unit tensor (3.2.7). Since C_{ijkl} and C^{o}_{ijkl} are elastic moduli tensors they obey the symmetry conditions (3.2.3) and thus their difference C'_{ijkl} obeys the same symmetry. The tensor I_{ijkl} obeys these symmetry conditions by definition. Therefore H_{ijkl} also has the same symmetry. Thus

$$H_{ijkl} = H_{jikl} = H_{ijlk} = H_{klij} \tag{3.7.11}$$

In view of (3.7.11), (3.7.9) can also be written in the form

$$\epsilon^{o}_{ij} + \epsilon'_{ij} = \epsilon^{o}_{ij} + \frac{1}{2}(u'_{i,j} + u'_{j,i}) = H_{ijkl}\, p_{kl} \tag{3.7.12}$$

Equations (3.7.8), (3.7.12) are 9 differential equations for the 9 quantities u'_i and p_{ij}. They may be taken as the governing differential equations for the C body if the solution for the C^o body is known.

For a multiphase body (3.7.8) and (3.7.12) have to be written for each phase. All quantities in these equations, except those with zero superscript, are then given a superscript m. The meaning of $H^{(m)}_{ijkl}$ is given by

$$H^{(m)}_{ijrs}\,(C^{(m)}_{rskl} - C^{o}_{rskl}) = I_{ijkl} \tag{3.7.13}$$

To complete the formulation of the problem in terms of u'_i and p_{ij} we have to consider the boundary and continuity conditions (3.7.2d,e,f). In view of (3.7.1d), (3.7.2f) and (3.7.4a) we have

$$u'_i\,(S) = 0 \tag{3.7.14}$$

as the boundary condition. Since u_i is continuous at interfaces and u^o_i is continuous everywhere, the difference u'_i is also continuous at interfaces, Similarly, $\sigma_{ij}\, n_j$ and $\sigma^o_{ij}\, n_j$ are continuous at interfaces and so their difference $\sigma'_{ij}\, n_j$ is also continuous there. This difference is expressed in terms of u'_i and p_{ij} by (3.7.7). Thus (3.7.2d,e) are replaced by

$$\left.\begin{array}{ll} u'_i & \text{(a)} \\ & \\ T'_i = C^{o}_{ijkl}\, u'_{k,l} + p_{ij})\, n_j & \text{(b)} \end{array}\right\} \text{continuous on } S_{int} \qquad (3.7.15)$$

It is seen that the elasticity problem for a heterogeneous C body has been formulated in terms of $u'^{(m)}_i$ and $p^{(m)}_{ij}$ which satisfy (3.7.8) and (3.7.12) in each phase, the boundary condition (3.7.14) and the continuity conditions (3.7.15).

Next we derive an interesting formula which expresses the strain energy U^{ϵ} of the C body in terms of p_{ij} and ϵ^{o}_{ij}. The strain energies of the C^{o} and C bodies are, respectively,

$$2U^{\epsilon}_{o} = \int_V \sigma^{o}_{ij}\, \epsilon^{o}_{ij}\, dV \qquad \text{(a)}$$

$$(3.7.16)$$

$$2U^{\epsilon} = \int_V \sigma_{ij}\, \epsilon_{ij}\, dV \qquad \text{(b)}$$

In view of (3.7.16a) and (3.7.4b,c), (3.7.16b) can be written as

$$2U^{\epsilon} = 2U^{\epsilon}_{o} + \int_V \sigma'_{ij}\, \epsilon^{o}_{ij}\, dV + \int_V \sigma_{ij}\, \epsilon'_{ij}\, dV \qquad (3.7.17)$$

Consider the last integral in (3.7.17). Because of (3.7.2a,e), (3.7.5) and (3.7.15), the theorem of virtual work is valid. Thus

$$\int_V \sigma_{ij}\, \epsilon'_{ij}\, dV = \int_V T_i\, u'_i\, dS = 0$$

which vanishes because of (3.7.14). Now consider the other integral in (3.7.17). Because of (3.7.7)

$$\int_V \sigma'_{ij}\, \epsilon^{o}_{ij}\, dV = \int_V C^{o}_{ijkl}\, \epsilon'_{kl}\, \epsilon^{o}_{ij}\, dV + \int_V p_{ij}\, \epsilon^{o}_{ij}\, dV$$

Because of ij, kl symmetry

$$\int_V C^{o}_{ijkl}\, \epsilon'_{kl}\, \epsilon^{o}_{ij}\, dV = \int_V C^{o}_{ijkl}\, \epsilon'_{ij}\, \epsilon^{o}_{kl}\, dV = \int_V \sigma^{o}_{ij}\, \epsilon'_{ij}\, dV$$

where the last step is due to (3.7.1b). By (3.7.1a), (3.7.5) and (3.7.15a) the theorem of virtual work applies to the last integral. We have

$$\int_V \sigma^{o}_{ij}\, \epsilon'_{ij}\, dV = \int_V T^{o}_{i}\, u'_i\, dV = 0$$

because of (3.7.14). Introduction of all of the preceding results into (3.7.17) yields the remarkable expression

$$U^{\epsilon} = U_o^{\epsilon} + \frac{1}{2} \int_V p_{ij}\, \epsilon^{o}_{ij}\, dV = U_o^{\epsilon} + \frac{1}{2} \sum_m \int_V p^{(m)}_{ij}\, \epsilon^{o}_{ij}\, dV \qquad (3.7.18)$$

This was first derived by Eshelby [3.27]. The extreme right side follows if the continuity condition (3.7.15b) is satisfied at interfaces.

3.7.2 Polarization Extremum Principles

We shall now derive an extremum principle which is a variational formulation of the elasticity problem for the C body as expressed by (3.7.8), (3.7.9) or (3.7.12) and (3.7.14-15). This extremum principle was first derived by Hashin and Shtrikman [3.28]. Another derivation was subsequently given by Hill [3.29], starting out from the classical extremum principles of par. 3.6.2. The present straightforward derivation follows a method used in [3.30]. As in our derivation of the classical extremum principles in par. 3.6.2 we shall again use finite differences rather than variations of fields.

We define admissible fields $\tilde{u}_i^{(m)}$ and $\tilde{p}_{ij}^{(m)}$ by the requirement that they satisfy (3.7.8) (in each phase) and (3.7.14-15), but not (3.7.9). Thus the last equation will assume the role of extremum condition. Accordingly, we have

$$C^o_{ijkl}\,\tilde{u}'_{k,lj} + \tilde{p}^{(m)}_{ij,j} = (C^o_{ijkl}\,\tilde{\varepsilon}'^{(m)}_{kl} + \tilde{p}^{(m)}_{ij})_{,j} = 0 \text{ in } R_m \qquad \text{(a)}$$

$$\tilde{u}'_i\,(S) = 0 \qquad \text{(b)}$$

$$\left.\begin{array}{l} \tilde{u}'_i \\ (C^o_{ijkl}\,\tilde{u}'_{k,l} + \tilde{p}_{ij})\,n_j \end{array}\right\} \text{ continuous on } S_{int} \qquad \begin{array}{l}\text{(c)}\\ \text{(d)}\end{array} \qquad (3.7.19)$$

where

$$\tilde{\epsilon}'^{(m)}_{ij} = \frac{1}{2}\left(\tilde{u}'^{(m)}_{i,j} + \tilde{u}'^{(m)}_{j,i}\right) \qquad \text{(e)}$$

We define the differences

$$\Delta u'^{(m)}_{i} = \tilde{u}'^{(m)}_{i} - u'^{(m)}_{i}$$

$$\Delta p^{(m)}_{ij} = \tilde{p}^{(m)}_{ij} - p^{(m)}_{ij} \qquad (3.7.20)$$

where $u'^{(m)}_{i}$ and $p^{(m)}_{ij}$ are the actual fields which also satisfy (3.7.9) or (3.7.12). Subtraction of corresponding equations in the group (3.7.8), (3.7.14-15) and the group (3.7.19) yields in view of (3.7.20)

$$C^{o}_{ijkl}\,\Delta u'^{(m)}_{k,lj} + \Delta p^{(m)}_{ij,j} = \left(C^{o}_{ijkl}\Delta\epsilon'^{(m)}_{kl} + \Delta p^{(m)}_{ij}\right)_{,j} = 0 \qquad \text{(a)}$$

$$\Delta u'_{i}\,(S) = 0 \qquad \text{(b)}$$

$$\left.\begin{array}{ll} \Delta u'_{i} & \text{(c)} \\ (C^{o}_{ijkl}\Delta\epsilon'_{kl} + \Delta p_{ij})\,n_{j} & \text{(d)} \end{array}\right\} \text{ continuous on } S_{int} \qquad (3.7.21)$$

where

$$\Delta\epsilon'^{(m)}_{ij} = \frac{1}{2}\left(\Delta u'^{(m)}_{i,j} + \Delta u'^{(m)}_{j,i}\right) \qquad \text{(e)}$$

In view of future derivations it is convenient to rewrite (3.7.19a,d) and (3.7.31a,d) in the following forms

$$C^{o}_{ijkl}\,\tilde{\epsilon}'^{(m)}_{kl} + \tilde{p}_{ij} = t^{(m)}_{ij} \qquad \text{(a)}$$

$$t^{(m)}_{ij,j} = 0 \qquad \text{(b)} \qquad (3.7.22)$$

$$t^{(m)}_{ij}\,n_j \text{ continuous on } S_{int} \qquad \text{(c)}$$

$$C^{o}_{ijkl}\,\Delta\epsilon'^{(m)}_{kl} + \Delta p^{(m)}_{ij} = \Delta t^{(m)}_{ij} \qquad \text{(a)}$$

$$\Delta t^{(m)}_{ij,j} = 0 \qquad \text{(b)} \qquad (3.7.23)$$

$$\Delta t^{(m)}_{ij}\,n_j \text{ continuous on } S_{int} \qquad \text{(c)}$$

Now (3.7.9), written for the mth phase, is multiplied by $\Delta p^{(m)}_{ij}$, then integrated over the mth phase volume and the resulting integrals are summed over all phases. The result is

$$\sum_m \int_{V_m} (-H^{(m)}_{ijkl}\,p^{(m)}_{kl}\,\Delta p^{(m)}_{ij} + \epsilon'^{(m)}_{ij}\,\Delta p^{(m)}_{ij} + \epsilon^{o}_{ij}\,\Delta p^{(m)}_{ij})\,dV = 0 \qquad (3.7.24)$$

Consider the middle integral in (3.7.24) and substitute for $\Delta p^{(m)}_{ij}$ from (3.7.23a). Then

$$\sum_m \int_{V_m} \epsilon'^{(m)}_{ij} \Delta p^{(m)}_{ij} dV = \sum_m \int_{V_m} (\epsilon'^{(m)}_{ij} \Delta t^{(m)}_{ij} - \epsilon'^{(m)}_{ij} C^o_{ijkl} \Delta \epsilon'^{(m)}_{kl}) \, dV \quad \text{(a)}$$

$$= \sum_m \int_{S_m} u'^{(m)}_i \Delta t^{(m)}_{ij} n_j \, dS - \sum_m \int_{V_m} \Delta \epsilon'^{(m)}_{ij} C^o_{ijkl} \epsilon'^{(m)}_{kl} \, dV \quad \text{(b)}$$

$$= \int_S u'_i \Delta t_{ij} n_j \, dS - \sum_m \int_{V_m} \Delta \epsilon'^{(m)}_{ij} (\sigma'^{(m)}_{ij} - p^{(m)}_{ij}) \, dV \quad \text{(c)}$$

(3.7.25)

$$= - \sum_m \int_{S_m} \Delta u'^{(m)}_i \sigma'^{(m)}_{ij} n_j \, dS + \sum_m \int_{V_m} \Delta \epsilon'^{(m)}_{ij} p^{(m)}_{ij} \, dV \quad \text{(d)}$$

$$= - \int_S \Delta u'_i \sigma'_{ij} n_j \, dS + \sum_m \int_{V_m} \Delta \epsilon'^{(m)}_{ij} p^{(m)}_{ij} \, dV \quad \text{(e)}$$

$$= \sum_m \int_{V_m} \Delta \epsilon'^{(m)}_{ij} p^{(m)}_{ij} \, dV \quad \text{(f)}$$

Here the conversion to surface integral in (b) follows from the theorem of virtual work which is valid because of (3.7.21e) and (3.7.23b). The interchange in the volume integral in (b) is permissible because of C^o_{ijkl} symmetry. The interface surface integrals in (b) cancel because of (3.7.15a) and (3.7.23c) and so only the integral over S remains in (c). The volume integral in (c) is obtained by

substituting for $C^{o}_{ijkl}\ \epsilon'^{(m)}_{kl}$ from (3.7.8) in (b). The surface integral in (c) vanishes because of (3.7.14). The first volume integral in (c) is converted to the surface integral in (d) by virtual work in view of (3.7.21e) and (3.7.22b). The interface integrals in (3.7.25d) cancel because of (3.7.21c) and (3.7.22c). Thus there remains the external surface integral in (e) which vanishes because of (3.7.21b).

Define the integrals

$$U' = \frac{1}{2} \sum_m \int_{V_m} \epsilon'^{(m)}_{ij}\, p^{(m)}_{ij}\, dV \qquad \text{(a)}$$

$$\tilde{U}' = \frac{1}{2} \sum_m \int_{V_m} \tilde{\epsilon}'^{(m)}_{ij}\, \tilde{p}^{(m)}_{ij}\, dV \qquad \text{(b)} \qquad (3.7.26)$$

The difference between these integrals is in view of (3.7.20) and (3.7.21e)

$$\Delta U' = \tilde{U}' - U' = \Delta^{(1)} U' + \Delta^{(2)} U' \qquad \text{(a)}$$

$$\Delta^{(1)} U' = \frac{1}{2} \sum_m \int_{V_m} (\Delta \epsilon'^{(m)}_{ij}\, p^{(m)}_{ij} + \epsilon'^{(m)}_{ij}\, \Delta p^{(m)}_{ij})\, dV \qquad \text{(b)}$$

$$(3.7.27)$$

$$\Delta^{(2)} U' = \frac{1}{2} \sum_m \int_{V_m} \Delta \epsilon'^{(m)}_{ij}\, \Delta p^{(m)}_{ij}\, dV \qquad \text{(c)}$$

In view of the equality of the first and last terms in (3.7.25), we have from (3.7.27a)

$$\frac{1}{2}\sum_m \int_{V_m} \epsilon'^{(m)}_{ij}\, \Delta p^{(m)}_{ij}\, dV = \frac{1}{2}\Delta^{(1)}U'$$

Substituting this into (3.7.24) it is seen that the resulting integral can be interpreted as $\Delta^{(1)}Q$ where

$$\Delta Q = \tilde{Q} - Q = \Delta^{(1)}Q + \Delta^{(2)}Q \qquad \text{(a)}$$

$$Q = \frac{1}{2}\sum_m \int_{V_m} (-H^{(m)}_{ijkl}\, p^{(m)}_{ij}\, p^{(m)}_{kl} + p^{(m)}_{ij}\, \epsilon'^{(m)}_{ij} + 2p^{(m)}_{ij}\, \epsilon^{o}_{ij})\, dV \qquad \text{(b)} \qquad (3.7.28)$$

$$\tilde{Q} = \frac{1}{2}\sum_m \int_{V_m} (-H^{(m)}_{ijkl}\, \tilde{p}^{(m)}_{ij}\, \tilde{p}^{(m)}_{kl} + \tilde{p}^{(m)}_{ij}\, \tilde{\epsilon}'^{(m)}_{ij} + 2\tilde{p}^{(m)}_{ij}\, \epsilon^{o}_{ij})\, dV \qquad \text{(c)}$$

and (3.7.11) has been used. So from (3.7.24)

$$\Delta^{(1)}Q = 0$$

For reasons of convenience and without loss of generality the known strain energy U_o^{ϵ} of the C^o body is added to (3.7.27b,c). Thus

$$U = U_o^{\epsilon} + Q = U_o^{\epsilon} - \frac{1}{2}\sum_m \int_{V_m} (H^{(m)}_{ijkl}\, p^{(m)}_{ij}\, p^{(m)}_{kl} - p^{(m)}_{ij}\, \epsilon'^{(m)}_{ij} - 2p^{(m)}_{ij}\, \epsilon^{o}_{ij})\, dV \qquad \text{(a)}$$

$$\tilde{U} = U_o^{\epsilon} + \tilde{Q} = U_o^{\epsilon} - \frac{1}{2}\sum_m \int_{V_m} (H^{(m)}_{ijkl}\, \tilde{p}^{(m)}_{ij}\, \tilde{p}_{kl} - \tilde{p}_{ij}\, \tilde{\epsilon}'^{(m)}_{ij} - 2\tilde{p}^{(m)}_{ij}\, \epsilon^{o}_{ij})\, dV \qquad \text{(b)}$$

$$(3.7.29)$$

$$\Delta U = \tilde{U} - U = \Delta^{(1)}U + \Delta^{(2)}U \qquad \text{(c)}$$

$$\Delta^{(1)}U = 0 \qquad \text{(d)}$$

The expression (3.7.29a) can be greatly simplified. We first observe that it can be written in the form

$$U = U_o^{\epsilon} + \frac{1}{2} \sum_m \int_{V_m} p_{ij}^{(m)} \epsilon_{ij}^{o} \, dV -$$

$$- \frac{1}{2} \sum_m \int_{V_m} (H_{ijkl}^{(m)} p_{kl}^{(m)} - \epsilon'^{(m)}_{ij} - \epsilon_{ij}^{o}) \, p_{ij}^{(m)} \, dV \tag{3.7.30}$$

It is seen that the parenthesis in the last integral in (3.7.30) vanishes in each phase because of (3.7.9). We thus have

$$U = U_o^{\epsilon} + \frac{1}{2} \sum_m \int_{V_m} p_{ij}^{(m)} \epsilon_{ij}^{o} \, dV = U^{\epsilon} \tag{3.7.31}$$

the last equality in (3.7.31) following from (3.7.18). It has thus been shown that U is the strain energy of the C body.

We shall call (3.7.31) and (3.7.29) polarization strain energy and polarization strain energy functional, respectively. So (3.7.29d) asserts that the first difference of the polarization strain energy functional vanishes.

We now examine $\Delta^{(2)}U$ in (3.7.29c). We have from (3.7.29a,b)

$$\Delta^{(2)}U = - \frac{1}{2} \sum_m \int_{V_m} (H_{ijkl}^{(m)} \Delta p_{ij}^{(m)} \Delta p_{kl}^{(m)} - \Delta p_{ij}^{(m)} \Delta \epsilon'^{(m)}_{ij}) \, dV \tag{3.7.32}$$

In order to establish an extremum principle for $\tilde{U}$ we have to examine the sign of (3.7.32). It will be shown that

$$\Delta^{(2)}U \leq 0 \qquad C'^{(m)}_{ijkl} = C^{(m)}_{ijkl} - C^{o}_{ijkl} \quad \text{positive definite} \quad (a)$$

$$(3.7.33)$$

$$\Delta^{(2)}U \geq 0 \qquad C'^{(m)}_{ijkl} = C^{(m)}_{ijkl} - C^{o}_{ijkl} \quad \text{negative definite} \quad (b)$$

To prove (3.7.33a) we substitute for $\Delta p^{(m)}_{ij}$ from (3.7.23a) in the second term in the integrand of (3.7.32), obtaining

$$\Delta^{(2)}U = -\frac{1}{2}\sum_m \int_{V_m} (H^{(m)}_{ijkl}\,\Delta p^{(m)}_{ij}\,\Delta p^{(m)}_{kl} + C^{o}_{ijkl}\,\Delta\epsilon'^{(m)}_{ij}\,\Delta\epsilon'^{(m)}_{kl}$$

$$- \Delta t^{(m)}_{ij}\,\Delta\epsilon'^{(m)}_{ij})dV \qquad (3.7.34)$$

The last integral in (3.7.34) vanishes by virtual work (by use of (3.7.23b,c) and (3.7.21b,e)). It is seen that the middle term in (3.7.34) is positive because C^{o}_{ijkl} is positive definite. The first term in (3.7.34) is positive if $H^{(m)}_{ijkl}$ is positive definite, so in that event (3.7.34) is negative. Now the matrices $H^{(m)}_{ijkl}$ and $C'^{(m)}_{ijkl}$ are inverses of one another, (3.7.13). Accordingly if $H^{(m)}_{ijkl}$ is positive definite so is $C'^{(m)}_{ijkl}$. This proves (3.7.33a).

To prove (3.7.33b) we first consider the integral

$$\delta = \sum_m \int_{V_m} S^{o}_{ijkl}\,\Delta p^{(m)}_{ij}\,\Delta p^{(m)}_{ij}\,\Delta p^{(m)}_{kl}\,dV \qquad (3.7.35)$$

where S^{o}_{ijkl} are the compliances, related to C^{o}_{ijkl} by (3.2.6-7). Again we substitute for $\Delta p^{(m)}_{ij}$, $\Delta p^{(m)}_{kl}$ from (3.7.23a), obtaining

$$\delta = \sum_m \int_{V_m} (S^o_{ijkl}\, \Delta t^{(m)}_{ij}\, \Delta t^{(m)}_{kl} + C^o_{ijkl}\, \Delta \epsilon'^{(m)}_{ij}\, \Delta \epsilon'^{(m)}_{kl}$$

$$- 2\, \Delta t^{(m)}_{ij}\, \Delta \epsilon'^{(m)}_{ij}\,)\, dV \qquad (3.7.36)$$

The last integral in (3.7.36) again vanishes by virtual work and it is seen that the remaining two terms in (3.7.36) are each positive definite. Therefore, if the first term in (3.7.36) is omitted the whole integral is decreased and thus becomes smaller than (3.7.35). Accordingly

$$\sum_m \int_{V_m} S^o_{ijkl}\, \Delta p^{(m)}_{ij}\, \Delta p^{(m)}_{kl}\, dV \geq \sum_m \int_{V_m} C^o_{ijkl}\, \Delta \epsilon'^{(m)}_{ij}\, \Delta \epsilon'^{(m)}_{kl}\, dV$$

Using this result in (3.7.34) we have

$$\Delta^{(2)} U \leq -\frac{1}{2} \sum_m \int_{V_m} (H^{(m)}_{ijkl} + S^o_{ijkl})\, \Delta p^{(m)}_{ij}\, \Delta p^{(m)}_{kl}\, dV \qquad (3.7.37)$$

To ensure that $\Delta^{(2)} U$ be positive we require that $H^{(m)}_{ijkl} + S^o_{ijkl}$ be negative definite. To show that this condition is equivalent to (3.7.33b) it is best to operate with matrices in symbolic form. Let the matrix of $H^{(m)}_{ijkl} + S^o_{ijkl}$ be written

$$\underline{\mathbf{A}} = \underline{H}^{(m)} + \underline{S}^o$$

It follows from (3.7.10), (3.7.13) and the definition of compliance that

$$\underline{A} = \frac{\underline{I}}{\underline{C}^{(m)} - \underline{C}^o} + \frac{\underline{I}}{\underline{C}^o}$$

Then

$$(\underline{C}^{(m)} - \underline{C}^o) \cdot \underline{C}^o \cdot \underline{A} = \underline{C}^{(m)}$$

Now since $\underline{C}^{(m)}$ and $\underline{C}^{o}$ are positive definite it follows that if $\underline{A}$ is negative definite then $\underline{C}^{(m)} - \underline{C}^{o}$ is also negative definite and vice versa. Therefore, if $\underline{C}^{(m)} - \underline{C}^{o}$ is negative definite the right side of (3.7.37) is positive. This proves (3.7.33b).

In view of (3.7.29c,d), (3.7.31) and (3.7.33) we have the following extremum principles

$$\tilde{U} \leq U^{\epsilon} \qquad C'^{(m)}_{ijkl} = C^{(m)}_{ijkl} - C^{o}_{ijkl} \text{ positive definite} \quad \text{(a)}$$

$$\tilde{U} \geq U^{\epsilon} \qquad C'^{(m)}_{ijkl} = C^{(m)}_{ijkl} - C^{o}_{ijkl} \text{ negative definite} \quad \text{(b)} \qquad (3.7.38)$$

In words: <u>The strain energy of the C body is the maximum/minimum of the polarization strain energy functional when the elastic moduli difference matrix $\underline{C}'^{(m)}_{ijkl}$ is everywhere positive/negative definite.</u>

The bounding of the strain energy U^{ϵ} and of EEM by the present extremum principle proceeds as follows: An admissible polarization tensor $\tilde{p}^{(m)}_{ij}$ is chosen at will. Then $p^{(m)}_{ij,j}$ may be regarded as the "input" in (3.7.19) and $\tilde{u}'^{(m)}_{i}$ as unknown functions ("output") to be determined. It is seen that (3.7.19) is an unusual elasticity problem for the <u>homogeneous</u> C^{o} body in which $\tilde{p}^{(m)}_{ij,j}$ are "body forces" and the "traction" $C^{o}_{ijkl}\,\tilde{u}'^{(m)}_{kl}\,n_j$ in (3.7.19d) is discontinuous at phase interfaces, if $\tilde{p}_{ij}\,n_j$ is discontinuous there.

It is easily shown that for any choice $\tilde{p}_{ij}$ there is a unique $\tilde{u}'_i$. For assume that there are two different $\tilde{u}'_i$ satisfying (3.7.19) with same $\tilde{p}_{ij}$. Then by subtraction and linearity the difference between these two $\tilde{u}_i$ satisfies (3.7.19) with zero $\tilde{p}_{ij}$. But this is a usual elasticity problem with zero boundary

values and zero body forces and, therefore, its solution vanishes. So the assumedly different $\tilde{u}'_i$ are the same.

Once $\tilde{u}'_i$ has been determined in terms of $\tilde{p}_{ij}$ the integral (3.7.26b) can be computed and is then carried back into (3.7.29b) and so $\tilde{U}$ provides bounds on U^{ϵ} because of (3.7.38). If the boundary conditions (3.7.1d), (3.7.2f) are homogeneous of form (3.3.1), then U^{ϵ} is expressible in terms of EEM and ϵ^{o}_{ij}, (3.3.14), and thus bounds on the EEM in terms of $\tilde{p}_{ij}$ and C^{o}_{ijkl} may be obtained. The $\tilde{p}_{ij}$ and C^{o}_{ijkl} are then determined by optimization of the bounds.

This procedure will be illustrated in the next paragraph for fibrous materials.

A dual set of extremum principles in terms of strain polarization has been derived in [3.28]. Additional extremum principles in terms of the polarization tensor have been derived in [3.30].

The extremum principles which were derived are now resummarized: An elastic homogeneous (C^{o}) body with elastic moduli C^{o}_{ijkl} and an elastic heterogeneous body (C) body with phase elastic moduli $C^{(m)}_{ijkl}$ are of identical external shapes and are subjected to the same boundary displacements on their entire bounding surfaces. Define the functional

$$\tilde{U} = U_o^{\epsilon} - \frac{1}{2} \sum_m \int_{V_m} (H^{(m)}_{ijkl} \tilde{p}^{(m)}_{ij} \tilde{p}^{(m)}_{kl} - \tilde{p}^{(m)}_{ij} \tilde{\epsilon}'^{(m)}_{ij} - 2\tilde{p}^{(m)}_{ij} \epsilon^{o}_{ij})\, dV \qquad (3.7.29b)$$

where

U_o^{ϵ} - strain energy of the C^o body

ϵ_{ij}^o - strain field in the C^o body

$$H_{ijrs}^{(m)} (C_{rskl}^{(m)} - C_{rskl}^o) = I_{ijkl} \tag{3.7.13}$$

$$\tilde{\epsilon}'^{(m)}_{ij} = \frac{1}{2} (\tilde{u}'^{(m)}_{i,j} + \tilde{u}'^{(m)}_{j,i}) \tag{3.7.19e}$$

The admissible fields $\tilde{p}_{ij}^{(m)}$ and $\tilde{u}'^{(m)}_i$ are related by the boundary value problem

$$\begin{aligned} &C^o_{ijkl} \tilde{u}'^{(m)}_{k,lj} + \tilde{p}^{(m)}_{ij,j} = 0 \quad \text{in } R_m && \text{(a)} \\ &\tilde{u}_i(S) = 0 && \text{(b)} \\ &\left.\begin{aligned} &\tilde{u}'_i && \text{(c)} \\ &(C^o_{ijkl} \tilde{u}'_{k,l} + \tilde{p}_{ij}) n_j && \text{(d)} \end{aligned}\right\} \text{continuous at } S_{int} \end{aligned} \tag{3.7.19}$$

Let U^{ϵ} be the strain energy of the C body. Then

$$\begin{aligned} &\tilde{U} \le U^{\epsilon} \quad \text{if} \quad C^{(m)}_{ijkl} - C^o_{ijkl} \quad \text{positive definite} && \text{(a)} \\ &\tilde{U} \ge U^{\epsilon} \quad \text{if} \quad C^{(m)}_{ijkl} - C^o_{ijkl} \quad \text{negative definite} && \text{(b)} \end{aligned} \tag{3.7.38}$$

Equality occurring if and only if $\tilde{p}_{ij}^{(m)}$ is the actual polarization $p_{ij}^{(m)}$ and $\tilde{u}'^{(m)}_i$ is the actual displacement difference $u'^{(m)}_i$, between the displacement fields in the C and C^o bodies.

3.7.3 Bounds for Fibrous Materials

We now proceed to use the polarization extremum principle to bound the EEM of macroscopically transversely isotropic FM. We shall first establish bounds for k^* and G_T^*. Bounds for E_A^* and ν_A^* are then easily constructed, in terms of the k^* bounds, by use of (3.6.66), (3.6.68), (3.4.117) and (3.4.118).

A separate treatment is necessary to establish G_A^* bounds.

We choose as the C^o body a transversely isotropic homogeneous cylinder of volume V, surface S and section A. This cylinder is henceforth called the C^o cylinder. The axis of elastic symmetry of the material is in generator direction and the elastic moduli are n_o, ℓ_o, k_o, G_{To}, G_{Ao} (see (3.4.86)).

We choose as the C body an externally identical cylinder of fibrous material, which is assumed to be macroscopically transversely isotropic with elastic symmetry axis in generator direction. This cylinder is called the C cylinder. We shall assume for simplicity that the phases are isotropic, bearing in mind that results for transversely isotropic phases are immediately obtainable from the isotropic phase results by use of the transverse-isotropy analogies which were described in par. 3.5.1. The variable moduli of the C cylinder are taken as $k=\lambda+G$ and G assuming the values k_m and G_m in the mth phase. The volume fraction of the mth phase is denoted v_m.

We are primarily interested in the two phase cylinder, but since the major part of the analysis is just as conveniently carried out for a multiphase cylinder we do not at present specify the number of phases.

Let the C^o and C cylinders be subjected to the homogeneous displacement boundary conditions

$$u_1^S = 0 \qquad \text{(a)}$$

$$u_2^S = \epsilon_{22}^o x_2 + \epsilon_{23}^o x_3 \qquad \text{(b)} \qquad (3.7.39)$$

$$u_3^S = \epsilon_{23}^o x_2 + \epsilon_{33}^o x_3 \qquad \text{(c)}$$

It is seen that (3.7.39) is a special case of (3.5.3). In view of (3.7.39a) both cylinders are in states of plane strain.

The fields u_i^o, ϵ_{ij}^o and σ_{ij}^o in the C^o cylinder can immediately be written down since $u_i^o(\underline{x})$ has the form (3.7.39) throughout the cylinder. Consequently ϵ_{ij}^o and σ_{ij}^o are homogeneous. It is convenient to separate strains and stresses in the x_2 x_3 plane into isotropic and deviatoric parts as was done in (3.4.69) and (3.4.71) for averages. We then have

$$\epsilon_{\alpha\beta}^o(\underline{x}) = \epsilon_{\alpha\beta}^o = \epsilon^o \delta_{\alpha\beta} + e_{\alpha\beta}^o \qquad \text{(a)}$$

$$\epsilon^o = \frac{1}{2}\epsilon_{\gamma\gamma}^o = \frac{1}{2}(\epsilon_{22}^o + \epsilon_{33}^o) \qquad \text{(b)} \qquad (3.7.40)$$

$$\epsilon_{11}^o = \epsilon_{12}^o = \epsilon_{13}^o = 0 \qquad \text{(c)}$$

$$\sigma^{o}_{\alpha\beta}(\underline{x}) = \sigma^{o}_{\alpha\beta} = \sigma^{o}\delta_{\alpha\beta} + s^{o}_{\alpha\beta} \qquad \text{(a)}$$

$$\sigma^{o} = \frac{1}{2}\sigma^{o}_{\gamma\gamma} \qquad \text{(b)}$$

$$\sigma^{o} = 2k_{o}\,\epsilon^{o} \qquad s^{o}_{\alpha\beta} = 2G_{o}\,e^{o}_{\alpha\beta} \qquad \text{(c)} \qquad (3.7.41)$$

$$\sigma^{o}_{11} = 2\ell_{o}\,\epsilon^{o} \qquad \sigma^{o}_{12} = \sigma^{o}_{13} = 0 \qquad \text{(d)}$$

$$\alpha, \beta, \gamma = 2, 3$$

In view of (3.6.44a) and (3.7.40-41) the strain energy U^{ϵ}_{o} stored in the C^{o} cylinder is

$$U^{\epsilon}_{o} = (2k_{o}\,\epsilon^{o^2} + G_{o}\,e^{o}_{\alpha\beta}\,e^{o}_{\alpha\beta})\,A \qquad (3.7.42)$$

per unit height of cylinder, where A is the section area and G_o is a simplified temporary notation for G_{To}.

The fields u_α and $\sigma_{\alpha\beta}$ in the C cylinder subjected to (3.7.39) are determined by the formulation given in par. 3.5.1 which reduces here to a plane strain problem. It follows from (3.5.9-12) that

$$u_1 = 0 \qquad u_\alpha = u_\alpha\,(x_2, x_3) \qquad \text{(a)}$$

$$\epsilon_{11} = \epsilon_{12} = \epsilon_{13} = 0 \quad , \quad \epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}(x_2, x_3) \qquad \text{(b)} \qquad (3.7.43)$$

$$\sigma_{12} = \sigma_{13} = 0 \quad , \quad \sigma_{11} \neq 0 \quad , \quad \sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_2, x_3) \qquad \text{(c)}$$

We consider (3.7.9) for the present case. Here C_{ijkl} represent the local phase moduli in the C cylinder and C^{o}_{ijkl} - the phase moduli of the transversely isotropic C^{o} cylinder. Consequently (3.7.9) has the form of a transversely isotropic stress-strain relation (see 3.4.86). In view of (3.7.43) we have

$$p_{11} = \ell' (\epsilon_{22} + \epsilon_{33}) \qquad \text{(a)}$$

$$p_{22} = (k' + G') \epsilon_{22} + (k' - G') \epsilon_{33} \qquad \text{(b)}$$

$$p_{33} = (k' - G') \epsilon_{22} + (k' + G') \epsilon_{33} \qquad \text{(c)} \qquad (3.7.44)$$

$$p_{23} = 2G' \epsilon_{23} \qquad \text{(d)}$$

$$p_{12} = p_{13} = 0 \qquad \text{(e)}$$

where

$$\ell' = \ell - \ell_o = \lambda - \ell_o \qquad \text{(a)}$$

$$k' = k - k_o = \lambda + G - k_o \qquad \text{(b)} \qquad (3.7.45)$$

$$G' = G - G_o = G - G_{To} \qquad \text{(c)}$$

In view of (3.7.43b) the p_{ij} are not functions of x_1.

It is convenient to separate the polarization and strain components in (3.7.44b,c,d) into isotropic and deviatoric parts. Thus

$$p_{\alpha\beta} = p\,\delta_{\alpha\beta} + q_{\alpha\beta} \qquad \text{(a)}$$

$$p = \frac{1}{2}\, p_{\gamma\gamma} \qquad \text{(b)} \qquad (3.7.46)$$

$$\epsilon_{\alpha\beta} = \epsilon\,\delta_{\alpha\beta} + e_{\alpha\beta} \qquad \text{(a)}$$

$$\epsilon = \frac{1}{2}\, \epsilon_{\gamma\gamma} \qquad \text{(b)} \qquad (3.7.47)$$

Then (3.7.44b,c,d) assumes the same form as (3.7.41c)

$$p = 2k'\,\epsilon \qquad \text{(a)}$$

$$q_{\alpha\beta} = 2G'\,e_{\alpha\beta} \qquad \text{(b)} \qquad (3.7.48)$$

We now proceed to evaluate a functional of type (3.7.29). In view of (3.7.9b) it is seen that

$$H_{ijkl}\, p_{ij}\, p_{kl} = p_{ij}\, \epsilon_{ij}$$

Because of (3.7.43b) and (3.7.44e) the summation extends only over 2,3 and thus reduces to $p_{\alpha\beta}\, \epsilon_{\alpha\beta}$. Using (3.7.48) we have

$$H_{ijkl}\, p_{ij}\, p_{kl} = \frac{p^2}{k'} + \frac{q_{\alpha\beta}\, q_{\alpha\beta}}{2G'} \qquad (3.7.49)$$

From (3.7.40c), (3.7.43) and (3.7.4) we have that

$$p_{ij}\, \epsilon'_{ij} = p_{\alpha\beta}\, \epsilon'_{\alpha\beta}$$

$$p_{ij}\, \overset{o}{\epsilon}_{ij} = p_{\alpha\beta}\, \overset{o}{\epsilon}_{\alpha\beta}$$

Again we separate into isotropic and deviatoric parts to find the expressions

$$p_{ij}\,\epsilon'_{ij} = 2p\epsilon' + q_{\alpha\beta}\,e'_{\alpha\beta}$$

$$p_{ij}\,\epsilon^{o}_{ij} = 2p\epsilon^{o} + q_{\alpha\beta}\,e^{o}_{\alpha\beta} \tag{3.7.50}$$

We now use (3.7.49-50) to write down the functional (3.7.29a). It is seen that the integrands are functions of x_2, x_3 only. Therefore, volume integration can be replaced by area integration over the section A. We have

$$U = U^{\epsilon} = U_o^{\epsilon} - \frac{1}{2}\sum_m \int_{A_m} \left(\frac{p^{(m)2}}{k'_m} + \frac{q^{(m)}_{\alpha\beta}\,q^{(m)}_{\alpha\beta}}{2G'_m} \right.$$

$$\left. - 2p^{(m)}\,\epsilon'^{(m)} - q^{(m)}_{\alpha\beta}\,e'^{(m)}_{\alpha\beta} - 4p^{(m)}\epsilon^{o} - 2q^{(m)}_{\alpha\beta}\,e^{o}_{\alpha\beta} \right) dA \tag{3.7.51}$$

where U and U_o^{ϵ} are strain energies per unit height of C and C^o cylinders, respectively, A_m are the phase areas, and k'_m and G'_m are (3.7.45b,c) for $k = k_m$ and $G = G_m$, respectively. Because of (3.7.39), (3.7.40), (3.4.70) and (3.3.26) the strain energy per unit height of C cylinder can also be written as

$$U^{\epsilon} = (2k^{*}\,\epsilon^{o2} + G^{*}_{T}\,e^{o}_{\alpha\beta}\,e^{o}_{\alpha\beta})\,A \tag{3.7.52}$$

The EEM k^{*} and G^{*}_{T} are the quantities which are to be bounded and for that purpose we shall bound (3.7.51) by use of admissible fields.

It is recalled that admissible polarizations $\tilde{p}_{ij}$ and admissible displacement deviations $\tilde{u}'_i$ must satisfy (3.7.19). In order to incorporate as many features of the actual fields as possible in the admissible fields we choose

$$\tilde{p}_{12} = \tilde{p}_{13} = 0 \qquad \tilde{p}_{11} = \tilde{p}_{11}\,(x_2, x_3) \qquad \tilde{p}_{\alpha\beta} = \tilde{p}_{\alpha\beta}\,(x_2, x_3) \tag{3.7.53}$$

$$\tilde{u}'_1 = 0 \qquad \tilde{u}'_\alpha = \tilde{u}'_\alpha\,(x_2, x_3)$$

Since C^o_{ijkl} is, by hypothesis, transversely isotropic and because of (3.7.53), (3.7.19a) assumes the form

$$k_o\,\tilde{u}'^{(m)}_{\beta,\beta\alpha} + G_o\,\tilde{u}'^{(m)}_{\alpha,\beta\beta} + \tilde{p}^{(m)}_{\alpha\beta,\beta} = 0 \tag{3.7.54}$$

in each phase. The $\tilde{u}_i$ differential expression in (3.7.54) has the same form as (3.5.14), for obvious reasons.

The rest of (3.7.19) assume the form

$$\tilde{u}'_\alpha = 0 \qquad \text{on } C \qquad \text{(a)}$$

$$\left.\begin{array}{ll} \tilde{u}'_\alpha & \text{(b)} \\ (k_o - G_o)\,\tilde{u}'_{\beta,\beta}\,n_\alpha + G_o\,(\tilde{u}'_{\alpha,\beta} + \tilde{u}'_{\beta,\alpha})\,n_\beta + \tilde{p}_{\alpha\beta} & \text{(c)} \end{array}\right\} \text{continuous on } C_{int} \tag{3.7.55}$$

It is seen that (3.7.54-55) is similar to the plane strain formulation of par. 3.5.1. In the present case there are "body forces" $\tilde{p}_{\alpha\beta,\beta}$. Equ. (3.7.55c) replaces traction continuity (3.5.19) . It is seen that the first two terms in

(3.7.55c) have the form of a traction expressed in terms of $\tilde{u}_i'$ displacement gradients. In general, $\tilde{p}_{\alpha\beta} n_\beta$ is not by itself continuous at the interface and, therefore, the "traction" is also discontinuous.

The functional (3.7.29b) can now be written down at once by replacement of actual fields in (3.7.51) by admissible fields (compare (3.7.29a) and (3.7.29b)). We have

$$\tilde{U} = U_o^{\epsilon} - \frac{1}{2} \sum_m \int_{A_m} \left(\frac{\tilde{p}^{(m)2}}{k'_m} + \frac{\tilde{q}^{(m)}_{\alpha\beta} \tilde{q}^{(m)}_{\alpha\beta}}{2G'_m} - 4\tilde{p}^{(m)} \epsilon^o \right.$$

$$\left. - 2\tilde{q}^{(m)}_{\alpha\beta} e^o_{\alpha\beta} \right) dA + \tilde{U}' \qquad \text{(a)}$$

$$\tilde{U}' = \frac{1}{2} \sum_m \int_{A_m} (2\tilde{p}^{(m)} \tilde{\epsilon}'^{(m)} + \tilde{q}_{\alpha\beta} \tilde{e}'^{(m)}_{\alpha\beta}) \, dA = \frac{1}{2} \sum_m \int_{A_m} \tilde{p}^{(m)}_{\alpha\beta} \tilde{\epsilon}'^{(m)}_{\alpha\beta} \, dA \qquad \text{(b)} \tag{3.7.56}$$

We now choose $\tilde{p}^{(m)}_{\alpha\beta}$ to have different <u>constant</u> values in each phase. Thus, $\tilde{p}_{\alpha\beta}$ is piecewise constant in the C cylinder. Then all terms in (3.7.56) except $\tilde{U}'$ can be readily computed. Assuming now that there are <u>two phases</u> we have

$$\tilde{U} = (2k_o \epsilon^{o^2} + G_o e^o_{\alpha\beta} e^o_{\alpha\beta}) A -$$

$$- \frac{1}{2} \left[\frac{\tilde{p}^{(1)2} v_1}{k_1 - k_o} + \frac{\tilde{p}^{(2)2} v_2}{k_2 - k_o} + \frac{\tilde{q}^{(1)}_{\alpha\beta} \tilde{q}^{(1)}_{\alpha\beta} v_1}{2(G_1 - G_o)} + \frac{\tilde{q}^{(2)}_{\alpha\beta} \tilde{q}^{(2)}_{\alpha\beta} v_2}{2(G_2 - G_o)} \right.$$

$$\left. - 4\bar{\tilde{p}} \epsilon^o - 2\bar{\tilde{q}}_{\alpha\beta} e^o_{\alpha\beta} \right] A + \tilde{U}' \qquad \text{(a)}$$

$$\bar{\tilde{p}} = \tilde{p}^{(1)} v_1 + \tilde{p}^{(2)} v_2 \qquad \text{(b)} \qquad (3.7.57)$$

$$\bar{\tilde{q}}_{\alpha\beta} = \tilde{q}^{(1)}_{\alpha\beta} v_1 + \tilde{q}^{(2)}_{\alpha\beta} v_2 \qquad \text{(c)}$$

The computation of $\tilde{U}'$ via the boundary value problem (3.7.54-55) for the present choice of $\tilde{p}_{\alpha\beta}$ and two phases is given in the appendix to this chapter. The result is

$$2\tilde{U}' = [a_o(\tilde{p}^{(2)} - \tilde{p}^{(1)})^2 + b_o(\tilde{q}^{(2)}_{\alpha\beta} - \tilde{q}^{(1)}_{\alpha\beta})(\tilde{q}^{(2)}_{\alpha\beta} - \tilde{q}^{(1)}_{\alpha\beta})] v_1 v_2 A \qquad \text{(a)}$$

$$a_o = -\frac{1}{k_o + G_o} \qquad \text{(b)} \qquad (3.7.58)$$

$$b_o = -\frac{k_o + 2G_o}{4G_o(k_o + G_o)} \qquad \text{(c)}$$

It is seen that (3.7.57-58) provide an expression for $\tilde{U}$ in terms of arbitrary piecewise constant polarization components and arbitrary elastic moduli of the C^o cylinder.

In order to examine the present meaning of the extremum conditions (3.7.38) we realize that the matrix of $C'_{ijkl} = C_{ijkl} - C^o_{ijkl}$ is now given by the moduli differences in (3.7.44). To examine positive or negative definiteness we form a strain energy type expression $C'_{ijkl}\,\epsilon_{ij}\,\epsilon_{kl}$ and examine its sign. Recalling that $\epsilon_{11} = 0$ we have from (3.7.44) and separation of $\epsilon_{\alpha\beta}$ into isotropic and deviatoric parts

$$C'_{ijkl}\,\epsilon_{ij}\,\epsilon_{kl} = 2\,(2k'\,\epsilon^2 + G'\,e_{\alpha\beta}\,e_{\alpha\beta})$$

Since all the strains appear squared, positive definiteness is ensured if, and only if, k' and G' are non-negative while negative definiteness is ensured if, and only if, k' and G' are non-positive. Recalling the definition of k' and G' by (3.7.45b,c) we have from (3.7.38)

$$\tilde{U} \geq U^{\epsilon} \qquad k_1, k_2 \geq k_o \qquad G_1, G_2 \geq G_o \qquad \text{(a)}$$

$$\tilde{U} \leq U^{\epsilon} \qquad k_1, k_2 \leq k_o \qquad G_1, G_2 \leq G_o \qquad \text{(b)} \qquad (3.7.59)$$

To find the best bound on U^{ϵ} with the present polarization choice we minimize $\tilde{U}$ as given by (3.7.57-58) with respect to $\tilde{p}^{(m)}$ and $\tilde{q}^{(m)}_{\alpha\beta}$ when (3.7.59) is fulfilled and we maximize when (3.7.59b) is fulfilled. Both extremum conditions are thus found by setting the derivatives of (3.7.57-58) with respect to polarization components equal to zero. Thus, the common extremum condition is

$$-\frac{\tilde{p}^{(m)}}{k_m - k_o} + a_o\,(\tilde{p}^{(m)} - \bar{\tilde{p}}) + 2\epsilon^o = 0 \qquad \text{(a)}$$

$$-\frac{\tilde{q}^{(m)}_{\alpha\beta}}{2\,(G_m - G_o)} + b_o\,(\tilde{q}^{(m)}_{\alpha\beta} - \bar{\tilde{q}}_{\alpha\beta}) + e^o_{\alpha\beta} = 0 \qquad \text{(b)} \qquad (3.7.60)$$

$$m = 1, 2$$

Computation of second derivatives of (3.7.57-58) easily shows that (3.7.60) are maximum conditions for (3.7.59a) and minimum conditions for (3.7.59b).

We carry (3.7.60) back into (3.7.57-58) and obtain the simple expression

$$\tilde{U}_{ext.} = U_o^{\epsilon} + \frac{1}{2}\left(2\bar{\tilde{p}}\,\epsilon^o + \bar{\tilde{q}}_{\alpha\beta}\, e^o_{\alpha\beta}\right) A \tag{3.7.61}$$

where $\bar{\tilde{p}}$ and $\bar{\tilde{q}}_{\alpha\beta}$ are the averages (3.7.57b,c) of the polarization components defined by (3.7.60). Averaging of (3.7.60) easily gives expressions of the averages in (3.7.61). We have

$$\bar{\tilde{p}} = \frac{2\mathcal{A}\,\epsilon^o}{1+a_o\mathcal{A}} \qquad \text{(a)}$$

$$\bar{\tilde{q}}_{\alpha\beta} = \frac{\mathcal{B}\,e^o_{\alpha\beta}}{1+b_o\mathcal{B}} \qquad \text{(b)}$$

$$\mathcal{A} = \sum_{m=1,2} \frac{v_m}{\frac{1}{k_m-k_o} - a_o} \qquad \text{(c)} \tag{3.7.62}$$

$$\mathcal{B} = \sum_{m=1,2} \frac{v_m}{\frac{1}{G_m-G_o} - b_o} \qquad \text{(d)}$$

Introduction of (3.7.6c) and (3.7.42) into (3.7.61) yields

$$\tilde{U}_{ext.} = 2\left(k_o + \frac{\mathcal{A}}{1+a_o\mathcal{A}}\right)\epsilon^{o^2} + \left(G_o + \frac{1}{2}\frac{\mathcal{B}}{1+b_o\mathcal{B}}\right)e^o_{\alpha\beta}\, e^o_{\alpha\beta}\ A \qquad (3.7.63)$$

We now insert (3.7.52) and (3.7.63) into (3.7.59). Since $\epsilon^o_{\alpha,\beta}$ can always be chosen as purely isotropic (in which case $e^o_{\alpha\beta} = 0$) or as purely deviatoric (in which case $\epsilon^o = 0$) we have

$$k^* \gtreqless k_o + \frac{\mathcal{A}}{1+a_o\mathcal{A}} \qquad \text{(a)}$$

$$G^* \gtreqless G_o + \frac{1}{2}\frac{\mathcal{B}}{1+b_o\mathcal{B}} \qquad \text{(b)} \qquad (3.7.64)$$

where the upper inequality signs apply for (3.7.59a) and the lower inequality signs apply for (3.7.59b). We now choose the moduli k_o, G_o so as to get the best bounds from (3.7.64). Let the right sides of (3.7.64a,b) be denoted $F_k(k_o, G_o)$ and $F_G(k_o, G_o)$ respectively. If these functions are explicitly written out by use of (3.7.58b,c) and (3.7.62b,d), it is easily shown by computation of first derivatives with respect to k_o and G_o that F_k and F_G are monotonically increasing functions of k_o and G_o. Therefore, to obtain the best lower bounds in (3.7.64) we have to choose the largest phase moduli which comply with (3.7.59a). Assume that

$$k_2 > k_1 \qquad\qquad G_2 > G_1 \qquad (3.7.65)$$

Then the best choice (3.7.59a) is

$$k_o = k_1 \qquad\qquad G_o = G_1 \tag{3.7.66}$$

Similarly, to obtain the best upper bounds from (3.7.64) we have to choose the smallest k_o, G_o which comply with (3.7.59b). This choice is

$$k_o = k_2 \qquad\qquad G_o = G_2 \tag{3.7.67}$$

Insertion of the conditions (3.7.66-67) into (3.7.64) yields the following bounds

$$\boxed{\begin{aligned}
k^*_{(-)} &= k_1 + \frac{v_2}{\dfrac{1}{k_2-k_1} + \dfrac{v_2}{k_1+G_1}} && \text{(a)} \\
k^*_{(+)} &= k_2 + \frac{v_1}{\dfrac{1}{k_1-k_2} + \dfrac{v_2}{k_2+G_2}} && \text{(b)} \\
G^*_{(-)} = G^*_{T(-)} &= G_1 + \frac{v_2}{\dfrac{1}{G_2-G_1} + \dfrac{(k_1+2G_1)v_1}{2G_1(k_1+G_1)}} && \text{(a)} \\
G^*_{(+)} = G^*_{T(+)} &= G_2 + \frac{v_1}{\dfrac{1}{G_1-G_2} + \dfrac{(k_2+2G_2)v_2}{2G_2(k_2+G_2)}} && \text{(b)}
\end{aligned}} \qquad \begin{matrix} (3.7.68) \\ \\ (3.7.69) \end{matrix}$$

It should be recalled that the bounds are based on the assumption (3.7.65). If the inequalities (3.7.65) reverse then obviously the upper bounds (3.7.68-69) become the lower bounds and vice versa, since the phases appear in the bounds in a completely unbiased manner.

The present treatment does not permit the cases

$$k_2 > k_1 \qquad G_2 < G_1$$

$$k_2 < k_1 \qquad G_2 > G_1$$

Such cases can be taken into account by a method given by Walpole [3.31], but they are hardly of practical interest since materials whose elastic properties are subject to such inequalities do not seem to exist.

It should be recalled that if the phases are isotropic then k in (3.7.68-69) is the plane strain bulk modulus as expressed by (3.4.89c) and G is the isotropic shear modulus. If the phases are transversely isotropic then k is the transverse bulk modulus and G is G_T, the transverse shear modulus.

We now recall the expression for k^*, of a composite cylinder assemblage and we observe the remarkable fact that (3.5.91) and (3.7.68a) are identical. Since (3.7.68b) may be obtained from (3.7.68a) by interchanging 1 with 2 we conclude that (3.7.68b) can also be interpreted as k^* of another, reversed, composite cylinder assemblage in which the fibers are of material 1 with volume fraction v_1 and the matrix is of material 2 with volume fraction v_2.

Two composite cylinders belonging to the two different assemblages are shown schematically in fig. 3.7.1. This identification of the bounds with composite cylinder assemblage results leads to some very important conclusions as will be now explained.

It is to be noted that (3.7.68a) is a lower bound for any transversely isotropic phase geometry, thus it is also in particular a lower bound for the composite cylinder assemblage. Let it be assumed that there exists a better general bound, i.e. higher, than (3.7.68a) in terms of volume fractions only. But it is clearly seen that this is impossible since k_c^* would be below it. It is, therefore, concluded that (3.7.68a) is the <u>best possible lower bound in terms of volume fractions</u>. It is similarly shown by identification of (3.7.68b) with the second, reversed, composite cylinder assemblage result that (3.7.68b) is the <u>best possible upper bound in terms of volume fractions</u>. The bounds (3.7.68) are called in short <u>best possible</u>.

It is thus seen that if in a transversely isotropic two phase FM the phase moduli and only the phase volume fractions are known, then this information has been exploited to the fullest extent by the bounds (3.7.68). Additional geometrical information is needed to improve the bounds.

This result has an important connection with statistical geometry considerations. It has been shown in chap. 2.2 that for a statistically

homogeneous body the volume fractions are one point probabilities. Thus, the present bounds may be considered as the fullest exploitation possible of this simple geometrical information.

It is of interest to note that if the phase shear moduli are equal while the transverse bulk moduli remain unequal the bounds (3.7.68) coincide. This is most easily seen by writing these bounds in the form (3.5.91a). We thus have the <u>exact result for arbitrary phase geometry</u>

$$k^* = k_1 + \frac{v_2}{\dfrac{1}{k_2-k_1} + \dfrac{v_1}{k_1+G}} = \frac{k_2(k_2+G)\,v_1 + k_2(k_1+G)\,v_2}{(k_2+G)\,v_1 + (k_1+G)\,v_2} \qquad \text{(a)}$$

(3.7.70)

if

$$G_2 = G_1 = G \qquad\qquad k_2 \neq k_1 \qquad \text{(b)}$$

This has been first shown by Hill [3.5] on the basis of a direct exact field solution for the case (3.7.70b). The result (3.7.70) can be added to the results of par. 3.5.2 as a FM exact solution for special relations between phase moduli. Unfortunately, the condition (3.7.70b) is not of practical interest.

At this time it is not known if (3.7.69) are best possible bounds in terms of volume fractions, since it has not been possible to identify these bounds with exact solutions for special phase geometry.

It is of some interest to note that for $v_2 << 1$, $v_1 \cong 1$, the bound (3.7.69a) coincides with the dilute reinforcement result (3.5.132). For $v_2 \cong 1$, $v_1 << 1$

the upper bound (3.7.69b) coincides with another dilute reinforcement result, for a small amount of circular fibers of material 1 which are imbedded in a matrix of material 2. Thus, the lower bound is best possible for very small v_2 while the upper bound is best possible for v_2 very close to unity.

It is easily shown that the bounds (3.7.68-69) are always closer together than the elementary bounds (3.6.63b), (3.6.65). As will be seen later, from numerical results, these new bounds are a substantial improvement of the elementary bounds. However, from a practical point of view, the bounds (3.7.68-69) by no means solve the problem of determination of k^* and G_T^*, for the margin between the bounds (3.6.68-69) increases with relative stiffness ratios of the phases, i.e. with the ratios k_2/k_1 and G_2/G_1. For elevated values of these ratios the bounds may become too far apart to provide good estimates on the EEM. This situation is not surprising for it should be recalled that the bounds are very general results for transversely isotropic fibrous materials, in which only the volume fractions are specified and the geometry is otherwise arbitrary. If phase 2 is very stiff and phase 1 is very compliant the bounds must in particular apply to the cases of (a) 2 - fibers, 1 - matrix (b) 1 - fibers, 2 - matrix. It is evident that FRM (b) is much stiffer than FRM (a) and, therefore, the bounds must be far apart.

This situation is also illustrated by the extreme cases of one rigid phase or one empty (cylindrical voids) phase. We have

<u>phase 2, rigid</u>

$$k_2/k_1 \to \infty, \; G_2/G_1 \to \infty \qquad \text{(a)}$$

$$k^*_{(-)} = k_1 + (k_1 + G_1)\,\frac{v_2}{v_1} \qquad \text{(b)}$$

$$G^*_{(-)} = G_1 \left[1 + \frac{2(k_1+G_1)}{k_1+2G_1}\;\frac{v_2}{v_1} \right] \qquad \text{(c)} \qquad (3.7.71)$$

$$k^*_{(+)} \to \infty \qquad G^*_{(+)} \to \infty \qquad \text{(d)}$$

<u>phase 2, empty</u>

$$k_2/k_1 \to 0, \qquad G_2/G_1 \to 0 \qquad \text{(a)}$$

$$k^*_{(-)} \to 0 \qquad G^*_{(-)} \to 0 \qquad \text{(b)}$$

$$k^*_{(+)} = k_1 \left[1 + \frac{(k_1+G_1)v_2}{G_1 + k_1 v_2} \right] \qquad \text{(c)} \qquad (3.7.72)$$

$$G^*_{(+)} = G_1 \left[1 - \frac{2(k_1+G_1)v_2}{2G_1 v_2 + k_1(1+v_2)} \right] \qquad \text{(d)}$$

Having obtained bounds for k^* we can immediately write down bounds for E_A^* and ν_A^* by insertion of (3.7.68) into (3.6.66), (3.6.68) for which, it is recalled, the functional relations between E_A^*, ν_A^* and k^* are defined by the general relations (3.4.117) and (3.4.118). Now it is to be remembered that (3.7.68) have been identified with k_c^* of two different composite cylinder assemblages and thus insertion of (3.7.68) into (3.6.66), (3.6.68) is equivalent to the derivation of E_A^* and ν_A^* of composite cylinder assemblages as has been done in par. 3.5.3. It is thus concluded that the resulting E_A^* and ν_A^* bounds are E_{Ac}^* and ν_{Ac}^* for the same assemblages connected with the k^* bounds. Accordingly, we can write down the bounds at once by use of (3.5.96-97)

$$E_{A(-)}^* = E_1 v_1 + E_2 v_2 + \frac{4(\nu_2-\nu_1)^2 v_1 v_2}{v_1/k_2 + v_2/k_1 + 1/G_1} \quad \text{(a)}$$

$$E_{A(+)}^* = E_1 v_1 + E_2 v_2 + \frac{4(\nu_2-\nu_1)^2 v_1 v_2}{v_1/k_2 + v_2/k_1 + 1/G_2} \quad \text{(b)} \qquad (3.7.73)$$

$$\nu_{A(\pm)}^* = \nu_1 v_1 + \nu_2 v_2 + \frac{(\nu_2-\nu_1)(1/k_1-1/k_2) v_1 v_2}{v_1/k_2 + v_2/k_1 + 1/G_1} \quad \text{(a)}$$

$$\nu_{A(\mp)}^* = \nu_1 v_1 + \nu_2 v_2 + \frac{(\nu_1-\nu_2)(1/k_2-1/k_1) v_1 v_2}{v_1/k_2 + v_2/k_1 + 1/G_2} \quad \text{(b)} \qquad (3.7.74)$$

The bounds are valid when the conditions (3.7.65) are fulfilled. (Actually, only the second of these is needed here). According to (3.6.68) the upper signs in the left of (3.7.74) apply for

$$\frac{\nu_1 - \nu_2}{k_2 - k_1} > 0$$

and the lower signs apply for

$$\frac{\nu_1 - \nu_2}{k_2 - k_1} < 0$$

If the phases are transversely isotropic then E of the phases becomes E_A, ν becomes ν_A and G becomes G_T.

The bounds are obviously <u>best possible in terms of volume fractions</u> since, by construction, they are composite cylinder assemblage results and at the same time bounds for arbitrary transversely isotropic phase geometry. In contrast to (3.7.68-69) the bounds (3.7.73-74) are close together. In particular the E_A^* bounds show that <u>$\bar{E}$ is an excellent approximation for E_A^* of any FM or FRM.</u>

With the bounds (3.7.68-69) and (3.7.73-74), bounds for E_T^* can now be constructed by use of the general results (3.6.107-108).

Bounds for n^* and ℓ^* can be obtained in similar fashion by use of the general relations (3.4.112). It is easily seen that if

$$\frac{\ell_2 - \ell_1}{k_2 - k_1} > 0$$

which is the case when 2 is stiffer than 1, both n^* and ℓ^* are monotonically increasing linear functions of k^*. Therefore an increase (decrease) of k^* produces an increase (decrease) of n^* and ℓ^*. Consequently, upper (lower) bounds for n^* and ℓ^* are obtained by introduction of upper (lower) bounds for k^* into (3.4.112). This results in

$$n^*_{(\pm)} = k^*_{(\pm)} \left(\frac{\ell_2 - \ell_1}{k_2 - k_1}\right)^2 - \bar{k}\left(\frac{\ell_2 - \ell_1}{k_2 - k_1}\right)^2 + \bar{n} \qquad \text{(a)}$$

$$\ell^*_{(\pm)} = k^*_{(\pm)} \frac{\ell_2 - \ell_1}{k_2 - k_1} + \frac{\ell_1 k_2 - \ell_2 k_1}{k_2 - k_1} \qquad \text{(b)} \qquad (3.7.75)$$

We now consider the axial shear modulus G^*_A. The C^o and C cylinders are both subjected to the boundary displacements.

$$u_1^S = \varepsilon^o_{12}\, x_2 \qquad \text{(a)}$$

$$u_2^S = \varepsilon^o_{12}\, x_1 \qquad \text{(b)} \qquad (3.7.76)$$

$$u_3^S = 0 \qquad \text{(c)}$$

which are a special case of (3.5.4). In the transversely isotropic C^o cylinder the displacement field is of the form (3.7.76) throughout. Therefore, the strains and stresses in the cylinder are homogeneous and are given by

$$[\epsilon^{o}_{ij}] = \begin{bmatrix} 0 & \epsilon^{o}_{12} & 0 \\ \epsilon^{o}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\sigma^{o}_{ij}] = \begin{bmatrix} 0 & 2G_o\epsilon^{o}_{12} & 0 \\ 2G_o\epsilon^{o}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where G_o is now the axial shear modulus G_{Ao}.

The strain energy U_o^{ϵ} per unit height is

$$U_o^{\epsilon} = 2G_o\epsilon^{o^2}_{12} A \tag{3.7.77}$$

The elastic fields in the C cylinder are determined by the axial shearing formulation of par. 3.5.1. We have from (3.5.33-34)

$$[\epsilon_{ij}] = \begin{bmatrix} 0 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & 0 & 0 \\ \epsilon_{13} & 0 & 0 \end{bmatrix} \qquad \text{(a)}$$

$$(3.7.78)$$

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 2G\epsilon_{12} & 2G\epsilon_{13} \\ 2G\epsilon_{12} & 0 & 0 \\ 2G\epsilon_{13} & 0 & 0 \end{bmatrix} \qquad \text{(b)}$$

where G is the phase shear modulus which assumes the values G_1, G_2 in the two phases. It follows from (3.7.9a), (3.7.78a) and the stress strain laws of the C^o and C cylinders that the polarization components are

$$[p_{ij}] = \begin{bmatrix} 0 & 2G'\epsilon_{12} & 2G'\epsilon_{13} \\ 2G'\epsilon_{12} & 0 & 0 \\ 2G'\epsilon_{13} & 0 & 0 \end{bmatrix} \qquad \text{(a)}$$

$$G' = G - G_o \qquad \text{(b)} \qquad (3.7.79)$$

Consequently, we choose an admissible polarization field of the form

$$[\tilde{p}_{ij}^{(m)}] = \begin{bmatrix} 0 & \tilde{p}_{12}^{(m)} & \tilde{p}_{13}^{(m)} \\ \tilde{p}_{12}^{(m)} & 0 & 0 \\ \tilde{p}_{13}^{(m)} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tau_2^{(m)} & \tau_3^{(m)} \\ \tau_2^{(m)} & 0 & 0 \\ \tau_3^{(m)} & 0 & 0 \end{bmatrix} \qquad (3.7.80)$$

which is constant in each phase, thus piecewise constant in the C cylinder.

Since the actual displacement field in the C cylinder is of the form (3.5.30-31) while the displacement field in the C^o cylinder has the form (3.7.76) we conclude, by taking their difference, that the actual u'_i has the form

$$u'_1 = u'_1 (x_2, x_3)$$

$$u'_2 = u'_3 = 0$$

Therefore $\tilde{u}'$ is chosen similarly as

$$\tilde{u}'_1 = \tilde{u}'_1 (x_2, x_3)$$
$$\tilde{u}'_2 = \tilde{u}'_3 = 0 \qquad (3.7.81)$$

The functional (3.7.29b) is now computed in terms of (3.7.80-81) for two phases. We find easily

$$\tilde{U} = U_o^{\epsilon} - \frac{1}{2}\left[\frac{\tau_2^{(1)^2} + \tau_3^{(1)^2}}{G_1 - G_o} v_1 + \frac{\tau_2^{(2)^2} + \tau_3^{(2)^2}}{G_2 - G_o} v_2 - \right.$$
$$\left. - 2\bar{\tau}_2 \epsilon^o_{12}\right] A + \tilde{U}' \qquad \text{(a)}$$

where

$$\tilde{U}' = \int_A (\tau_2 \tilde{\epsilon}'_{12} + \tau_3 \tilde{\epsilon}'_{13})\, dA \qquad \text{(b)} \qquad (3.7.82)$$

$$\bar{\tau}_2 = \bar{\tau}_2^{(1)} v_1 + \bar{\tau}_2^{(2)} v_2 \qquad \text{(c)}$$

and $\tilde{\epsilon}_{12}$, $\tilde{\epsilon}_{13}$ are derived from (3.7.81).

Furthermore, $\tilde{u}'_i$ and τ_i are connected by the boundary value problem (3.7.19) which, because of (3.7.80-81), now assumes the form

$$G_o \nabla^2 \tilde{u}'_1 (x_2, x_3) + \tau_{2,2} + \tau_{3,3} = 0 \qquad \text{(a)}$$

$$\tilde{u}_1 = 0 \qquad \text{on } C \qquad \text{(b)}$$

(3.7.83)

$$\left.\begin{array}{ll} \tilde{u}_1 & \\ G_o\,(\tilde{u}'_{1,2}\,n_2 + \tilde{u}'_{1,3}\,n_3) + \tau_2\,n_2 + \tau_3\,n_3 & \end{array}\right\} \begin{array}{ll} \text{continuous} & \text{(c)} \\ \text{on } C_{int} & \text{(d)} \end{array}$$

The computation of $\tilde{U}'$ in (3.7.82) must now be performed by insertion of (3.7.80) into (3.7.83), computation of $\tilde{u}_1$ and subsequent computation of (3.7.82b). The details are given in the appendix further below. The result is

$$\tilde{U}' = -\frac{1}{2G_o}\left[(\tau_2^{(1)} - \tau_2^{(2)})^2 + (\tau_3^{(1)} - \tau_3^{(2)})^2\right] v_1\,v_2\,A \qquad (3.7.84)$$

We now carry (3.7.84) back into (3.7.82) and thus this expression is now given in terms of $\tau_2^{(m)}$, $\tau_3^{(m)}$, G_o and known quantities. The rest of the bounding procedure is entirely analogous to the one previously used for k^* and G_T^* bounding. Therefore, only a brief outline will be given: In view of (3.6.47) the actual strain energy per unit height of C cylinder is

$$U^e = 2G_A^*\,\epsilon_{12}^{o\,2}\,A \qquad (3.7.85)$$

For the present case (3.7.38) assume the simple form

$$\begin{array}{lll} \tilde{U} \le U^e & G_1,\,G_2 \ge G_o & \text{(a)} \\ \tilde{U} \ge U^e & G_1,\,G_2 \le G_o & \text{(b)} \end{array} \qquad (3.7.86)$$

Now (3.7.84) and (3.7.77) are introduced into (3.7.82). The resulting expression for $\tilde{U}$ and (3.7.85) are then introduced into (3.7.86) and then $\tilde{U}$ is optimized

with respect to $\tau_2^{(m)}$, $\tau_3^{(m)}$ in order to approach U^{ϵ} as closely as possible. Thus are obtained bounds for G_A^* in terms of the unknown G_o. These are optimized with respect to G_o subject to the right side restrictions (3.7.86). The result is

$$\boxed{\begin{aligned} G^*_{A(-)} &= G_1 + \frac{v_2}{\dfrac{1}{G_2-G_1} + \dfrac{v_1}{2G_1}} = G_1 \frac{G_1 v_1 + G_2 (1+v_2)}{G_1(1+v_2) + G_2 v_1} \quad \text{(a)} \\ G^*_{A(+)} &= G_2 + \frac{v_1}{\dfrac{1}{G_1-G_2} + \dfrac{v_2}{2G_2}} = G_2 \frac{G_2 v_2 + G_1 (1+v_1)}{G_2 (1+v_1) + G_1 v_2} \quad \text{(b)} \end{aligned}} \qquad (3.7.87)$$

when

$$G_2 > G_1 \qquad \text{(c)}$$

Comparison of (3.7.87a) with (3.5.111) shows that it is identical to the composite cylinder assemblage G^*_{Ac}. Similarly, (3.7.87b) can be identified with another, reversed composite cylinder assemblage G^*_{Ac}. (Compare, similar discussion for k^* and G^*_T bounds, above). Therefore, the bounds (3.7.87) are <u>best possible in terms of volume fractions</u>.

If the phases are transversely isotropic G_1, G_2 in (3.7.87) are to be interpreted as the axial phase shear moduli.

The margin between the bounds depends upon the ratio G_2/G_1 and in this respect their behavior is similar to that of the bounds (3.7.68-69). For extreme cases of one rigid phase or one empty phase we obtain the following results

<u>phase 2, rigid</u>

$$G_2/G_1 \to \infty$$

$$G^*_{A(-)} = G_1 \frac{1+v_2}{1-v_2} \tag{3.7.88}$$

$$G^*_{A(+)} \to \infty$$

<u>phase 2, empty</u>

$$G_2/G_1 \to 0$$

$$G^*_{A(-)} \to 0 \tag{3.7.89}$$

$$G^*_{A(+)} = G_1 \frac{1-v_2}{1+v_2}$$

The bounds (3.7.68-69), (3.7.73-74) and (3.7.87) bracket five EEM which completely describe the macroscopic elastic behavior of macroscopically transversely isotropic two phase fibrous materials, when the phases are isotropic or transversely isotropic. All the bounds except (3.7.69) are known to be best possible in terms of volume fractions.

Bounds for E^*_T may be obtained by use of (3.6.108-109).

Bounds (3.7.73-74) were derived by Hill [3.5], bounds (3.7.69), (3.7.87) were derived by Hashin [3.13] and (3.7.68) were derived independently in both of these references.

Hill's method of derivation is entirely different from the present one. Its starting point is the direct proof of (3.7.70), from which were deduced the bounds (3.7.68). Then the bounds (3.7.73-74) follow as was described above. While Hill stated that the bounds are best possible he did not actually show this. He did identify bounds with "effective moduli" of a single composite cylinder, but the composite cylinder assemblage model is needed to identify the bounds with effective moduli expressions of a composite material.

The bounds given here are very important results as they clearly define the restrictions placed upon EEM by specification of volume fractions only. By their general nature the bounds obviously also apply for FRM. If the fibers are much stiffer than the matrix then the upper bounds (3.7.68b), (3.7.69b) and (3.7.87b) are generally not useful from a practical point of view. In this case the lower bounds are much more important since four of them coincide with composite cylinder assemblage results and are thus in good agreement with experimental results for circular fibers, as has been shown in par. 3.5.3.

Physical reasoning why the k^* and G_T^* bounds must be far apart for a two phase material, in which one phase is very much stiffer than the other, has been given previously. The same reasoning also applies for G_A^* and E_T^* bounds. For all of these EEM it is crucial whether the stiff phase is in the form of matrix or fibers.

The situation is different, however, for E_A^* and ν_A^*, for these are defined by uniaxial loading, in which case it does not matter too much which material is matrix and which is fibers. Thus the E_A^* bounds (3.7.73) are found to be always extremely close while the ν_A^* bounds (3.7.74) are quite close.

These observations will now be illustrated by some numerical examples. Plots of some two phase bounds are shown in figs. 3.7.2 - 3.7.4 for a Boron-Aluminum fibrous material. The E_T^* bounds have been obtained by use of (3.6.107-108). It is seen that in the present case the bounds are fairly close together and thus provide valuable estimates for the EEM. It should be borne in mind that because of the generality of the bounds they apply for any transversely isotropic FRM with fibers of any shape.

Fig. 3.7.2 also shows, by comparison, plots of the elementary shear modulus bounds (3.6.64).

The E_A^* bounds are practically coincident and may for all practical purposes be represented by $\bar{E}$. A list of ν_A^* bounds for various volume fractions is given below. It is seen that these bounds are also very close.

v_2	$\nu^*_{A(-)}$	$\nu^*_{A(+)}$
0.2	0.276	0.288
0.4	0.249	0.263
0.6	0.228	0.240
0.8	0.213	0.219

It should be borne in mind that the G^*_T, E^*_T and G^*_A bounds for the present material are reasonably close since the phase stiffness ratios are not too large. The Young's moduli ratio, for example, is $E_2/E_1 = 5.6$ for the present material. The situation is different for such materials as Boron-Epoxy and Glass-Epoxy in which the fibers are very much stiffer than the matrix. As an example we consider the Glass-Epoxy material whose phase properties are listed in table 3.6.1, par. 3.6.1. We compute fibrous materials bounds for $v_1 = v_2 = 0.5$ and compare them in the table below to the previously computed elementary bounds, given on page 224.

Table 3.7.1

$v_1 = v_2 = 0.5$

Elastic moduli in 10^6 psi

Lower Bound Elementary	Lower Bound Improved		Upper Bound Improved	Upper Bound Elementary
0.92	1.04	k^*	2.49	3.90
5.45	5.45	E^*_A	5.47	5.47
0.210	0.221	ν^*_A	0.266	0.275
0.29	0.41	G^*_A	1.59	2.26
0.29	0.36	G^*_T	1.19	2.26

The table shows that while the present bounds are a marked improvement of the elementary bounds for k^*, G_A^* and G_T^*, they are too far apart for the present material to provide good estimates. On the other hand, the improved E_A^* bounds are excellent but are numerically the same as the elementary bounds. The ν_A^* improved bounds are better than the elenentary ones, but not significantly.

The method of derivation used above to obtain bounds for two phase FM is easily applicable for any number of cylindrical phases. Bounds for k^*, G_T^* and G_A^* for this case were given in [3.13]. These results are

$$k^*_{(-)} = k_1 + (k_1 + G_1)\left\{\left[\sum_{m=2}^{m=M} \frac{k_m - k_1}{k_m + G_1} v_m\right]^{-1} - 1\right\}^{-1}$$

$$k^*_{(+)} = k_M + (k_M + G_M)\left\{\left[\sum_{m=1}^{m=M-1} \frac{k_m - k_M}{k_m + G_M} v_m\right]^{-1} - 1\right\}^{-1} \tag{3.7.90}$$

$$G^*_{T(-)} = G_1 + \frac{2G_1(k_1+G_1)}{k_1+2G_1}\left\{\left[\sum_{m=2}^{m=M} \frac{(G_m - G_1)\, v_m}{G_m + k_1 G_1/(k_1+2G_1)}\right]^{-1} - 1\right\}^{-1}$$

$$G^*_{T(+)} = G_M + \frac{2G_M(k_M+G_M)}{k_M + 2G_M}\left\{\left[\sum_{m=1}^{m=M-1} \frac{(G_m - G_M)\, v_m}{G_m + k_M G_M/(k_M+2G_M)}\right]^{-1} - 1\right\}^{-1} \tag{3.7.91}$$

$$G^*_{A(-)} = G_1 \left\{ 1 + 2 \left\{ \left[\sum_{m=2}^{m=M} \frac{G_m - G_1}{G_m + G_1} v_m \right]^{-1} - 1 \right\}^{-1} \right\} \tag{3.7.92}$$

$$G^*_{A(+)} = G_M \left\{ 1 + 2 \left\{ \left[\sum_{m=1}^{m=M-1} \frac{G_m - G_M}{G_m + G_M} v_m \right]^{-1} - 1 \right\}^{-1} \right\}$$

Here M is the number of phases, G_1, k_1 are the smallest phase moduli and G_M, k_M are the largest phase moduli.

For transversely isotropic phases k is the transverse bulk modulus, G is the transverse shear modulus in (3.7.91) and the axial shear modulus is (3.7.92).

Bounds for E^*_A and ν^*_A cannot be obtained from (3.7.90) by use of the relations (3.4.117) and (3.4.118) since these are restricted to the two phase case. It is, however, to be expected that $\bar{E}$ remains an excellent approximation for E^*_A, while $\bar{\nu}$ should be a fair approximation for ν^*_A.

It is not known whether or not any of the multiphase bounds is best possible in terms of volume fractions.

3.7.4 Bounds for Circular Fiber Reinforcement

The bounds for EEM of fibrous materials which were developed in paragraph 3.7.3 are very general results which are valid for any fibrous material whose phase geometry is statistically transversely isotropic and whose phase elastic moduli and phase volume fractions are specified. Consequently, the bounds are certainly also valid for transversely isotropic fiber reinforced materials, where the fiber cross sections may be of any shape. Furthermore, the identification of lower bounds with composite cylinder assemblage EEM shows that this configuration is a FRM with minimum EEM for given phase properties and volume fractions.

It will be recalled that in the treatment of the composite cylinder assemblage model the EEM G^*_{Tc} has been bounded from above and below by application of the principles of minimum potential energy and of minimum complementary energy, (3.6.104). Since the composite cylinder assemblage has transversely isotropic geometry, the general bounds (3.7.69) are certainly also bounds for G^*_{Tc} and there arises the question which bounds are more restrictive?

It may be shown (by tedious algebra) that when the fibers are stiffer than the matrix, i.e. when condition (3.7.65) is fulfilled, the lower bound (3.7.69a) is always above the lower bound (3.6.104); therefore the latter is not needed. On the other hand the upper bound (3.7.69b) is below the upper bound (3.6.104). Consequently, optimum bounds on G^*_{Tc} are given by (3.7.69a) (lower) and (3.6.104), upper. The last bound can be put into the form (3.5.113) and so there are obtained the bounds (3.5.113-114).

As has been stated before, recent work by Hashin and Rosen indicates that the upper bound (3.5.113) may actually be the G^*_{Tc} expression for a composite cylinder assemblage in which the fibers are stiffer than the matrix.

If the matrix is stiffer than the fibers it may be shown that the upper bound (3.7.69b) is below the upper bound (3.6.104), thus superseding the latter, while the lower bound (3.7.69a) is below the lower bound (3.6.104). Consequently, optimum bounds consist of the lower bound (3.6.104) and of the upper (3.7.69b). But note that according to the previous convention, where 2 was stiffer than 1, the matrix must now be denoted 2 and the fibers 1 to obtain the upper bound from (3.7.69). To avoid confusion let the matrix be denoted by subscript m and the fibers by subscript f. We then have for

$$k_m > k_f \qquad\qquad G_m > G_f$$

$$G_m \left[1 + \frac{2(1-\nu_m)}{1-2\nu_m} A_4^{\sigma}(v_f)\, v_f \right] \le G_T^* \le G_m + \frac{v_f}{\dfrac{1}{G_f - G_m} + \dfrac{k_m + 2G_m}{2G_m(k_m + G_m)} v_m} \tag{3.7.93}$$

where $A_4^{\sigma}(v_f)$ is defined by (3.6.97).

It is not clear whether the arbitrary fibrous geometry bounds of par. 3.7.3 are valid bounds for regular arrays of circular fibers, such as square or hexagonal arrays. It is recalled that the hexagonal array is elastically transversely isotropic while the square array has this property with respect to axial shearing, par. 3.4.3. But it should be remembered that the derivation of the bounds in par. 3.7.3 was based on the condition (2.2.14) which was used in (29), Appendix to chap. 3.7. This condition implies that the <u>phase geometry</u> is statistically transversely isotropic. Evidently, square and hexagonal arrays do not obey such a geometrical restriction.

On the other hand, it is empirically observed that for fibers which are stiffer than matrix, lower arbitrary fibrous geometry bounds of par. 3.7.3 are below numerical results for EEM of such arrays and above the lower bounds (3.6.78-80) for such arrays.

Figs. 3.7.5-6 show numerical examples for G_A^* of square and hexagonal arrays, respectively. In fig. 3.7.5, the upper bound is the right side of (3.6.80) with v_c and v_{2c} defined by (3.6.75a) and (3.6.76), respectively. The "lower bound" is (3.7.87a). Also shown is the numerical analysis result of ref. [3.22] for this case. Similar bounds are shown in fig. 3.7.6 for G_A^* of a hexagonal array of circular fibers. The upper bound in this case is the right side of (3.6.80) with v_c and v_{2c} defined by (3.6.75b), (3.6.76). The "lower bound" is again (3.7.87a). Numerical results of ref. [3.6] are also shown and it is seen that the lower bound is practically indistinguishable from these up to 70% fiber volume fraction. As the lower bound is also a composite cylinder assemblage result it is recalled that this numerical coincidence has already been described in table 3.5.

The question of the validity of the arbitrary fibrous geometry bound for periodic arrays merits further investigation.

3.7.5 Bounds for Effective Elastic Moduli: Randomly Oriented Fibers

We consider the case of a statistically homogeneous and isotropic two phase material with isotropic phases. A particular case of this is a FRM in which the fibers are randomly oriented, all directions being equally likely, fig. 3.4.4.

General bounds for K^* and G^* of statistically isotropic composites in which only phase properties and phase volume fractions are specified have been derived in [3.33] on the basis of the polarization extremum principles which were derived in par. 3.7.2 . The method of derivation is analogous to the one employed in par. 3.7.3 to derive bounds for k^* and G_T^* of FRM.

Let the two phases be isotropic with elastic moduli K_1, G_1 and K_2, G_2, respectively, for each phase. Also $K_2 > K_1$, $G_2 > G_1$. The bounds are then

$$K^*_{(-)} = K_1 + \frac{v_2}{\dfrac{1}{K_2-K_1} + \dfrac{3v_1}{3K_1+4G_1}} \qquad \text{(a)}$$

$$K^*_{(+)} = K_2 + \frac{v_1}{\dfrac{1}{K_1-K_2} + \dfrac{3v_2}{3K_2+4G_2}} \qquad \text{(b)}$$

(3.7.94)

$$G^*_{(-)} = G_1 + \frac{v_2}{\dfrac{1}{G_2-G_1} + \dfrac{6(K_1+2G_1)\,v_1}{5G_1(3K_1+4G_1)}} \qquad \text{(a)}$$

$$G^*_{(+)} = G_2 + \frac{v_1}{\dfrac{1}{G_1-G_2} + \dfrac{6(K_2+2G_2)v_2}{5G_2(3K_2+4G_2)}} \qquad \text{(b)}$$

(3.7.95)

The bounds (3.7.94) are best possible in terms of the information available. It is not known whether or not the bounds (3.7.95) are best possible.

The numerical margins between the bounds (3.7.94) and (3.7.95) are

of the same order as in the case of the previously considered k^* and G_T^* bounds (3.7.68-69). Thus in the case of not too large phase relative stiffness such as Boron-Aluminum the bounds (3.7.94-95) provide good estimates. In the case of large phase relative stiffness such as Boron-Epoxy , this is not the case.

APPENDIX

Computation of $\tilde{U}'$

Establishment of the results (3.7.58) and (3.7.84) for $\tilde{U}'$, for the case of piecewise constant polarization, is not simple. The results were originally derived in [3.13] by use of statistical methods for differential equations with random forcing functions, the forcing functions being in this case the body force type terms $\tilde{p}_{\alpha\beta,\beta}$ in (3.7.54). Subsequently, Green's function methods where used in [3.31] to derive a related result in the case of statistically isotropic two phase composites. The present method of derivation contains ingredients of both methods mentioned above.

(a) $\tilde{U}'$ for Transverse Bulk and Shear Modulus Bounds

We introduce the simplified notation

$$\tilde{u}'_\alpha = v_\alpha \tag{1}$$

The polarization $\tilde{p}_{\alpha\beta}$ is split into average $\bar{\tilde{p}}_{\alpha\beta}$ and deviation from average $\tau_{\alpha\beta}$, thus

$$\tilde{p}_{\alpha\beta} = \bar{\tilde{p}}_{\alpha\beta} + \tau_{\alpha\beta} \qquad \text{(a)}$$

$$\bar{\tilde{p}}_{\alpha\beta} = \bar{\tilde{p}}^{(1)}_{\alpha\beta} v_1 + \bar{\tilde{p}}^{(2)}_{\alpha\beta} v_2 \qquad \text{(b)} \tag{2}$$

At present, $\tilde{p}_{\alpha\beta}$ is a general tensor. It will be taken as piecewise constant in a later stage of the development.

In view of (2a) the expression (3.7.56b) for $\tilde{U}'$ can be written as

$$\tilde{U}' = \frac{1}{2}\,\bar{\tilde{p}}_{\alpha\beta}\int \tilde{\epsilon}'_{\alpha\beta}\,dA + \frac{1}{2}\int \tau_{\alpha\beta}\,\tilde{\epsilon}'_{\alpha\beta}\,dA \tag{3}$$

since $\bar{\tilde{p}}_{\alpha\beta}$ is a constant tensor. Because of the boundary condition (3.7.55a) and the average strain theorem, the first integral on the right side of (3) vanishes. Also, since $\tau_{\alpha\beta}$ is symmetric we can rewrite (3) using notation (1) in the form

$$\tilde{U}' = \frac{1}{2}\int \tau_{\alpha\beta}\,v_{\alpha,\beta}\,dA \tag{4}$$

Next, (2) is introduced into the boundary value problem (3.7.54-55). Since $\bar{\tilde{p}}_{\alpha\beta}$ is a constant tensor in all phases it contributes neither to (3.7.54) nor to (3.7.55c). Accordingly, the boundary value problem can be rewritten as follows

$$k_o\,v^{(m)}_{\beta,\beta\alpha} + G_o\,v^{(m)}_{\alpha,\beta\beta} + \tau^{(m)}_{\alpha\beta,\beta} = 0 \qquad \text{in } R_m \qquad \text{(a)}$$

$$v_\alpha = 0 \qquad \text{on } C \qquad \text{(b)}$$

$$v_\alpha \qquad \text{(c)}$$

$$\left.\begin{matrix} \overset{v}{T}_\alpha + \tau_{\alpha\beta}\,n_\beta \end{matrix}\right\} \quad \text{continuous on } C_{int} \qquad \text{(d)} \tag{5}$$

Here $\overset{v}{T}_\alpha$ denotes the traction type term

$$\overset{v}{T}_\alpha = (k_o - G_o)\,v_{\beta,\beta}\,n_\alpha + G_o\,(v_{\alpha,\beta} + v_{\beta,\alpha})\,n_\beta$$

which is the usual plane elasticity traction associated with a plane displacement field v_α.

It is our purpose to construct a solution of problem (5) in terms of plane elasticity Green's functions. In this respect it is first noted that since the cylinder cross section A, which is bounded by C, can be taken to be of infinite extent (in comparison to size of phase regions), the boundary condition (5b) can be replaced by

$$v_\alpha \to 0 \qquad \text{at infinity} \tag{6}$$

This replacement merely introduces the usual boundary layer effect at C which quickly becomes insignificant at points removed from C.

Next we consider the auxiliary problem

$$k_o v_{\beta,\beta\alpha} + G_o v_{\alpha,\beta\beta} + F_\alpha = 0 \qquad \text{(a)}$$

$$v_\alpha \to 0 \qquad \text{at infinity} \qquad \text{(b)}$$

$$\left.\begin{array}{l} v_\alpha \\ \\ v \\ T_\alpha \end{array}\right\} \text{continuous everywhere} \qquad \begin{array}{l}\text{(c)} \\ \\ \text{(d)}\end{array} \tag{7}$$

where F_α is some body force distribution. The solution to this problem is given by

$$v_\alpha(\underline{x}) = \int G_{\alpha\beta}(\underline{x},\underline{x}')\, F_\beta(\underline{x}')\, d\underline{x}' \tag{8}$$

where $G_{\alpha\beta}$ is the Green's tensor whose functional form will be discussed below, $d\underline{x}'$ is the area element

$$d\underline{x}' = dx_2'\, dx_3'$$

and the integral extends over infinite two dimensional $\underline{x}'$ space. Similar notation for integrals will be employed from now on: all elements of area shall be written in the above form with all integrations extending over infinite two dimensional space.

If we now compare problem (5) (with (5b) replaced by (6)) with problem (7) we see that the body force distribution F_α can be simply replaced by $\tau^{(m)}_{\alpha\beta,\beta}$, but there is a fundamental difference between (5d) and (7d). For simplicity the case of two phases is considered; then (5d) can be written as

$$\overset{v}{T}{}^{(2)}_{\alpha} - \overset{v}{T}{}^{(1)}_{\alpha} = (\tau^{(1)}_{\alpha\beta} - \tau^{(2)}_{\alpha\beta})\, n^{(2)}_{\beta} \quad \text{on } C_{12} \tag{9}$$

where $n^{(2)}_{\beta}$ is the normal pointing <u>outward</u> from R_2. From now on this normal is taken as positive and is simply written n_β for the interface.

The traction discontinuity (9) can be interpreted as a body force layer in the interface whose resultant per unit interface area is given by the right side of (9), [3.27]. Accordingly, from (8), the displacement due to this body force is

$$_{12}v_\alpha(\underline{x}) = \int_{C_{12}} G_{\alpha\beta}(\underline{x}, \underline{x}'_{12})\, (\tau^{(1)}_{\beta\gamma} - \tau^{(2)}_{\beta\gamma})\, n_\gamma\, ds \tag{10}$$

where $\underline{x}'_{12}$ denotes interface points, n_γ is the interface normal (outward to R_2) and ds is element of arc of C_{12}. Therefore, the total displacement v_α in (5) which is produced by volume and interface body forces is

$$v_\alpha(\underline{x}) = {}_{12}v_\alpha(\underline{x}) + \int G_{\alpha\beta}(\underline{x}, \underline{x}')\, \tau_{\beta\gamma,\gamma}(\underline{x}')\, d\underline{x}' \tag{11}$$

where the integral extends over all phase regions. The integrand may be rewritten as

$$G_{\alpha\beta}\,\tau_{\beta\gamma,\gamma} = (G_{\alpha\beta}\,\tau_{\beta\gamma})_{,\gamma} = G_{\alpha\beta,\gamma}\,\tau_{\beta\gamma}$$

Application of the plane divergence theorem brings the integral of the first term into the form

$$\int_{C_\infty} G_{\alpha\beta}\,\tau_{\beta\gamma}\,n_\gamma\,ds + \int_{C_{12}} G_{\alpha\beta}\,(\underline{x}',\,\underline{x}'_{12})\,(\tau^{(2)}_{\beta\gamma} - \tau^{(1)}_{\beta\gamma})\,n_\gamma\,ds \qquad (12)$$

It may be shown on the basis of the functional form of $G_{\alpha\beta}$ and the fact that the average of $\tau_{\beta\gamma}$ vanishes, (2), that the integral over C_∞, the boundary at infinity, vanishes. Thus introduction of the preceding results and of (10) into (11) cancels the interface contribution and brings the solution of (5) into the form

$$v_\alpha(\underline{x}) = -\int \frac{\partial}{\partial x'_\gamma}[G_{\alpha\beta}(\underline{x},\,\underline{x}')]\,\tau_{\beta\gamma}(\underline{x}')\,d\underline{x}' \qquad (13)$$

Equ. (13) provides a formal solution for v_α for a given distribution $\tau_{\alpha\beta}$ provided that (13) goes to zero at infinity in order to comply with (6). Expression (4) is now known in principle; actual computation, however, is not simple.

We proceed by first obtaining the Green's tensor $G_{\alpha\beta}$ for the present case. The simplest way of doing this is to obtain its Fourier Transform (FT) by use of (7). We define the two dimensional FT of a function φ in the form

$$\hat{\varphi}(\kappa_2,\kappa_3) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x_2,x_3)\,e^{\iota(\kappa_2 x_2+\kappa_3 x_3)}\,dx_2\,dx_3$$

where $\iota = \sqrt{-1}$. In simplified notation

$$\mathcal{F}(\varphi) = \hat{\varphi}(\underline{\kappa}) = \left(\frac{1}{2\pi}\right)^2 \int \varphi(\underline{x})\, e^{\iota \underline{\kappa} \cdot \underline{x}} d\underline{x} \qquad \text{(a)}$$

$$\varphi(\underline{x}) = \int \hat{\varphi}(\underline{\kappa})\, e^{-\iota \underline{\kappa} \cdot \underline{x}} d\underline{\kappa} \qquad \text{(b)} \tag{14}$$

where (b) is the inversion of (a).

The FT of derivatives is given by

$$\mathcal{F}(\varphi_{,\alpha}) = -\iota \kappa_\alpha \hat{\varphi} \tag{15}$$

provided that φ vanishes at infinity, [3.32]. Accordingly, the FT of (9) is given by

$$k_o \kappa_\alpha \kappa_\beta \hat{v}_\beta + G_o \kappa^2 \hat{v}_\alpha - \hat{F}_\alpha = 0 \qquad \text{(a)}$$

$$\kappa^2 = \kappa_\alpha \kappa_\alpha = \kappa_2^{\ 2} + \kappa_3^{\ 2} \qquad \text{(b)} \tag{16}$$

Solving for v_α we have

$$\hat{v}_\alpha = \left(\frac{1}{G_o} \frac{\delta_{\alpha\beta}}{\kappa^2} - \frac{k_o}{G_o(k_o+G_o)} \frac{\kappa_\alpha \kappa_\beta}{\kappa^4} \right) \hat{F}_\beta \tag{17}$$

Inversion of (17) by use of the convolution theorem for FT, [3.32], yields

$$v_\alpha(\underline{x}) = \int G_{\alpha\beta}(\underline{x} - \underline{x}')\, F_\beta(\underline{x}')\, d\underline{x}' \tag{18}$$

where $G_{\alpha\beta}(\underline{x})$ is the FT inverse of the parenthesis in (18). Accordingly

$$\hat{G}_{\alpha\beta}(\underline{\kappa}) = \frac{1}{G_o} \frac{\delta_{\alpha\beta}}{\kappa^2} - \frac{k_o}{G_o(k_o+G_o)} \frac{\kappa_\alpha \kappa_\beta}{\kappa^4} \tag{19}$$

The Green's tensor $G_{\alpha\beta}$ can now be found by inversion, but it will not be needed for subsequent development. It is seen that (18) is a reproduction (indeed a proof) of (8) and shows that the argument of the Green's tensor in the integrand is $\underline{x} - \underline{x}'$. It follows from (18) and (13) that

$$v_{\alpha,\beta}(\underline{x}) = \frac{\partial}{\partial x_\beta} v_\alpha = -\int \frac{\partial^2}{\partial x_\beta \partial x'_\gamma} [G_{\alpha\delta}(\underline{x}-\underline{x}')] \tau_{\delta\gamma}(\underline{x}')\, d\underline{x}' \qquad (20)$$

We introduce the variable

$$\underline{x} - \underline{x}' = \underline{r} \qquad (21)$$

Consequently

$$\frac{\partial}{\partial x'_\alpha} = -\frac{\partial}{\partial r_\alpha} \qquad \text{(a)}$$

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial r_\alpha} \qquad \text{(b)} \qquad (22)$$

$$d\underline{x}' = dx'_2\, dx'_3 = dr'_2\, dr'_3 = d\underline{r} \qquad \text{(c)}$$

Introducing (21-22) into (20) we have

$$v_{\alpha,\beta}(\underline{x}) = \int \frac{\partial^2}{\partial r_\beta \partial r_\gamma} [G_{\alpha\delta}(\underline{r})]\, \tau_{\delta\gamma}(\underline{x} - \underline{r})\, d\underline{r} \qquad (23)$$

$$\tilde{U}' = \frac{1}{2} \int \frac{\partial^2}{\partial r_\beta \partial r_\gamma} G_{\alpha\delta}(\underline{r}) \left\{ \int \tau_{\alpha\beta}(\underline{x})\, \tau_{\gamma\delta}(\underline{x} - \underline{r})\, d\underline{x} \right\} d\underline{r} \qquad (24)$$

Equ.(24) formally determines $\tilde{U}'$ for any choice of $\tau_{\alpha\beta}$, but the integral may be extremely difficult to evaluate.

We consider the case of a two phase material and piecewise constant $\tilde{p}_{\alpha\beta}$. Then

$$\tilde{p}_{\alpha\beta} = \begin{cases} \tilde{p}^{(1)}_{\alpha\beta} = \text{const}, & \text{in } R_1 \\ \tilde{p}^{(2)}_{\alpha\beta} = \text{const}, & \text{in } R_2 \end{cases} \tag{25}$$

and from (2)

$$\tau_{\alpha\beta} = \begin{cases} \tau^{(1)}_{\alpha\beta} = (\tilde{p}^{(1)}_{\alpha\beta} - \tilde{p}^{(2)}_{\alpha\beta})\, v_2 & \text{in } R_1 \\ \tau^{(2)}_{\alpha\beta} = (\tilde{p}^{(2)}_{\alpha\beta} - \tilde{p}^{(1)}_{\alpha\beta})\, v_1 & \text{in } R_2 \end{cases} \tag{26}$$

$$\bar{\tilde{p}}_{\alpha\beta} = \tilde{p}^{(1)}_{\alpha\beta}\, v_1 + \tilde{p}^{(2)}_{\alpha\beta}\, v_2 \tag{27}$$

Consider the two point correlation integral

$$P_{\alpha\beta\gamma\delta}(\underline{r}) = \int \tau_{\alpha\beta}(\underline{x})\, \tau_{\gamma\delta}(\underline{x}+\underline{r})\, d\underline{x} \tag{28}$$

For any $\underline{x}$, $\tau_{\alpha\beta}(\underline{x})$ assumes only either one of the values given in (26), the same being true of $\tau_{\gamma\delta}(\underline{x}+\underline{r})$. The integrand can thus assume the values $\tau^{(1)}_{\alpha\beta}\tau^{(1)}_{\gamma\delta}$, $\tau^{(1)}_{\alpha\beta}\tau^{(2)}_{\gamma\delta}$, $\tau^{(2)}_{\alpha\beta}\tau^{(1)}_{\gamma\delta}$, $\tau^{(2)}_{\alpha\beta}\tau^{(2)}_{\gamma\delta}$ only, according to the phases in which the points $\underline{x}$ and $\underline{x}+\underline{r}$ are situated. In terms of the two point probability functions (2.2.8) of chap. 2.2 which have the form (2.2.14), because of statistical transverse isotropy, the integral (28) can be written as

$$P_{\alpha\beta\gamma\delta}(\underline{r}) = [\tau^{(1)}_{\alpha\beta}\,\tau^{(1)}_{\gamma\delta}\,P_{11}(r) + \tau^{(1)}_{\alpha\beta}\,\tau^{(2)}_{\gamma\delta}\,P_{12}(r) +$$

$$+ \tau^{(2)}_{\alpha\beta}\,\tau^{(1)}_{\gamma\delta}\,P_{21}(r) + \tau^{(2)}_{\alpha\beta}\,\tau^{(2)}_{\gamma\delta}\,P_{22}(r)\,]\,A \tag{29}$$

This shows that

$$P_{\alpha\beta\gamma\delta}(\underline{r}) = P_{\alpha\beta\gamma\delta}(r) \tag{30}$$

where r is the magnitude of $\underline{r}$. It will be seen that this fact alone will enable us to evaluate (24).

It follows from (28) that (24) can be written in the form

$$\tilde{U}' = \frac{1}{2}\int \mathcal{F}\left[\frac{\partial^2}{\partial r_\beta \partial r_\gamma}\,G_{\alpha\delta}(\underline{r})\right]\hat{P}_{\alpha\beta\gamma\delta}(\underline{\kappa})\,d\underline{\kappa} \tag{31}$$

By the rule (16) for FT of derivatives we have

$$\mathcal{F}\left[\frac{\partial^2}{\partial r_\beta \partial r_\gamma}\,G_{\alpha\delta}(\underline{r})\right] = -\,\kappa_\beta\,\kappa_\gamma\,\hat{G}_{\alpha\delta}(\underline{\kappa}) \tag{32}$$

Also, because of (30)

$$\hat{P}_{\alpha\beta\gamma\delta}(\underline{\kappa}) = \hat{P}_{\alpha\beta\gamma\delta}(\kappa) \tag{33}$$

where κ is given by (17b). Introducing (33), (32) and (20) into (31) we have

$$2\tilde{U}' = -\frac{1}{G_o}\int \frac{\kappa_\alpha \kappa_\beta}{\kappa^2}\,\hat{P}_{\alpha\gamma\gamma\beta}(\kappa)\,d\underline{\kappa} + \frac{k_o}{G_o(k_o+G_o)}\int \frac{\kappa_\alpha \kappa_\beta \kappa_\gamma \kappa_\delta}{\kappa^4}\,\hat{P}_{\alpha\beta\gamma\delta}(\kappa)\,d\underline{\kappa} \tag{34}$$

We now introduce polar coordinates in κ space i.e.

$$\kappa_2 = \kappa \cos\theta$$
$$\kappa_3 = \kappa \sin\theta \tag{35}$$

and integrate (34) over an infinite circle. Then, since $\hat{P}_{\alpha\beta\gamma\delta}$ is a function of κ only, the angular integration can be carried out. The result is

$$2\tilde{U}' = \frac{k_o}{8G_o(k_o+G_o)} \int_o^\infty \hat{P}_{\alpha\alpha\beta\beta}(\kappa)\, 2\pi\kappa d\kappa - \frac{k_o+2G_o}{4G_o(k_o+G_o)} \int_o^\infty \hat{P}_{\alpha\beta\alpha\beta}(\kappa)\, 2\pi\kappa d\kappa \tag{36}$$

Now, from the FT inversion theorem (14b) we have

$$P_{\alpha\beta\gamma\delta}(\underline{r}) = \int \hat{P}_{\alpha\beta\gamma\delta}(\underline{\kappa})\, e^{-\iota\underline{\kappa}\cdot\underline{r}}\, d\underline{\kappa} \qquad \text{(a)}$$
$$P_{\alpha\beta\gamma\delta}(o) = \int \hat{P}_{\alpha\beta\gamma\delta}(\underline{\kappa})\, d\underline{\kappa} \qquad \text{(b)} \tag{37}$$

Because of (33), (37b) reduces to

$$P_{\alpha\beta\gamma\delta}(o) = \int_o^\infty \hat{P}_{\alpha\beta\gamma\delta}(\kappa)\, 2\pi\kappa\, d\kappa \tag{38}$$

We have from (38), (28) and (26)

$$P_{\alpha\beta\gamma\delta}(o) = \int \tau_{\alpha\beta}(\underline{x})\, \tau_{\gamma\delta}(\underline{x})\, d\underline{x}$$
$$= [\tau^{(1)}_{\alpha\beta}\, \tau^{(1)}_{\gamma\delta}\, v_1 + \tau^{(2)}_{\alpha\beta}\, \tau^{(2)}_{\gamma\delta}\, v_2]\, A \tag{39}$$

Note that (39) also follows from (29) and from (2.2.10), chap. 2. Insertion of (38, 39) in terms of (26) into (36) yields

$$2\tilde{U}' = \left[\frac{k_o}{8G_o(k_o+G_o)} (\tilde{p}^{(2)}_{\alpha\alpha} - \tilde{p}^{(1)}_{\alpha\alpha})(\tilde{p}^{(2)}_{\beta\beta} - \tilde{p}^{(1)}_{\beta\beta}) - \right. \tag{40}$$

$$\left. - \frac{k_o + 2G_o}{4G_o(k_o+G_o)} (\tilde{p}^{(2)}_{\alpha\beta} - \tilde{p}^{(1)}_{\alpha\beta})(\tilde{p}^{(2)}_{\alpha\beta} - \tilde{p}^{(1)}_{\alpha\beta}) \right] v_1 v_2 A$$

If the polarization components in (40) are separated into isotropic and deviatoric parts according to (3.7.46), the result (3.7.58) follows at once.

(b) $\tilde{U}'$ for Axial Shear Modulus Bound

The method is entirely the same as the one previously employed. The boundary value problem is now (3.7.83) and $\tilde{U}'$ is defined by (3.7.82b). We use the notation

$$\tilde{u}'_1 = \psi \tag{41}$$

and we split the polarization vector τ_α into average and deviation, thus

$$\tau_\alpha = \bar{\tau}_\alpha + \tau'_\alpha \tag{42}$$

Then (3.7.82b) assumes the form

$$\tilde{U}' = \int \tau'_\alpha \psi_{,\alpha} \, d\underline{x} \tag{43}$$

and the boundary value problem (3.7.83) becomes

$$G_o \nabla^2 \psi^{(m)} + \tau'^{(m)}_{\alpha,\alpha} = 0 \qquad \text{(a)}$$

$$\psi\,(C) = 0 \qquad \text{(b)}$$

$$\left.\begin{array}{ll} \psi & \text{(c)} \\ G_o \dfrac{\partial \psi}{\partial n} + \tau'_\alpha & \text{(d)} \end{array}\right\} \text{ continuous at } C_{int} \qquad (44)$$

In analogy to (13) the solution may be written as

$$\psi\,(\underline{x}) = - \int \frac{\partial}{\partial x'_\alpha} [H(\underline{x} - \underline{x}')]\, \tau'_\alpha\,(\underline{x}')\, d\underline{x}' \qquad (45)$$

where H is the Green's function which solves the problem

$$G_o \nabla^2 \psi\,(\underline{x}) + F\,(\underline{x}) = 0$$

$$\psi \to 0 \text{ at infinity} \qquad (46)$$

$$\psi,\ \frac{\partial \psi}{\partial n} \text{ continuous everywhere,}$$

in the form

$$\psi\,(\underline{x}) = \int H(\underline{x} - \underline{x}')\, F\,(\underline{x}')\, d\underline{x}' \qquad (47)$$

The FT of H is easily shown to be

$$\hat{H}\,(\underline{\kappa}) = \frac{1}{G_o\, \kappa^2} \qquad (48)$$

Proceeding now just as in the derivation of (24) we find

$$\tilde{U}' = \int \frac{\partial^2}{\partial r_\alpha \partial r_\beta} H(\underline{r}) \left\{ \int \tau'_\alpha(\underline{x}) \tau'_\beta(\underline{x} - \underline{r}) d\underline{x} \right\} d\underline{r} \tag{49}$$

Next for piecewise constant polarization we have

$$\tau'_\alpha = \begin{cases} (\tau_\alpha^{(1)} - \tau_\alpha^{(2)}) v_2 & \text{in } R_1 \\ (\tau_\alpha^{(2)} - \tau_\alpha^{(1)}) v_1 & \text{in } R_2 \end{cases} \tag{50}$$

$$T_{\alpha\beta}(\underline{r}) = \int \tau'_\alpha(\underline{x}) \tau'_\beta(\underline{x} + \underline{r}) d\underline{x} = T_{\alpha\beta}(r) \tag{51}$$

Proceeding as previously the analogue of (32) becomes

$$\tilde{U}' = \int \mathcal{F}\left[\frac{\partial^2}{\partial r_\alpha \partial r_\beta} H(\underline{r})\right] \hat{T}_{\alpha\beta}(\underline{\kappa}) d\kappa \tag{52}$$

which leads in view of (48) and (16) to

$$\tilde{U}' = -\frac{1}{G_o} \int \frac{\kappa_\alpha \kappa_\beta}{\kappa^2} \hat{T}_{\alpha\beta}(\kappa) d\underline{\kappa} \tag{53}$$

Introducing now the polar coordinates (35) and carrying out the angular integration, (53) becomes

$$\tilde{U}' = -\frac{1}{2G_o} \int_o^\infty \hat{T}_{\alpha\alpha}(\kappa) 2\pi\kappa d\kappa = -\frac{1}{2G_o} T_{\alpha\alpha}(o) \tag{54}$$

where a relation of type (38) has been used. Then from (50-51) and (54)

$$\tilde{U}' = -\frac{1}{2G_o}\left(\tau_\alpha^{(2)} - \tau_\alpha^{(1)}\right)\left(\tau_\alpha^{(2)} - \tau_\alpha^{(1)}\right) v_1 v_2 A \tag{55}$$

which is the result (3.7.84)

3.8 FIBER EMBEDDING APPROXIMATION

We have so far considered only rigorous methods of prediction of EEM of FRM. Many approximate treatments can be found in the literature. Discussion of all of these is certainly not within the scope of the present work. In view of the respectable available number of rigorous theoretical results, and their good agreement with experiments, it has also to be questioned whether such approximations serve much useful purpose.

We shall here be concerned with only one method of approximation which is often called the "self consistent scheme." This method is here called: fiber embedding approximation (FEA) since it is believed that this name is more descriptive.

The method has been applied in different ways and with different end results by Hill [3.21] and Hermans [3.34]. The FEA will here be generalized so as to include the previous two different approaches as special cases. The present treatment is similar to one given by Hashin [3.35], for particulate composites.

Let a transversely isotropic FRM with circular fibers be subjected to the usual homogeneous displacement boundary conditions (3.3.1). By the results

of par. 3.4.6 we know that it is sufficient to determine strain averages in one phase, which is here chosen as the fibers, in order to compute EEM. The FEA may be interpreted as a method by which the strain averages in the fibers are estimated and are then used in the general EEM expressions which were derived in par. 3.4.6.

For illustrative purposes we consider first the problem of computation of the effective transverse bulk modulus k*. Appropriate homogeneous displacement boundary conditions on a cylindrical specimen are (3.4.59), which are here rewritten

$$u_1(S) = 0 \qquad u_2(S) = \epsilon^o x_2 \qquad u_3(S) = \epsilon^o x_3 \qquad (3.8.1)$$

A typical fiber of radius a is imagined to be surrounded by a concentric cylindrical surface of radius b, which lies wholly in the matrix and does not include any other fiber. There has thus been formed a composite cylinder of radii a and b. The fundamental approximation to be introduced now is that the composite material outside the composite cylinder is replaced by a <u>homogeneous</u> transversely isotropic material whose elastic moduli are the EEM of the composite. Before we proceed further it is worthwhile to examine this fundamental approximation. It is recalled that the concept of EEM is based on averages taken over RVE which must necessarily contain many fibers. The present approximation departs rather severely from this concept in that the immediate vicinity of <u>one</u> composite cylinder is replaced by an equivalent homogeneous material. This can certainly not be correct. All that can be said about this immediate vicinity is that the elastic moduli assume erratically either fiber or matrix values.

Bearing all this in mind we proceed with the development. The boundary conditions (3.8.1) are applied at a distance $R \gg a, b$ from the composite cylinder center, fig. 3.8.1, and at the interfaces $r = a, b$ the usual displacement and traction continuity conditions must be fulfilled. Thus there has been formulated a well defined elasticity boundary value problem of a three layer composite cylinder which enables us to compute the strains in the fiber.

The solution of this problem is very similar to that of the two layer composite cylinder which was considered in par. 3.5.3 and the solutions in all layers are again of the form (3.5.73). Instead of solving the boundary value problem to find the strain in the fiber, we shall develop a different method of solution. We recall from par. 3.5.3 that a composite cylinder behaves as a homogeneous cylinder with apparent transverse bulk modulus given by (3.5.80). Denote

$$(a/b)^2 = v_{2c} \tag{3.8.2}$$

and denote the function on the right side of (3.5.80) by $k_c(v_{2c})$. Thus

$$k_c(v_{2c}) = \frac{k_1(k_2 + G_1)(1 - v_{2c}) + k_2(k_1+G_1)\, v_{2c}}{(k_2+ G_1)(1-v_{2c}) + (k_1+G_1)\, v_{2c}} \tag{3.8.3}$$

Then the inner composite cylinder is replaced by a homogeneous cylinder with transverse bulk modulus $k_c(v_{2c})$. The strain $\epsilon = \frac{1}{2}(\epsilon_{22}+\epsilon_{33})$ in this equivalent cylinder is the average strain in the composite cylinder and is denoted $\bar{\epsilon}_c$. Such a problem has been solved in par. 3.5.4 in conjunction with dilute

reinforcement theory, the pertinent result being (3.5.124). In the present interpretation and by the first isotropy-transverse isotropy analogy, par. 3.5.1 we have

$$\bar{\varepsilon}_c = \frac{k^* + G_T^*}{k_c(v_{2c}) + G_T^*} \varepsilon^o \tag{3.8.4}$$

where k^* and G_T^* are the unknown EEM of the FRM.

The displacement of the surface of the inner composite cylinder is radial and is given by

$$u_c(b) = \bar{\varepsilon}_c b$$

To find the strain in the fiber we consider the problem of the inner composite cylinder with this boundary condition. This has been solved in par. 3.5.3 but again we use a different approach. We apply (3.4.123) to the composite cylinder. Consequently

$$k_c(v_{2c}) = k_1 + (k_2 - k_1) \frac{\bar{\varepsilon}^{(2)}}{\bar{\varepsilon}_c} v_{2c} \tag{3.8.5}$$

where $\bar{\varepsilon}_c$ is the average strain in the fiber. From (3.8.5)and (3.8.3)

$$\bar{\varepsilon}^{(2)} = \frac{k_c(v_{2c}) - k_1}{k_2 - k_1} \cdot \frac{k^* + G_T^*}{k_c(v_{2c}) + G_T^*} \frac{\varepsilon^o}{v_{2c}} \tag{3.8.6}$$

The same computation can be carried out for any fiber and as long as <u>a/b is always the same</u> the result (3.8.6) remains the same. Thus (3.8.6) is the average strain in all fibers.

Finally, we use the estimate (3.8.6) in the general equation (3.4.123) for the FRM. We then have

$$k^* = k_1 + [k_c(v_{2c}) - k_1] \frac{k^* + G_T^*}{k_c(v_{2c}) + G_T^*} \cdot \frac{v_2}{v_{2c}} \tag{3.8.7}$$

where v_2 is the fibers volume fraction relative to the composite. It is seen that (6.8.7) involves both k^* and G_T^* and is thus not in general sufficient to determine either one of them. Another equation which involves both k^* and G_T^* is thus needed. This will be discussed later below. It is also noted that k^* and G_T^* in (3.8.7) are both functions of v_{2c} which is not known and can be chosen in many ways. Thus v_{2c} has to be regarded as some unknown parameter of uncertain status.

In Hill's [3.21] approach it is assumed that

$$v_{2c} = 1 \tag{3.8.8}$$

Because of (3.8.2) this implies that the fiber is directly imbedded in the equivalent composite without an intermediate matrix shell. Since by (3.5.80)

$$k_c(1) = k_2$$

we have from (3.8.7)

$$k^*(1) = k_1 + (k_2 - k_1) \frac{k^*(1) + G_T^*(1)}{k_2 + G_T^*(1)} v_2 \tag{3.8.9}$$

which is in agreement with Hill's result.

On the other hand Hermans [3.34] assumed

$$v_{2c} = v_2 \tag{3.8.10}$$

This implies that the fiber and matrix volume fractions in the composite cylinder are the same as in the FRM. If (3.8.10) is introduced into (3.8.7) it is easily shown after some algebra that without regard to the value of G_T^*,

$$k^*(v_2) = k_c(v_2) \tag{3.8.11}$$

In view of (3.8.3), (3.8.11) implies that in this case <u>the FEA predicts the composite cylinder assemblage results (3.5.80), (3.5.91).</u>

Next we consider the effective axial Young's modulus E_A^* and Poisson's ratio ν_A^* on the basis of the FEA. Hill [3.21] who assumed (3.8.8), showed that in that case his general relations between k^*, E_A^* and ν_A^*, (3.4.117), (3.4.118), remained valid for the FEA predicted results. It is not difficult to show by his reasoning that the relations are also valid for E_A^* and ν_A^* predicted by the present generalized FEA. Therefore we have

$$E_A^*(v_{2c}) = \bar{E} + \frac{4(\nu_2 - \nu_1)^2}{1/k_1 - 1/k_2} \left[\frac{v_1}{k_1} + \frac{v_2}{k_2} - \frac{1}{k^*(v_{2c})} \right] \tag{3.8.12}$$

$$\nu_A^*(v_{2c}) = \frac{k_1 k_2(\nu_1 - \nu_2)}{k_2 - k_1} \cdot \frac{1}{k^*(v_{2c})} + \frac{\nu_2 k_2 - \nu_1 k_1}{k_2 - k_1} \tag{3.8.13}$$

where $k^*(v_{2c})$ is given by (3.8.7). It is seen that (3.8.12-13) are functions of v_{2c} whose value has to be assumed and of $G_T^*(v_{2c})$ which is not yet known. If (3.8.8) is assumed we obtain Hill's results. If (3.8.10) is assumed we obviously obtain the composite cylinder assemblage results (3.5.96-97), since in that event $k^*(v_{2c})$ reduces to (3.5.91).

The problem of G_T^* is more difficult, though the principle of the method is the same as in the case of k^*. The three layered cylinder, fig. 3.8.1, is now subjected to the boundary conditions (3.4.13). Then G_T^* is given by (3.4.15) or by (3.4.125), the latter being more convenient for our purposes. It is seen that it is now necessary to find the strain $\epsilon_{23}^{(2)}$ in the fiber and for this purpose it is necessary to solve the elasticity problem of the three layered cylinder under boundary conditions (3.4.13) with the usual traction and displacement continuity conditions at the interfaces $r = a, b$. The solution to this problem is of the general form (3.6.85), par. 3.6.4 which was there utilized for the problem of a two layer composite cylinder. There are here three different solutions for the equivalent material, matrix shell and fiber respectively. These involve eight arbitrary constants which are defined by a system of eight linear equations in eight unknowns. Analytical solution of the problem has not yet been carried out and we can therefore not establish here a relation of type (3.8.7) for G_T^* as a function of any v_{2c}. Instead, we shall

consider only the special cases (3.8.8) and (3.8.10). In the first case the fiber is directly imbedded in the equivalent material and there is no matrix shell. In that event the solution for a single fiber in a large cylinder which has been given in par. 3.5.4 for dilute reinforcement is immediately applicable here. We have from (3.5.133) and the first isotropy - transverse isotropy analogy, par. 3.5.1

$$\bar{\epsilon}^{(2)}_{23} = 2\,\frac{k^*(1) + G^*_T(1)}{k^*(1) + [k^*(1) + 2G^*_T(1)]\,G_2/G^*_T(1)}\,\epsilon^{o}_{23} \tag{3.8.14}$$

Insertion of (3.8.14) into (3.4.125) gives

$$G^*_T(1) = G_1 + 2(G_2 - G_1)\,\frac{k^*(1) + G^*_T(1)}{k^*(1) + [k^*(1) + 2G^*_T(1)]\,G_2/G^*_T(1)}\,v_2 \tag{3.8.15}$$

This is equivalent to the result obtained by Hill [3.21]. Eqns. (3.8.9) and (3.8.15) determine the unknowns $k^*(1)$ and $G^*_T(1)$.

Analysis on the basis of the assumption (3.8.10) has been attempted by Hermans [3.34]. The continuity conditions at $r = b$, in the three layer cylinder problem have, however, been disregarded in his analysis and the meaning of his result in the frame of the FEA is therefore not clear. It is a curious fact that Hermans' G^*_T result is the same as the general fibrous material lower bound (3.7.69a). The reason for this is not known to the writer.

It thus appears that at the present time (3.8.15) is the only available FEA result for G_T^*.

Finally we consider the axial effective shear modulus G_A^*. To obtain a FEA for this modulus we can utilize the same method which led to the expression (3.8.7) for k^*. For this purpose we make use of eqns. (3.4.124), (3.5.111), (3.5.130) and (3.5.106). We dispense with the details and give only the final result

$$G_A^*(v_{2c}) = G_1 + 2G_A^*(v_{2c}) \frac{G_c(v_{2c}) - G_1}{G_c(v_{2c}) + G_A^*(v_{2c})} \cdot \frac{v_2}{v_{2c}} \tag{3.8.16}$$

where

$$G_c(v_{2c}) = G_1 \frac{G_1(1-v_{2c})+G_2(1+v_{2c})}{G_1(1+v_{2c})+G_2(1-v_{2c})} \tag{3.8.17}$$

It is seen that (3.8.16) directly determines $G_A^*(v_{2c})$ as the solution of a quadratic. In the special case (3.8.8) we obtain

$$G_A^{*2}(1) - 2G_A^*(1)\,(G_2 - G_1)\,(v_2 - v_1) - G_2\,G_1 = 0 \tag{3.8.18}$$

which is equivalent to Hill's result. In the special case (3.8.10) we obtain

$$G_A^*(v_2) = G_c(v_2) \tag{3.8.19}$$

which because of (3.8.17) implies that (3.8.19) <u>is the composite cylinder assemblage result</u>, (3.5.111). This again agrees with the result obtained in [3.34].

We shall now attempt to assess the importance of the FEA method on the basis of the results which were obtained above. It is recalled that only in the special case (3.8.8) is a complete set of FEA expressions available. They are, however, not convenient results from the computational point of view, since k^* and G_T^* must first be obtained by solution of the two simultaneous equations (3.8.9) and (3.8.15).

It has been shown by Hill [3.21] that his FEA results, which are based on the assumption (3.8.8), are always bracketed by the general fibrous material bounds which were given in par. 3.7.3. It can moreover be shown that if the bounds and Hill's FEA results for any EEM are plotted as function of fiber volume fraction the FEA result starts out tangent to the lower bound and terminates tangent to the upper bound. A schematic example is shown in fig. 3.8.2. Since for the stiff fibers and soft matrix which are used in practice the upper bounds are much higher than the actual values of EEM, as has been explained in par. 3.7.3, it follows that for that case Hill's FEA results also considerably overestimate the EEM values for appreciable fiber volume fractions (50% - 70%) as used in practice.

In the event of a FRM in which the fiber to matrix stiffness ratios are not too high (e.g. metal fibers in different metal matrix) the general fibrous material bounds may become quite close. But in that event the FEA is not needed, since the bounds themselves provide good estimates.

As has been shown above the FEA in Hermans' version, which is based on (3.8.10) does not provide new information since in four cases his expressions reduce to the composite cylinder assemblage results, which are also general fibrous material lower bounds, and in the fifth case (G_T^*) the result appears to be incorrect. But it would seem that the composite cylinder assemblage interpretation of the results is much preferable to their FEA interpretation, since the former is based on rigorous analysis of a model while the latter is based on an approximation of uncertain validity.

Neither of the assumptions (3.8.8) or (3.8.10) has fundamental significance, since as has been pointed out the FEA can be carried out for any value of v_{2c}. (It is most likely, however, that v_{2c} must be bracketed between the values (3.8.8) and (3.8.10), compare [3.35]). Accordingly there exists an infinity of FEA predictions and it is not clear why any of these should be preferred to another.

It may therefore be concluded that the FEA is a method of quite limited value and that available results which are based on rigorous analyses of FRM models are much to be preferred.

3.9 COMMENTS ON MACRO-MECHANICS OF COMPOSITES

In all of the preceding treatment we have been solely concerned with effective elastic moduli of composites. It is recalled that EEM are defined by linearity relations between stress and strain averages of statistically homogeneous fields of stress and strain, and it should be emphasized that such fields are an exception rather than a rule in heterogeneous media. This may be better understood if it is pointed out that a SH field in a SH body is the statistical generalization of a homogeneous field in a homogeneous body. In homogeneous elastic bodies, homogeneous fields of stress and strain arise only in the case of homogeneous boundary conditions in static conditions. In all other cases, and in particular in all dynamic cases, the fields of stress and strain are non-homogeneous, i.e. space variable.

The statistical generalization of a non-homogeneous field is a statistically non-homogeneous (SNH) field. Such fields no longer have the property that body averages are equal to RVE averages.(See fig. 3.2.3). Simple examples of SNH fields are internal stress and strain fields in heterogeneous cylinders in torsion or bending and in vibrating heterogeneous cylinders.

Now the chief interest of the engineer in prediction of EEM or other effective physical properties is to use them in the analysis of structures or parts which are made of composite materials. Since, as has been pointed out, the stress and strain fields in such structures and parts are mostly SNH, there apparently arises a difficulty in the practical use of EEM. A general approach to the problems mentioned would consist of efforts to establish

macroscopic equations of composites in terms of some average quantities which would describe the behavior of the composite in some global sense. This is at present an active research area and a definitive theory does not seem to be available at the present time. It is possible that macroscopic continuum equations of a composite are of multipolar type. For examples of such investigations the reader is referred e.g. to [3.36-37]. Here we shall only give some simple analytical reasoning with the aim of providing some justification for use of classical equations in terms of the effective elastic moduli.

Consider a composite body whose phase geometry is SH. The fields of strain and stress in the body are, however, SNH. We choose a RVE as previously defined which has some specified shape, e.g. a cube. The position of the RVE is determined by the position vector $\underline{x}$ from the body coordinate system to a specified point, e.g. the centroid, in the RVE. The RVE may be located at any place in the body, its sides remaining parallel to the body's coordinate planes. Then the position vector $\underline{x}$ may cover all points within the composite.

Within the RVE in any position define a local coordinate system y_i whose origin is the end point of the position vector $\underline{x}$. Consider the average of displacements taken over RVE. These are given by

$$\bar{u}_i(\underline{x}) = \frac{1}{V'} \int_{V'} u_i(\underline{x} + \underline{y})\, d\underline{y} \quad (*) \qquad (3.9.1)$$

$$d\underline{y} = dy_1\, dy_2\, dy_3$$

(*) Such an average is sometimes called a moving average, since it may be imagined that the RVE moves throughout the body with the average being taken instantaneously.

where V' is the RVE volume. It is assumed that the average is a continuous function of $\underline{x}$. The displacement $u_i(\underline{x}+\underline{y})$ is called the micro-displacement while the average $\bar{u}_i(\underline{x})$ is called the macro-displacement. Similar averages can of course be written down for any quantity.

Next consider the gradient of $\bar{u}_i(\underline{x})$. We have

$$\frac{\partial \bar{u}_i}{\partial x_j} = \frac{1}{V'} \int_{V'} \frac{\partial u_i(\underline{x}+\underline{y})}{\partial x_j} d\underline{y} = \frac{1}{V'} \int_{V'} \frac{\partial u_i(\underline{x}+\underline{y})}{\partial y_j} d\underline{y} \qquad (3.9.2)$$

The left side of (3.9.2) may be called the macro-gradient while $\frac{\partial u_i(\underline{x}+\underline{y})}{\partial y_j}$ may be called the micro-gradient. Equ. (3.9.2) states that the macro-gradient is the average of the micro-gradient. Analogously, we define a macro-strain by

$$\bar{\epsilon}_{ij}(\underline{x}) = \frac{1}{2}\left(\frac{\partial \bar{u}_i(\underline{x})}{\partial x_j} + \frac{\partial u_j(\underline{x})}{\partial x_i}\right) \qquad (3.9.3)$$

and a micro-strain by

$$\epsilon_{ij}(\underline{x}+\underline{y}) = \frac{1}{2}\left(\frac{\partial u_i(\underline{x}+\underline{y})}{\partial y_j} + \frac{\partial u_j(\underline{x}+\underline{y})}{\partial y_i}\right) \qquad (3.9.4)$$

It follows from (3.9.3-4) that

$$\bar{\epsilon}_{ij}(\underline{x}) = \frac{1}{V'} \int_{V'} \epsilon_{ij}(\underline{x}+\underline{y})\, d\underline{y} \qquad (3.9.5)$$

which implies that the macro-strain is the average of the micro-strain.

We define macro-stresses and macro-body forces by

$$\bar{\sigma}_{ij}(\underline{x}) = \frac{1}{V'} \int_{V'} \sigma_{ij}(\underline{x} + \underline{y})\, d\underline{y}$$

$$\bar{F}_i(\underline{x}) = \frac{1}{V'} \int_{V'} F_i(\underline{x} + \underline{y})\, d\underline{y} \qquad (3.9.6)$$

Averaging of the equilibrium equations for micro-stresses

$$\frac{\partial \sigma_{ij}(\underline{x} + \underline{y})}{\partial y_i} + F_i(\underline{x} + \underline{y}) = 0$$

as done in (3.9.2), easily yields similar equilibrium equations for macro-stresses. Thus

$$\frac{\bar{\partial} \sigma_{ij}(\underline{x})}{\partial x_j} + \bar{F}_i(\underline{x}) = 0 \qquad (3.9.7)$$

All results derived up to now are exact. It is our purpose now to establish differential equations for the macro-displacements $\bar{u}_i$. The simplest approach that we can take is to assume that the stresses and strains in any RVE may be considered to be statistically homogeneous. Note carefully that this does not imply that stresses and strains are SH <u>throughout</u> the body, since the averages over RVE vary with position.

The assumption made is certainly reasonable when the space variation of RVE averages, e.g. of macro-stresses and macro-strains is not drastic through typical RVE dimensions. If this assumption is adopted we can write down the macro-stress-strain relation

$$\bar{\sigma}_{ij}(\underline{x}) = C^*_{ijkl}\,\bar{\epsilon}_{kl}(\underline{x}) \tag{3.9.8}$$

where C^*_{ijkl} are the usual EEM.

It is now seen that equs. (3.9.3), (3.9.7-8) have just the form of the usual elasticity equation of par. 3.2.1. Indeed, substitution of (3) into (8) and substitution of the resulting macro-stresses into (7) yields

$$C^*_{ijkl}\,\bar{u}_{k,lj} + \bar{F}_i = 0 \tag{3.9.9}$$

which are of the form of the classical elasticity equations, (3.2.13).

Boundary conditions for $\bar{u}_i$ are obtained by performance of the average (3.9.1) over RVE's near the boundary, thus in a boundary layer. It is intuitively plausible that actual deterministic boundary conditions

$$\begin{aligned} u_i(S) &= u^o_i \quad \text{on } S_u \\ T_i(S) &= T^o_i \quad \text{on } S_T \end{aligned} \tag{3.9.10}$$

prescribed over the surface of the composite, can be approximated by specification of the boundary layer conditions

$$\begin{aligned} \bar{u}_i(S) &= u^o_i \\ \bar{T}_i(S) &= T^o_i \end{aligned} \tag{3.9.11}$$

where overbars denote local RVE averages. This completes the mathematical formulation of the problem for the determination of the macro-displacements $\bar{u}_i(\underline{x})$.

It is recalled that the crucial assumption used is contained in equ. (3.9.8). More complicated theories can presumably be established by generalization of (3.9.8).

Since the present theory is completely analogous to classical elasticity theory it is immediately concluded that all classical static elasticity solutions, and also elastic strength of materials results, generate similar results for macro-displacements, strains and stresses as defined here, simply by replacement of homogeneous elastic moduli by effective moduli.

The present theory may be called the first approximation to macro-elasticity theory of composites. The writer believes that on the basis of our experience with composite structures and parts it should provide reasonably accurate answers for most elastostatic problems of composite bodies.

To give an example for the first approximation consider the deflection of a uniaxially fiber reinforced beam under transverse central concentrated load, with the fibers in beam axis direction. For a homogeneous transversely isotropic beam

$$\delta = \frac{P\ell^3}{48E_A I}$$

where

P	- load
I	- moment of inertia
ℓ	- span
E_A	- axial Young's modulus
δ	- deflection under load.

If the beam is fiber reinforced with effective axial Young's modulus E_A^*, then according to the first approximation

$$\delta = \frac{P\ell^3}{48E_A^* I}$$

Note that the last result is based on an Euler-Bernoulli assumption for macro-displacements.

Similarly, all static solutions for homogeneous structures may now be interpreted as first approximation solutions for structures made of composite materials, simply by replacement of the elastic moduli in the classical results by effective elastic moduli.

Establishment of a similar first order approximation for elasto-dynamics of composites, in general, and of FRM, in particular, is not so simple and straightforward. It has proved possible to derive such a theory for two phase materials. The resulting macro-differential equations are similar to classical equation of elastodynamics and contain in their coefficients static EEM and effective densities which are <u>not</u> the average densities, (to be published).

The problem of dispersion of elastic waves due to material heterogeneity is at present an active research area. For examples of such work the reader is referred to [3.37-38].

3.10 BIAXIAL AND MULTIAXIAL FIBER REINFORCEMENT: LAMINATES

3.10.1 Introduction

Elastic analyses of conventional uniaxial FRM which were given in preceding chapters, as well as experimental results, show that stiffness in fiber direction can be much larger than stiffness in transverse direction. To give an example we consider the ratio $E^*_A/E^*_{T(+)}$ on the basis of the numerical results given in tables 3.2-4. For $v_1 = v_2 = 0.5$ we obtain for glass/epoxy, boron/epoxy and boron/aluminum the values 3.64, 17.70 and 1.55, respectively, for this ratio. It is seen that the ratio is largest for boron/epoxy the reason being, of course, that this material has the largest phase stiffness ratio of the three FRM considered.

A similar even more pronounced difference exists between strengths in axial and transverse directions.

These anisotropic stiffness and strength properties lead to important engineering conclusions. Evidently, the uniaxial FRM is a suitable material for structural members which are predominantly uniaxially stressed; thus for structural members subject to axial forces and bending in one plane. Such structural members are bars, struts, beams, frames and rings. On the other hand, plate and shell structures are subjected to forces and moments in all directions within their surfaces and thus uniaxial reinforcement is not

suitable, for it is liable to expose the weak transverse direction of the material to unbearable stresses. (It should be borne in mind that a material as boron/aluminum is less limited in this respect.) For such structures it is therefore advantageous to employ fiber reinforcement in more than one direction. This is usually done by construction of FRM which consist of parallel uniaxially reinforced layers. Fig. 3.10.1a shows a biaxially reinforced material in which the angle of reinforcement alternates from layer to layer. Triaxially and multiaxially reinforced materials may be constructed in similar fashion.

It is necessary to distinguish between two fundamentally different cases. In the first case, each layer contains only one or two planar sets of fibers, fig. 3.10.1b. Such a material must be considered as a three dimensional fiber reinforced material. Its analysis is exceedingly difficult and is at the present time an open subject. Methods of analysis for uniaxial FRM which were discussed in preceding chapters are not applicable.

In the second case, each layer contains many uniaxially oriented fibers, fig. 3.10.1c, and therefore the layer itself may be considered as a uniaxial FRM whose properties were discussed in preceding chapters. There is therefore introduced the fundamental approximation that each layer may be replaced by a homogeneous material whose elastic moduli are the EEM of the uniaxial FRM. If this approximation is accepted, the theory of biaxially (or multiaxially) reinforced plates and shells

reduces at once to theory of laminated plates and shells in which the laminae or layers are anisotropic.

3.10.2 Laminae Stress-Strain Relations

Since the elastic axes of the laminae are differently oriented, it is necessary to refer their elastic properties to one common coordinate system. Taking as an example the biaxial laminate(*), it is advantageous to refer the elastic properties of a layer or lamina to a cartesian coordinate system one of whose axes is perpendicular to the layer surface while the other two axes bisect the angles produced by the alternating directions of reinforcement, fig. 3.10.2. In the case of a plate this is the coordinate system to which the entire laminated plate is referred. In the case of a shell the coordinate system described defines the local directions of a curvilinear system one of whose surfaces is the shell surface.

Let it be assumed that the laminae are made of the same FRM which is transversely isotropic around its fiber direction. The fixed coordinate system to which the biaxial material is referred is denoted $x_1\ x_2\ x_3$, the material coordinate system of layers with reinforcement angle θ is denoted $x'_1\ x'_2\ x'_3$ and the corresponding coordinate system of layers with reinforcement angle

(*) Such a laminate is sometimes called: angle-ply. In the special case of perpendicular fiber directions it is called: cross-ply.

$-\theta$ is denoted $x''_1\ x''_2\ x''_3$. In these layers x'_1, x''_1 are in respective fiber direction, fig. 3.10.2.

The elastic moduli of the laminae, referred to their material coordinate systems, are by hypothesis the effective elastic moduli of the same transversely isotropic FRM and are denoted $C_{ijk\ell}$. It is necessary to transform these moduli to the $x_1\ x_2\ x_3$ system. The transformed moduli in the $x'_1\ x'_2\ x'_3$ and $x''_1\ x''_2\ x''_3$ systems are denoted $C'_{ijk\ell}$ and $C''_{ijk\ell}$, respectively.

By the laws of tensor transformation

$$C'_{ijk\ell} = \ell'_{ip}\,\ell'_{jq}\,\ell'_{kr}\,\ell'_{\ell s}\,C_{pqrs} \qquad \text{(a)}$$

$$C''_{ijk\ell} = \ell''_{ip}\,\ell''_{jq}\,\ell''_{kr}\,\ell''_{\ell s}\,C_{pqrs} \qquad \text{(b)} \qquad (3.10.1)$$

where ℓ'_{ij} and ℓ''_{ij} are the sets of direction cosines which define the position of the $x'_1\ x'_2\ x'_3$ and $x''_1\ x''_2\ x''_3$ system. The relative position of these systems are defined by rotations of θ and $-\theta$, respectively, in the $x_1\ x_2$ plane. Therefore,

$$\ell'_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(a)}$$

$$(3.10.2)$$

$$\ell''_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(b)}$$

Similarly, for compliances

$$S'_{ijk\ell} = \ell'_{ip}\,\ell'_{jq}\,\ell'_{kr}\,\ell'_{\ell s}\,S_{pqrs} \qquad \text{(a)}$$

$$S''_{ijk\ell} = \ell''_{ip}\,\ell''_{jq}\,\ell''_{kr}\,\ell''_{\ell s}\,S_{pqrs} \qquad \text{(b)} \qquad (3.10.3)$$

From (3.4.47-49), (3.4.51-55) and (3.4.78) single lamina moduli and compliances are given by

$$[C_{pqrs}] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1122} & C_{2233} & C_{2222} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \qquad \text{(a)}$$

$$C_{2323} = \frac{1}{2}\,(C_{2222} - C_{2233}) \qquad \text{(b)} \qquad (3.10.4)$$

or in "engineering" notation

$$[C_{pq}] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \qquad \text{(a)}$$

$$C_{55} = \frac{1}{2}\,(C_{22} - C_{23}) \qquad \text{(b)} \qquad (3.10.5)$$

with similar matrices for compliances.

The moduli and compliance components may also be written as

$$
\begin{aligned}
C_{1111} &= C_{11} = n \\
C_{2222} &= C_{22} = k+G_T \\
C_{1122} &= C_{22} = \ell \\
C_{2233} &= C_{23} = k-G_T \\
C_{1212} &= C_{44} = G_A \\
C_{2323} &= C_{55} = G_T
\end{aligned}
\tag{3.10.6}
$$

$$
\begin{aligned}
S_{1111} &= S_{11} = \frac{1}{E_A} \\
S_{2222} &= S_{22} = \frac{1}{E_T} \\
S_{1122} &= S_{12} = -\frac{\nu_A}{E_A} \\
S_{2233} &= S_{23} = -\frac{\nu_T}{E_T} \\
S_{1212} &= S_{44} = \frac{1}{4G_A} \\
S_{2323} &= S_{55} = \frac{1}{4G_T}
\end{aligned}
\tag{3.10.7}
$$

and all others vanish. It is recalled that all moduli and compliances in (3.10.6-7) are effective.

It follows from (3.10.1), (3.10.2) and (3.10.3) that the moduli C'_{pqrs} are given by the following symmetric matrix

$$[C'_{pqrs}] = \begin{bmatrix} C'_{1111} & C'_{1122} & C'_{1133} & C'_{1112} & 0 & 0 \\ C'_{1122} & C'_{2222} & C'_{2233} & C'_{2212} & 0 & 0 \\ C'_{1133} & C'_{2233} & C'_{3333} & C'_{3312} & 0 & 0 \\ C'_{1112} & C'_{2212} & C'_{3312} & C'_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C'_{2323} & C'_{2331} \\ 0 & 0 & 0 & 0 & C'_{2331} & C'_{3131} \end{bmatrix} \qquad (3.10.8)$$

In the engineering notation, (3.10.8) assumes the form

$$[C'_{pq}] = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & C'_{14} & 0 & 0 \\ C'_{12} & C'_{22} & C'_{23} & C'_{24} & 0 & 0 \\ C'_{13} & C'_{23} & C'_{33} & C'_{34} & 0 & 0 \\ C'_{14} & C'_{24} & C'_{34} & C'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C'_{55} & C'_{56} \\ 0 & 0 & 0 & 0 & C'_{56} & C'_{66} \end{bmatrix} \qquad (3.10.9)$$

Obviously, the C''_{pqrs}, S'_{pqrs} and S''_{pqrs} matrices have the same form as (3.10.6-7).

The C'_{pq} components are given by

$$
\begin{aligned}
C'_{11} &= C_{11}\cos^4\theta + C_{22}\sin^4\theta + 2C_{12}\cos^2\theta\sin^2\theta + 4C_{44}\cos^2\theta\sin^2\theta \\
C'_{12} &= (C_{11}+C_{22})\cos^2\theta\sin^2\theta + C_{12}(\cos^4\theta + \sin^4\theta) - 4C_{44}\cos^2\theta\sin^2\theta \\
C'_{22} &= C_{11}\sin^4\theta + C_{22}\cos^4\theta + 2C_{12}\cos^2\theta\sin^2\theta + 4C_{44}\cos^2\theta\sin^2\theta \\
C'_{13} &= C_{12}\cos^2\theta + C_{23}\sin^2\theta \\
C'_{23} &= C_{12}\sin^2\theta + C_{23}\cos^2\theta \\
C'_{33} &= C_{22} \qquad\qquad (3.10.10) \\
C'_{14} &= -C_{11}\cos^3\theta\sin\theta + C_{22}\cos\theta\sin^3\theta + C_{12}(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta) \\
&\quad + 2C_{44}(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta) \\
C'_{24} &= -C_{11}\cos\theta\sin^3\theta + C_{22}\cos^3\theta\sin\theta + C_{12}(\cos\theta\sin^3\theta - \cos^3\theta\sin\theta) \\
&\quad + 2C_{44}(\cos\theta\sin^3\theta - \cos^3\theta\sin\theta) \\
C'_{34} &= (C_{23} - C_{12})\cos\theta\sin\theta \\
C'_{44} &= (C_{11} + C_{22})\cos^2\theta\sin^2\theta - 2C_{12}\cos^2\theta\sin^2\theta + C_{44}(\cos^2\theta - \sin^2\theta)^2 \\
C'_{55} &= C_{44}\sin^2\theta + C_{55}\cos^2\theta \\
C'_{56} &= (C_{55} - C_{44})\cos\theta\sin\theta \\
C'_{66} &= C_{44}\cos^2\theta + C_{55}\sin^2\theta
\end{aligned}
$$

For a lamina whose fibers are oriented at $-\theta$ the C''_{pqrs} or C''_{pq} moduli are obtained by replacing $\sin\theta$ by $-\sin\theta$ in (3.10.10). It is seen therefore that C''_{pq} is given by

$$[C''_{pq}] = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & -C'_{14} & 0 & 0 \\ C'_{12} & C'_{22} & C'_{23} & -C'_{24} & 0 & 0 \\ C'_{13} & C'_{23} & C'_{33} & -C'_{34} & 0 & 0 \\ -C'_{14} & -C'_{24} & -C'_{34} & -C'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C'_{55} & -C'_{56} \\ 0 & 0 & 0 & 0 & -C'_{56} & -C'_{66} \end{bmatrix} \qquad (3.10.11)$$

The results (3.10.10-11) obviously are also valid for compliances by replacement of moduli by corresponding compliances everywhere.

It should be noted that all of the preceding developments remain valid for square symmetric layers if the relations (3.10.4b) and (3.10.5b) are omitted and thus C_{2323} or C_{55} becomes an independent modulus.

Finally, it is mentioned that in the case of a general laminate where each lamina is oriented at some angle, relative to an appropriate common coordinate system, equs. (3.10.6-7), (3.10.9-10) define the elastic properties of any lamina with respect to the common coordinate system.

3.10.3 Laminated Plates

Theory of laminated plates and shells is a subject of considerable engineering importance which has received great impetus by the advent of

fiber reinforced materials. A sample of important references is given by [3.39-45].

Here we shall limit ourselves to establishment of differential equations for a thin laminated plate, which is subjected to bending and in-plane forces. The plate is referred to a fixed cartesian coordinate system $x_1\ x_2\ x_3$, the x_1, x_2 plane being the "reference surface" which need not be the middle plane of the plate. Fig. 3.10.1a may be regarded as a typical element of such a plate. Reinforcement direction of the laminae is not specified at the present time and the elastic properties of the mth lamina, referred to the plate coordinates, are denoted $C^{(m)}_{ijk\ell}$ in general fashion.

Conventional analysis of laminated plates is based on the Kirchhoff-Love assumption according to which: A normal to a reference surface of the undeformed plate remains straight and normal to the deformed reference surface. The mathematical expression of this assumption is

$$\begin{aligned} u_3 &= u_3(x_1, x_2) = w(x_1, x_2) \qquad \text{(a)} \\ u_\alpha &= u^o_\alpha(x_1, x_2) - x_3\, w_{,\alpha} \end{aligned} \qquad (3.10.12)$$

where w and u_α are transverse and in-plane displacements, respectively, u^o_α are in-plane displacements of the reference surface and α - as well as other greek subscripts from now on - ranges over 1, 2.

The strains $\epsilon_{\alpha\beta}$ are then given by

$$\epsilon_{\alpha\beta} = \epsilon^{o}_{\alpha\beta}(x_1, x_2) + x_3\ \kappa_{\alpha\beta}(x_1, x_2) \qquad \text{(a)}$$

$$\epsilon^{o}_{\alpha\beta} = \frac{1}{2}(u^{o}_{\alpha,\beta} + u^{o}_{\beta,\alpha}) \qquad \text{(b)} \qquad (3.10.13)$$

$$\kappa_{\alpha\beta} = -\ w_{,\beta\alpha} \qquad \text{(c)}$$

where $\kappa_{\alpha\beta}$ are the curvatures and $\epsilon^{o}_{\alpha\beta}$ are the reference surface strains.

We consider the plane part $\sigma^{(m)}_{\alpha\beta}$ of the stress $\sigma^{(m)}_{ij}$ in the mth layer. In view of (3.10.13a) this stress is given by

$$\sigma^{(m)}_{\alpha\beta} = C^{(m)}_{\alpha\beta\gamma\delta}\ \epsilon_{\gamma\delta} = C^{(m)}_{\alpha\beta\gamma\delta}\ \epsilon^{o}_{\gamma\delta} + x_3\ C^{(m)}_{\alpha\beta\gamma\delta}\ \kappa_{\gamma\delta} \qquad (3.10.14)$$

Since the subscripts range over 1, 2 only, it follows from (3.10.8) that $C^{(m)}_{\alpha\beta\gamma\delta}$ are given by the following matrix

$$[C^{(m)}_{\alpha\beta\gamma\delta}] = \begin{bmatrix} C^{(m)}_{1111} & C^{(m)}_{1122} & C^{(m)}_{1112} \\ C^{(m)}_{1122} & C^{(m)}_{2222} & C^{(m)}_{2212} \\ C^{(m)}_{1112} & C^{(m)}_{2212} & C^{(m)}_{1212} \end{bmatrix} \qquad (3.10.15)$$

or in engineering notation

$$[C^{(m)}_{pq}] = \begin{bmatrix} C^{(m)}_{11} & C^{(m)}_{12} & C^{(m)}_{14} \\ C^{(m)}_{12} & C^{(m)}_{22} & C^{(m)}_{24} \\ C^{(m)}_{14} & C^{(m)}_{24} & C^{(m)}_{44} \end{bmatrix} \qquad (3.10.16)$$

$$pq = 1, 2, 4$$

Let the laminae be labeled by numbers 1, 2, ... M starting with the uppermost lamina. The mth lamina is between the planes $x_3 = h_m$, $x_3 = h_{m-1}$, the m plane always being further away from the upper face than the m-1 plane, fig. 3.10.3. The membrane forces $N_{\alpha\beta}$ and internal moments $M_{\alpha\beta}$ of the plate are then defined by

$$N_{\alpha\beta} = \int_{h_o}^{h_M} \sigma_{\alpha\beta}\, dx_3 \qquad \text{(a)}$$

$$M_{\alpha\beta} = \int_{h_o}^{h_M} x_3\, \sigma_{\alpha\beta}\, dx_3 \qquad \text{(b)} \qquad (3.10.17)$$

Insertion of (3.10.14) into (3.10.17) yields the results

$$N_{\alpha\beta} = A_{\alpha\beta\gamma\delta}\, \overset{o}{\epsilon}_{\gamma\delta} + B_{\alpha\beta\gamma\delta}\, \kappa_{\gamma\delta} \qquad \text{(a)}$$

$$M_{\alpha\beta} = B_{\alpha\beta\gamma\delta}\, \overset{o}{\epsilon}_{\gamma\delta} + D_{\alpha\beta\gamma\delta}\, \kappa_{\gamma\delta} \qquad \text{(b)} \qquad (3.10.18)$$

where

$$A_{\alpha\beta\gamma\delta} = \sum_m C^{(m)}_{\alpha\beta\gamma\delta}\, (h_m - h_{m-1}) \qquad \text{(a)}$$

$$B_{\alpha\beta\gamma\delta} = \frac{1}{2} \sum_m C^{(m)}_{\alpha\beta\gamma\delta}\, (h_m^2 - h_{m-1}^2) \qquad \text{(b)} \qquad (3.10.19)$$

$$D_{\alpha\beta\gamma\delta} = \frac{1}{3} \sum_m C^{(m)}_{\alpha\beta\gamma\delta}\, (h_m^3 - h_{m-1}^3) \qquad \text{(c)}$$

In engineering notation

$$A_{pq} = \sum_m C_{pq}^{(m)} (h_m - h_{m-1}) \qquad \text{(a)}$$

$$B_{pq} = \frac{1}{2} \sum_m C_{pq}^{(m)} (h_m^2 - h_{m-1}^2) \qquad \text{(b)} \qquad (3.10.20)$$

$$D_{pq} = \frac{1}{3} \sum_m C_{pq}^{(m)} (h_m^3 - h_{m-1}^3) \qquad \text{(c)}$$

It should be carefully noted that the h_m in (3.10.19-20) are to be taken with their proper signs.

Insertion of (3.10.13 b,c) into (3.10.18) gives the expressions

$$N_{\alpha\beta} = A_{\alpha\beta\gamma\delta}\, u^o_{\gamma,\delta} - B_{\alpha\beta\gamma\delta}\, w_{,\alpha\beta} = N_{\beta\alpha} \qquad \text{(a)}$$

$$(3.10.21)$$

$$M_{\alpha\beta} = B_{\alpha\beta\gamma\delta}\, u^o_{\gamma,\delta} - D_{\alpha\beta\gamma\delta}\, w_{,\alpha\beta} = M_{\alpha\beta} \qquad \text{(b)}$$

The plate resultant equilibrium equations are

$$N_{\alpha\beta,\beta} + p_\alpha = 0 \qquad \text{(a)}$$

$$(3.10.22)$$

$$M_{\alpha\beta,\alpha\beta} + p = 0 \qquad \text{(b)}$$

where p_α and p are loads per unit area in the x_1, x_2 and x_3 direction, respectively. (The contribution of membrane forces to x_3 equilibrium, through plate curvature, has been neglected). Insertion of (3.10.21) into (3.10.22) yields the set of three differential equations

$$A_{\alpha\beta\gamma\delta}\, u^{o}_{\gamma,\beta\delta} - B_{\alpha\beta\gamma\delta}\, w_{,\beta\gamma\delta} + p_{\alpha} = 0 \qquad \text{(a)}$$

$$B_{\alpha\beta\gamma\delta}\, u^{o}_{\gamma,\alpha\beta\delta} - D_{\alpha\beta\gamma\delta}\, w_{,\alpha\beta\gamma\delta} + p = 0 \qquad \text{(b)} \qquad (3.10.23)$$

Typical boundary conditions consist of prescription of displacements and/or force and moment resultants on the plate boundary. The force and moment boundary conditions are expressed in terms of displacement derivatives via (3.10.21).

Consider for example the case of a simply supported rectangular plate with sides a_1, a_2 and rigid horizontal support. The boundary conditions are in this case:

$$u^{o}_{\alpha}(0,x_2) = u^{o}_{\alpha}(a_1,x_2) = u^{o}_{\alpha}(x_1, 0) = u^{o}_{\alpha}(x_1, a_2) = 0$$
$$w(0,x_2) = w(a_1, x_2) = w(x_1, 0) = w(x_1, a_1) = 0$$
$$M_{11}(0,x_2) = M_{11}(a_1, x_2) = M_{22}(x_1, 0) = M_{22}(x_1, a_2) = 0$$

where the moments M_{11}, M_{22} and M_{12} should be expressed by (3.10.21b).

The complexity of equations (3.10.23) makes it necessary, in most cases, to resort to numerical methods of solutiom. When the displacements are known, the strains $\epsilon^{o}_{\alpha\beta}$ and curvatures $\kappa_{\alpha\beta}$ follow from (3.10.13b). Then the plane stresses $\sigma^{(m)}_{\alpha\beta}$ in all layers are determined by (3.10.14).

It should be noted that shear stresses $\sigma^{(m)}_{3\alpha}$ may be determined from $\sigma^{(m)}_{\alpha\beta}$ by equilibrium considerations, while $\sigma^{(m)}_{33}$ is assumed to be negligibly small in conventional plate theory.

The problem may be differently formulated by use of an Airy stress function to satisfy (3.10.22a) identically (compare e.g. [3.45]).

It is seen that in the general formulation given there is coupling between in-plane stretching-shearing and transverse bending-twisting. Thus, for example, a plate which is loaded by in-plane forces, only, will in general also experience bending and twisting.

We now consider some simplified specific cases of lamination, as used in practice, and the ensuing simplifications of the plate boundary value problem. Let it be first assumed that the laminated plate is geometrically symmetric with respect to its middle surface. By this is meant that to each lamina on one side of the middle surface corresponds another lamina which is its mirror image in the middle surface. Such a lamina pair is shown in fig. 3.10.4. In this event it is convenient to choose the middle surface as the plate reference surface. The upper lamina is labeled m and is bounded by the planes $x_3 = h_{m-1}, h_m$, while the lower lamina is labeled n and is bounded by the planes $x_3 = h_n, h_{n-1}$. The m,n subscripts conform to the convention employed in (3.10.19-20).

By the assumed symmetry

$$h_n = h_{m-1} \qquad (3.10.24)$$

$$h_{n-1} = - h_m$$

It follows from (3.10.24) that

$$h_m - h_{m-1} = h_n - h_{n-1} \qquad \text{(a)}$$

$$h_m^2 - h_{m-1}^2 = -(h_n^2 - h_{n-1}^2) \qquad \text{(b)} \qquad (3.10.25)$$

$$h_m^3 - h_{m-1}^3 = h_n^3 - h_{n-1}^3 \qquad \text{(c)}$$

Let it be further assumed that the laminate is also elastically symmetric with respect to the middle surface, by which is meant that

$$C^{(m)}_{\alpha\beta\gamma\delta} = C^{(n)}_{\alpha\beta\gamma\delta} \qquad \text{(a)}$$

$$C^{(m)}_{pq} = C^{(n)}_{pq} \qquad p,q = 1,\ 2,\ 4 \qquad \text{(b)} \qquad (3.10.26)$$

A laminate which is both geometrically and elastically symmetric with respect to its middle surface is henceforth simply called: symmetric laminate.

It follows from (3.10.19b), (3.10.20b), (3.10.25b) and (3.10.26) that for a symmetric laminate

$$B_{\alpha\beta\gamma\delta} = 0 \qquad \text{(a)}$$

$$B_{pq} = 0 \qquad p,q = 1,\ 2,\ 4 \qquad \text{(b)} \qquad (3.10.27)$$

Introduction of (3.10.27) into (3.10.21) and (3.10.23) simplifies these equations to

$$N_{\alpha\beta} = A_{\alpha\beta\gamma\delta}\, u^o{}_{\gamma,\delta} \qquad \text{(a)}$$

$$M_{\alpha\beta} = -D_{\alpha\beta\gamma\delta}\, w_{,\gamma\delta} \qquad \text{(b)} \qquad (3.10.28)$$

$$A_{\alpha\beta\gamma\delta}\, u^{o}{}_{\gamma,\beta\delta} + p_{\alpha} = 0 \qquad \text{(a)}$$

$$D_{\alpha\beta\gamma\delta}\, w_{,\alpha\beta\gamma\delta} - p = 0 \qquad \text{(b)} \qquad (3.10.29)$$

It is seen that for the symmetric laminate the previous general formulation "decouples", i.e. in-plane displacements $u^{o}{}_{\alpha}$ and membrane forces $N_{\alpha\beta}$ are determined independently of transverse deflection w and bending moments $M_{\alpha\beta}$.

Equs. (3.10.29b), (3.10.28b), written out in detail, in the notation (3.10.20), are

$$D_{11}\, w_{,1111} + 4D_{14}\, w_{,1112} + 2(D_{12} + 2D_{44})\, w_{,1122} + 4D_{24}\, w_{,1222} + D_{22}\, w_{,2222} = 0 \qquad \text{(a)}$$

$$M_{11} = -\,(D_{11}\, w_{,11} + D_{12}\, w_{,22} + 2D_{14}\, w_{,12}) \qquad \text{(b)}$$

$$M_{22} = -\,(D_{12}\, w_{,11} + D_{22}\, w_{,22} + 2D_{24}\, w_{,12}) \qquad \text{(c)} \qquad (3.10.30)$$

$$M_{12} = -\,(D_{14}\, w_{,11} + D_{22}\, w_{,22} + 2D_{44}\, w_{,12}) \qquad \text{(d)}$$

The symmetric laminate is now further specialized by the assumption that the laminae are uniaxially fiber reinforced, the reinforcement direction alternating over θ, $-\theta$ with respect to the plate x_1 axis, from lamina to lamina. The relation between elastic moduli of adjacent laminae is then defined by (3.10.11). In the present case only the moduli (3.10.15-16) are of interest. It follows from (3.10.11) that

$$C''_{pq} = \begin{bmatrix} C'_{11} & C'_{12} & -C'_{14} \\ C'_{12} & C'_{22} & -C'_{24} \\ -C'_{14} & C'_{24} & C'_{44} \end{bmatrix} \tag{3.10.31}$$

where C'_{pq} (θ orientation) are given by (3.10.10). Since C_{11}, C_{12}, C_{22}, C_{44} are the same for both kinds of laminae, it follows from (3.10.20 a,c) and (3.10.25 a,c) that

$$\begin{aligned} [A_{11}, A_{12}, A_{22}, A_{44}] &= h[C'_{11}, C'_{12}, C'_{22}, C'_{44}] \quad \text{(a)} \\ [D_{11}, D_{12}, D_{22}, D_{44}] &= \frac{h^3}{12}[C'_{11}, C'_{12}, C'_{22}, C'_{44}] \quad \text{(b)} \end{aligned} \tag{3.10.32}$$

where h is the plate thickness.

The components A_{14}, A_{24}, D_{14} and D_{24} depend on the laminae thicknesses, cannot be expressed in such simple form as (3.10.32) and should be computed from (3.10.20). However, in the event that the sum of the θ laminae thicknesses is equal to the sum of the $-\theta$ laminae thicknesses, it follows from (3.10.20a), (3.10.25a) and (3.10.31) that

$$A_{14} = A_{24} = 0 \tag{3.10.33}$$

In this event equs. (3.10.28a) and (3.10.29a) assume orthotropic form.

An important kind of laminate is an odd number of equally thick laminae with θ, $-\theta$ alternating reinforcement direction. The laminate is evidently symmetric and so the results (3.10.27-30) and (3.10.32) apply. However, (3.10.33) is not valid.

A related important laminate is an even number of θ, $-\theta$ alternating laminae of equal thickness. This laminate is geometrically symmetric with respect to its middle surface, but it is not elastically symmetric, for in each pair of symmetrically located laminae one is at θ orientation while the other is at $-\theta$ orientation. Labeling again the upper lamina in the pair by m and the lower by n, the relations (3.10.25) remain valid. Also, in view of (3.10.31)

$$[C^{(n)}_{pq}] = \begin{bmatrix} C^{(m)}_{11} & C^{(m)}_{12} & -C^{(m)}_{14} \\ C^{(m)}_{12} & C^{(m)}_{22} & -C^{(m)}_{24} \\ -C^{(m)}_{14} & -C^{(m)}_{24} & C^{(m)}_{44} \end{bmatrix} \tag{3.10.34}$$

Combining (3.10.34) with (3.10.20) and (3.10.25) we have

$$[A_{pq}] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{44} \end{bmatrix} = h \begin{bmatrix} C'_{11} & C'_{12} & 0 \\ C'_{12} & C'_{22} & 0 \\ 0 & 0 & C'_{44} \end{bmatrix} \tag{a}$$

$$[B_{pq}] = \begin{bmatrix} 0 & 0 & B_{14} \\ 0 & 0 & B_{24} \\ B_{14} & B_{24} & 0 \end{bmatrix} \tag{b}$$

(3.10.35)

$$[D_{pq}] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{44} \end{bmatrix} = \frac{h^3}{12} \begin{bmatrix} C'_{11} & C'_{12} & 0 \\ C'_{12} & C'_{22} & 0 \\ 0 & 0 & C'_{44} \end{bmatrix} \tag{c}$$

The components B_{14}, B_{24} depend on the laminae thicknesses, are not given by such simple relations as (3.10.35 a,c) and should be computed from (3.10.20b).

Establishment of theories of laminated shells on the basis of the Kirchhoff-Love assumption is similar to development of theory of laminated plates, though more complicated. The interested reader is referred e.g. to [3.42-43], [3.45].

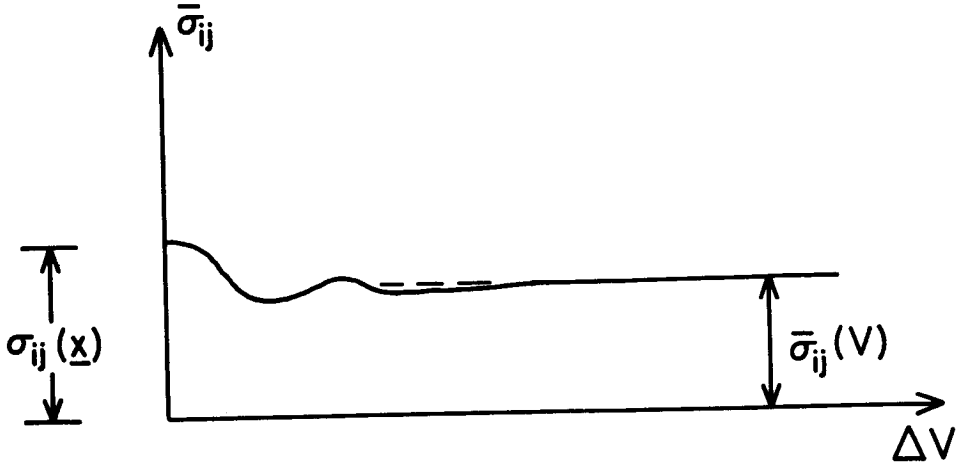


FIG. 3.2.1 - VARIATION OF AVERAGE STRESS WITH SIZE OF VOLUME ELEMENT

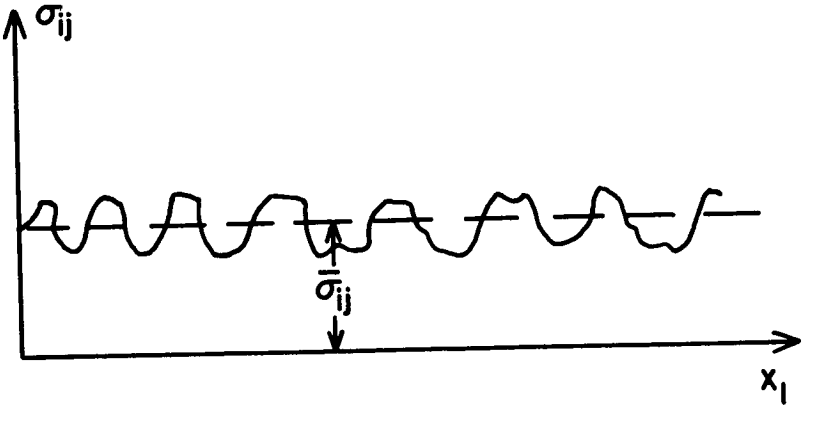


FIG. 3.2.2 - VARIATION OF STATISTICALLY HOMOGENEOUS STRESS IN A COMPOSITE

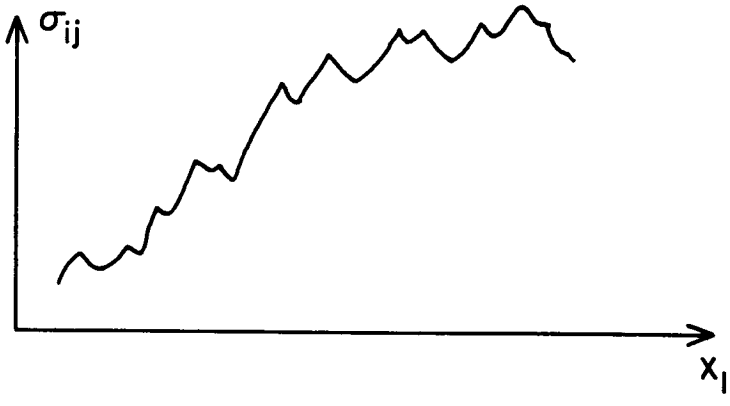


FIG. 3.2.3 - VARIATION OF STATISTICALLY NONHOMOGENEOUS STRESS IN A COMPOSITE

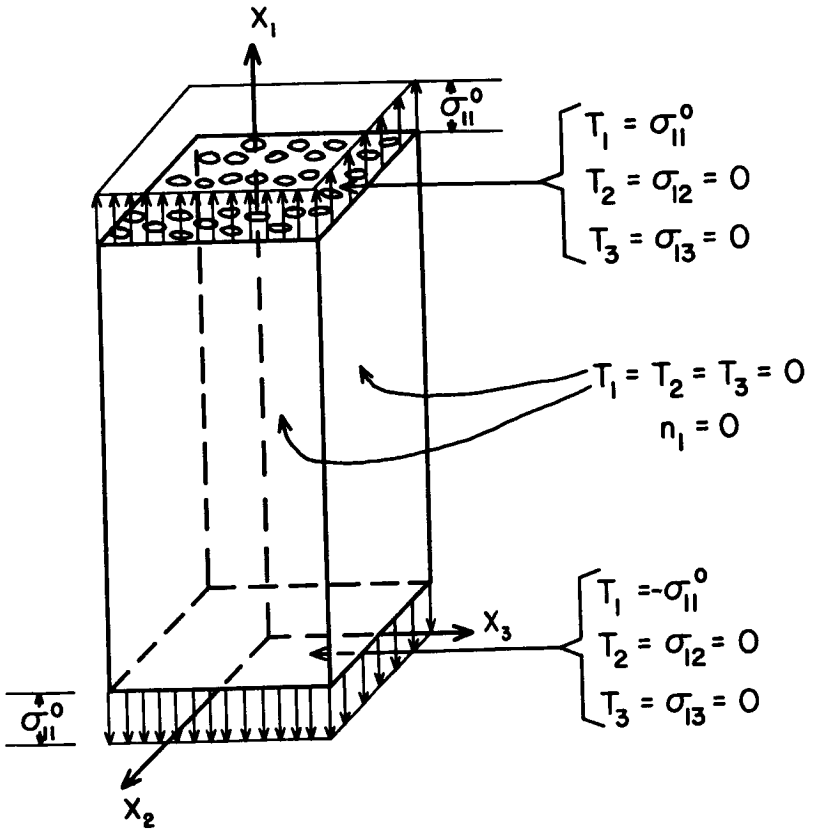


FIG. 3.4.1 - UNIAXIAL STRESSING OF FIBER REINFORCED CYLINDER

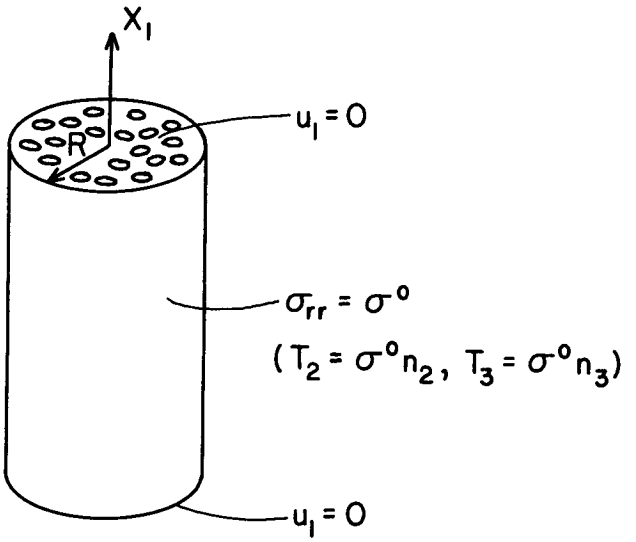
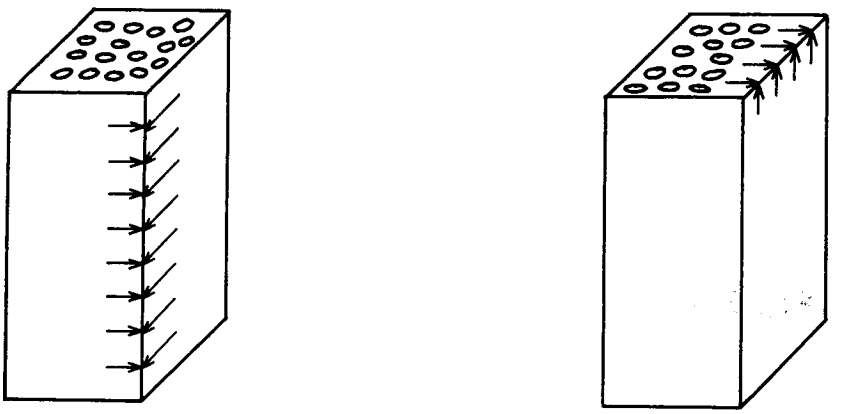


FIG. 3.4.2 - TRANSVERSELY ISOTROPIC STRESSING OF CIRCULAR FIBER REINFORCED CYLINDER



TRANSVERSE SHEARING

AXIAL SHEARING

FIG. 3.4.3 - SHEARING OF FIBER REINFORCED CYLINDER

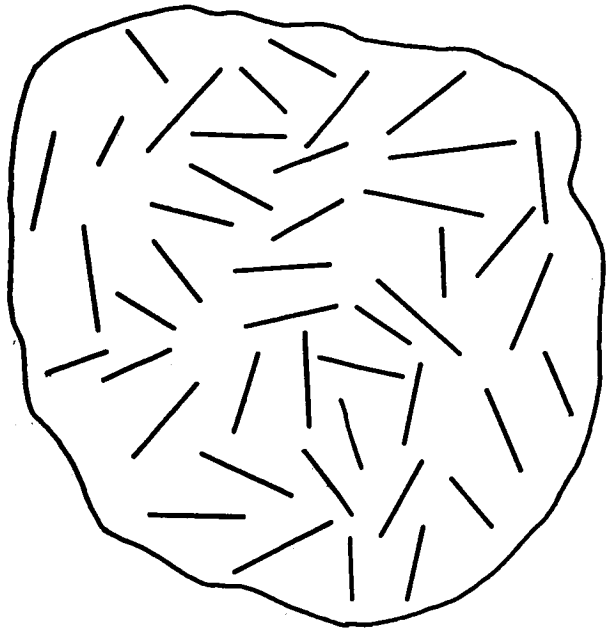


FIG. 3.4.4 - STATISTICALLY ISOTROPIC FIBER REINFORCED MATERIAL

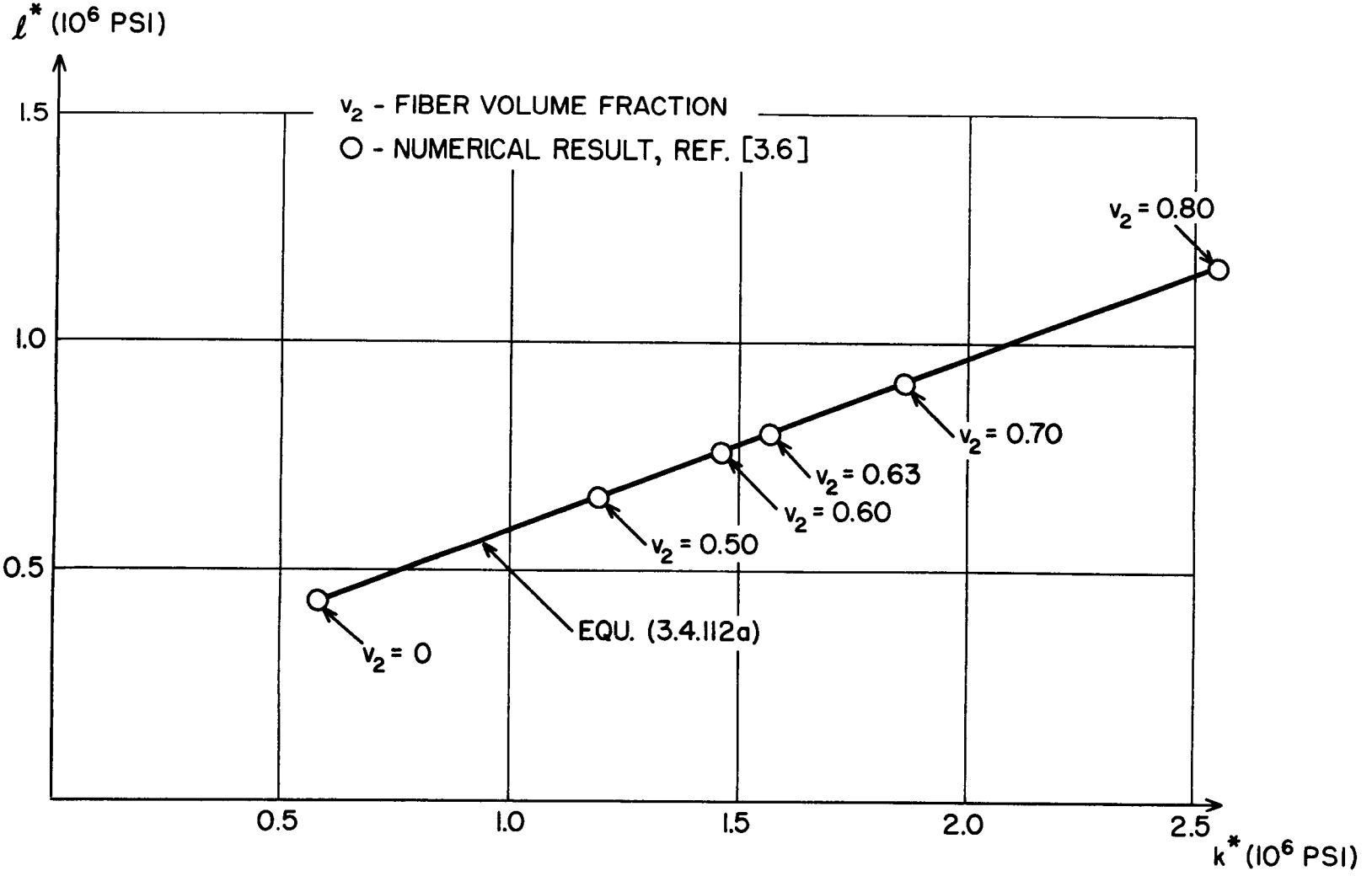
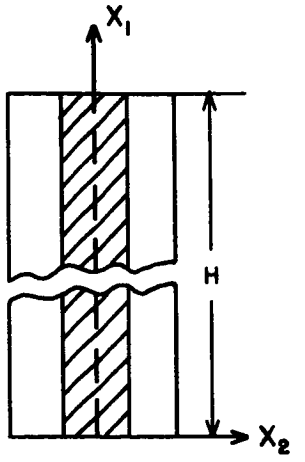
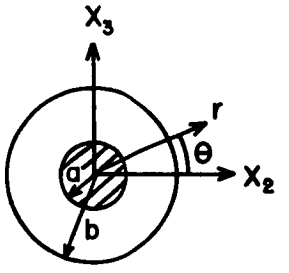


FIG. 3.4.5 - VERIFICATION OF LINEAR RELATION BETWEEN ℓ^* and k^* FOR NUMERICAL RESULTS; HEXAGONAL CIRCULAR FIBER ARRAY



AXIAL CENTER CUT



TRANSVERSE CUT

FIG. 3.5.1 - COMPOSITE CYLINDER

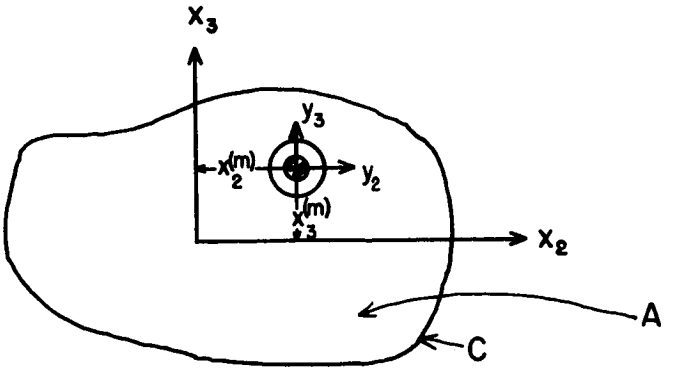


FIG. 3.5.2 - LOCAL COORDINATE SYSTEM OF COMPOSITE CYLINDER

E-GLASS FIBERS $E_2 = 10.5 \times 10^6$ PSI

$\nu_2 = 0.20$

EPOXY MATRIX $E_1 = 0.5 \times 10^6$ PSI

$\nu_1 = 0.35$

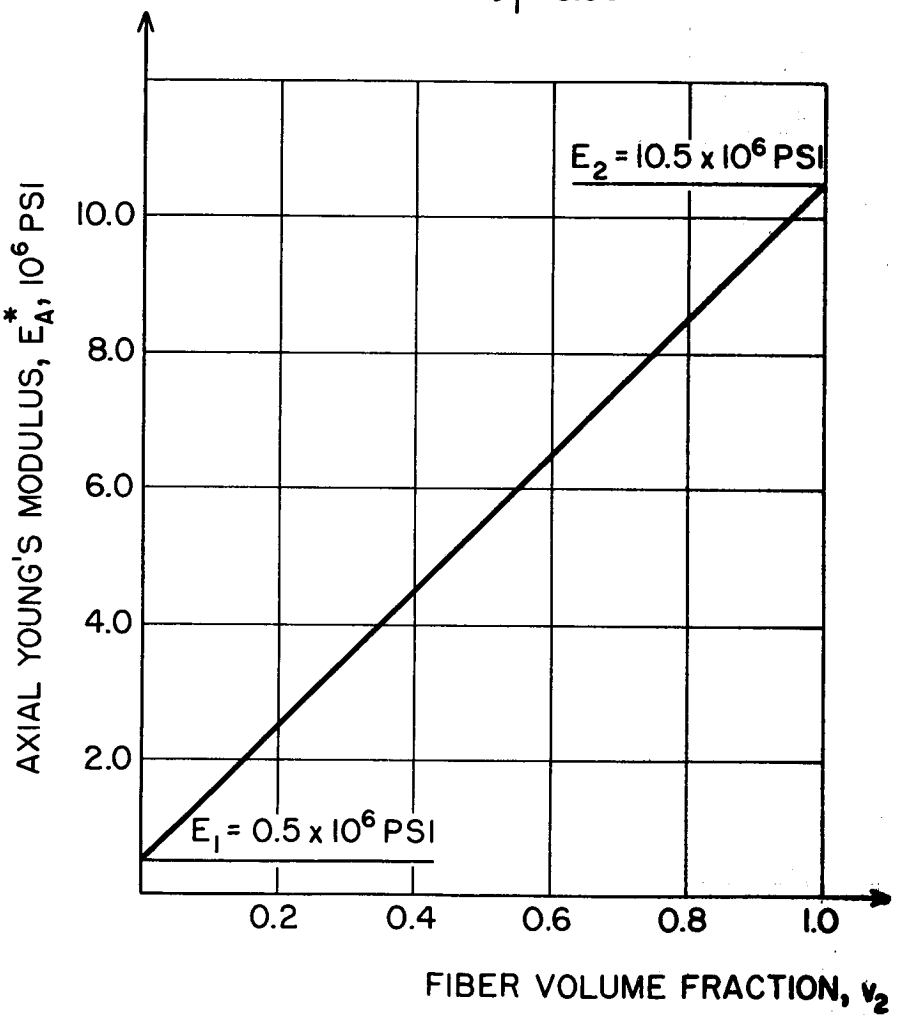


FIG. 3.5.3 - AXIAL YOUNG'S MODULUS, E_A^*

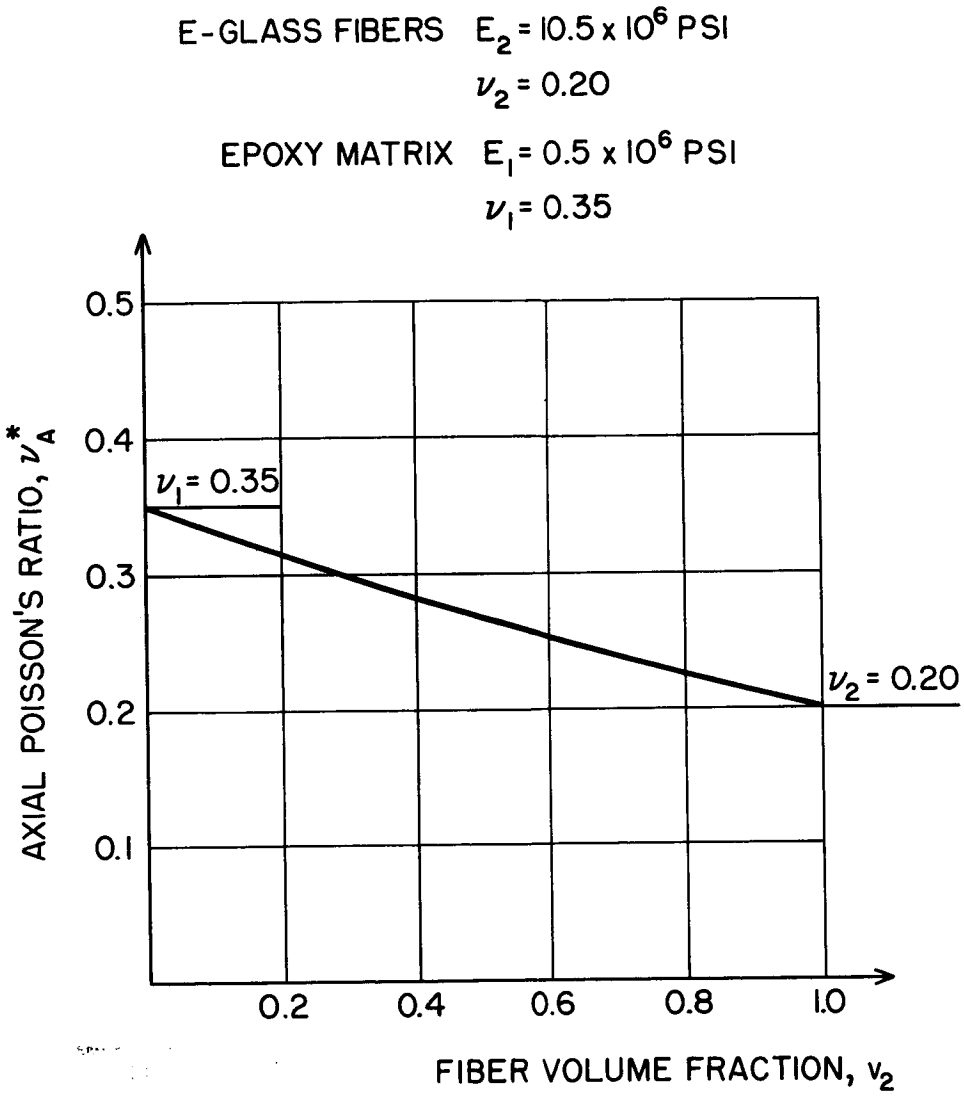


FIG. 3.5.4 - AXIAL POISSON'S RATIO, ν_A^*, CCA MODEL

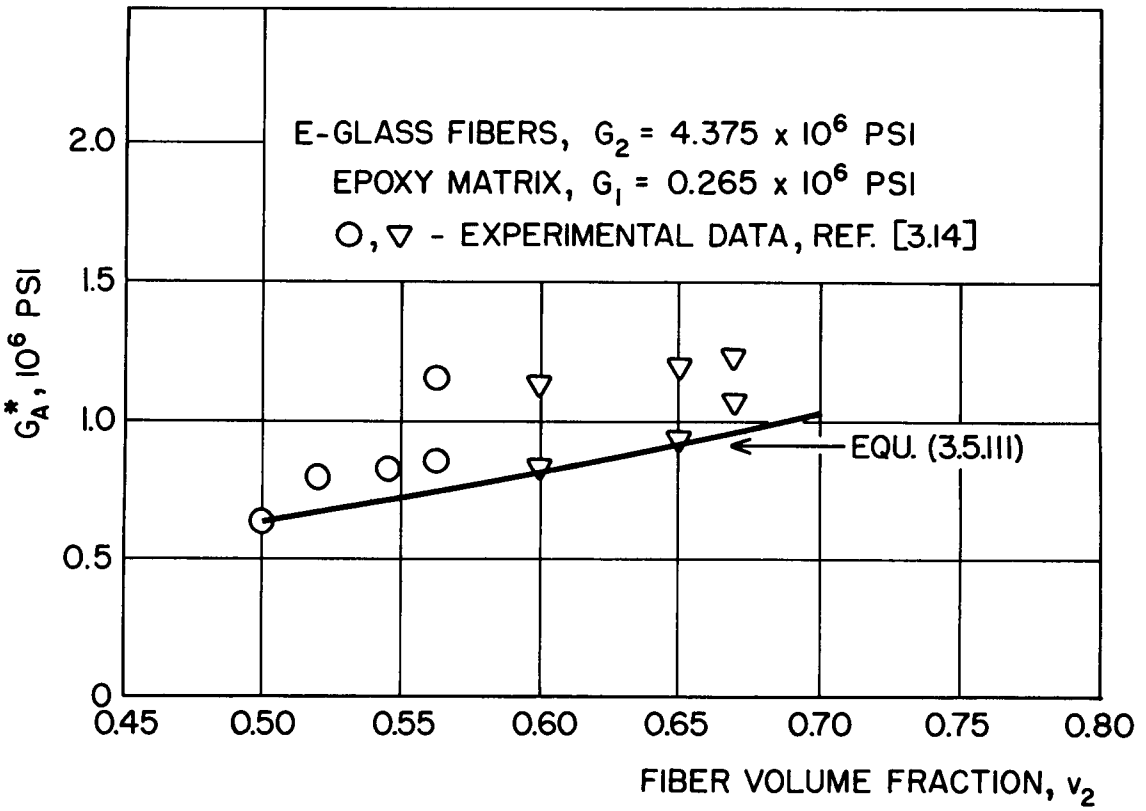


FIG. 3.5.5 - AXIAL SHEAR MODULUS G_A^*, CCA MODEL AND EXPERIMENTAL RESULTS

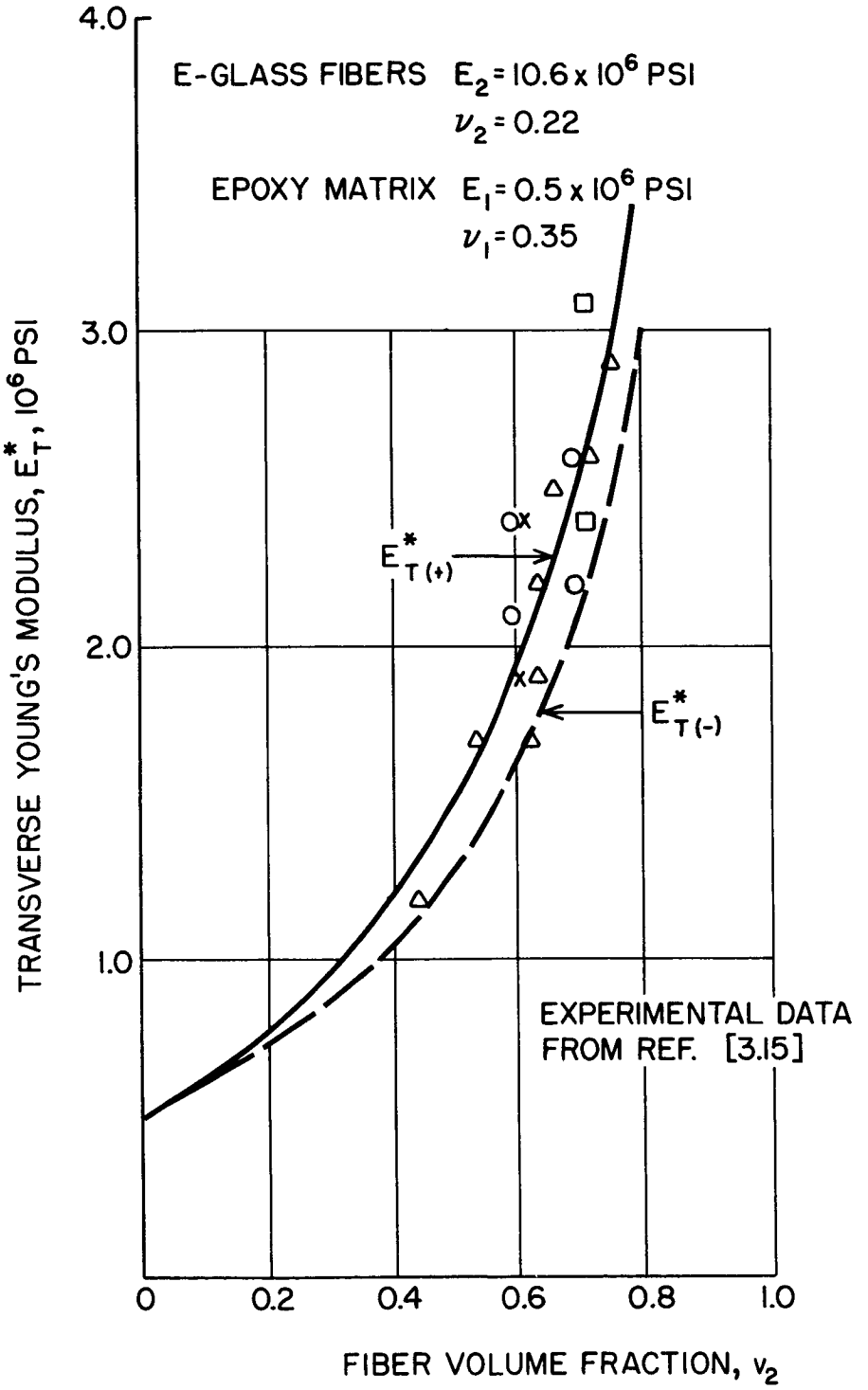


FIG. 3.5.6 - TRANSVERSE YOUNG'S MODULUS, E^*_T, OF GLASS REINFORCED EPOXY - COMPARISON OF THEORY AND EXPERIMENT

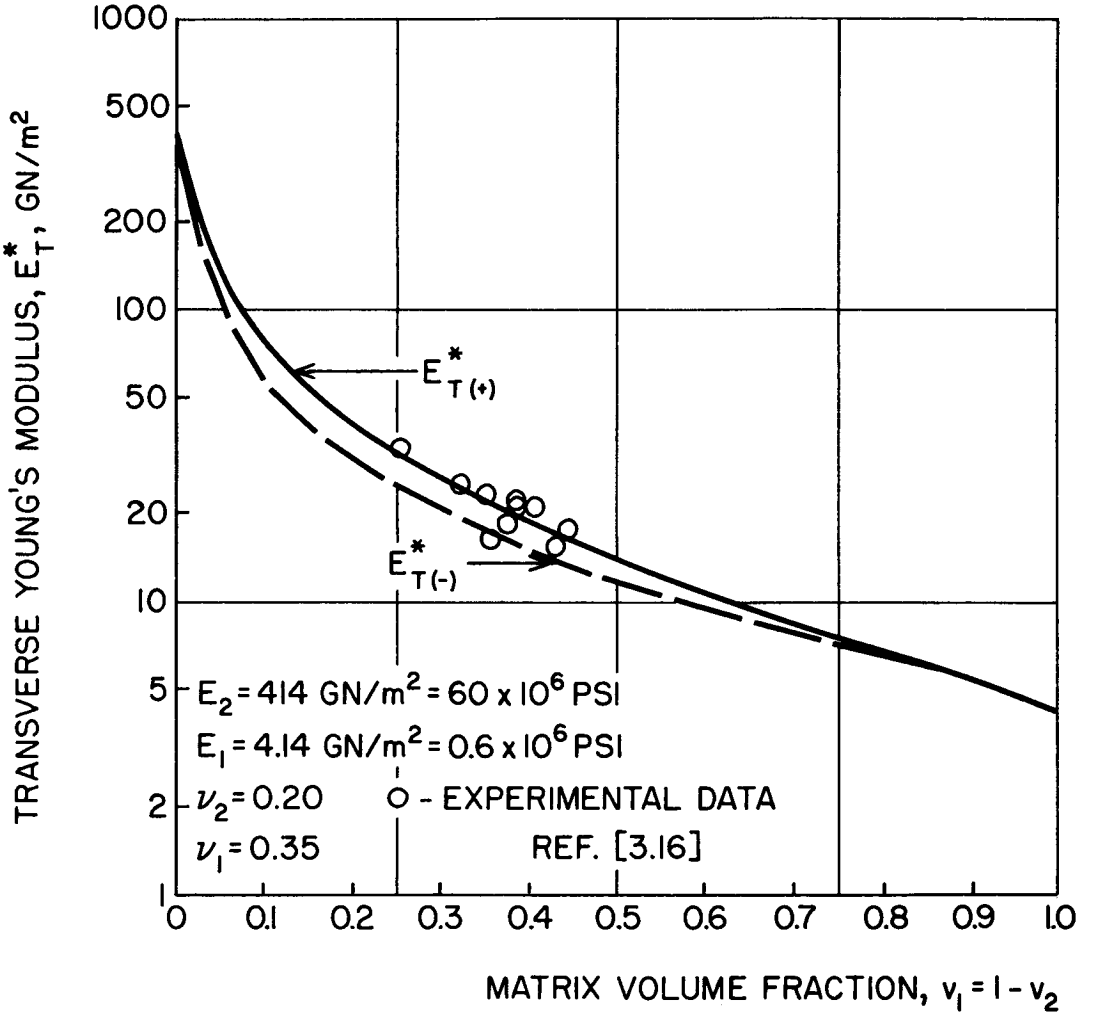


FIG. 3.5.7 - TRANSVERSE YOUNG'S MODULUS, E_T^*, OF BORON-REINFORCED EPOXY - COMPARISON OF THEORY AND EXPERIMENT

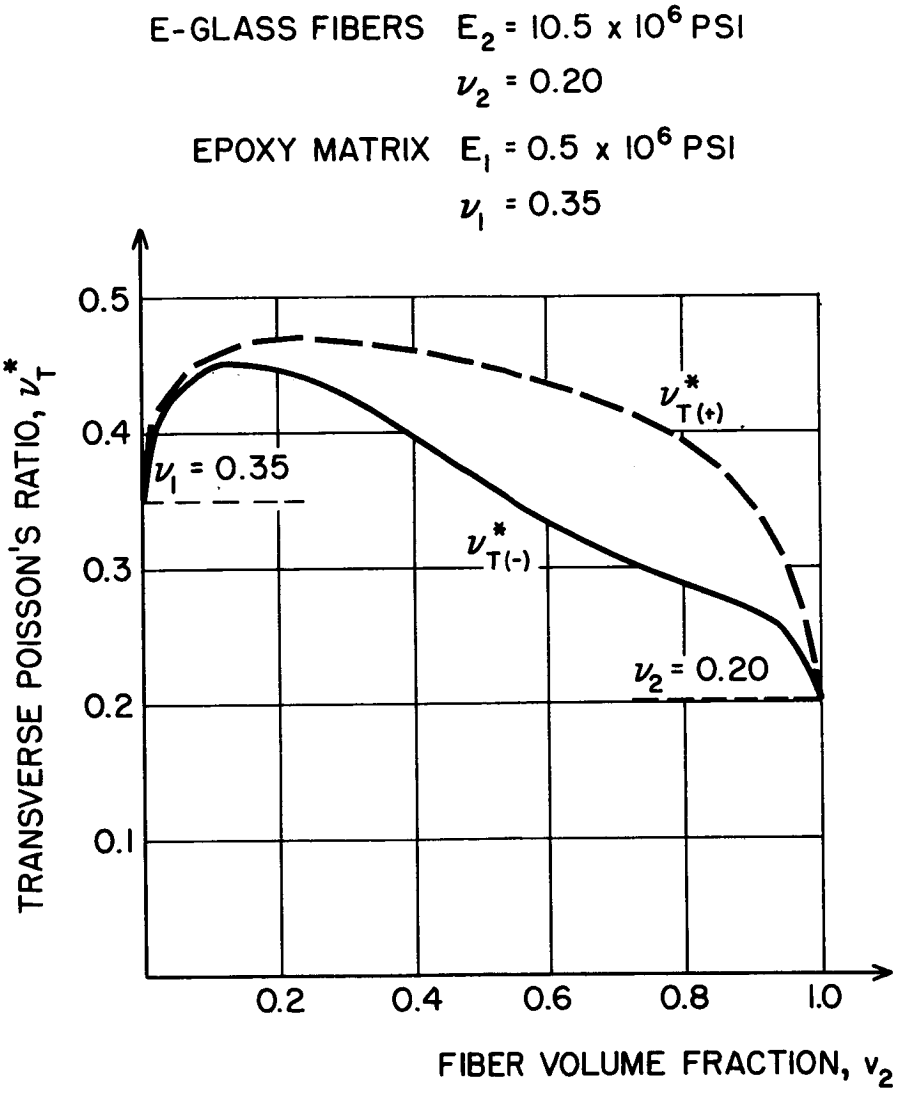
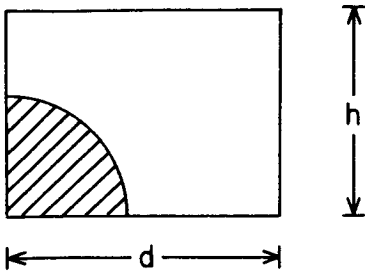
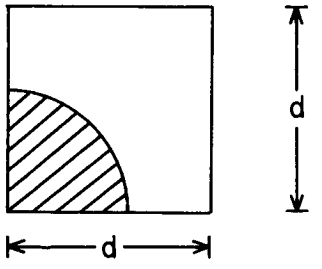


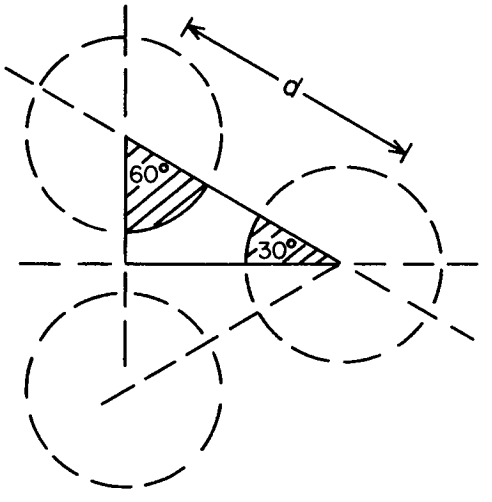
FIG. 3.5.8 - BOUNDS FOR ν_T^*, CCA MODEL



(a) RECTANGULAR ARRAY



(b) SQUARE ARRAY



(c) HEXAGONAL ARRAY

FIG. 3.5.9 - REPEATING ELEMENTS FOR PERIODIC FIBER ARRAYS

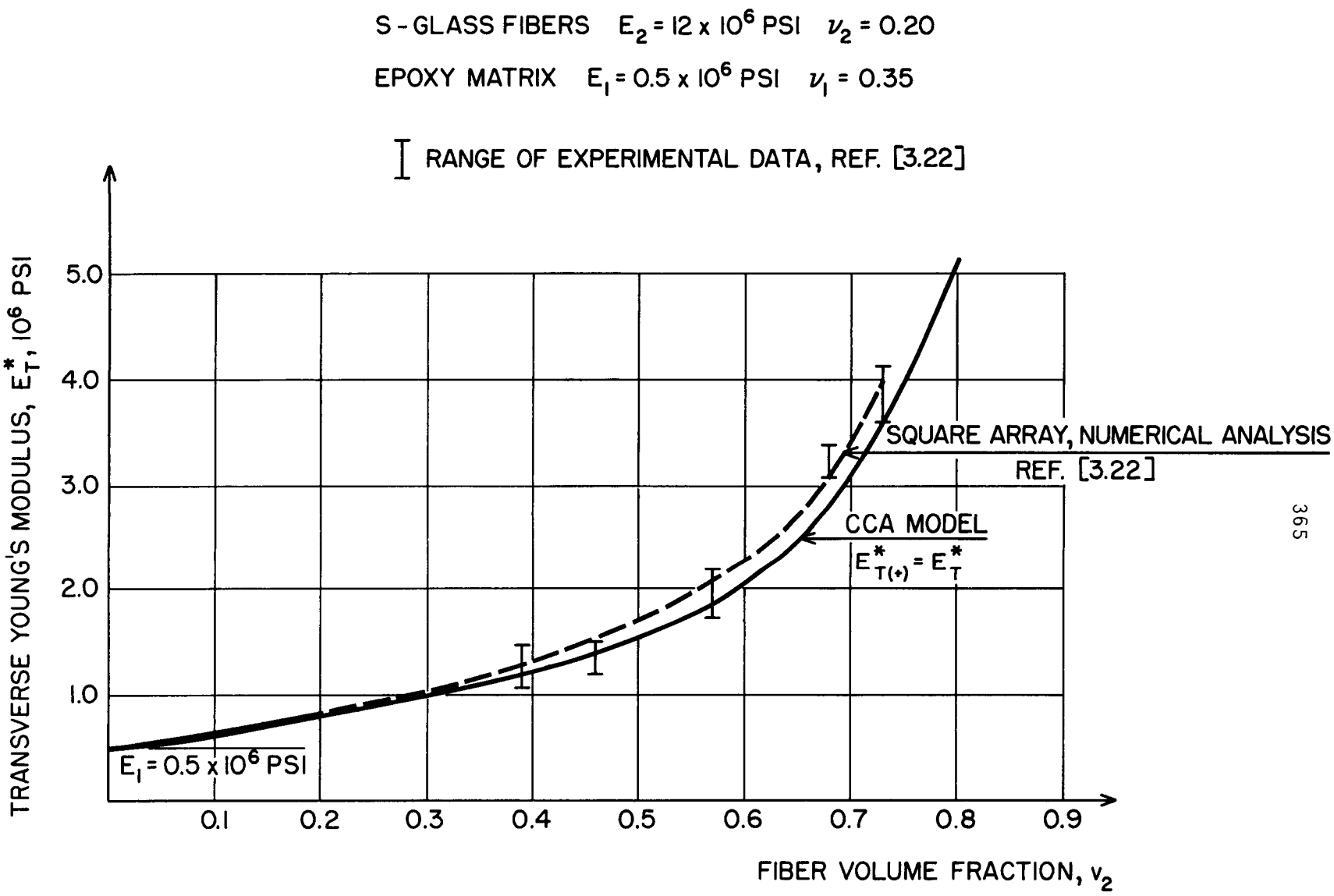


FIG. 3.5.10 - COMPARISON OF SQUARE ARRAY, CCA MODEL AND EXPERIMENTAL RESULTS, TRANSVERSE YOUNG'S MODULUS, E_T^*

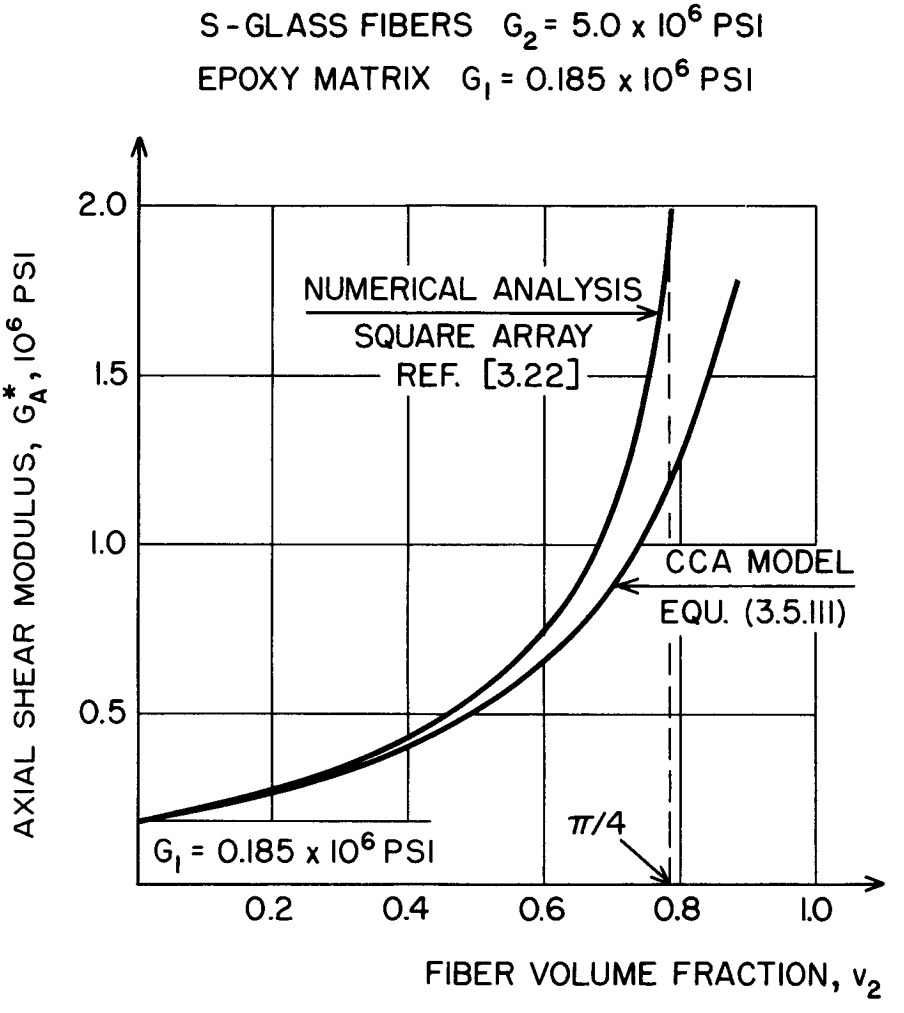


FIG. 3.5.11 - COMPARISON OF SQUARE ARRAY AND CCA MODEL RESULTS: AXIAL SHEAR MODULUS G_A^*

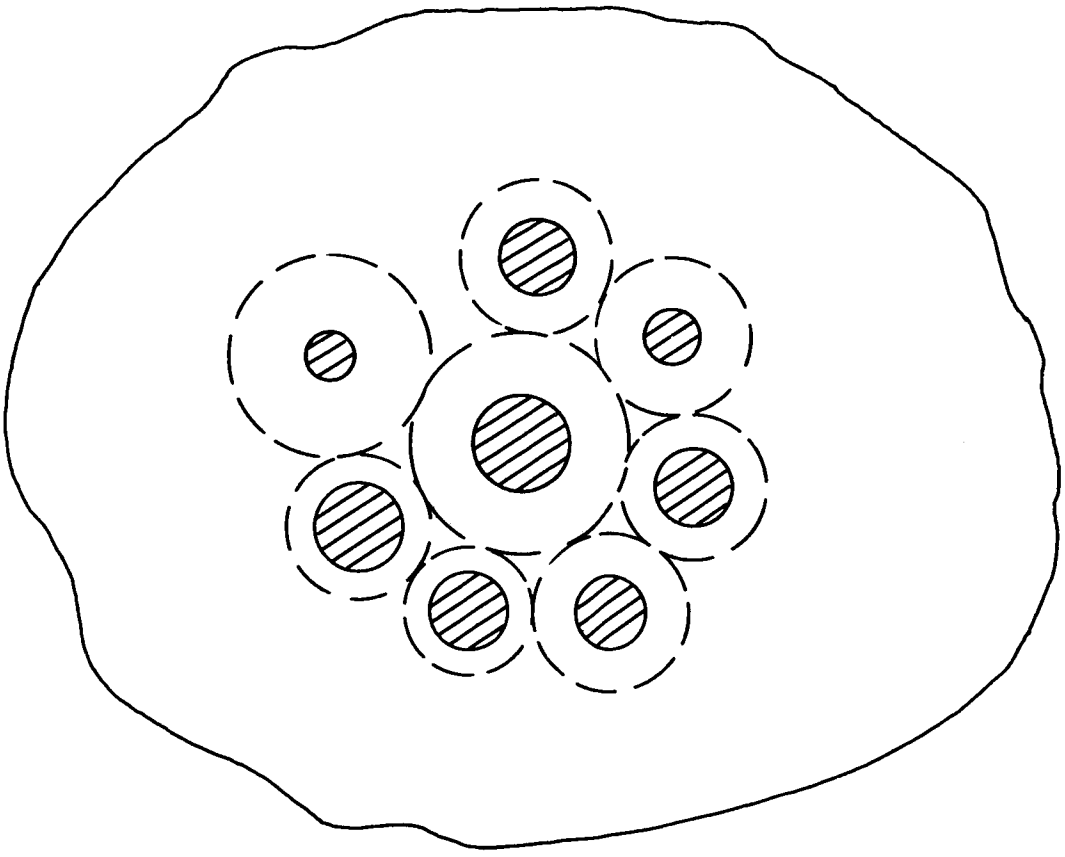


FIG. 3.6.1 - COMPOSITE CYLINDER CONSTRUCTION

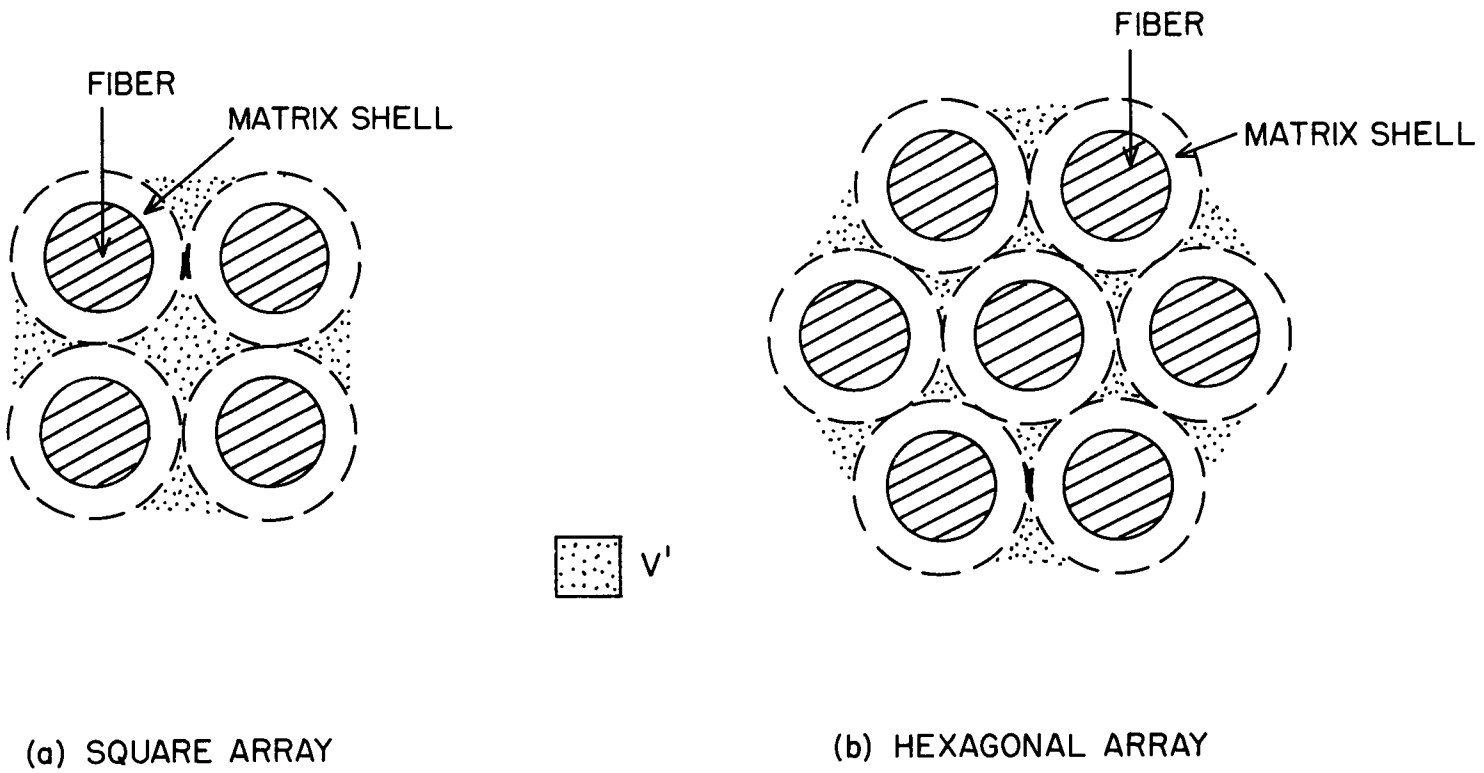


FIG. 3.6.2 - COMPOSITE CYLINDER CONSTRUCTION FOR PERIODIC ARRAY BOUNDING

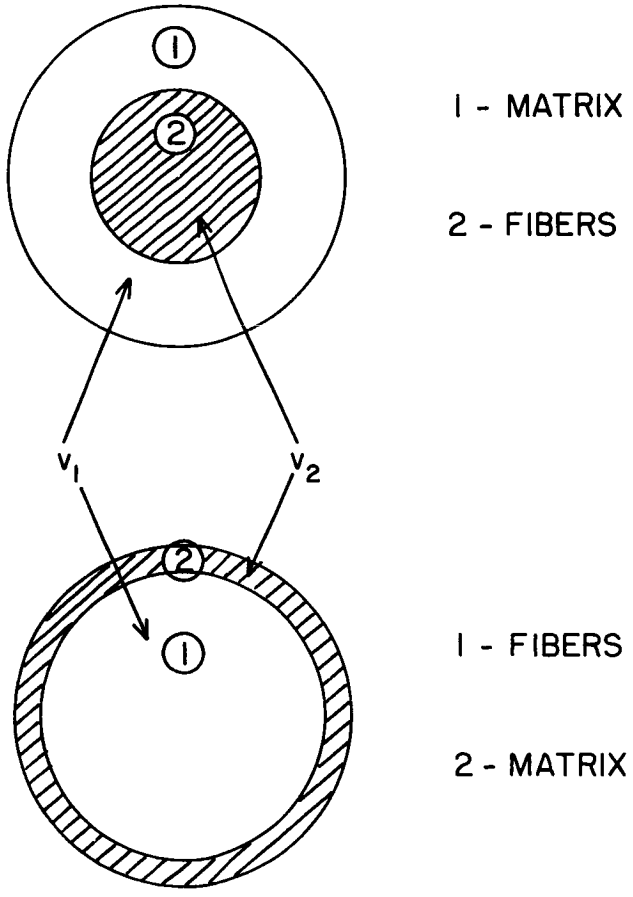


FIG. 3.7.1 - TYPICAL COMPOSITE CYLINDERS OF DIFFERENT ASSEMBLAGES WITH SAME PHASE VOLUME FRACTIONS

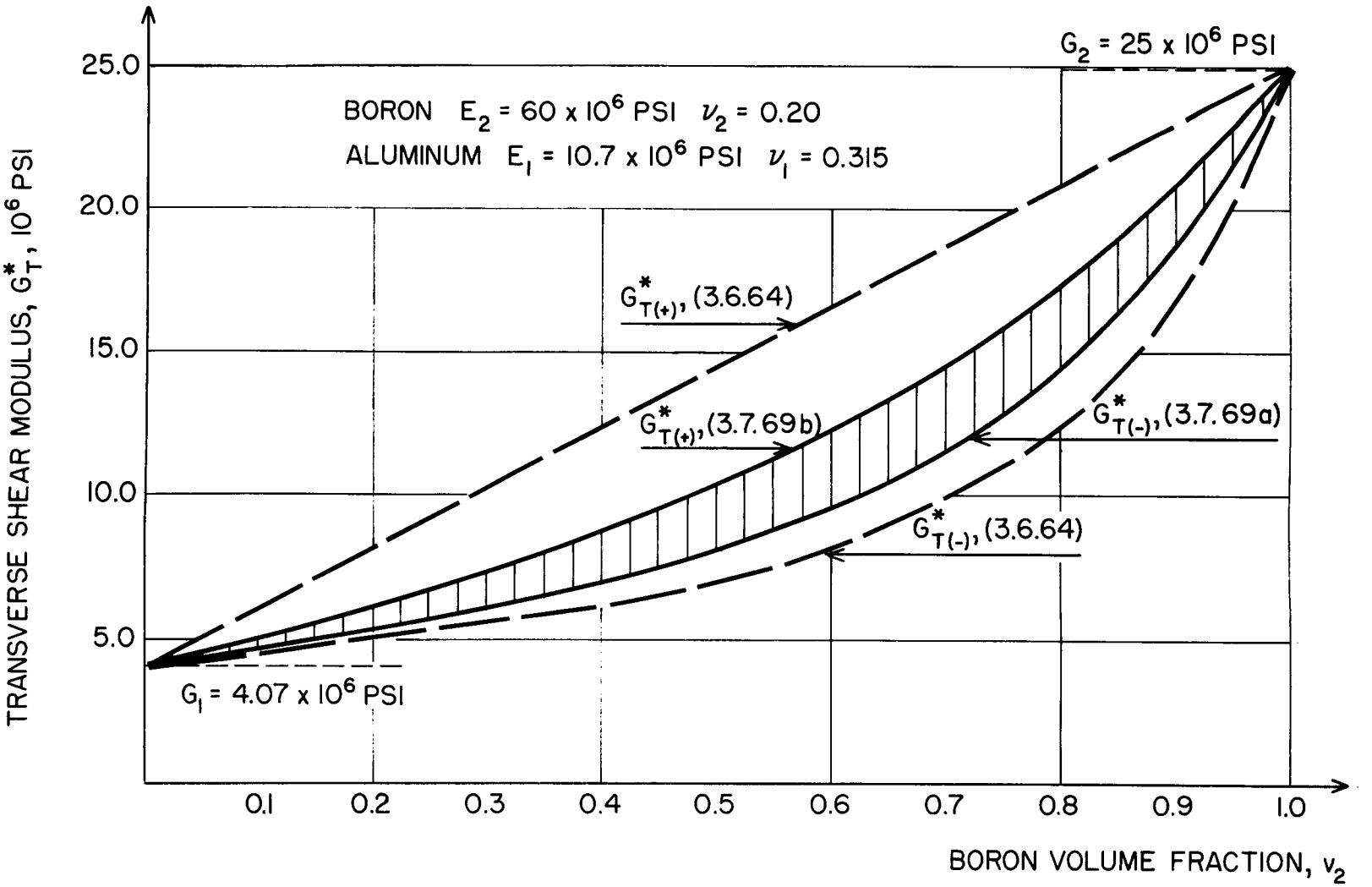


FIG. 3.7.2 - BOUNDS FOR G^*_T, BORON-ALUMINUM FIBROUS MATERIAL

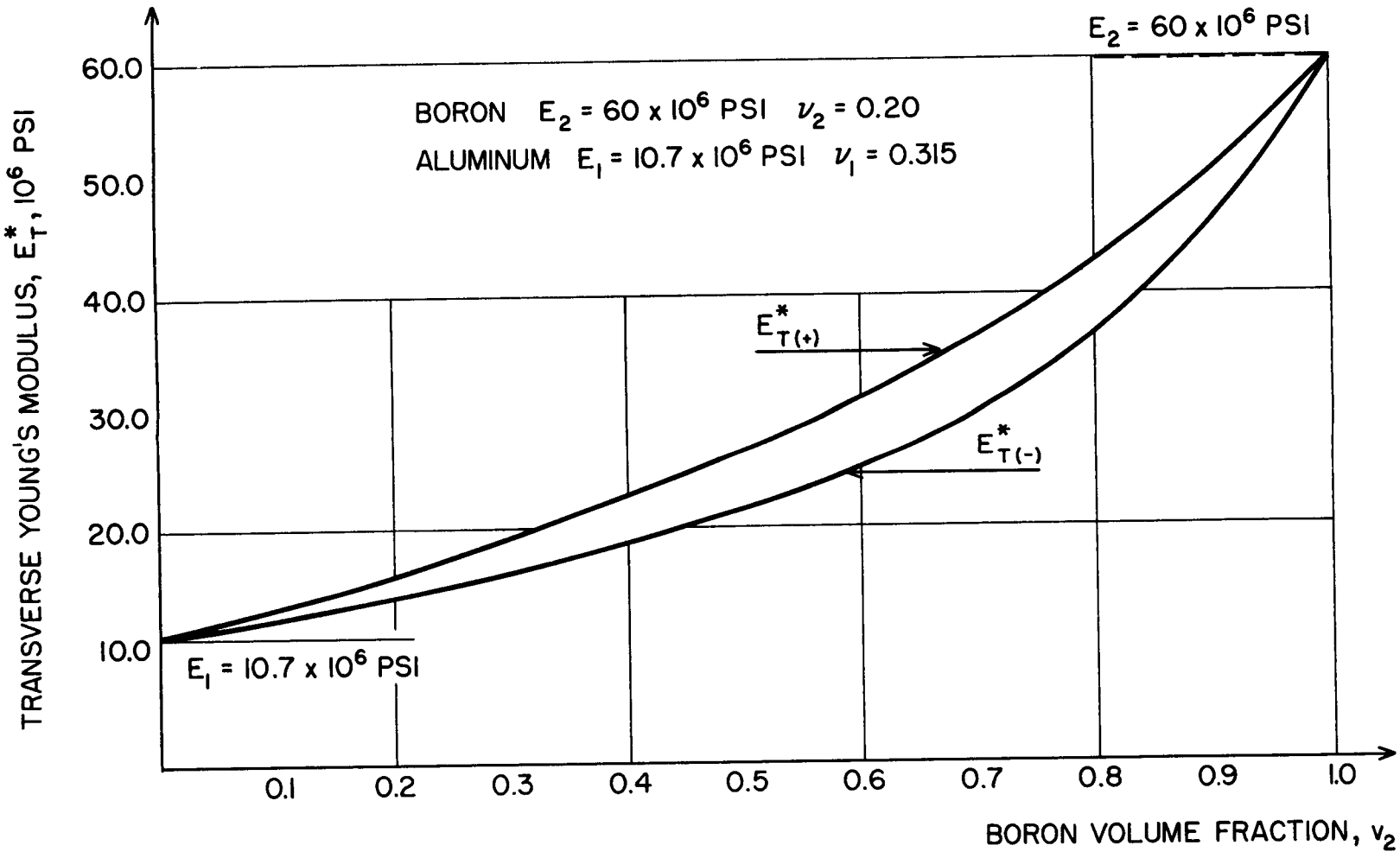


FIG. 3.7.3 - BOUNDS FOR E^*_T, BORON - ALUMINUM FIBROUS MATERIAL

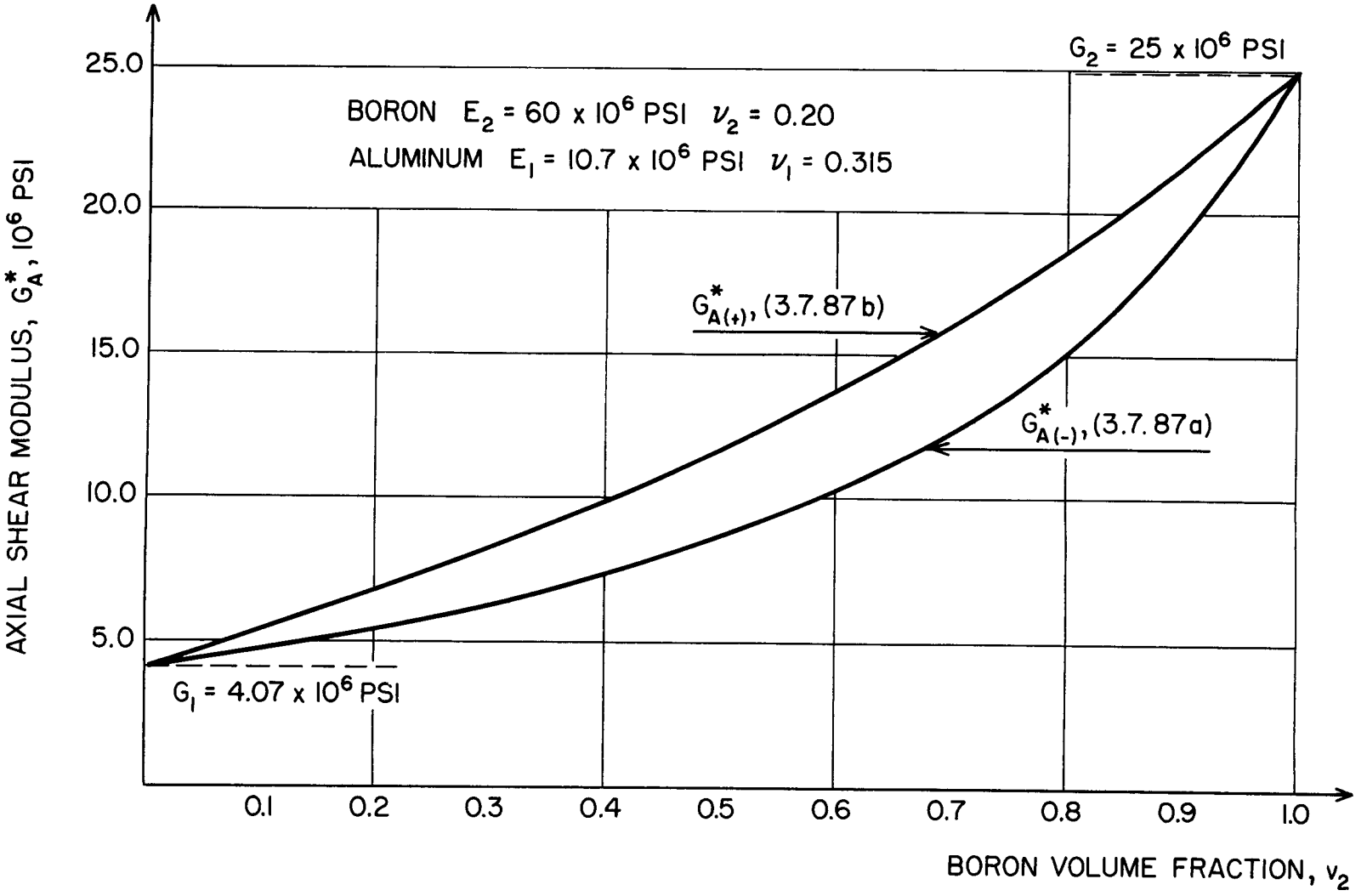


FIG. 3.7.4 - BOUNDS FOR G_A^*, BORON-ALUMINUM FIBROUS MATERIAL

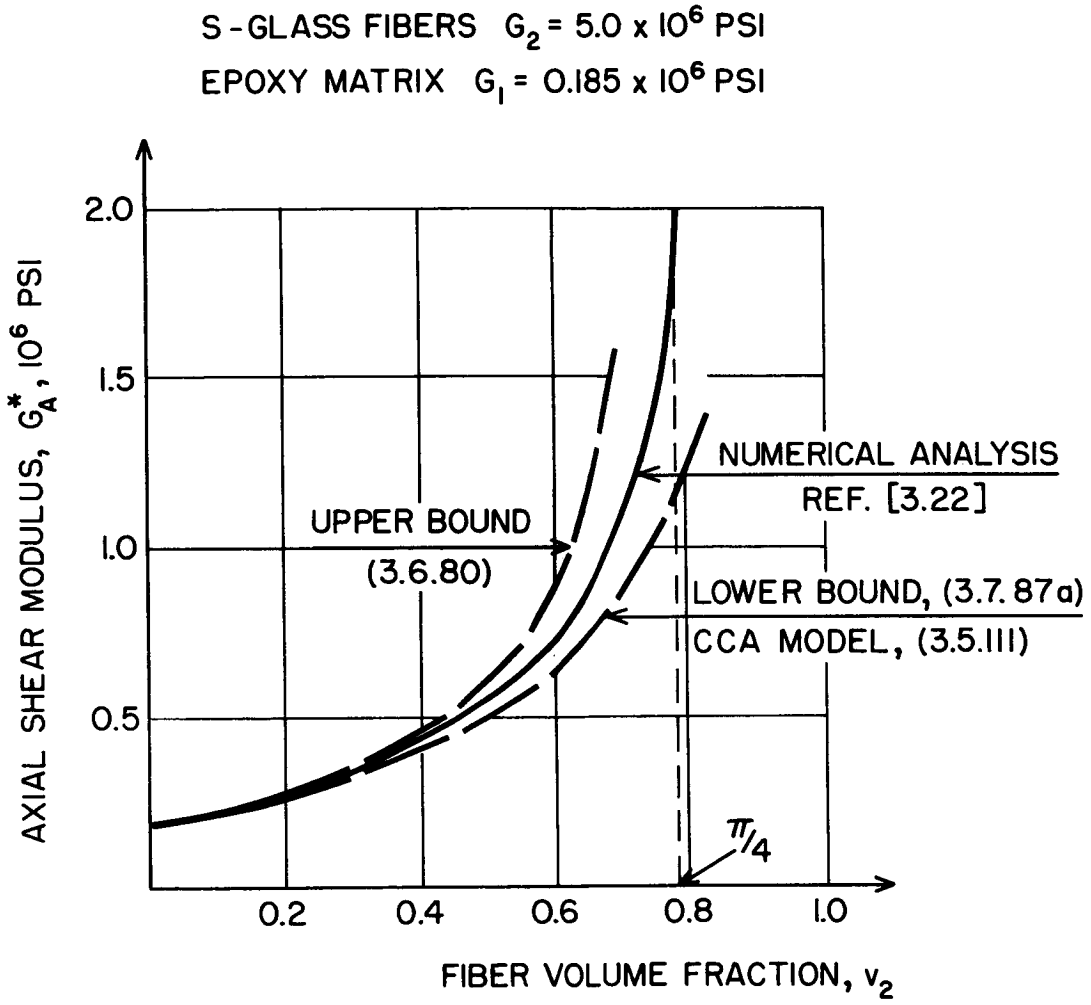


FIG. 3.7.5 - BOUNDS FOR AXIAL SHEAR MODULUS G_A^* OF SQUARE ARRAY OF CIRCULAR FIBERS

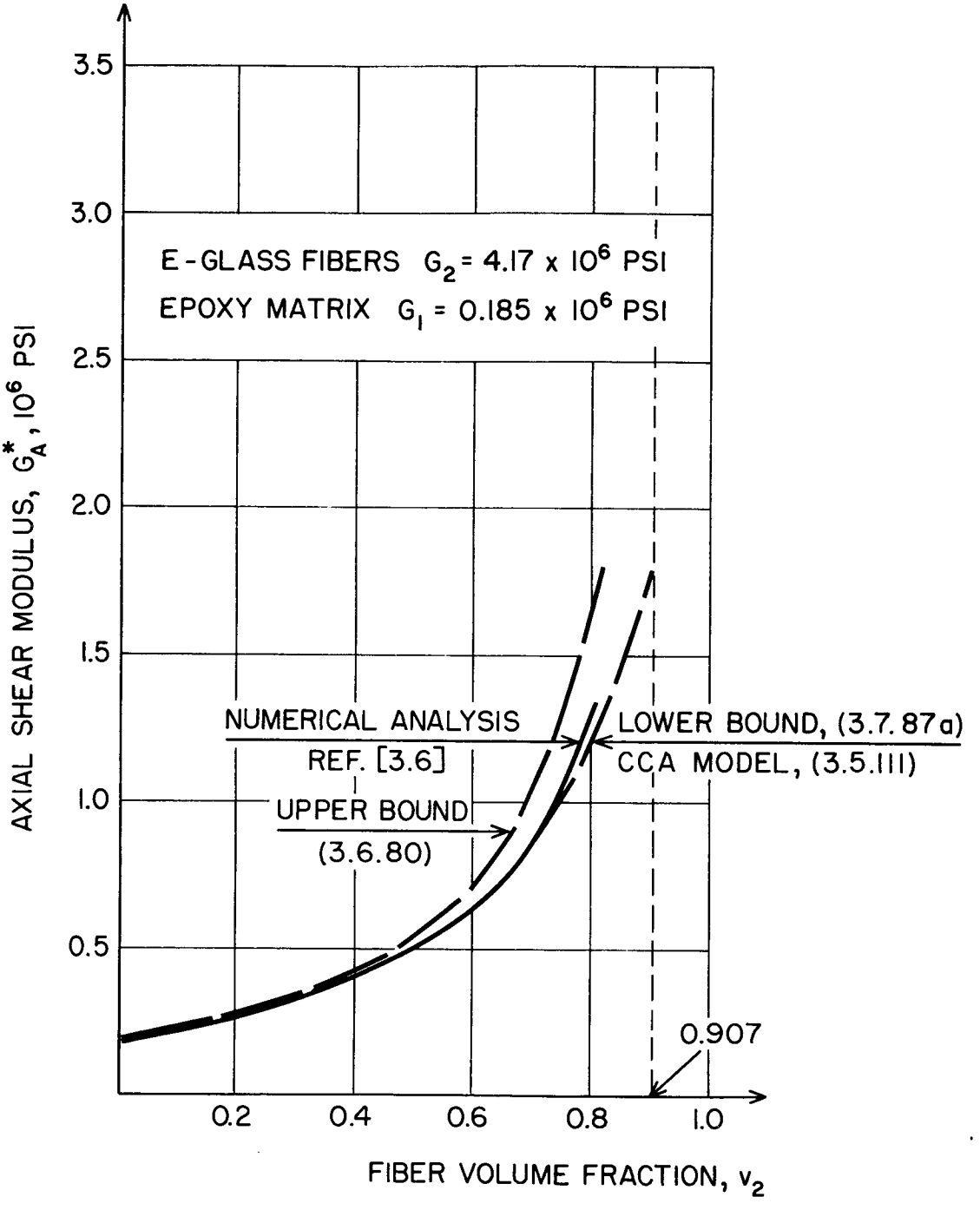


FIG. 3.7.6 - BOUNDS FOR AXIAL SHEAR MODULUS G_A^* OF HEXAGONAL ARRAY OF CIRCULAR FIBERS

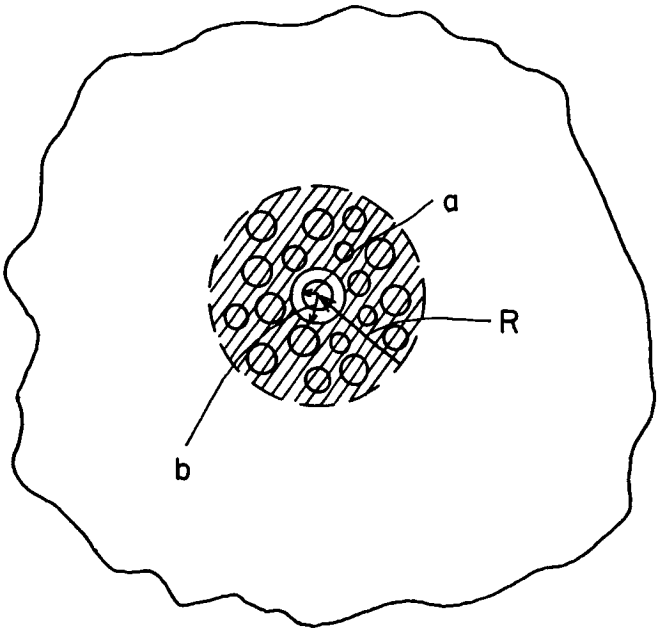


FIG. 3.8.1 - GEOMETRY OF FIBER EMBEDDING APPROXIMATION

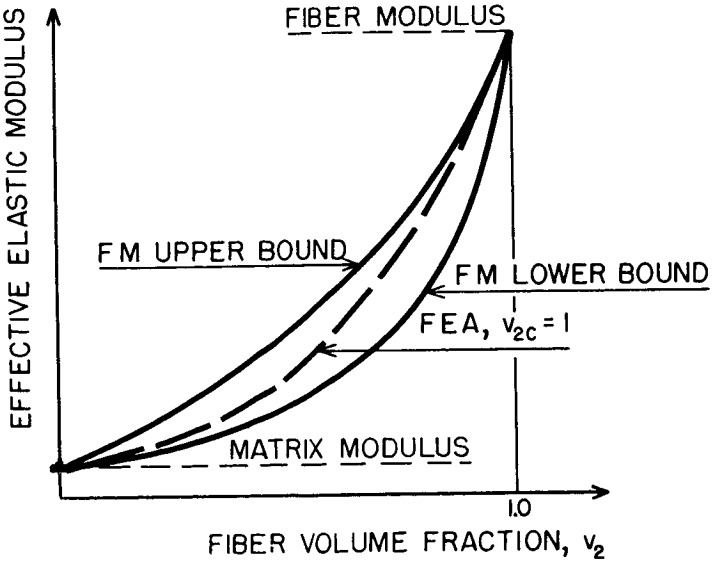


FIG. 3.8.2 - RELATION BETWEEN FIBROUS MATERIAL BOUNDS AND FEA APPROXIMATION (SCHEMATIC)

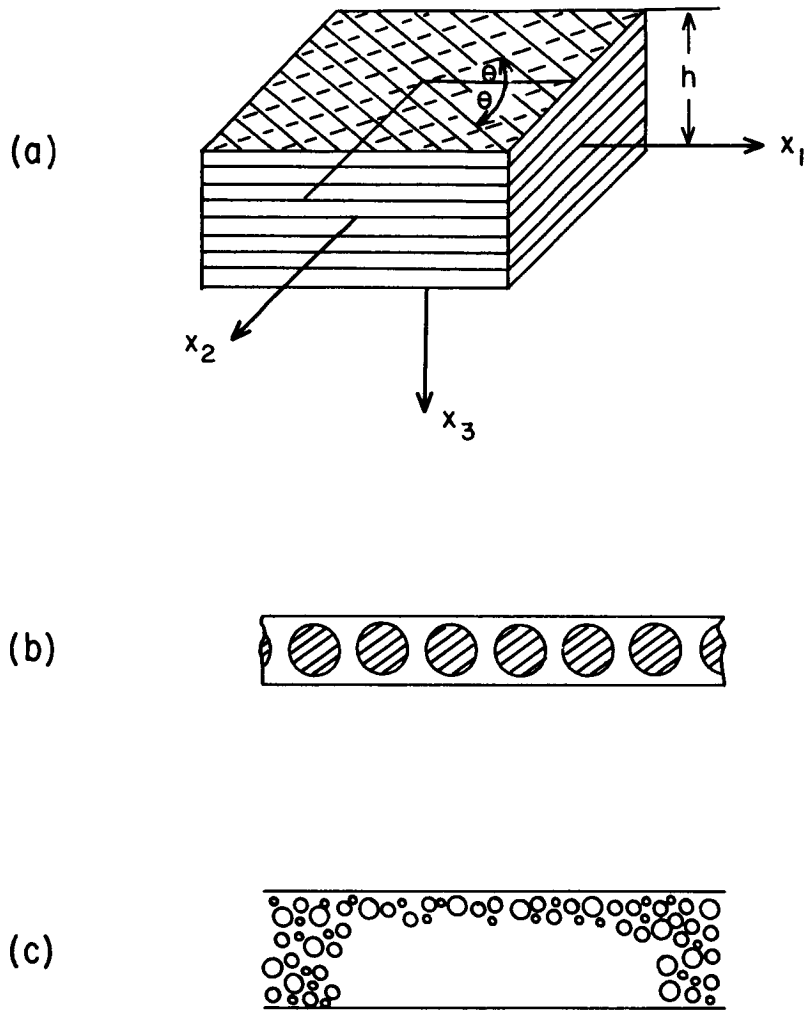


FIG. 3.10.1 - BIAXIALLY REINFORCED MATERIAL

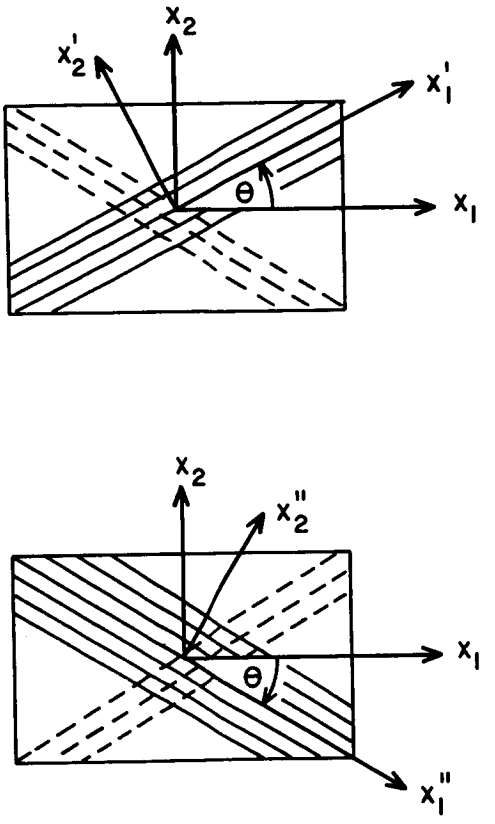


FIG. 3.10.2 - COORDINATE SYSTEMS OF LAMINAE

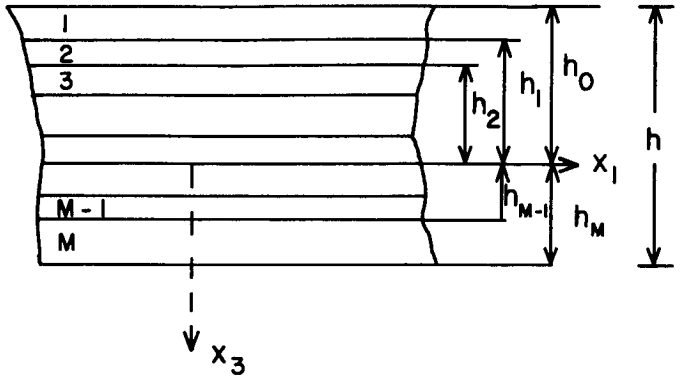


FIG. 3.10.3 - LAMINA CONVENTION

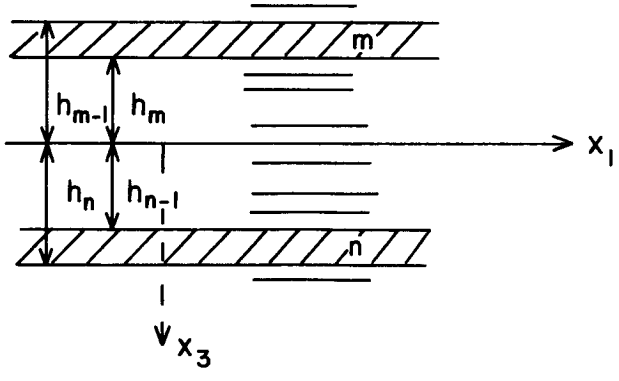


FIG. 3.10.4 - SYMMETRIC LAMINAE PAIR

Table 3.5.1 Elastic Properties of Fiber and Matrix Materials

Elastic Moduli in 10^6 psi, ν - nondimensional

Fiber	Material	E	ν	G	k
	E-Glass	10.5	0.20	4.38	7.29
	S-Glass	12.4	0.20 (?)		
	Beryllium	44.0	0.09	20.18	24.83
	Carbon	55.0	(?)		
	Boron	60.0	0.20	25.00	41.67
Matrix					
(typical)	Epoxy	0.50	0.35	0.185	0.617
	Magnesium	6.5	0.35	2.41	8.03
	Aluminum	10.3	0.33	3.87	10.73

Table 3.5.2- Effective Elastic Properties, E-Glass Fibers - Epoxy Matrix CCA Model

Effective Moduli in 10^6 psi , v_2 - Fiber vol. fraction

v_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
k^*	0.696	0.792	0.911	1.063	1.264	1.543	1.954	2.622	3.897
E_A^*	1.501	2.502	3.503	4.504	5.504	6.503	7.503	8.502	9.501
ν_A^*	0.331	0.314	0.297	0.281	0.266	0.252	0.238	0.225	0.212
G_A^*	0.223	0.269	0.326	0.400	0.500	0.640	0.853	1.213	1.955
$G_{T(-)}^*$	0.216	0.254	0.302	0.364	0.448	0.568	0.752	1.071	1.761
$G_{T(+)}^*$	0.219	0.269	0.337	0.429	0.554	0.726	0.974	1.361	2.073
$E_{T(-)}^*$	0.629	0.747	0.887	1.065	1.302	1.634	2.136	2.987	4.743
$E_{T(+)}^*$	0.636	0.778	0.960	1.197	1.511	1.938	2.550	3.509	5.277
$\nu_{T(-)}^*$	0.450	0.447	0.425	0.395	0.363	0.334	0.309	0.289	0.272
$\nu_{T(+)}^*$	0.458	0.469	0.468	0.462	0.452	0.439	0.421	0.395	0.346

Table 3.5.3- Effective Elastic Properties, Boron Fibers - Epoxy Matrix.

CCA Model

Effective Moduli in 10^6 psi, v_2 - Fiber vol. fraction

v_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
k^*	0.704	0.813	0.952	1.135	1.390	1.765	2.375	3.541	6.659
E_A^*	6.451	12.40	18.35	24.30	30.25	36.20	42.15	48.10	54.05
ν_A^*	0.331	0.313	0.296	0.281	0.265	0.251	0.237	0.224	0.212
G_A^*	0.226	0.276	0.341	0.426	0.544	0.721	1.008	1.564	3.085
$G_{T(-)}^*$	0.218	0.259	0.312	0.382	0.479	0.623	0.860	1.320	2.601
$G_{T(+)}^*$	0.222	0.276	0.354	0.464	0.621	0.855	1.232	1.938	3.760
$E_{T(-)}^*$	0.659	0.782	0.936	1.139	1.420	1.837	2.517	3.831	7.435
$E_{T(+)}^*$	0.668	0.819	1.027	1.311	1.710	2.294	3.231	4.984	9.536
$\nu_{T(-)}^*$	0.503	0.483	0.451	0.414	0.377	0.342	0.311	0.285	0.268
$\nu_{T(+)}^*$	0.510	0.507	0.500	0.491	0.482	0.473	0.463	0.451	0.429

Table 3.5.4 Effective Elastic Properties, Boron Fibers - Aluminum Matrix

CCA Model

Effective Moduli in 10^6 psi , v_2 - Fiber vol. fraction

v_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
k^*	12.48	13.73	15.19	16.92	18.99	21.52	24.68	28.73	34.13
E_A^*	15.29	20.28	25.26	30.23	35.20	40.17	45.13	50.09	55.05
ν_A^*	0.314	0.300	0.285	0.272	0.258	0.246	0.234	0.222	0.211
G_A^*	4.484	5.200	6.051	7.077	8.341	9.934	12.01	14.81	18.81
$G_{T(-)}^*$	4.390	5.003	5.740	6.641	7.770	9.225	11.17	13.91	18.04
$G_{T(+)}^*$	4.431	5.165	6.095	7.242	8.632	10.31	12.38	15.02	18.68
$E_{T(-)}^*$	11.98	13.77	15.81	18.23	21.17	24.86	29.66	36.15	45.48
$E_{T(+)}^*$	12.06	14.08	16.48	19.33	22.72	26.77	31.71	37.97	46.47
$\nu_{T(-)}^*$	0.361	0.363	0.352	0.334	0.316	0.298	0.281	0.264	0.244
$\nu_{T(+)}^*$	0.365	0.377	0.378	0.372	0.362	0.348	0.327	0.300	0.260

Table 3.5.5- Comparison of Effective Elastic Properties Predicted by CCA Analysis and Hexagonal Array Numerical Analysis, Ref [3.6, 3.23]

$E_2 = 10.0 \times 10^6$ psi

$\nu_2 = 0.20$

$E_1 = 0.496 \times 10^6$ psi

$\nu_1 = 0.34$

ν_2	0.5	0.5	0.6	0.6	0.7	0.7	0.8	0.8
k^*	1.194	1.194	1.459	1.461	1.851	1.865	2.487	2.563
E_A^*	5.255	5.253	6.206	6.210	7.157	7.160	8.107	8.104
ν_A^*	0.261	0.262	0.248	0.248	0.235	0.235	0.223	0.223
G_A^*	0.497	0.495	0.635	0.693	0.844	0.852	1.196	1.308
$G_{T(-)}^*$	0.437		0.553		0.729		1.060	
G_T^*		0.463		0.611		0.850		1.264
$G_{T(+)}^*$	0.540		0.705		0.944		1.323	
	CCA	HA	CCA	HA	CCA	HA	CCA	HA

4.1 QUASI-STATIC LINEAR VISCOELASTICITY

4.1.1 Stress-Strain Relations

In the present chapter we shall give a brief discussion of linear viscoelasticity theory. For more detailed expositions the reader is referred to [4.1-3].

A linear viscoelastic material may be defined as a time dependent material in which the displacements and strains are small and effects may be superposed in time according to the Boltzmann superposition principle.

Let the displacements, strains and stresses be space and time dependent and be denoted $u_i(\underline{x},t)$, $\epsilon_{ij}(\underline{x}, t)$ and $\sigma_{ij}(\underline{x}, t)$ respectively. The strains and displacements are related by

$$\epsilon_{ij}(\underline{x},t) = \frac{1}{2}\left(u_{i,j}(\underline{x},t) + u_{j,i}(\underline{x},t)\right) \tag{4.1.1}$$

The most general anisotropic stress-strain relations are

$$\sigma_{ij}(\underline{x},t) = \int_{-\infty}^{t} C_{ijkl}(\underline{x}, t-\tau)\,\frac{\partial \epsilon_{kl}(\underline{x},\tau)}{\partial \tau}\,d\tau \qquad \text{(a)}$$

$$\epsilon_{ij}(\underline{x},t) = \int_{-\infty}^{t} S_{ijkl}(\underline{x},t-\tau)\,\frac{\partial \sigma_{kl}(\underline{x},\tau)}{\partial \tau}\,d\tau \qquad \text{(b)} \tag{4.1.2}$$

The first of these will be referred to as the viscoelastic stress-strain relation and the second will be referred to as the viscoelastic strain-stress relation.

The functions C_{ijkl} and S_{ijkl} define the material behavior. The former are known as relaxation moduli and the latter as creep compliances. Both are components of fourth rank tensors. If C_{ijkl} and S_{ijkl} are not space dependent the viscoelastic body is homogeneous; otherwise it is nonhomogeneous.

Symmetry of the stress and strain tensors leads to

$$C_{ijkl}(t) = C_{jikl}(t) = C_{ijlk}(t) \qquad \text{(a)}$$

$$S_{ijkl}(t) = S_{jikl}(t) = S_{ijlk}(t) \qquad \text{(b)} \qquad (4.1.3)$$

Biot [4.4] (see also [3.25], chap. 13) has shown that if Onsager's principle is assumed valid for viscoelastic materials, then

$$C_{ijkl}(t) = C_{klij}(t) \qquad S_{ijkl}(t) = S_{klij}(t) \qquad (4.1.4)$$

It appears, however, that the symmetry relations (4.1.4) are not as universally accepted as their elastic analogues (last equalities in (3.2.3) and (3.2.5)). For discussion of this subject see e.g. [4.5]. In the present work the validity of (4.1.4) shall be assumed.

The functions C_{ijkl} and S_{ijkl} are obviously related, for if σ_{ij} as given by (4.1.2) is introduced into the right side of (4.1.2b) the strains on both sides must be the same. It is, however, inconvenient to relate C_{ijkl} and S_{ijkl} in this fashion. A better way of doing this will be described later.

A physical interpretation of relaxation moduli and creep compliances is given in terms of responses to step function inputs. Suppose that we impose the strain

$$\epsilon_{ij}(t) = {}^{o}\epsilon_{ij}\, H(t) \tag{4.1.5}$$

where $H(t)$ is the Heaviside step function which is defined as

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{4.1.6}$$

Insertion of (4.1.6) into (4.1.2) and taking cognizance of the fact that the derivative of $H(t)$ is the delta function, $\delta(t)$, yields

$$\sigma_{ij}(t) = C_{ijkl}(t)\, {}^{o}\epsilon_{kl} \tag{4.1.7}$$

Dually, if we impose the stress variation

$$\sigma_{ij}(t) = {}^{o}\sigma_{ij}\, H(t) \tag{4.1.8}$$

then (4.1.3) yields

$$\epsilon_{ij}(t) = S_{ijkl}(t)\, {}^{o}\sigma_{kl} \tag{4.1.9}$$

It is seen that relaxation moduli are given by the stresses produced by time constant strains, and creep compliances are given by strains which are produced by time constant stresses. This indicates the type of experiments which have to be performed to measure relaxation moduli and creep compliances.

The stress-strain relations (4.1.2) may be derived by starting out with (4.1.7), (4.1.9) by use of the Boltzmann superposition principle. Such a procedure will be used later below in order to derive the effective stress-strain relations of viscoelastic heterogeneous materials.

We now examine the important cases of discontinuities in time of the strain or stress variations in (4.1.2-3). Since at this time we are only interested in time variation at a typical point there is no need to incorporate the space variation into the stress-strain relations.

Consider the case of a strain which is suddenly applied at time t=0, and varies continuously from there on. Such a strain may be written

$$\epsilon_{ij}(-\infty,t) = \begin{cases} 0 & t<0 \\ \epsilon_{ij}(t) & t \geq 0 \end{cases} \tag{4.1.10}$$

It is seen that there is a discontinuity at t=0. Let the strain variation be written

$$\epsilon_{ij}(-\infty,t) = \epsilon_{ij}(o)\, H(t) + \epsilon'_{ij}(t)$$

where ϵ'_{ij} is continuous and vanishes for $t < 0$. Then

$$\frac{\partial \epsilon_{ij}}{\partial t} = \epsilon_{ij}(o)\, \delta(t) + \frac{\partial \epsilon'_{ij}}{\partial t} = \epsilon_{ij}(o)\, \delta(t) + \frac{\partial \epsilon_{ij}}{\partial t}\Big|_{t \geq 0_+} \tag{4.1.11}$$

Inserting (4.1.11) into (4.1.2) we have

$$\sigma_{ij}(t) = C_{ijkl}(t)\ \epsilon_{kl}(o) + \int_{o_+}^{t} C_{ijkl}(t-\tau)\ \frac{\partial \epsilon_{kl}}{\partial \tau}\ d\tau \tag{4.1.12}$$

If the last integral in (4.1.12) is integrated by parts we obtain the equivalent form

$$\sigma_{ij}(t) = C_{ijkl}(o)\ \epsilon_{kl}(t) + \int_{o_+}^{t} \frac{\partial C_{ijkl}(\tau)}{\partial \tau}\ \epsilon_{kl}(t-\tau)\ d\tau$$

We shall find it more convenient here to use the form (4.1.12).

If we impose the discontinuous stress variation

$$\sigma_{ij(-\infty,t)} = \begin{cases} 0 & t < 0 \\ \sigma_{ij}(t) & t \geq 0 \end{cases} \tag{4.1.13}$$

then by similar reasoning

$$\epsilon_{ij}(t) = S_{ijkl}(t)\ \sigma_{kl}(o) + \int_{o_+}^{t} S_{ijkl}(t-\tau)\ \frac{\partial \sigma_{kl}}{\partial \tau}\ d\tau \tag{4.1.14}$$

It is seen from (4.1.11) and (4.1.14) that

$$\sigma_{ij}(o) = C_{ijkl}(o)\ \epsilon_{kl}(o) \qquad \text{(a)}$$

$$\epsilon_{ij}(o) = S_{ijkl}(o)\ \sigma_{kl}(o) \qquad \text{(b)} \tag{4.1.15}$$

These expressions define the initial (elastic) responses of the viscoelastic material.

It is often convenient to write (4.1.10) and (4.1.14) in a more simplified form. It is easily seen that these relations may formally be written

$$\sigma_{ij}(t) = \int_0^t C_{ijkl}(t-\tau)\frac{\partial \epsilon_{kl}}{\partial \tau}\,d\tau \qquad \text{(a)}$$

$$\epsilon_{ij}(t) = \int_0^t S_{ijkl}(t-\tau)\frac{\partial \sigma_{kl}}{\partial \tau}\,d\tau \qquad \text{(b)} \qquad (4.1.16)$$

Insertion of (4.1.10) into (4.1.16a) and of (4.1.13) into (4.1.16b) recovers (4.1.12) and (4.1.14). Thus (4.1.16) are general viscoelastic stress-strain relations for strain or stress inputs which vanish for $t < 0$, with the understanding that delta functions are permitted to occur in the integrands. The representations (4.1.16) also remain formally valid for a number of discontinuities at different times.

Integrals of type (4.1.16) are called <u>hereditary integrals</u> or <u>convolutions</u>. For simplicity of writing, such integrals or their more detailed versions (4.1.11) and (4.1.14) are frequently expressed in the forms

$$\sigma_{ij} = C_{ijkl} \otimes d\epsilon_{kl}$$

$$\epsilon_{ij} = S_{ijkl} \otimes d\sigma_{kl} \qquad (4.1.17)$$

We shall use this notation whenever convenient.

If the viscoelastic material is isotropic the stress-strain laws (4.1.16) are reduced exactly as for isotropic elastic materials. It is convenient in this case to split stresses and strains into isotropic and deviatoric parts in the usual fashion. Thus

$$\epsilon_{ij} = \epsilon\,\delta_{ij} + e_{ij} \quad , \qquad \epsilon = \frac{1}{3}\,\epsilon_{kk} \qquad \text{(a)}$$

$$\sigma_{ij} = \sigma\delta_{ij} + s_{ij} \quad , \qquad \sigma = \frac{1}{3}\,\sigma_{kk} \qquad \text{(b)} \qquad (4.1.18)$$

Then (4.1.16) for the isotropic case assume the forms

$$\sigma(t) = 3\int_0^t K(t-\tau)\,\frac{\partial\epsilon}{\partial\tau}\,d\tau \qquad \text{(a)}$$

$$s_{ij}(t) = 2\int_0^t G(t-\tau)\,\frac{\partial e_{ij}}{\partial\tau}\,d\tau \qquad \text{(b)}$$

$$\epsilon(t) = \frac{1}{3}\int_0^t I(t-\tau)\,\frac{\partial\sigma}{\partial\tau}\,d\tau \qquad \text{(c)} \qquad (4.1.19)$$

$$e_{ij}(t) = \frac{1}{2}\int_0^t J(t-\tau)\,\frac{\partial s_{ij}}{\partial\tau}\,d\tau \qquad \text{(d)}$$

Here $K(t)$ and $G(t)$ are the bulk and shear relaxation moduli, respectively, and $I(t)$ and $J(t)$ are the bulk and shear creep conpliances, respectively.

The stress-strain relations of a transversely isotropic viscoelastic material are similar to (3.4.86). Let x_1 be the axis of elastic symmetry. We then have in the notation (4.1.17)

$$\sigma_{11} = n \odot d\epsilon_{11} + \ell \odot d\epsilon_{22} + \epsilon \odot d\epsilon_{33} \qquad \text{(a)}$$

$$\sigma_{22} = \ell \odot d\epsilon_{11} + (k + G_T) \odot d\epsilon_{22} + (k - G_T) \odot d\epsilon_{33} \qquad \text{(b)}$$

$$\sigma_{33} = \ell \odot d\epsilon_{11} + (k - G_T) \odot d\epsilon_{22} + (k + G_T) \odot d\epsilon_{33} \qquad \text{(c)}$$

$$\sigma_{12} = 2G_A \odot d\epsilon_{12} \qquad \text{(d)} \qquad (4.1.20)$$

$$\sigma_{23} = 2G_T \odot d\epsilon_{23} \qquad \text{(e)}$$

$$\sigma_{13} = 2G_A \odot d\epsilon_{13} \qquad \text{(f)}$$

where $n(t)$, $\ell(t)$, $k(t)$, $G_T(t)$ and $G_A(t)$ are the five relaxation moduli of the material. It is similarly possible to write a strain-stress relation in terms of five creep compliances but this will not be done at the present time.

For reasons of mathematical expedience viscoelastic stress-strain relations are frequently expressed in terms of differential time operators. Limiting ourselves to the isotropic case we can write such stress-strain relations in the form

$$R(D)\,\sigma = S(D)\,\epsilon \qquad \text{(a)}$$

$$P(D)\,s_{ij} = Q(D)\,e_{ij} \qquad \text{(b)} \qquad (4.1.21)$$

where

$$D \equiv \frac{d}{dt}$$

$$R(D) = \sum_{m=0}^{r} R_m D^m$$

$$S(D) = \sum_{m=0}^{s} S_m D^m \qquad (4.1.22)$$

$$P(D) = \sum_{m=0}^{p} P_m D^m$$

$$Q(D) = \sum_{m=0}^{q} Q_m D^m$$

Here R_m, S_m, P_m, Q_m are constants and

$$D^m = \frac{d^m}{dt^m}$$

To each differential operator stress-strain law belongs a set of initial conditions. Methods to derive these may be found in [4.6], par. 15.6.

The simplest examples of (4.1.21-22) are the so-called Maxwell and Kelvin models, whose deviatoric stress-strain relations are, repectively,

$$De_{ij} = \frac{s_{ij}}{2\eta} + \frac{1}{2G} Ds_{ij} \qquad \text{(a)}$$

$$(4.1.23)$$

$$s_{ij} = 2Ge_{ij} + 2\eta\, De_{ij} \qquad \text{(b)}$$

Any viscoelastic differential operator stress-strain relation can be represented by an appropriate spring-dashpot model.

It should be carefully noted that (4.1.19) are much more fundamental than (4.1.21) since they are based on measured viscoelastic functions. It can be shown that relations of type (4.1.21) can always be brought into the form (4.1.19) while the converse is not necessarily true. To see this we define relaxation moduli associated with (4.1.21) by the stress variations due to strain step functions. Similarly, creep compliances are defined by strain responses to stress step functions. Thus the relaxation moduli and creep compliances are given by certain solutions of the differential equation (4.1.21). Having obtained these viscoelastic functions the responses to any strain or stress variations are found by superposition as integrals of form (4.1.19).

Evidently, the relaxation moduli and creep compliances associated with (4.1.21) will contain the constants R_m, S_m, P_m and Q_m. To find the constants the differential operators must be arbitrarily terminated at some m and the constants are then found by fitting of the theoretical viscoelastic functions to the experimental results at a number of points. To obtain a realistic presentation the differential operators may become lengthy. The simple operators contained in (4.1.23) can certainly not be expected to represent real materials with sufficient accuracy and their use can only lead to qualitative results.

It should be mentioned that elastic stress-strain laws are contained as special cases in viscoelastic stress-strain laws. Elastic moduli and compliances may be interpreted as relaxation moduli and creep compliances in the form of step functions. Thus

$$C_{ijkl}(t) = {}^{e}C_{ijkl}\, H(t) \qquad \text{(a)}$$

$$S_{ijkl}(t) = {}^{e}S_{ijkl}\, H(t) \qquad \text{(b)} \qquad (4.1.24)$$

where elastic properties are here and henceforward denoted by a left e superscript. Insertion of (4.1.24) into (4.1.16) yields the elastic stress-strain laws

$$\sigma_{ij}(t) = {}^{e}C_{ijkl}\, \epsilon_{kl}(t)$$

$$\epsilon_{ij}(t) = {}^{e}S_{ijkl}\, \sigma_{kl}(t) \qquad (4.1.25)$$

For stress-strain relations of type (4.1.21), elasticity implies that all constants except R_o, S_o, P_o and Q_o vanish in the differential operators.

In many isotropic viscoelastic materials viscoelastic behavior is predominant in shear and negligible in dilatation. In this case (4.1.19a,c) and (4.1.21a) simply become

$$\sigma(t) = 3\,{}^{e}K\, \epsilon(t) \qquad (4.1.26)$$

while (4.1.19b,d) and (4.1.21b) stay as they are.

A most important tool in linear viscoelasticity theory is the Laplace transform. We shall use the notations

$$\mathcal{L}\,[\varphi(t)] = \hat{\varphi}(p) = \int_0^\infty e^{-pt}\,\varphi(t)\,dt \tag{4.1.27}$$

for the Laplace transform, abbreviated LT from now on. We recall that the LT of a convolution is given by

$$\mathcal{L}\,(\varphi \circledcirc d\psi) = \mathcal{L}\,[\int_0^t \varphi(t-\tau)\,\psi(\tau)\,d\tau] = \hat{\varphi}(p)\,\hat{\psi}(p) \tag{4.1.28}$$

The LT of the stress-strain laws (4.1.11), (4.1.14) or (4.1.16) then assume the form

$$\begin{aligned} &\hat{\sigma}_{ij}(p) = p\,\hat{C}_{ijkl}(p)\,\hat{\epsilon}_{kl}(p) && \text{(a)} \\ &\hat{\epsilon}_{ij}(p) = p\,\hat{S}_{ijkl}(p)\,\hat{\sigma}_{kl}(p) && \text{(b)} \end{aligned} \tag{4.1.29}$$

where (4.1.28) and the rule for the LT of a derivative have been used. It is seen from (4.1.29) that the tensors $p\hat{C}_{ijkl}$ and $p\hat{S}_{ijkl}$ are reciprocal. Thus

$$p^2\,\hat{C}_{ijrs}\,\hat{S}_{rskl} = I_{ijkl} \tag{4.1.30}$$

where I_{ijkl} is the unit tensor given by (3.2.7). (Provided that $C_{ijkl}(t)$ and $S_{ijkl}(t)$ are ij,kl symmetric). Expression (4.1.30) provides a relation between relaxation moduli and creep compliances which assumes a complicated form in the time domain. Suffice it is to say that if the relaxation moduli are known the creep compliances must be found by solution of integral equations.

We introduce the notation

$$p\,\hat{C}_{ijkl}(p) = \Gamma_{ijkl}(p) \tag{4.1.31}$$

$$\hat{\sigma}_{ij}(p) = \Gamma_{ijkl}(p)\,\hat{\epsilon}_{kl}(p) \tag{4.1.32}$$

which is similar to an elastic stress-strain law. Therefore $\Gamma_{ijkl}(p)$ will be called <u>transform domain</u> (abbreviated-TD) <u>moduli</u>.

The LT of (4.1.19) is given by

$$\begin{aligned} &\hat{\sigma} = 3p\,\hat{K}\,\hat{\epsilon} = 3\,\kappa\,\hat{\epsilon} && \text{(a)} \\ &\hat{s}_{ij} = 2p\,\hat{G}\,\hat{e}_{ij} = 2\Gamma\,\hat{e}_{ij} && \text{(b)} \\ &\hat{\epsilon} = \frac{1}{3}p\,\hat{I}\,\hat{\sigma} && \text{(c)} \\ &\hat{e}_{ij} = \frac{1}{2}p\,\hat{J}\,\hat{s}_{ij} && \text{(d)} \end{aligned} \tag{4.1.33}$$

where

$$\begin{aligned} &\kappa(p) = p\,\hat{K}(p) && \text{(a)} \\ &\Gamma(p) = p\,\hat{G}(p) && \text{(b)} \end{aligned} \tag{4.1.34}$$

are the TD bulk and shear modulus. It follows from (4.1.33) that

$$\begin{aligned} &p^2\,\hat{K}\,\hat{I} = 1 && \text{(a)} \\ &p^2\,\hat{G}\,\hat{J} = 1 && \text{(b)} \end{aligned} \tag{4.1.35}$$

Consequently the TD bulk and shear compliances are $1/\kappa$ and $1/\Gamma$.

We can define other TD moduli just as in elasticity. As an example we consider the TD Young's modulus, which is denoted $p\hat{E}$ and $\hat{E}$ is the LT of the Young's relaxation modulus. Suppose that a cylinder is subjected to the space constant uniaxial state of stress

$$[\sigma_{ij}(t)] = \begin{bmatrix} \sigma_{11}(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with transform

$$[\hat{\sigma}_{ij}(p)] = \begin{bmatrix} \hat{\sigma}_{11}(p) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We define $p\hat{E}$ as

$$p\hat{E} = \frac{\hat{\sigma}_{11}}{\hat{\epsilon}_{11}} \tag{4.1.36}$$

Separation of the transformed uniaxial stress into isotropic and deviatoric parts and insertion into (4.1.33a,b) easily yields $\hat{\epsilon}_{11}$. Then we have from (4.1.36).

$$\hat{E} = \frac{9\hat{K}\hat{G}}{3\hat{K} + \hat{G}} \tag{4.1.37}$$

Inversion of (4.1.36) into the time domain yields in view of (4.1.28)

$$\sigma_{11}(t) = \int_0^t E(t-\tau) \frac{\partial \epsilon_{11}}{\partial \tau} d\tau \tag{4.1.38}$$

which is the uniaxial stress-strain relation. Note that the relation (4.1.37) is very complicated in the time domain.

We can formally define a TD Poisson's ratio by analogy with elasticity, in the form

$$\nu(p) = -\frac{\hat{\epsilon}_{22}}{\hat{\epsilon}_{11}} = -\frac{\hat{\epsilon}_{33}}{\hat{\epsilon}_{11}}$$

It follows just as in elasticity that

$$\nu(p) = \frac{3\hat{K}(p) - 2\hat{G}(p)}{2[3\hat{K}(p)+\hat{G}(p)]} = \frac{3\kappa(p)-2\Gamma(p)}{2[3\kappa(p)+\Gamma(p)]} \tag{4.1.39}$$

The LT of (4.1.20) leads to the TD stress-strain relations

$$\hat{\sigma}_{11} = p\hat{n}\,\hat{\epsilon}_{11} + p\hat{\ell}\,\hat{\epsilon}_{22} + p\hat{\ell}\,\hat{\epsilon}_{33} \tag{a}$$

$$\hat{\sigma}_{22} = p\hat{\ell}\,\hat{\epsilon}_{11} + p(\hat{k}+\hat{G}_T)\,\hat{\epsilon}_{22} + p(\hat{k}-\hat{G}_T)\,\hat{\epsilon}_{33} \tag{b}$$

$$\hat{\sigma}_{33} = p\hat{\ell}\,\hat{\epsilon}_{11} + p(\hat{k}-\hat{G}_T)\,\hat{\epsilon}_{22} + p(\hat{k}+\hat{G}_T)\,\hat{\epsilon}_{33} \tag{c}$$

$$\hat{\sigma}_{12} = 2p\,\hat{G}_A\,\hat{\epsilon}_{12} \tag{d}$$

$$\hat{\sigma}_{23} = 2p\,\hat{G}_T\,\hat{\epsilon}_{23} \tag{e}$$

$$\hat{\sigma}_{13} = 2p\,\hat{G}_A\,\hat{\epsilon}_{13} \tag{f}$$

(4.1.40)

The LT of differential operator stress-strain relations of type (4.1.21-22) are

$$R(p)\,\hat{\sigma} = S(p)\,\hat{\epsilon} \qquad \text{(a)}$$

$$P(p)\,\hat{s}_{ij} = Q(p)\,\hat{e}_{ij} \qquad \text{(b)} \qquad (4.1.41)$$

The residual terms at t=0 in the evaluation of the LT of the derivatives on both sides of (4.1.21) cancel out because of the initial conditions, [4.6], par. 15.6. Comparing (4.1.41) with (4.1.33) we see that in the present case

$$\kappa(p) = p\hat{K} = \frac{1}{3}\,\frac{S(p)}{R(p)} \qquad \text{(a)}$$

$$\Gamma(p) = p\hat{G} = \frac{1}{2}\,\frac{Q(p)}{P(p)} \qquad \text{(b)} \qquad (4.1.42)$$

Equations (4.1.42) define in a simple manner the relaxation moduli associated with the differential operator stress-strain law.

We illustrate the use of (4.1.42) to find the relaxation shear modulus and creep compliance for the Maxwell stress-strain relation (4.1.23a). It follows from (4.1.23a) and (4.1.42b) that

$$\hat{G}_M(p) = \frac{\eta}{1+Tp} \qquad (4.1.43a)$$

where M denotes Maxwell and $T = \eta/G$ is a characteristic time which is called relaxation time. It follows from (4.1.35b) and the result for $\hat{G}_M(p)$ that

$$\hat{J}_M(p) = \frac{1}{Gp} + \frac{1}{\eta p^2} \qquad (4.1.43b)$$

Inversion of (4.1.43a,b) yields

$$G_M(t) = G\, e^{-t/T} \qquad \text{(a)}$$

$$J_M(t) = \frac{1}{G} H(t) + \frac{t}{\eta} \qquad \text{(b)} \qquad (4.1.44)$$

4.1.2 Boundary Value Problems

We consider the case of a viscoelastic body in the absence of body forces. Since the stresses and displacements are time dependent there are in principle inertia terms in the equilibrium equations. We adopt the usual assumption that changes in time are so slow that these inertia terms can be neglected. Then the equilibrium equations are

$$\sigma_{ij,j}(\underline{x},t) = 0 \qquad (4.1.45)$$

in which the time appears only as a parameter. This kind of approximate static equilibrium state is called quasi-static.

To obtain differential equations for the displacements, (4.1.16a) are substituted into (4.1.45). The result may be written

$$(C_{ijkl} \otimes du_{k,l})_{,j} = 0 \qquad (4.1.46)$$

where ε_{kl} have been replaced by (4.1.1), the symmetry (4.1.4) has been taken into account and the notation (4.1.7a) has been used. If the body is homogeneous, C_{ijkl} are not space dependent. Equ. (4.1.46) can be written as

$$\frac{\partial^2}{\partial x_l \, \partial x_j} \left[\int_0^t C_{ijkl}(t-\tau) \frac{\partial u_k(\underline{x},\tau)}{\partial \tau} d\tau \right] = 0 \qquad (4.1.47)$$

Boundary conditions associated with (4.1.46) or (4.1.47) may be of the type

$$\begin{aligned} u_i(S,t) &= u_i^o \quad \text{on } S_u \quad \text{(a)} \\ T_i(S,t) &= T_i^o \quad \text{on } S_T \quad \text{(b)} \end{aligned} \qquad (4.1.48)$$

or we may have the mixed kind of boundary conditions of type (3.6.18).

If the body is heterogeneous then (4.1.46) or (4.1.47) must be satisfied in each phase with relaxation moduli $C_{ijkl}^{(m)}$, and there are added the interface conditions

$$\left.\begin{aligned} &u_i(\underline{x},t) \\ &T_i(\underline{x},t) \end{aligned}\right\} \text{ continuous on } S_{int} \qquad (4.1.49)$$

The boundary value problem as expressed above is exceedingly complicated if considered in the space-time domain. Great simplification is achieved if all quantities are Laplace transformed with respect to time. Listing the LT of (4.1.47-49) we have

$$p\,\hat{C}_{ijkl}\,\hat{u}_{k,lj} = 0 \qquad (4.1.50)$$

$$\begin{aligned} \hat{u}_i(S,p) &= \hat{u}_i^o \quad \text{on } S_u \quad \text{(a)} \\ \hat{T}_i(S,p) &= \hat{T}_i^o \quad \text{on } S_T \quad \text{(b)} \end{aligned} \qquad (4.1.51)$$

$$\left.\begin{array}{l}\hat{u}_i\ (\underline{x},p) \\ \\ \hat{T}_i\ (\underline{x},p)\end{array}\right\}\ \text{continuous on } S_{int} \qquad \begin{array}{l}(c)\\ \\ (d)\end{array}$$

where

$$\hat{T}_i = p\,\hat{C}_{ijkl}\,\hat{u}_{k,l}\,n_j \tag{4.1.52}$$

Comparison with the formulation of elasticity problems in chapter 3.2 shows that we have formally obtained an elastic problem for $\hat{u}_i$ in space - p domain in which the TD moduli $p\hat{C}_{ijkl} = \Gamma_{ijkl}$ assume the role of elastic moduli. If more convenient, $p\hat{S}_{ijkl}$ can be taken as the TD compliances, in view of the reciprocity relation (4.1.30). This "elastic" problem is generally called the <u>associated elastic problem</u>. Thus a convenient method of solving linear quasi-static viscoelastic problems is to solve the associated elastic problem which yields the transforms $\hat{u}_i\ (\underline{x},\ p)$ and $\hat{\sigma}_{ij}\ (\underline{x},\ p)$ of the viscoelastic solution. Then inversion of the LT gives the viscoelastic solution.

It should be emphasized that it has been tacitly assumed that the temporal change of external and internal boundaries can be neglected.

The analogy described above is known as the <u>correspondence principle.</u> We shall refer to it as the <u>static correspondence principle</u> since we shall later describe a dynamic correspondence principle. The static correspondence principle has been gradually developed by a number of writers (see [4.7]).

If the material is isotropic the associated elastic problem is also concerned with isotropic materials which are now described by the TD moduli (4.1.34).

If the viscoelastic material behavior is described by differential operator stress-strain relations of type (4.1.21), the TD moduli in the associated elastic problem are given by (4.1.42).

We shall now exploit the correspondence principle to extend the solution for homogeneous elastic bodies of arbitrary shape, under homogeneous boundary conditions, par. 3.2.2, to viscoelasticity. Let a homogeneous viscoelastic body be subjected to the boundary displacements

$$u_i(S,t) = \begin{cases} \epsilon^{o}_{ij}(t)\, x_j & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{4.1.53}$$

Then

$$\hat{u}_i\,(S,p) = \hat{\epsilon}^{o}_{ij}\,(p)\, x_j \tag{4.1.54}$$

The associated elastic problem is now to find the elastic fields in an elastic body with "moduli" $p\hat{C}_{ijkl}$, subjected to (4.1.54). By par. 3.2.2 the solution is

$$\hat{u}_i\,(\underline{x},p) = \hat{\epsilon}^{o}_{ij}\,(p)\, x_j \qquad \text{(a)}$$

$$\hat{\sigma}_{ij}\,(\underline{x},p) = p\hat{C}_{ijkl}(p)\;\hat{\epsilon}^{o}_{ij}\,(p) \qquad \text{(b)} \tag{4.1.55}$$

Inversion into the time domain gives

$$\left.\begin{aligned} u_i(\underline{x}, t) &= \epsilon^{o}_{ij}(t)\, x_j \\ \epsilon_{ij}(\underline{x}, t) &= \epsilon^{o}_{ij}(t) \end{aligned}\right\} \quad t \geq 0$$

$$\sigma_{ij}(\underline{x},t) = \int_{o}^{t} C_{ijkl}(t-\tau)\, \frac{d\epsilon_{kl}(\tau)}{d\tau}\, d\tau = \sigma_{ij}(t) \qquad (4.1.56)$$

$$u_i(\underline{x}, t),\ \epsilon_{ij}(\underline{x}, t),\ \sigma_{ij}(\underline{x}, t) = 0 \qquad t < 0$$

It is seen that the strain and stresses are space constant (homogeneous) and time variable.

Next, let the viscoelastic body be subjected to the boundary tractions

$$T_i(S,t) = \begin{cases} \sigma^{o}_{ij}(t)\, n_j & t \geq 0 \\ 0 & \end{cases} \qquad (4.1.57)$$

Then

$$\hat{T}_i(S, p) = \hat{\sigma}^{o}_{ij}(p)\, n_j \qquad (4.1.58)$$

It is now convenient to consider the associated elastic problem as an elastic body with TD compliances $p\hat{S}_{ijkl}(p)$. By par. 3.2.2 the solution is

$$\hat{\sigma}_{ij}(\underline{x}, p) = \hat{\sigma}^{o}_{ij}(p) \qquad \text{(a)}$$

$$\hat{\epsilon}_{ij}(\underline{x},p) = p\hat{S}_{ijkl}(p)\, \hat{\sigma}^{o}_{ij}(p) \qquad \text{(b)} \qquad (4.1.59)$$

The inverse transform is

$$\sigma_{ij}(\underline{x},t) = \sigma^{o}_{ij}(t) \qquad \text{(a)}$$

$$\epsilon_{ij}(\underline{x},t) = \int_{o}^{t} S_{ijkl}(t-\tau)\,\frac{d\sigma^{o}_{kl}(\tau)}{d\tau}\,d\tau = \epsilon_{ij}(t) \qquad \text{(b)} \qquad (4.1.60)$$

with displacements

$$u_i(\underline{x},t) = \epsilon_{ij}(t)\,x_j \qquad \text{(c)}$$

Expressions (4.1.60) are defined for $t \geq 0$ and vanish for $t < 0$. Thus again, the strain and stresses are spatially homogeneous and time variable. Accordingly, boundary conditions (4.1.54) and (4.1.57) may be called homogeneous boundary conditions for viscoelastic bodies.

4.2 GENERAL QUASI-STATIC THEORY OF VISCOELASTIC COMPOSITES(*)

4.2.1 Definition of Effective Viscoelastic Properties of Composites

The definition of effective viscoelastic properties of composites is closely related to that of effective elastic moduli of composites, as discussed in chapter 3.3.

Suppose that a composite of volume V and surface S consists of two or more homogeneous viscoelastic phases. The properties of the mth phase are specified by relaxation moduli $C^{(m)}_{ijkl}(t)$ or by creep compliances $S^{(m)}_{ijkl}(t)$. It is assumed that the phase geometry is statistically homogeneous and that internal and external geometry changes remain small.

The composite will be subjected to homogeneous boundary conditions of type (4.1.53) and (4.1.57). We recall the fundamental postulate of the theory of heterogeneous media as given in par. 3.2.2 and we extend it to the visco-elastic case: <u>The stress and strain fields in a very large SH viscoelastic heterogeneous body, subjected to homogeneous boundary conditions are spatially SH, except for a narrow boundary layer near the external surface</u>. It will be shown later below that the postulate for the viscoelastic case actually follows from the postulate for elastic heterogeneous media.

The composite is first subjected to a special homogeneous boundary condition of type (4.1.53)

(*) The theory presented in chapter 4.2 is based on references [4.8-9].

$$u_i(S,t) = \epsilon^o_{ij} H(t)\, x_j \tag{4.2.1}$$

where H(t) has been defined by (4.1.6) and ϵ^o_{ij} are constant. It follows from the average strain theorem, (3.1.18), that in the present case

$$\bar{\epsilon}_{ij}(t) = \epsilon^o_{ij} H(t) \tag{4.2.2}$$

Because of SH (4.2.2) are also the local RVE strain averages.

Since the differential equations of the viscoelastic phases, (4.1.47), are spatially linear it follows by the same kind of argument as given in par. 3.3.1 that the average stresses are linearly related to ϵ^o_{ij}. In general

$$\bar{\sigma}_{ij}(t) = C^*_{ijkl}(t)\, \epsilon^o_{kl} = C^*_{ijkl}(t)\, \bar{\epsilon}_{kl} \qquad t \geq 0 \tag{4.2.3}$$

The coefficients $C^*_{ijkl}(t)$ are defined as the <u>effective relaxation moduli</u> (ERM) of the composite, [4.8]. It is seen that this definition is completely analogous to (4.1.5-7) for homogeneous viscoelastic media.

If the homogeneous boundary conditions are (4.1.53) then by the average strain theorem (3.1.18)

$$\bar{\epsilon}_{ij}(t) = \epsilon^o_{ij}(t) \tag{4.2.4}$$

For this case $\bar{\sigma}_{ij}(t)$ can be constructed from (4.2.3) and the Boltzmann superposition principle. We have

$$\boxed{\sigma_{ij}(t) = \int_0^t C^*_{ijkl}(t-\tau)\, \frac{d\epsilon^o_{kl}(\tau)}{d\tau}\, d\tau = \int_0^t C^*_{ijkl}(t-\tau)\, \frac{d\bar{\epsilon}_{kl}(\tau)}{d\tau}\, d\tau} \tag{4.2.5}$$

which is the effective viscoelastic stress-strain relation of the composite. For a SH composite the averages in (4.2.5) are local over RVE as well as body averages. It should be noted that (4.2.5) also holds for any heterogeneous viscoelastic body, not necessarily SH. In that case the averages are just body averages.

Dually, the composite is subjected to a special homogeneous traction boundary condition

$$T_i\,(S,t) = \sigma^o_{ij}\,H(t)\,n_j \tag{4.2.6}$$

It follows from the average stress theorem, (3.1.35), that

$$\bar{\sigma}_{ij}(t) = \sigma^o_{ij}\,H(t) \tag{4.2.7}$$

and by SH these are also RVE averages. It follows by linearity that the average strains are linearly related to the average stresses. This is expressed as

$$\bar{\epsilon}_{ij}(t) = S^*_{ijkl}(t)\,\sigma^o_{kl} = S^*_{ijkl}(t)\,\bar{\sigma}_{kl} \qquad t \geq 0 \tag{4.2.8}$$

The $S^*_{ijkl}(t)$ are defined in analogy to (4.1.8-9) as the <u>effective creep compliances</u> (ECC) of the composite.

If the traction boundary condition is (4.1.57), it follows from (4.2.7-8) and the Boltzmann superposition principle that

$$\boxed{\bar{\epsilon}_{ij}(t) = \int_0^t S^*_{ijkl}(t-\tau)\,\frac{d\sigma^o_{kl}(\tau)}{d\sigma}\,d\tau = \int_0^t S^*_{ijkl}(t-\tau)\,\frac{d\bar{\sigma}_{kl}(\tau)}{d\tau}\,d\tau} \tag{4.2.9}$$

which is the effective strain-stress relation of the composite. This is again a perfectly general result for any heterogeneous body under homogeneous boundary conditions. For SH the body averages in (4.2.9) are also local RVE averages.

Equs. (4.2.5) and (4.2.9) are the most general stress-strain relations for viscoelastic SH composites. Using arguments similar to the ones given in par. 3.3.2 for reciprocity of EEM and EEC tensors, it follows in the present case that $\bar{\epsilon}_{ij}$ and $\bar{\sigma}_{ij}$ in (4.2.5) and (4.2.9) may be taken as the same quantities. Consequently $C^*_{ijkl}(t)$ and $S^*_{ijkl}(t)$ are related as in a homogeneous viscoelastic body. Again it is convenient to express this relation by LT as was done in (4.1.30). This is postponed to the next paragraph.

Because of the symmetry of the average stress and strain tensors it follows from (4.2.5) and (4.2.9) that

$$\boxed{\begin{aligned} &C^*_{ijkl}(t) = C^*_{jikl}(t) = C^*_{ijlk}(t) && \text{(a)} \\ &S^*_{ijkl}(t) = S^*_{jikl}(t) = S^*_{ijlk}(t) && \text{(b)} \end{aligned}} \qquad (4.2.10)$$

If (4.1.4) is accepted then it is easy to show that also

$$\boxed{\begin{aligned} &C^*_{ijkl}(t) = C^*_{klij}(t) && \text{(a)} \\ &S^*_{ijkl}(t) = S^*_{klij}(t) && \text{(b)} \end{aligned}} \qquad (4.2.11)$$

The proof is very similar to the one given in par. 3.3.2 for EEM and EEC. We express the average stress in the left side of (4.2.3) by (4.1.16a) and then equate to the right side of (4.2.3).

$$\bar{\sigma}_{ij}(t) = \int_0^t \overline{C_{ijkl}(\underline{x}, t-\tau) \frac{\partial}{\partial \tau} \epsilon_{kl}(\underline{x},\tau)}\, d\tau = C^*_{ijkl}(t)\, \bar{\epsilon}_{kl}$$

where $C_{ijkl}(\underline{x},t)$ denotes the variable (piecewise constant) relaxation moduli of the composite. It follows that

$$(C^*_{ijkl} - C^*_{klij})\, \bar{\epsilon}_{kl} = \int_0^t \overline{[C_{ijkl}(\underline{x},t-\tau) - C_{klij}(\underline{x},t-\tau)] \frac{\partial}{\partial \tau} \epsilon_{kl}(\underline{x},\tau)}\, d\tau$$

Now the integrand on the right side vanishes because of (4.1.4) and therefore the left side vanishes. But since $\bar{\epsilon}_{kl}$ are arbitrary this is possible only if each coefficient of $\bar{\epsilon}_{ij}$ vanishes and this proves (4.2.11a). The proof of (4.2.11b) is evidently analogous.

For various cases of macroscopic symmetry such as orthotropy, square symmetry and transverse isotropy, the general stress-strain relations (4.2.5) and (4.2.9) may be simplified just as effective elastic stress-strain relations. Thus for macroscopic orthotropy (4.2.5) assume by analogy with (3.4.1-2) the form

$$\bar{\sigma}_{11}(t) = C^*_{11} \otimes d\bar{\epsilon}_{11} + C^*_{12} \otimes d\bar{\epsilon}_{22} + C^*_{13} \otimes d\bar{\epsilon}_{33} \qquad \text{(a)}$$

$$\bar{\sigma}_{22}(t) = C^*_{12} \otimes d\bar{\epsilon}_{11} + C^*_{22} \otimes d\bar{\epsilon}_{22} + C^*_{23} \otimes d\bar{\epsilon}_{33} \qquad \text{(b)}$$

$$\bar{\sigma}_{33}(t) = C^*_{13} \otimes d\bar{\epsilon}_{11} + C^*_{23} \otimes d\bar{\epsilon}_{22} + C^*_{33} \otimes d\bar{\epsilon}_{33} \qquad \text{(c)}$$

(4.2.12)

$$\bar{\sigma}_{12}(t) = 2C^*_{44} \otimes d\bar{\epsilon}_{12} \qquad \text{(d)}$$

$$\bar{\sigma}_{23}(t) = 2C^*_{55} \otimes d\bar{\epsilon}_{23} \qquad \text{(e)}$$

$$\bar{\sigma}_{13}(t) = 2C^*_{66} \otimes d\bar{\epsilon}_{13} \qquad \text{(f)}$$

Here $C^*_{11}(t)$, $C^*_{12}(t)$ etc. are the ERM and we have used the shortened convolution notation (4.1.17).

The strain-stress relations may be simply written by analogy with (3.4.5-6) in terms of ECC $S^*_{11}(t)$, S^*_{12} etc.

For macroscopic transverse isotropy we have by analogy with (3.4.47-48) and (3.4.51-55)

$$\begin{aligned}
\bar{\sigma}_{11}(t) &= n^* \otimes d\bar{\epsilon}_{11} + \ell^* \otimes d\bar{\epsilon}_{22} + \ell^* \otimes d\bar{\epsilon}_{33} \\
\bar{\sigma}_{22}(t) &= \ell^* \otimes d\bar{\epsilon}_{11} + (k^* + G^*_T) \otimes d\bar{\epsilon}_{22} + (k^* - G^*_T) \otimes d\bar{\epsilon}_{33} \\
\bar{\sigma}_{33}(t) &= \ell^* \otimes d\bar{\epsilon}_{11} + (k^* - G^*_T) \otimes d\bar{\epsilon}_{22} + (k^* + G^*_T) \otimes d\bar{\epsilon}_{33} \\
\bar{\sigma}_{12}(t) &= 2G^*_A \otimes d\bar{\epsilon}_{12} \\
\bar{\sigma}_{23}(t) &= 2G^*_T \otimes d\bar{\epsilon}_{23} \\
\bar{\sigma}_{13}(t) &= 2G^*_A \otimes d\bar{\epsilon}_{13}
\end{aligned} \qquad (4.2.13)$$

where $n^*(t)$, $\ell^*(t)$, $k^*(t)$, $G_T^*(t)$ and $G_A^*(t)$ are the five ERM of the material.

Homogeneous boundary conditions appropriate for computation of the ERM in (4.2.12-13) are of type (4.2.1). Thus all the boundary conditions in chapter 3.4 are transcribed to the viscoelastic case by multiplying their right sides by $H(t)$. For example: For $k^*(t)$ definition (see (3.4.59-61))

$$u_1(S,t) = 0 \qquad u_2(S,t) = \epsilon^o H(t)\, x_2 \qquad u_3(S,t) = \epsilon^o H(t)\, x_3$$

$$[\bar{\epsilon}_{ij}(t)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon^o & 0 \\ 0 & 0 & \epsilon^o \end{bmatrix} H(t)$$

$$\bar{\sigma}_{22}(t) = \bar{\sigma}_{33}(t) = 2k^*(t)\, \epsilon^o$$

If $\epsilon^o H(t)$ in the boundary conditions is replaced by $\epsilon^o(t)$, $t \geq 0$, then

$$\bar{\sigma}_{22}(t) = \bar{\sigma}_{33}(t) = 2\int_o^t k^*(t-\tau)\, \frac{d\epsilon^o(\tau)}{d\tau}\, d\tau$$

4.2.2 Static Correspondence Principle for Viscoelastic Composites

We now proceed to derive a correspondence principle for viscoelastic composites which will enable us to find effective relaxation moduli and effective creep compliances on the basis of known effective elastic moduli. The theory developed also leads to the general results of par. 4.2.1 in independent fashion.

Consider a viscoelastic composite which is subjected to the homogeneous boundary conditions (4.1.53). We summarize the Laplace transformed problem in the manner of (4.1.51-52).

$$p\,\hat{C}^{(m)}_{ijkl}\,\hat{u}^{(m)}_{k,lj} = 0 \qquad \text{in } R_m \qquad \text{(a)}$$

$$\hat{u}_i(S,p) = \hat{\epsilon}^{o}_{ij}(p)\,x_j \qquad \text{(b)}$$

$$\left.\begin{array}{ll} \hat{u}_i & \text{(c)} \\ p\,\hat{C}_{ijkl}\,\hat{u}_{k,l}\,n_j & \text{(d)} \end{array}\right\} \quad \text{continuous on } S_{int}. \qquad (4.2.14)$$

The transformed stress is given by

$$\hat{\sigma}^{(m)}_{ij}(\underline{x},p) = p\,\hat{C}^{(m)}_{ijkl}(p)\,\hat{\epsilon}^{(m)}_{kl}(\underline{x},p) \qquad \text{in } R_m \qquad (4.2.15)$$

Suppose that we wish to compute the average of $\hat{\sigma}_{ij}(\underline{x},p)$. It is seen that this is precisely the problem which arose in the computation of average stresses in elastic composites for purpose of computation of EEM, pars. 3.3.1-2. We may regard (4.2.14) as a problem for an associated elastic composite whose phase "elastic moduli" are the previously defined TD moduli

$$\Gamma^{(m)}_{ijkl}(p) = p\,\hat{C}^{(m)}_{ijkl}(p) \qquad (4.2.16)$$

By the linearity argument of par. 3.3.1 we can write the average of (4.2.15) as

where $n^*(t)$, $\ell^*(t)$, $k^*(t)$, $G_T^*(t)$ and $G_A^*(t)$ are the five ERM of the material.

Homogeneous boundary conditions appropriate for computation of the ERM in (4.2.12-13) are of type (4.2.1). Thus all the boundary conditions in chapter 3.4 are transcribed to the viscoelastic case by multiplying their right sides by $H(t)$. For example: For $k^*(t)$ definition (see (3.4.59-61))

$$u_1(S,t) = 0 \qquad u_2(S,t) = \epsilon^o H(t)\, x_2 \qquad u_3(S,t) = \epsilon^o H(t)\, x_3$$

$$[\bar{\epsilon}_{ij}(t)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon^o & 0 \\ 0 & 0 & \epsilon^o \end{bmatrix} H(t)$$

$$\bar{\sigma}_{22}(t) = \bar{\sigma}_{33}(t) = 2k^*(t)\, \epsilon^o$$

If $\epsilon^o H(t)$ in the boundary conditions is replaced by $\epsilon^o(t)$, $t \geq 0$, then

$$\bar{\sigma}_{22}(t) = \bar{\sigma}_{33}(t) = 2\int_0^t k^*(t-\tau)\, \frac{d\epsilon^o(\tau)}{d\tau}\, d\tau$$

4.2.2 Static Correspondence Principle for Viscoelastic Composites

We now proceed to derive a correspondence principle for viscoelastic composites which will enable us to find effective relaxation moduli and effective creep compliances on the basis of known effective elastic moduli. The theory developed also leads to the general results of par. 4.2.1 in independent fashion.

$$\hat{\bar{\sigma}}_{ij}(p) = p\ \hat{C}^*_{ijkl}(p)\ \hat{\epsilon}^o_{kl}(p) = p\ \hat{C}^*_{ijkl}(p)\ \hat{\bar{\epsilon}}_{kl}(p) \tag{4.2.17}$$

where $p\ \hat{C}^*_{ijkl}(p)$ are for the time being arbitrary coefficients and the last equality in (4.2.17) is due to the average strain theorem applied to the transformed strains in the associated elastic composite.

If the viscoelastic composite is SH then the associated elastic composite is also SH since SH is a property of the phase geometry only and the Laplace transformation operates only on the time variable. Consequently, $p\hat{C}^*_{ijkl}$ may be defined as the "effective elastic moduli" of the associated elastic composite by the arguments of par. 3.3.2. They may also be called the <u>effective TD moduli</u>. Thus

$$\Gamma^*_{ijkl}(p) = p\ \hat{C}^*_{ijkl}(p) \tag{4.2.18}$$

If the homogeneous boundary condition is chosen in the special form (4.2.1) then the average strains are $\epsilon^o_{ij}\ H(t)$ with LT ϵ^o_{ij}/p. Insertion of the transformed average strains into (4.2.17) yields

$$\bar{\sigma}_{ij}(p) = \hat{C}^*_{ijkl}(p)\ \epsilon^o_{kl}$$

with inverse transform

$$\bar{\sigma}_{ij}(t) = C^*_{ijkl}(t)\ \epsilon^o_{kl} = C^*_{ijkl}(t)\ \bar{\epsilon}_{kl} \qquad t \geq 0$$

which is the same as (4.2.3). Thus $C^*_{ijkl}(t)$ are recognized as the ERM. The inversion of (4.2.17) is

$$\bar{\sigma}_{ij}(t) = \int_0^t C^*_{ijkl}(t-\tau) \frac{d\bar{\epsilon}_{kl}(\tau)}{d\tau} d\tau$$

which is the same as (4.2.5).

Effective viscoelastic stress-strain relations for various cases of macroscopic symmetry have been given in par. 4.2.1. Such reductions may also be easily performed on the basis of present developments in terms of transformed quantities. The effective TD stress-strain relation (4.2.15) may be reduced on the basis of symmetry considerations precisely as an effective elastic stress-strain relation and so the results of chapter 3.4 are directly applicable. Inversion then produces viscoelastic stress-strain relations as written in par. 4.2.1.

For example: in the case of transverse isotropy (4.2.17) involves only five effective TD moduli and may be written

$$\hat{\bar{\sigma}}_{11} = p\hat{n}^* \hat{\bar{\epsilon}}_{11} + p\hat{\ell}^* \hat{\bar{\epsilon}}_{22} + p\hat{\ell}^* \hat{\bar{\epsilon}}_{33} \qquad \text{(a)}$$

$$\hat{\bar{\sigma}}_{22} = p\hat{\ell}^* \hat{\bar{\epsilon}}_{11} + p(\hat{k}^* + \hat{G}^*_T) \hat{\bar{\epsilon}}_{22} + p(\hat{k}^* - \hat{G}^*_T) \hat{\bar{\epsilon}}_{33} \qquad \text{(b)}$$

$$\hat{\bar{\sigma}}_{33} = p\hat{\ell}^* \hat{\bar{\epsilon}}_{11} + p(\hat{k}^* - \hat{G}^*_T) \hat{\bar{\epsilon}}_{22} + p(\hat{k}^* + \hat{G}^*_T) \hat{\bar{\epsilon}}_{33} \qquad \text{(c)}$$

(4.2.19)

$$\hat{\bar{\sigma}}_{12} = 2p\,\hat{G}^*_A\,\hat{\bar{\epsilon}}_{12} \qquad \text{(d)}$$

$$\hat{\bar{\sigma}}_{23} = 2p\,\hat{G}^*_T\,\hat{\bar{\epsilon}}_{23} \qquad \text{(e)}$$

$$\hat{\bar{\sigma}}_{13} = 2p\,\hat{G}^*_A\,\hat{\bar{\epsilon}}_{13} \qquad \text{(f)}$$

where $p\hat{n}^*$, $p\hat{\ell}^*$, $p\hat{k}^*$, $p\hat{G}^*_T$ and $p\hat{G}^*_A$ are TD effective moduli. Inversion of (4.2.19) produces (4.2.13).

We shall now derive a correspondence principle which directly relates effective elastic and viscoelastic properties, [4.8]. Let the EEM of an elastic composite and the phase moduli be denoted ${}^eC^*_{ijkl}$ and ${}^eC^{(m)}_{ijkl}$, respectively. We write symbolically

$${}^eC^*_{ijkl} = {}^eC^*_{ijkl}\,({}^e\underline{C}^{(m)},\ \{g\}) \qquad (4.2.20)$$

where the first term in the parenthesis denotes phase moduli and the second term denotes phase geometry. It follows for the associated composite that

$$\boxed{\Gamma^*_{ijkl}(p) = {}^eC^*_{ijkl}\,[\,\underline{\Gamma}^{(m)}(p)\ ,\ \{g\}\ ]} \qquad (4.2.21)$$

Equ. (4.2.21) states the static correspondence principle for viscoelastic composites. In words: <u>The effective TD moduli of a viscoelastic composite are obtained by replacement of phase elastic moduli by TD phase moduli in the expressions for the effective elastic moduli of a composite with identical phase geometry</u>.

Once Γ^*_{ijkl} in (4.2.21) are known it follows from (4.2.18) that

$$\boxed{C^*_{ijkl}(t) = \mathcal{L}^{-1}\left[\frac{1}{p}\,\Gamma^*_{ijkl}(p)\right]} \tag{4.2.22}$$

where $\mathcal{L}^{-1}$ denotes the inversion of the LT .

The correspondence principle described above is valid for a general anisotropic composite with general anisotropic phases. If the phases are isotropic the phase TD moduli have the form (4.1.34) if the phases are described by stress-strain relations of type (4.1.19). If the phase stress-strain relations are of type (4.1.21-22) then the phase TD moduli are given by (4.1.42).

It should also be noted that if the displacement, strains and stresses in an elastic composite are known then the "displacements", "strains"and "stresses" in the associated elastic composite are also known in terms of TD moduli. These are then the transforms of the displacements, strains and stresses in a viscoelastic composite with identical phase geometry.

The theory developed above applies equally for effective creep compliances. Equ. (4.2.9) may be rederived on the basis of associated elastic composite theory just as (4.2.5) has been rederived above. The LT of (4.2.9) is

$$\hat{\bar{\epsilon}}_{ij}(p) = p\,\hat{S}^*_{ijkl}\,\hat{\bar{\sigma}}_{kl}(p) \tag{4.2.23}$$

Since by SH (4.2.22) and (4.2.17) must apply for the same transformed average stresses and strains we conclude that $p\,\hat{C}^*_{ijkl}$ and $p\,\hat{S}^*_{ijkl}$ are reciprocal. Thus

$$\boxed{p^2 \hat{C}^*_{ijrs} \hat{S}^*_{rskl} = I_{ijkl}} \tag{4.2.24}$$

and $p\hat{S}^*_{ijkl}$ may be identified as TD effective compliances. Once $\hat{C}^*_{ijkl}$ are known, $\hat{S}^*_{ijkl}$ can be computed from (4.2.24) and can then be inverted into the time domain.

Since in applications one is primarily concerned with the case of viscoelastic matrix and elastic fibers it is of importance to investigate the form of the correspondence principle for the case of a composite consisting of viscoelastic and elastic phases. It has been stated before, (4.1.24) , that in viscoelasticity theory elastic moduli are interpreted as relaxation moduli in terms of Heaviside unit functions. We can also see this if we take the LT of (4.1.25a) for an elastic phase with moduli ${}^eC^{(m)}_{ijkl}$ and compare the result with (4.2.15) . It is seen that

$$p\hat{C}^{(m)}_{ijkl}(p) = {}^eC^{(m)}_{ijkl} \tag{4.2.25}$$

from which it follows immediately that

$$C^{(m)}_{ijkl}(t) = {}^eC^{(m)}_{ijkl} H(t)$$

which is the same as (4.1.24). More important, (4.2.25) shows that the TD moduli of an elastic phase are just the elastic moduli. This shows that the correspondence principle as stated above remains unchanged in the case of the presence of viscoelastic and elastic phases. It may be emphasized that in this case the moduli of the elastic phases are left unchanged in the replacement scheme which leads to the effective TD moduli.

The transform inversion in (4.2.22) is frequently very difficult to carry out. But it is fortunately possible to obtain information about the behavior of $C^*_{ijkl}(t)$ at $t = 0$ and at $t \to \infty$ without performing a transform inversion. This is done by the use of initial and final value theorems (Abel-Tauber Theorems) for Laplace transforms.

Suppose that $\hat{\varphi}(p)$ is the LT of a function $\varphi(t)$. Then

$$\lim_{p \to \infty} p\hat{\varphi}(p) = \varphi(o) \qquad \text{(a)}$$

$$\lim_{p \to o} p\hat{\varphi}(p) = \varphi(\infty) \qquad \text{(b)} \qquad (4.2.26)$$

provided that the limits exist. For proof see e.g. [4.10]. The first of (4.2.26) is the initial value theorem and the second is the final value theorem.

Now from (4.2.18), (4.2.21) and (4.2.26)

$$C^*_{ijkl}(o) = \lim_{p \to \infty} p\,\hat{C}^*_{ijkl}(p) = \lim_{p \to \infty} {}^eC^*_{ijkl}[\underline{\Gamma}^{(m)}(p), \{g\}] \qquad \text{(a)}$$

$$C^*_{ijkl}(\infty) = \lim_{p \to o} p\,\hat{C}^*_{ijkl}(p) = \lim_{p \to o} {}^eC^*_{ijkl}[\underline{\Gamma}^{(m)}(p), \{g\}] \qquad \text{(b)} \qquad (4.2.27)$$

From (4.2.16) and (4.2.26)

$$\lim_{p \to \infty} \Gamma^{(m)}_{ijkl} = \lim_{p \to \infty} p\,\hat{C}^{(m)}_{ijkl}(p) = C^{(m)}_{ijkl}(o) \qquad \text{(a)}$$

$$\lim_{p \to o} \Gamma^{(m)}_{ijkl} = \lim_{p \to o} p\,\hat{C}^{(m)}_{ijkl}(p) = C^{(m)}_{ijkl}(\infty) \qquad \text{(b)} \qquad (4.2.28)$$

Insertion of (4.2.28) into (4.2.27) yields

$$\boxed{\begin{aligned} C^*_{ijkl}(o) &= {}^eC^*_{ijkl}\,[\underline{C}^{(m)}(o),\ \{g\}\ ] \qquad \text{(a)} \\ C^*_{ijkl}(\infty) &= {}^eC^*_{ijkl}\,[\underline{C}^{(m)}(\infty),\ \{g\}\ ] \qquad \text{(b)} \end{aligned}} \qquad (4.2.29)$$

We shall call $C^*_{ijkl}(o)$ and $C^{(m)}_{ijkl}(o)$ <u>initial</u> effective and phase relaxation moduli, respectively. Similarly, we shall call $C^*_{ijkl}(\infty)$ and $C^{(m)}_{ijkl}(\infty)$ <u>final</u> effective and phase relaxation moduli, respectively. The theorem which we have proved is: <u>The initial (final) effective relaxation moduli of a viscoelastic composite are obtained by replacement of phase elastic moduli by initial (final) phase relaxation moduli in the expressions for the effective elastic moduli of a composite with identical phase geometry</u>. This theorem will be called the <u>initial (final) value correspondence principle</u>.

The theorem which has been proved is of considerable practical importance as will be seen in applications further below. It provides the possibility to compute the initial (elastic) and final (after long time) values of ERM. In some cases these values are quite close and this then indicates that the viscoelastic effect is not of importance, <u>if it is assumed that ERM and ECC vary monotonically with time</u>.

Initial and final effective creep compliances are most easily obtained in the following manner. From (4.2.24) and (4.2.26)

$$\lim_{p\to\infty} p^2\,\hat{C}^*_{ijrs}(p)\,\hat{S}^*_{rskl}(p) = \lim_{p\to\infty}[p\,\hat{C}^*_{ijrs}]\lim[p\,\hat{S}^*_{rskl}] =$$

$$= C^*_{ijrs}(o)\,S^*_{rskl}(o) = I_{ijkl}$$

with a similar result for $p \to 0$. Summarizing, the results are

$$\boxed{\begin{aligned} C^*_{ijrs}(o)\, S^*_{rskl}(o) &= I_{ijkl} \\ C^*_{ijrs}(\infty)\, S^*_{rskl}(\infty) &= I_{ijkl} \end{aligned}} \qquad (4.2.30)$$

In words: <u>The initial (final) ERM and ECC are reciprocal</u>.

Evidently the same proof also applies to phase moduli. Thus

$$\begin{aligned} C^{(m)}_{ijrs}(o)\, S^{(m)}_{rskl}(o) &= I_{ijkl} \\ C^{(m)}_{ijrs}(\infty)\, S^{(m)}_{rskl}(\infty) &= I_{ijkl} \end{aligned} \qquad (4.2.31)$$

Consequently, initial and final ECC are computed from initial and final ERM just as in elasticity. Also, phase final (initial) creep compliances may be used instead of phase initial (final) relaxation moduli in the initial (final) value correspondence principle, if convenient.

We note that the correspondence principle for viscoelastic composites applies only to <u>expressions</u> for effective properties. No conclusion can be drawn at the present time for relations between bounds for EEM and bounds for ERM. It is possible by means of the elasticity-viscoelasticity analogy to derive bounds on TD moduli just as in elasticity, but it is not known what these bounds imply for the inverse transforms in the time domain.

Schapery [4.11] has discussed some relations between transform bounds and inverse transform bounds on the basis of approximate transform inversion, but the magnitude of error introduced by his approximation does not seem to be known.

4.3 VISCOELASTIC BEHAVIOR OF FIBER REINFORCED MATERIALS(*)

4.3.1 Effective Transform Domain (TD) Moduli

We now proceed to exploit the correspondence principle for viscoelastic composites and the results for EEM of FRM, given in part 3, to derive ERM and ECC of FRM. It is recalled that the correspondence principle applies to elastic and viscoelastic composites of same phase geometry and that the principle can be used only if explicit expressions for EEM are known. Fortunately, we have expressions for EEM of transversely isotropic FRM described by the composite cylinder assemblage model. Therefore we can use the correspondence principle to study the viscoelastic behavior of viscoelastic composite cylinder assemblages. It has been seen that the EEM computed for this model agree quite well with experimental results. It would seem, therefore, that similar good agreement can be expected in the viscoelastic case.

The analysis to be given is based on the following assumptions:

(a) The fibers are perfectly elastic, isotropic or transversely isotropic.

(b) The matrix is viscoelastic, isotropic.

(c) The viscoelasticity of the matrix can be neglected in dilatation.

These assumptions imply that fiber (phase 2) stress-strain relations are (3.4.88) or (3.4.86-87) and the matrix (phase 1) stress-strain relation is a simplification of (4.1.19), given by

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*The theory presented in chapter 4.3 is based on ref. [4.9].

$$\sigma^{(1)}(\underline{x},t) = 3K_1\ \epsilon^{(1)}(\underline{x},t) \qquad \text{(a)}$$

$$s_{ij}^{(1)}(\underline{x},t) = 2\int_0^t G_1(t-\tau)\ \frac{\partial}{\partial\tau}\ e_{ij}^{(1)}(\underline{x},\tau)\ d\tau \qquad \text{(b)} \qquad (4.3.1)$$

$$e_{ij}^{(1)}(\underline{x},t) = \frac{1}{2}\int_0^t J_1\ (t-\tau)\ \frac{\partial}{\partial\tau}\ s_{ij}^{(1)}\ (\underline{x},t)\ d\tau \qquad \text{(c)}$$

It is noted that G_1 and J_1 are related by (4.1.35b).

If it is desired to use differential operator stress-strain relations of type (4.1.21-22), then in the present case

$$\sigma^{(1)}(\underline{x},t) = 3K_1\ \epsilon^{(1)}(\underline{x},t) \qquad \text{(a)}$$

$$(4.3.2)$$

$$P(D)\ s_{ij}^{(1)}\ (\underline{x},t) = Q(D)\ e_{ij}^{(1)}(\underline{x},t) \qquad \text{(b)}$$

Equs. (4.3.1a), (4.3.2a) are a consequence of assumption (c) which is a frequently used approximation.

Note that neither of the assumptions (a-c) is theoretically needed. The following treatment may be easily extended to transversely isotropic matrix, viscoelastic fibers and matrix dilatational viscoelasticity.

To apply the correspondence principle we use the expressions for composite cylinder assemblage EEM which were derived in par. 3.5.3. We note that since the fibers are elastic their TD moduli are also their elastic moduli, as has been shown in par. 4.2.2, and they are thus left unchanged in the replacement scheme.

For matrix moduli we have to incorporate the assumption that the dilatational stress-strain relation is elastic with bulk modulus K_1. To do this we first write the matrix moduli appearing in composite cylinder assemblage EEM expressions in terms of K_1 and G_1. To avoid confusion with relaxation moduli we assign to elastic matrix moduli a left superscript e, except for K_1 where this is not needed. We recall the well known relation for isotropic elasticity

$$K = \lambda + \frac{2}{3} G$$

We then have from (3.4.89)

$$^{e}\eta_1 = K_1 + \frac{4}{3}\, {}^{e}G_1 \qquad \text{(a)}$$

$$^{e}\lambda_1 = K_1 - \frac{2}{3}\, {}^{e}G_1 \qquad \text{(b)} \qquad (4.3.3)$$

$$^{e}k_1 = K_1 + \frac{1}{3}\, {}^{e}G_1 \qquad \text{(c)}$$

The Young's modulus $^{e}E_1$ and the Poisson's ratio $^{e}\nu_1$ are given by the well known expressions

$$^{e}E_1 = \frac{9K_1\, {}^{e}G_1}{3K_1 + {}^{e}G_1} \qquad \text{(a)}$$

$$(4.3.4)$$

$$^{e}\nu_1 = \frac{3K_1 - 2\, {}^{e}G_1}{2(3K_1 + {}^{e}G_1)} \qquad \text{(b)}$$

Now the TD moduli replacing (4.3.3-4) in EEM expression are simply obtained by replacement of ${}^{e}G_1$ by $\Gamma_1(p)$, everywhere. Therefore the replacement rule is

$$ {}^{e}G_1 \rightarrow \Gamma_1(p) = p\hat{G}_1(p) \qquad \text{(a)} $$

$$ {}^{e}\eta_1 \rightarrow p\hat{\eta}_1(p) = K_1 + \frac{4}{3}\Gamma_1(p) \qquad \text{(b)} $$

$$ {}^{e}\lambda_1 \rightarrow p\hat{\lambda}_1(p) = K_1 - \frac{2}{3}\Gamma_1(p) \qquad \text{(c)} $$

$$ (4.3.5) $$

$$ {}^{e}k_1 \rightarrow p\hat{k}_1(p) = K_1 + \frac{1}{3}\Gamma_1(p) \qquad \text{(d)} $$

$$ {}^{e}E_1 \rightarrow p\hat{E}_1(p) = \frac{9K_1\,\Gamma_1(p)}{3K_1+\Gamma_1(p)} \qquad \text{(e)} $$

$$ {}^{e}\nu_1 \rightarrow \nu_1(p) = \frac{3K_1-2\Gamma_1(p)}{2[3K_1+\Gamma_1(p)]} \qquad \text{(f)} $$

Here (4.3.5a) is to be interpreted as (4.1.42) for stress-strain relations of type (4.3.2), which also define (4.3.5b-f) for this case.

Use of the replacement scheme (4.3.5) in the expressions (3.5.91-92), (3.5.96-97), (3.5.99) and (3.5.111) yields the corresponding TD effective moduli. The case of the transverse effective shear modulus G_T^* is unfortunately more complicated. It will be recalled that bounds, (3.5.113-114), for this EEM

were given, but it is not known at the present time how these bounds can be generalized to the present viscoelastic case. As has been stated before recent work by Hashin and Rosen indicates that (3.5.113) may be the actual result for G_T^* of a composite cylinder assemblage if the fibers are stiffer than the matrix. It has also been seen that (3.5.113) is in good agreement with experimental results. We consequently regard (3.5.113) as an ad-hoc expression for G_T^* of the composite cylinder assemblage and also transform it into effective TD modulus form as the other moduli.

The resulting TD effective moduli are now summarized.

$$p\hat{n}^*(p) = p\hat{n}_1 v_1 + n_2 v_2 - \frac{(p\hat{\lambda}_1 - \lambda_2)^2 v_1 v_2}{p\hat{k}_1 v_2 + k_2 v_1 + p\hat{G}_1} \qquad \text{(a)}$$

$$p\hat{\ell}^*(p) = \frac{p\hat{\lambda}_1 (k_2 + p\hat{G}_1) v_1 + \lambda_2 (p\hat{k}_1 + p\hat{G}_1) v_2}{(k_2 + p\hat{G}_1) v_1 + (p\hat{k}_1 + p\hat{G}_1) v_2} \qquad \text{(b)}$$

$$p\hat{k}^*(p) = \frac{p\hat{k}_1 (k_2 + p\hat{G}_1) v_1 + k_2 (p\hat{k}_1 + p\hat{G}_1) v_2}{(k_2 + p\hat{G}_1) v_1 + (p\hat{k}_1 + p\hat{G}_1) v_2} \qquad \text{(c)}$$

$$p\hat{E}^*(p) = p\hat{E}_1 v_1 + E_2 v_2 + \frac{4[\nu_2 - \nu_1(p)]^2 v_1 v_2}{v_1/k_2 + v_2/p\hat{k}_1 + 1/p\hat{G}_1} \qquad \text{(d)} \qquad (4.3.6)$$

$$\nu_A^*(p) = \nu_1(p)\, v_1 + \nu_2\, v_2 + \frac{[\nu_2 - \nu_1(p)](1/p\hat{k}_1 - 1/k_2)\, v_1\, v_2}{v_1/k_2 + v_2/p\hat{k}_1 + 1/p\hat{G}_1} \qquad \text{(e)}$$

$$p\hat{G}_A^*(p) = p\hat{G}_1 \frac{p\hat{G}_1\, v_1 + G_2\,(1+v_2)}{p\hat{G}_1(1+v_2) + G_2\, v_1} \qquad \text{(f)}$$

$$p\hat{G}_T^*(p) = p\hat{G}_1 \frac{[1+\alpha(p)\, v_2^3][\rho(p) + \beta_1(p)\, v_2] - 3v_2\, v_1^2 \beta_1^2(p)}{[1+\alpha(p) v_2^3][\rho(p) - v_2] \qquad - 3v_2\, v_1^2 \beta_1^2(p)} \qquad (4.3.7)$$

where in (4.3.7)

$$\alpha(p) = \frac{\beta_1(p) - \gamma(p)\, \beta_2}{1 + \gamma(p)\, \beta_2} \qquad \text{(a)}$$

$$\rho(p) = \frac{\gamma(p) + \beta_1(p)}{\gamma(p) - 1} \qquad \text{(b)}$$

$$\gamma(p) = \frac{G_2}{p\hat{G}_1(p)} \qquad \text{(c)}$$

$$\beta_1(p) = \frac{1}{3 - 4\nu_1(p)} \qquad \text{(d)}$$

$$\beta_2 = \frac{1}{3 - 4\nu_2} \qquad \text{(e)}$$

The TD modulus $p\hat{E}^*_T$ and the TD Poisson's ratio $\nu^*_T(p)$ are defined just as for elastic composites. Thus from (3.4.82-84)

$$\boxed{\begin{aligned} p\hat{E}^*_T(p) &= \frac{4p\hat{k}^*\, p\hat{G}^*_T}{p\hat{k}^* + m(p)\hat{G}^*_T} && \text{(a)} \\ \nu^*_T(p) &= \frac{p\hat{k}^* - m(p)p\hat{G}^*_T}{p\hat{k}^* + m(p)p\hat{G}^*_T} && \text{(b)} \qquad (4.3.8) \\ m(p) &= 1 + \frac{4p\hat{k}^*\, \nu^{*2}_A(p)}{p\hat{E}^*_A} && \text{(c)} \end{aligned}}$$

A few comments on equs. (4.3.6-8) are in order. Firstly, it will be noticed that in many equations p factors have not been cancelled. The reason for this is that because of the presence of K_1 there is no common p factor in the right sides of (4.3.5) except for (4.3.5a) and so cancellation of p factors would provide no advantage.

Secondly, the occurrence of TD Poisson's ratios (4.3.6d) and (4.3.8b) calls for some explanation. These "Poisson's ratios" are only of formal significance in relating TD moduli (for example (4.3.7a,c)). The inverse transforms of these "Poisson's ratios" have no intrinsic physical meaning.

In order to obtain an effective strain-stress law we write first the TD analogue of (3.4.78) which is given by

$$\hat{\bar{\epsilon}}_{11} = \frac{1}{p\hat{E}^*_A}\,\hat{\bar{\sigma}}_{11} - \frac{\nu^*_A(p)}{p\hat{E}^*_A}\,\hat{\bar{\sigma}}_{22} - \frac{\nu^*_A(p)}{p\hat{E}^*_A}\,\hat{\bar{\sigma}}_{33} \qquad \text{(a)}$$

$$\hat{\bar{\epsilon}}_{22} = -\frac{\nu^*_A(p)}{p\hat{E}^*_A}\,\hat{\bar{\sigma}}_{11} + \frac{1}{p\hat{E}^*_T}\,\hat{\bar{\sigma}}_{22} - \frac{\nu^*_T(p)}{p\hat{E}^*_T}\,\hat{\bar{\sigma}}_{33} \qquad \text{(b)} \qquad (4.3.9)$$

$$\hat{\bar{\epsilon}}_{33} = -\frac{\nu^*_A(p)}{p\hat{E}^*_A}\,\hat{\bar{\sigma}}_{11} - \frac{\nu^*_T(p)}{p\hat{E}^*_T}\,\hat{\bar{\sigma}}_{22} + \frac{1}{p\hat{E}^*_T}\,\hat{\bar{\sigma}}_{33} \qquad \text{(c)}$$

For the shears we merely invert (4.2.19d,e,f) to read

$$\hat{\bar{\epsilon}}_{12} = \frac{\hat{\bar{\sigma}}_{12}}{2p\hat{G}^*_A} \qquad \text{(a)}$$

$$\hat{\bar{\epsilon}}_{23} = \frac{\hat{\bar{\sigma}}_{23}}{2p\hat{G}^*_T} \qquad \text{(b)} \qquad (4.3.10)$$

$$\hat{\bar{\epsilon}}_{13} = \frac{\hat{\bar{\sigma}}_{13}}{2p\hat{G}^*_A}$$

Inversion of (4.3.8-9) yields strain-stress relations of type (4.2.9) and thus the effective creep compliances.

Often relaxation moduli reduce to a small fraction of their initial values after long time, (theoretically infinite), has elapsed. This has important bearing on the calculation of final values of ERM. It is recalled that according to the initial and final value correspondence principle initial and final value ERM are found by substitution of initial and final value phase relaxation moduli, respectively, into EEM expressions. The initial values of the relaxation moduli are written simply as the left sides of (4.3.3-4) with zero time argument. The final values of the relaxation counterparts of (4.3.3-4) are found by substitution of $G_1(\infty)$ into these expressions while K_1 is left unchanged, since it is by hypothesis time invariant. Now $G_1(\infty)$ may often be neglected in comparison to K_1. In this case we have from (4.3.3-4)

$$\eta_1(\infty) = \lambda_1(\infty) = k_1(\infty) = K_1 \qquad \text{(a)}$$

$$E_1(\infty) = 3G_1(\infty) \qquad \text{(b)} \qquad (4.3.11)$$

$$\nu_1(\infty) = \frac{1}{2} \qquad \text{(c)}$$

It is, of course, not necessary to neglect $G_1(\infty)$ with respect to K_1. If $G_1(\infty)$ can be estimated from experiment its value can be easily used.

4.3.2 Effective Relaxation Moduli and Creep Compliances

4.3.2.1 Methods of investigation

The ERM and ECC of the viscoelastic FRM may now be obtained by transform inversion of (4.3.6-7), but examination of the expressions to be inverted reveals that this is a formidable undertaking. An extensive investigation of the resulting viscoelastic properties is not within the scope of the present work. We shall limit ourselves to study of some typical cases and some conclusions of general interest.

The methods of investigation at our disposal are:

(a) Representation of matrix shear relaxation modulus $G_1(t)$ by an arbitrary function which is determined by experiment.

In this case the inversion must be performed numerically by either one of the following methods: numerical solution of an integral equation or approximate numerical transform inversion.

(b) Representation of the matrix shear stress-strain relation by differential operators of type (4.3.2b).

In this case $p\hat{G}_1(p)$ is given by the expression (4.1.42b) which is a ratio of two polynomials in p. All of (4.3.6-7) then also become ratios of polynomials in p and the inversion can be performed by conventional methods. The necessary calculations may, however, become extremely laborious.

(c) Simplification of fiber and matrix properties in order to obtain easier transform inversions.

Great simplification is achieved if it is assumed that fibers are rigid or (and) that the matrix is incompressible. Expressions for composite cylinder assemblage EEM for such extreme cases have been given in par. 3.5.3 and the correspondence principle can be applied directly to these expressions. The simplification is primarily due to disappearance of matrix Poisson's ratio whose TD counterpart (4.3.58), is largely responsible for transform complexity. It should, however, be realized that the rigid fiber and/or incompressible matrix approximation may introduce significant errors. Consequently, viscoelastic results obtained on the basis of these approximations should be regarded as qualitative.

(d) Use of the initial and final value correspondence principles, of par. 4.2.2, to obtain initial and final values of ERM and ECC.

This method is of significant practical importance since it uncovers those cases where the viscoelastic effect is unimportant. If the final value of an ERM is close to its initial value and if the ERM is a monotonic function of time, then this implies insignificant time effect. It is not, however, known to the author how monotonicity can be demonstrated without finding the ERM for the whole time range. It seems physically reasonable to assume that this is the case if matrix relaxation moduli are monotonic functions of time, which is a well established physical fact. So <u>the monotonic time character of ERM is at present regarded as a conjecture</u>.

4.3.2.2 Axial and isotropic-plane stressing and straining

Suppose that a fiber reinforced cylinder is subjected to average strains

$$[\bar{\epsilon}_{ij}(t)] = \begin{bmatrix} \bar{\epsilon}_{11}(t) & 0 & 0 \\ 0 & \bar{\epsilon}_T(t) & 0 \\ 0 & 0 & \bar{\epsilon}_T(t) \end{bmatrix} \tag{4.3.12}$$

by means of the homogeneous boundary conditions

$$u_1(S,t) = \bar{\epsilon}_{11}(t)\, x_1 \qquad u_2(S,t) = \epsilon_T(t)\, x_2 \qquad u_3(S,t) = \epsilon_T(t)\, x_3 \tag{4.3.13}$$

In that event the TD stress-strain law (4.2.19) becomes

$$\begin{aligned} \hat{\bar{\sigma}}_{11}(p) &= p\hat{n}^*(p)\, \hat{\bar{\epsilon}}_{11}(p) + 2p\hat{\ell}^*(p)\, \hat{\bar{\epsilon}}_T(p) \\ \hat{\bar{\sigma}}_{22} = \hat{\bar{\sigma}}_{33} &= \hat{\bar{\sigma}}_T(p) = p\hat{\ell}^*(p) + 2p\hat{k}^*(p)\, \hat{\bar{\epsilon}}_T(p) \end{aligned} \tag{4.3.14}$$

The inversion of (4.3.14) into the time domain is

$$\begin{aligned} \bar{\sigma}_{11}(t) &= n^* \otimes d\bar{\epsilon}_{11} + 2\ell^* \otimes d\bar{\epsilon}_T \\ \bar{\sigma}_T(t) &= \ell^* \otimes d\bar{\epsilon}_{11} + 2k^* \otimes d\bar{\epsilon}_T \end{aligned} \tag{4.3.15}$$

We wish to study the ERM $n^*(t)$, $\ell^*(t)$ and $k^*(t)$ entering into (4.3.15).

We start with application of the initial and final value correspondence principle. The initial moduli are then given by replacement of elastic matrix moduli in (3.5.99), (3.5.91) and (3.5.92) by the corresponding initial values of the phase relaxation moduli. Therefore

$$\boxed{\begin{aligned} n^*(o) &= n_1(o)\,v_1 + n_2\,v_2 - \frac{[\lambda_1(o)-\lambda_2]^2 v_1 v_2}{k_1(o)v_2+k_2v_1+G_1(o)} && \text{(a)} \\ \ell^*(o) &= \frac{\lambda_1(o)[k_2+G_1(o)]v_1+\lambda_2[k_1(o)+G_1(o)]v_2}{[k_2+G_1(o)]v_1+[k_1(o)+G_1(o)]v_2} && \text{(b)} \\ k^*(o) &= \frac{k_1(o)[k_2+G_1(o)]v_1+k_2[k_1(o)+G_1(o)]v_2}{[k_2+G_1(o)]v_1+[k_1(o)+G_1(o)]v_2} && \text{(c)} \end{aligned}} \qquad (4.3.16)$$

where because of isotropy [compare (3.4.89)]

$$n_1(o) = \lambda_1(o) + 2G_1(o) = k_1(o) + G_1(o)$$

$$n_2 = \lambda_2 + 2G_2 = k_2 + G_2$$

The final values of $n^*(t)$, $\ell^*(t)$, $k^*(t)$ are found by substitution of (4.3.11) into (3.5.99), (3.5.91-92). This yields

$$n^*(\infty) = K_1 v_1 + n_2 v_2 - \frac{(K_1 - \lambda_2)^2 v_1 v_2}{K_1 v_2 + k_2 v_1} \qquad \text{(a)}$$

$$\ell^*(\infty) = \frac{K_1(k_2 v_1 + \lambda_2 v_2)}{k_2 v_1 + K_1 v_2} \qquad \text{(b)} \qquad (4.3.17)$$

$$k^*(\infty) = \frac{K_1 k_2}{k_2 v_1 + K_1 v_2} \qquad \text{(c)}$$

Note that $G_1(\infty)$ has been neglected with respect to fiber moduli.

We perform a sample calculation for a FRM consisting of epoxy matrix and glass fibers. The elastic moduli of the epoxy are taken as the initial values of the epoxy relaxation moduli. The epoxy and glass elastic properties are given in table 3.6.1, p.223 . The results for $n^*(o)$ and $n^*(\infty)$ are given for various fiber volume fractions in the table below.

v_2	$n^*(o)$	$n^*(\infty)$ $(10^6$ psi)
0	0.64	0.44
0.2	2.66	2.54
0.4	4.69	4.63
0.6	6.71	6.74
0.8	8.88	8.91
1.0	11.68	11.68

It is seen that the results are so close that the time dependence of $\eta^*(t)$ can be safely ignored for this kind of FRM.

Results for $\ell^*(o)$, $\ell^*(\infty)$ and $k^*(o)$, $k^*(\infty)$ are shown in figs. 4.3.1-2. It is seen that the time dependence of $\ell^*(t)$ and $k^*(t)$ is not very significant. It should be borne in mind that to study this time dependence it is necessary to resort to methods (a) or (b) described in sub-par. 4.3.2.1. Since the mathematics involved is laborious and complicated this hardly seems to be a worthwhile undertaking in view of the small difference between initial and final values shown in the figures.

If the fibers are assumed rigid it is possible to establish a very simple expression for $k^*(t)$ which expresses this ERM explicitly in terms of arbitrary matrix properties. For rigid fibers ${}^e k^*$ is given by (3.5.115). The corresponding TD modulus is found by replacement of k_1, G_1 by (4.3.5a,d). We find

$$p\hat{k}^*(p) = K_1 + \frac{1}{3}\, p\hat{G}_1(p) + [K_1 + \frac{4}{3}\, p\,\hat{G}_1(p)]\,\frac{v_2}{1-v_2} \qquad (4.3.18)$$

Division of both sides by p and subsequent inversion yields

$$k^*(t) = \frac{K_1}{1-v_2} + \frac{1-3v_2}{3(1-v_2)}\, G_1(t) \qquad t \geq 0 \qquad (4.3.19)$$

It is easily realized that (4.3.19) can be generalized to the case of a matrix which is viscoelastic in shear <u>and</u> dilatation. In that event (4.3.19) becomes

$$\boxed{k^*(t) = \frac{K_1(t)}{1-v_2} + \frac{1+3v_2}{3(1-v_2)} G_1(t)} \qquad (4.3.20)$$

Expressions (4.3.19-20) <u>permit the direct use of experimentally obtained matrix viscoelastic properties</u>. Such expressions are a happy and unusual occurrence in viscoelastic composite theory. It is, however, emphasized again that the underlying rigid fiber approximation has to be viewed with caution. A possible way of checking the validity of (4.3.19) for a FRM with stiff fibers is to compute from it $k^*(o)$ and $k^*(\infty)$ and to compare these with $k^*(o)$, $k^*(\infty)$ results obtained by the previously used method for non-rigid elastic fibers. If the results are close it may be assumed that (4.3.19) can be used. We have for rigid fibers

$$k^*(o) = k_1(o) + [k_1(o) + G_1(o)] \frac{v_2}{1-v_2} \qquad \text{(a)}$$

$$(4.3.21)$$

$$k^*(\infty) = \frac{K_1}{1-v_2} \qquad \text{(b)}$$

where (4.3.21) follows from (3.5.115) and (4.3.21b) follows from (4.3.18) by neglect of $G_1(\infty)$ with respect to K_1. A numerical comparison of the values of $k^*(o)$ and $k^*(\infty)$ based on (4.3.16c), (4.3.17c) for epoxy matrix and glass fibers and based on (4.3.21) for epoxy matrix is given in the table below

	rigid fibers		elastic E-glass fibers		
v_2	$k^*(o)$	$k^*(\infty)$	$k^*(o)$	$k^*(\infty)$	(10^6 psi)
0	0.49	0.44	0.49	0.44	
0.2	0.65	0.56	0.64	0.55	
0.4	0.92	0.74	0.87	0.70	
0.6	1.46	1.11	1.26	1.02	
0.8	3.51	2.22	2.21	1.78	

It is seen that for the present material for which $k_2/k_1(o) = 14.8$ the values in the range $0 \leq v_2 \leq 0.6$ are reasonably close and therefore the use of (4.3.18) for the whole time range seems justified.

We now proceed to study ECC and for this purpose we consider average stresses of the form

$$[\bar{\sigma}_{ij}(t)] = \begin{bmatrix} \bar{\sigma}_{11}(t) & 0 & 0 \\ 0 & \bar{\sigma}_T(t) & 0 \\ 0 & 0 & \bar{\sigma}_T(t) \end{bmatrix} \quad t \geq 0 \qquad (4.3.22)$$

which are imposed by means of the homogeneous traction system

$$T_1(S,t) = \bar{\sigma}_{11}(t)\, n_1 \qquad T_2(S,t) = \bar{\sigma}_T(t)\, n_2 \qquad T_3(S,t) = \bar{\sigma}_T(t)\, n_3$$

The TD strain-stress relations (4.3.9) now assume the form

$$\hat{\bar{\epsilon}}_{11} = \frac{1}{pE_A^*}\hat{\bar{\sigma}}_{11} - \frac{2\nu_A^*(p)}{p\hat{E}_A^*}\hat{\bar{\sigma}}_T \qquad \text{(a)}$$

$$(4.3.23)$$

$$\hat{\bar{\epsilon}}_{22} = \hat{\bar{\epsilon}}_{33} = \hat{\bar{\epsilon}}_T = -\frac{\nu_A^*(p)}{p\hat{E}_A^*}\hat{\bar{\sigma}}_{11} + \frac{1-\nu_T^*(p)}{p\hat{E}_T^*}\hat{\bar{\sigma}}_T \qquad \text{(b)}$$

Since TD moduli are related just as elastic moduli we recognize from (3.4.85b), (3.4.82-84) that

$$\frac{1-\nu_T^*(p)}{p\hat{E}_T^*} = \frac{1}{2p\hat{k}^*} + \frac{2[\nu_A^*(p)]^2}{p\hat{E}_A^*} \tag{4.3.24}$$

We now introduce the notation

$$p\,\hat{e}_A^* = \frac{1}{p\hat{E}_A^*} \qquad \text{(a)}$$

$$p\,\hat{\kappa}^* = \frac{1}{p\hat{k}^*} \qquad \text{(b)}$$

$$p\,\hat{s}^* = \frac{\nu_A^*(p)}{p\hat{E}_A^*} \qquad \text{(c)} \tag{4.3.25}$$

$$p\,\hat{r}^* = \frac{[\nu_A^*(p)]^2}{p\hat{E}_A^*} \qquad \text{(d)}$$

Insertion of (4.3.24-25) into (4.3.23) and inversion yields

$$\bar{\varepsilon}_{11}(t) = e_A^* \circledast d\bar{\sigma}_{11} + 2\,s^* \circledast d\bar{\sigma}_T \qquad \text{(a)}$$

$$\bar{\varepsilon}_T(t) = s^* \circledast d\bar{\sigma}_{11} + \left(\frac{1}{2}\kappa^* + 2r^*\right) \circledast d\bar{\sigma}_T \qquad \text{(b)} \tag{4.3.26}$$

The functions $e_A^*(t)$, $s^*(t)$, $\kappa^*(t)$ and $r^*(t)$ are effective creep compliances.

If $\bar{\sigma}_T$ is zero then the specimen is in a state of uniaxial average stress. We then have from (4.3.23a) and (4.3.26a), respectively

$$\bar{\sigma}_{11}(t) = \int_0^t E_A^*(t-\tau)\,\frac{d\bar{\epsilon}_{11}(\tau)}{d\tau}\,d\tau \qquad \text{(a)}$$

$$\bar{\epsilon}_{11}(t) = \int_0^t e_A^*(t-\tau)\,\frac{d\bar{\sigma}_{11}(\tau)}{d\tau}\,d\tau \qquad \text{(b)} \qquad (4.3.27)$$

Equ. (4.3.27a) determines the uniaxial stress variation (relaxation) if an axial strain history is given and (4.3.27b) determines the axial strain variation (creep) for uniaxial stress history.

If $\bar{\epsilon}_{11}(t)$ vanishes then the specimen is in a state of plane strain. In this event it follows from (4.3.23), (4.3.24), (4.3.25b) and (4.3.26b) that

$$\bar{\sigma}_T(t) = 2\int_0^t k^*(t-\tau)\,\frac{d\bar{\epsilon}_T(\tau)}{d\tau}\,d\tau \qquad \text{(a)}$$

$$\bar{\epsilon}_T(t) = \frac{1}{2}\int_0^t \kappa^*(t-\tau)\,\frac{d\bar{\sigma}_T(\tau)}{d\tau}\,d\tau \qquad \text{(b)} \qquad (4.3.28)$$

The physical interpretation of (4.3.28) is similar to that of (4.3.27). Equ. (4.3.28a) determines average stress relaxation for given average strain history, while (4.3.28b) determines average strain (creep) for given average stress history.

The creep compliances $s^*(t)$ and $r^*(t)$ appearing in (4.3.26) determine the coupling effects on deformation when $\bar{\sigma}_{11}$ and $\bar{\sigma}_T$ act simultaneously. They are of the nature of time dependent Poisson's ratios.

We proceed to study the behavior of the ERM $E_A^*(t)$, which is of great practical importance. We have from (3.5.96) and from the initial and final value correspondence principles with (4.3.11) used in the latter,

$$\boxed{\begin{aligned} E_A^*(o) &= E_1(o)v_1 + E_2 v_2 + \frac{4[\nu_2 - \nu_1(o)]^2 v_1 v_2}{v_1/k_2 + v_2/k_1 + 1/G_1(o)} && \text{(a)} \\ E_A^*(\infty) &= E_2 v_2 && \text{(b)} \end{aligned}} \qquad (4.3.29)$$

The simple form of (4.3.29b) is obtained because $G_1(\infty)$ is neglected with respect to fiber moduli.

Plots of (4.3.29) for glass and epoxy properties, previously used, are shown in fig. 4.3.3. The third term in (4.3.27a) is numerically insignificant and so $E_A^*(o)$ is practically given by the first two terms. This phenomenon has been repeatedly noted in discussion of elastic E_A^*. It is seen that only for small fiber volume fraction is there an appreciable difference between $E^*(o)$ and $E^*(\infty)$. For the fiber volume fractions used in practice which are generally higher than 0.4, the difference is negligible. Thus the time dependence of $E_A^*(t)$ can be neglected for such materials.

The vanishing of $E_A^*(\infty)$ for no fibers should not be taken too literally. Since in this case $E_A^*(\infty)$ is $E_1(\infty)$, this merely implies that $E_1(\infty)$ is regarded as a small number.

It is recalled that it has been shown previously that $\bar{E}$ is an excellent approximation for ${}^eE_A^*$, i.e. the effective axial Young's modulus for any fibrous or fiber reinforced material. It is reasonable to expect that therefore the TD modulus $p\hat{E}_A^*$ can be represented with excellent approximation by the average of the phase TD moduli, thus

$$p\hat{E}_A^* \cong p\hat{E}_1 v_1 + E_2 v_2$$

from which by inversion

$$\boxed{E_A^*(t) \cong E_1(t)\, v_1 + E_2\, H(t)} \qquad (4.3.30)$$

The initial and final values of the compliances $e_A^*(t)$ and $k^*(t)$ are also easily found. We have from (4.3.25a,b) and (4.2.26)

$$e_A^*(o) = \lim_{p\to\infty} p\hat{e}_A^*(p) = \lim_{p\to\infty} \left[\frac{1}{p\hat{E}_A^*(p)}\right] = \frac{1}{E_A^*(o)} \qquad \text{(a)}$$

$$(4.3.31)$$

$$e_A^*(\infty) = \lim p\hat{e}_A^*(p) = \lim \left[\frac{1}{p\hat{E}_A^*(p)}\right] = \frac{1}{E_A^*(\infty)} \qquad \text{(b)}$$

and similarly

$$\mathcal{k}^*(o) = \frac{1}{k^*(o)} \qquad \text{(a)}$$

$$\mathcal{k}^*(\infty) = \frac{1}{k^*(\infty)} \qquad \text{(b)} \qquad (4.3.32)$$

The results (4.3.31-32) are really examples of the general relations (4.2.30) and can be directly derived from the latter.

It follows from (4.3.29), (4.3.16c), (4.3.17c) and (4.3.31-32) that

$$e_A^*(o) \cong \frac{1}{E_1(o)\, v_2 + E_2\, v_1} \qquad \text{(a)}$$

$$e_A^*(\infty) = \frac{1}{E_2\, v_2} \qquad \text{(b)} \qquad (4.3.33)$$

$$\mathcal{k}^*(o) = \frac{[k_2+G_1(o)]\, v_1 + [k_1(o) + G_1(o)]\, v_2}{k_1(o)[k_2+G_1(o)]v_1+k_2[k_1(o)+G_1(o)]\, v_2} \qquad \text{(a)}$$

$$\mathcal{k}^*(\infty) = \frac{v_1}{K_1} + \frac{v_2}{k_2} \qquad \text{(b)} \qquad (4.3.34)$$

where the third term in the right side of (4.3.29a) has been neglected.

Plots of (4.3.33-34) for Epoxy-glass FRM are shown in figs. 4.3.4-5. It is seen that for sufficiently large fiber volume fractions, as used in practice.

$e_A^*(o)$ and $e_A^*(\infty)$ are very close together, indicating again the insignificance of the viscoelastic effect for a uniaxial stress in fiber direction. There is a significant difference at low fiber volume fractions. Indeed $e_A^*(\infty)$ for the matrix without fibers becomes theoretically unbounded. This, of course, merely implies that $E_1(\infty)$ is small.

The values of $\kappa^*(o)$ and $\kappa^*(\infty)$ are also not significantly different. Evaluation of $\kappa^*(t)$ for the whole time range is of course as laborious as evaluation of $k^*(t)$ and must again be done by numerical methods or on the basis of assumed differential operator stress-strain relations for the matrix.

It is to be noted that there is no such simple result as (4.3.19) for the creep compliance $\kappa^*(t)$, when the fibers are assumed rigid. To see this we use (4.3.25b) and (4.3.18) to obtain

$$\frac{K_1}{1-v_2}\,\frac{\hat{\kappa}^*(p)}{p} + \frac{1+3v_2}{3(1-v_2)}\,\hat{G}_1(p)\,\hat{\kappa}^*(p) = \frac{1}{p^2}$$

The inversion of this equation into the time domain is

$$\frac{K_1}{1-v_2}\int_o^t \kappa^*(\tau)\,d\tau + \frac{1+3v_2}{3(1-v_2)}\int_o^t G_1(t-\tau)\,\kappa^*(\tau)\,d\tau = t \qquad (4.3.35)$$

Here $G_1(t)$ is a numerically known function and $\kappa^*(t)$ is the unknown function. Another form of the equation may be obtained by differentiation of (4.3.35) with respect to t. We then have

$$\left[\frac{K_1}{1-v_2} + G_1(o)\right]\kappa^*(t) + \frac{1+3v_2}{3(1-v_2)}\int_o^t \frac{\partial G_1(t-\tau)}{\partial \tau}\,\kappa^*(\tau)\,d\tau = 1$$

Such integral equations can be solved numerically by methods described in [4.15].

We now consider the ECC $s^*(t)$ which enters into (4.3.26). It follows from (4.3.25c) and the theorems (4.2.26) that

$$s^*(o) = -\frac{\nu_A^*(o)}{E_A^*(o)} \qquad \text{(a)}$$

$$s^*(\infty) = -\frac{\nu_A^*(\infty)}{E_A^*(\infty)} \qquad \text{(b)} \qquad (4.3.36)$$

The denominators of (4.3.36) are given by (4.3.29). The numerators are obtained from (3.5.97) with the replacement (4.3.11) for $\nu_A^*(\infty)$. We then have

$$\boxed{\begin{aligned} \nu_A^*(o) &= \nu_1(o)\, v_1 + \nu_2\, v_2 + \frac{[\nu_2 - \nu_1(o)][1/k_1(o) - 1/k_2]\, v_1\, v_2}{v_1/k_2 + v_2/k_1(o) + 1/G_1(o)} \qquad \text{(a)} \\ \nu_A^*(\infty) &= \frac{1}{2}\, v_1 + \nu_2\, v_2 \qquad \text{(b)} \end{aligned}} \qquad (4.3.37)$$

Plots of $s^*(o)$ and $s^*(\infty)$ are shown in fig. 4.3.6. It is seen that for fiber volume fractions larger than 0.3 the difference between the two is insignificant. We observe that $s^*(\infty) - s^*(o)$ is positive for fiber volume fractions between 0 to about 0.55 and then becomes negative. The same curious phenomenon was also found for $\ell^*(t)$, fig. 4.3.1.

Treatment of $r^*(t)$ is, of course, completely analogous and it is again found that for the present material the viscoelastic effect is insignificant.

4.3.2.3 Axial and transverse shearing

We shall now discuss the viscoelastic properties of FRM in axial or transverse shear. It will be seen that in contrast to the viscoelastic properties discussed in sub-par. 4.3.2.2 there is a significant viscoelastic effect for both of these shears.

Suppose that a fiber reinforced cylinder is subjected to average axial shear

$$[\bar{\epsilon}_{ij}(t)] = \begin{bmatrix} 0 & \bar{\epsilon}_{12}(t) & 0 \\ \bar{\epsilon}_{12}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad t \geq 0 \tag{4.3.38}$$

by means of the homogeneous boundary conditions

$$u_1(S,t) = \bar{\epsilon}_{12}(t)\, x_2 \qquad u_2(S,t) = \bar{\epsilon}_{12}(t)\, x_1 \qquad u_3(S,t) = 0$$

It follows from (4.2.13d) that the only surviving average stress is $\bar{\sigma}_{12}$ which is given by

$$\bar{\sigma}_{12}(t) = 2\int_0^t G_A^*(t-\tau)\,\frac{d\bar{\epsilon}_{12}(\tau)}{d\tau}\,d\tau \tag{4.3.39}$$

where G_A^* is the effective axial shear modulus, whose transform is defined by (4.3.6f).

Dually, the specimen is subjected to the average stress

$$[\sigma_{ij}(t)] = \begin{bmatrix} 0 & \bar{\sigma}_{12}(t) & 0 \\ \bar{\sigma}_{12}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad t \geq 0$$

by means of the homogeneous traction boundary conditions

$$T_1(S,t) = \bar{\sigma}_{12}(t)\, n_2 \qquad T_2(S,t) = \bar{\sigma}_{12}(t)\, n_1 \qquad T_3(S,t) = 0$$

To find the strain response we consider (4.3.10a). We define the quantity $\hat{g}^*_A(p)$ by

$$p\hat{g}^*_A(p) = \frac{1}{p\hat{G}^*_A(p)} \tag{4.3.40}$$

Insertion of (4.3.40) into (4.3.10a) and inversion yields

$$\bar{\epsilon}_{12}(t) = \frac{1}{2}\int_0^t g^*_A(t-\tau)\,\frac{d\bar{\sigma}_{12}(t)}{d\tau}\,d\tau \tag{4.3.41}$$

Thus g^*_A is recognized as the effective axial shear compliance.

Obviously, similar results for transverse shearing can be written down at once. For average transverse shear strain

$$[\bar{\epsilon}_{ij}(t)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\epsilon}_{23}(t) \\ 0 & \bar{\epsilon}_{23}(t) & 0 \end{bmatrix} \quad t \geq 0$$

the stress response is

$$\bar{\sigma}_{23}(t) = 2 \int_0^t G_T^*(t-\tau) \frac{d\bar{\epsilon}_{23}(\tau)}{d\tau} d\tau \tag{4.3.42}$$

where the LT of the effective transverse shear relaxation modulus G_T^* is defined by (4.3.6g).

For average transverse shear stress

$$[\bar{\sigma}_{ij}(t)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\sigma}_{23}(t) \\ 0 & \bar{\sigma}_{23}(t) & 0 \end{bmatrix} \quad t \geq 0$$

the strain response is

$$\bar{\epsilon}_{23}(t) = \frac{1}{2} \int_0^t g_T^*(t-\tau) \frac{d\bar{\sigma}_{12}(\tau)}{d\tau} d\tau \tag{4.3.43}$$

where the LT of the effective transverse shear creep compliance $g_T^*(t)$ is defined by

$$p\hat{g}_T^*(p) = \frac{1}{p\hat{G}_T^*(p)} \tag{4.3.44}$$

We start the investigation of shear properties with application of the initial and final value correspondence principle. The elastic axial shear modulus ${}^{e}G_A^*$ is given by (3.5.111). The form of $G_A^*(o)$ is immediate and $G_A^*(\infty)$ is obtained by replacement of G_1 by $G_1(\infty)$ in (3.5.111). We then neglect $G_1(\infty)$ in comparison to G_2. The results are then

$$\boxed{\begin{aligned} G_A^*(o) &= G_1(o)\,\frac{G_1(o)\,v_1 + G_2(1+v_2)}{G_1(o)(1+v_2)+G_2 v_1} \qquad \text{(a)} \\ G_A^*(\infty) &= G_1(\infty)\,\frac{1+v_2}{v_1} \qquad \text{(b)} \end{aligned}} \qquad (4.3.45)$$

It is seen by the form of (4.3.45) that $G_A^*(\infty)$ is of the order of $G_1(\infty)$. Therefore the reduction of $G_A^*(t)$ in the time range (o, ∞) is considerably larger than even that of $G_1(t)$. This implies that the viscoelastic effect is significant and the initial and final values do not give good estimates.

The initial and final values of the creep compliance $g_A^*(t)$, entering into (4.3.42) are simply found by use of theorem (4.2.36) in the relation (4.3.40). It follows that

$$\boxed{\begin{aligned} g_A^*(o) &= \frac{1}{G_A^*(o)} \\ g_A^*(\infty) &= \frac{1}{G_A^*(\infty)} \end{aligned}} \qquad (4.3.46)$$

and thus (4.3.45-46) define the initial and final values of $g_A^*(t)$. It is seen that $g_A^*(\infty)$ is very large since $G_A^*(\infty)$ is very small. This again indicates the significance of the viscoelastic effect.

The procedure for $G_T^*(t)$ is entirely analogous. The elastic ${}^eG_T^*$ is assumed to be given by (3.5.113). The initial value $G_T^*(o)$ is given by replacement of G_1 and ν_1 by $G_1(o)$ and $\nu_1(o)$, respectively. The final value $G_T^*(\infty)$ is given by replacement of G_1 by $G_1(\infty)$ and of ν_1 by $\frac{1}{2}$, (4.3.11). If $G_1(\infty)$ is also neglected with respect to G_2 we have

$$G_T^*(\infty) = G_1(\infty)\frac{1-v_2+5v_2^2+v_2^3}{(1-v_2)^3} \tag{4.3.47}$$

It is again seen that $G_T^*(\infty)$ is of the order of $G_1(\infty)$ which indicates significant viscoelastic effect in transverse shear. Comparison of (4.3.47) with (3.5.119) reveals identical forms. This is not surprising since as has been mentioned (4.3.47) is based on final Poisson's ratio $\frac{1}{2}$ and negligible $G_1(\infty)$ with respect to G_2, which is equivalent to incompressible matrix - rigid fiber situation.

Again, it follows from (4.2.36) and (4.3.44) that

$$g_T^*(o) = \frac{1}{G_T^*(o)}$$

$$g_T^*(\infty) = \frac{1}{G_T^*(\infty)} \tag{4.3.48}$$

and $g_T^*(\infty)$ becomes very large since $G_T^*(\infty)$ is very small.

Having satisfied ourselves that the initial and final value correspondence principle does not provide useful information for $G_A^*(t)$ and $G_T^*(t)$, we are forced to look for other methods of investigation. We start with derivation of some very simple and general results for $G_A^*(t)$ when the fibers are rigid, and for $G_T^*(t)$ when the fibers are rigid and the matrix is incompressible.

Suppose that the material is fibrous or fiber reinforced, with arbitrary phase geometry. It has been shown in par. 3.5.1 that the elastic axial shear solution depends only on the phase elastic moduli ${}^eG_1$ and ${}^eG_2$. Consequently, we can write the general expression

$$ {}^eG_A^* = {}^eG_A^* \left({}^eG_1, {}^eG_2, \{g\}\right) \tag{4.3.49} $$

where $\{g\}$ stands for phase geometry. By dimensional argument (4.3.49) can be written in the form

$$ {}^eG_A^* = {}^eG_1 \, F\left(\frac{{}^eG_2}{{}^eG_1}, \{g\}\right) \tag{4.3.50} $$

where F is some function.

Now let phase 2 be rigid and so ${}^eG_2/{}^eG_1$ becomes infinitely large. It is assumed, however, that ${}^eG_A^*$ remains finite. Such a situation will arise when the rigid phase is in the form of fibers, 2, which are surrounded by matrix, 1. Then (4.3.50) can be rewritten

$$ {}^eG_A^* = {}^eG_1 \, F_A \tag{4.3.51} $$

where F_A is a function of the phase geometry only.

We now apply the correspondence principle to (4.3.51). This implies that ${}^{e}G_A^*$ and ${}^{e}G_1$ are replaced by $p\hat{G}_A^*$ and $p\hat{G}_1$ in (4.3.51). This procedure yields

$$\hat{G}_A^*(p) = \hat{G}_1(p)\, F_A \tag{4.3.52}$$

which immediately inverts into

$$G_A^*(t) = G_1(t)\, F_A \tag{4.3.53}$$

In view of (4.3.51) we can also write (4.3.53) in the form

$$\boxed{\frac{G_A^*(t)}{G_1(t)} = \frac{{}^{e}G_A^*}{{}^{e}G_1}} \tag{4.3.54}$$

Expressions (4.3.53-54) are very interesting results. They show that the ratio between the effective axial shear relaxation modulus and the matrix shear relaxation modulus remainsconstant for all times and is given by the ratio between effective axial elastic shear modulus and elastic matrix shear modulus.

Inserting (4.3.52) into (4.3.41) and using the relation (4.1.35) for the matrix material, we find after inversion

$$g_A^*(t) = J_1(t)/F_A \tag{4.3.55}$$

where $J_1(t)$ is the shear creep compliance of the matrix.

We now apply the general results obtained to the composite cylinder assemblage model. Equ. (3.5.115c) gives ${}^{e}G_A^*$ for rigid fibers. Using this result in (4.3.53) we have

$$G_A^*(t) = G_1(t)\,\frac{1 + v_2}{1 - v_2} \tag{4.3.56}$$

This equation makes it possible to find $G_A^*(t)$ when $G_1(t)$ is known only as a numerical experimental result. Similarly, (4.3.54) for the composite cylinder assemblage yields the simple result

$$g_A^*(t) = J_1(t)\,\frac{1 - v_2}{1 + v_2} \tag{4.3.57}$$

Similar considerations can be applied for establishment of an expression for $G_T^*(t)$ when one phase is rigid and the other is viscoelastic. But here it is necessary to make the additional assumption that the viscoelastic phase is incompressible, since ${}^{e}G_T^*$ depends on all phase elastic properties. We can then derive a relation similar to (4.3.51)

$${}^{e}G_T^* = {}^{e}G_1\, F_T \tag{4.3.58}$$

where F_T is a function of phase geometry only. It follows by the same arguments used before that

$$\boxed{\begin{aligned} \frac{G_T^*(t)}{G_1(t)} &= \frac{{}^eG_T^*}{{}^eG_1} \qquad \text{(a)} \\ \frac{g_T^*(t)}{J_1(t)} &= \frac{{}^eG_1}{{}^eG_T^*} \qquad \text{(b)} \end{aligned}} \qquad (4.3.59)$$

The right sides of (4.3.59) are given by (3.5.119) . Consequently

$$G_T^*(t) = G_1(t)\,\frac{1-v_2 + 5v_2^3 + v_2^3}{(1-v_2)^3} \qquad \text{(a)}$$

$$(4.3.60)$$

$$g_T^*(t) = J_1(t)\,\frac{(1-v_2)^3}{1-v_2 + 5v_2^2 + v_2^3} \qquad \text{(b)}$$

It should be remembered that (4.3.60) has to be regarded with much more caution than (4.3.56-57), because of the incompressible matrix assumption involved in the former.

We continue the investigation of viscoelastic shear properties with the derivation of expressions for $G_A^*(t)$ and $g_A^*(t)$ when the matrix behavior is expressed by a differential operator stress-strain relation. We choose the simple Maxwell relation (4.3.10a) . The associated TD shear modulus is given by

$$\Gamma_1(p) = p\hat{G}_1(p) = \frac{\eta_1 p}{1+T_1 p} \tag{4.3.61}$$

where

$$T_1 = \eta_1/G_1$$

and η_1 and G_1 are the viscosity coefficient and shear modulus of the matrix, respectively. Insertion of this TD modulus into (4.3.6f) yields the result

$$\hat{G}_A^*(p) = \frac{\eta_1}{1+T_1 p} \cdot \frac{\gamma(1+v_2) + T_1[v_1 + \gamma(1+v_2)]p}{\gamma v_1 + T_1(1+v_2 + \gamma v_1)p} \tag{4.3.62}$$

where

$$\gamma = G_2/G_1$$

The LT (4.3.62) is easily inverted by the method of partial fractions. The result is

$$\frac{G_A^*(t)}{G_1} = \frac{v_1}{1+v_2}\exp(-t/T_1) + \frac{4\gamma v_2}{(1+v_2)(1+v_2+\gamma v_1)}\exp\left(-\frac{\gamma v_1}{1+v_2+\gamma v_1}\cdot\frac{t}{T_1}\right) \tag{4.3.63}$$

When the fibers are rigid $\gamma \to \infty$, and (4.3.63) reduces to

$$G_A^*(t) = G_1\frac{1+v_2}{1-v_2}\exp(-t/T_1) \tag{4.3.64}$$

It has been previously shown, (4.1.44a), that the relaxation modulus of the Maxwell material is

$$G_1(t) = G_1\exp(-t/T_1)$$

Comparison of the last result with (4.3.64) verifies the general formula (4.3.56) for the present case.

The creep compliance $g_A^*(t)$ is also easily computed. It follows from (4.3.62) and (4.3.41) that

$$\hat{g}_A^*(p) = \frac{1 + T_1 p}{\eta_1 p^2} \cdot \frac{\gamma v_1 + T_1(1+v_2+\gamma v_1)\, p}{\gamma(1+v_2) + T_1[v_1 + \gamma(1+v_2)]\, p} \tag{4.3.65}$$

Again inversion is easily carried out and the result is

$$G_1\, g_A^*(t) = \frac{v_1}{1+v_2}\,[H(t) + t/T_1] + \frac{4v_2}{\gamma(1+v_2)^2}\, H(t) -$$

$$- \frac{4v_1 v_2}{(1+v_2)^2 \gamma\,[\gamma(1+v_2) + v_1]} \exp\left(- \frac{\gamma(1+v_2)}{\gamma(1+v_2)+v_1} \cdot \frac{t}{T_1} \right) \tag{4.3.66}$$

where $H(t)$ is the Heaviside step function. For rigid fibers, (4.3.66) reduces to

$$g_A^*(t) = \frac{1 - v_2}{1 + v_2}\left[\frac{H(t)}{G_1} + \frac{t}{\eta_1} \right] \tag{4.3.67}$$

Comparing this with the shear creep compliance of the Maxwell matrix, (4.1.44b) it is seen that the general formulat (4.3.57) is verified.

The results obtained here can be used to obtain some idea about the γ value for which the rigid fiber approximation becomes permissible. Fig. 4.3.7

shows plots of the relaxation modulus (4.3.63) for different values of γ , in the case $v_1 = v_2 = 0.5$. It is seen that for $\gamma \geq 25$ the rigid fiber approximation becomes quite accurate. Typical values of γ are about 30 for glass-epoxy FRM and 60-90 for boron-epoxy FRM. It is, therefore, to be expected that the general results (4.3.54-55), for any geometry, and (4.3.56-57) for the composite cylinder assemblage model , should be applicable for such materials.

In order to perform a similar investigation for transverse shear properties it is necessary to use the transform (4.3.7) in which $\nu_1(p)$ is to be interpreted as (4.3.5f). It is seen that even for the simple Maxwell matrix the inversion is extremely cumbersome, though there is no theoretical difficulty involved. Such an inversion will not be performed here. It would seem more advisable to resort to approximate transform inversion methods. See e.g. Schapery [4.11].

Another method of investigation mentioned in sub-par. 4.3.2.1 is that of integral equations, (method (a)). As an example we derive the integral equation for the ERM $G_A^*(t)$. We rewrite the transform (4.3.6f) in the form.

$$(1 + v_2)\, G_1\, G_A^* + G_2\, v_1 \frac{1}{p} G_A^* = v_1\, G_1^2 + G_2 (1 + v_2) \frac{1}{p}\, G_1$$

Inversion of this expression gives

$$(1 + v_2) \int_0^t G_1\,(t-\tau)\, G_A^*(\tau)\, d\tau + G_2\, v_1 \int_0^t G_A^*(\tau)\, d\tau =$$

$$= v_1 \int_0^t G_1\,(t-\tau)\, G_1(\tau)\, d\tau + G_2\,(1+v_2) \int_0^t G_1(\tau)\, d\tau \qquad (4.3.68)$$

Another form is obtained by differentiation of (4.3.68)

$$(1+v_2) \int_0^t \frac{\partial}{\partial t}\, G_1(t-\tau)\, G_A^*(\tau)\, d\tau = v_1 \int_0^t \frac{\partial}{\partial t}\, G_1(t-\tau)\, G_1(\tau)\, d\tau +$$

$$+ 2\,[G_2 - G_1(o)\,]\, G_A^*(t) \qquad (4.3.69)$$

These integral equations must in general be solved numerically and this can be done for matrix relaxation modulus $G_1(t)$ which is known only numerically from experiment.

Again, the situation is much more complicated for $G_T^*(t)$. An integral equation can in principle be derived from (4.3.6g) but its establishment, let alone its solution, is a very laborious affair. We must again conclude that approximate transform inversion seems to be the best method indicated in this case.

4.3.2.4 <u>Transverse stressing and straining</u>

Let a fiber reinforced specimen be subjected to the average time dependent stress system

$$[\bar{\sigma}_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{\sigma}_{22}(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad t \geq 0$$

by application of the appropriate homogeneous traction boundary conditions. It then follows from (4.3.9) that

$$\bar{\varepsilon}_{22} = \frac{\bar{\sigma}_{22}}{pE^*_T} \tag{4.3.70}$$

We define the transverse TD compliance pe^*_T by

$$pe^*_T = \frac{1}{pE^*_T} \tag{4.3.71}$$

Insertion of (4.3.71) into (4.3.70) and subsequent inversion yields the creep relation

$$\bar{\varepsilon}_{22}(t) = \int_0^t e^*_T(t-\ ) \frac{d\bar{\sigma}_{22}(\ )}{d\ } d\ \tag{4.3.72}$$

and it is seen that $e^*_T(t)$ is the effective transverse Young's creep compliance.

Next, the specimen is subjected to the average transverse strain $\bar{\varepsilon}_{22}(t)$ and all other $\bar{\sigma}_{ij}(t)$ are kept zero, except for $\bar{\sigma}_{22}(t)$. This is achieved by the mixed boundary conditions

$$\left.\begin{aligned} T_1(S) &= 0 \\ u_2(S) &= \bar{\varepsilon}_{22}(t)\, x_2 \\ T_3(S) &= 0 \end{aligned}\right\} \quad t \geq 0$$

Again (4.3.70) is obtained from (4.3.9) and is now inverted into the relaxation relation

$$\bar{\sigma}_{22}(t) = \int_0^t E_T^*(t-\tau)\,\frac{d\bar{\epsilon}_{22}(\tau)}{d\tau}\,d\tau \qquad (4.3.73)$$

where $E_T^*(t)$ is the effective transverse Young's relaxation modulus.

To obtain results for E_T^* and e_T^* it is necessary to use (4.3.8) which defines the transform of E_T^*, and by virtue of (4.3.71) also the transform of e_T^*, in terms of the transforms of other ERM, which are given for the composite cylinder assemblage by (4.3.6c,d,e) and by (4.3.7). It is seen that the resulting transforms $\hat{E}_T^*$ and $\hat{e}_T^*$ are exceedingly complex expressions. Because of the appearance of $\hat{G}_T^*$ in the results, E_T^* and e_T^* vary significantly with time and so the initial and final value theorems do not yield practical results. Again, it is concluded that the practical method for obtaining E_T^* and e_T^* variations must be based on numerical transform inversion.

If the matrix is nearly incompressible and the fibers are very stiff, (4.3.8a) becomes approximately

$$E_T^* \cong 4G_T^*$$

which inverts into

$$E_T^*(t) \cong 4G_T^*(t) \qquad (4.3.74)$$

This result should, however, be regarded with great caution because of the incompressibility assumption.

APPENDIX

Internal Stresses

The general correspondence principle which relates elastic and viscoelastic solutions, as discussed in par. 4.1.2 , makes it possible to write down Laplace Transforms of viscoelastic stresses if expressions for elastic stress are known. Inversion of the transforms gives the stresses as functions of space coordinates and time.

It should be noted that numerical elastic results cannot be converted into viscoelastic results by means of the correspondence principle. In order to obtain numerical viscoelastic results it is necessary to analyze numerically the viscoelastic boundary value problem of a composite in the space-time domain, which is a very complex undertaking. It appears that no such numerical work has been carried out to date.

Let it be assumed that a viscoelastic FRM is subjected to the boundary condition

$$T_i(S,t) = \sigma^{o}_{ij}\, n_j\, H(t) \tag{1}$$

If the stresses which result from (1) are found and are momentarily denoted $\sigma^{H}_{ij}(\underline{x},t)$, then the stresses in the case where $H(t)$ is replaced by a general function $F(t)$, $t \geq 0$, are given simply by

$$\sigma_{ij}(\underline{x},t) = \int_{o}^{t} \sigma^{H}_{ij}(\underline{x},t-\tau)\, \frac{dF(\tau)}{d\tau}\, d\tau \tag{2}$$

The LT of (1) is given by

$$\hat{T}_i(S,p) = \frac{1}{p}\,\sigma^{o}_{ij}\,n_j \tag{3}$$

this being the boundary condition for the associated "elastic" problem with TD phase moduli $\Gamma_{ijkl}(p) = p\hat{C}_{ijkl}(p)$.

Consider an elastic FRM, of entirely identical phase geometry, which is subjected to the boundary conditions

$$T_i(S) = \sigma^{o}_{ij}\,n_j \tag{4}$$

Let the internal stresses under (4) be denoted

$$^{e}\sigma_{ij}\,(\underline{x}) = f_{ij}\,[{}^{e}\underline{C}^{(m)},\ \{g\}] \tag{5}$$

where ${}^{e}\underline{C}^{(m)}$ is a compact notation for elastic phase moduli and $\{g\}$ denotes the phase geometry. It follows from (3), (5) and the correspondence principle that the stresses in the associated "elastic" problem, thus the LT of the viscoelastic stresses, are given by

$$\hat{\sigma}_{ij}\,(\underline{x},p) = \frac{1}{p}\,f_{ij}\,[p\hat{\underline{C}}^{(m)}(p),\ \{g\}] \tag{6}$$

where $\hat{C}^{(m)}(p)$ is a compact notation for the LT of the phase relaxation moduli and $\{g\}$ indicates the <u>same</u> geometry.

It follows that if expressions (5) for elastic stresses are known, expressions (6) for LT of viscoelastic stresses can be written down at once by replacement of all phase elastic moduli in (5) by TD phase moduli and division by p. The remaining, and major problem is then to invert (6) into the time domain.

Again the initial and final value correspondence principles, which were proved in par. 4.2.2 for effective viscoelastic properties, are very helpful. It follows by similar reasoning that initial and final values of viscoelastic stresses under boundary conditions (1) are given by

$$\sigma_{ij}(\underline{x},o) = {}^{e}\sigma_{ij}[\underline{C}^{(m)}(o), \{g\}]$$

$$\sigma_{ij}(\underline{x},\infty) = {}^{e}\sigma_{ij}[\underline{C}^{(m)}(\infty), \{g\}] \tag{7}$$

which implies that initial/final stresses are simply found by replacement of phase moduli in the elastic stress expressions by initial/final values of phase relaxation moduli.

The general approach outlined here will now be applied to obtain some results for internal stresses in viscoelastic composite cylinder assemblages. It is recalled that results for internal elastic stresses were given in Appendix 2, Chap. 3.5. From the point of view of conversion to viscoelastic results, the simplest case is that of axial shearing.

Let a fiber reinforced cylindrical specimen with viscoelastic matrix and elastic fibers be subjected to the axial shearing boundary condition

$$T_1(S,t) = \sigma^{o}_{12}\, n_2\, H(t)$$

$$T_2(S,t) = \sigma^{o}_{12}\, n_1\, H(t) \tag{8}$$

$$T_3(S,t) = 0$$

in which event the only surviving average stress is σ^{o}_{12} H(t). In this case the LT of the stresses can be obtained from Appendix 2, (21), (26-27) and the present (6) in the following form

$$\hat{\sigma}^{(1)}_{rz}(r,\theta,p) = \sigma^{o}_{12} \frac{\hat{G}_1(p)}{p\hat{G}^{*}_{A}(p)} \frac{\gamma(p)+1+[\gamma(p)-1]a^2/r^2}{\gamma(p)\,v_1+1+v_2} \cos\theta$$

$$\hat{\sigma}^{(1)}_{\theta z}(r,\theta,p) = -\sigma^{o}_{12} \frac{\hat{G}_1(p)}{p\hat{G}^{*}_{A}(p)} \frac{\gamma(p)+1-[\gamma(p)-1]a^2/r^2}{\gamma(p)v_1+1+v_2} \sin\theta$$

$$\hat{\sigma}^{(2)}_{rz}(r,\theta,p) = 2\sigma^{o}_{12} \frac{G_2}{p^2\hat{G}^{*}_{A}(p)} \frac{\cos\theta}{\gamma(p)v_1+1+v_2} \tag{9}$$

$$\hat{\sigma}^{(2)}_{\theta z}(r,\theta,p) = -2\sigma^{o}_{12} \frac{G_2}{p^2\hat{G}^{*}_{A}(p)} \frac{\sin\theta}{\gamma(p)v_1+1+v_2}$$

$$\gamma(p) = \frac{G_2}{p\hat{G}_1(p)}$$

where $\hat{G}^{*}_{A}(p)$ is given by (4.3.6f).

If the matrix viscoelastic behavior is idealized by a differential operator stress-strain relation, then (9) can be inverted in elementary fashion. The necessary calculations become heavy unless the matrix is represented by very simple viscoelastic models.

Application of the initial and final value correspondence principles to the elastic axial shearing results easily yields the following expressions

$$\sigma^{(1)}_{rz}(r,\theta,o) = \sigma^{o}_{12}\,\frac{\gamma(o)+1+[\gamma(o)-1]\,a^2/r^2}{\gamma(o)(1+v_2)+v_1}\cos\theta$$

$$\sigma^{(1)}_{\theta z}(r,\theta,o) = \sigma^{o}_{12}\,\frac{\gamma(o)+1-[\gamma(o)-1]\,a^2/r^2}{\gamma(o)(1+v_2)+v_1}\sin\theta$$

$$\sigma^{(2)}_{rz}(r,\theta,o) = 2\sigma^{o}_{12}\,\frac{\gamma(o)}{\gamma(o)(1+v_2)+v_1}\cos\theta \qquad (10)$$

$$\sigma^{(2)}_{\theta z}(r,\theta,o) = -2\sigma^{o}_{12}\,\frac{\gamma(o)}{\gamma(o)(1+v_2)+v_1}\sin\theta$$

$$\gamma(o) = \frac{G_2}{G_1(o)}$$

$$\sigma_{rz}^{(1)}(r,\theta,\infty) = \sigma_{12}^{o}\,\frac{1+a^2/r^2}{1+v_2}\cos\theta$$

$$\sigma_{\theta z}^{(1)}(r,\theta,\infty) = -\,\sigma_{12}^{o}\,\frac{1-a^2/r^2}{1+v_2}\sin\theta$$

$$\sigma_{rz}^{(2)}(r,\theta,\infty) = \frac{2\sigma_{12}^{o}\cos\theta}{1+v_2} \tag{11}$$

$$\sigma_{\theta z}^{(2)}(r,\theta,\infty) = -\,\frac{2\sigma_{12}^{o}\sin\theta}{1+v_2}$$

It has been assumed in (11) that $G_1(\infty)$ is infinitely smaller than G_2.

The maximum matrix shear stress is obtained at $r = a$, $\theta = o$. We have from (10-11)

$$\frac{\tau_{max}(\infty)}{\tau_{max}(o)} = 1 + \frac{1-v_2}{1+v_2}\,\frac{G_1(o)}{G_2} \tag{12}$$

which shows that $\tau_{max}(\infty)$ is larger than $\tau_{max}(o)$.

If it is assumed that the fibers are rigid, it is found that the matrix stresses are given by (11) for the whole time range $t \geq o$. This easily shown by first specializing the elastic stresses to the rigid fiber case by setting $\gamma \to \infty$, and then applying the correspondence principle.

It is also not difficult to show that for <u>any</u> FRM with rigid fibers, internal stresses under boundary condition (8) are the same for elastic and viscoelastic matrix.

A similar conclusion can be established for transverse shearing, but only if the matrix is also incompressible.

Next we consider the case of uniaxial stressing. The appropriate boundary conditions are

$$T_1(S,t) = \sigma^o_{11}\, n_1\, H(t)$$
$$T_2(S,t) = T_3(S,t) = 0 \tag{13}$$

Important elastic stresses are given by (14-18), Appendix 2, Chap. 3.5. The initial and final values of the axial stresses in fiber and matrix are found to be with very accurate approximation.

$$\sigma^{(1)}_{zz}(o) \cong \sigma^o_{11} \frac{E_1(o)}{E^*_A(o)}$$
$$\sigma^{(2)}_{zz}(o) \cong \sigma^o_{11} \frac{E_2}{E^*_A(o)} \tag{14}$$

$$\sigma^{(1)}_{zz}(\infty) \cong 0$$
$$\sigma^{(2)}_{zz}(\infty) \cong \frac{\sigma^o_{11}}{v_2} \tag{15}$$

where $E_A^*(o)$ in (14) is given by (4.3.29a) (the third term of which may be disregarded) and (4.3.11) and (4.3.29b) have been used to obtain (15).

It is seen from (15) that after long time the fibers are taking the entire axial load.

The methods described here may evidently be applied to obtain the viscoelastic counterparts of other elastic stresses in FRM such as those produced by plane-isotropic loading. There is also no difficulty to treat displacement boundary conditions (relaxation) and to find viscoelastic deformations.

4.4 DYNAMIC LINEAR VISCOELASTICITY

4.4.1 General Dynamic Problem

We shall now consider linear viscoelastic bodies, as defined in par. 4.1.1, under dynamic conditions, in which case the inertia terms have to be incorporated into the equilibrium equations. In this event we have instead of (4.1.45), the equations of motion

$$\sigma_{ij,j}(\underline{x},t) = \rho \ddot{u}_i(\underline{x},t) \tag{4.4.1}$$

where ρ is the density which is here assumed constant and the dots above u_i denote partial time derivative.

To obtain differential equations for the displacements it is customary to substitute (4.1.2a) into (4.4.1), Then are obtained the equations

$$\int_{-\infty}^{t} [C_{ijkl}(t-\tau) \frac{\partial}{\partial \tau} u_{k,l}(\underline{x},t)]_{,j} d\tau = \rho \ddot{u}_i(\underline{x},t) \tag{4.4.2}$$

The use of (4.1.2a) in the present dynamic case can be interpreted in two ways. Firstly, it may be assumed that the C_{ijkl} in (4.4.2) are the relaxation moduli obtained from a quasi-static experiment. Secondly, it may be assumed that this is not the case, but that the Boltzmann superposition principle is valid also for the present dynamic situation. Then (4.1.2) are dynamic constitutive relations in which C_{ijkl} have to be obtained from some dynamic experiment.

Appropriate boundary conditions are again

$$u_i(S,t) = u^o_i \quad \text{on } S_u \qquad \text{(a)}$$

$$T_i(S,t) = T^o_i \quad \text{on } S_T \qquad \text{(b)} \qquad (4.4.3)$$

These have to be supplemented by the initial conditions

$$u_i(\underline{x},o) = f_i(\underline{x}) \qquad \text{(a)}$$

$$\dot{u}_i(\underline{x},o) = g_i(\underline{x}) \qquad \text{(b)} \qquad (4.4.4)$$

Equs. (4.4.2-4) mathematically formulate a general class of problems in dynamic viscoelasticity.

Extension to heterogeneous bodies is obvious. Equs. (4.4.2) apply separately for the different phase regions with appropriate phase C_{ijkl} and there are added to (4.4.3-4) the interface conditions (4.1.49).

We shall not be concerned with dynamic problems of such generality. It is our purpose to consider the case of steady state vibrations. This is done in the next paragraph.

4.4.2 Viscoelastic Vibrations: Complex Moduli

Let the time variation of strain at a typical point be sinusoidal. Rather than consider the cases

$$\epsilon_{ij}(\underline{x},t) = \begin{cases} \tilde{\epsilon}_{ij}(\underline{x}) \sin \omega t^{(*)} \\ \tilde{\epsilon}_{ij}(\underline{x}) \cos \omega t \end{cases}$$

separately, we use the more convenient complex representation

$$\epsilon_{ij}(\underline{x},t) = \tilde{\epsilon}_{ij}(\underline{x})\, e^{\iota \omega t} \tag{4.4.5}$$

where $\iota = \sqrt{-1}$, ω is the frequency and $\tilde{\epsilon}_{ij}$ may be real or complex. In the former case $\tilde{\epsilon}_{ij}$ is the amplitude. Suppose that $\tilde{\epsilon}_{ij}$ is real. If (4.4.5) is inserted into the usual elastic stress-strain relation we find

$$\sigma_{ij}(\underline{x},t) = \tilde{\sigma}_{ij}(\underline{x})\, e^{\iota \omega t} \qquad \text{(a)}$$

$$\tilde{\sigma}_{ij}(\underline{x}) = {}^{e}C_{ijkl}\, \tilde{\epsilon}_{kl}(\underline{x}) \qquad \text{(b)} \tag{4.4.6}$$

It is seen that stress and strain vibrate in phase and that their amplitudes $\tilde{\sigma}_{ij}$ and $\tilde{\epsilon}_{ij}$ are related by the elastic stress-strain law.

We now adopt a similar procedure for viscoelastic materials and insert (4.4.5) into (4.1.2a). This yields

$$\sigma_{ij}(t) = \tilde{\epsilon}_{kl}\, \iota\omega \int_{-\infty}^{t} C_{ijkl}(t-\tau)\, e^{\iota \omega t}\, d\tau \tag{4.4.7}$$

(*)The ~ sign above a quantity, as will be used here and henceforward, should not be confused with a similar notation in chaps. 3.6, 3.7 where it denoted admissible fields.

where the $\underline{x}$ argument has been suppressed for convenience. The change of variable $y = t - \tau$ in (4.4.7) gives the result

$$\sigma_{ij}(t) = \tilde{\epsilon}_{kl} \; e^{\iota\omega t} \; \iota\omega \int_0^\infty C_{ijkl}(y) \, e^{-\iota\omega y} \, dy \tag{4.4.8}$$

We define

$$D_{ijkl}(\iota\omega) = \iota\omega \int_0^\infty C_{ijkl}(y) \, e^{-\iota\omega y} \, dy \tag{4.4.9}$$

It is seen that (4.4.9) is a one sided complex Fourier transform. There may arise problems with respect to the existence of the integral (4.4.9) which shall not be discussed here. The reader is referred to [4.1].

In view of (4.4.5) and (4.4.9), (4.4.8) may be rewritten in the form

$$\sigma_{ij}(t) = \tilde{\sigma}_{ij} \, e^{\iota\omega t} \qquad \text{(a)}$$

$$\tilde{\sigma}_{ij} = D_{ijkl} \, \tilde{\epsilon}_{kl} \qquad \text{(b)} \tag{4.4.10}$$

It is seen that (4.4.10b) has the appearance of an elastic stress-strain law. Therefore D_{ijkl} are called the <u>complex moduli</u> of the viscoelastic material. They may also be written in the form

$$D_{ijkl}(\iota\omega) = D^R_{ijkl}(\omega) + \iota D^I_{ijkl}(\omega) \tag{4.4.11}$$

where D^R_{ijkl} and D^I_{ijkl} are the real and imaginary parts, respectively. It should be noted that $\tilde{\sigma}_{ij}$ is complex, even when $\tilde{\epsilon}_{ij}$ is real, and is thus not the amplitude of σ_{ij}.

The physical meaning of (4.4.10) is best obtained by insertion of (4.4.11) into (4.4.10b) and computation of stress from (4.4.10a). We then have for $\tilde{\epsilon}_{ij}$ real

$$\sigma_{ij}(t) = (D^R_{ijkl} \cos \omega t - D^I_{ijkl} \sin \omega t)\, \tilde{\epsilon}_{kl} +$$

$$+ \; \iota\, (D^R_{ijkl} \sin \omega t + D^I_{ijkl} \cos \omega t)\, \tilde{\epsilon}_{kl} \qquad (4.4.12)$$

It is easily seen that the real part of (4.4.12) is the response to a strain vibration $\tilde{\epsilon}_{kl} \cos \omega t$ while the imaginary part is the response to a strain vibration $\tilde{\epsilon}_{kl} \sin \omega t$. Unlike elastic vibrations, (4.4.6), the stress and strain vibrations in the present viscoelastic case are not in phase.

We can define complex compliances in completely analogous fashion. We assume a stress vibration

$$\sigma_{ij}(t) = \tilde{\sigma}_{ij}\, e^{\iota \omega t} \qquad (4.4.13)$$

and we insert (4.4.13) into (4.1.2b). We then obtain

$$\epsilon_{ij}(t) = \tilde{\epsilon}_{ij}\, e^{\iota \omega t} \qquad \text{(a)}$$

$$\tilde{\epsilon}_{ij} = R_{ijkl}\, \tilde{\sigma}_{kl} \qquad \text{(b)} \qquad (4.4.14)$$

$$R_{ijkl}(\iota \omega) = \iota \omega \int_{-\infty}^{\infty} S_{ijkl}(y)\, e^{-\iota \omega y}\, dy \qquad \text{(c)}$$

where S_{ijkl} are the creep compliances. The tensor components R_{ijkl} are called <u>complex compliances</u>.

If it is assumed that $C_{ijkl}(t)$ and $S_{ijkl}(t)$ are ij, kl symmetric then the same follows, because of (4.4.9) and (4.4.14c), for the complex moduli D_{ijkl} and the complex compliances R_{ijkl}. Evidently, because of $\tilde{\sigma}_{ij}$ and $\tilde{\epsilon}_{ij}$ symmetry, D_{ijkl} and R_{ijkl} are also i,j and k,l symmetric. In summary

$$D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij} \qquad \text{(a)}$$

$$R_{ijkl} = R_{jikl} = R_{ijlk} = R_{klij} \qquad \text{(b)} \qquad (4.4.15)$$

Furthermore, since (4.4.10b) and (4.4.14b) are one and the same equation, we conclude that the complex moduli and complex compliances tensors are reciprocal. Thus

$$D_{ijrs} R_{rskl} = I_{ijkl} \qquad (4.4.16)$$

Next, we note an important connection between Laplace transforms of relaxation moduli (creep compliances) and complex moduli (complex compliances). Recall the definition (4.1.31) of TD moduli which may be written

$$\Gamma_{ijkl}(p) = p \int_0^\infty C_{ijkl}(t)\, e^{-pt}\, dt$$

Comparison with (4.4.9) shows that

$$\Gamma_{ijkl}(\iota\omega) = D_{ijkl}(\iota\omega) \qquad \text{(a)}$$

$$D_{ijkl}(p) = \Gamma_{ijkl}(p) \qquad \text{(b)} \qquad (4.4.17)$$

which implies that if $\Gamma_{ijkl}(p)$ is known, replacement of p by $\iota\omega$ yields the complex moduli, and if $D_{ijkl}(\iota\omega)$ is known replacement of $\iota\omega$ by p yields the TD moduli. Similar relations may be obtained between TD compliances and complex compliances.

We have so far been concerned only with generally anisotropic bodies and we shall now consider the case of material symmetry. It is seen that the complex stress-strain law (4.4.10b) has the same form as an elastic stress-strain law. Thus, symmetry reductions for elastic stress-strain relations are immediately applicable in the present case.

For an isotropic material (4.4.10b) assumes the form

$$\tilde{\sigma}_{ij}(\iota\omega) = \tilde{\lambda}(\iota\omega)\,\tilde{\varepsilon}_{kk}\,\delta_{ij} + 2\tilde{G}(\iota\omega)\,\tilde{\varepsilon}_{ij} \qquad (4.4.18)$$

or equivalently

$$\tilde{\sigma} = 3\tilde{K}(\iota\omega)\,\tilde{\varepsilon} \qquad \text{(a)}$$

$$\tilde{s}_{ij} = 2\tilde{G}(\iota\omega)\,\tilde{e}_{ij} \qquad \text{(b)} \qquad (4.4.19)$$

where $\tilde{K}$ and $\tilde{G}$ are the complex bulk and shear moduli, respectively; $\tilde{\sigma}$ and $\tilde{s}_{ij}$ are the isotropic and deviatoric parts, respectively, of $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}$ and $\tilde{e}_{ij}$ are the isotropic and deviatoric parts, respectively, of $\tilde{\varepsilon}_{ij}$.

Other complex moduli are defined just as in elasticity and all the relations between elastic moduli as listed in (3.4.88-90) carry over unchanged to complex moduli. Similarly, the complex stress-strain relations for a transversely isotropic material have the forms (3.4.86-87) with appropriate complex moduli.

Relations of type (4.4.17) between TD moduli and complex moduli can also be written down at once. We have for the isotropic case, for example

$$\tilde{K}(\iota\omega) = \kappa(\iota\omega) \qquad \text{(a)}$$

$$\tilde{G}(\iota\omega) = \Gamma(\iota\omega) \qquad \text{(b)} \qquad (4.4.20)$$

where the right sides of (4.4.20) are defined by the p functions (4.1.34).

For viscoelastic stress-strain relations in differential operator form, (4.1.21), we have from (4.1.42) and (4.4.20)

$$\tilde{K}(\iota\omega) = \frac{1}{3}\,\frac{S(\iota\omega)}{R(\iota\omega)} \qquad \text{(a)}$$

$$\tilde{G}(\iota\omega) = \frac{1}{2}\,\frac{P(\iota\omega)}{Q(\iota\omega)} \qquad \text{(b)} \qquad (4.4.21)$$

As an example consider the case of the simple Maxwell model. We have from (4.4.21b) and (4.1.43a)

$$\tilde{G}_M(\iota\omega) = \frac{\iota\omega\eta}{1+\iota\omega T} \qquad (4.4.22)$$

Complex moduli are often written in the form (4.4.11). The complex shear modulus, for example, may be written

$$\tilde{G}(\iota\omega) = G^R(\omega) + \iota G^I(\omega) \tag{4.4.23}$$

Another frequently used form is

$$\tilde{G}(\iota\omega) = |G| \; e^{\iota\delta} \qquad \text{(a)}$$

$$|G| = \sqrt{(G^R)^2 + (G^I)^2} \qquad \text{(b)} \qquad (4.4.24)$$

$$\tan\delta = \frac{G^I}{G^R} \qquad \text{(c)}$$

The angle δ is called the loss angle and $\tan\delta$ is called the loss tangent.

Finally we note by comparison of (4.4.6b) and (4.4.10b) that for an elastic material the complex moduli are just the elastic moduli. In that case the real part of the complex modulus is the frequency independent elastic modulus and the imaginary part vanishes as also does the loss angle δ.

4.4.3 Viscoelastic Vibrations: Boundary Value Problems and Correspondence Principle

We consider a viscoelastic body which is subjected to displacement or traction boundary conditions which vary sinusoidally in time. A general form of such boundary conditions is

$$u_i(S,t) = \tilde{u}^o_i(S)\, e^{\iota\omega t} \qquad \text{(a)}$$

$$T_i(S,t) = \tilde{T}^o_i(S)\, e^{\iota\omega t} \qquad \text{(b)} \qquad (4.4.25)$$

where $\tilde{u}^o_i(S)$ and $\tilde{T}^o_i(S)$ are time independent. By taking real or imaginary parts of (4.4.25) we obtain the cases of cosine and sine vibrations, respectively. It is seen that the class of problems considered is that of forced vibrations of viscoelastic bodies. To obtain a mathematical formulation it is assumed, and later justified, that the internal displacements have the form

$$u_i(\underline{x},t) = \tilde{u}_i(\underline{x})\, e^{\iota\omega t} \qquad (4.4.26)$$

It follows that the strains have the form (4.4.5) where

$$\tilde{\epsilon}_{ij} = \frac{1}{2}\left(\tilde{u}_{i,j} + \tilde{u}_{j,i}\right) \qquad (4.4.27)$$

and that the stresses are given by (4.4.10).

To obtain differential field equations, the stresses (4.4.10a) are expressed in terms of (4.4.10b) and (4.4.27) and are then substituted into

the left side of the equation of motion (4.4.1), while (4.4.26) is substituted into the right side. Cancelling the common factor $e^{\iota\omega t}$ we obtain

$$D_{ijkl}(\iota\omega)\tilde{u}_{k,lj} + \rho\omega^2 \tilde{u}_i = 0 \tag{4.4.28}$$

From (4.4.26) and (4.4.25) we obtain

$$\begin{aligned} &\tilde{u}_i(S) = \tilde{u}_i^o && \text{(a)} \\ &\tilde{T}_i(S) = \tilde{T}_i^o && \text{(b)} \end{aligned} \tag{4.4.29}$$

where

$$\tilde{T}_i(S) = D_{ijkl}(\iota\omega)\tilde{u}_{k,l}\, n_j \tag{4.4.30}$$

This completes the mathematical formulation of the viscoelastic vibration problem. The space dependent parts $\tilde{u}_i$ of the displacements are uniquely determined by (4.4.28-29) which fact justifies the assumption (4.4.26). They are in general complex since D_{ijkl} is complex. The space dependent parts of the stresses follow from (4.4.27) and (4.4.10b) and actual complex displacement and stresses are obtained by multiplication of space dependent parts by $e^{\iota\omega t}$. The real parts of displacements and stresses obtained give the solution for a real boundary cosine input while the imaginary parts give the solution for a real boundary sine input.

Note that in specialization of the general dynamic viscoelastic problem of par. 4.4.1 to the present case of vibrations the initial conditions (4.4.4)

have been disregarded. The reason for this is that the form (4.4.26) of the displacements already specifies $f_i(\underline{x})$ and $g_i(\underline{x})$ in (4.4.4).

Evidently, the boundary value problem (4.4.28-29) includes as a special case the problem of vibrations of an elastic body. In that case the complex moduli in (4.4.28) have merely to be replaced by the elastic moduli ${}^{e}C_{ijkl}$, since as has been pointed out in par. 4.4.2 complex moduli reduce to elastic moduli for elastic materials. Consequently, the problems of elastic and viscoelastic vibrations are mathematically analogous for the complex formulation of the latter. This analogy forms the basis of a <u>correspondence principle for viscoelastic vibrations</u> which may be summarized as follows: Let the solution of an elastic vibration problem be ${}^{e}\tilde{u}_i(\underline{x})\, e^{\iota\omega t}$, ${}^{e}\tilde{\sigma}_{ij}(\underline{x})\, e^{\iota\omega t}$. The complex solution of the geometrically identical viscoelastic problem is then $\tilde{u}_i(\underline{x})\, e^{\iota\omega t}$, $\tilde{\sigma}_{ij}(\underline{x})\, e^{\iota\omega t}$ where $\tilde{u}_i(\underline{x})$, $\tilde{\sigma}_{ij}(\underline{x})$ are obtained by replacing in ${}^{e}\tilde{u}_i(\underline{x})$, ${}^{e}\sigma_{ij}(\underline{x})$ the elastic moduli by the viscoelastic complex moduli. (compare e.g. [4.15])

4.5 EFFECTIVE COMPLEX MODULI(*)

4.5.1 Definition and Correspondence Principle

In attempting to formulate a theory of dynamic behavior, in general, or of vibrations, in particular, of viscoelastic composites, we encounter the same kind of difficulties which were discussed in chap. 3.9 with respect to elastic composites. We therefore confine ourselves to the first approximation whose fundamental assumption is that macro-stress and macro-strain, i.e. local averages over RVE, are related by a classical type stress-strain relation, in terms of the usual effective physical constants. If this approximation is adopted it is possible to establish a set of differential equations for the space dependent parts of the macro-displacement (local averages) in two phase materials, which are similar to equs. (4.4.28). These equations contain as coefficients the effective complex moduli to be discussed here and also effective densities which are not the average densities and are moreover complex numbers (to be published).

In the elasto-static or elasto-dynamic case the relation between macro-stress and macro-strain is expressed by equ. (3.9.8). Accordingly, in the present case such a relation will have the form

$$\tilde{\bar{\sigma}}_{ij}(\underline{x}) = D^{*}_{ijkl}(\iota\omega)\,\tilde{\bar{\epsilon}}_{kl}(\underline{x}) \tag{4.5.1}$$

where $\tilde{\bar{\sigma}}_{ij}$ and $\tilde{\bar{\epsilon}}_{ij}$ are defined by

(*) The theory and results given in this chapter are based on refs. [4.16-18].

$$\bar{\sigma}_{ij}(\underline{x},t) = \tilde{\bar{\sigma}}_{ij}(\underline{x})e^{\iota\omega t} \qquad \text{(a)}$$

$$\bar{\epsilon}_{ij}(\underline{x},t) = \tilde{\bar{\epsilon}}_{ij}(\underline{x})\, e^{\iota\omega t} \qquad \text{(b)} \qquad (4.5.2)$$

Here $\bar{\sigma}_{ij}$ and $\bar{\epsilon}_{ij}$ are macro-stresses and macro-strains which are space and time dependent and D^*_{ijkl} are defined as the <u>effective complex moduli</u> (ECM) of the viscoelastic material.

To obtain information about D^*_{ijkl} we proceed as in par. 4.4.2 in the derivation of (4.4.9). We insert (4.6.2b) into the effective visco-elastic stress-strain relation (4.2.5), with an hereditary integral whose lower limit of integration is $-\infty$. We then obtain exactly as in par. 4.4.2

$$D^*_{ijkl}(\iota\omega) = \iota\omega \int_0^\infty C^*_{ijkl}(y)\, e^{-\iota\omega y}\, dy \qquad (4.5.3)$$

which relates the effective complex moduli to the effective relaxation moduli.

In completely similar fashion we can obtain the form of effective complex compliances R^*_{ijkl}. Thus

$$R^*_{ijkl}(\iota\omega) = \iota\omega \int_0^\infty S^*_{ijkl}(y)\, e^{-\iota\omega y}\, dy \qquad (4.5.4)$$

It follows from (4.2.10-11) and (4.5.3-4) that

$$\boxed{\begin{aligned} D^*_{ijkl} &= D^*_{jikl} = D^*_{ijlk} = D^*_{klij} \qquad \text{(a)} \\ R^*_{ijkl} &= R^*_{jikl} = R^*_{ijlk} = R^*_{klij} \qquad \text{(b)} \end{aligned}} \qquad (4.5.5)$$

We also have the reciprocity relation

$$\boxed{D^*_{ijrs}\, R^*_{rskl} = I_{ijkl}} \tag{4.5.6}$$

which is proved just as (4.4.16).

We now recall the definition of TD effective moduli (4.2.18) which is here rewritten in the form

$$\Gamma^*_{ijkl}(p) = p\int_0^\infty C^*_{ijkl}(t)\, e^{-pt}\, dt \tag{4.5.7}$$

Comparison of (4.5.7) with (4.5.3) shows that Γ^*_{ijkl} depends functionally on p just as D^*_{ijkl} depends functionally on ω. We, therefore, conclude that

$$\boxed{\begin{aligned} &D^*_{ijkl}(\iota\omega) = \Gamma^*_{ijkl}(\iota\omega) && \text{(a)} \\ &\Gamma^*_{ijkl}(p) = D^*_{ijkl}(p) && \text{(b)} \end{aligned}} \tag{4.5.8}$$

This implies that if $\Gamma^*_{ijkl}(p)$ are known, $D^*_{ijkl}(\iota\omega)$ is obtained by replacement of p by $\iota\omega$. Conversely, if $D^*_{ijkl}(\iota\omega)$ is known, $\Gamma^*_{ijkl}(p)$ may be obtained by replacement of $\iota\omega$ by p.

Next we recall the static correspondence principle for viscoelastic composites, equ. (4.2.21). It follows from this principle and from (4.5.8) that

$$D^*_{ijkl}(\iota\omega) = {}^{e}C^*_{ijkl}[\underline{\Gamma}^{(m)}(\iota\omega), \{g\}] \tag{4.5.9}$$

where $\underline{\Gamma}^{(m)}(\iota\omega)$ is a symbolic notation for the left side of (4.4.17a), associated with the phases and {g} denotes phase geometry. But by (4.4.17a) $\Gamma^{(m)}_{ijkl}(\iota\omega)$ are the phase complex moduli $D^{(m)}_{ijkl}(\iota\omega)$. Consequently, (4.5.9) may be rewritten as

$$\boxed{D^{*}_{ijkl}(\iota\omega) = {}^{e}C^{*}_{ijkl}[\underline{D}^{(m)}(\iota\omega), \{g\}]} \tag{4.5.10}$$

Equ. (4.5.10) expresses the <u>correspondence principle for vibrations of viscoelastic composites</u>. In words: <u>The effective complex moduli of a viscoelastic composite are obtained by replacement of phase elastic moduli by phase complex moduli in the expressions for the effective elastic moduli of a composite with identical phase geometry</u>.

In the event that a certain phase is elastic, its phase elastic moduli are left unchanged in the ${}^{e}C^{*}_{ijkl}$ expressions, since the complex moduli of an elastic material are its elastic moduli.

Note that the present correspondence principle is much easier to use than the one for static viscoelasticity of composites, since the complex phase moduli are directly obtainable from experiment. Thus, expressions for effective complex moduli are obtained in a rather simple fashion in terms of experimentally measured quantities. It is recalled, in contrast, that in the static case the correspondence principle leads only to Laplace transforms of effective viscoelastic properties, whose inversion may lead to considerable difficulty.

On the other hand, it should be noted that the present concept of effective complex moduli is based on a physical approximation, which was called the first approximation . This limits the applicability of the present theory to some unknown frequency range.

It should be further noted that the existence of the present powerful correspondence principle was to be expected on the basis of the first approximation and the correspondence principle for viscoelastic vibrations as formulated in par. 4.4.3. The first approximation for vibrations of elastic composites states that a macro-theory in terms of local averages may be approximately formulated in the form of a classical theory with the usual effective elastic moduli. In view of the correspondence principle for viscoelastic vibrations it is not surprising that the effective complex moduli which enter into the first approximation theory for vibrations of viscoelastic composites depend upon phase complex moduli, just as effective elastic moduli depend upon phase elastic moduli.

For purposes of illustration and subsequent usage, we specialize the principle (4.5.10) to the case of a composite which consists of two isotropic phases, but is otherwise macroscopically anisotropic (e.g. fiber-reinforced). Let the EEM of a composite with identical phase geometry be written

$$^{e}C^{*}_{ijkl} = {}^{e}C^{*}_{ijkl}\,({}^{e}K_1,\ {}^{e}G_1,\ {}^{e}K_2,\ {}^{e}G_2,\ \{g\})$$

Then the effective complex moduli are

$$D^{*}_{ijkl}(\iota\omega) = {}^{e}C^{*}_{ijkl}\,[\tilde{K}_1\ (\iota\omega),\ \tilde{G}_1(\iota\omega),\ \tilde{K}_2(\iota\omega), \tilde{G}_2(\iota\omega), \{g\}) \qquad (4.5.11)$$

If the phases are assumed elastic in dilatation, then $\tilde{K}_1(\iota\omega)$ and $\tilde{K}_2(\iota\omega)$ become the real and frequency independent elastic K_1 and K_2.

In FRM the fibers are usually elastic and the viscoelastic effect is thus confined to the matrix. For reasons of simplification it shall be mostly assumed that the matrix is elastic in dilatation. Then (4.5.12) assumes the form

$$\boxed{D^{*}_{ijkl}(\iota\omega) = {}^{e}C^{*}_{ijkl}\,[K_1,\ G_1\ (\iota\omega),\ K_2,\ G_2\ ]} \qquad (4.5.12)$$

4.5.2 Effective Complex Moduli of Fiber Reinforced Materials

The correspondence principle for effective complex moduli (ECM) which was derived in par. 4.5.1 enables us to write down at once expressions for ECM of FRM in all cases where analytical expressions for EEM are known. It is not possible to thus exploit the correspondence principle for numerical results for EEM.

The viscoelastic matrix properties may be characterized by the complex bulk and shear moduli and associated loss angles. Thus

$$\tilde{K}_1(\iota\omega) = K_1^R(\omega) + \iota K_1^I(\omega) \qquad \text{(a)}$$

$$\tan\delta_K = \frac{K_1^I}{K_1^R} \qquad \text{(b)} \qquad (4.5.13)$$

$$\tilde{G}_1(\iota\omega) = G_1^R(\omega) + \iota G_1^I(\omega) \qquad \text{(a)}$$

$$(4.5.14)$$

$$\tan\delta_G = \frac{G_1^I}{G_1^R} \qquad \text{(b)}$$

Other matrix complex moduli are related to (4.5.13a) and (4.5.14b) by elastic type relations. Thus

$$\tilde{E}_1(\iota\omega) = \frac{9\tilde{K}_1(\iota\omega)\,\tilde{G}_1(\iota\omega)}{3\tilde{K}_1(\iota\omega) + \tilde{G}_1(\iota\omega)} \qquad \text{(a)}$$

$$\tilde{\nu}_1(\iota\omega) = \frac{3\tilde{K}_1(\iota\omega) - 2\tilde{G}_1(\iota\omega)}{2[3\tilde{K}_1(\iota\omega) + \tilde{G}_1(\iota\omega)} \qquad \text{(b)} \qquad (4.5.15)$$

$$\tilde{k}_1(\iota\omega) = \tilde{K}_1(\iota\omega) + \frac{1}{3}\tilde{G}_1(\iota\omega) \qquad \text{(c)}$$

According to the correspondence principle of par. 4.5.1 any ECM is obtained from a corresponding EEM expression by replacement of the matrix elastic moduli by the appropriate matrix complex moduli, while the fiber elastic moduli are left unchanged. The resulting expression for the ECM has then to be separated into real and imaginary parts. The ratio between the latter and the former defines the effective loss angle.

We may thus exploit the expressions for EEM of the composite cylinder assemblage model, par. 3.5.3, to write down corresponding ECM . Results for the more important ECM, for elastic fibers and visco-elastic matrix are listed below:

$$\tilde{k}^*(\iota\omega) = \frac{\tilde{k}_1(\iota\omega)[k_2+\tilde{G}_1(\iota\omega)]\, v_1 + k_2[\tilde{k}_1(\iota\omega) + \tilde{G}_1(\iota\omega)]v_2}{[k_2+\tilde{G}_1(\iota\omega)]v_1 + [\tilde{k}_1(\iota\omega) + \tilde{G}_1(\iota\omega)]v_2} \tag{4.5.16}$$

$$\tilde{E}_A^*(\iota\omega) = \tilde{E}_1(\iota\omega)\, v_1 + E_2\, v_2 + \frac{4[\nu_2 - \tilde{\nu}_1(\iota\omega)]^2\, v_1\, v_2}{v_1/k_2 + v_2/\tilde{k}_1(\iota\omega) + 1/\tilde{G}_1(\iota\omega)} \tag{4.5.17}$$

$$\tilde{\nu}_A^*(\iota\omega) = \tilde{\nu}_1(\iota\omega)\, v_1 + \nu_2\, v_2 + \frac{[\nu_2 - \tilde{\nu}_1(\iota\omega)][1/\tilde{k}_1(\iota\omega) - 1/k_2]\, v_1\, v_2}{v_1/k_2 + v_2/\tilde{k}_1(\iota\omega) + 1/\tilde{G}_1(\iota\omega)} \tag{4.5.18}$$

$$\tilde{G}_A^*(\iota\omega) = \tilde{G}_1(\iota\omega)\, \frac{\tilde{G}_1(\iota\omega)\, v_1 + G_2(1+v_2)}{\tilde{G}_1(\iota\omega)\,(1+v_2) + G_2\, v_1} \tag{4.5.19}$$

$$\tilde{G}_T^*(\iota\omega) = \tilde{G}_1(\iota\omega)\, \frac{[1+\tilde{\alpha}(\iota\omega)\, v_2^{\,3}][\tilde{\rho}(\iota\omega) + \tilde{\beta}_1(\iota\omega)\, v_2] - 3 v_2 v_1^{\,2} \tilde{\beta}_1^{\,2}(\iota\omega)}{[1+\tilde{\alpha}(\iota\omega)\, v_2^{\,3}]\,[\tilde{\rho}(\iota\omega) - v_2] - 3 v_2 v_1^{\,2} \tilde{\beta}_1^{\,2}(\iota\omega)} \tag{a}$$

where

$$\tilde{\alpha}(\iota\omega) = \frac{\tilde{\beta}_1(\iota\omega) - \tilde{\gamma}(\iota\omega)\, \beta_2}{1 + \tilde{\gamma}(\iota\omega)\, \beta_2} \qquad \tilde{\rho}(\iota\omega) = \frac{\tilde{\gamma}(\iota\omega) + \tilde{\beta}_1(\iota\omega)}{\tilde{\gamma}(\iota\omega) - 1} \tag{b}$$

(4.5.20)

$$\tilde{\beta}(\iota\omega) = \frac{1}{3 - 4\,\tilde{\nu}_1(\iota\omega)} \qquad \beta_2 = \frac{1}{3 - 4\nu_2} \tag{c}$$

$$\tilde{\gamma}(\iota\omega) = \frac{G_2}{\tilde{G}_1(\iota\omega)} \tag{d}$$

Again (3.5.113) has been taken as an ad-hoc expression for ${}^{e}G_T^*$ of the composite cylinder assemblage, thus yielding (4.5.20) as an ad-hoc $\tilde{G}_T^*$. In this respect it is noted that Christensen [4.19] has given a method by which effective shear modulus bounds for the composite spheres assemblage model, [3.11], can be transformed into bounds for the effective complex shear modulus real and imaginary parts of the same model. No doubt, the method can also be applied to derive similar bounds for the composite cylinder assemblage.(*)

In view of (3.4.83-84), the transverse complex Young's modulus $\tilde{E}_T^*(\iota\omega)$ is given by

$$\boxed{\begin{aligned} \tilde{E}_T^*(\iota\omega) &= \frac{4\tilde{k}^*(\iota\omega)\ \tilde{G}_T^*(\iota\omega)}{\tilde{k}^*(\iota\omega)+\tilde{m}(\iota\omega)\tilde{G}_T^*(\iota\omega)} && \text{(a)} \\ \tilde{m}(\iota\omega) &= 1+\frac{4\tilde{k}^*(\iota\omega)[\tilde{\nu}_A^*(\iota\omega)]^2}{\tilde{E}_A^*(\iota\omega)} && \text{(b)} \end{aligned}} \qquad (4.5.21)$$

The matrix complex moduli appearing in (4.5.16-21) are given by (4.5.13-15). Each of these has to be written in the complex number forms (4.5.13a-14a). The resulting ECM expressions may then be separated into real and imaginary parts. While this presents no fundamental difficulty, the required algebra is very heavy in many cases.

(*) Simple general ECM bounds, for isotropic composites of arbitrary phase geometry, have been obtained by Roscoe [4.13].

In order to simplify the results we idealize the matrix viscoelastic behavior in the following fashion:

(a) The matrix behaves elastically in dilatation. This implies that (4.6.13a) reduces to

$$\tilde{K}_1^*(\iota\omega) = K_1 \tag{4.5.22}$$

where K_1 is frequency independent. (Evidently the loss tangent (4.5.13b) then vanishes).

(b) The shear loss tangent of the matrix is so small that its square can be neglected with respect to unity. This implies that in (4.5.14b)

$$\tan^2 \delta_G << 1 \tag{4.5.23}$$

The viscoelastic behavior of a respectable number of polymeric materials may be well approximated by (4.5.22-23), at moderate temperatures. It should, however, be borne in mind that (4.5.23) may be seriously in error beyond the glass transition temperature.

Separation of the ECM expression (4.5.16-21) into real and imaginary parts is now greatly facilitated by the following observations. Firstly, it is easily proved that if the condition (4.5.23) applies, the loss tangents of any of the complex moduli (4.5.15) also obey a similar condition. Secondly, it may be proved (to be published) that for the conditions (4.5.22-23) the real part of any ECM is obtained by replacement of elastic phase moduli in the corresponding effective elastic modulus expression by the real parts of the complex phase moduli. In symbols, let M^* be any effective elastic

modulus which is written as

$$M^* = F(K_1, G_1, K_2, G_2, \{g\}) \tag{4.5.24}$$

where $\{g\}$ denotes phase geometry. Under the conditions (4.5.22-23) the real part M^{*R} of the corresponding ECM is then given by

$$\boxed{M^{*R} = F[K_1, G_1^R(\omega), K_2, G_2, \{g\}]} \tag{4.5.25}$$

Note that since G_1^R is a function of the frequency ω, so is M^{*R}.

If in the EEM expression there appear the elastic moduli, E_1, ν_1 and k_1, then these have to be replaced by E_1^R, ν_1^R and k_1^R to obtain the ECM expression. These real parts are given in view of (4.5.15) and (4.5.22-23) by elastic type relations

$$E_1^R(\omega) = \frac{9K_1G_1^R(\omega)}{3K_1 + G_1^R(\omega)} \qquad \text{(a)}$$

$$\nu_1^R(\omega) = \frac{3K_1 - 2G_1^R(\omega)}{2[3K_1 + G_1^R(\omega)]} \qquad \text{(b)} \qquad (4.5.26)$$

$$k_1^R(\omega) = K_1 + \frac{1}{3}\, G_1^R(\omega) \qquad \text{(c)}$$

Thirdly, it can be proved (to be published) that the imaginary part M^{*I} is given by

$$M^{*I} = G_1^{\,I} \frac{\partial M^{*R}}{\partial G_1^{\,R}} \tag{4.5.27}$$

It follows from (4.5.25), (4.5.27), that the loss angle is

$$\boxed{\tan \delta_M^* = \frac{M^{*I}}{M^{*R}} = \frac{G_1^{\,R}}{M^{*R}} \frac{\partial M^{*R}}{\partial G_1^{\,R}} \tan \delta_G} \tag{4.5.28}$$

where $\tan \delta_G$ is the matrix shear loss tangent.

We now proceed to apply the above given results to obtain ECM expressions.

The real and imaginary parts of $\tilde{k}^*(\iota\omega)$ are easily found to be

$$\boxed{\begin{aligned} k^{*R} &= \frac{k_1^{\,R}(k_2 + G_1^{\,R})v_1 + k_2(k_1 + G_1^{\,R})v_2}{(k_2 + G_1^{\,R})v_1 + (k_1^{\,R} + G_1^{\,R})v_2} && \text{(a)} \\ k^{*I} &= \frac{1}{3} G_1^{\,I} \left\{ 1 - \frac{(k_1^{\,R} + G_1^{\,R})^2 - 4v_1(k_2 - k_1^{\,R})^2}{\left[(k_2 + G_1^{\,R})v_1 + (k_1^{\,R} + G_1^{\,R})v_2\right]^2} \right\} && \text{(b)} \end{aligned}} \tag{4.5.29}$$

The real part of $\tilde{E}_A^*(\iota\omega)$ is given by (3.5.96) with matrix elastic moduli replaced by the real parts of corresponding matrix complex moduli which are interpreted as (4.5.26). Then the third term in the expression for E_A^{*R} is negligible, just as in the case of the elastic E_A^*. It is also easily shown by use of (4.5.27) that the third term contributes very little to E_A^{*I}. Accordingly we can approximate (4.5.17) with high accuracy by

$$\tilde{E}_A^*(\iota\omega) \cong \tilde{E}_1(\iota\omega)\, v_1 + E_2\, v_2 \tag{4.5.30}$$

It follows very simply that

$$E_A^{*R} = E_1^R \, v_1 + E_2 \, v_2 \qquad \text{(a)}$$

$$E_A^{*I} = E_1^I \, v_1 \qquad \text{(b)} \qquad (4.5.31)$$

$$\tan \delta_E^* = \frac{E_A^{*I}}{E_A^{*R}} = \frac{\tan \delta_E}{1 + \dfrac{E_2 \, v_2}{E_1^R v_1}} \qquad \text{(c)}$$

where

$$\tan \delta_E = \frac{E_1^I}{E_1^R}$$

Note that in the usual fiber reinforced materials E_2/E_1^R is a large number of the order of 25-60, while v_2 and v_1 are of comparable magnitudes (i.e. $v_2 = 0.5 - 0.7$, $v_1 = 0.5 - 0.3$). It follows that $\tan \delta_E^*$ is a much smaller number than $\tan \delta_E$, which implies that a fiber reinforced cylinder displays much less viscoelastic effect in oscillatory axial stressing and straining than a similar matrix cylinder. This is, of course, physically plausible since the stiff elastic fibers inhibit the axial deformation which would develop without presence of fibers. It is recalled that similar conclusions were reached in the static case, par. 4.3.2, sub. par. 4.3.2.2.

The real part of the complex Poisson's ratio (4.6.18) is given by

$$\nu_A^{*R} = \nu_1^R v_1 + \nu_2 v_2 + \frac{(\nu_2 - \nu_1^R)(1/k_1^R - 1/k_2)v_1 v_2}{v_1/k_2 + v_2/k_1^R + 1/G_1^R} \tag{4.5.32}$$

where ν_1^R and k_1^R are defined by (4.5.26b,c). The computation of ν_A^{*I} is complicated because of the third term in the right sides of (4.5.18) or (4.5.32). The contribution of this term is not substantial though its neglect leads to a larger error than in the case of $\tilde{E}_A^*(\iota\omega)$. We do not take into account this term here and we can write accordingly

$$\nu_A^{*I} \sim \nu_1^I v_1 \tag{4.5.33}$$

where

$$\nu_1^I = -\frac{9K_1 G_1^I}{2(3K_1+G_1^I)^2}$$

the last expression following from (4.6.15b), (4.6.22) and (4.6.23).

Next we consider the ECM $\tilde{G}_A^*(\iota\omega)$. It is easily found that

$$\boxed{\begin{aligned} G_A^{*R} &= G_1^R \frac{\gamma_R(1+v_2)+v_1}{\gamma_R v_1+v_2} && \text{(a)} \\ G_A^{*I} &= G_1^I \frac{[(\gamma_R+1)^2 + v_2(\gamma_R - 1)^2]\, v_1}{(\gamma_R v_1 + 1 + v_2)^2} && \text{(b)} \end{aligned}} \tag{4.5.34}$$

where

$$\gamma_R = \frac{G_2}{G_1^R}$$

The ECM $\tilde{G}_T^*(\iota\omega)$ is unfortunately very complicated as can be seen by the form of (4.5.20). Still the real part G_T^{*R} is easily obtained by replacement of all matrix moduli in the expression for the elastic G_T^*, (3.5.113), by the corresponding real parts of matrix complex moduli, G_1^R and ν_1^R as given by (4.5.26b). We thus obtain

$$G_T^{*R} = G_1^R \frac{(1+\alpha v_2^3)(\rho + \beta_1 v_2) - 3v_2 v_1^2 \beta_1^2}{(1+\alpha v_2^3)(\rho - v_2) - 3v_2 v_1^2 \beta_1^2} \qquad \text{(a)}$$

where

$$\alpha = \frac{\beta_1 - \gamma_R \beta_2}{1 + \gamma_R \beta_2} \qquad \rho = \frac{\gamma_R + \beta_1}{\gamma_R - 1} \qquad \text{(b)} \qquad (4.5.35)$$

$$\beta_1 = \frac{1}{3 - 4\nu_1^R} \qquad \beta_2 = \frac{1}{3 - 4\nu_2} \qquad \text{(c)}$$

To find an expression for G_T^{*I} it is necessary to carry out the differentiation in (4.5.27) or to separate out the imaginary part of (4.5.20). This requires extremely cumbersome calculations which will not be performed

here. Instead we consider the simplified case of perfectly rigid fibers. In that event $\gamma_R \to \infty$ in (4.5.35) and we obtain a result analogous to (3.5.115d)

$$G_T^{*R} = G_1^R \frac{(1-v_2^3)(1+\beta_1 v_2) - 3v_2 v_1^2 \beta_1^2}{(1-v_2^3)(1-v_2) - 3v_2 v_1^2 \beta_1^2} \tag{4.5.36}$$

where β_1 is given by the first of (4.5.35c). We compute the transverse shear loss angle by use of (4.5.28) and (4.5.36). The result is

$$\tan\delta^*_{G_T} = \left\{ 1 - \frac{4}{3}(1+\nu_1^R)(1-2\nu_1^R)\beta_1^2 v_1^3 v_2 \cdot \right.$$

$$\left. \cdot \frac{1-v_2^3 + 3v_2 v_1 \beta_1(\beta_1+2)}{(1-v_2^3 - 3v_2 v_1 \beta_1^2)[(1-v_2^3)(1+\beta_1 v_2) - 3v_2 v_1^2 \beta_1^2]} \right\} \tan\delta_G \tag{4.5.37}$$

Several numerical calculations have shown that the second term in the parenthesis of (4.5.37) is a small number of the order of 0.05. It may therefore be concluded that for rigid fibers

$$\boxed{\tan\delta^*_{G_T} \sim \tan\delta_G} \tag{4.5.38}$$

and it is very likely that such a conclusion is also valid for fibers which are very much stiffer than the matrix, as is the case in many FRM. Consequently, the approximate relation (4.5.38) enables us to compute G_T^{*I} in the following approximate fashion. First G_T^{*R} is computed from (4.5.35) and then

$$G_T^{*I} \sim G_T^{*R} \tan\delta_G \tag{4.5.39}$$

The approximations (4.5.38-39) are related to some rigorous results for the shear loss angle of FRM with rigid fibers, which will be derived further below.

Equ. (4.5.21) may be now exploited to obtain the real and imaginary parts of $\tilde{E}_T^*(\iota\omega)$. It is readily shown that for the simplifying assumptions (4.5.22-23),

$$\boxed{\begin{aligned} E_T^{*R} &= \frac{4k^{*R}G_T^{*R}}{k^{*R} + m_R G_T^{*R}} \qquad \text{(a)} \\ m_R &= 1 + \frac{4k^{*R}(\nu_A^{*R})^2}{E_A^{*R}} \qquad \text{(b)} \end{aligned}} \qquad (4.5.40)$$

There is no special difficulty to obtain E_T^{*I} but the result is cumbersome and will not be given here.

It will be recalled that in the discussion of static viscoelastic properties of FRM, par. 4.32, great simplification was achieved in some cases by the assumption of rigid fibers. We now consider similar simplifications for ECM. Assumptions (4.5.22-23) are discarded at the present time.

Results for EEM in the case of rigid fibers are given by (3.5.115). We use the present correspondence principle to transform these into ECM.

It follows from (3.5.115a) and (4.5.15c) that

$$\tilde{k}^*(\iota\omega) = \tilde{K}_1(\iota\omega) + \frac{1}{3}\tilde{G}_1(\iota\omega) + [\tilde{K}_1(\iota\omega) + \frac{4}{3}\tilde{G}_1(\iota\omega)]\frac{v_2}{1-v_2} \qquad (4.5.41)$$

Separation into real and imaginary parts is immediate and yields very simply

$$k^{*R} = K_1^R + \frac{1}{3}G_1^R + (K_1^R + \frac{4}{3}G_1^R)\frac{v_2}{1-v_2} \qquad \text{(a)}$$

$$(4.5.42)$$

$$k^{*I} = K_1^I + \frac{1}{3}G_1^I + (K_1^I + \frac{4}{3}G_1^I)\frac{v_2}{1-v_2} \qquad \text{(b)}$$

If assumption (4.5.22) is introduced, (4.5.42) reduces to

$$k^{*R} = K_1 + \frac{1}{3}G_1^R + (K_1 + \frac{4}{3}G_1^R)\frac{v_2}{1-v_2} \qquad \text{(a)}$$

$$(4.5.43)$$

$$k^{*I} = G_1^I\frac{1+3v_2}{(1-v_2)} \qquad \text{(b)}$$

If (4.5.23) is valid then G_1^I is much smaller than G_1^R or K_1, and it is seen that in this event the loss tangent $\tan\delta_k^*$ becomes much smaller than the matrix shear loss angle.

Next we consider the axial shear modulus as given by (3.5.115c). The correspondence principle yields at once

$$\tilde{G}_A^*(\iota\omega) = \tilde{G}_1(\iota\omega)\frac{1+v_2}{1-v_2} \qquad (4.5.44)$$

From which it follows that

$$G_A^{*R} = G_1^R \frac{1+v_2}{1-v_2} \qquad \text{(a)}$$

$$G_A^{*I} = G_1^I \frac{1+v_2}{1-v_2} \qquad \text{(b)} \qquad (4.5.45)$$

$$\tan \delta_{G_A}^{*} = \tan \delta_G \qquad \text{(c)}$$

The last result is of particular interest since it implies that the FRM axial shear loss tangent is the same as the matrix shear loss tangent. This result resembles (4.3.54) and just as in that case it can be generalized to any FRM with rigid fibers. To show this we use the general result (4.3.51) from which it follows at once by the correspondence principle that

$$\boxed{\tilde{G}_A^{*}(\iota\omega) = \tilde{G}_1(\iota\omega)\,\frac{{}^{e}G_A^{*}}{{}^{e}G_1}}$$

or in real and imaginary parts

$$\frac{G_A^{*R}}{G_1^R} = \frac{G_A^{*I}}{G_1^I} = \frac{{}^{e}G_A^{*}}{{}^{e}G_1} \qquad (4.5.46)$$

The results (4.5.46) show that the ratio of real/imaginary part of axial complex shear modulus to real/imaginary part of matrix shear modulus are the same as the ratio of effective elastic axial shear modulus to matrix elastic shear modulus and this relation holds for any frequency. It also follows from (4.5.46)

that

$$\boxed{\tan \delta^*_{G_A} = \tan \delta_G} \tag{4.5.47}$$

this relation thus being valid for any FRM with rigid fibers

Similar relations may be derived for $\tilde{G}^*_T(\iota\omega)$, by use of the general relation (4.3.58). But it should be carefully noted that here it is necessary to assume in addition that the matrix is incompressible. In this case it follows that

$$\boxed{\tilde{G}^*_T(\iota\omega) = \tilde{G}_1(\iota\omega)\,\frac{e_{G^*_T}}{e_{G_1}}}$$

from which we have

$$\frac{G^{*R}_T}{G^R_1} = \frac{G^{*I}_T}{G^*_1} = \frac{e_{G^*_T}}{e_{G_1}} \qquad \text{(a)}$$

(4.5.48)

$$\boxed{\tan \delta^*_{G_T} = \tan \delta_G} \qquad \text{(b)}$$

It has been previously noted, (4.5.38), that (4.5.48b) is approximately fulfilled for stiff non-rigid fibers and for polymeric compressible matrix (Poisson's ratio of order 0.35 - 0.40). On the other hand calculations show that (4.5.48a) is liable to lead to significant errors when applied to such phase materials.

It would seem therefore that it is best to estimate $\tilde{G}_T^*(\iota\omega)$ by use of (4.5.35) and (4.5.39).

The expressions for ECM given in this chapter easily permit numerical computation in terms of measured matrix complex moduli. To give a simple example we consider the case of a polyisobutilene matrix. Figs. (4.5.1-2) show plots of real and imaginary parts of the effective complex axial shear modulus $G_A^*(\iota\omega)$ as a function of log frequency, for different fiber volume concentrations. It has been assumed that the fibers are rigid and thus eqn. (4.5.45) has been used. The complex shear modulus variation of polyisobutilene (plots for $v_2 = 0$) have been taken from [4.20].

4.6 STRUCTURAL APPLICATIONS

4.6.1 Quasi-Static Theory and Examples

If we intend to use the effective viscoelastic properties, which were derived in preceding chapters, for a theory of viscoelastic structures which are made of composite materials, we encounter the same kind of difficulty which was discussed in chapter 3.9 for elastic composites, since the stress and strain fields in structures are in general not statistically homogeneous.

The difficulty is resolved, or rather mitigated, as for elastic composites by establishment of a first order approximation, in terms of local averages, based on the assumption of local statistical homogeneity. Indeed, the establishment of such an approximation is in all respects similar to the development given in chapter 3.9. The local averages (3.9.1), (3.9.3) and (3.9.6) now depend on space and time and the macro-equilibrium equations (3.9.7) remain in the same form and now represent quasi-static equilibrium. Thus for viscoelastic composites

$$\frac{\partial \bar{\sigma}_{ij}(\underline{x},t)}{\partial x_j} + \bar{F}_i(\underline{x}, t) = 0 \tag{4.6.1}$$

The elastic macro-stress-strain relation (3.9.8) is now replaced by its viscoelastic macro-counterparts

$$\bar{\sigma}_{ij}(\underline{x},t) = \int_{-\infty}^{t} C^{*}_{ijkl}(t-\tau)\,\frac{\partial}{\partial\tau}\,\bar{\epsilon}_{kl}(\underline{x},\tau)\,d\tau \qquad \text{(a)}$$

$$(4.6.2)$$

$$\bar{\epsilon}_{ij}(\underline{x},t) = \int_{-\infty}^{t} S^{*}_{ijkl}(t-\tau)\,\frac{\partial}{\partial\tau}\,\bar{\sigma}_{kl}(\underline{x},\tau)\,d\tau \qquad \text{(b)}$$

where $C^{*}_{ijkl}(t)$ and $S^{*}_{ijkl}(t)$ are the effective viscoelastic properties which were discussed in preceding chapters.

Boundary conditions of form

$$u_i(S,t) = u^{o}_{i} \qquad \text{on } S_u$$

$$T_i(S,t) = T^{o}_{i} \qquad \text{on } S_T$$

are replaced by the boundary layer conditions

$$\bar{u}_i(S,t) = u^{o}_{i} \qquad \text{on } S_u$$

$$(4.6.3)$$

$$\bar{T}_i(S,t) = T^{o}_{i} \qquad \text{on } S_T$$

It is seen that (4.6.1-3) define a boundary value problem for $\bar{u}_i(\underline{x},t)$ which is in all respects analogous to a typical quasi-static boundary value problem for homogeneous viscoelastic bodies, the field variables and viscoelastic properties of the latter being replaced by local averages and effective viscoelastic properties of the former. We conclude that <u>all classical quasi-static viscoelasticity</u>

solutions and also viscoelastic strength of materials solutions generate similar results for viscoelastic macro-displacements, strains and stresses of viscoelastic composites, simply by replacement of homogeneous viscoelastic properties by effective viscoelastic properties.

For illustrative purposes we consider two simple examples: bending-shear deflection of a fiber reinforced beam and torsion of a fiber reinforced cylinder.

(a) Bending-shear deflection of beam

Consider a homogeneous elastic beam as shown in fig. 4.6.1. Let the material be transversely isotropic with material axis of symmetry along the beam axis. Then the bending and shear deflections obey the differential equations

$$\frac{d^2 w^b}{dx^2} = -\frac{M(x)}{E_A I} \qquad \text{(a)}$$

$$\frac{d^2 w^s}{dx^2} = -\frac{\alpha q(x)}{G_A A} \qquad \text{(b)} \qquad (4.6.4)$$

$$w(x) = w^b(x) + w^s(x) \qquad \text{(c)}$$

w^b - bending deflection

w^s - shear deflection

w - total deflection

M - bending moment

q - load per unit length

I - section moment of inertia

A - section area

α - geometrical factor defined as first moment of area above neutral axis divided by section width at neutral axis

E_A - axial Young's modulus

G_A - axial shear modulus

Equs. (4.6.4) may be found in any strength of materials book where they are usually derived for isotropic materials. It is easily realized that they also hold for the present case of transverse isotropy.

If we consider instead a viscoelastic transversely isotropic beam, we have by the usual correspondence principle which was discussed in par. 4.1.2

$$\frac{d^2\hat{w}^b}{dx^2} = -\frac{\hat{M}}{p\hat{E}_A I} \qquad \text{(a)}$$

$$\frac{d^2\hat{w}^s}{dx^2} = -\frac{\alpha\hat{q}}{p\hat{G}_A A} \qquad \text{(b)} \qquad (4.6.5)$$

$$\hat{w} = \hat{w}^b + \hat{w}^s \qquad \text{(c)}$$

where $\hat{w}^b$, $\hat{w}^s$, $\hat{w}$, $\hat{M}$ and $\hat{q}$ are Laplace transforms of $w^b(x,t)$, $w^s(x,t)$, $w(x,t)$, $M(x,t)$ and $q(x,t)$, respectively, and $p\hat{E}_A$ and $p\hat{G}_A$ are the TD (transform domain) moduli.

Consider the basic case

$$q(x,t) = q(x)\ H(t) \qquad \text{(a)}$$

$$M(x,t) = M(x)\ H(t) \qquad \text{(b)} \tag{4.6.6}$$

where $H(t)$ is the Heaviside step function. This describes a load which is applied at $t = 0$, and is then left unchanged. (Note that (b) follows from (a).) Then

$$\hat{q}(x,p) = \frac{q(x)}{p}$$

$$\hat{M}(x,p) = \frac{M(x)}{p} \tag{4.6.7}$$

Recall also the relations (4.3.25a) and (4.3.40) which are certainly valid for homogeneous materials. Thus

$$\hat{e}_A = \frac{1}{p^2 \hat{E}_A}$$

$$\hat{g}_A = \frac{1}{p^2 \hat{G}_A} \tag{4.6.8}$$

where $\hat{e}_A$ and $\hat{g}_A$ are the LT of the Young's and axial shear creep compliances, $e_A(t)$ and $g_A(t)$, respectively. Introduction of (4.6.7-8) into (4.6.5) leads to expressions which may be inverted at once, the results being

$$\frac{\partial^2 w^b(x,t)}{\partial x^2} = -\frac{M(x)}{I}\, e_A(t) \qquad \text{(a)}$$

$$\frac{\partial^2 w^s(x,t)}{\partial x^2} = -\frac{\alpha q(x)}{A}\, g_A(t) \qquad \text{(b)} \qquad (4.6.9)$$

$$w(x,t) = w^b(x,t) + w^s(x,t) \qquad \text{(c)}$$

Comparison of (4.6.4) and (4.6.9) shows that if the solution of an elastic bending-shear deflection problem is known then the solution of the analogous viscoelastic problem, with corresponding boundary conditions is simply obtained by replacement of $1/E_A$ and $1/G_A$ in the elastic solution by $e_A(t)$ and $g_A(t)$, respectively.

The macro-bending stresses are simply

$$\bar{\sigma}_{xx} = -\frac{M(x)z}{I}\, H(t)$$

It should be noted that it follows from superposition that if the load (4.6.6a) is replaced by

$$q(x,t) = q(x)\, f(t)$$

where $f(t)$ is any function, and the beam end conditions are kept the same, then

$$w(x,t) = \int_{-\infty}^{t} w_H(x,t-\tau)\, f(\tau)\, d\tau$$

where w_H is the deflection under (4.6.6a).

The macro-bending stresses become

$$\bar{\sigma}_{xx} = -\frac{M(x)z}{I} f(t)$$

Consider for example the case of uniform load per unit length of a simply supported beam, in which case q=const. in (4.6.4b) and in (4.6.6a). Then the deflection at the center is

$$w(\frac{\ell}{2}) = \delta = \frac{5q\ell^4}{384IE_A} + \frac{\alpha q\ell^2}{8AG_A} \tag{4.6.10}$$

(see e.g. [4.21]). Then for a viscoelastic beam

$$w(\frac{\ell}{2}, t) = \delta(t) = \frac{5q\ell^4}{384I} e_A(t) + \frac{\alpha q\ell^2}{8A} g_A(t) \tag{4.6.11}$$

Consider the same beam, except that the material is now fiber reinforced with fibers parallel to beam axis. By the first approximation, $e_A(t)$ and $g_A(t)$ merely have to be replaced by $e_A^*(t)$ and $g_A^*(t)$. Thus

$$\delta(t) = \frac{5q\ell^4}{384I} e_A^*(t) + \frac{\alpha q\ell^2}{8A} g_A^*(t) \tag{4.6.12}$$

It is now recalled that it was shown in par. 4.3.2 , sub-par. 4.3.2.2. that for stiff fibers the time variation of $e_A^*(t)$ is negligible and therefore

$$e_A^*(t) \approx \frac{1}{E_A^*(o)} \tag{4.6.13}$$

$$E_A^*(o) \approx E_1(o)v_1 + E_2 v_2$$

In contrast, it was shown in sub-par. 4.3.2.3 that the time variation of $g_A^*(t)$ is considerable. Indeed for rigid fibers, $g_A^*(t)$ is given by (4.3.57) in the form

$$g_A^*(t) = J_1(t) \frac{1-v_2}{1+v_2}$$

where $J_1(t)$ is the shear creep compliance of the matrix, which varies considerably. It is thus seen there is a fundamental difference between the two parts of (4.6.12). The first part which is the bending deflection varies negligibly with time, while the second, which is the shear deflection, increases with time. It is well known that for elastic beams the shear deflection is generally very small in comparison to the bending deflection (unless the beam is very short, in which case the validity of the approximate theory is doubtful). It is seen by the present example that this may not be the case for a viscoelastic fiber reinforced beam, in which after the elapse of sufficient time the bending and shear deflections may become of same order of magnitude. This phenomenon is easily understood on physical grounds: Bending is produced by stresses normal to the section and in this case the stiff fibers deform very little and constrain the matrix

deformation. The shear stresses, however, act in the matrix parallel to the fibers and in this case the matrix easily deforms by relative sliding, carrying the fibers with it.

(b) Torsion of Cylinder

We consider the torsion of a fiber reinforced cylinder in which the fibers are parallel to the axis, fig. 4.11. If the cylinder were elastic homogeneous and transversely isotropic, with material axis of symmetry parallel to the cylinder axis then

$$\theta = \frac{T}{G_A I'}$$

where

θ - angle of twist per unit axial length

T - torque

G_A - axial shear modulus

$$I' - \int_A \left(x_2^{\,2} + x_3^{\,2} + x_2 \frac{\partial\varphi}{\partial x_3} - x_3 \frac{\partial\varphi}{\partial x_2}\right) dx_2\, dx_3 \qquad (4.6.14)$$

φ - torsion function

If the cylinder is viscoelastic and subjected to torque $T(t)$, then by the correspondence principle

$$\hat{\theta}(p) = \frac{\hat{T}(p)}{p\hat{G}_A(p) I'} \qquad (4.6.15)$$

Let

$$T(t) = T_o H(t) \tag{4.6.16}$$

then

$$\hat{T}(p) = \frac{T_o}{p} \tag{4.6.17}$$

Introducing (4.6.17) into (4.6.15), using (4.6.8b) and inverting we have

$$\theta(t) = \frac{T_o}{I'} g_A(t) \tag{4.6.18}$$

If instead of (4.6.16)

$$T(t) = T_o f(t)$$

then from (4.6.18) and superposition

$$\theta(t) = \frac{T_o}{I'} \int_{-\infty}^{t} g_A(t-\tau) f(\tau) d\tau \tag{4.6.19}$$

Suppose that the cylinder is fiber reinforced, with axial shear compliance $g_A^*(t)$. Then by the first approximation (4.6.18-19) simply becomes

$$\theta(t) = \frac{T_o}{I'} g_A^*(t) \qquad \text{(a)}$$

$$\theta(t) = \frac{T_o}{I'} \int_{-\infty}^{t} g_A^*(t-\tau) f(\tau) d\tau \qquad \text{(b)} \tag{4.6.20}$$

(c) General comments on viscoelastic fiber reinforced structures under static loads

The results obtained in this chapter, in conjunction with the results of chap. 4.3, indicate an approximate rule for analysis of viscoelastic fiber reinforced structures which are subjected to loads which are constant in time: The solution for an elastic structure of same geometry and under the same loads is first obtained. To obtain the LT of solution for the viscoelastic structure effective elastic moduli are replaced by effective TD moduli. Any of the group of EEM ${}^{e}n^{*}$, ${}^{e}\ell^{*}$, ${}^{e}k^{*}$, ${}^{e}E_{A}^{*}$, ${}^{e}\nu_{A}^{*}$ may be replaced by the initial values $n^{*}(o)$, $\ell^{*}(o)$, $k^{*}(o)$, $E_{A}^{*}(o)$, $\nu_{A}^{*}(o)$ because of their small time variation. Any of the group ${}^{e}G_{A}^{*}$, ${}^{e}G_{T}^{*}$, ${}^{e}E_{T}^{*}$, ${}^{e}\nu_{T}^{*}$ should be replaced by the corresponding effective TD modulus of the group $p\hat{G}_{A}^{*}$, $p\hat{G}_{T}^{*}$, $p\hat{E}_{T}^{*}$, $\nu_{T}^{*}(p)$.

Once the solution for loads constant in time is known the problem of variable loads is solved by an hereditary superposition integral.

4.6.2 Torsional Forced Vibrations of Fiber Reinforced Cylinder

As has been mentioned in par. 4.5.1 vibration problems of two phase composites may be treated on the basis of a first approximation theory in terms of effective complex moduli and effective densities. Since this subject is still in the development stage, we treat the present example by classical vibration theory of homogeneous cylinders, with complex moduli and average density, respectively. It is not clear to what extent the analysis is meaningful.

The cylinder shown in fig. 4.6.2 is built in at $x_1 = x = 0$ and is subjected to a sinusoidal forcing torque at the end $x = \ell$. If the cylinder is elastic homogeneous transversely isotropic with axis of material symmetry in x direction then the governing differential equation of the problem is

$$\frac{\partial^2 \vartheta}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \vartheta}{\partial t^2} \tag{4.6.21}$$

where

$\vartheta(x,t)$ - angle of twist

$$c^2 = \frac{G_A I'}{\rho I} \tag{4.6.22}$$

where I' is given by (4.6.14) and

I - polar moment of inertia of section

ρ - density

The boundary conditions are

$$\vartheta(o,t) = 0 \qquad \text{(a)}$$

$$G_A I' \frac{\partial \vartheta(\ell,t)}{\partial x} = T_o e^{\iota \omega t} \qquad \text{(b)} \tag{4.6.23}$$

where the right side of (4.6.23) is the forcing torque of frequency ω.

The solution of (4.6.21) with (4.6.23) is (see e.g. [4.15])

$$\vartheta(x,t) = \frac{cT_o}{\omega G_A} \frac{\sin(\frac{\omega x}{c})}{\cos(\frac{\omega \ell}{c})} e^{\iota\omega t} \tag{4.6.24}$$

Suppose the cylinder is homogeneous viscoelastic transversely isotropic with complex axial shear modulus $\tilde{G}_A(\iota\omega)$. In view of the classical correspondence principle of par. 4.5.3 the solution can then be written down on the basis of (4.6.24) in the following fashion

$$\vartheta(x,t) = \tilde{\vartheta}(x)\, e^{\iota\omega t} \qquad \text{(a)}$$

$$\tilde{\vartheta}(x) = \frac{\tilde{c}T_o}{\omega \tilde{G}_A} \frac{\sin(\frac{\omega x}{c})}{\cos(\frac{\omega \ell}{c})} \qquad \text{(b)} \qquad (4.6.25)$$

$$\tilde{c}^2 = \frac{\tilde{G}_A I'}{\rho I} \qquad \text{(c)}$$

The forcing torques

$$T = T_o \cos \omega t \qquad \text{(a)}$$

$$T = T_o \sin \omega t \qquad \text{(b)} \qquad (4.6.26)$$

produce the solutions

$$\vartheta_c(x,t) = \mathrm{Re}\,[\,\vartheta(x,t)\,] \qquad \text{(a)}$$

$$\vartheta_s(x,t) = \mathrm{Im}\,[\,\vartheta(x,t)\,] \qquad \text{(b)} \qquad (4.6.27)$$

respectively.

Now for a fiber reinforced cylinder, $\tilde{G}_A$ in (4.6.25) is replaced by $\tilde{G}_A^*$ and ρ is replaced by the average density $\bar{\rho}$. The last replacement, in particular, cannot be correct. As mentioned in chapter 3.9 the effective density is not the average density and in the present case it is to be expected that the effective density will be a complex number. Proceeding nevertheless with the simple assumptions adopted here we find after straightforward calculations (for details see [4.16]) that for a cosine forcing torque (4.7.27a) the angle of twist at the extremity $x = \ell$ is given by

$$\vartheta_c(\ell,t) = \frac{c^* T_o}{\omega G_A^{*R}(\omega) I'} \frac{\sin^2 2\alpha + \sinh^2 2\beta}{\cos 2\alpha + \cosh 2\beta} \cos(\omega t - \delta/2 - \psi) \qquad (4.6.28)$$

where

$$G_A^{*R}(\omega) = G_1^R(\omega) \frac{1+v_2}{1-v_2} \qquad \text{(a)}$$

$$c^{*2} = \frac{G_A^{*R}(\omega) I'}{\bar{\rho} I} \qquad \text{(b)}$$

$$\alpha = \frac{\omega \ell}{c^*} \cos \delta/2 \qquad \text{(c)}$$

$$\beta = \frac{\omega \ell}{c^*} \sin \delta/2 \qquad \text{(d)}$$

$$\tan \delta = \frac{G_1^I}{G_1^R} \qquad \text{matrix shear loss tangent} \quad (e)$$

$$\tan^2 \delta << 1 \qquad (f)$$

$$\tan \psi = \frac{\sinh 2\beta}{\sin 2\alpha} \qquad (g)$$

It has here been assumed that the fibers are rigid and that the FRM is described by the composite cylinder assemblage; hence the form (a) for G_A^{*R}. It is seen that the factor multiplying the cosine in (4.6.28) is the amplitude and $\psi + \delta/2$ in the cosine argument is the phase lag.

A plot of the amplitude of ϑ_c, as given by (4.6.28), as a function of frequency ω, is shown in fig. 4.6.3. Also shown is Amp ϑ_c in the case of torsional vibrations of an elastic fiber reinforced cylinder. The data used are:

$\ell = 5.0$ ft.

$d = 4.0$ in. diameter of circular section

$\bar{\rho} = 3.0$ density relative to water

$$G_1^R(\omega) = G_1^R(o)\ (1 + \frac{1}{4} \log_{10} \omega)$$

$$G_1^R(o) = 0.5 \times 10^6 \text{ psi}$$

$\tan \delta = 0.1$

$v_1 = 0.4$ $\qquad$ $v_2 = 0.6$

In the elastic analysis it has been assumed that $G_1^R(o)$ is the elastic matrix shear modulus.

The elastic amplitude diagram shows typical resonances. It is seen that for the viscoelastic cylinder the resonances are quickly damped out.

Other cases of vibration analysis of fiber reinforced viscoelastic structures have been analyzed in [4.16].

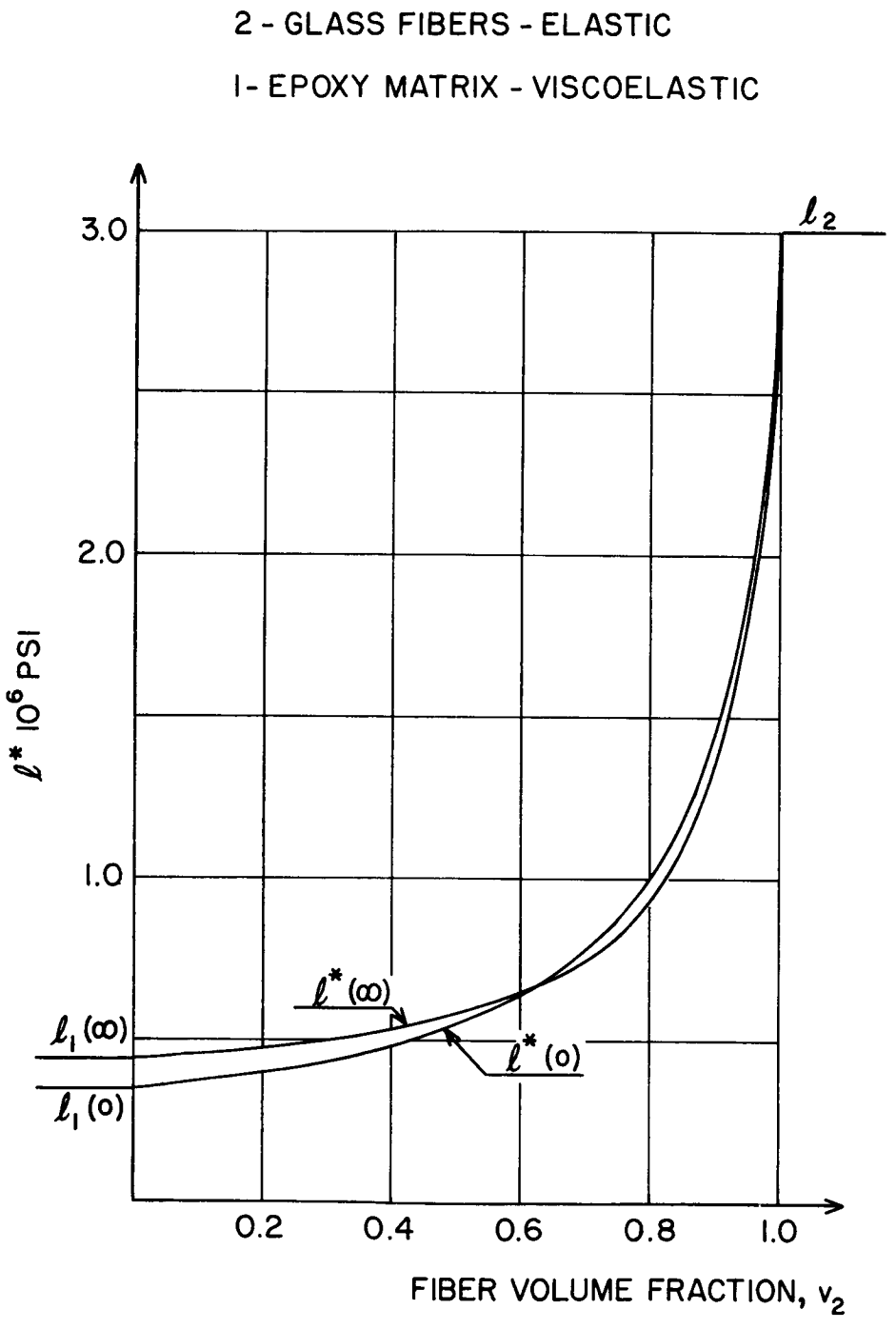


FIG. 4.3.1 - INITIAL AND FINAL RELAXATION MODULUS, ℓ^*

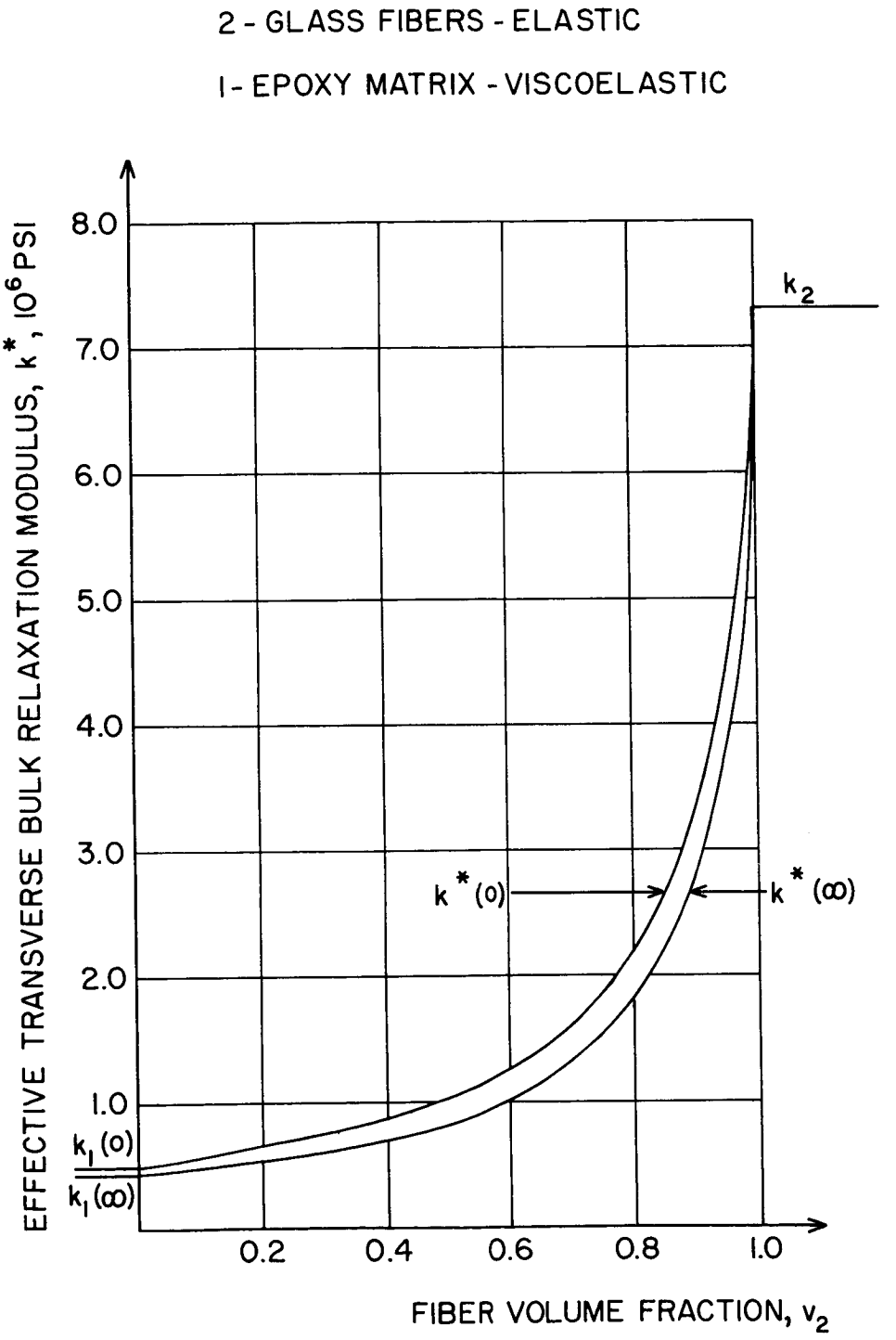


FIG. 4.3.2 - INITIAL AND FINAL TRANSVERSE BULK RELAXATION MODULUS, k^*

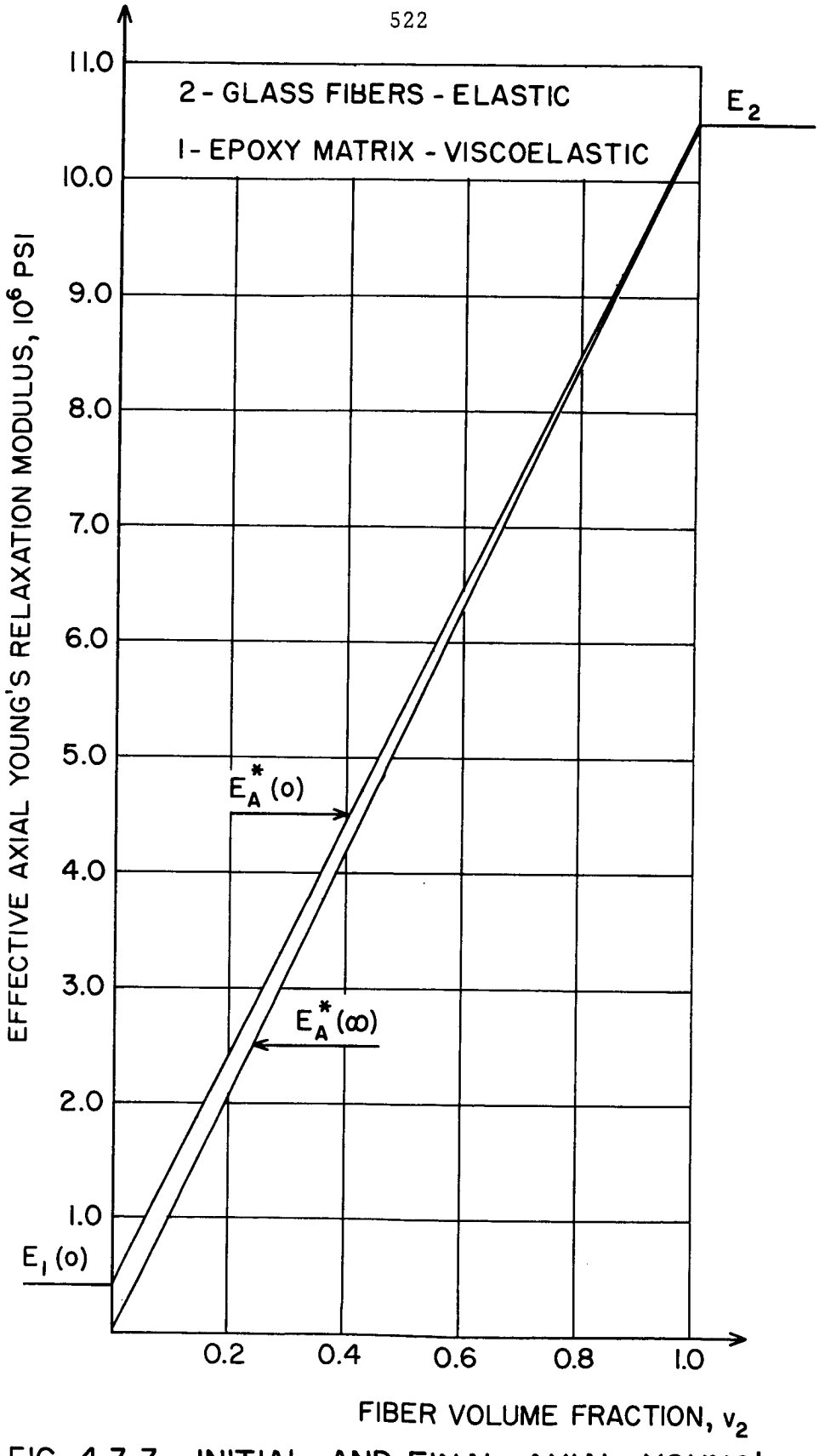


FIG. 4.3.3 - INITIAL AND FINAL AXIAL YOUNG'S RELAXATION MODULUS, E_A^*

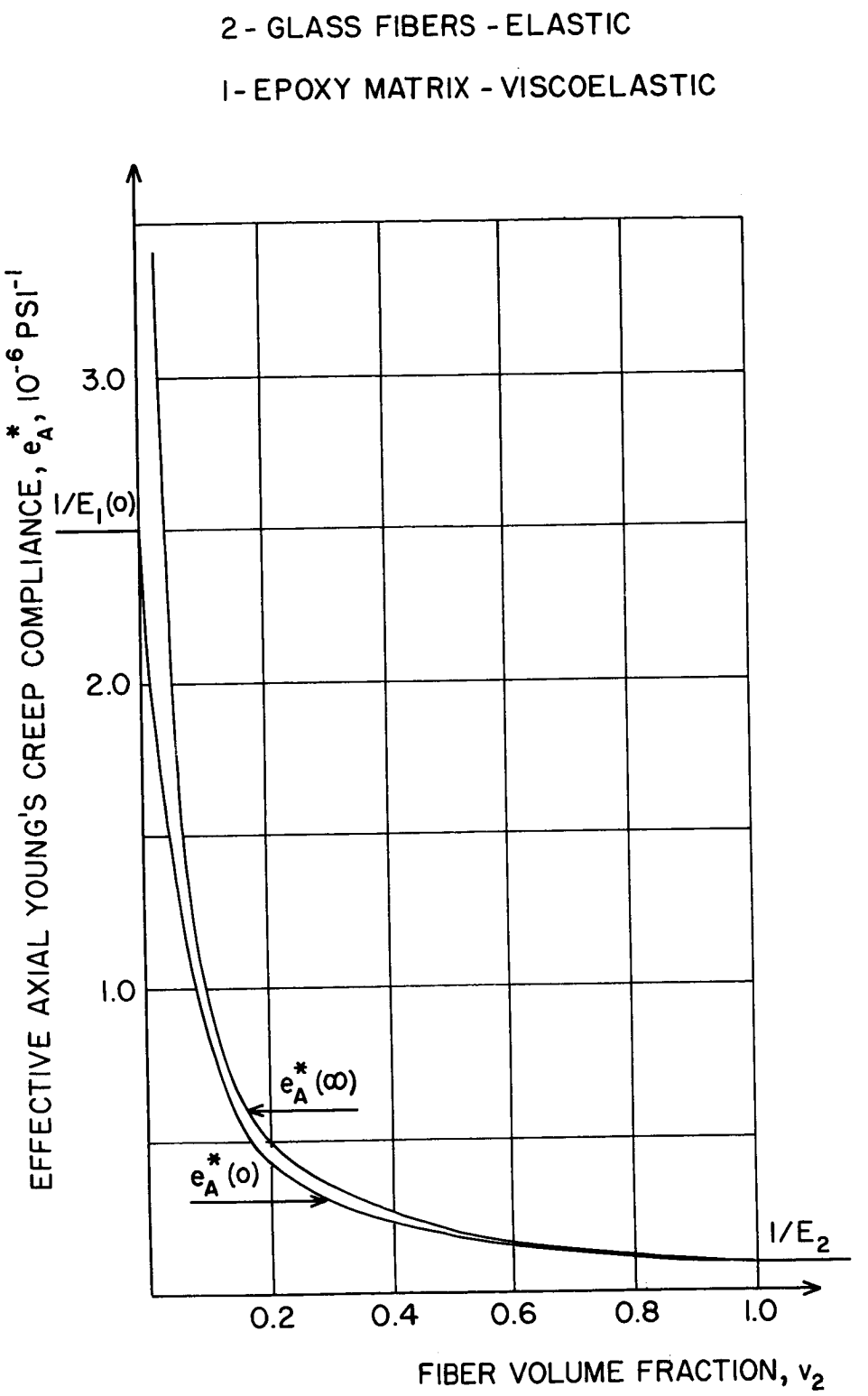


FIG. 4.3.4 - INITIAL AND FINAL AXIAL YOUNG'S CREEP COMPLIANCE, e_A^*

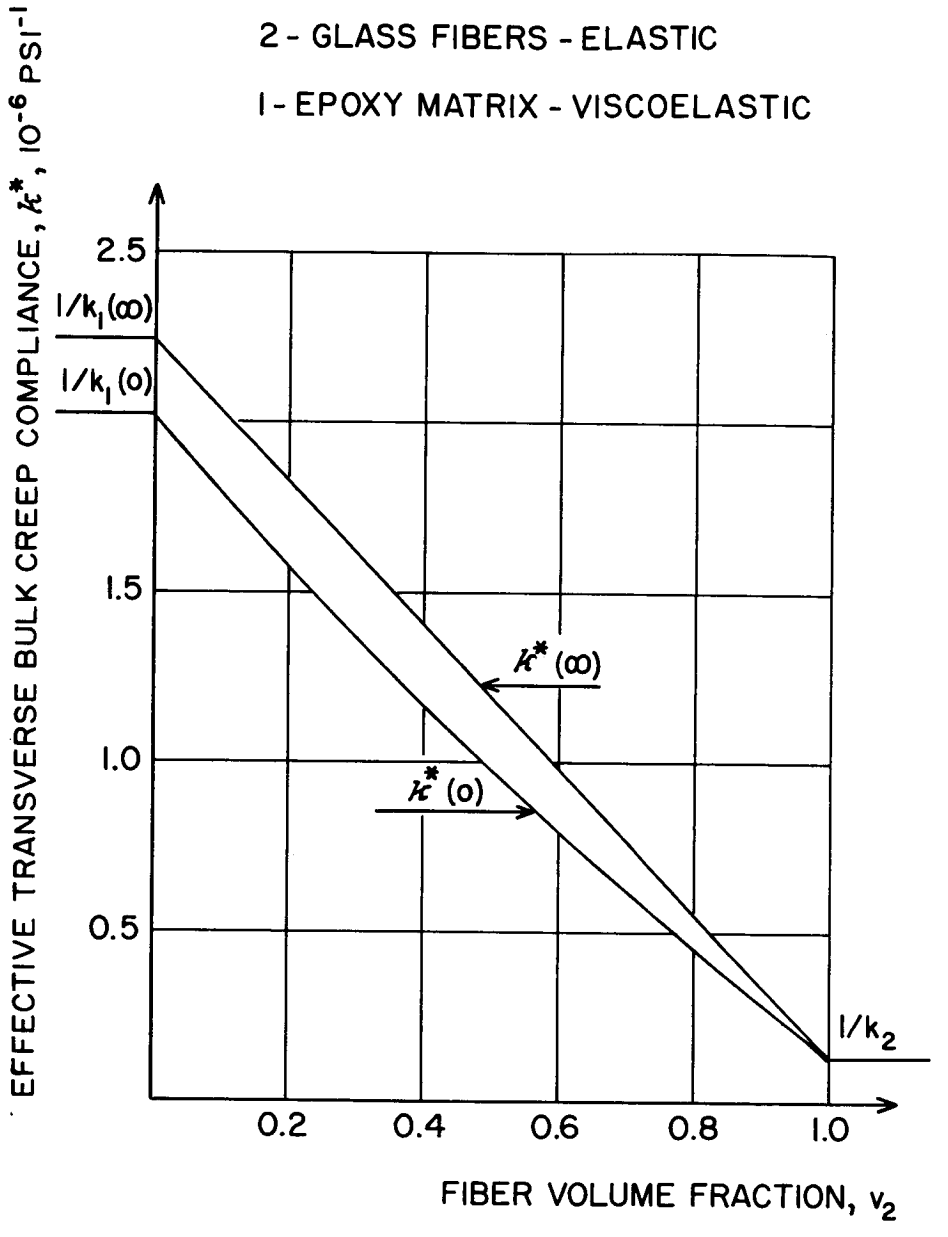


FIG. 4.3.5 - INITIAL AND FINAL TRANSVERSE BULK CREEP COMPLIANCE, k^*

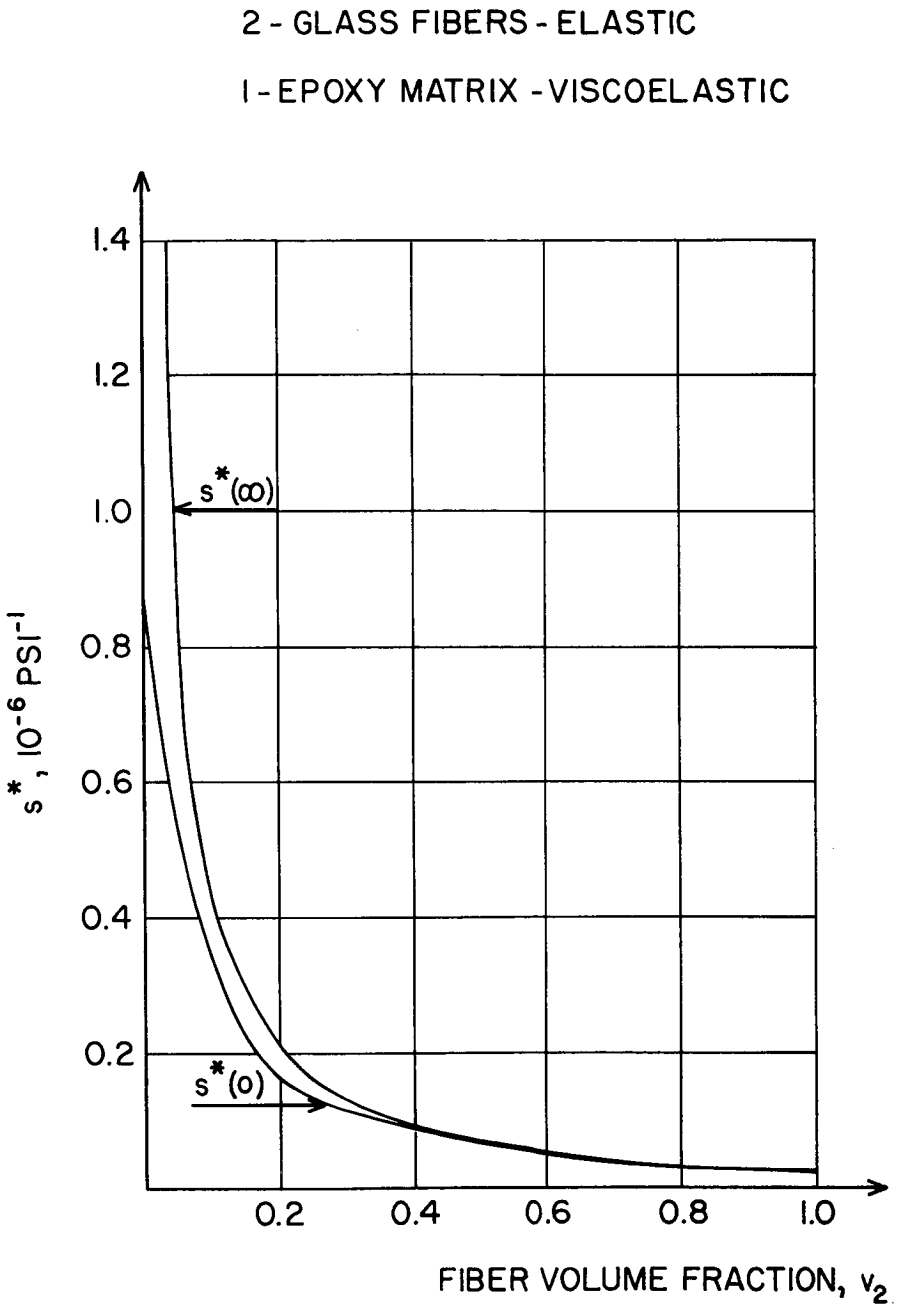


FIG. 4.3.6 - INITIAL AND FINAL CREEP COMPLIANCE, s^*

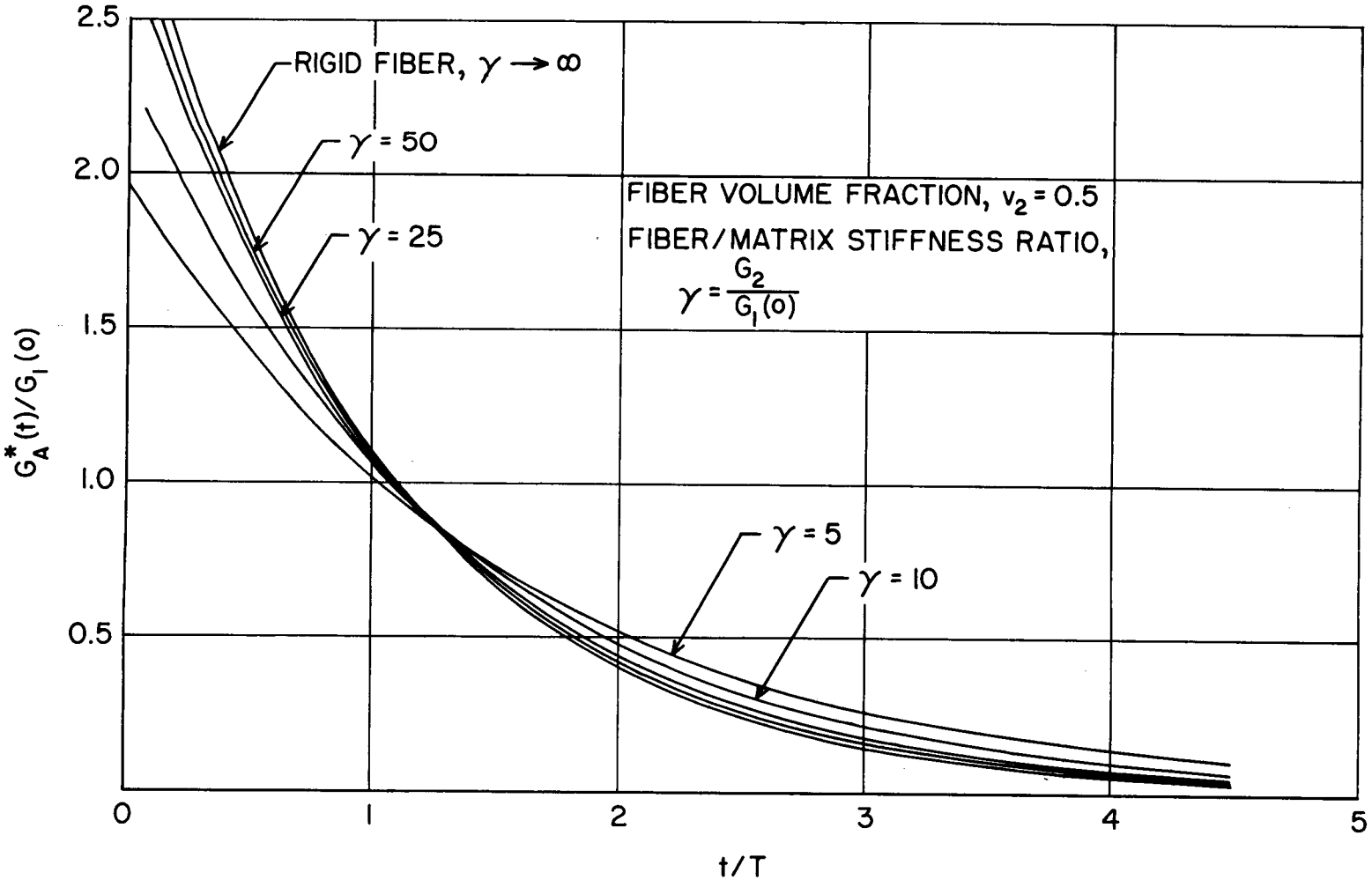


FIG. 4.3.7 - DEPENDENCE OF AXIAL SHEAR RELAXATION MODULUS $G_A^*(t)$ ON FIBER/MATRIX STIFFNESS RATIO. MAXWELL MODEL MATRIX

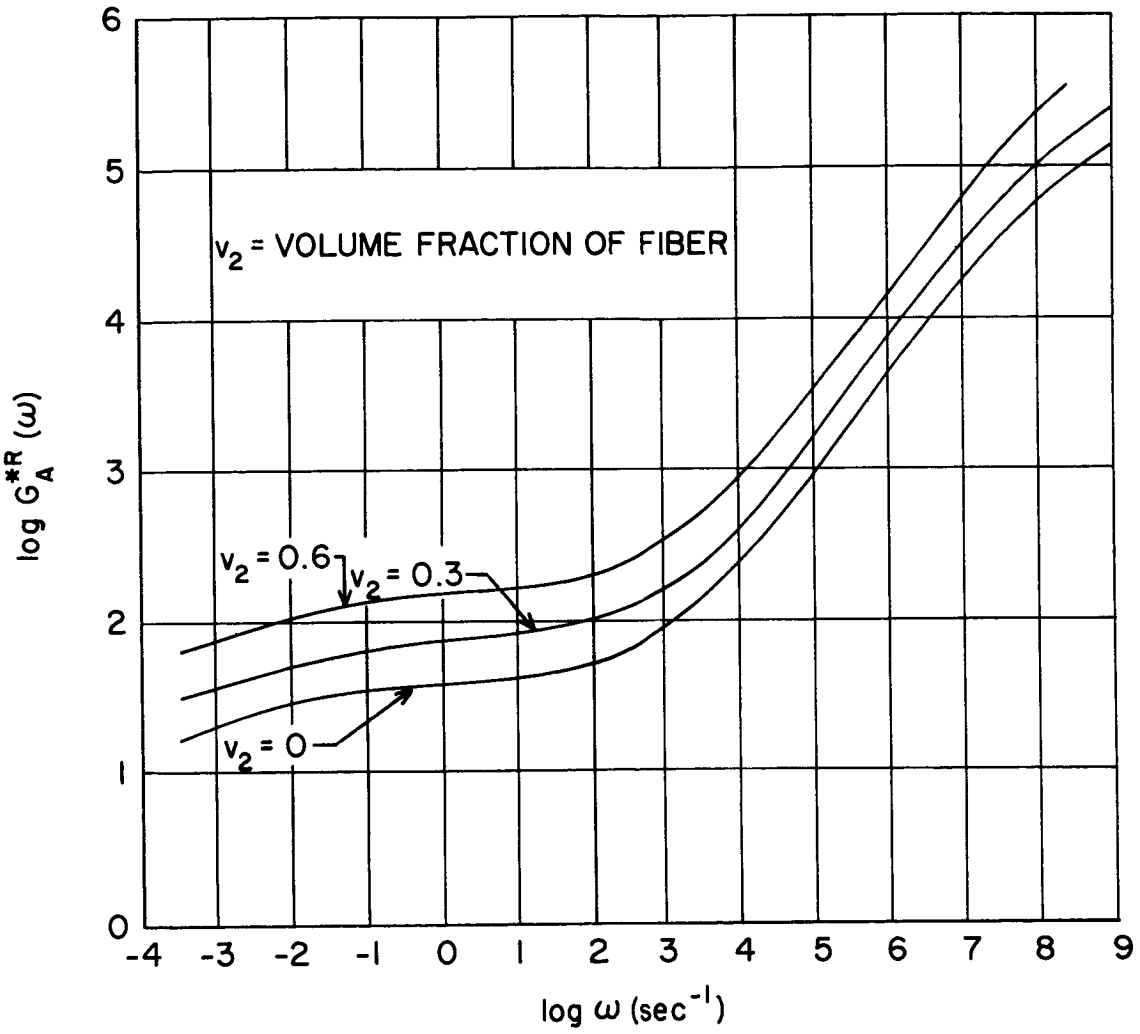


FIG. 4.5.1 - REAL PART OF COMPLEX AXIAL SHEAR MODULUS, $G_A^{*R}(\omega)$, POLYISOBUTYLENE MATRIX, RIGID FIBERS

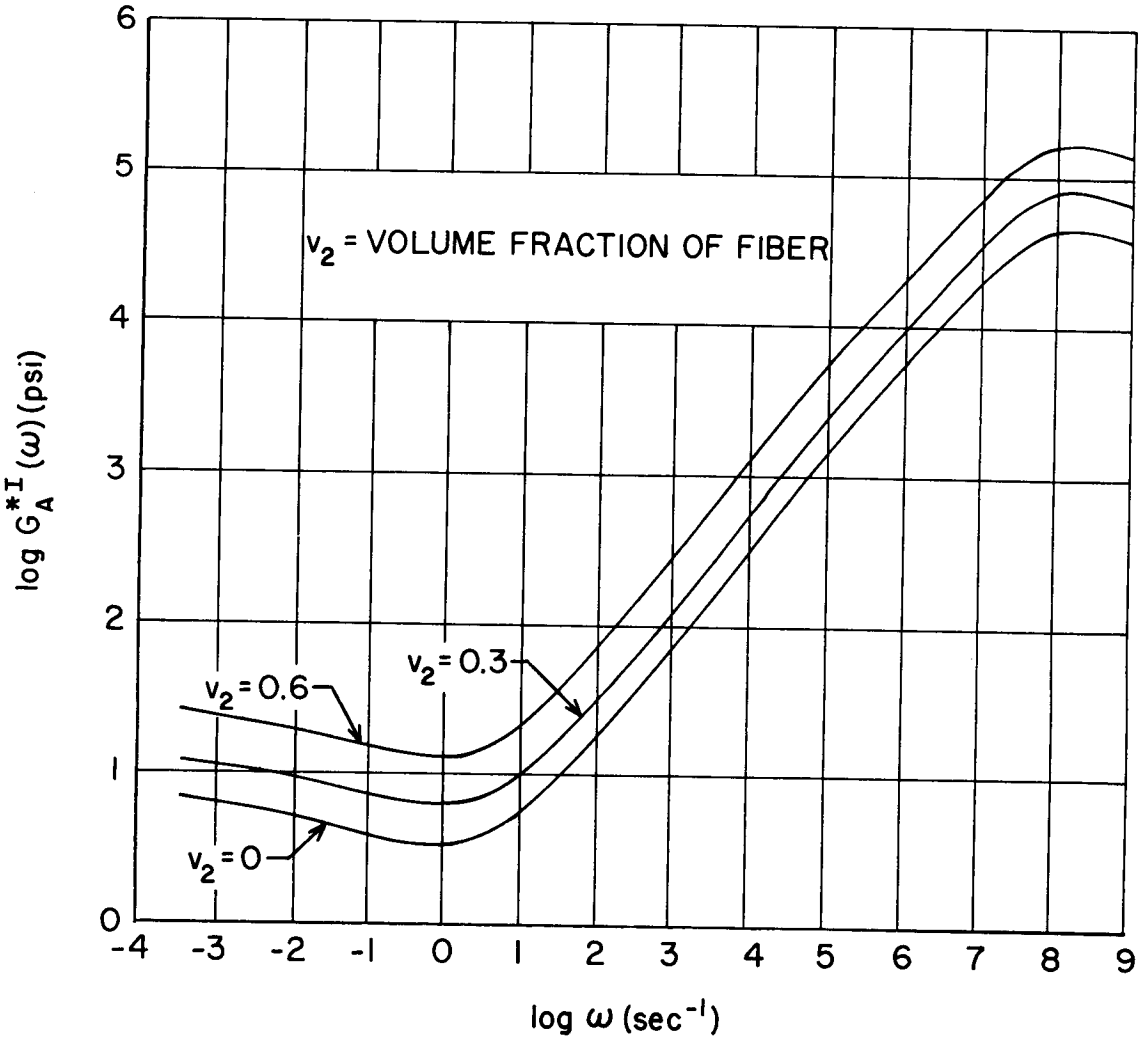


FIG. 4.5.2 - IMAGINARY PART OF COMPLEX AXIAL SHEAR MODULUS $G_A^{*I}(\omega)$, POLYISOBUTYLENE MATRIX, RIGID FIBERS

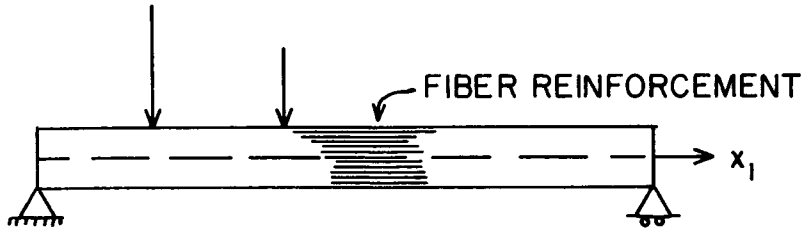


FIG. 4.6.1 - BENDING OF FIBER REINFORCED BEAM

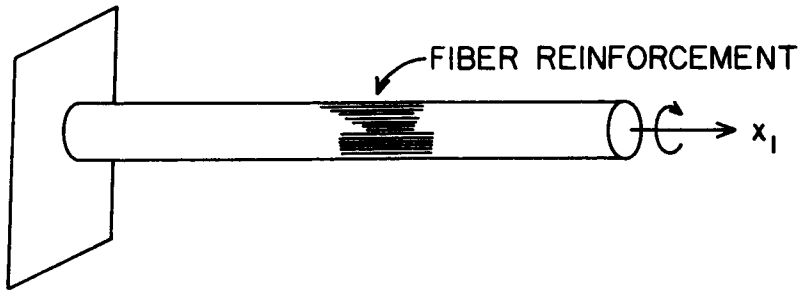


FIG. 4.6.2 - TORSION OF FIBER REINFORCED CYLINDER

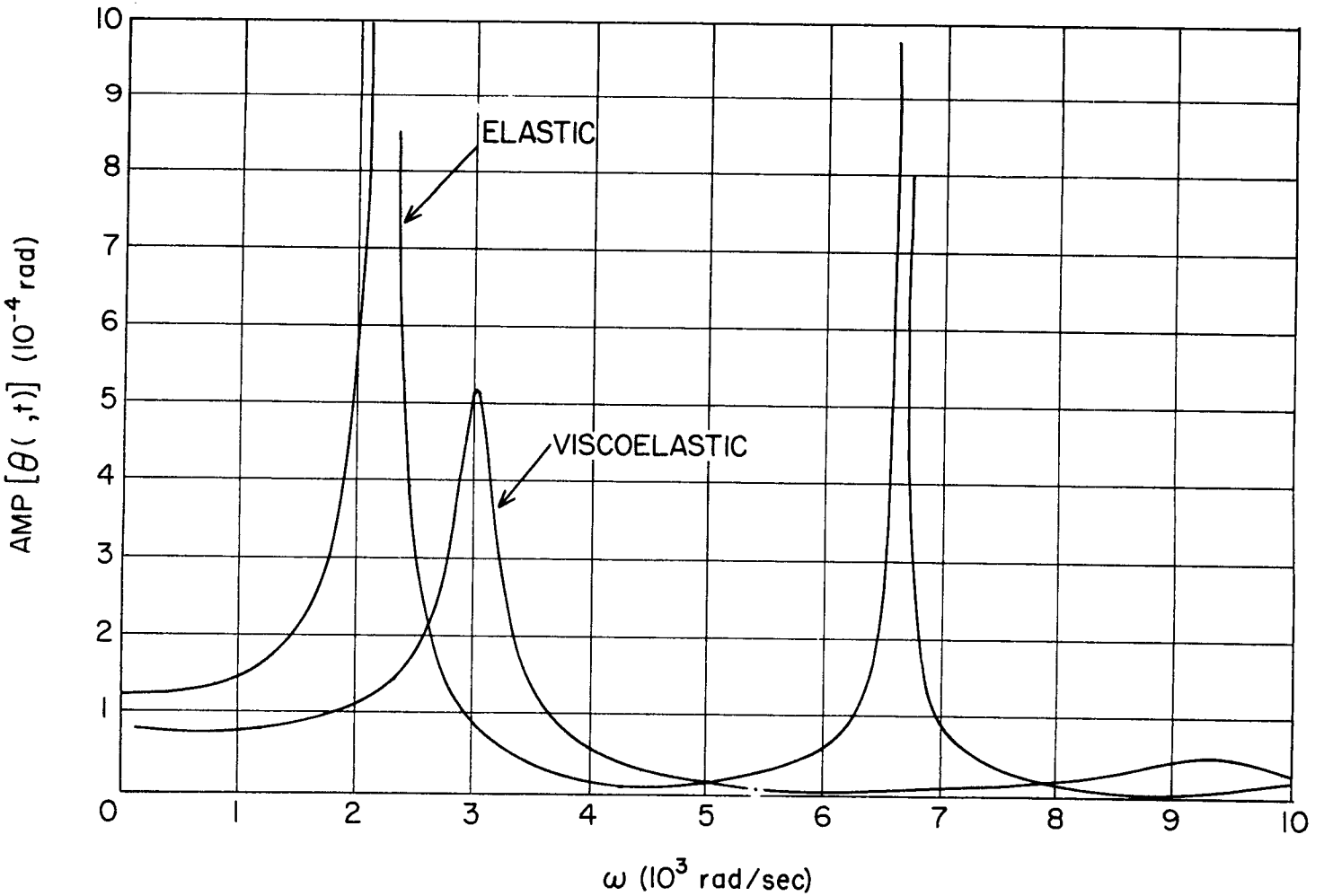


FIG. 4.6.3 - AMPLITUDE OF ANGLE OF TWIST IN TORSIONAL VIBRATIONS

5. CONDUCTION, DIELECTRICS AND MAGNETICS

INTRODUCTION

In the present part, we shall be concerned with steady state thermal and electrical conduction and magnetic and dielectric behavior of FRM. The reason for grouping these subjects together is that the problems involved are mathematically completely analogous, as will be shown below.

The structure of the theory to be developed in this part is identical to the structure of the theory for elastic behavior developed in part 3. Moreover, the problems to be solved are much simpler than the elastic problems since the governing equations in the present physical subjects are much simpler than elasticity equations.

At a certain stage of the development, it will be shown that the problems to be solved here are mathematically analogous to problems which have arisen in elastic axial shearing theory and this analogy will enable us to immediately convert axial shearing results into corresponding results for the present subject.

5.1 VECTORIAL AVERAGE THEOREMS

5.1.1 Average Gradient or Intensity Theorem

The vectorial average theorems to be derived in this chapter are the vector analogues of the tensorial average theorems which were given in chap. 3.1. The theorems are of general mathematical nature and do not presuppose any specific physical behavior.

Consider a two-phase body of volume V and with external surface S. The phase volumes are V_1 and V_2 occupying regions R_1 and R_2, respectively, and the phase interfaces are denoted S_{12}.

Define a continuous scalar function φ, called the potential which has the following properties

$$\varphi(S) = \varphi^{o} \qquad \text{(a)}$$

$$\varphi(\underline{x}) = \begin{cases} \varphi^{(1)}(\underline{x}) \\ \varphi^{(2)}(\underline{x}) \end{cases} \quad \text{in} \quad \begin{cases} R_1 \\ R_2 \end{cases} \qquad \text{(b)} \qquad (5.1.1)$$

$$\varphi^{(1)}(S_{12}) = \varphi^{(2)}(S_{12}) \qquad \text{(c)}$$

The gradient $\nabla\varphi$ is given by

$$(\nabla\varphi)_i = \varphi_{,i} \tag{5.1.2}$$

For reasons of convenience we shall define an <u>intensity vector</u> $\underline{H}$ as the negative of the gradient (5.1.2), thus

$$\begin{aligned} \underline{H}(\underline{x}) &= -\nabla\varphi \\ H_i(\underline{x}) &= -\varphi_{,i} \end{aligned} \tag{5.1.3}$$

The average of the intensity is defined by

$$\begin{aligned} \bar{\underline{H}} &= \frac{1}{V}\int_V \underline{H}\, dV = -\frac{1}{V}\int_V \nabla\varphi\, dV \\ \bar{H}_i &= \frac{1}{V}\int_V H_i dV = -\frac{1}{V}\int_V \varphi_{,i}\, dV \end{aligned} \tag{5.1.4}$$

The average intensity or gradient theorem asserts that

$$\boxed{\begin{aligned} \bar{\underline{H}} &= -\frac{1}{V}\int_S \overset{o}{\varphi}\,\underline{n}\, dS \\ \bar{H}_i &= -\frac{1}{V}\int_S \overset{o}{\varphi}\, n_i\, dS \end{aligned}} \tag{5.1.5}$$

The proof is immediate: The volume integral in (5.1.3) is converted to two phase region surface integrals by means of the divergence theorem. The surface integrals on the interface cancel because of (5.1.1c) and the reversal of the interface normal. Then (5.1.5) follows at once.

From (5.1.5) we obtain the following special result

<u>Corollary</u>

If

$$\boxed{\varphi(S) = \overset{o}{H}_i x_i = -\underline{\overset{o}{H}} \cdot \underline{x}} \tag{5.1.6}$$

where H_i^o are arbitrary constants and $\underline{H}^o$ is a constant vector with components $\underline{H}_i^o$, then

$$\boxed{\begin{aligned} \bar{\underline{H}} &= \underline{H}^o \\ \bar{H}_i &= H_i^o \end{aligned}} \qquad (5.1.7)$$

Proof:

Insert (5.1.6) into (5.1.5) to obtain

$$\bar{H}_i = \frac{1}{V} H_j^o \int_S x_j n_i dS$$

Now by the divergence theorem

$$\frac{1}{V} H_j^o \int_S x_j n_i dS = \frac{1}{V} H_j^o \int_V x_{j,i} dV = \frac{1}{V} H_j^o \delta_{ij} V = H_i^o$$

which proves the corollary.

Evidently, the theorems proved hold for any number of phases and they are also easily generalized to time dependent fields to hold for intensities and rates of intensity, as has been done for strain rates in par. 3.1.1.

By comparison with par. 3.1.1, it is noted that φ is the scalar analogue of the displacement vector u_i and the intensity H_i (or the negative of the gradient $\nabla\varphi$) is the vector analogue of the strain tensor ϵ_{ij}.

5.1.2 Average Flux Theorem

We define a vector $\underline{D}$ with components D_i which has the following properties

$$\underline{D}_n(S) = \underline{D} \cdot \underline{n} = D_i(S)n_i = D_n^o \qquad \text{(a)}$$

$$D_i(\underline{x}) = \begin{cases} D_i^{(1)}(\underline{x}) \\ D_i^{(2)}(\underline{x}) \end{cases} \quad \text{in} \quad \begin{cases} R_1 \\ R_2 \end{cases} \qquad \text{(b)}$$

$$\nabla \cdot \underline{D}^{(1)} = D_{i,i}^{(1)} = 0 \quad \text{in} \quad R_1$$

$$\nabla \cdot \underline{D}^{(2)} = D_{i,i}^{(2)} = 0 \quad \text{in} \quad R_2 \qquad \text{(c)}$$

$$D_n^{(1)}(S_{12}) = D_n^{(2)}(S_{12}) \qquad \text{(d)}$$

(5.1.8)

The vector $\underline{D}$ is called the <u>flux vector</u>.

Equ. (5.1.8a) states that the normal component of $\underline{D}$ is prescribed on S, (5.1.8d) expresses continuity of the normal component at the phase interface and (5.1.8c) states that the divergence of $\underline{D}$ vanishes everywhere.

The average flux vector $\overline{\underline{D}}$ is defined by

$$\overline{\underline{D}} = \frac{1}{V}\int_V \overline{\underline{D}}(\underline{x})\, dV$$

$$\overline{D}_i = \frac{1}{V}\int_V \overline{D}_i(\underline{x})\, dV \qquad (5.1.9)$$

The <u>average flux vector theorem</u> asserts that

$$\boxed{\begin{aligned} \bar{\underline{D}} &= \frac{1}{V}\int_S (\underline{D}\cdot\underline{n})\,\underline{x}\,dS = \frac{1}{V}\int_S D_n^o \underline{x}\,dS \\ \bar{D}_i &= \frac{1}{V}\int_S D_j n_j x_i\,dS = \frac{1}{V}\int_S D_n^o x_i\,dS \end{aligned}} \qquad (5.1.10)$$

Proof:

We use the identity

$$(D_j x_i)_{,j} = D_i$$

which is true for any vector satisfying (5.1.8c). Substituting this expression for D_i into (5.1.9) we have

$$\bar{D}_i = \frac{1}{V}\int_V (D_j x_i)_{,j}\,dV$$

We convert to surface integrals on the surfaces S_1 and S_2, enclosing V_1 and V_2. Then the integrals on the interface S_{12} cancel because of (5.1.8d). (The same problem with the normal arises here as in the proof of (3.9.29) and is resolved in the same way by proper consideration of the sign convention of the normal.) The result (5.1.10) then follows at once.

From (5.1.10) we have the following special result

<u>Corollary</u>

If

$$\boxed{D_n(S) = \underline{D}^o\cdot\underline{n} = D_i^o n_i} \qquad (5.1.11)$$

where $\underline{D}^o$ is a constant vector with constant components D_i^o, then

$$\boxed{\begin{aligned} \bar{\underline{D}} &= \underline{D}^o \\ \bar{D}_i &= D_i^o \end{aligned}} \qquad (5.1.12)$$

Proof:

Insert (5.1.11) into (5.1.10) to obtain

$$\bar{D}_i = \frac{1}{V} D_j \int_S x_i n_j dS$$

The surface integral which appears here is equal to $\delta_{ij}V$ as has been shown in the proof of (5.1.7). So (5.1.12) follows at once.

Again the theorems proven are easily extended to bodies with any number of phases and also to flux vector rates in the case of time dependence.

By comparison with par. 3.1.2 it is noted that the average flux theorems are the vector analogues of the average stress theorems. The flux vector takes the place of the stress tensor, the normal flux component takes the place of the traction and vanishing of the divergence is analogous to stress equilibrium without body forces.

5.1.3 Average Virtual Work Theorems

Suppose that in the two-phase body there is defined a scalar function

φ which has the properties (5.1.1) and a flux vector $\underline{D}$ which has the properties (5.1.8). The integral

$$Q = \int_V D_i H_i \, dV = - \int_V D_i \varphi_{,i} \, dV \tag{5.1.13}$$

is defined as the virtual work. It should be emphasized that the vectors $\underline{D}$ and $\underline{H}$ are at present unrelated and Q has no physical meaning.

The virtual work theorem asserts that

$$Q = \int_V D_i H_i \, dV = - \int_S D_n \varphi \, dS \tag{5.1.14}$$

To prove this theorem we note that at any point

$$D_i H_i = - D_i \varphi_{,i} = - (D_i \varphi)_{,i} + D_{i,i} \varphi = - (D_i \varphi)_{,i}$$

since $D_{i,i}$ vanishes according to (5.1.8c). We introduce this result into the left integral in (5.1.4), integrate separately over the phase volumes V_1 and V_2, and convert the volume integrals into surface integrals by use of the divergence theorem. We then have

$$Q = - \int_{S_1} D_i \, n_i \, \varphi \, dS - \int_{S_2} D_i \, n_i \, \varphi \, dS$$

The surface integrals on the interface cancel and the only remaining contribution is from the external surface S which gives the surface integral in (5.1.14) and thus proves the theorem.

Obviously, a similar result holds for any number of phases.

We now consider theorem (5.1.4) for the cases when either one of (5.1.6) or (5.1.11) is given on the boundary. We first split $\underline{D}$ and $\underline{H}$ at any point into average and deviation. Thus

$$\begin{aligned} D_i(\underline{x}) &= \bar{D}_i + D_i'(\underline{x}) \\ H_i(\underline{x}) &= \bar{H}_i + H_i'(\underline{x}) \end{aligned} \tag{5.1.15}$$

Consequently,

$$Q = \bar{D}_i \bar{H}_i V + \int_V D_i' H_i' dV \tag{5.1.16}$$

Evidently, it is always possible to express $\bar{H}_i$ in the form

$$\bar{H}_i = (\bar{H}_j x_j)_{,i}$$

Because of (5.1.3) it then follows that

$$\begin{aligned} H_i'(\underline{x}) &= -\varphi'_{,i} && \text{(a)} \\ \varphi'(\underline{x}) &= \varphi(\underline{x}) + \bar{H}_j x_j && \text{(b)} \end{aligned} \tag{5.1.17}$$

Since $\bar{D}_i$ in (5.1.15a) is a constant vector it trivially satisfies both (5.1.8c) and (5.1.8d). Since $D_i(\underline{x})$ also satisfies these conditions, the same follows for $D_i'(\underline{x})$. Thus

$$\begin{aligned} D'_{i,i} &= 0 && \text{(a)} \\ D'_n &\quad \text{continuous on } S_{12} && \text{(b)} \end{aligned} \tag{5.1.18}$$

In view of (5.1.17a) and (5.1.18) the theorem of virtual work applies to the second integral in (5.1.16)

$$\int_V D'_i H'_i \, dV = - \int_V D'_n \varphi' \, dS \tag{5.1.19}$$

The development up to this point is perfectly general. We now consider specifically the boundary conditions (5.1.6) and (5.1.11). In the first case we have from (5.1.7) and (5.1.17b) that

$$\varphi'(S) = 0$$

Consequently, (5.1.19) vanishes and only the first term remains in the right side of (5.1.16).

If (5.1.11) holds it follows from (5.1.12) and (5.1.15) that

$$D'_n(S) = 0$$

and again the integral (5.1.19) vanishes. We may thus summarize

$$\boxed{\int_V D_i H_i dV = \bar{D}_i H_i V = \begin{cases} \bar{D}_i H^o_i V & \text{(a)} \\ D^o_i \bar{H}_i V & \text{(b)} \end{cases}} \tag{5.1.20}$$

where (5.1.20a) refers to (5.1.6) and (5.1.20b) refers to (5.1.11).

5.2 STEADY STATE BOUNDARY VALUE PROBLEMS

5.2.1 Formulation

Suppose that the intensity $\underline{H}$ and the flux $\underline{D}$, defined in chap. 3.1, are connected by the general linear relation

$$D_i = \mu_{ij} H_j = - \mu_{ij} \varphi_{,j} \qquad \text{(a)}$$
$$\mu_{ij} = \mu_{ji} \qquad \text{(b)} \qquad (5.2.1)$$

where μ_{ij} may be space dependent. Equs. (5.2.1) may be regarded as a constitutive relation. The inverse of (5.2.1a) may be written as

$$H_i = \xi_{ij} D_j \qquad \text{(a)}$$
$$\xi_{ik} \mu_{kj} = \delta_{ij} \qquad \text{(b)} \qquad (5.2.2)$$

It follows from (5.2.16) and (5.2.2b) that

$$\xi_{ij} = \xi_{ji} \qquad (5.2.3)$$

It is seen that μ_{ij} is the analogue of the elastic moduli tensor C_{ijkl}, ξ_{ij} is the analogue of the elastic compliance tensor S_{ijkl} and (5.2.1a), (5.2.2a) are analogues of Hooke's law.

Inserting the extreme right of (5.2.1a) into the zero divergence

condition (5.1.8c) we obtain the differential equation

$$(\mu_{ij}\varphi_{,j})_{,i} = 0 \tag{5.2.4}$$

If μ_{ij} are constant, (5.2.4) reduces to

$$\mu_{ij}\varphi_{,ij} = 0 \tag{5.2.5}$$

In an orthotropic material

$$[\mu_{ij}] = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \qquad \text{(a)}$$

$$[\xi_{ij}] = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{bmatrix} \qquad \text{(b)} \qquad (5.2.6)$$

$$\xi_1 = \frac{1}{\mu_1} \qquad \xi_2 = \frac{1}{\mu_2} \qquad \xi_3 = \frac{1}{\mu_3} \qquad \text{(c)}$$

Since μ_{ij} is a symmetric tensor it can always be brought into the form (5.2.6a) by referring it to its principal axes.

If the material is transversely isotropic, with x_1 axis of symmetry, we have

$$\mu_1 = \mu_A \quad , \qquad \xi_1 = \xi_A = \frac{1}{\mu_A} \qquad \text{(a)}$$

$$\mu_2 = \mu_3 = \mu_T, \qquad \xi_2 = \xi_3 = \xi_T = \frac{1}{\mu_T} \qquad \text{(b)} \qquad (5.2.7)$$

If the material is completely isotropic we have

$$\mu_1 = \mu_2 = \mu_3 = \mu$$

and consequently

$$\mu_{ij} = \mu\delta_{ij} \qquad \text{(a)}$$

$$\xi_{ij} = \xi\delta_{ij} = \frac{1}{\mu}\delta_{ij} \qquad \text{(b)} \qquad (5.2.8)$$

Then (5.2.5) reduces to

$$\nabla^2\varphi = \varphi_{,ii} = 0 \qquad (5.2.9)$$

which is the Laplace equation.

Boundary conditions to be considered are

$$\varphi = \overset{o}{\varphi} \quad \text{on} \quad S_\varphi \qquad \text{(a)}$$

$$D_n = \overset{o}{D}_n \quad \text{on} \quad S_D \qquad \text{(b)} \qquad (5.2.10)$$

which imply that the potential φ is prescribed on part of the external boundary and the normal flux component on the remaining part of the boundary. Since the formulation is in terms of φ, (5.2.10b) should be expressed in terms of this quantity. It is easily seen that

$$D_n = -\mu_{ij}\varphi_{,j}n_i \qquad (5.2.11)$$

For a homogeneous or continuously non-homogeneous body (5.2.5)

or (5.2.4) subject to (5.2.10) define a unique potential function φ if μ_{ij} is positive definite. In the event that S_φ vanishes, i.e. (5.2.10b) is prescribed over the entire boundary, φ is unique except for an arbitrary additive constant.

If the body is two-phase or multiphase the formulation has to be modified. To be specific we consider two homogeneous phases. Then

$$\mu_{ij} = \begin{cases} \mu_{ij}^{(1)} & \text{in} \quad R_1 \\ \mu_{ij}^{(2)} & \text{in} \quad R_2 \end{cases} \tag{5.2.12}$$

The field equations (5.2.5) become

$$\begin{aligned} &\mu_{ij}^{(1)} \varphi_{,ij}^{(1)} = 0 \quad \text{in} \quad R_1 && \text{(a)} \\ &\mu_{ij}^{(2)} \varphi_{,ij}^{(2)} = 0 \quad \text{in} \quad R_2 && \text{(b)} \end{aligned} \tag{5.2.13}$$

The boundary conditions (5.2.10) with (5.2.11) may be left as they stand, with the understanding that the quantities in them must be given indices 1 or 2 in the boundary regions which belong to the phases 1 or 2.

At the phase interfaces there are now imposed the continuity conditions

$$\left.\begin{aligned} \varphi^{(1)} &= \varphi^{(2)} \\ D_n^{(1)} &= D_n^{(2)} \end{aligned}\right\} \quad \text{on } S_{12} \qquad \begin{aligned} &\text{(a)} \\ &\text{(b)} \end{aligned} \tag{5.2.14}$$

the last of which may be written in view of (5.2.11) as

$$\mu_{ij}^{(1)} \varphi_{,j}^{(1)} n_i = \mu_{ij}^{(2)} \varphi_{,j}^{(2)} n_i \tag{5.2.15}$$

The problem thus formulated also has a unique solution if $\mu_{ij}^{(1)}$ and $\mu_{ij}^{(2)}$ are positive definite. There is of course no difficulty whatsoever to generalize the formulation (5.2.10), (5.2.13-15) to bodies with any number of phases.

The mathematical formulation given above applies to the physical subjects of thermal and electrical conduction, electrostatics and magnetostatics. We list below the physical interpretation of the various quantities defined.

Physical Subject	φ	$\underline{H} = -\nabla\varphi$
Thermal conduction	temperature	temperature gradient
Electric conduction	electric potential	electric field intensity
Electrostatics	electric potential	electric field intensity
Magnetostatics	magnetic potential	magnetic field intensity
$\underline{D}$	μ_{ij}	ξ_{ij}
heat flux	heat conductivities	resistivities
current density	electric conductivities	resistivities
electric induction, electric displacement	dielectric constants, permittivities	
magnetic induction	magnetic permeabilities	

5.2.2 Elementary Solutions for Homogeneous Bodies of Arbitrary Shape. Homogeneous Boundary Conditions

We now derive solutions analogous to the ones derived in par. 3.2.2 for elastic bodies.

Consider a homogeneous body with conductivities u_{ij} and let it first be subjected to the boundary condition (5.1.6) which is given again below

$$\varphi(S) = - H_i^o x_i \tag{5.2.16}$$

We try a potential solution of the form

$$\varphi(\underline{x}) = - H_i^o x_i$$

It is seen that this function trivially satisfies (5.2.5). It obviously satisfies (5.2.16) and it is therefore the correct solution by the uniqueness theorem. Consequently, by (5.1.13) and (5.2.1)

$$\begin{aligned} H_i(\underline{x}) &= H_i^o && \text{(a)} \\ D_i(\underline{x}) &= u_{ij} H_j^o && \text{(b)} \end{aligned} \tag{5.2.17}$$

which are homogeneous vector fields. Therefore (5.2.16) is called: <u>homogeneous potential boundary condition.</u>

Now let the boundary condition (5.1.11) be imposed, which is given again below

$$D_n(S) = -\mu_{ij}\varphi_{,j}n_i = D_i^o n_i \tag{5.2.18}$$

We try a solution of the form

$$\varphi(\underline{x}) = -\xi_{ij}D_i^o x_i + \text{const} \tag{5.2.19}$$

where ξ_{ij} are defined by (5.2.2b). Since (5.2.19) is a linear function it satisfies (5.2.5) trivially. It follows from (5.2.19), (5.1.3), (5.2.1) and (5.2.2b) that

$$\begin{aligned} H_i(\underline{x}) &= \xi_{ij}D_j^o && \text{(a)} \\ D_i &= D_i^o && \text{(b)} \end{aligned} \tag{5.2.20}$$

Forming the normal component of (5.2.20b) it is seen that (5.2.18) is satisfied. Therefore (5.2.19) is the solution and (5.2.20) are the homogeneous intensity and flux fields within the body. Consequently, (5.2.18) is called: <u>Homogeneous flux boundary condition.</u>

5.3 EFFECTIVE CONDUCTIVITIES OF GENERAL COMPOSITES

5.3.1 Heterogeneous Bodies with Homogeneous Boundary Conditions

In the present chapter effective conductivities (EC) will be defined and discussed for general SH composite materials. We shall use the following expressions

φ - potential

$\underline{H}$ - intensity

$\underline{D}$ - flux

u_{ij} - conductivities

ξ_{ij} - resistivities

with the understanding that the theory applies to the subjects of thermal and electrical conduction, electrostatics and magnetostatics.

Let a composite body be subjected to the homogeneous potential boundary condition (5.2.16), which is here written out in detail

$$\varphi(S) = -(H_1^o x_1 + H_2^o x_2 + H_3^o x_3) \tag{5.3.1}$$

Because of the linearity of the governing equations the solution may be regarded as the sum of three different solutions, each of which is defined by the application of a single term of the right side of (5.3.1) on the boundary. Consider the case when

$$\varphi(S) = - x_1$$

and denote the corresponding field solution by $\varphi_{(1)}(\underline{x})$. If instead there is imposed on the boundary $\varphi(S) = - H_1^o x_1$ then by linearity the solution is $H_1^o \varphi_{(1)}(\underline{x})$. Defining, similarly, unit solutions $\varphi_{(2)}(\underline{x})$ and $\varphi_{(3)}(\underline{x})$ corresponding to $\varphi(S) = - x_2$, $\varphi(S) = - x_3$, respectively, the solution for the general case (5.3.1) can be written, by superposition, in the form

$$\varphi(\underline{x}) = H_1^o \varphi_{(1)}(\underline{x}) + H_2^o \varphi_{(2)}(\underline{x}) + H_3^o \varphi_{(3)}(\underline{x}) = H_k^o \varphi_{(k)}(\underline{x}) \tag{5.3.2}$$

It follows from (5.3.2), (5.1.3) and (5.2.1a) that the intensity and flux at any point have the forms

$$\begin{aligned} H_i(\underline{x}) &= - H_k^o \varphi_{(k),i} && \text{(a)} \\ D_i(\underline{x}) &= - \mu_{ik}(\underline{x}) H_j^o \varphi_{(j),k} && \text{(b)} \end{aligned} \tag{5.3.3}$$

where $\mu_{ik}(\underline{x})$ denotes the space variable (piecewise constant) conductivities of the heterogeneous body. Consequently the average of (5.3.3b) can be written in the form

$$\overline{D}_i = \mu^*_{ij} H_j^o \tag{5.3.4}$$

where

$$\mu^*_{ij} = - \overline{\mu_{ik}(\underline{x}) \varphi_{(j),k}} \tag{5.3.5}$$

In view of the average gradient or intensity theorem (5.1.6-7) it is seen that (5.3.4) can also be written in the form

$$\boxed{\bar{D}_i = \mu^*_{ij}\bar{H}_j} \tag{5.3.6}$$

There is thus a linear relation between the average flux and intensity components.

Next, the homogeneous flux boundary condition (5.2.16) is applied to the boundary. It follows by similar linearity arguments that

$$\bar{H}_i = \xi^*_{ij}D^o_j \tag{5.3.7}$$

where ξ^*_{ij} are some averages of field quantities. In the present case D^o_j are the flux averages because of theorem (5.1.11-12). Therefore (5.3.7) can be written in the form

$$\boxed{\bar{H}_i = \xi^*_{ij}\bar{D}_j} \tag{5.3.8}$$

It is to be noted that (5.3.6) and (5.3.8) are general results for any body, homogeneous or heterogeneous, under homogeneous boundary conditions. If the body is homogeneous (5.3.6) and (5.3.8) merely reduce to the results (5.2.15b), (5.2.23a), respectively.

It is easily shown that μ^*_{ij} and ξ^*_{ij} are symmetric tensors if μ_{ij} is symmetric. To see this we average (5.2.1a) and equate to (5.3.6). Thus

$$\mu^*_{ij}\bar{H}_j = \overline{\mu_{ij}H_j}$$

and also

$$\mu^*_{ji}\bar{H}_j = \overline{\mu_{ji}H_j}$$

It is seen that the right sides of these equations are equal because of $\mu_{ij} = \mu_{ji}$. Therefore

$$(\mu^*_{ij} - \mu^*_{ji})\,\bar{H}_j = 0$$

But $\bar{H}_j = H^o_j$ is an arbitrary vector and therefore

$$\boxed{\mu^*_{ij} = \mu^*_{ji}} \tag{5.3.9}$$

The proof for ξ^*_{ij} symmetry is evidently analogous. Thus

$$\boxed{\xi^*_{ij} = \xi^*_{ji}} \tag{5.3.10}$$

We now consider the volume integral

$$L = \frac{1}{2}\int_V D_i(\underline{x})H_i(\underline{x})dV \tag{5.3.11}$$

taken over a heterogeneous body. In view of (5.2.1-2) this integral can be expressed the the alternative forms

$$\begin{aligned} L &= \frac{1}{2}\int_V \mu_{ij}H_iH_j dV \\ L &= \frac{1}{2}\int_V \xi_{ij}D_iD_j dV \end{aligned} \tag{5.3.12}$$

The physical interpretation of (5.3.11-12) is rate of entropy production for thermal conduction, half the electric power for electric conduction, the electrostatic energy for electrostatics and the magnetostatic energy for magnetostatics. (There may be a different factor before the integral according to the physical units which are used). To be specific we shall refer to (5.3.11) simply as the energy integral . This integral is obviously the analogue of the elastic energy (strain or stress energy).

From the average theorem of virtual work, it follows at once that for homogeneous boundary conditions, (5.3.11) is rigorously given by

$$L = \frac{1}{2} \bar{D}_i \bar{H}_i V \tag{5.3.13}$$

If the boundary condition is (5.1.6) it follows from (5.1.20a) and (5.3.4) that

$$\boxed{L = \frac{1}{2} \mu^*_{ij} H^o_i H^o_j V} \tag{5.3.14}$$

If the boundary condition is (5.1.11) it follows from (5.1.20b) and (5.3.7) that

$$\boxed{L = \frac{1}{2} \xi^*_{ij} D^o_i D^o_j V} \tag{5.3.15}$$

5.3.2 <u>Statistically Homogeneous Composites</u>

The interpretation of the general results (5.3.4, (5.3.6-8) and

(5.3.14-15) for SH composites is exactly as in elasticity, par. 3.3.2. We first state the fundamental postulate of theory of conductivity of heterogeneous media: <u>The intensity and flux fields in a very large SH body, subjected to homogeneous boundary conditions are SH, except in a narrow boundary layer near the external surface</u>. It follows that the body averages which enter into the results obtained in par. 5.3.1 are now also RVE averages. Hence (5.3.6) and (5.3.8) become the effective constitutive relations, and μ^*_{ij} are the effective conductivities and ξ^*_{ij} are the effective resistivities. By the same arguments as given in par. 3.3.2 these tensors are reciprocal for a SH body. Thus

$$\boxed{\mu^*_{ik}\,\xi^*_{kj} = \delta_{ij}} \tag{5.3.16}$$

The effective physical constants μ^*_{ij} and ξ^*_{ij} may also, alternatively and equivalently, be defined by the energy expressions (5.3.14-15) For a SH body these expressions can also be interpreted as energy densities per unit volume RVE. We write

$$\boxed{\begin{aligned} M_H &= \frac{1}{2}\,\mu^*_{ij}\,\bar{H}_i\,\bar{H}_j \\ M_D &= \frac{1}{2}\,\xi^*_{ij}\,\bar{D}_i\,\bar{D}_j \end{aligned}} \tag{5.3.17}$$

These are the analogues of elastic strain energy and stress energy densities, respectively.

5.3.3 Effective Physical Constants in Terms of Phase Averages

We shall now derive expressions for μ^*_{ij} or ξ^*_{ij} of a two-phase body in terms of averages over a single phase. We decompose intensity and flux averages as follows:

$$\bar{H}_i = \bar{H}_i^{(1)} v_1 + \bar{H}_i^{(2)} v_2 \qquad \text{(a)}$$
$$\bar{D}_i = \bar{D}_i^{(1)} v_1 + \bar{D}_i^{(2)} v_2 \qquad \text{(b)} \qquad (5.3.18)$$

where the averages in the right sides of (5.3.18) are taken over phase volumes. For homogeneous phases we have

$$\bar{D}_i^{(1)} = \mu_{ij}^{(1)} \bar{H}_j^{(1)} \qquad \text{(a)}$$
$$\bar{D}_i^{(2)} = \mu_{ij}^{(2)} \bar{H}_j^{(2)} \qquad \text{(b)} \qquad (5.3.19)$$

$$\bar{H}_i^{(1)} = \xi_{ij}^{(1)} \bar{D}_j^{(1)} \qquad \text{(a)}$$
$$\bar{H}_i^{(2)} = \xi_{ij}^{(2)} \bar{D}_j^{(2)} \qquad \text{(b)} \qquad (5.3.20)$$

Consider the equations (5.3.6), (5.3.18) and (5.3.19). We eliminate from these the quantities $\bar{D}_i$, $\bar{D}_i^{(1)}$, $\bar{D}_i^{(2)}$ and $\bar{H}_i^{(1)}$. This leaves the equation

$$\mu^*_{ij} \bar{H}_j = \mu_{ij}^{(1)} \bar{H}_j + (\mu_{ij}^{(2)} - \mu_{ij}^{(1)}) \bar{H}_j^{(2)} v_2 \qquad (5.3.21)$$

If the body is subjected to the homogeneous potential boundary condition

(5.2.16) then $\overline{H}_i = H_i^o$. In this event we can also write the linear influence relation

$$\overline{H}_i^{(2)} = A_{ij}^{(2)} H_j^o \tag{5.3.22}$$

where $A_{ij}^{(2)}$ is an intensity average influence tensor for phase 2. Inserting (5.3.22) into (5.3.21) we obtain

$$\mu^*_{ij} - [\mu_{ij}^{(1)} + (\mu_{ik}^{(2)} - \mu_{ik}^{(1)}) A_{kj}^{(2)}] \quad H_j^o = 0$$

Since H_j^o is an arbitrary vector, each coefficient of H_j^o must vanish separately. Consequently

$$\mu^*_{ij} = \mu_{ij}^{(1)} + (\mu_{ik}^{(2)} - \mu_{ik}^{(1)}) A_{kj}^{(2)} v_2 \tag{5.3.23}$$

Evidently (5.3.23) remains valid if 1 and 2 are interchanged.

The same procedure may be repeated for fluxes and resistivities. The counterpart of (5.3.21) is then

$$\xi^*_{ij} \overline{D}_j = \xi_{ij}^{(1)} \overline{D}_j + (\xi_{ij}^{(2)} - \xi_{ij}^{(1)}) \overline{D}_j^{(2)} v_2 \tag{5.3.24}$$

For homogeneous flux boundary conditions (5.2.18) we have

$$\overline{D}_i^{(2)} = B_{ij}^{(2)} D_j^o \tag{5.3.25}$$

where $B_{ij}^{(2)}$ is a flux average influence tensor. Then (5.3.24) can be brought into the form

$$\xi^*_{ij} = \xi_{ij}^{(1)} + (\xi_{ik}^{(2)} - \xi_{ik}^{(1)}) B_{kj}^{(2)} v_2 \tag{5.3.26}$$

5.4 EFFECTIVE CONSTITUTIVE RELATIONS OF FIBER REINFORCED MATERIALS

5.4.1 Symmetry Reductions

We consider the implications of various symmetry properties for the general constitutive relations (5.3.6) and (5.3.8). Suppose that the material is fibrous and SH. The system of axes is as usual x_1 in generator direction of cylindrical phase regions or fibers, and x_2, x_3 in a transverse plane. It is clear that the x_2 x_3 plane is a plane of symmetry for the constitutive relations if it is also a plane of constitutive symmetry for the phase materials. From this it follows easily that

$$\boxed{\mu^*_{12} = \mu^*_{13} = 0} \tag{5.4.1}$$

To see this suppose that there is only an average intensity component $\bar{H}_1$ in x_1 direction while $\bar{H}_2$ and $\bar{H}_3$ vanish. In this event (5.3.6) assumes the form

$$\bar{D}_1 = \mu^*_{11} \bar{H}_1 \qquad \text{(a)}$$

$$\bar{D}_2 = \mu^*_{12} \bar{H}_1 \qquad \text{(b)} \qquad (5.4.2)$$

$$\bar{D}_3 = \mu^*_{13} \bar{H}_1 \qquad \text{(c)}$$

Let the coordinate system be rotated in the following fashion: the x_2 axis is a fixed axis of rotation while x_1 and x_3 are rotated counter clockwise by 180° to assume positions x_1' and x_3'. The direction cosines of the rotation are

$$\cos(x_1', x_1) = \ell_{11} = -1$$

$$\cos(x_2', x_2) = \ell_{22} = 1$$

$$\cos(x_3', x_3) = \ell_{33} = -1$$

and the rest of the direction cosines vanish. By the laws of vector transformation the vector $\underline{\bar{D}}$ in (5.4.2) transforms into a vector $\underline{\bar{D}}'$ with the components

$$[\underline{\bar{D}}'] = [-\bar{D}_1, \bar{D}_2, -\bar{D}_3] \tag{5.4.3}$$

in the new system, while the axial vector $[\underline{\bar{H}}] = [\bar{H}_1, 0, 0]$ in (5.4.2) transforms into

$$[\underline{\bar{H}}'] = [-\bar{H}_1, 0, 0] \tag{5.4.4}$$

Since $x_2\ x_3$ is a plane of symmetry the components of $\underline{\bar{D}}'$ and $\underline{\bar{H}}'$ must be related just as (5.4.2) It follows in particular from (5 4.3-4) and (5.4.2b) that

$$\bar{D}_2 = -\mu_{12}^* \bar{H}_1 \tag{5.4.5}$$

Comparison of (5.4 2b) and (5 4.5) reveals a contradiction which can only be resolved by letting μ^*_{12} vanish.

We may similarly perform a 180° rotation of x_1 and x_2 axes around the x_3 axis. This makes μ^*_{13} vanish by the same kind of argument. This completes the proof of (5.4 1).

Consequently for <u>any</u> SH fibrous material

$$\boxed{\begin{aligned} \bar{D}_1 &= \mu^*_{11}\,\bar{H}_1 \\ \bar{D}_2 &= \mu^*_{22}\,\bar{H}_2 + \mu^*_{23}\,\bar{H}_3 \\ \bar{D}_3 &= \mu^*_{23}\,\bar{H}_2 + \mu^*_{33}\,\bar{H}_3 \end{aligned}} \qquad \begin{aligned} &\text{(a)} \\ &\text{(b)} \\ &\text{(c)} \end{aligned} \qquad (5.4.6)$$

Obviously the reciprocal relation (5.3.8) will also be of the same form as (5.4.6).

The constitutive relations (5.4.6) are physically plausible. They imply that for a fibrous material an intensity component in x_1 direction does not induce fluxes in the transverse directions. There is thus no "Poisson" effect in the present case.

The relations (5.4.6) may be further reduced by referring the two dimensional $\mu^*_{\alpha\beta}$ tensor ($\alpha, \beta = 2,3$) in (5.4.6b,c) to its principal axes. In this event there will only remain the principal conductivities μ^*_2 and μ^*_3. This, however, is not very helpful since the values of $\mu^*_{\alpha\beta}$ must be known in order to find the principal axes, and these are precisely the quantities we wish to determine.

Next, suppose that the composite is geometrically orthotropic with respect to the x_1 x_2 x_3 system and that the phase materials are also orthotropic with respect to the same system of axes. Then the composite is macroscopically orthotropic and (5.4.6) reduce to

$$\bar{D}_1 = \mu^*_1 \bar{H}_1 \qquad \text{(a)}$$

$$\bar{D}_2 = \mu^*_2 \bar{H}_2 \qquad \text{(b)} \qquad (5\ 4.7)$$

$$\bar{D}_3 = \mu^*_3 \bar{H}_3 \qquad \text{(c)}$$

where

$$\mu^*_1 = \mu^*_{11} \qquad \mu^*_2 = \mu^*_{22} \qquad \mu^*_3 = \mu^*_{33} \qquad \text{(d)}$$

If the composite is macroscopically transversely isotropic then

$$\mu^*_2 = \mu^*_3 = \mu^*_T$$

to which we adjoin the notation

$$\mu^*_1 = \mu^*_A$$

Then (5.4.7) simplifies to

$$\bar{D}_1 = \mu^*_A \bar{H}_1 \qquad \text{(a)}$$

$$\bar{D}_2 = \mu^*_T \bar{H}_2 \qquad \text{(b)} \qquad (5.4\ 8)$$

$$\bar{D}_3 = \mu^*_T \bar{H}_3 \qquad \text{(c)}$$

Examples of orthotropic and transversely isotropic FRM have been discussed in chaps. 2.2 and 3.4. It is easily seen that in the present physical subject a geometrically square symmetric material is also transversely isotropic with respect to constitutive relations.

This completes the discussion of symmetry reduction of the macroscopic constitutive relations. The present simple situation should be contrasted with the complexities of elastic stress-strain relations, chap. 3.4.

5.4.2 Axial Conductivity

We proceed to establish a general formula for the axial conductivity μ^*_{11} of fibrous materials, of any transverse geometry. Let a cylindrical composite specimen be subjected to the homogeneous potential boundary condition

$$\varphi(S) = - H_1^o x_1 \tag{5.4.9}$$

Then by the average intensity theorem (5.1.7), and (5.4.1)

$$\bar{D}_1 = \mu^*_{11} H_1^o \tag{5.4.10}$$

The general formulation of the conduction problem of a two-phase body is contained in equs. (5.2.10), (5.2.13-15). We assume that for the present problem

$$\varphi^{(1)}(\underline{x}) = \varphi^{(2)}(\underline{x}) = -H_1^o x_1 \tag{5.4.11}$$

It is seen that (5.4.11) trivially satisfies (5.4.9), (5.2.13) and (5.2.14a) Thus the only remaining condition is continuity of normal flux component (5.2.14b) or (5.2.15) at the interfaces.

We recall that the phase interfaces are here cylindrical and consequently the interface normal $\underline{n}$ has the form

$$[\underline{n}] = [0,\ n_2,\ n_3]$$

Furthermore, we have from (5.4.11)

$$[\nabla\varphi^{(1)}] = [\nabla\varphi^{(2)}] = [H_1^o,\ 0,\ 0]$$

Therefore, (5.2.15) assumes the form

$$\mu_{\alpha 1}^{(1)} n_\alpha = \mu_{\alpha 1}^{(2)} n_\alpha \qquad \alpha = 2,3$$

This condition is evidently satisfied when

$$\begin{aligned} \mu_{12}^{(1)} &= \mu_{12}^{(2)} = 0 \\ \mu_{13}^{(1)} &= \mu_{13}^{(2)} = 0 \end{aligned} \tag{5.4.12}$$

i.e. when the phase constitutive relations have the very general form (5.4.6). (Strictly speaking, the equality to zero in (5.4.12) is not even necessary.) Consequently (5.4.11) is indeed the potential for a very general class of phases.

Now from (5.2.1), (5.4.11) and (5.4.12)

$$\bar{D}_1 = (\mu_{11}^{(1)} v_1 + \mu_{11}^{(2)} v_2) H_1^o$$

where v_1, v_2 are the volume fractions. Comparing this to (5.4.10) we have

$$\boxed{\mu_{11}^* = \mu_{11}^{(1)} v_1 + \mu_{11}^{(2)} v_2 = \overline{\mu_{11}}} \qquad (5.4.13)$$

For orthotropic, transversely isotropic or isotropic phases (5.4.13) reduces to

$$\boxed{\begin{aligned} \mu_{11}^* &= \bar{\mu}_1 && \text{(a)} \\ \mu_{11}^* &= \bar{\mu}_A && \text{(b)} \\ \mu_{11}^* &= \bar{\mu} && \text{(c)} \end{aligned}} \qquad (5.4.14)$$

respectively.

Evidently the axial resistivity ξ_{11}^* is given by the reciprocal of (5.4.13), in view of the form of (5.4.1a).

The present simple and general result is the analogue of the results (3.5.55), (3.5.60a) for the axial Young's modulus of a fibrous material in the case of equal phase Poisson's ratios.

5.5 TRANSVERSE CONDUCTION

5.5.1 Formulation and Axial Shearing-Transverse Conduction Analogy

To investigate transverse conduction of a fibrous or fiber reinforced material we impose upon a cylindrical specimen, fig. 2.1.1, the homogeneous potential boundary condition

$$\varphi(S) = -(H_2^o x_2 + H_3^o x_3) \tag{5.5.1}$$

It is assumed that the potential $\varphi(\underline{x})$ is not a function of x_1. Thus

$$\varphi^{(1)}(\underline{x}) = \varphi^{(1)}(x_2, x_3) \qquad \text{(a)}$$

$$\varphi^{(2)}(\underline{x}) = \varphi^{(2)}(x_2, x_3) \qquad \text{(b)} \tag{5.5.2}$$

Consequently the boundary condition (5.5.1) may be written

$$\varphi(C) = -(H_2^o x_2 + H_3^o x_3) \qquad \text{(a)}$$

$$\varphi(0, x_2, x_3) = \varphi(H, x_2, x_3) = -(H_2^o x_2 + H_3^o x_3) \qquad \text{(b)} \tag{5.5.3}$$

where C is the contour of any transverse section. Thus (5.5.3a) is a boundary condition for the curved cylinder surface and (5.5.3b) - for the terminal sections.

Let it be assumed that the two phases are transversely isotropic with axes of symmetry in x_1 direction. In this event, (see (5.2.7)) the only non-vanishing phase conductivities are

$$\mu_{11}^{(1)} = \mu_{A}^{(1)} , \qquad \mu_{22}^{(1)} = \mu_{33}^{(1)} = \mu_{T}^{(1)}$$

$$\mu_{11}^{(2)} = \mu_{A}^{(2)} , \qquad \mu_{22}^{(2)} = \mu_{33}^{(2)} = \mu_{T}^{(2)} \tag{5.5.4}$$

The phase potentials (5.5.3) must satisfy equations (5.2.13). In view of (5.5.2) and (5.5.4) we obtain simply

$$\nabla^2 \varphi^{(1)} = 0 \qquad \text{in } R_1 \qquad \text{(a)}$$

$$\nabla^2 \varphi^{(2)} = 0 \qquad \text{in } R_2 \qquad \text{(b)} \tag{5.5.5}$$

$$\nabla^2 = \frac{\partial^2}{\partial x_2^{\,2}} + \frac{\partial^2}{\partial x_3^{\,2}} \qquad \text{(c)}$$

The continuity condition (5.2.14a) is

$$\varphi^{(1)} = \varphi^{(2)} \qquad \text{on } C_{12} \tag{5.5.6}$$

where C_{12} is a transverse section through the phase interfaces.

In view of (5.5.2), (5.5.4), and the fact that the interface normal has no component in x_1 direction, (5.2.14b) reduces to

$$\mu_T^{(1)} (\varphi_{,2}^{(1)} n_2 + \varphi_{,3}^{(1)} n_3) = \mu_T^{(2)} (\varphi_{,2}^{(2)} n_2 + \varphi_{,3}^{(2)} n_3) \text{ on } C_{12} \tag{5.5.7}$$

It is seen that the parentheses in (5.5.7) contain normal derivatives. We adopt for simplicity the notation

$$\mu_T^{(1)} = \mu_1 \qquad \text{(a)}$$

$$\mu_T^{(2)} = \mu_2 \qquad \text{(b)} \tag{5.5.8}$$

Consequently (5.5.7) may be written in the form

$$\mu_1 \frac{\partial \varphi^{(1)}}{\partial n} = \mu_2 \frac{\partial \varphi^{(2)}}{\partial n} \qquad \text{on } C_{12} \tag{5.5.9}$$

The problem formulated by (5.5.2a), (5.5.5), (5.5.6) and (5.5.9) is a plane potential theory problem for a plane two-phase region. It has a unique solution if μ_1 and μ_2 in (5.5.8) are positive. (As they certainly are for physical reasons.)

It is seen that the boundary conditions (5.5.3b) cannot be satisfied by the present solution. However, for a cylinder which is very long in comparison to typical cross section dimension this is of no consequence.

We shall now show that there is complete mathematical analogy between the axial shearing formulation , par. 3.5.1 and the present transverse conduction formulation. We observe that the governing differential equations (3.5.39), (3.5.41) and (5.5.6) , (5.5.8) , and the boundary conditions (3.5.37) and (5.5.3a) have the same form. Comparison of all of these shows the mathematical equivalence of the following quantities

$$\begin{aligned} &G \longleftrightarrow \mu \qquad \text{i.e. } G_A \longleftrightarrow \mu_T \\ &2\epsilon^{o}_{1\alpha} \longleftrightarrow -H^{o}_{\alpha} \qquad \alpha = 2,3 \end{aligned} \tag{5.5.10}$$

Comparison of (3.5.33b,c) and (3.5.34b,c) with the conduction constitutive relations reveals the further equivalence

$$2\epsilon_{1\alpha}(x_2,x_3) \longleftrightarrow -H_\alpha(x_2,x_3) = \varphi_{,\alpha} \qquad \text{(a)}$$

$$\sigma_{12}(x_2,x_3) \longleftrightarrow D_\alpha(x_2,x_3) \qquad \text{(b)} \qquad (5.5.11)$$

The scheme (5.5.10-11) expresses the mathematical equivalence of the axial shearing and transverse conduction problems. If a solution for one of them is available the solution for the other can be immediately written down. We shall call this analogy the <u>axial shearing-transverse conduction analogy</u>.

This analogy has far reaching consequences for the computation and bounding of transverse conductivities of FRM. The effective axial shear modulus G_A^* is defined by

$$\bar{\sigma}_{1\alpha} = 2G_A^* \, \bar{\epsilon}_{1\alpha} \qquad (5.5.12)$$

while the effective transverse conductivity μ_T^* is defined by

$$\bar{D}_\alpha = \mu_T^* \, \bar{H}_\alpha \qquad (5.5.13)$$

It is seen that (5.5.12-13) are mathematically analogous in view of (5.5.10-11). Now G_A^* is a function of phase geometry and phase shear moduli only. Consequently μ_T^* is the <u>same</u> function of phase geometry and phase conductivities. Thus

$$\boxed{\begin{aligned} G_A^* &= F(G_1, G_2, \{g\}) \\ \mu_T^* &= F(\mu_1, \mu_2, \{g\}) \end{aligned}} \qquad (5.5.14)$$

This permits us to write down effective transverse conductivity expressions simply by replacement of phase shear moduli by transverse phase conductivities (for transversely isotropic phases) in effective axial shear modulus expressions.

It is not difficult to realize that the same analogy is valid not only for exact expressions for G_A^* but also for bounds on G_A^*. It is recalled that G_A^* bounds were obtained by application of elasticity theory extremum principles. If there are substituted into these extremum principles the special axial shearing forms (3.5.33-34), there are obtained extremum principles in terms of φ of the shearing formulation. (The reader may verify this.) All the G_A^* bounds can be obtained on the basis of these modified extremum principles. But because of axial shearing-transverse conduction analogy the modified extremum principles also provide bounds for μ_T^* and so the conclusion is reached that G_A^* bounds are transformed into μ_T^* bounds by replacement of phase shear moduli by phase transverse conductivities.

It is easily shown that the axial shearing transverse conduction analogy also remains valid for orthotropic phases.

5.5.2 Expressions and Bounds

On the basis of the analogy proved in par. 5.5.1 we can now transform all the G_A^* results into corresponding μ_T^* results.

From (3.5.111) we have for the composite cylinder assemblage

$$\boxed{\begin{aligned} \mu_T^* &= \mu_1 + \frac{v_2}{\dfrac{1}{\mu_2-\mu_1} + \dfrac{v_1}{2\mu_1}} && \text{(a)} \\ \mu_T^* &= \mu_1 \frac{\mu_1 v_1 + \mu_2 (1+v_2)}{\mu_1 (1+v_2) + \mu_2 v_1} && \text{(b)} \end{aligned}} \qquad (5.5.15)$$

where 1 denotes matrix and 2 denotes fibers. This result has been given in [5.1-2].

For dilute reinforcement of circular fibers we have from (3.5.131)

$$\mu_T^* = \mu_1 \left(1 + 2 \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} c\right) \qquad (5.5.16)$$

where

$$c = v_2 << 1$$

Bounds for any transversely isotropic fibrous material are provided by (3.6.63) and (3.7.87).

Elementary bounds

$$\frac{\overline{1}}{\mu} \le \mu_T^* \le \bar{\mu} \qquad (5.5.17)$$

Improved bounds

$$\boxed{\mu_{T(-)}^* = \mu_1 \frac{\mu_1 v_1 + \mu_2 (1+v_2)}{\mu_1 (1+v_2) + \mu_2 v_1}} \qquad \text{(a)}$$

$$\boxed{\mu^*_{T(+)} = \mu_2 \frac{\mu_2 v_2 + \mu_1(1+v_1)}{\mu_2(1+v_1) + \mu_1 v_2}} \qquad \text{(b)} \qquad (5.5.18)$$

The conduction analogue of a rigid phase is a phase with infinite (very large)conductivity. The conduction analogue of an empty phase is an insulating phase. Evidently the special bounds (3.6.30) are also applicable for transverse conductivity. Furthermore, all axial shearing numerical results can also be interpreted as corresponding results for transverse conductivity. (This has been pointed out in [5.3] for the case of periodic arrays).

Experimental values of thermal conductivities of FRM have been reported in [5.4] . There is not good agreement between these and the theoretical results obtained. In particular, experimental values reported are consistently below the lower bound (5.5.18a).

On the other hand, similar theoretical results for two phase isotropic media (see appendix) do agree very well with measured values of effective magnetic permittivity, dielectric constant and electrical conductivity of two phase media, [5.5]. It is consequently of importance to explain the disagreement in the thermal conduction case and to perform further experiments.

APPENDIX

Conductivity of Isotropic Composites

We consider a statistically isotropic composite which consists of two isotropic phases. If the only information available is phase conductivities μ_1, μ_2 and phase volume fractions v_1, v_2, then elementary bounds for μ^* of the composite are given by

$$\left(\frac{v_1}{\mu_1} + \frac{v_2}{\mu_2}\right)^{-1} \leq \mu^* \leq \mu_1 v_1 + \mu_2 v_2 \tag{1}$$

Wiener, [5.5].

Improved best, possible bounds, are

$$\boxed{\mu_1 + \frac{v_2}{\dfrac{1}{\mu_2-\mu_1} + \dfrac{v_1}{3\mu_1}} \leq \mu^* \leq \mu_2 + \frac{v_1}{\dfrac{1}{\mu_1-\mu_2} + \dfrac{v_2}{3\mu_2}}} \tag{2}$$

Hashin and Shtrikman, [5.6]. $\mu_2 > \mu_1$

The bounds apply to FRM with randomly oriented fibers as well as to isotropic particulate composites. In the latter case the lower bound (2) is appropriate as a composite spheres assemblage expression with matrix 1 and particles 2.

For further results and discussion see e.g. [5.7-8].

Table 5.5.1 Thermal Conductivities of Fiber and Matrix Materials

	Material	Thermal Conductivity Cal/cm-sec-^{o}K	Thermal Conductivity Btu-in/hr-ft^{2}-^{o}F
Fiber	E-Glass	0.00214	6.2
	Al_2O_3	0.08	232
	SiC	1.7	4930
Matrix	Magnesium	0.38	1090
	Aluminum	0.53	1520
	Epoxy (typical)	0.00045	1.3

6.1 UNCOUPLED THERMO-ELASTICITY

6.1.1 Formulation

We are concerned with an elastic body in which there are stresses, strains and temperature changes. As a consequence of the temperature changes there are expansions or contractions which affect the strains and stresses that would have been present under isothermal conditions.

The simplest constitutive relations for a generally anisotropic thermoelastic body are

$$\sigma_{ij} = S_{ijkl}\epsilon_{kl} + \alpha_{ij}\varphi \tag{6.1.1}$$

the inverse of which is

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} + \Gamma_{ij}\varphi \qquad \text{(a)}$$

where (6.1.2)

$$\Gamma_{ij} = - C_{ijkl}\,\alpha_{kl} \qquad \text{(b)}$$

Here φ is the temperature rise relative to a reference temperature which is arbitrarily chosen as zero, C_{ijkl} and S_{ijkl} are elastic moduli and compliances, respectively, at reference temperature, and α_{ij} are the thermal expansion coefficients. Because of the assumption that elastic properties remain unchanged by the temperature change, (6.1.1) cannot be expected to hold for large temperature changes.

It is seen from (6.1.1) that when $\sigma_{ij} = 0$, then

$$\epsilon_{ij} = \alpha_{ij}\,\varphi \tag{6.1.3}$$

It follows that α_{ij} are the stress free-strains per unit temperature change. Also, in view of strain symmetry

$$\alpha_{ij} = \alpha_{ij} \tag{6.1.4}$$

There are thus six expansion coefficients in the general anisotropic case. As in the case of conductivity, par. 5.2.1 , it is always possible to find a set of material axes (principal axes of α_{ij}) in which the mixed α_{ij} will vanish, and thus α_{ij} will be orthotropic.

If the material is elastically <u>and</u> thermally isotropic, (6.1.2) reduces to the following form

$$\sigma_{ij} = \lambda\,\epsilon_{kk}\,\delta_{ij} + 2G\epsilon_{ij} - 3K\alpha\,\varphi\delta_{ij} \tag{6.1.5}$$

Here λ and G are the usual Lamé and shear modulus, K is the bulk modulus and α is the isotropic expansion coefficient.

Equs. (6.1.5) may be inverted to obtain the isotropic form of (6.1.1) but the resulting expressions are inconvenient. Equivalent more convenient expressions are

$$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\,\sigma_{ij} - \nu\sigma_{kk}\right] + \alpha\,\varphi \qquad i = j \qquad \text{(a)}$$

$$\epsilon_{ij} = \frac{\sigma_{ij}}{2G} \qquad i \neq j \qquad \text{(b)} \tag{6.1.6}$$

If the material is elastically <u>and</u> thermally transversely isotropic, with x_1 axis of symmetry, then the elastic stress-strain laws are given by (3.4.86-87) while the thermal expansion matrix assumes the form

$$[\alpha_{ij}] = \begin{bmatrix} \alpha_A & 0 & 0 \\ 0 & \alpha_T & 0 \\ 0 & 0 & \alpha_T \end{bmatrix} \tag{6.1.7}$$

In this event the constitutive relations (6.1.2) for normal strains and stresses assume the form

$$\begin{aligned} \sigma_{11} &= n\epsilon_{11} + \ell\,(\epsilon_{22} + \epsilon_{33}) - (n\alpha_A + 2\ell\,\alpha_T)\,\varphi \\ \sigma_{22} &= \ell\,\epsilon_{11} + (k + G_T)\,\epsilon_{22} + (k - G_T)\,\epsilon_{33} - (\ell\alpha_A + 2k\alpha_T)\,\varphi \\ \sigma_{33} &= \ell\epsilon_{11} + (k - G_T)\,\epsilon_{22} + (k + G_T)\,\epsilon_{33} - (\ell\,\alpha_A + 2k\alpha_T)\,\varphi \end{aligned} \tag{6.1.8}$$

while (6.1.1) assume the form

$$\begin{aligned} \epsilon_{11} &= \frac{1}{E_A}\,\sigma_{11} - \frac{\nu_A}{E_A}\,(\sigma_{22} + \sigma_{33}) + \alpha_A\,\varphi \\ \epsilon_{22} &= -\frac{\nu_A}{E_A}\,\sigma_{11} + \frac{1}{E_T}\,\sigma_{22} - \frac{\nu_T}{E_T}\,\sigma_{33} + \alpha_T\,\varphi \\ \epsilon_{33} &= -\frac{\nu_A}{E_A}\,\sigma_{11} - \frac{\nu_T}{E_T}\,\sigma_{22} + \frac{1}{E_T}\,\sigma_{33} + \alpha_T\,\varphi \end{aligned} \tag{6.1.9}$$

The shear stress-strain relations remain, of course, as in isothermal elasticity

$$\sigma_{12} = 2G_A \epsilon_{12}$$

$$\sigma_{23} = 2G_T \epsilon_{23} \qquad (6.1.10)$$

$$\sigma_{13} = 2G_A \epsilon_{13}$$

Suppose that the body is in static equilibrium without body forces. Then

$$\sigma_{ij,j} = 0 \qquad (6.1.11)$$

We consider first a body which is homogeneous and isotropic. To obtain differential equations for displacements u_i and temperature φ, the strains in (6.1.5) are expressed in terms of displacement gradients and the resulting expressions are substituted into (6.1.11). We then obtain

$$(\lambda + G)\, u_{j,ji} + G\, u_{i,jj} - 3K\alpha\, \varphi_{,i} = 0 \qquad (6.1.12)$$

Another differential equation is needed for the temperature φ. It is the fundamental premise of uncoupled thermoelastic theory that the temperature φ may be determined from a heat conduction problem without consideration of mechanical deformation. We adopt this assumption and we furthermore assume steady state conduction. The problem of steady state conduction has been discussed in chap. 5.2; for an isotropic homogeneous body a general class of problems is formulated by (5.2.9-10). We thus regard φ in (6.1.12) as a known function

which has been found by solution of the heat conduction problem. Consequently, the term $-3K\alpha\varphi_{,i}$ in (6.1.12) is of the nature of a body force.

In the general anisotropic homogeneous case we obtain the differential equations by substitution of (6.1.1a) into (6.1.11). Using also (6.1.4) we find

$$C_{ijkl}\,(u_{k,lj} - \alpha_{kl}\,\varphi_{,j}) = 0 \tag{6.1.13}$$

and the heat conduction problem is now defined by (5.2.5) and (5.2.10).

Appropriate boundary conditions are prescribed tractions or displacements on the boundary S. An expression for tractions is provided by substitution of (6.1.2) or (6.1.5) into $T_i = \sigma_{ij}\,n_j$. We thus have for the generally anisotropic and isotropic cases, respectively

$$T_i = C_{ijkl}\,(u_{k,l} - \varphi\alpha_{kl})\,n_j \qquad \text{(a)}$$

$$T_i = (\lambda\,\epsilon_{kk} - 3K\alpha\varphi)\,n_i + 2G\epsilon_{ij}\,n_j \qquad \text{(b)} \tag{6.1.14}$$

A general form of boundary conditions is

$$u_i = u^o_i \qquad \text{on } S_u \qquad \text{(a)}$$

$$T_i = T^o_i \qquad \text{on } S_T \qquad \text{(b)} \tag{6.1.15}$$

where T_i in (6.1.15) is to be interpreted in the forms (6.1.14).

The thermoelastic problem thus formulated has a unique solution if the matrix of elastic moduli is positive definite. A proof for the isotropic body may be found in [4.6] , chap. 2. Proof for the anisotropic body is analogous.

Extension of the formulation to heterogeneous bodies consisting of homogeneous phases is immediate. As an example we consider a body consisting of two anisotropic phases. The phase differential equations are

$$C^{(1)}_{ijkl}\,(u^{(1)}_{kl,j} - \alpha^{(1)}_{kl}\,\varphi^{(1)}_{,j}) = 0 \qquad \text{(a)}$$

$$C^{(2)}_{ijkl}\,(u^{(2)}_{kl,j} - \alpha^{(2)}_{kl}\,\varphi^{(2)}_{,j}) = 0 \qquad \text{(b)} \qquad (6.1.16)$$

where 1 and 2 denote the phases. To the boundary conditions (6.1.15) there are now adjoined the interface conditions

$$\left.\begin{aligned} u^{(1)}_i &= u^{(2)}_i \qquad \text{(a)} \\ T^{(1)}_i &= T^{(2)}_i \qquad \text{(b)} \end{aligned}\right\} \quad \text{on } S_{12} \qquad (6.1.17)$$

where (6.1.17) are to be taken in the forms (6.1.14a).

Uniqueness for two or multiphase bodies is also easily proved, it being necessary that the matrix of elastic moduli be positive definite, everywhere.

The associated heat condition problem to determine $\varphi^{(1)}$ and $\varphi^{(2)}$ has been formulated in par. 5.2.1.

6.1.2 Elementary Solutions

We consider first a heat conduction problem whose solution will be needed later. Let a heterogeneous body, consisting of homogeneous anisotropic phases, be subjected to a constant boundary temperature.

$$\boxed{\varphi(S) = \varphi^{o}} \tag{6.1.18}$$

Then the temperature everywhere in the body is also φ^{o}. Thus

$$\boxed{\varphi(\underline{x}) = \varphi^{o}} \tag{6.1.19}$$

To prove this we note that (6.1.19) trivially satisfies the phase conduction differential equations (5.2.13). The boundary condition (6.1.18) and the continuity condition (5.2.14a) are obviously satisfied by (6.1.19). The heat flux vector associated with (6.1.19) vanishes everywhere and so (5.2.14b) is also satisfied; this completes the proof. Obviously, the result is valid for any number of phases.

Next we consider a homogeneous anisotropic body which is subjected to the temperature boundary condition (6.1.18) and the usual homogeneous displacement boundary condition

$$u_i(S) = \overset{o}{\varepsilon}_{ij}\, x_j \tag{6.1.20}$$

The solution of the thermoelastic problem under boundary conditions (6.1.18) and (6.1.20) is (6.1.19) and

$$u_i(\underline{x}) = \overset{o}{\varepsilon}_{ij}\, x_j \tag{6.1.21}$$

Proof: Obviously (6.1.18) is also the temperature solution for the present homogeneous body. The differential equations for the displacements are given by (6.1.13) and since the temperature φ^o is constant its gradient vanishes and so (6.1.13) reduce to the elasticity differential equations (3.2.18). Now (6.1.21) satisfies these equations and certainly also the boundary conditions (6.1.20). This completes the proof.

The strains in the body are homogeneous and are given by

$$\epsilon_{ij}(\underline{x}) = \epsilon^o_{ij} \tag{6.1.22}$$

and the homogeneous stresses follow from (6.1.2), (6.1.4) and (6.1.19) as

$$\sigma_{ij}(\underline{x}) = C_{ijkl}(\epsilon^o_{kl} - \alpha_{kl}\varphi^o) \tag{6.1.23}$$

Dually, the temperature (6.1.18) and homogeneous traction boundary conditions

$$T_i(S) = \sigma^o_{ij} n_j \tag{6.1.24}$$

are applied to the boundary. It is easily shown that the stresses and strains are again homogeneous and are given by

$$\sigma_{ij}(\underline{x}) = \sigma^o_{ij} \qquad \text{(a)}$$

$$\epsilon_{ij}(\underline{x}) = S_{ijkl}\sigma^o_{kl} + \alpha_{ij}\varphi^o \qquad \text{(b)} \tag{6.1.25}$$

The solutions (6.1.22-23) and (6.1.25) are the thermo-elastic extensions of the elastic solutions obtained in par. 3.2.2.

6.2 EFFECTIVE THERMO-ELASTIC STRESS-STRAIN RELATIONS

6.2.1 Establishment of Stress-Strain Relations

A statistically homogeneous heterogeneous body of volume V and with bounding surface S, which consists of any number of phases, is subjected to the boundary conditions (6.1.18) and (6.1.21) , which are here rewritten for reasons of convenience.

$$\varphi(S) = \varphi^{o} \qquad \text{(a)}$$

$$u_i(S) = \epsilon^{o}_{ij} x_j \qquad \text{(b)} \qquad (6.2.1)$$

The boundary conditions (6.2.1) are thermoelastic homogeneous boundary conditions since, as has been shown in par. 6.1.2, they produce uniform fields of temperature, strain and stress in homogeneous bodies of arbitrary shape. As has been done previously we postulate that <u>the stress and strain fields in a large SH heterogeneous body subjected to thermoelastic homogeneous boundary conditions are SH, except for a narrow boundary layer near the external surface</u>.

Evidently, the average strain theorems derived in par. (3.1.1) remain valid in the present case and therefore

$$\bar{\epsilon}_{ij} = \epsilon^{o}_{ij}$$

Since the thermoelastic problem is linear we may consider instead of (6.2.1) the two separate cases

$$\varphi(S) = 0 \qquad \text{(a)}$$

$$u_i(S) = \epsilon^o_{ij} x_j \qquad \text{(b)} \qquad (6.2.2)$$

$$\varphi(S) = \varphi^o \qquad \text{(a)}$$

$$u_i(S) = 0 \qquad \text{(b)} \qquad (6.2.3)$$

The superposition of the fields due to (6.2.2) and (6.2.3), respectively, yields the field due to (6.2.1). It follows from the result (6.1.19) that for (6.2.2) applied the temperature vanishes throughout the heterogeneous body. Consequently, the body under (6.2.2) applied is isothermally elastic and the theory of elastic heterogeneous bodies is valid. Therefore the average stress $\bar{\sigma}^I_{ij}$ for boundary conditions (6.2.2) is given by

$$\bar{\sigma}^I_{ij} = C^*_{ijkl}\, \epsilon^o_{kl} = C^*_{ijkl}\, \bar{\epsilon}_{kl} \qquad (6.2.4)$$

where C^*_{ijkl} are the effective elastic moduli at zero temperature (i.e. reference temperature).

Next we consider the boundary conditions (6.2.3). The physical situation expressed by these is heating with boundary deformation prevented by a bonded rigid enclosure. It is known from the result (6.1.19) that the temperature

throughout the heterogeneous body is uniform and equal to φ^o. It is our purpose here to show that the average stress, $\bar{\sigma}^{II}_{ij}$, when (6.2.3) is applied is proportional to φ^o.

To see this we set up the thermoelastic equations in this case. We assume for simplicity two phases only which are, however, generally anisotropic. We have from (6.1.16) the phase differential equations

$$C^{(1)}_{ijkl}\, u^{(1)}_{k,lj} = 0 \qquad \text{(a)}$$

$$C^{(2)}_{ijkl}\, u^{(2)}_{k,lj} = 0 \qquad \text{(b)} \qquad (6.2.5)$$

The phase interface continuity conditions (6.1.17) assume the form

$$u^{(1)}_i = u^{(2)}_i \qquad \text{(a)}$$

$$(C^{(2)}_{ijkl}\, u^{(2)}_{k,l} - C^{(1)}_{ijkl}\, u^{(1)}_{k,l})\, n_j = \varphi^o (C^{(2)}_{ijkl}\, \alpha^{(2)}_{kl} - C^{(1)}_{ijkl}\, \alpha^{(1)}_{kl})\, n_j \qquad \text{(b)} \qquad (6.2.6)$$

where (6.1.14a) has been used to obtain (6.2.6b).

Suppose that $\varphi^o = 1$ and denote the corresponding displacement solution, as defined by (6.2.5-6) and (6.2.3b), $w_i(\underline{x})$. It is easily seen that the vector field $\varphi^o w_i(\underline{x})$ satisfies equs. (6.2.5-6) and (6.2.4b) for $\varphi^o \neq 1$. Hence

$$u_i(\underline{x}) = \varphi^o w_i(\underline{x}) \qquad (6.2.7)$$

where $u_i(\underline{x})$ is to be interpreted as $u_i^{(1)}(\underline{x})$ and $u_i^{(2)}(\underline{x})$ in the different phases, respectively.

Now from (6.1.1b) and (6.2.7)

$$\sigma_{ij}(\underline{x}) = \varphi^o \left[C_{ijkl}(\underline{x})\, w_{k,l} + \Gamma_{ij}(\underline{x})\right] \tag{6.2.8}$$

where $C_{ijkl}(\underline{x})$ and $\Gamma_{ij}(\underline{x})$ denote the piecewise constant variation of those quantities throughout the heterogeneous body. Hence, the average of (6.2.8) may be expressed as

$$\bar{\sigma}_{ij}^{II} = \Gamma_{ij}^{*}\, \varphi^o \tag{6.2.9}$$

where

$$\Gamma_{ij}^{*} = \overline{C_{ijkl}\, w_{k,l}} + \bar{\Gamma}_{ij}$$

This proves our assertion that $\bar{\sigma}_{ij}^{II}$ is proportional to φ^o.

Superposition of (6.2.5) and (6.2.9) yields

$$\bar{\sigma}_{ij}^{I} + \bar{\sigma}_{ij}^{II} = \bar{\sigma}_{ij} = C_{ijkl}^{*}\, \epsilon_{kl}^{o} + \Gamma_{ij}^{*}\, \varphi^o \tag{6.2.10}$$

Since φ^o is uniform its average is certainly also φ^o. Thus (6.2.10) may be rewritten as

$$\boxed{\bar{\sigma}_{ij} = C_{ijkl}^{*}\, \bar{\epsilon}_{kl} + \Gamma_{ij}^{*}\, \bar{\varphi}} \tag{6.2.11}$$

all averages being body averages as well as RVE averages.

A dual set of thermoelastic homogeneous boundary conditions is provided by (6.1.18) and (6.1.24). Thus

$$\varphi\,(S) = \varphi^{o} \qquad \text{(a)}$$

$$T_i(S) = \sigma^{o}_{ij}\, n_j \qquad \text{(b)} \qquad (6.2.12)$$

By the same kind of reasoning which led to (6.2.10-11) it is easily proved that for (6.2.12) applied

$$\boxed{\begin{aligned} \bar{\epsilon}_{ij} &= S^{*}_{ijkl}\,\sigma^{o}_{kl} + \alpha^{*}_{ij}\,\varphi^{o} \qquad \text{(a)} \\ \bar{\epsilon}_{ij} &= S^{*}_{ijkl}\,\bar{\sigma}_{kl} + \alpha^{*}_{ij}\,\bar{\varphi} \qquad \text{(b)} \end{aligned}} \qquad (6.2.13)$$

where S^{*}_{ijkl} are the isothermal effective elastic compliances at zero temperature, α^{*}_{ij} are some constants expressed by field averages, $\bar{\sigma}_{ij}$ are the average stresses σ^{o}_{ij} and $\bar{\varphi}$ is the average of the uniform temperature φ^{o}, thus φ^{o} itself.

The constants α^{*}_{ij} are defined by analogy with (6.1.1b) as the <u>effective thermal expansion coefficients</u> (ETEC) of the heterogeneous body. We also adopt the usual assumption (compare par. 3.3.2) that equs. (6.2.11) and (6.2.13b) are identical for a SH body. Since C^{*}_{ijkl} and S^{*}_{ijkl} are reciprocal it follows easily by substitution of (6.2.13b) into (6.6.11) that

$$\Gamma^{*}_{ij} = -\,C^{*}_{ijkl}\,\alpha^{*}_{kl} \qquad (6.2.14)$$

which is the analogue of (6.1.2b).

In analogy with (6.1.3) we can interpret the α^*_{ij} as average stress free-average strains. To see this we consider (6.2.12) in the special case when $\sigma^o_{ij} = 0$. The resulting boundary conditions are

$$\varphi(S) = \varphi^o$$
$$T_i(S) = 0 \tag{6.2.15}$$

In this case the average stresses evidently vanish and we have from (6.2.13)

$$\bar{\epsilon}_{ij} = \alpha^*_{ij}\,\varphi^o = \alpha^*_{ij}\,\bar{\varphi} \tag{6.2.16}$$

which shows that α^*_{ij} are the average strains for a body with traction free boundary and unit temperature rise.

Also because of $\bar{\epsilon}_{ij}$ symmetry we have from (6.2.16) that

$$\boxed{\alpha^*_{ij} = \alpha^*_{ji}} \tag{6.2.17}$$

The present derivation of thermoelastic effective stress-strain relations has been carried out for two phase bodies but, evidently , similar relations are valid for heterogeneous bodies consisting of any number of phases.

Finally, it should be noted that relations as (6.2.10-11) and (6.2.13) are not restricted to SH bodies but are valid for any heterogeneous body. However, if the body is not SH C^*_{ijkl} and S^*_{ijkl} will not in general be reciprocal, and the relation (6.2.14) will not be valid and (6.2.10-11) and (6.2.13) will not be effective constitutive relations but simply linearity relations.

6.2.2. Relations Between Effective Elastic Properties and Effective Thermal Expansion Coefficients

In par. 6.2.1 effective thermal expansion coefficients have been defined in terms of field averages. It would seem at first sight that in order to obtain specific results for ETEC it would be necessary to find internal fields for specific internal geometries (such as, for example, the composite cylinder assemblage) and on the basis of these compute averages and thus obtain the α^*_{ij}. Fortunately this is not necessary. We shall derive in the present paragraph fundamental relations between effective elastic properties and ETEC which will enable us to determine the ETEC of any two phase body on the basis of its effective elastic properties, phase elastic properties and phase expansion coefficients.

The theory to be developed is based on a remarkable paper by Levin [6.1].

We consider a heterogeneous multiphase body which is subjected to constant temperature change on the boundary and to zero surface tractions, the internal fields in this case being denoted φ', ϵ'_{ij} and σ'_{ij}. Then

$$\varphi'(S) = \varphi^o = \varphi'(\underline{x}) \qquad \text{(a)}$$

$$T'_i(S) = 0 \qquad \text{(b)} \qquad (6.2.18)$$

In view of (6.1.1) and (6.1.19) the strains, stresses and temperature at any interior point are then related by

$$\epsilon'_{ij} = S_{ijkl}\,\sigma'_{kl} + \alpha_{ij}\,\varphi^o \qquad (6.2.19)$$

where S_{ijkl} are piecewise constant.

Next we consider the <u>same</u> heterogeneous body, subjected to zero surface temperature and homogeneous non-vanishing traction boundary conditions. The field variables for this case are written unprimed. Then

$$\varphi(S) = 0 \qquad \text{(a)}$$

$$T_i(S) = \sigma^o_{ij}\, n_j \qquad \text{(b)} \qquad (6.2.20)$$

In view of (6.2.20) and (6.1.19) the internal temperature vanishes everywhere. Therefore

$$\epsilon_{ij} = S_{ijkl}\, \sigma_{kl} \qquad (6.2.21)$$

which is the usual isothermal elasticity stress strain relation.

We note that by the average stress theorem, (3.1.35), we have, because of the homogeneous boundary conditions, (6.2.18b) and (6.2.20b), respectively.

$$\bar{\sigma}'_{ij} = 0 \qquad \text{(a)}$$

$$\bar{\sigma}_{ij} = \sigma^o_{ij} \qquad \text{(b)} \qquad (6.2.22)$$

By the average theorem of virtual work, (3.1.50), and by (6.2.22)

$$\int_V \sigma_{ij}\, \epsilon'_{ij}\, dV = \sigma^o_{ij}\, \bar{\epsilon}'_{ij}\, V \qquad \text{(a)}$$

$$\int_V \sigma'_{ij}\, \epsilon_{ij}\, dV = 0 \qquad \text{(b)} \qquad (6.2.23)$$

Insertion of (6.2.21) into (6.2.23b) yields the alternative form

$$\int_V S_{ijkl}\, \sigma_{ij}\, \sigma'_{kl}\, dV = 0 \tag{6.2.24}$$

In view of (6.2.18), $\bar{\epsilon}'_{ij}$ in (6.2.23a) can be replaced by (6.2.16). We also replace ϵ'_{ij} in the left side of (6.2.23a) by (6.2.19) and make use of (6.2.24). The resulting equation is

$$\int_V \sigma_{ij}\, \alpha_{ij}\, dV = \sigma^{o}_{ij}\, \alpha^{*}_{ij}\, V \tag{6.2.25}$$

Suppose that the body consists of M homogeneous phases, the expansion coefficients of the mth phase being $\alpha^{(m)}_{ij}$. Then

$$\int_V \sigma_{ij}\, \alpha_{ij}\, dV = \sum_m \alpha^{(m)}_{ij} \int_{V_m} \sigma_{ij}\, dV = \sum_m \alpha^{(m)}_{ij}\, \bar{\sigma}^{(m)}_{ij}\, V_m \tag{6.2.26}$$

where $\bar{\sigma}^{(m)}_{ij}$ is the average of σ_{ij} over the mth phase region, which has the volume V_m .

Now it should be recalled that σ_{ij} are stresses in an isothermal elastic body with traction boundary conditions (6.2.20b). By linearity

$$\bar{\sigma}^{(m)}_{ij} = B^{(m)}_{ijkl}\, \sigma^{o}_{kl} \tag{6.2.27}$$

where $B^{(m)}_{ijkl}$ is a stress-average influence tensor for the mth phase (compare a similar relation in (3.3.38)). We insert (6.2.27) into the extreme right of (6.2.26) and then the resulting expression into (6.2.25) . After rearrangement we obtain

$$[\alpha^*_{kl} - \sum_m \alpha^{(m)}_{ij} B^{(m)}_{ijkl} v_m] \sigma^o_{kl} = 0 \tag{6.2.28}$$

where v_m is the volume fraction of the mth phase. Since σ^o_{kl} is an arbitrary tensor the parentheses in (6.2.28) must all vanish. Interchanging subscripts we have

$$\sum_{m=1}^{M} \alpha^{(m)}_{kl} B^{(m)}_{klij} v_m = \alpha^*_{ij} \tag{6.2.29}$$

Next we consider the average $\bar{\epsilon}_{ij}$ which can be written in two ways. Firstly,

$$\bar{\epsilon}_{ij} = S^*_{ijkl} \sigma^o_{kl} \tag{6.2.30}$$

where S^*_{ijkl} are the effective elastic compliances . Secondly, in view of (6.2.21)

$$\bar{\epsilon}_{ij} = \frac{1}{V} \int_V S_{ijkl} \sigma_{kl} dV = \frac{1}{V} \sum_m S^{(m)}_{ijkl} \int_V \sigma_{kl} dV$$

$$= \sum_m S^{(m)}_{ijkl} \bar{\sigma}^{(m)}_{kl} v_m \tag{6.2.31}$$

We insert (6.2.27) into the last of (6.2.31) and equate to the right side of (6.2.30). Rearrangement yields an expression of the form (6.2.28) and again the coefficients of σ^o_{kl} must vanish. This yields

$$\sum_{m=1}^{M} S^{(m)}_{ijrs} B^{(m)}_{rskl} v_m = S^*_{ijkl} \tag{6.2.32}$$

Finally, we consider the average of σ_{ij} which may be written

$$\bar{\sigma}_{ij} = \sum_m \bar{\sigma}^{(m)}_{ij} v_m = \sum_m B^{(m)}_{ijkl} v_m \sigma^o_{kl} \tag{6.2.33}$$

where (6.2.27) has been used. Equating the extreme right of (6.2.33) to the right side of (6.2.22b) we find after rearrangement

$$\sum_{m=1}^{M} B^{(m)}_{ijkl} v_m = I_{ijkl} \tag{6.2.34}$$

where I_{ijkl} is the unit tensor defined by (3.2.7).

It is our purpose to establish a relation between S^*_{ijkl}, α^*_{ij} and known phase properties $S^{(m)}_{ijkl}$ and $\alpha^{(m)}_{ij}$. To this end the unknown influence coefficients $B^{(m)}_{ijkl}$ must be eliminated from equs. (6.2.29), (6.2.32) and (6.2.34). <u>This can be done only when the heterogeneous body consists of no more than two phases</u>. In that event there are two sets of influence coefficients $B^{(1)}_{ijkl}$ and $B^{(2)}_{ijkl}$ and we have three sets of equations (6.2.29), (6.2.32), (6.2.34) which serve to eliminate them. The result of the elimination can be written in the following equivalent forms

$$\boxed{\begin{aligned} \alpha^*_{ij} &= \alpha^{(1)}_{ij} + (\alpha^{(2)}_{kl} - \alpha^{(1)}_{kl})\, P_{klrs}\, (S^*_{rsij} - S^{(1)}_{rsij}) && \text{(a)} \\ \alpha^*_{ij} &= \alpha^{(2)}_{ij} + (\alpha^{(2)}_{kl} - \alpha^{(1)}_{kl})\, P_{klrs}\, (S^*_{rsij} - S^{(2)}_{rsij}) && \text{(b)} \\ \alpha^*_{ij} &= \bar{\alpha}_{ij} + (\alpha^{(2)}_{kl} - \alpha^{(1)}_{kl})\, P_{klrs}\, (S^*_{rsij} - \bar{S}_{rsij}) && \text{(c)} \end{aligned}} \qquad (6.2.35)$$

where the tensor P_{klrs} is defined by

$$\boxed{P_{klrs}\, (S^{(2)}_{rsij} - S^{(1)}_{rsij}) = I_{klij}} \qquad (6.2.36)$$

and $\bar{\alpha}_{ij}$, $\bar{S}_{rsij}$ are defined as usual by

$$\bar{\alpha}_{ij} = \alpha^{(1)}_{ij}\, v_1 + \alpha^{(2)}_{ij}\, v_2$$

$$\bar{S}_{rsij} = S^{(1)}_{rsij}\, v_1 + S^{(2)}_{rsij}\, v_2$$

We have thus obtained explicit expressions for α^*_{ij} in terms of S^*_{ijkl} and phase properties. A relation of type (6.2.35) was first given implicitly in [6.1] for a macroscopically anisotropic two phase body with isotropic phases. The present more general and explicit relation was given in [6.2] .

When there are more than two phases the number of influence coefficient sets $B^{(m)}_{ijkl}$ is larger than two but the number of equation sets available is still three. Therefore, elimination is not possible and a relation of type (6.2.35)

cannot be established. It is, however, possible in that case to bound α^*_{ij} by variational methods. This subject will not be discussed here and the interested reader is referred to [6.2].

It should be emphasized that (6.2.35) is a general result for a SH two phase body which is macroscopically anisotropic and consists of anisotropic phases. As an example for the use of (6.2.35) we consider the simple case of a statistically isotropic body, consisting of isotropic phases. In this event

$$\alpha^*_{ij} = \alpha^* \delta_{ij} \qquad \text{(a)}$$

$$\alpha^{(1)}_{ij} = \alpha_1 \delta_{ij}, \qquad \alpha^{(2)}_{ij} = \alpha_2 \delta_{ij} \qquad \text{(b)} \qquad (6.2.37)$$

$$\bar{\alpha}_{ij} = \bar{\alpha} \delta_{ij}, \qquad \bar{\alpha} = \alpha_1 v_1 + \alpha_2 v_2 \qquad \text{(c)}$$

Insertion of (6.2.37) into (6.2.35c) and contraction of the resulting equation over all free subscripts yields

$$\alpha^* = \bar{\alpha} + \frac{1}{3} (\alpha_2 - \alpha_1) P_{kkrs} (S^*_{rsii} - \bar{S}_{rsii}) \qquad (6.2.38)$$

Similar contraction of (6.2.36) yields

$$P_{kkrs} (S^{(2)}_{rsii} - S^{(1)}_{rsii}) = 3 \qquad (6.2.39)$$

It is not difficult to show that for an isotropic material

$$S_{ijkl} = -\frac{\lambda}{6GK}\,\delta_{ij}\,\delta_{kl} + \frac{1}{4G}\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) \tag{6.2.40}$$

Consequently

$$S_{rsii} = \frac{\delta_{rs}}{3K}$$

and therefore

$$S^{*}_{rsii} = \frac{\delta_{rs}}{3K^{*}} \qquad \bar{S}_{rsii} = \frac{\delta_{rs}}{3}\,\frac{\bar{1}}{K} \qquad \text{(a)}$$

(6.2.41)

$$S^{(2)}_{rsii} = \frac{\delta_{rs}}{3K_2} \qquad S^{(1)}_{rsii} = \frac{\delta_{rs}}{3K_1} \qquad \text{(b)}$$

where

$$\frac{\bar{1}}{K} = \frac{v_1}{K_1} + \frac{v_2}{K_2}$$

Insertion of (6.2.41a) into (6.2.38), of (6.2.41b) in (6.2.39), and combination of the resulting expressions, yields

$$\boxed{\alpha^{*} = \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{\frac{1}{K_2} - \frac{1}{K_1}}\left(\frac{1}{K^{*}} - \frac{\bar{1}}{K}\right)} \tag{6.2.42}$$

Forms equivalent to (6.2.42) are obtained by use of (6.2.35a,b) instead of (6.2.35c). These are

$$\alpha^* = \alpha_1 + \frac{\alpha_2 - \alpha_1}{\frac{1}{K_2} - \frac{1}{K_1}} \left(\frac{1}{K^*} - \frac{1}{K_1} \right) \qquad \text{(a)}$$

$$\alpha^* = \alpha_2 + \frac{\alpha_2 - \alpha_1}{\frac{1}{K_2} - \frac{1}{K_1}} \left(\frac{1}{K^*} - \frac{1}{K_2} \right) \qquad \text{(b)} \qquad (6.2.43)$$

The fundamental results (6.2.42-43) explicitly express α^* in terms of K^* and phase properties. Thus, if K^* is known either from theory or experiment, α^* is determined. These results have first been derived in [6.1] in different form and have also been independently derived in [6.2-4].

An interesting result is obtained if

$$K_1 \alpha_1 = K_2 \alpha_2 \qquad (6.2.44)$$

Such a relationship is approximately valid for many materials. In that event (6.2.42) reduces to

$$K^* \alpha^* = K_1 \alpha_1 = K_2 \alpha_2 \qquad (6.2.45)$$

It has been shown in [6.2] that a relation of type (6.2.45) is also valid for isotropic composites with an arbitrary number of phases, if the product $K\alpha$ is the same for all phases.

Furthermore, bounds on α^* can be easily obtained from (6.2.42) in terms of bounds on K^*. It is to be noted that the right side of (6.2.42) varies monotonically with K^*, the sense of the variation being determined by the sign of

$$\psi = \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1} \tag{6.2.46}$$

Consequently, we have

$$\psi > 0 \quad \text{i.e.} \quad (K_2 - K_1)(\alpha_2 - \alpha_1) < 0 \qquad \text{(a)}$$

$$\alpha^*_{(\pm)} = \bar{\alpha} + \psi \left(\frac{1}{K^*_{(\mp)}} - \frac{\bar{1}}{K} \right) \qquad \text{(b)}$$

$$\tag{6.2.47}$$

$$\psi < 0 \quad \text{i.e.} \quad (K_2 - K_1)(\alpha_2 - \alpha_1) > 0 \qquad \text{(c)}$$

$$\alpha^*_{(\pm)} = \bar{\alpha} + \psi \left(\frac{1}{K^*_{(\pm)}} - \frac{\bar{1}}{K} \right) \qquad \text{(d)}$$

Since stiffer materials have generally smaller thermal expansion coefficients than more compliant materials, the relation (6.2.47a) is mostly fulfilled. This is in accordance with the previously noted approximate validity of $K\alpha$ = const. for many materials.

Best possible K^* bounds for macroscopically isotropic two phase materials have been previously mentioned, (3.7.94). If these bounds are inserted into (6.2.46) there are obtained best possible bounds for α^*. These are

$$\boxed{\frac{4\,G_1\,(K_2 - K_1)(\alpha_2 - \alpha_1)\,v_1\,v_2}{3K_1\,K_2 + 4\,G_1\,\bar{K}} \le \alpha^* - \bar{\alpha} \le \frac{4\,G_2\,(K_2 - K_1)(\alpha_2 - \alpha_1)\,v_1\,v_2}{3K_1\,K_2 + 4\,G_2\,\bar{K}}} \tag{6.2.48}$$

provided that

$$K_2 \geq K_1 \qquad\qquad G_2 \geq G_1$$

and (6.2.47a) is valid.

The bounds (6.2.48) are valid for a FRM with randomly oriented fibers as well as for any statistically isotropic two phase material.

6.3 THERMO-ELASTIC FIBER REINFORCED MATERIALS

6.3.1 Effective Thermal Expansion Coefficients

It is recalled that equs. (6.2.35-36) determine the ETEC of any two phase composite in terms of its EEC and phase properties. It is a straightforward matter to exploit (6.2.35-36) to derive expressions for α^*_{ij} of a two phase FRM in terms of its EEC.

For the sake of simplicity it shall be assumed that the two phases are elastically and thermally isotropic while the composite itself is macroscopically transversely isotropic or square symmetric. Other cases, such as macroscopic orthotropy or transversely isotropic phases, can also be treated in a straightforward fashion; however, the necessary calculations are somewhat heavy.

We first proceed to simplify (6.2.35) for the case of isotropic phases, while the composite itself is macroscopically anisotropic in the most general sense.

Introduction of (6.2.37b,c) into (6.2.35c) yields

$$\alpha^*_{ij} = \bar{\alpha}\,\delta_{ij} + (\alpha_2 - \alpha_1)\,P_{kkrs}\,(S^*_{rsij} - \bar{S}_{rsij}) \tag{6.3.1}$$

From (6.2.36)

$$P_{kkrs}\,(S^{(2)}_{rsij} - S^{(1)}_{rsij}) = \delta_{ij} \tag{6.3.2}$$

Since the phases are assumed elastically isotropic, $S^{(1)}_{rsij}$ and $S^{(2)}_{rsij}$ are isotropic tensors of form (6.2.40) and therefore their difference which enters into (6.2.36) is also an isotropic tensor. It therefore follows from (6.2.36) that P_{klrs} is also isotropic and thus has the form

$$P_{klrs} = P' \delta_{kl} \delta_{rs} + P'' (\delta_{kr} \delta_{ls} + \delta_{ks} \delta_{lr})$$

where P' and P" are two scalars. Therefore

$$P_{kkrs} = (3P' + 2P'') \delta_{rs} \tag{6.3.3}$$

Insertion of (6.3.3) into (6.3.2) yields

$$3P' + 2P'' = \frac{1}{S^{(2)}_{rrii} - S^{(1)}_{rrii}} = \frac{1}{1/K_2 - 1/K_1} \tag{6.3.4}$$

the last equation following from (6.2.41b). Insertion of (6.3.3-4) into (6.3.1) and using the result

$$\bar{S}_{rrij} = \frac{1}{3} \frac{\bar{1}}{K} \delta_{ij}$$

which follows from (6.2.40) , it is found that

$$\boxed{\alpha^*_{ij} = \bar{\alpha} \delta_{ij} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1} (3S^*_{kkij} - \frac{\bar{1}}{K} \delta_{ij})} \tag{6.3.5}$$

Equ. (6.3.5) is valid for the ETEC of any two phase composite with isotropic phases.

For purposes of application to FRM it is now assumed that the composite is transversely isotropic or square symmetric. In that event

$$[\alpha^*_{ij}] = \begin{bmatrix} \alpha^*_A & 0 & 0 \\ 0 & \alpha^*_T & 0 \\ 0 & 0 & \alpha^*_T \end{bmatrix} \tag{6.3.6}$$

and also

$$S^*_{kk22} = S^*_{kk33} \tag{6.3.7}$$

Therefore, (6.3.5) reduces to

$$\alpha^*_A = \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1} \left(3\, S^*_{kk11} - \frac{\bar{1}}{K}\right) \qquad \text{(a)}$$

$$\alpha^*_T = \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1} \left(3\, S^*_{kk22} - \frac{\bar{1}}{K}\right) \qquad \text{(b)} \tag{6.3.8}$$

The EEC of a transversely isotropic or square symmetric FRM which appear in (6.3.8) are given in (3.4.78). We easily find

$$S^*_{kk11} = \frac{1-2\nu^*_A}{E^*_A} \qquad \text{(a)}$$

$$S^*_{kk22} = \frac{1}{2k^*} - \frac{(1-2\nu^*_A)\nu^*_A}{E^*_A} \qquad \text{(b)} \tag{6.3.9}$$

where (3.4.82-83) have been used to obtain (6.3.9b). Inserting (6.3.9) into (6.3.8) we find

$$\boxed{\begin{aligned} \alpha_A^* &= \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\left[\frac{3(1-2\nu_A^*)}{E_A^*} - \frac{\bar{1}}{K}\right] \quad \text{(a)} \\ \alpha_T^* &= \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\left[\frac{3}{2k^*} - \frac{3(1-2\nu_A^*)\nu_A^*}{E_A^*} - \frac{\bar{1}}{K}\right] \quad \text{(b)} \end{aligned}} \quad (*) \qquad (6.3.10)$$

It is recalled that K is the phases three dimensional bulk modulus and E_A^*, ν_A^* and k^* are the effective axial Young's modulus , axial Poisson's ratio and transverse bulk modulus respectively of the FRM. Expressions equivalent to (6.3.10) may be obtained by use of (6.2.35a,b) instead of (6.2.35c). These are

$$\alpha_A^* = \alpha_m + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\left[\frac{3(1-2\nu_A^*)}{E_A^*} - \frac{1}{K_m}\right] \quad \text{(a)}$$

(6.3.11)

$$\alpha_T^* = \alpha_m + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\left[\frac{3}{2k^*} - \frac{3(1-2\nu_A^*)\nu_A^*}{E_A^*} - \frac{1}{K_m}\right] \quad \text{(b)}$$

$m = 1,2$

These results were implicitly given in [6.1] and explicitly in [6.2](*).

(*) In [6.2] there is a misprint in the equation corresponding to (6.3.10a).

The results (6.3.10-11) determine the ETEC of any transversely isotropic or square symmetric fibrous or fiber reinforced material in terms of effective elastic properties. This is a rather unusual situation for it should have been expected that in order to find ETEC, it would have been necessary to find thermo-elastic fields in composites and then average them.

It is to be noted that because of the general relations between k^*, E_A^* and ν_A^*, par. 3.4.5, equs. (3.4.117-118), it is possible to express α_A^* and α_T^* in terms of one EEM only, e.g. k^*. This, however, results in complicated expressions which serve no particular advantage. Another possibility is to express (6.3.10) in terms of the group k^*, n^*, ℓ^* by use of (3.4.80-81). The resulting expressions are

$$\alpha_A^* = \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1} \left[3 \frac{k^* - \ell^*}{k^* n^* - \ell^{*2}} - \frac{\bar{1}}{K} \right] \qquad \text{(a)}$$

$$\alpha_T^* = \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1} \left[\frac{3}{2} \frac{n^* - \ell^*}{k^* n^* - \ell^{*2}} - \frac{\bar{1}}{K} \right] \qquad \text{(b)} \qquad (6.3.12)$$

These expressions have a certain theoretical advantage for bounding purposes as will be explained further below.

Any analytical, numerical or experimental results for k^*, E_A^* and ν_A^* may be introduced into (6.3.10) to find the associated α_A^* and α_T^*. We consider here a few cases of interest.

We recall the simple results of par. 3.5.2 for arbitrary fibrous materials, when the phase Poisson's ratios are equal. Introduction of (3.5.55-56 into (6.3.10) shows that it thus becomes possible to determine α_A^* for arbitrary FM, but not α_T^*. Thus

$$\boxed{\alpha_A^* = \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\left[\frac{3(1-2\nu)}{\bar{E}} - \frac{\bar{1}}{K}\right]} \qquad (6.3.13)$$

when $\nu_2 = \nu_1$, <u>for arbitrary fibrous geometry.</u>

Next, we recall that composite cylinder assemblage analysis of par. 3.5.3 yielded rigorous closed form results for E_A^*, ν_A^* and k^*, (3.5.96-97), (3.5.91). <u>Introduction of these results into (6.3.10) therefore gives the ETEC of the composite cylinder assemblage model</u>. Thus the ETEC of the composite cylinder assemblage may be written as

$$\boxed{\begin{aligned} \alpha_{Ac}^* &= \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\left[\frac{3(1-2\nu_{Ac}^*)}{E_{Ac}^*} - \frac{\bar{1}}{K}\right] && \text{(a)} \\ \alpha_{Tc}^* &= \bar{\alpha} + \frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\left[\frac{3}{2k_c^*} - \frac{3(1-2\nu_{Ac}^*)\nu_{Ac}^*}{E_{Ac}^*} - \frac{\bar{1}}{K}\right] && \text{(b)} \end{aligned}} \qquad (6.3.14)$$

Equivalent results have already been obtained previously, [6.5], by direct composite cylinder analysis, before the general results (3.6.10) were known.

Fig. 6.3.1 shows plots of α_A^* and α_T^* on the basis of the composite cylinder assemblage model, for a typical glass-epoxy FRM.

It will now be shown that the results (6.3.14) can be used to obtain best possible bounds, in terms of phase properties and volume fractions, for any transversely isotropic two phase fibrous or fiber reinforced material. To see this it is advantageous to initially consider the ETEC expressions in the form (6.3.12) and to express in these ℓ^* and n^* in terms of k^* by use of the general relations (3.4.112), par. 3.4.5. Then the terms containing k^*, ℓ^* and n^* in the brackets assume in each case the form

$$\frac{Ak^* + B}{Ck^* + D}$$

where A, B, C, D are some constants which are expressed in terms of phase properties and phase volume fractions. Now it follows by differentiation that the above given expression is a monotonic function of k^*. It therefore follows that (6.3.12) are monotonic functions of k^* and since (6.3.10) are the equivalent of (6.3.12), the former are also monotonic functions of k^*.

We now recall the best possible bounds (3.7.68) for k^*, each of these bounds also being a composite cylinder assemblage results with associated E_A^* and ν_A^*. It follows that if a set of k^*, E_A^* and ν_A^* for a composite cylinder assemblage is inserted into (6.3.10) then the resulting expressions (6.3.14) are best possible bounds for the ETEC of any transversely isotropic fibrous or fiber reinforced material whose phase properties and phase volume fractions are the same as those of the assemblage. Thus, best possible bounds are defined by (6.3.14) in following fashion: Best possible bounds for the ETEC of transversely

isotropic FM or FRM consisting of isotropic phases 1 and 2 with volume fractions v_1, v_2 are given by (3.6.14) evaluated, (a) for a composite cylinder assemblage in which 1 is matrix and 2 is fibers, (b) for an assemblage in which 2 is matrix and 1 is fibers, the phase volume fractions of both assemblages being the same.

It is possible to give a complex analytical criterion as to which of the bounds are lower or upper, but it is probably best to ascertain the nature of the bounds on the basis of the numerical results obtained. An example for this procedure is shown for a glass-epoxy composite in fig. 3.6.2 where lower bounds correspond to an assemblage with glass matrix and epoxy fibers and upper bounds are the composite cylinder assemblage results of fig. 3.6.1.

6.3.2 Internal Stresses

The unusual aspect of theory of ETEC as developed above is the successful avoidance of the problem of thermo-elastic field computation in composites. This in contrast to theory of other effective physical properties.

If it is desired to obtain some information about internal stresses due to temperature changes, the thermoelastic problem of the composite has to be considered in full detail. It is recalled that the general problem of two phase composites has been formulated in par. 6.1.1. It is our present purpose to consider the problem, as was done in par. 3.5.1 for isothermal elastic fibrous or fiber reinforced specimens, with the aim of reducing the three dimensional formulation to a two dimensional one.

Consider a long cylindrical fibrous or fiber reinforced specimen, fig. 2.1.1, whose phases are thermo-elastically transversely isotropic about the x_1 direction. The specimen is subjected to the previously considered boundary conditions

$$\varphi(S) = \varphi^o = \text{const.} \qquad \text{(a)}$$

$$u_i(S) = \epsilon^o_{ij}\, x_j \qquad \text{(b)} \qquad (6.3.15)$$

or

$$T_i(S) = \sigma^o_{ij}\, n_j \qquad \text{(c)}$$

Proceeding as in par. 6.2.1, the boundary conditions (6.3.15a,b) are split into (6.2.2) and (6.2.3) which are applied separately. Now when (6.2.2) is applied the temperature vanishes throughout the specimen by (6.1.18-19). Thus (6.2.2) lead to an isothermal problem which has been fully considered in par. 3.5.1. Consequently, it is sufficient to consider the problem

$$\varphi(S) = \varphi^o = \text{const.} \qquad \text{(a)}$$

$$(6.3.16)$$

$$u_i(S) = 0 \qquad \text{(b)}$$

and superpose on the solution the isothermal solution under (6.3.15b).

Similarly the problem (6.3.15a,c) may be split into

$$\varphi(S) = 0 \qquad \text{(a)}$$

$$(6.3.17)$$

$$T_i(S) = \sigma^o_{ij}\, n_j \qquad \text{(b)}$$

$$\varphi(S) = \varphi^{o} = \text{const.} \qquad \text{(a)}$$

$$T_i(S) = 0 \qquad \text{(b)} \qquad (6.3.18)$$

Again, (6.3.17a,b) is an isothermal elastic problem which has been fully discussed in par. 3.5.1 and so it is only necessary to consider (6.3.18).

We consider initially the problem (6.3.16) . Adopting the semi-inverse approach of par. 3.5.1 the displacements are assumed to be of form (3.5.9). Since, however, in the present case $u_1(S)$ vanishes, because of (6.3.16b), it follows that ϵ^{o}_{11} in (3.5.9) vanishes. The internal displacements and strains thus assume the plane strain form

$$u_1(\underline{x}) = 0 \qquad \text{(a)}$$

$$u_2(\underline{x}) = u_2(x_2, x_3) \qquad \text{(b)} \qquad (6.3.19)$$

$$u_3(\underline{x}) = u_3(x_2, x_3) \qquad \text{(c)}$$

$$\epsilon_{\alpha\beta}(\underline{x}) = \epsilon_{\alpha\beta}(x_2, x_3) \qquad \alpha, \beta = 2, 3 \qquad \text{(a)}$$

$$\epsilon_{11} = \epsilon_{12} = \epsilon_{13} = 0 \qquad \text{(b)} \qquad (6.3.20)$$

It is furthermore recalled that because of (6.1.18-19) and (6.3.16a) the temperature φ is constant and equal to φ^{o} throughout the specimen. Using this result together with the strains (6.3.20) in the transversely isotropic stress-strain law (6.1.8), (6.1.10) , we have for the internal stresses

$$\sigma_{11}(x_2,x_3) = \ell\,(\epsilon_{22} + \epsilon_{33}) - (n\alpha_A + 2\ell\,\alpha_T)\,\varphi^o \qquad \text{(a)}$$

$$\sigma_{\alpha\beta}(x_2,x_3) = (k-G_T)\,\epsilon_{\gamma\gamma}\,\delta_{\alpha\beta} + 2G_T\,\epsilon_{\alpha\beta} - (\ell\alpha_A + 2k\alpha_T)\varphi^o\delta_{\alpha\beta} \qquad \text{(b)} \qquad (6.3.21)$$

$$\sigma_{12} = \sigma_{13} = 0 \qquad \text{(c)}$$

where elastic moduli and thermal expansion coefficients assume different values in each phase. An interesting consequence of (6.3.21) is that a uniform temperature rise does not produce axial shear but does produce transverse shear.

Evidently, the equilibrium equations (3.5.13) remain unchanged. Insertion of (6.3.21) into (3.5.13) leads again to the differential equations (3.5.14) which are rewritten

$$k_1\,u^{(1)}_{\beta,\beta\alpha} + G_{T1}\,u^{(1)}_{\alpha,\beta\beta} = 0 \qquad \text{in } R_1 \qquad \text{(a)}$$

$$(6.3.22)$$

$$k_2\,u^{(2)}_{\beta,\beta\alpha} + G_{T2}\,u^{(2)}_{\alpha,\beta\beta} = 0 \qquad \text{in } R_2 \qquad \text{(b)}$$

The displacements must satisfy continuity at interfaces. Thus

$$u^{(1)}_{\alpha} = u^{(2)}_{\alpha} \qquad \text{on } C_{12} \qquad (6.3.23)$$

Traction continuity requirement leads to

$$[(k_2 - G_{T2})\, u^{(2)}_{\beta,\beta} - (k_1 - G_{T1})\, u^{(1)}_{\beta,\beta}]\, n_\alpha + [G_{T2}(u^{(2)}_{\alpha,\beta} + u^{(2)}_{\beta,\alpha}) -$$

$$- G_{T1}(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha})]\, n_\beta =$$

$$= [(\ell_1 \alpha_{A1} + 2k_1 \alpha_{T1}) - (\ell_2 \alpha_{T2} + 2k_2 \alpha_{T2})]\, \varphi^o n_\alpha \quad \text{on } C_{12} \qquad (6.3.24)$$

The problem may thus be summarized as follows: find the plane displacements $u^{(1)}_\alpha(x_2, x_3)$, $u^{(2)}_\alpha(x_2, x_3)$ which satisfy the differential equations (6.3.22), the boundary conditions

$$u_\alpha(C) = 0 \qquad (6.3.25)$$

and the continuity conditions (6.3.23-24)

It is easily seen that the mixed problem

$$\varphi(S) = \varphi^o = \text{const.} \qquad \text{(a)}$$

$$u_1(S) = 0 \qquad \text{(b)} \qquad (6.3.26)$$

$$T_\alpha(S) = \sigma^o_{\alpha\beta}\, n_\beta \qquad \text{(c)}$$

also falls under the present category. In this case the isothermal problem is (6.3.26) with $\varphi^o = 0$, which is a plane strain problem to be solved by the method given in par. 3.5.1. The remaining thermo-elastic problem is (6.3.26) with $T_\alpha(S) = 0$. The formulation of this problem is the same as the previously considered thermo-elastic problem with (6.3.25) replaced by

$$T_\alpha(C) = 0 \qquad (6.3.27)$$

Next we consider the problem (6.3.18). Again the displacements are assumed to be of form (3.5.9), but in the present case there is no plane strain condition, so ϵ^{o}_{11} does not vanish and is unknown. Consequently, strains have the form (3.5.10) and use of (6.1.8), (6.1.10) leads to the following internal stresses

$$\sigma_{11}(x_2,x_3) = n\,\epsilon^{o}_{11} + \ell\,(\epsilon_{22} + \epsilon_{33}) - (n\alpha_A + 2\ell\,\alpha_T)\,\varphi^{o} \qquad \text{(a)}$$

$$\sigma_{\alpha\beta}(x_2,x_3) = [\ell\,\epsilon^{o}_{11} + (k-G_T)\,\epsilon_{\gamma\gamma}]\,\delta_{\alpha\beta} + 2G_T\,\epsilon_{\alpha\beta} - (\ell\alpha_A + 2k\alpha_T)\varphi^{o}\delta_{\alpha\beta} \qquad \text{(b)} \qquad (6.3.28)$$

$$\sigma_{12} = \sigma_{13} = 0 \qquad \text{(c)}$$

It is seen that also in the present there is no axial shear throughout the specimen.

Stress equilibrium equations remain of form (3.5.13) and it is easily seen that insertion of (6.3.27) into (3.5.15) again recovers the differential equations (6.3.22).

Continuity conditions at the interface consist again of (6.3.23) while (6.3.24) is modified and assumes the form

$$[(k_2-G_{T2})\,u^{(2)}_{\beta,\beta} - (k_1-G_{T1})u^{(1)}_{\beta,\beta}]\,n_\alpha + [G_{T2}(u^{(2)}_{\alpha,\beta} + u^{(2)}_{\beta,\alpha}) - G_{T1}(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha})]\,n_\beta =$$

$$= \{(\ell_2 - \ell_1)\,\epsilon^{o}_{11} + [(\ell_1\alpha_{A1} + 2k_1\alpha_{T1}) - (\ell_2\alpha_{A2} + 2k_2\alpha_{T2})]\,\varphi^{o}\}\,n_\alpha \quad \text{on } C_{12} \qquad (6.3.29)$$

It remains to consider the boundary conditions (6.3.18b). On the terminal sections A_o, A_H of the cylinder the traction components are $\sigma_{11}, \sigma_{12}, \sigma_{13}$, the last two of which vanish by (6.3.28c). Therefore the remaining condition is

$$\sigma_{11}(x_2, x_3) = 0 \qquad x_1 = o, H \tag{6.3.30}$$

On the lateral surface of the cylinder the traction T_1 vanishes since n_1, σ_{12} and σ_{13} vanish. Therefore, the traction conditions to be satisfied there reduce to

$$T_\alpha(C) = 0 \tag{6.3.31}$$

which written out in terms of displacement gradients, by use of (6.3.28), assume the form

$$(k - G_T)\, u_{\beta,\beta}\, n_\alpha + G_T(u_{\alpha,\beta} + u_{\beta,\alpha})\, n_\beta =$$

$$= [-\ell\, \epsilon^o_{11} + (\ell\, \alpha_A + 2k\alpha_T)\, \varphi^o]\, n_\alpha \tag{6.3.32}$$

As in par. 3.5.1, the unknown strain ϵ^o_{11} is to be found by satisfaction of (6.3.30) which can only be achieved in the Saint Venant sense and thus the cylinder must be much longer than cross section diameter. Instead of (6.3.30) we write

$$\int_A \sigma_{11}\, dA = 0 \qquad \text{(a)}$$

$$\int_A x_2\, \sigma_{11}\, dA = \int_A x_3\, \sigma_{11}\, dA = 0 \qquad \text{(b)} \tag{6.3.33}$$

Insertion of (6.3.28a) into (6.3.33a) leads to

$$\epsilon^{o}_{11} = \frac{\overline{n\alpha_A + 2k\alpha_T \varphi^{o}} - \overline{\ell(\epsilon_{22} + \epsilon_{33})}}{\bar{n}} \qquad (6.3.34)$$

where an overbar denotes section average.

It is seen that in order to carry out a solution a displacement field which satisfies (6.3.22) subject to (6.3.25), (6.3.29) and (6.3.32), with arbitrary ϵ^{o}_{11}, must first be found. Then ϵ^{o}_{11} is determined by (6.3.34).

There is of course no guarantee that (6.3.33b) are also satisfied. As in the isothermal elastic case they are automatically satisfied for phase geometries with x_2, x_3 axes of symmetry, statistically homogeneous phase geometry (in the limit), and also for composite cylinder assemblages (since in that case (6.3.33b) are satisfied by symmetry for any one composite cylinder).

The procedure described is not feasible for numerical analysis since it would be impossible to cope with an arbitrary ϵ^{o}_{11}. In that event it is necessary to split the displacement field u_α into two parts, u'_α and u''_α. The first satisfies (6.3.29) and (6.3.32) with zero φ^{o}, is thus proportional to ϵ^{o}_{11} and may be written $\epsilon^{o}_{11} v'_\alpha$, where v'_α corresponds to unit ϵ^{o}_{11}. Similarly, u''_α satisfies (6.3.29), (6.3.32) with zero ϵ^{o}_{11}, is thus proportional to φ^{o} and is written $\varphi^{o} v''_\alpha$, where v''_α corresponds to unit φ^{o}. Then the complete solution is

$$u_\alpha = \epsilon^{o}_{11} v'_\alpha + \varphi^{o} v''_\alpha \qquad (6.3.35)$$

where v'_α, v''_α are numerically known functions. Now φ^o is generally prescribed, so insertion of (6.3.35) into (6.3.34) determines ϵ^o_{11} numerically.

As a simple and pertinent example of thermo-elastic stress analysis we consider a FRM which is described by the composite cylinder assemblage model and is subjected to uniaxial stress in fiber direction and a temperature rise. Thus for a cylindrical fiber reinforced specimen

$$\varphi(S) = \varphi^o = \text{const.} \qquad \text{(a)}$$

$$T_1(S) = \sigma^o_{11}\, n_1 \qquad \text{(b)} \qquad (6.3.36)$$

$$T_2(S) = T_3(S) = 0 \qquad \text{(c)}$$

In accordance with (6.3.17-18) we consider the two separate boundary conditions

$$\varphi(S) = 0 \qquad \text{(a)}$$

$$T_1(S) = \sigma^o_{11}\, n_1 \qquad \text{(b)} \qquad (6.3.37)$$

$$T_2(S) = T_3(S) = 0 \qquad \text{(c)}$$

and

$$\varphi(S) = \varphi^o \qquad \text{(a)}$$

$$(6.3.38)$$

$$T_1(S) = T_2(S) = T_3(S) = 0 \qquad \text{(b)}$$

The first problem is isothermal and its solution for the present model has been given in chapter 3.5, appendix 2, important stresses being (14-17) with ϵ^o replaced according to (18) by σ^o_{11}/E^*_A .

To solve (6.3.38) we consider a single composite cylinder, fig. 3.5.1, which is subjected to (6.3.38). In cylindrical coordinates

$$x_1 = z, \qquad x_2 = r\cos\theta, \qquad x_3 = r\sin\theta$$

we have

$$\sigma_{zz}(H,r) = \sigma_{zz}(o,r) = \sigma^o_{11} \qquad \text{(a)}$$

$$\sigma^{(1)}_{rr}(z,a) = 0 \qquad \text{(b)} \qquad (6.3.39)$$

and the temperature is φ^o throughout the cylinder. The problem is axially symmetric and so $\sigma_{r\theta}$ and u_θ vanish throughout the cylinder and the displacements u_z and u_r are not functions of θ. Since in the previous general formulation it was shown that u_2, u_3 are not functions of $x_1 = z$ and since u_r depends only on u_2 and u_3 , it follows that u_r is not a function of z. Consequently,

$$u_r = u_r(r) \qquad (6.3.40)$$

Furthermore, by the general previous formulation

$$u_z = \epsilon^o_{11} z \qquad (6.3.41)$$

where ϵ^o_{11} is as yet unknown.

Computation of strains from (6.3.40-41) and insertion into the thermoelastic stress strain law yields

$$\epsilon_{rr} = \frac{du_r}{dr} \qquad \epsilon_{\theta\theta} = \frac{u_r}{r} \qquad \epsilon_{zz} = \epsilon^{o}_{11} \qquad \text{(a)}$$

(6.3.42)

$$\epsilon_{r\theta} = \epsilon_{\theta z} = \epsilon_{rz} = 0 \qquad \text{(b)}$$

$$\sigma_{zz} = n\,\epsilon^{o}_{11} + \ell(\epsilon_{rr} + \epsilon_{\theta\theta}) - (n\alpha_A + 2\ell\,\alpha_T)\,\varphi^{o} \qquad \text{(a)}$$

$$\sigma_{rr} = \ell\,\epsilon^{o}_{11} + (k + G_T)\,\epsilon_{rr} + (k - G_T)\,\epsilon_{\theta\theta} - (\ell\,\alpha_A + 2k\alpha_T)\,\varphi^{o} \qquad \text{(b)}$$

(6.3.43)

$$\sigma_{\theta\theta} = \ell\,\epsilon^{o}_{11} + (k - G_T)\,\epsilon_{rr} + (k + G_T)\,\epsilon_{\theta\theta} - (\ell\,\alpha_A + 2k\alpha_T)\,\varphi^{o} \qquad \text{(c)}$$

$$\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{rz} = 0 \qquad \text{(d)}$$

The only surviving equilibrium equation is

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad (6.3.44)$$

Insertion of (6.3.42-43) into (6.3.44) yields the following differential equation for u_r

$$r^2 \frac{d^2 u_r}{dr^2} + r\frac{du_r}{dr} - u_r = 0 \qquad (6.3.45)$$

whose general solution is $Br + C/r$. Thus

$$u_r = \begin{cases} B_1 r + C_1/r & a \le r \le b, \text{ matrix} \quad \text{(a)} \\ B_2 r & 0 \le r \le a, \text{ fiber} \quad \text{(b)} \end{cases} \tag{6.3.46}$$

Then matrix and fiber stresses are from (6.3.46) and (6.3.42-43)

$$\sigma_{zz}^{(1)} = n_1 \epsilon_{11}^{o} + 2\ell_1 B_1 - (n_1 \alpha_{A1} + 2\ell_1 \alpha_{T1}) \varphi^{o} \quad \text{(a)}$$

$$\sigma_{rr}^{(1)} = \ell_1 \epsilon_{11}^{o} + 2k_1 B_1 - 2G_{T1} C_1/r^2 - (\ell_1 \alpha_{A1} + 2k_1 \alpha_{T1})\varphi^{o} \quad \text{(b)}$$

$$\sigma_{\theta\theta}^{(1)} = \ell_1 \epsilon_{11}^{o} + 2k_1 B_1 - 2G_{T1} C_1/r^2 - (\ell_1 \alpha_{A1} + 2k_1 \alpha_{T1}) \varphi^{o} \quad \text{(c)} \quad (6.3.47)$$

$$\sigma_{zz}^{(2)} = n_2 \epsilon_{11}^{o} + 2\ell_2 B_2 - (n_2 \alpha_{A2} + 2\ell_2 \alpha_{T2}) \varphi^{o} \quad \text{(d)}$$

$$\sigma_{rr}^{(2)} = \sigma_{\theta\theta}^{(2)} = \ell_2 \epsilon_{11}^{o} + 2k_2 B_2 - (\ell_2 \alpha_{A2} + 2k_2 \alpha_{T2}) \varphi^{o} \quad \text{(e)}$$

To find the constants B_1, C_1, B_2 the boundary condition (6.3.39b) and the continuity conditions

$$\begin{aligned} u_r^{(1)}(a) &= u_r^{(2)}(a) \\ \sigma_{rr}^{(1)}(a) &= \sigma_{rr}^{(2)}(a) \end{aligned} \tag{6.3.48}$$

are utilized. Insertion of (6.3.46-47) into these gives the equations

$$2k_1 B_1 - 2G_{T1} C_1/b^2 \qquad + \ell_1 \epsilon^o_{11} = (\ell_1 \alpha_{A1} + 2k_1 \alpha_{T1})\varphi^o \qquad \text{(a)}$$

$$B_1 + \qquad C_1/a^2 \quad - B_2 \qquad = 0 \qquad \text{(b)} \quad (6.3.49)$$

$$2k_1 B_1 - 2G_{T1} C_1/a^2 \quad - 2k_2 B_2 + (\ell_1 - \ell_2)\, \epsilon^o_{11} =$$

$$= [(\ell_1 \alpha_{A1} + 2k_1 \alpha_{T1}) - (\ell_2 \alpha_{A2} + 2k_2 \alpha_{T2})]\varphi^o \qquad \text{(c)}$$

which are three equations for the four unknowns B_1, C_1, B_2 and ϵ^o_{11}. To find a fourth equation (6.3.34) is used. Since in the present case

$$\epsilon_{22} + \epsilon_{33} = \epsilon_{rr} + \epsilon_{\theta\theta} = \begin{cases} 2B_1 & \text{matrix} \\ 2B_2 & \text{fiber} \end{cases}$$

the last result following from (6.3.46) and (6.3.42a), equ. (6.3.34) assumes the form

$$2\ell_1 v_1 B_1 + 2\ell_2 v_2 B_2 + \bar{n}\, \epsilon^o_{11} = \overline{n\alpha_A + 2\kappa\alpha_T} \quad \varphi^o \qquad (6.3.50)$$

where

$$\bar{n} = n_1 v_1 + n_2 v_2 \qquad \text{(a)}$$

$$\overline{n\alpha_A + 2k\alpha_T} = (n_1 \alpha_{A1} + 2k_1 \alpha_{T1})\, v_1 + (n_2 \alpha_{A2} + 2k_2 \alpha_{T2})\, v_2 \qquad \text{(b)}$$

(6.3.51)

$$v_1 = 1 - (a/b)^2 \qquad \text{(c)}$$

$$v_2 = (a/b)^2 \qquad \text{(d)}$$

Thus B_1, C_1, B_2, ϵ^o_{11} are determined by the four linear equations (6.3.49), (6.3.51) and all the stresses (6.3.47) become known. Analytical expressions for the stresses are, however, cumbersome. It is better to solve (6.3.49), (6.3.51), numerically.

It is quite easy to realize that the stresses in the composite cylinder are the correct stresses in any cylinder of the assemblage. To see this it is pointed out that on the surface $r = b$, $\sigma^{(1)}_{rr}$, (6.3.47b), is a constant stress and $u^{(1)}_r$, (6.3.46a), is a purely radial displacement, of the form $u_r(b) = \epsilon_r b$, where ϵ_r and $\sigma^{(1)}_{rr}$ are the same for any cylinder with same a/b. Therefore, the composite cylinder appears to an external observer as some homogeneous cylinder with certain thermo-elastic properties. Now the argument of par. 3.5.3, whereby all the cylinders can be fitted together by matching displacements and tractions can be repeated verbatim.

It is also to be noted that by the present argument it is easily possible to obtain the ETEC of the composite cylinder assemblage as apparent expansion coefficients of any one composite cylinder, [6.5]. However, the general method given previously, par. 6.3.1, is certainly much more attractive.

APPENDIX

Specific Heat

The effective specific heat of a composite is defined as the amount of heat necessary to raise the temperature of unit mass (in the RVE sense) by unit temperature. It is necessary to distinguish between the specific heat at constant average strain c_v^*, and the specific heat at constant average stress c_p^*.

It has been shown, [6.6], that the two effective specific heats of any composite are related by

$$c_v^* - c_p^* = C_{ijkl}^* \alpha_{ij}^* \alpha_{kl}^* \varphi_o \tag{1}$$

where φ_o is the absolute temperature and C_{ijkl}^* and α_{ij}^* are EEM and ETEC at that temperature. This relation is analogous to the relation between c_v and c_p of a homogeneous elastic material.

For transverse isotropy or square symmetry, (1) assumes the alternative forms

$$c_v^* - c_p^* = \begin{cases} (n^* \alpha_A^{*2} + 4\ell^* \alpha_A^* \alpha_T^* + 4k^* \alpha_T^{*2}) \varphi_o \\ [E_A^* \alpha_A^{*2} + 4k^* (\nu_A^* \alpha_A^* + \alpha_T^*)^2] \varphi_o \end{cases} \tag{2}$$

For a two phase transversely isotropic or square symmetric FM or FRM, with isotropic phases, c_p^* is given by

$$\frac{c_p^* - \bar{c}_p}{\varphi_o} = 3\,\frac{\alpha_2 - \alpha_1}{1/K_2 - 1/K_1}\,(3\,\bar{\alpha} - \alpha_A^* - 2\,\alpha_T^*) \tag{3}$$

where α_A^*, α_T^* are given by (6.3.10). This is simply shown on the basis of the results given in [6.2]. With this expression, c_v^* becomes known from (2).

Numerical values of c_p^* and c_v^* are very accurately given by

$$c_p^* \cong c_v^* \cong \bar{c}_p \cong \bar{c}_v \tag{4}$$

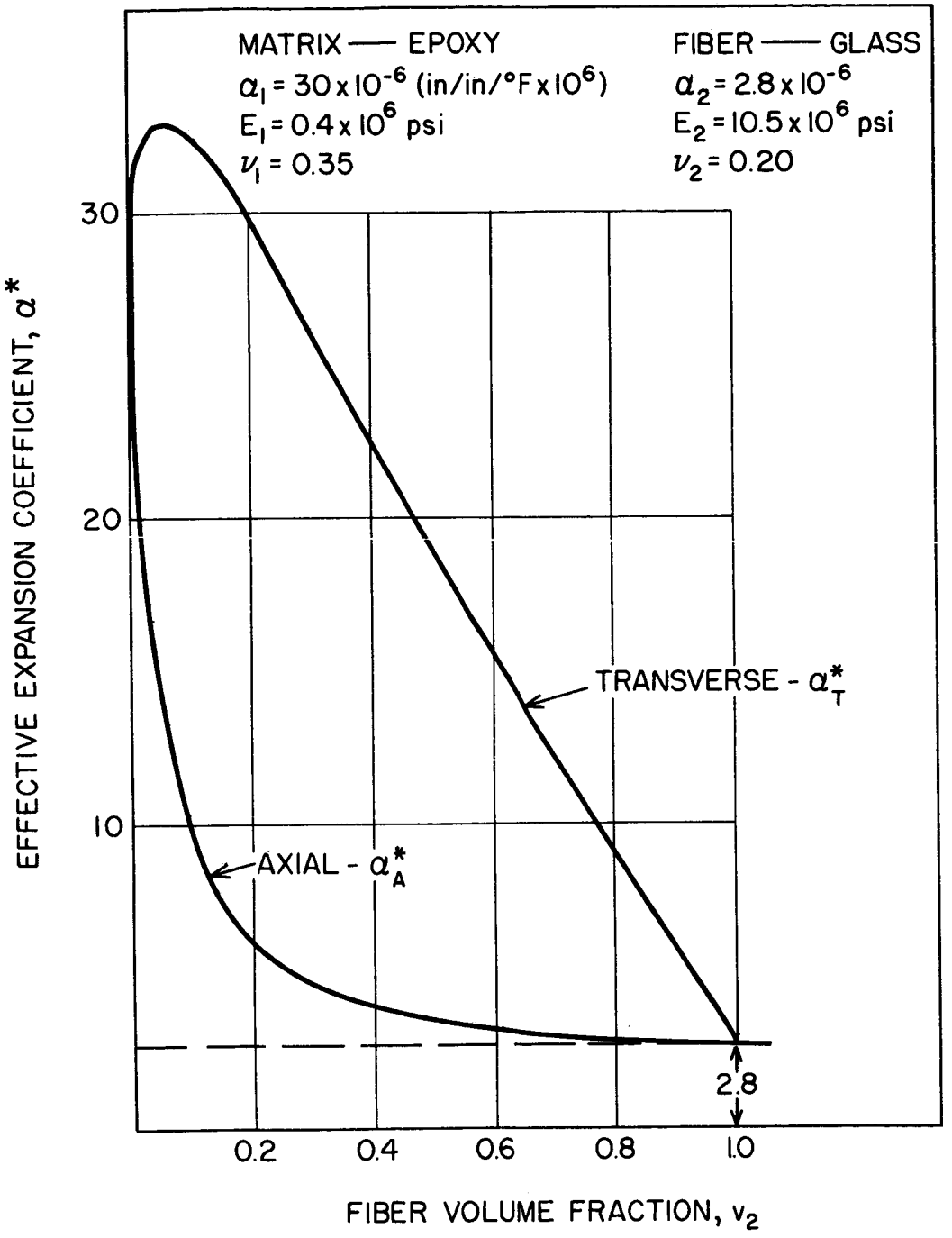


FIG. 6.3.1 - EFFECTIVE THERMAL EXPANSION COEFFICIENTS
CCA MODEL

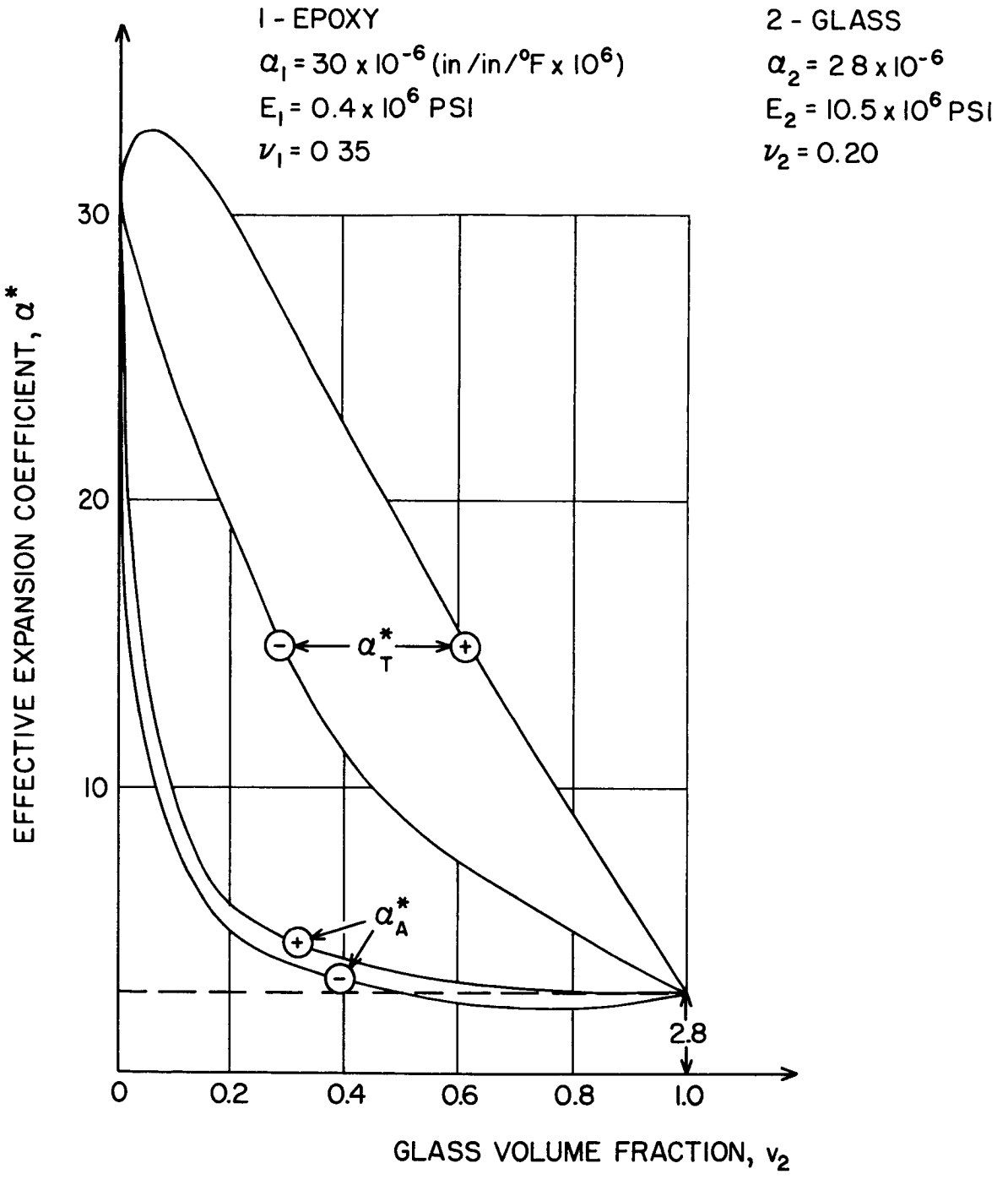


FIG. 6.3.2 - BEST POSSIBLE BOUNDS FOR EFFECTIVE THERMAL EXPANSION COEFFICIENTS

Table 6.3.1 Thermal Expansion Coefficients of Fiber and Matrix Materials

	Material	Thermal Expansion Coefficient ($x10^6$)	
		^oK	^oF
Fiber	E-Glass	5.0	2.8
	S-Glass	4.0	2.2
	Carbon	?	?
	Boron	8.3	4.6
	Al_2O_3	8.7	4.8
	SiC'	4.8	2.7
Matrix	Magnesium	25.7	14.3
	Aluminum	24.6	13.7
	Epoxy (typical)	63	35

7.1 INTRODUCTION

One of the principal advantages of uniaxial FRM as engineering materials is their potential to exploit the very high strength of fibers, such as glass, boron and carbon fibers, to produce materials of very high axial strength. Moreover, the fibers and matrix have low specific weights and thus the specific weight of the FRM is also low, as low as one third of the specific weight of steel. (List of strengths and specific weights is given in table 7.1.)

This attractive combination of high axial strength and specific weight is often expressed by the specific (axial) strength which is defined as the axial strength-to-specific weight ratio and has thus the dimension of length. Indeed, the specific strength is the length of a vertically suspended cylinder which fails under its own weight.

On the other hand, the transverse strength of uniaxial FRM is in general only of the order of the strength of the matrix in which the fibers are embedded and is therefore by an order of magnitude smaller than the axial strength. It follows that the axial strength potential of uniaxial FRM can be exploited to advantage in structural members which are predominantly uniaxially stressed, such as bars, struts, beams, rings and frames. In the case of two dimensional structures, such as plates and shells, uniaxial reinforcement is not adequate since the weak transverse direction would be exposed to unbearable stresses. For such structures it is therefore advantageous to employ biaxial or multiaxial reinforcement, e.g. in the form of laminates.

Prediction of strength of FRM is evidently of foremost practical importance. Unfortunately, however, the subject is of such difficulty that on the basis of present state of knowledge it does not seem possible to present a coherent, reasonably rigorous development, as has been done so far in this work. Consequently, the presentation of this subject here will be of more qualitative than quantitative nature, ideas rather than mathematical developments will be emphasized and detailed derivations will not be given.

To appreciate the difficulties inherent in the problem of strength prediction it is instructive to contrast this problem with that of prediction of effective physical properties, such as effective elastic moduli, which has been discussed in preceding parts. For the latter kind of problems the material undergoes a process throughout which it may be assumed that the phase stress-strain relations and the phase geometry do not change. This makes it possible to define these problems mathematically. In the case of strength, however, the process continues until failure which implies fundamental changes in material behavior and phase geometry, e.g. plastification , large deformations and crack formation. It is not generally known where and how these drastic internal changes occur. Even if the local stress fields in the specimen were known in minute detail during all stages of a loading process there would still remain the problem of devising a meaningful failure criterion in terms of these stresses and there would, of course, be no assurance that other variables, in addition to stresses, would not have decisive influence.

In order to perform any kind of failure analysis it is necessary to assume a failure mechanism or failure mode, which describes the basic changes inside the material which lead to failure. Such failure modes are chosen on the basis of reasoning and/or experimental information. It should be emphasized that the same specimen will have quite different failure modes for different kinds of loading. Thus the failure modes of a uniaxial FRM are very different for axial tension, axial compression and transverse loading, respectively. Reasonable failure modes for such loadings of uniaxial FRM, and their associated failure analyses, are discussed in this presentation.

There arises the additional severe difficulty that reliable failure modes for individual simple loadings cannot in general be simply combined to lead to failure modes for combined loadings. Because of the absence of sufficient information on such failure modes, present discussion of this subject is limited to assumption of simple quadratic failure criteria whose coefficients have to be determined in terms of the assumedly known failure stresses under simple loadings.

Finally, it is noted that the present discussion is confined to static strength. Important topics such as impact and fatigue strength are omitted since only a very small amount of reliable information on these subjects is available at the present time.

7.2 AXIAL STRENGTH

The axial strength of a uniaxial FRM is defined as the average uniaxial stress in fiber direction which results in failure of a cylindrical specimen, whose generators are parallel to fiber direction. There is a fundamental difference between the failure modes for tensile and compressive loading since in the latter case the fibers may buckle while in the former case they do not. Therefore, tensile and compressive axial strengths are to be discussed separately.

7.2.1 Tensile Strength

Let the usual fiber reinforced cylindrical specimen, with axis and fibers in x_1 direction, be subjected to uniform tensile stress $\sigma_{11} = \sigma^o$ on its end sections. There are no shear stresses on these sections and no load on the lateral surface.

A popular formula for the failure stress $\sigma_{11} = \bar{\sigma}_u$ is given by the so-called "rule of mixtures" which asserts that

$$\bar{\sigma}_u = \sigma_u^{(m)} v_m + \sigma_u^{(f)} v_f \tag{7.2.1}$$

where here and from now on, m and f denote matrix and fibers, respectively, and $\sigma_u^{(m)}$ and $\sigma_u^{(f)}$ are individually determined uniaxial tensile failure stresses of matrix and fibers respectively.

The following discussion will serve to provide some justification for (7.2.1) under certain conditions. By the average stress theorem it follows that for any σ^o

$$\bar{\sigma}_{11} = \sigma^o = \bar{\sigma}^{(m)} v_m + \bar{\sigma}^{(f)} v_f \tag{7.2.2}$$

where 11 subscripts on phase axial stresses have been suppressed. The averages in (7.2.2) are volume averages. It may be safely assumed that phase stress <u>surface</u> averages are the same in any transverse section. Therefore, all averages in (7.2.2) may be interpreted as surface averages.

Now it should be borne in mind that in usual FRM the tensile strength of fibers is very much higher than the matrix tensile strength, while volume fractions are of same order of magnitude (see table 7.1 for some strength values). If all the fibers in a section fail, the matrix will follow suit at once and the specimen will fail. In that event the failure stress is

$$\bar{\sigma}_u = \bar{\sigma}^{(m)} v_m + \bar{\sigma}^{(f)}_u v_f \tag{7.2.3}$$

where $\bar{\sigma}^{(f)}_u$ is the section average fiber stress at fiber failure and $\bar{\sigma}^{(m)}$ is the average matrix stress at fiber failure.

If the matrix part of the section fails first, the load release is carried by the fibers and the external load can be further increased until the fibers also fail. Then

$$\bar{\sigma}_u = \bar{\sigma}^{(f)}_u v_f \tag{7.2.4}$$

Since the fibers which are usually used are by an order of magnitude stronger than the matrix, the numerical difference between (7.2.3) and (7.2.4) is insignificant.

Now it may be assumed that the variation of fiber axial stress over fiber areas can be neglected. (Compare e.g. composite cylinder assemblage analysis, appendix 2, chapter 3.5, where fiber and matrix axial stress were found to be constant). If there exists in addition a reasonably constant (little scatter) fiber failure stress $\sigma_u^{(f)}$ of individually tested fibers, then $\bar{\sigma}_u^{(f)}$ in (7.2.4) can be replaced by $\sigma_u^{(f)}$.

Finally, since the first term in the right side of (7.2.3) is very small in comparison to the second term, no significant error can result from replacement of $\bar{\sigma}^{(m)}$ by $\sigma_u^{(m)}$, the matrix failure stress. Thus there is obtained the result (7.2.1).

The simple formula (7.2.1) is well verified experimentally in the case of fibers which have reasonably constant strength, such as metal fibers. See [7.1-3]. It should be borne in mind that for small amount of fiber reinforcement the matrix contribution may become important and then (7.2.1) does not apply. For discussion of this effect see e.g. [7.3].

It is of some interest to note that in the event that both fibers and matrix are ideally plastic, it follows by limit analysis methods that (7.2.1) is an upper bound on the tensile strength, which in this event is defined as the average axial stress at which average axial strain increases with no increase in average stress.

Of all the assumptions made in the establishment of (7.2.1) the most crucial one is the assumption of constant fiber failure stress. For the usual brittle fibers, such as glass, boron and carbon, which are used in practice,

such an assumption is not realistic since it is found that the tensile strength of such fibers is a random variable which is a decreasing function of the fiber gage length. An explanation of this phenomenon is usually given in terms of the occurrence of local fiber defects. Evidently, the expected number of defects increases with fiber length and so a longer fiber has a higher failure probability at some stress level than a shorter one.

The stochastic strength-length relationship of fibers has been incorporated into a cumulative damage-strength theory by Rosen [7.4-5], whose main ideas will be here discussed.

At the outset it should be borne in mind that (7.2.3-4) certainly remain valid for any kind of fibers and that the numerical difference between them is still negligible. But since $\sigma_u^{(f)}$ is now a random variable, $\bar{\sigma}_u^{(f)}$ is unknown. It is not permissible to take as $\bar{\sigma}_u^{(f)}$ the experimentally obtained average strength of fibers of arbitrary gage length since, as will be seen later, the FRM contains fiber segments of very different lengths during the loading process and thus the probability of further breakage will be different for segments of different lengths.

A fundamental aspect of the theory is possible consequences of a fiber break. Suppose that a fiber has broken; then the ensuing loss of tensile load must be taken by the remainder of the section, and, clearly, the neighboring fibers will experience most of the increase in load. Suppose that the neighboring fibers break because of the load increase, thus producing increased load and failure in their neighbors, etc. The result of such a chain reaction would be an

expanding transverse crack which would lead to immediate failure of the specimen. Such a transverse crack failure would imply a very undesirable material in the case of brittle fiber reinforcement; for in such fibers some fiber breaks will always occur at low stress levels and so the material would fail at low axial stress. Whether or not such a failure mode ever occurs is at present a matter of conjecture. It is assumed in the present treatment that this does not happen.

A more important consequence of the occurrence of a fiber break is the ensuing axial shear stress appearance at the fiber-matrix interface. It should be noted that in an elastic specimen with continuous fibers, such shear stresses do not occur; see par. 3.5.1. They must, however, appear near a fiber break from equilibrium considerations. For with increasing distance from the break the tensile stress in the fiber increases from zero until at sufficient distance it builds up to the original fiber stress before the break (because of Saint Venants principle). Consideration of a fiber end, extending from break to some axial distance, as a free body clearly requires axial interface shear stress for equilibrium and, evidently, this stress decreases with increasing distance from the break, fig. 7.2.1. The actual computation of the shear stress, however, is an exceedingly difficult problem because of the presence of neighboring fibers which must be taken into account. A number of approximate treatments have been given. For listing and description see e.g. [7.3],[7.8].

The shear stress concentration can cause debonding of the fiber from the matrix. If the fiber becomes completely debonded its break renders it ineffective,

since it can no longer transmit axial load. Such a situation would imply that any fiber becomes ineffective after having sustained one break and the specimen fails when all fibers are broken once. A specimen which fails according to this debonding failure mechanism is not very strong (though stronger than the one governed by the previously considered transverse crack failure mechanism) since it merely acts as an ordinary bundle of fibers. Such debonding should be prevented by insurance of proper matrix-fiber adhesion and it is henceforth assumed that this is the case.

In the absence of transverse crack and fiber debonding failure modes the fibers will react to load increase by progressive cracking.(*) (For experimental verification see [7.4].) Thus there are formed fiber segments of different lengths whose further breakage probability is governed by the (experimentally known) strength-length probability distribution of fibers. A Weibull distribution has been chosen in [7.4] to represent this distribution function.

The actual length of a broken fiber segment should not, however, be used in the distribution function since, as previously explained, the end parts of the segment are in a state of variable tensile stress because of the vanishing of the stress at the ends, while in individually tested fibers the tensile stress is uniform along the gage length. The broken fiber segment thus has an "effective length" which is smaller than its actual length.

(*) Such a failure mechanism has apparently first been suggested by Parratt [7.6].

This effective length has been chosen in [7.4] as $\ell - 2\delta$ where ℓ is the actual length and δ is the distance from the broken end at which the fiber tensile stress attains some fraction (e.g. 90%) of its original value. It is possible to give other, less arbitrary, definitions of the effective length.

Statistical failure analysis has been carried out in [7.4] by subdivision of the composite specimen into a chain of parallel layers (links) of equal thickness δ, following a general idea of Gücer and Gurland [7.7]. Assuming that in any such layer the matrix axial strength is negligible relative to fiber strength and that axial load concentration in fibers adjacent to a broken fiber can be ignored, the layer becomes a simple bundle of fibers of length δ and the probability of failure of a layer can therefore be expressed in terms of probability of fiber failure by statistical bundle theory. Thus the composite specimen is now a chain of links (layers) whose individual failure probabilities are known. The probability of failure of the composite is thus the probability of failure of some link in the chain, which is established by statistical chain theory.

Let the strength probability distribution of fibers of length ℓ be given by the Weibull distribution

$$f(\sigma) = \ell \alpha \beta \sigma^{\beta - 1} \exp(-\ell \alpha \sigma^{\beta}) \qquad (7.2.5)$$

where α, β are two parameters to be obtained from fit to experimental data and $f(\sigma)$ is the probability that a fiber of length ℓ fails within a stress interval $[\sigma, \sigma + d\sigma]$. (See [7.9] for theoretical reasoning for applicability of Weibull distribution to fiber strength). The probability of failure of a fiber of length ℓ

at stress smaller or equal to σ is then given by the cumulative distribution

$$F(\sigma) = \int_{o}^{\sigma} f(\sigma)\, d\sigma$$

Average fiber failure stress and standard deviation at length ℓ are

$$\bar{\sigma} = (\alpha \ell)^{-1/\beta}\, \Gamma(1 + 1/\beta) \qquad \text{(a)}$$

$$s = (\alpha \ell)^{-1/\beta}\, [\, (1+2/\beta) - \Gamma^2 (1+1/\beta)\,] \qquad \text{(b)} \qquad (7.2.6)$$

where Γ denotes the gamma function.

The most probable failure stress σ_u^o of the composite specimen, predicted by the analysis, is

$$\sigma_u^o = v_f (\alpha \beta \delta e)^{-1/\beta} \qquad (7.2.7)$$

where e is the base of natural logarithms and δ is infinitely smaller than specimen length, which is an accurate assumption for usual FRM.

The above described approach to analysis of failure of FRM, though approximate, is of major conceptual importance. Unlike the simplistic approach leading to (7.2.1) , it recognizes the fact that brittle fiber strength cannot be defined independently of fiber length. It is indeed seen that no strength value appears in (7.2.7) . Instead there are the two parameters α and β which describe the probability strength-length distribution (7.2.5).

Comparison between strength values predicted by (7.2.1) and (7.2.7) has been given in [7.5] in following fashion: Suppose that mean strength of

fibers of some specified gage length ℓ , which has been obtained by experiment, is used in (7.2.1). (This would appear reasonable to somebody who believes in the universal validity of (7.2.1)). Neglecting for convenience the matrix strength, the result is from (7.2.1) and (7.2.6a).

$$\sigma_u^r = v_f \, (\alpha \ell)^{-1/\beta} \Gamma(1+1/\beta) \tag{7.2.8}$$

A quantity which expresses the amount of scatter in fiber strength measurement is the coefficient of variation u which is defined by

$$\mu = \frac{s}{\bar{\sigma}}$$

and may be computed in terms of (7.2.6). The ratio σ_u^o/σ_u^r as defined by (7.2.7-8) is plotted against s of the previously considered fibers of length δ, for different ratios of ℓ/δ , fig. 7.2.2 (taken from [7.5]). It is seen that for small coefficient of variation, (7.2.7) and (7.2.8) are very close. This is reasonable, for a small coefficient of variation implies that fiber strength is more or less constant, in which event (7.2.1) should be applicable. On the other hand, for substantial coefficients of variation, thus considerable scatter in fiber strength, (7.2.7) can become much larger than (7.2.8) and thus the "rule of mixtures" (7.2.1) underestimates the composite's strength considerably.

For further work concerning the cumulative damage model the reader is referred to [7.10-12] .

Other aspects of tensile strength, such as fracture mechanics and crack propagation, are discussed in [7.13-14] , the case of discontinuous fibers-in [7.3], [7.15], and interface effects on strength-in [7.16].

7.2.2 Compressive Strength

Analysis of compressive axial strength of uniaxial FRM has been carried out on the basis of the assumption that a specimen fails in compression when the fibers buckle. For experimental evidence to support this assumption see e.g. [7.5] .

Analytical prediction of fiber buckling is evidently an extremely complicated problem. Fiber lateral deflection is resisted in unknown fashion by the matrix material in which they are embedded, this phenomenon being reminiscent but not analogous to buckling of a column on elastic foundation. Furthermore, fiber buckling modes are certainly influenced by the presence of neighboring fibers.

Approximate two dimensional analyses by elastic energy methods have been given in [7.5],[7.17]. Two idealized two dimensional buckling configurations were assumed. In the first all fibers buckle in sine wave pattern, the pattern of adjacent fibers being antisymmetric, or 180° out of phase. This pattern was called "extensional mode." In the second configuration the sinusoidal patterns of all fibers are in phase. This was called the "shear mode." The results of these approximate analyses are

$$\sigma_u^o = v_f \sqrt{\frac{2E_m E_f v_f}{3(1-v_f)}} \qquad \text{Extensional mode (a)}$$

$$\sigma_u^o = \frac{G_m}{1-v_f} \qquad \text{Shear mode (b)} \qquad (7.2.9)$$

7.3 TRANSVERSE STRENGTH

The transverse strength of a uniaxial FRM may be defined as the average plane stress combination $\bar{\sigma}_{\alpha\beta}$, in the transverse x_2x_3 plane which produces failure. Such average stress may be produced by application of the boundary conditions

$$u_1(S) = 0 \qquad \text{(a)}$$

or

$$T_1(S) = 0 \qquad \text{(b)} \qquad (7.3.1)$$

$$T_\alpha(S) = \sigma^o_{\alpha\beta} n_\beta \qquad \alpha, \beta = 2,3 \qquad \text{(c)}$$

to the usual cylindrical specimen.

There is a fundamental difference between (7.3.1a,c) and (7.3.1b,c). In the former case the specimen is in plane strain and the prevention of axial deformation produces an average axial stress $\bar{\sigma}_{11}$. In that event application of the average stress theorem, taking cognizance of the cylindrical external geometry, shows that the average state of stress in the specimen is

$$[\bar{\sigma}_{ij}] = \begin{bmatrix} \bar{\sigma}_{11} & 0 & 0 \\ 0 & \sigma^o_{22} & \sigma^o_{23} \\ 0 & \sigma^o_{23} & \sigma^o_{33} \end{bmatrix} \qquad (7.3.2)$$

In the latter case $\bar{\sigma}_{11}$ vanishes and the average stress in the specimen is

$$\bar{\sigma}_{\alpha\beta} = \sigma^o_{\alpha\beta} \qquad (7.3.3)$$

If it is assumed that failure for a certain composite specimen is uniquely produced by some combination of the average stresses then the <u>failure criterion</u> of the material may be written for the cases (7.3.2), (7.3.3), respectively, as

$$f(\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{23}, \bar{\sigma}_{33}) = \text{const.} \qquad \text{(a)}$$

$$(7.3.4)$$

$$f(\bar{\sigma}_{22}, \bar{\sigma}_{23}, \bar{\sigma}_{33}) = \text{const.} \qquad \text{(b)}$$

where f are some unknown functions.

It should be noted that $\bar{\sigma}_{11}$ in (7.3.4a) is not an independent stress. It would be independent in the event that the specimen were subjected to (7.3.1c) and in addition to a uniaxial stress in fiber direction. In the plane strain case, however, $\bar{\sigma}_{11}$ depends on other average stresses and on material effective properties. Thus for a transversely isotropic specimen which is elastic until failure there is the relation

$$\bar{\sigma}_{11} = \nu_A^* \ (\bar{\sigma}_{22} + \bar{\sigma}_{33})$$

since $\bar{\epsilon}_{11}$ vanishes. See (3.4.78a). Thus (7.3.4a) is in reality only a function of $\bar{\sigma}_{\alpha\beta}$. Consequently, both failure criteria are of the form (7.3.4b), the functional forms being different, however.

We now consider specifically the case of transversely isotropic specimens. The concept of macroscopic transverse isotropy has here to be somewhat revised in that it is assumed to apply to failure . By this is meant that a failure criterion of type (7.3.4b) cannot depend upon the orientation

of the plane coordinate system to which $\bar{\sigma}_{\alpha\beta}$ are referred. Consequently, (7.3.4b) must be invariant with respect to any coordinate rotation around the x_1 axis and therefore only the invariants of $\bar{\sigma}_{\alpha\beta}$ can appear in (7.3.4b). These invariants are

$$I_1 = \bar{\sigma}_{22} + \bar{\sigma}_{33} \qquad \text{(a)}$$

$$I_2 = I_3 = \bar{\sigma}_{22}\,\bar{\sigma}_{33} - \bar{\sigma}_{23}^{2} \tag{7.3.5}$$

We thus conclude that (7.3.4b) has the form

$$f(I_1, I_2) = \text{const.} \tag{7.3.6}$$

The principal average shear stress $\bar{\tau}$, at 45° between the principal axes of $\bar{\sigma}_{\alpha\beta}$ is given by

$$\bar{\tau} = \pm \frac{1}{2} \sqrt{(\bar{\sigma}_{22} - \bar{\sigma}_{33})^2 + 4\,\bar{\sigma}_{23}^{\;2}} \tag{7.3.7}$$

Define as usual

$$\bar{\sigma} = \frac{1}{2}\,\bar{\sigma}_{\alpha\alpha} = \frac{1}{2}\,(\bar{\sigma}_{22} + \bar{\sigma}_{33}) \tag{7.3.8}$$

It follows from (7.3.5) and (7.3.7-8) that

$$I_1 = 2\,\bar{\sigma}$$

$$I_2 = \sqrt{\bar{\sigma}^2 - \bar{\tau}^2} \tag{7.3.9}$$

Thus (7.3.6) may be written as

$$f(\bar{\sigma}, \bar{\tau}) = \text{const.} \tag{7.3.10}$$

It is thus seen that as a consequence of the macroscopic transverse isotropy the failure criterion depends only on the two stress variables $\bar{\sigma}$ and $\bar{\tau}$.

The preceding argument is of course familiar from the theory of ideal plasticity for establishment of the form of the yield function (see e.g. [7.18]). In that case the argument is carried further (in three dimensions) by the assumption that isotropic stress does not produce plastic deformations. A similar, much less justified, assumption in the present case would be that plane-isotropic average stress i.e.

$$\bar{\sigma}_{\alpha\beta} = \bar{\sigma}\,\delta_{\alpha\beta} \tag{7.3.11}$$

does not produce failure, or rather, that it must attain a much higher value than shear stress $\bar{\tau}$ to produce failure. It is also to be noted that such an assumption would be more plausible for isotropic compression than for isotropic tension. If such an assumption were adopted it would follow that $\bar{\sigma}$ does not enter into the failure criterion, and so (7.3.10) would reduce to the simple form

$$\bar{\tau} = \text{const.} \tag{7.3.12}$$

At the present time it is difficult to assess the accuracy of the approximation (7.3.12).

It is worthwhile to note that if the specimen is subjected to pure average shear there can be no significant difference between the plane strain and plane stress failure criteria since $\bar{\sigma}_{11}$ would in that case be zero or negligible.

The quantitative establishment of a failure criterion is a matter of considerable difficulty, even for the simple case of pure transverse shear. Local elastic stress fields can be found for simple geometries, e.g. by numerical analysis of regular square and hexagonal arrays. Analysis for elasto-plastic stress fields does not seem to be available at all. The primary question is, however, how to devise a failure criterion on the basis of computed local stresses, and the answer to this does not seem to be known at the present time.

A powerful well known method to find or to estimate the failure loads of ideally plastic bodies is <u>limit analysis</u>. The method is applicable to FRM if it is assumed that the matrix is incompressible ideally plastic and that the fibers are rigid (*). It is thus primarily appropriate for the case of ductile metal matrices, with sharp yield points, and for fibers which are sufficiently stiff to be considered as rigid relative to a plastically yielding matrix. While for matrices such as epoxy limit analysis is less justified, the results predicted by it should nevertheless be of qualitative importance.

The FRM failure mode assumed by limit analysis is unrestricted plastic flow of the matrix at constant load, this load being the failure load.

(*) It is also possible to apply limit analysis when matrix and fibers are ideally plastic.

The chief advantage of the method is in that it permits bounding of failure loads by use of the theorems of limit analysis, see e.g. [7.19-20], without it being necessary to determine actual local stress fields. This approach is thus reminiscent of bounding of effective elastic moduli by use of variational methods as discussed in chapters 3.6-7.

We consider a FRM with ideally plastic matrix and rigid fibers which is subjected to (7.3.1b,c). According to limit analysis we define a statically admissible stress field which has to satisfy the following conditions

(a) Traction boundary conditions (7.3.1c)

(b) Equilibrium in the matrix.

(c) Stress resultants on any fiber vanish.

(d) Not violate the plastic yield condition of the matrix.

The boundary tractions associated with such an admissible stress field are lower bounds on the boundary tractions which produce unrestricted plastic flow.

It is easy to show that boundary tractions which produce failure in an unreinforced plastic specimen are lower bounds on the failure tractions of a reinforced specimen, [7.21-22]. As an example, let the matrix obey the Mises yield condition

$$s_{ij}\, s_{ij} = 2k^2 = \frac{2}{3}\, \sigma_y^{\,2} \tag{7.3.13}$$

where s_{ij} are the deviatoric components of the stress tensor and σ_y is the yield stress of the matrix in simple tension. Let the tractions (7.3.1c) which produce plastic failure be denoted

$$T_\alpha^u = \sigma_{\alpha\beta}^u \, n_\beta \tag{7.3.14}$$

Then by the lower bound theorem

$$s_{\alpha\beta}^u \, s_{\alpha\beta}^u \geq 2k^2 = \frac{2}{3} \sigma_y^2 \tag{7.3.15}$$

where $s_{\alpha\beta}^u$ are the deviatoric components of $\sigma_{\alpha\beta}^u$.

If, in particular, the specimen is subjected to uniaxial stress in x_2 direction with axial deformation prevented, thus

$$u_1(S) = 0 \qquad \text{(a)}$$

$$T_2(S) = \sigma_{22}^o \, n_2 \qquad T_3(S) = 0 \qquad \text{(b)} \tag{7.3.16}$$

then it follows from (7.3.15) that

$$\sigma_{22}^u \geq \sigma_y \tag{7.3.17}$$

If the specimen is subjected instead of (7.3.16b) to pure transverse shear

$$T_2(S) = \sigma_{23}^o \, n_3 \qquad T_3(S) = \sigma_{23}^o \, n_2$$

Then

$$\sigma_{23}^u \geq \sigma_y/2 \tag{7.3.18}$$

The results (7.3.17-18) are evidently quite trivial. Unfortunately, however, better lower bounds do not seem to be available. It is also to be noted that the bounds (7.3.15), (7.3.17-18) become invalid in the case of fiber-matrix separation.

We now proceed to the problem of construction of <u>upper bounds</u> on the failure loads and for this purpose it is necessary to define a <u>kinematically admissible velocity field</u>, $\tilde{v}_\alpha$, which has to obey the following conditions

(a) Incompressibility in the matrix.

(b) The normal component of $\tilde{v}_\alpha$ on any internal surface must be continuous while the tangential component may be discontinuous.

(c) The normal component of $\tilde{v}_\alpha$ on fiber-matrix interface vanishes.

Next there is defined a kinematically admissible multiplier m_k as the ratio between the internal plastic dissipation, as computed from $\tilde{v}_\alpha$, and the external (virtual) rate of work of the tractions (7.3.1c) on the velocities $\tilde{v}_\alpha$, [7.19]. Then $m_k \sigma^o_{\alpha\beta} n_\beta$ are <u>upper bounds</u> on the failure tractions (7.3.14).

Following a treatment by Drucker, [7.21], it has been shown by Hashin, [7.22], that under certain circumstances the transverse plastic strength of a FRM is precisely equal to the plastic strength of the matrix. To describe this situation consider the principal shear stress direction associated with $\sigma^o_{\alpha\beta}$ in (7.3.1c). <u>Whenever it is possible to put a longitudinal plane through the FRM which cuts the transverse x_2 x_3 plane</u>

in the principal shear direction and does not pass through any fiber , the FRM strength is equal to the matrix strength. Thus fiber reinforcement does not produce increased strength in this case.

Fig. (7.3.1a) shows such a situation schematically. It should be emphasized that the underlying geometrical restriction on fiber placement is severe and can hardly occur in FRM with randomly placed fibers and substantial fiber volume fractions. On the other hand, in regular arrays such as hexagonal and square arrays of identical fibers, there are many possibilities to pass such a shear failure plane through the matrix. Fig. (7.3.1b) shows this for an hexagonal array and fig. (7.3.2) for two cases of square array. The reader will have no difficulty to construct other such shear failure planes for different geometries and different loadings.

The simple results obtained above support the experimentally observed phenomenon that the transverse strength of a FRM with stiff fibers is of the order of magnitude of matrix strength.

It is to be noted that in all cases described above the FRM fails in shear. Thus the failure criterion is of type (7.3.11) . It is easy to realize that this must always be the case for plane strain limit analysis of FRM, since by hypothesis the matrix is incompressible and the fibers are rigid. Thus a plane-isotropic external stress, (7.3.11), cannot, theoretically, produce deformation and thus no plastic failure.

Drucker, [7.23], has given a plastic failure analysis for a special two dimensional hexagonal geometry which may be interpreted as a FRM in which the fibers are hexagonal and are surrounded by very thin layers of matrix. The analysis is restricted to high fiber volume fractions. The results show that for 70% fibers the FRM transverse uniaxial strength is only about twice the matrix yield stress. Since the material with the ideal geometry considered is much more resistant to plastic flow than a real FRM, this result also indicates that stiff fiber reinforcement does not lead to substantial transverse strength.

Shu and Rosen [7.24] and Shu [7.25] have applied limit analysis methods to bound the transverse strength of FRM described by the composite cylinder assemblage model. Upper bounds were constructed on the basis of kinematically admissible velocities whose functional form is that of elastic displacements in composite cylinders, (3.6.85), par. 3.6.4. The simple results (7.3.15), (7.3.17-18) were used as lower bounds.

The bounds obtained were not close, as may be seen by the following numerical examples for pure transverse shear

$$v_2 = 0.5 \qquad 1 \le \frac{\sigma_{23}^u}{\sigma_y/2} \le 2.45$$

$$v_2 = 0.7 \qquad 1 \le \frac{\sigma_{23}^u}{\sigma_y/2} \le 5.0$$

where v_2 is the fibers volume fraction.

7.4 AXIAL SHEAR STRENGTH

The most general case of axial shear is defined by the boundary tractions

$$T_1(S) = \sigma^o_{12}\, n_2 + \sigma^o_{13}\, n_3$$

$$T_2(S) = \sigma^o_{12}\, n_1 \qquad (7.4.1)$$

$$T_3(S) = \sigma^o_{13}\, n_1$$

in which case the average stress tensor is given by

$$[\bar{\sigma}_{ij}] = \begin{bmatrix} 0 & \sigma^o_{12} & \sigma^o_{13} \\ \sigma^o_{12} & 0 & 0 \\ \sigma^o_{13} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \bar{\sigma}_{12} & \bar{\sigma}_{13} \\ \bar{\sigma}_{12} & 0 & 0 \\ \bar{\sigma}_{13} & 0 & 0 \end{bmatrix} \qquad (7.4.2)$$

If it is again assumed that failure is uniquely produced by some combination of average stress, then the failure criterion in the present case is

$$f(\bar{\sigma}_{12}, \bar{\sigma}_{13}) = \text{const.} \qquad (7.4.3)$$

If the material is assumed to be transversely isotropic in failure then (7.4.3) can depend only upon the invariants of (7.4.2) which are

$$I_1 = I_3 = 0$$

$$(7.4.4)$$

$$I_2 = -(\bar{\sigma}^2_{12} + \bar{\sigma}^2_{13})$$

Since the principal shear stress $\bar{\tau}$ in the x_2x_3 plane is given by

$$\bar{\tau} = \sqrt{\bar{\sigma}_{12}^2 + \bar{\sigma}_{13}^2} \tag{7.4.5}$$

it is seen that (7.4.3) reduces to

$$\bar{\tau} = \text{const.} \tag{7.4.6}$$

for transverse isotropy.

It is again advantageous to use limit analysis methods. It may be shown, [7.22], that also in the present case the matrix plastic shear strength is a lower bound on the FRM shear strength. Thus

$$\sigma_y/2 \leq \bar{\tau}_u \tag{7.4.7}$$

for any geometry.

It can also be shown that if a principal shear plane, thus an axial plane in which $\bar{\tau}$ is situated, can be passed entirely through matrix, the FRM shear strength is precisely the matrix shear strength. Thus in that case

$$\bar{\tau}_u = \sigma_y/2 \tag{7.4.8}$$

Examples for such situations can again be illustrated by figs. 7.3.1-2 where the principal shear plane is now to be interpreted as the plane of principal axial shear.

Shu and Rosen [7.24] have analyzed axial plastic shear strength on the basis of the composite cylinder assemblage model and have obtained close bounds. An example is given below

$$v_2 = 0.7 \qquad 1 \le \frac{\bar{\tau}_u}{\sigma_y/2} \le 1.19$$

All the results quoted here indicate that axial shear strength of FRM is essentially the same as axial shear strength of the unreinforced matrix.

7.5 FAILURE CRITERIA FOR COMBINED LOADING

In the preceding chapters we have considered FRM under separate simple states of average stress . We consider now the case of general homogeneous traction boundary conditions

$$T_i(S) = \sigma^o_{ij} n_j \tag{7.5.1}$$

which may thought to be applied to some cylindrical specimen. Then the average stresses are

$$\bar{\sigma}_{ij} = \sigma^o_{ij} \tag{7.5.2}$$

If we again adopt the assumption that failure is due to some combination of average stress, then the failure criterion has the form

$$f(\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{33}, \bar{\sigma}_{12}, \bar{\sigma}_{23}, \bar{\sigma}_{13}) = \text{const.} \tag{7.5.3}$$

It is again assumed that the material is transversely isotropic in failure, the axis of transverse isotropy and fiber direction being x_1. This transverse isotropy implies that if the same numerical values of $\bar{\sigma}_{ij}$ in (7.5.3), which produce failure, are applied in reference to a coordinate system $x_1, x_2' x_3'$ which is defined by any rotation about the x_1 axis, the specimen will fail. If the failure stresses are transformed to the $x_1 x_2 x_3$ coordinate system, $\bar{\sigma}_{11}$ will remain the same but all others will change. These transformed stresses must however satisfy the failure criterion (7.5.3) since failure can evidently not depend on the choice of coordinate system to which the failure stresses are referred. Since this condition must be met for any rotation about the x_1 axis

it is concluded that (7.5.3) can be at most a function of average stress expressions which are invariant under a rotation about x_1. Those invariants are the previously considered (7.3.5) and (7.4.4) which may be taken in the forms (7.3.9) and (7.3.5), respectively. It is therefore concluded that the transversely isotropic form of the failure criterion (7.5.3) is given by

$$f(\bar{\sigma}_A, \bar{\sigma}, \bar{\tau}_T, \bar{\tau}_A) = \text{const.} \tag{7.5.4}$$

where

$$\bar{\sigma}_A = \bar{\sigma}_{11} \qquad \text{axial stress} \qquad \text{(a)}$$

$$\bar{\sigma} = \frac{1}{2}(\bar{\sigma}_{22} + \bar{\sigma}_{33}) \qquad \text{transverse plane - isotropic stress} \qquad \text{(b)}$$

$$\bar{\tau}_T = \pm \frac{1}{2}\sqrt{(\bar{\sigma}_{22} - \bar{\sigma}_{33})^2 + 4\bar{\sigma}_{23}^2} \qquad \text{transverse principal shear stress} \qquad \text{(c)} \tag{7.5.5}$$

$$\bar{\tau}_A = \pm\sqrt{\bar{\sigma}_{12}^2 + \bar{\sigma}_{13}^2} \qquad \text{axial principal shear stress} \qquad \text{(d)}$$

On the basis of the discussion given in chap. 7.2 it is seen that a FRM has different uniaxial tensile and compressive failure stresses. The same phenomenon can be expected, on physical grounds, for failure under transverse plane-isotropic stress. Such inequality of tensile and compressive failure stresses may be called Bauschinger effect in accordance with plasticity theory nomenclature. In contrast the shear failure will evidently not depend

upon the directions of the shear stresses. It is therefore concluded that (7.5.4) is an even function of $\bar{\tau}_T$ and $\bar{\tau}_A$, but not of $\bar{\sigma}_A$ and $\bar{\sigma}$.

The functional form of (7.5.4a) is of course unknown and its establishment on the basis of material structure can be regarded at the present time as a prohibitively difficult problem.

In the rather special case of elastic-ideally plastic fibers and matrix the problem is more tractable. For this case Hill [7.26] has given an approximate analysis of uniaxial FRM under combined uniaxial and plane-isotropic loading. McLaughlin and Batterman [7.27] have investigated general yield and failure conditions, including possibility of fiber buckling and pull out. The most important case is, however, brittle fibers in which case, as has been seen previously, transverse and axial shearing failure modes are fundamentally different than axial tensile and compressive failure modes. Therefore, the case of combined such loadings is one of very great difficulty. Consequently, we shall limit ourselves to discussion of the possible form of such a failure criterion on the basis of purely macroscopic considerations.

It is recalled that the fundamental assumption underlying the present failure criterion is (7.5.3) which implies that failure is produced by some combination of the average stresses. With this assumption the problem becomes analogous to that of the establishment of failure criteria or initial yield criteria in "homogeneous" bodies, as long as these are assumed to be

dependent on the stresses only. For a recent comprehensive discussion of such failure and yield criteria the reader is referred to Paul,[7.28].

A popular form of criterion is a quadratic stress polynomial. Such a criterion for a generally anisotropic material has been proposed by Mises [7.29] in the context of plastic yielding and may be equally postulated for failure. For a macroscopically <u>orthotropic</u> material the Mises criterion may be written in the normalized form

$$A_{11}\bar{\sigma}_{11}^{2} + A_{22}\bar{\sigma}_{22}^{2} + A_{33}\bar{\sigma}_{33}^{2} + A_{12}\bar{\sigma}_{11}\bar{\sigma}_{22} + A_{23}\bar{\sigma}_{22}\bar{\sigma}_{33}$$
$$+ A_{13}\bar{\sigma}_{11}\bar{\sigma}_{33} + A_{44}\bar{\sigma}_{12}^{2} + A_{55}\bar{\sigma}_{23}^{2} + A_{66}\bar{\sigma}_{13}^{2} = 1 \qquad (7.5.6)$$

This form indicates that the quadratic stress polynomial is equal to a constant. Then both sides are divided by the constant to obtain the present normalized form.

The constants in (7.5.6) must be found by subjecting the material to various simple failure tests. Thus A_{44}, A_{55}, A_{66}, can be determined from pure shear failure experiments; A_{11}, A_{22}, A_{33} can be found from uniaxial failure tests while A_{12}, A_{23}, A_{13} must be determined from biaxial loading failure tests. The last kind of tests are unfortunately quite difficult to perform.

If it is desired to incorporate a normal stress Bauschinger effect into (7.5.6) there are two possibilities to do this. Firstly, the coefficients A_{11},..... A_{13} may be interpreted as having different values for different kinds of loadings such as uniaxial tension, uniaxial compression, biaxial tension-tension,

tension-compression, etc. Secondly, linear normal stress terms may be added to (7.5.6), which then assumes the form

$$A_{11}\bar{\sigma}_{11}^{2} + A_{22}\bar{\sigma}_{22}^{2} + A_{33}\bar{\sigma}_{33}^{2} + A_{12}\bar{\sigma}_{11}\bar{\sigma}_{22} + A_{23}\bar{\sigma}_{22}\bar{\sigma}_{33}$$

$$+ A_{13}\bar{\sigma}_{11}\bar{\sigma}_{33} + B_1\bar{\sigma}_{11} + B_2\bar{\sigma}_{22} + B_3\bar{\sigma}_{33} + A_{44}\bar{\sigma}_{12}^{2} + A_{55}\bar{\sigma}_{23}^{2} + A_{66}\bar{\sigma}_{13}^{2} = 1 \tag{7.5.7}$$

Evaluation of (7.5.7) for failure in uniaxial tension or compression in x_1, x_2, x_3 directions, respectively, and for 12, 23, 13, shear failures, respectively, easily yields the results

$$A_{11} = \frac{1}{{}_{(+)}\bar{\sigma}_{11}^{u} \cdot {}_{(-)}\bar{\sigma}_{11}^{u}} \qquad B_1 = \frac{1}{{}_{(+)}\bar{\sigma}_{11}^{u}} - \frac{1}{{}_{(-)}\bar{\sigma}_{11}^{u}}$$

$$A_{22} = \frac{1}{{}_{(+)}\bar{\sigma}_{22}^{u} \cdot {}_{(-)}\bar{\sigma}_{22}^{u}} \qquad B_2 = \frac{1}{{}_{(+)}\bar{\sigma}_{22}^{u}} - \frac{1}{{}_{(-)}\bar{\sigma}_{22}^{u}}$$

$$A_{33} = \frac{1}{{}_{(+)}\bar{\sigma}_{33}^{u} \cdot {}_{(-)}\bar{\sigma}_{33}^{u}} \qquad B_3 = \frac{1}{{}_{(+)}\bar{\sigma}_{33}^{u}} - \frac{1}{{}_{(-)}\bar{\sigma}_{33}^{u}} \tag{7.5.8}$$

$$A_{44} = \left(\frac{1}{\bar{\sigma}_{12}^{u}}\right)^2 \qquad A_{55} = \left(\frac{1}{\bar{\sigma}_{23}^{u}}\right)^2 \qquad A_{66} = \left(\frac{1}{\bar{\sigma}_{13}^{u}}\right)^2$$

where ${}_{(+)}\bar{\sigma}^{u}_{11}$, ${}_{(-)}\bar{\sigma}^{u}$ etc. are tensile and compressive uniaxial failure stresses, respectively, and $\bar{\sigma}^{u}_{12}$ etc. are shear failure stresses in pure shear loadings.

Unfortunately, A_{12}, A_{23} and A_{13} still have to be evaluated in terms of biaxial stress failure values and each must assume four different values for different loading regimes, in order to cope with biaxial Bauschinger effect.

The failure criteria (7.5.6-7) are frequently simplified by the assumption that isotropic (e.g. hydrostatic) stress does not produce failure. With this assumption (7.5.7) assumes the form

$$C_{12}(\bar{\sigma}_{11} - \bar{\sigma}_{22})^2 + C_{23}(\bar{\sigma}_{22} - \bar{\sigma}_{33})^2 + C_{13}(\bar{\sigma}_{11} - \bar{\sigma}_{33})^2 + \\ + B_1\bar{\sigma}_{11} + B_2\bar{\sigma}_{22} + B_3\bar{\sigma}_{33} + A_{44}\bar{\sigma}^2_{12} + A_{55}\bar{\sigma}^2_{23} + A_{66}\bar{\sigma}^2_{13} = 1 \qquad (7.5.9)$$

where

$$2C_{12} = A_{11} + A_{22} - A_{33} \qquad \text{(a)}$$

$$2C_{23} = A_{22} + A_{33} - A_{11} \qquad \text{(b)}$$

$$2C_{13} = A_{11} + A_{33} - A_{22} \qquad \text{(c)} \qquad (7.5.10)$$

$$B_1 + B_2 + B_3 = 0 \qquad \text{(d)}$$

It should, however, be borne in mind that the assumption that failure does not occur under isotropic stress is much less justified for anisotropic materials than for isotropic materials. In the latter case justification may be provided by the fact that isotropic stress produces isotropic strain and vice versa.

There is thus no shear stress or shear strain anywhere, and if it is assumed that failure is primarily associated with shearing on some plane, the assumption is not unreasonable. In an anisotropic material, however, the situation is different. Here isotropic stress produces a nonisotropic state of strain. Thus shearing strains do occur on some planes and it is possible that they may lead to failure for values of isotropic stress which are not extremely elevated. It would seem therefore that the validity of the simplification (7.5.9-10) merits further investigation.

It is seen that the advantage of (7.5.9) over (7.5.7) is in that it is no longer necessary to determine coefficients of mixed products from biaxial tests. On the other hand, there arises the difficulty that (7.5.10d) will not in general be fulfilled by the values of B_1, B_2, B_3 in (7.5.8).

Hoffman [7.30] has shown good agreement between (7.5.9) and experimental data. He has, however, disregarded the requirement (7.5.10d), without which (7.5.9) is inconsistent.

Hill [7.31] has used (7.5.9) without the linear terms as yield criterion for orthotropic ideally plastic materials. Tsai [7.32] and Azzi and Tsai [7.33] have used the same criterion for failure of FRM and have reported good agreement with experimental results. They have chosen to account for Bauschinger effect by assigning to coefficients different values in tension and compression regimes. If this approach is adopted then (7.5.9) assumes the form

$$C_{12}(\bar{\sigma}_{11} - \bar{\sigma}_{22})^2 + C_{23}(\bar{\sigma}_{22} - \bar{\sigma}_{33})^2 + C_{13}(\bar{\sigma}_{11} - \bar{\sigma}_{33})^2$$

$$+ A_{44}\ \bar{\sigma}_{12}^2 + A_{55}\ \bar{\sigma}_{23}^2 + A_{66}\ \bar{\sigma}_{13}^2 = 1 \tag{7.5.11}$$

The coefficients A_{44}, A_{55}, A_{66} are still given by the last of (7.5.8) but the coefficients C_{12}, C_{23}, C_{13} must now be assigned a variety of different values in order to cope with all possibilities of combined normal stress failure, e.g. tensile-tensile-tensile, tensile-tensile-compressive, etc.

Returning now to the failure criterion (7.5.4) for macroscopically transversely isotropic materials, it is easily realized that it is a special case of failure criteria for orthotropic materials. Thus if it is assumed quadratic, its most general form for x_1 axis of transverse isotropy, is obtained from (7.5.7-8) by setting the pairs ${}_{(+)}\bar{\sigma}^u_{22}$, ${}_{(+)}\bar{\sigma}^u_{33}$; ${}_{(-)}\bar{\sigma}^u_{22}$, ${}_{(-)}\bar{\sigma}^u_{33}$; $\bar{\sigma}^u_{12}$, $\bar{\sigma}^u_{13}$ equal. Then (7.5.7) reduces to

$$A_{11}\ \bar{\sigma}_{11}^2 + A_{22}(\bar{\sigma}_{22}^2 + \bar{\sigma}_{33}^2) + A_{12}\ \bar{\sigma}_{11}\ (\bar{\sigma}_{22} + \bar{\sigma}_{33}) + A_{23}\ \bar{\sigma}_{22}\ \bar{\sigma}_{33}$$

$$+ B_1\ \bar{\sigma}_{11} + B_2(\bar{\sigma}_{22} + \bar{\sigma}_{33}) + A_{44}(\bar{\sigma}_{12}^2 + \bar{\sigma}_{13}^2) + A_{55}\bar{\sigma}_{23}^2 = 1 \tag{7.5.12}$$

It is easily seen that if (7.5.4) is assumed quadratic in the shear stresses and quadratic and linear in $\bar{\sigma}_A$ and $\bar{\sigma}$, the same form as (7.5.11) will result on substitution of (7.5.5).

The simplification to transverse isotropy does not mitigate the problems which arose in fitting orthotropic quadratic failure criteria to experimental data

with Bauschinger effect. The choices are again: use of (7.5.12) with multiple values of A_{12}, A_{23}; simplification of form (7.5.9) which would imply that $B_1 + 2B_2 = 0$, which condition is not generally fulfilled; or choice of form (7.5.11) which leads to a multiplicity of values for normal stress coefficients.

The form of the failure criterion for a uniaxial FRM enters naturally into the problem of the determination of the strength of a laminate. As has been seen in chap. 3.10, a lamina in a laminate is generally in a complex state of stress. As a first approximation the stress normal to the lamina, σ_{33}, can be neglected and as a second approximation the shear stresses σ_{13}, σ_{23} can be considered of secondary nature. So a single lamina is at least in the plane state of stress σ_{11}, σ_{22}, σ_{12} which varies linearly through its thickness, according to plate and shell theory.

The fundamental problem which arises is: given the failure criteria of single laminae (it is well to remember that we know little about them), to establish the failure criterion of the laminate.

Two approximate approaches, of similar philosophy, to this problem have been given by Tsai and Azzi [7.34] and by Dow and Rosen [7.35]. It is assumed in both approaches that when a certain lamina, or group of laminae, fails, its further response to continued loading is changed in some fashion. In [7.34] the changed response was treated by modification of lamina elastic stiffnesses, in that it was assumed that certain stiffnesses reduce to zero after lamina failure while others remain operative. In [7.35] it was assumed that after lamina failure certain stress components in the lamina remain at their

failure levels while others can continue to increase. The predictions of both approaches are in good agreement with experimental data for laminate failure.

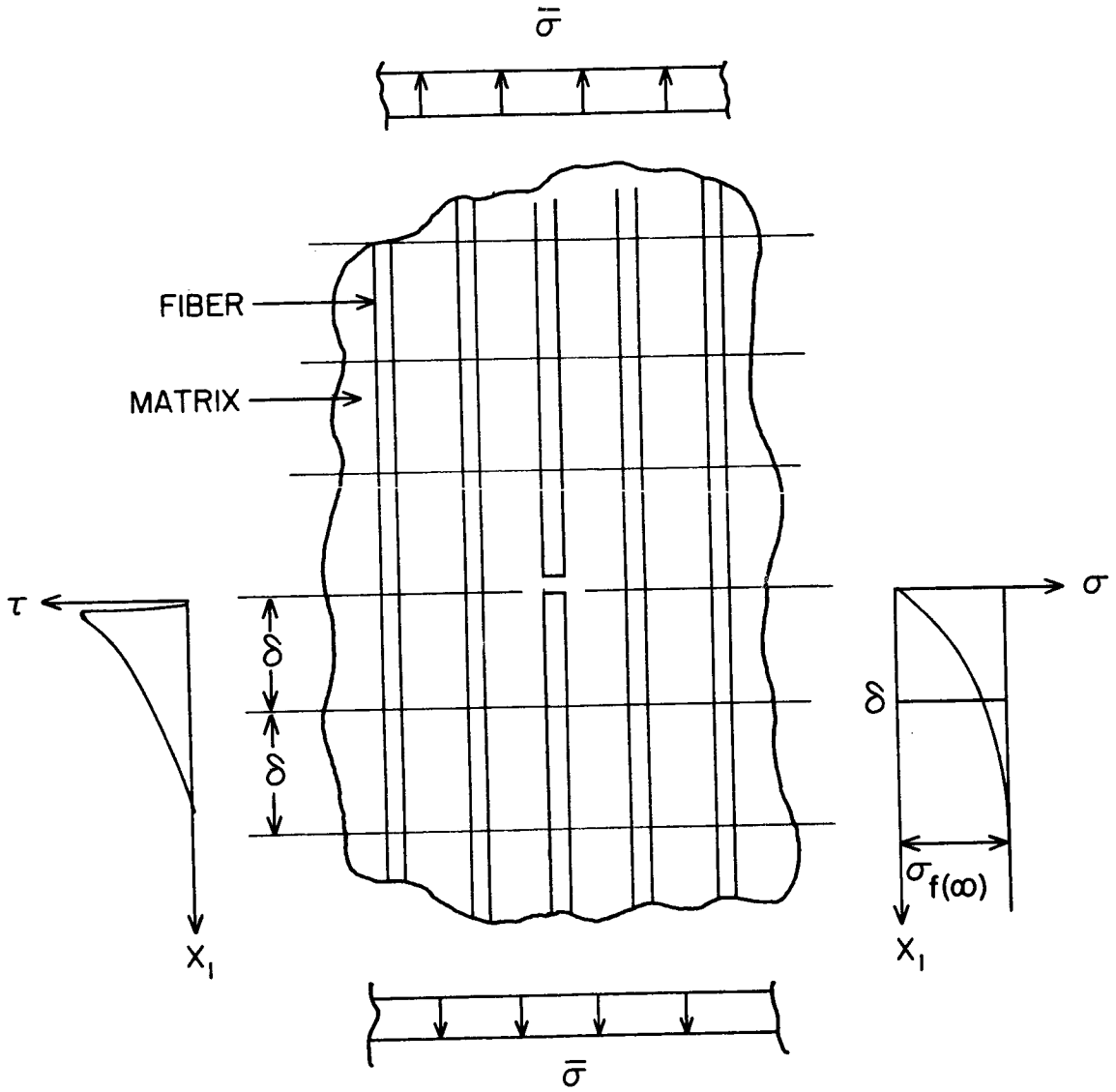


FIG. 7.2.1 - STRESSES NEAR FIBER BREAKS - SCHEMATIC

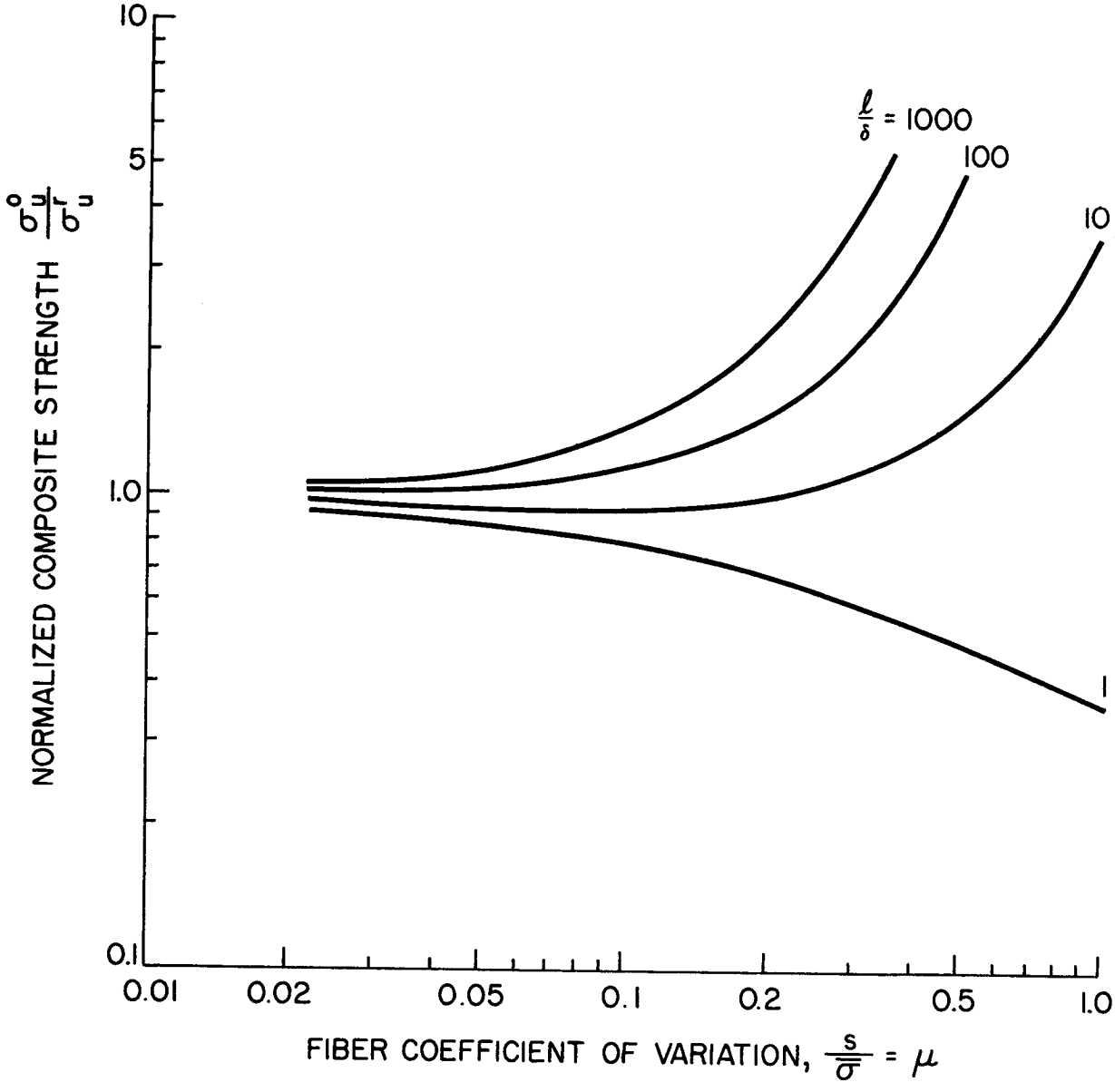
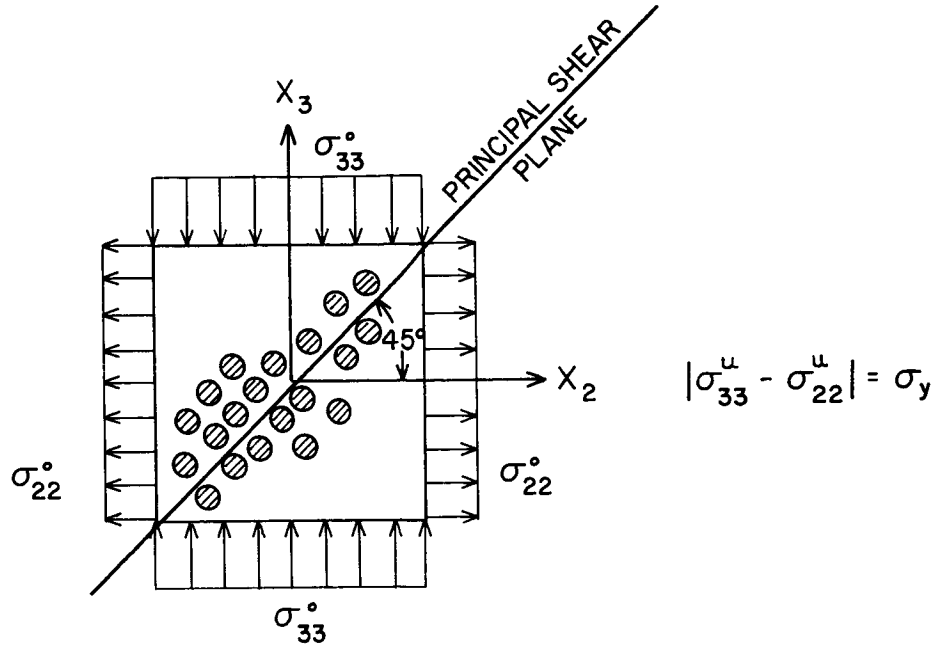
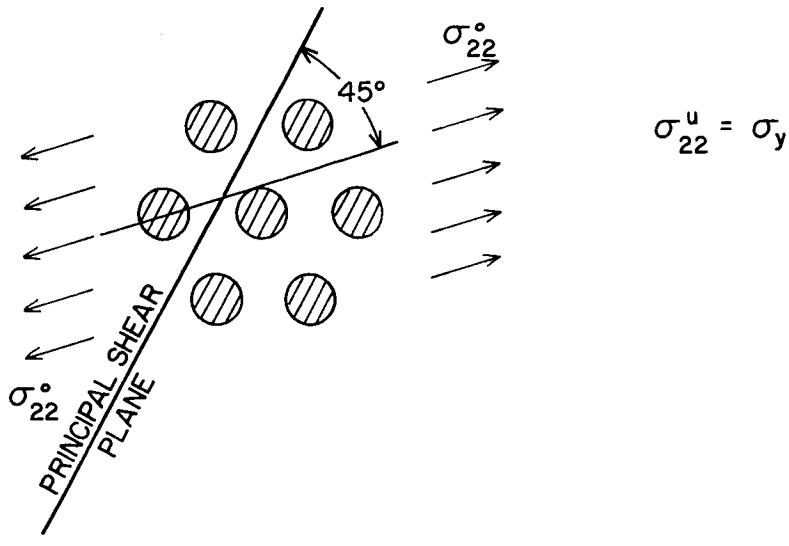


FIG. 7.2.2 - EFFECT OF FIBER COEFFICIENT OF VARIATION AND REFERENCE FIBER STRENGTH UPON THE RATIO OF COMPOSITE TENSILE STRENGTH (CUMULATIVE DAMAGE MODE) TO MEAN FIBER STRENGTH

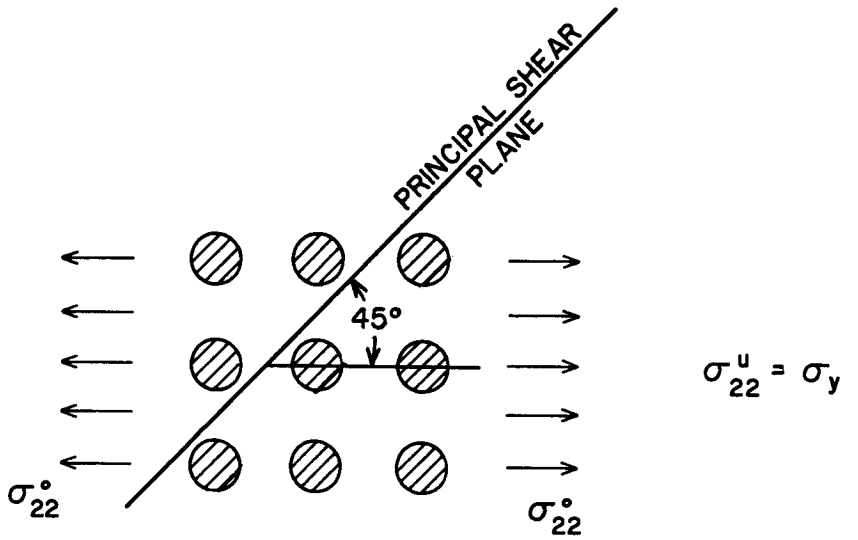


(a) BIAXIAL STRESS

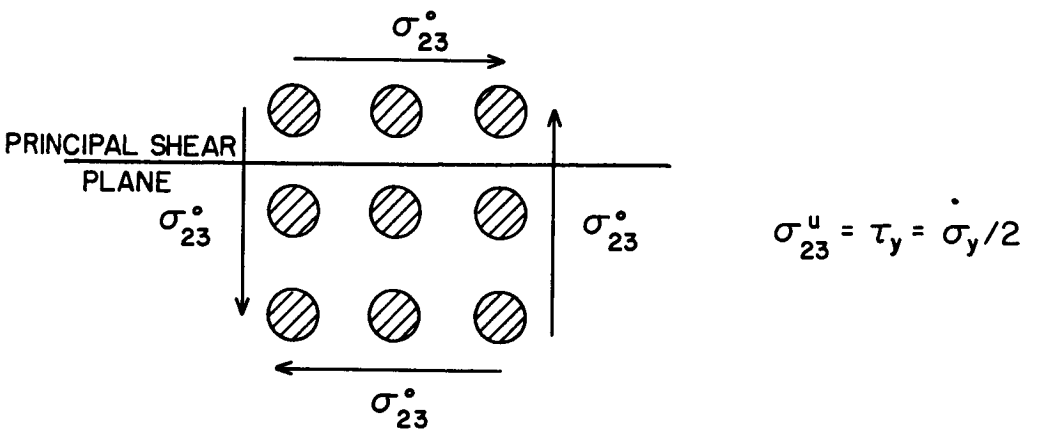


(b) HEXAGONAL ARRAY; UNIAXIAL STRESS

FIG. 7.3.1 - PLASTIC TRANSVERSE STRENGTH FOR SOME GEOMETRIES



(a) UNIAXIAL TRANSVERSE STRESS



(b) TRANSVERSE SHEAR

FIG. 7.3.2 - PLASTIC TRANSVERSE STRENGTH OF SOME SQUARE ARRAYS

Table 7.1.1 Specific Weights and Tensile Strengths of Fiber and Matrix Materials

	Material	Specific Weight gm/cm^3	Specific Weight lb/in^3	Tensile Strength ksi
Fiber	E-Glass	2.54	.0917	500
	S-Glass	2.48	.0895	650
	Carbon	1.9	.0686	300
	Boron	2.53	.0915	400
	Al_2O_3	4.6	.143	1200
	Si C	3.2	.123	1400
Matrix	Magnesium	1.75	.063	35
	Aluminum	2.72	.098	70
	Epoxy (typical)	1.38	.050	10

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PRINCIPAL NOTATION

Latin Symbols

a -	radius of circular fiber
A -	area of transverse cross section of cylindrical specimen
$A_{\alpha\beta\gamma\delta}, A_{pq}$ -	membrane elastic stiffness tensor of laminated plate
b -	radius of composite cylinder
$B_{\alpha\beta\gamma\delta}, B_{pq}$ -	membrane/flexure coupling stiffness tensor of laminated plate
c -	fiber volume fraction; wave velocity, pt. 4
c_p -	specific heat at constant stress
c_v -	specific heat at constant strain
C -	boundary of transverse section of cylinder
C_{12} -	phases 1, 2 interfaces in transverse section of cylinder
C_{ijkl} -	elastic moduli tensor
C_{ij} -	elastic moduli in six-by-six "engineering" notation
${}^{e}C_{ijkl}$ -	elastic moduli tensor, pt. 4
$C_{ijkl}(t)$ -	relaxation moduli tensor
D -	time derivative operator, pt. 4
$\underline{D}, D_i$ -	flux vector, (thermal, electric, magnetic)

D_{ijkl} -	complex moduli tensor
$D_{pq}, D_{\alpha\beta\gamma\delta}$ -	flexural stiffness tensor of laminated plate
e_{ij} -	deviatoric part of strain tensor
$e(t)$ -	isotropic Young's creep compliance
$e_A(t)$ -	axial Young's creep compliance } transverse isotropy
$e_T(t)$ -	transverse Young's creep compliance } transverse isotropy
E -	isotropic Young's modulus
$\dot{E}(t)$ -	isotropic Young's relaxation modulus
E_A -	axial Young's modulus } transverse isotropy
E_T -	transverse Young's modulus } transverse isotropy
E_1, E_2, E_3 -	Young's moduli, orthotropy
$f(\ )$ -	probability density function, pt. 7
$F(\ )$ -	cumulative probability function, pt. 7
F_i -	body force vector
$\mathcal{F}$ -	Fourier transform operator
$\{g\}$ -	phase geometry, symbolic
$g_A(t)$ -	axial shear creep compliance } transverse isotropy
$g_T(t)$ -	transverse shear creep compliance } transverse isotropy

G -	isotropic elastic shear modulus	
$G(t)$ -	isotropic shear relaxation modulus	
${}^{e}G$ -	isotropic elastic shear modulus, pt. 4	
G_A -	axial shear modulus	transverse isotropy
$G_A(t)$ -	axial shear relaxation modulus	transverse isotropy
G_T -	transverse shear modulus	transverse isotropy
$G_T(t)$ -	transverse shear relaxation modulus	transverse isotropy
G'_T -	second shear modulus, square symmetry	
G_{12}, G_{23}, G_{13} -	shear moduli, orthotropy	
h -	plate thickness	
h_m, h_n -	reference heights of layers in laminate	
H -	height of cylindrical specimen	
$H(t)$ -	Heaviside unit step function	
$\underline{H}$, H_i -	intensity vector (thermal, electric, magnetic)	
I -	moment of inertia of section	
I' -	torsional rigidity divided by shear modulus	
I_1, I_2, I_3 -	invariants of stress tensor	
$\underline{I}$, I_{ijkl} -	fourth rank symmetric unit tensor	
$I(t)$ -	isotropic bulk creep compliance	

$J(t)$ -	isotropic shear creep compliance
k -	transverse bulk modulus, transverse isotropy; plane strain bulk modulus, isotropy
${}^{e}k$ -	same as above, pt. 4
$\kappa(t)$ -	transverse bulk creep compliance
K -	isotropic elastic bulk modulus
$K(t)$ -	isotropic bulk relaxation modulus
ℓ -	elastic modulus, transverse isotropy; length of beam or fiber
$\mathcal{L}$ -	Laplace transform operator
M -	number of phases; elastic modulus
$M_{\alpha\beta}$ -	moment tensor in plate
n_i -	components of outward normal
n -	elastic modulus, transverse isotropy
$N_{\alpha\beta}$ -	membrane force tensor
p -	isotropic part of stress polarization tensor; transverse load per unit area of plate; Laplace transform variable
p_α -	in plane loads per unit area of plate
p_{ij} -	stress polarization tensor
$\tilde{p}_{ij}$ -	admissible stress polarization

$P(\quad)$ -	probability
P_i -	one point probability
P_{ij} -	two point probability
q -	load per unit length of beam
q_{ij} -	deviatoric part of stress polarization tensor
r -	radial coordinate
$\underline{r}$ -	radial vector
$r(t)$ -	axial/transverse creep compliance, chap. 4.3
R -	region
R_{ijkl} -	complex compliances tensor
s -	standard deviation
$s(t)$ -	transverse/axial creep compliance, chap. 4.3
s_{ij} -	deviatoric part of stress tensor
S -	bounding surface
S_{12} -	phases 1, 2 interfaces
S_{ijkl} -	elastic compliances tensor
S_{ij} -	elastic compliances tensor in six-by-six "engineering" notation
$S_{ijkl}(t)$ -	creep compliances tensor

U^{σ} - stress energy

$\tilde{U}^{\sigma}$ - stress energy functional

v - volume fraction

v_{2c} - fiber volume fraction relative to composite cylinder

v_c - volume fraction of composite cylinders relative to composite

v_i - velocity vector

V - volume

w - transverse deflection of beam or plate

W^{ϵ} - elastic strain energy density

W^{σ} - elastic stress energy density

x_i - cartesian coordinates

$\underline{x}$ - position vector from origin

y_i - local cartesian coordinates

z - axial coordinate, cylindrical system

Greek Symbols

α - isotropic thermal expansion coefficient

α_{ij} - thermal expansion tensor

α_A - axial thermal expansion coefficient } transverse isotropy

α_T - transverse thermal expansion coefficient } transverse isotropy

γ - ratio, fiber to matrix shear modulus

γ_R - ratio, fiber shear modulus to real part of matrix complex shear modulus

γ_{ij} - strain rate tensor

Γ - transform domain (TD) shear modulus; gamma function

Γ_{ijkl} - transform domain (TD moduli tensor

δ - deflection; loss angle; ineffective fiber length

$\delta(t)$ - time dependent deflection; time delta function

δ_{ij} - three dimensional Kronecker delta

$\delta_{\alpha\beta}$ - two dimensional Kronecker delta

∇ - del operator

∇^2 - Laplace operator

ϵ - isotropic part of strain tensor

ϵ_{ij} - strain tensor

η - viscosity coefficient

θ angular coordinate, cylindrical system

ι - square root of minus one

κ - transform domain (TD) bulk modulus

κ_α - plane Fourier transform variables

$\kappa_{\alpha\beta}$ - plate curvature tensor

λ - elastic Lame' modulus, isotropy

μ - isotropic conductivity; statistical coefficient of variation

μ_{ij} - conductivity tensor

μ_A - axial conductivity } transverse isotropy

μ_T - transverse conductivity

ν - isotropic elastic Poisson's ratio

ν_A - axial Poisson's ratio } transverse isotropy

ν_T - transverse Poisson's ratio

ν_{ij} - Poisson's ratios of orthotropic material
$i \neq j$

ξ - isotropic resistivity

ξ_A - axial resistivity } transverse isotropy

ξ_T - transverse resistivity

ρ - density

σ - isotropic part of stress tensor

σ_{ij} - stress tensor

$\tilde{\sigma}_{ij}$ - admissible stress field

τ - time; shear stress

φ - plane harmonic function; potential; temperature

ω - circular frequency

Subscripts

c - composite cylinder assemblage

$i, j, k, l, \ell, m, n, r, s$ - range over 1,2,3

$\alpha, \beta, \gamma, \delta$ - range over 2,3;
over 1,2 in chap. 3.10

1 - phase 1, indicates matrix in the case of Fiber Reinforced Material

2 - phase 2, indicates fibers in the case of Fiber Reinforced Material

(-) - lower bound

(+) - upper bound

,i - partial derivative with respect to x_i

Superscripts

I - imaginary part of

R - real part of

(m) - m^{th} phase

(1) - phase 1

(2) - phase 2

* - effective property

Overscripts

$\dot{}$ - time derivative

$\bar{}$ - average

^ - Fourier transform, Append. to Chap. 3.7;
Laplace transform, Pt. 4

~ - admissible field, Pt. 3;
complex, Pt. 4

LIST OF ABBREVIATIONS

CCA - Composite Cylinder Assemblage

EC - Effective Conductivity

ECC - Effective Creep Compliance

ECM - Effective Complex Modulus

EEC - Effective Elastic Compliance

EEM - Effective Elastic Modulus

ERM - Effective Relaxation Modulus

ETEC - Effective Thermal Expansion Coefficient

FEA - Fiber Embedding Approximation

FM - Fibrous Material

FRM - Fiber Reinforced Material

FT - Fourier Transform

LT - Laplace Transform

RVE - Representative Volume Element

SH - Statistically Homogeneous

SI - Statistically Isotropic

SNH - Statistically NonHomogeneous

STI - Statistically Transversely Isotropic

TD - Transform Domain