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CONCAVE HOLOGRAPHIC GRATING FOR SPECTROGRAPHIC APPLICATIONS

Study of the aberrations

by G. PIEUCHARD

and J. FLAMAND

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by Mr G. Pieuchard
Mr J. Flamand.

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STATEMENT OF THE GENERAL METHOD USED

In this first chapter we essentially introduce the general method that we intend to use.

I -1 - BASIC PRINCIPLE OF CALCULATION

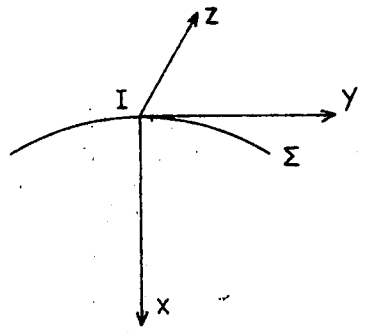

Fig. 1

The optical surface is given in the X Y Z coordinate system. The axis are : $\overrightarrow{IX}$ perpendicular to the surface Σ at I. $\overrightarrow{IY}$ and $\overrightarrow{IZ}$ rectangular axis located on the plane tangent to Σ at I.

Consequently the equation of a sphere is the following :

$$X^2 + Y^2 + Z^2 - 2R\,X = 0$$

From a more general point of view the equation of second degree surface will be :

$$u\,X^2 + v\,Y^2 + w\,Z^2 - 2R\,X = 0$$

A point in space will be located by three coordinates x y z and so we have :

$$\ell^2 = x^2 + y^2 + z^2$$

I - 2 - GENERAL CONDITIONS FOR STIGMATISM

From the principle of Fermat we know that B is the stigmatic image of the object point A if the relation A M + M B = cst is satisfied. (1)

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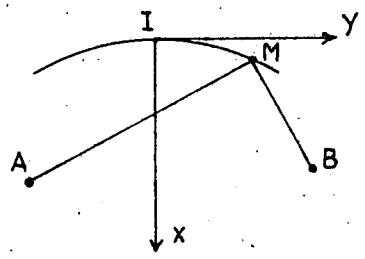

Fig. 2

In this relation M is a point of the surface Σ.

When the relation is not satisfied we may describe the defects of B image of A by a difference between the optical path A M + M B and a constant that is generally chosen egal to A I + I B , I being the origin of the coordinates system.

Therefore, we are studying now the aberrant optical path

(2) $\Delta = (A\,M + M\,B) - (A\,I + I\,B)$

I - 3 - STUDY OF THE ABERRANT OPTICAL PATH

This study of Δ leads us to obtain a general series expansion

(3) $\Delta = \mathcal{A}\,Y^2 + \mathcal{A}'\,Z^2 + CY^3 + DY^2\,Z + EY^4$

Conventionally we designate :

astigmatism : the terms of second order with regard to Y and Z,

coma : the terms of third order with regard to Y and Z

spherical aberration : the terms of forth order with regard to Y and Z.

We shall demonstrate in our last report how it is possible to evaluate the widening of the image B from Δ .

Our work is based on the study of B image of the point A.

.../..

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Generally we whall choose A in the plan (I X Y).

II - HOLOGRAPHIC GRATING PROPERTIES

An holographic grating is made by recording on a sensitive surface Σ the interference pattern of two coherent waves emitted by two points C and D.

The surfaces of same phase are revolution hyperboloids of axis C D and focus C and D.

Under such conditions, we see that the relation (2) can be written as follows :

$$(4) \quad M\,A + M\,B - (I\,A + I\,B) - k\left[(M\,C - M\,D) - (IC - ID)\right]\frac{\lambda}{\lambda_o} = 0$$

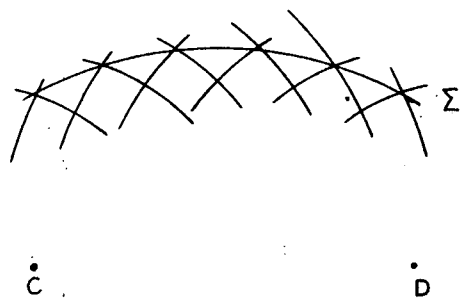

Fig. 3

In this relation ,

λ_o represents the wawelength of the radiation issued from the recording points C and D,

λ represents the wawelength at B image of A,

k represents the order in which the holographic grating operates.

II - 1 - ABSOLUTE STIGMATISM OF HOLOGRAPHIC GRATINGS GENERAL CONDITIONS

We can write $\mathcal{P} = I\,A + I\,B - \frac{k\lambda}{\lambda_o}\,(I\,C - I\,D)$ (5)

.../..

.../..

$\mathcal{P}$ is a specific constant of the group of homologus points A B C D under study.

Then the stigmatism condition is

$$MA + MB - \frac{k\lambda}{\lambda_o}(MC - MD) = \mathcal{P} = \text{constant} . \qquad (6)$$

If the points A B C and D are different such a relation determines a surface of the sixteenth degree.

More generally, a relation of that kind with n points determines a surface of the 2^n th degree.

To keep solutions physically feasible we must consider only surfaces of the second or forth order.

We have a solution when the relation degenerate so as to use two distinct points only.

The relation (6) can be written as followed :

(7) $\nu MP + \mu MQ = \text{cste}$ $\qquad$ ν and μ are constant.

Then the relation (7) determines a surface Σ on which the grating is recorded.

II- 2- APPLICATION OF THE GENERAL CONDITION

The stigmatism condition is the following one :

$$\boxed{MA + MB - k\frac{\lambda}{\lambda_o}(MC - MD) = \mathcal{P} = \text{cst}} \qquad (6)$$

- In this case we cannot have C = D
- Some cases of degeneration may happen

1° A = B = C

Then we have : $MA\left(2 - \frac{k\lambda}{o}\right) + k\frac{\lambda}{\lambda_o} MD = \mathcal{P}$

We have a rigorous stigmatism in the case of autocollimation on A

.../..

.../..

with a given wavelength that can be chosen everywhere in the whole spectrum if Σ is the surface of Descartes given by this relation.

2°
$$\begin{cases} A = B = C \\ \lambda = \dfrac{2\lambda_o}{k} \qquad k > o \\ MD = Cst \end{cases}$$

$\longrightarrow$ MD = cst imply that Σ is a sphere of centre D.

Therefore, there is a rigorous stigmatism for $\dfrac{2\lambda_o}{k}$ in the case of autocollimation on the source A with A on C.

3° We shall write A = D In this case

(6) becomes : $MA\left(1 + \dfrac{k\lambda}{\lambda_o}\right) + MB - \dfrac{k\lambda}{\lambda_o} MC = \mathcal{P}$

a) More than if we put $\lambda = -\dfrac{\lambda_o}{k}$ with $k < o$

We can write $MB + MC = \mathcal{P}$

$\longrightarrow$ Σ must be an ellipsoid with focus in B and C.

If the surface Σ is the ellipsoid with focus in B and C (C is a recording point source) we have rigorous stigmatism at the wavelength $-\dfrac{\lambda_o}{k}$

in the negative orders when the source is put on D.

b) If, furthermore MA = cst with B = C

Then Σ is a sphere with centre A, D.
We observe at point C.

There is stigmatism on C when the wavelength is $\dfrac{\lambda_o}{k}$ $k > o$

c) We always have $\begin{cases} A = D \\ MA = cst. \end{cases}$

A and D are on the centre of a portion of a sphere.

If furthermore we have

(8) $\boxed{MB - \dfrac{k\lambda}{\lambda_o} MC = o \quad \text{whatever is M.}}$

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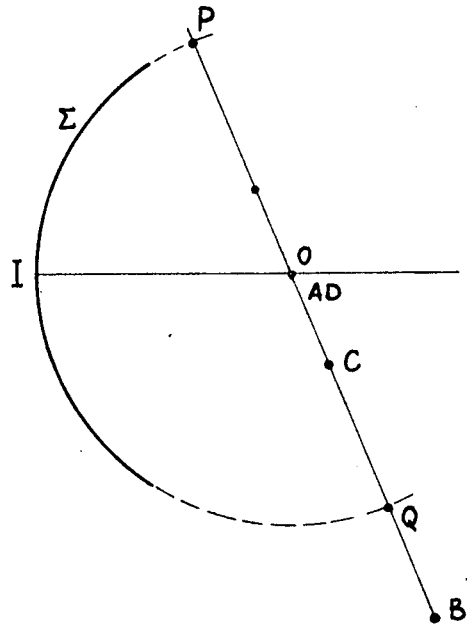

$$\frac{BP}{BQ} = -\frac{CP}{CQ}$$

Fig: 4

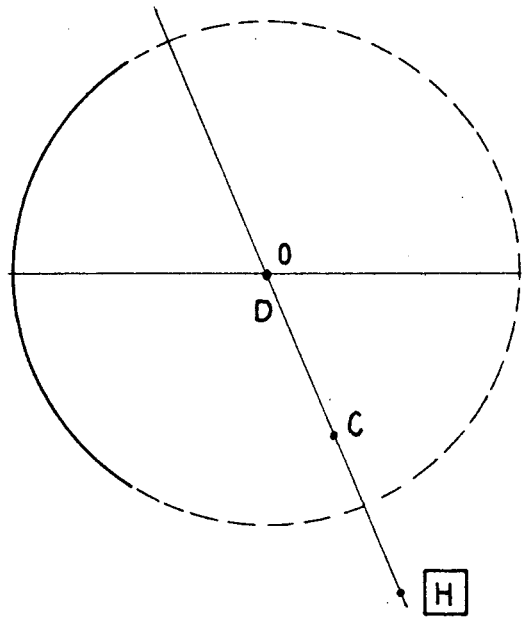

Fig: 5

.../..

We have on B a perferctly stigmatic image for any given wavelength.

Let us write $\boxed{\lambda = m \cdot \frac{\lambda_o}{k}}$

the relation 8 becomes :

$$\boxed{MB - m\,MC = 0} \quad \text{whatever M is.}$$

Its significance (Fig. 4) is that points B and C must be situated on a diameter P, Q of the sphere Σ and that P and Q must divide the segment BC in the harmonic ratio m

$$\frac{PB}{PC} = \frac{QB}{QC} = m.$$

Using this last equation we obtain :

$$(9) \quad \boxed{\frac{DP}{DC} = \frac{DB}{DP} = m} \Rightarrow \boxed{DC = \frac{R}{m}} \quad \boxed{DB = mR}$$

From these ratios we can obtain the position of C and of B.

To summarize if Σ is a sphere with centre D (MD = cst) and if one consider the harmonic conjuguate of the point C with regard to a diameter of the sphere (MB = m MC)

- when the source A is at the centre MA = MD = cst
- there is a rigorous stigmatism
  - . at C with the wavelength $\frac{\lambda_o}{k}$
  - . at B with the wavelength $m \frac{\lambda_o}{k}$
- as the stigmatism express the fact that the light-paths are stationary one must obtain the stigmatism for other wavelength when, using the same grating, one put the source at C or B and observes at one of the points D, C or B.

4° - <u>let us analyse these different cases</u> :

The hypothesis are :

spherical surface with centre at D : MD = cst

point $\boxed{H}$ harmonically conjuguate of C

with regard to a diameter : MH = m MC

(voir Fig. 5)

.../..

.../..

1° Let us put the source at C : MA = MC
We observe at H for what
wavelength is the stigmatism : MB = MH

The condition (6) $\boxed{MA + MB - k \frac{\lambda}{\lambda_o} (MC - MD) = \mathcal{P} = cst.}$

becomes : $MC + MH - k \frac{\lambda}{\lambda_o} (MC - MD) = \mathcal{P} = cst.$

that can be written :

$$MC \left(1 + m - k \frac{\lambda}{\lambda_o}\right) + k \frac{\lambda}{\lambda_o} MD = \mathcal{P} = cst.$$

This condition is satisfied if the term with MC is zero,
that can be written $\boxed{\lambda = (m + 1) \frac{\lambda_o}{k}}$ $k > o$

In another hand we have already demonstrate that for the same position of the source there is stigmatism

at D for $\frac{\lambda_o}{k}$

at C for $\frac{2\lambda_o}{k}$

2° Let us put the source at H : MA = MH
a) we observe at D : MB = MD

The condition (6) becomes :
$MH + MD - k \frac{\lambda}{\lambda_o} (MC - MD) = \mathcal{P} = cst.$
that can be written :
$MC \left(m - k \frac{\lambda}{\lambda_o}\right) + MD \left(1 + k \frac{\lambda}{\lambda_o}\right) = \mathcal{P} = cst.$
This condition is satisfied for $\boxed{\lambda = m \frac{\lambda_o}{k}}$ $k > o$

b) source at H : MA = MH
we observe at C : MB = MC
The condition (6) becomes :
$MH + MC - k \frac{\lambda}{\lambda_o} (MC - MD) = \mathcal{P} = cst.$

.../..

.../..

$$MC\left(1 + m - k\frac{\lambda}{\lambda_o}\right) + k\frac{\lambda}{\lambda_o}MD = cst.$$

There is stigmatism for $\boxed{\lambda = (m+1)\frac{\lambda_o}{k}}$ $k > o$

c) source at H : MA = MH
we observe at H : MB = MH

The condition (6) becomes :

$$2\,MH - k\frac{\lambda}{\lambda_o}(MC - MD) = \mathcal{P} = cst.$$

that can be written :

$$MC\left(2m - k\frac{\lambda}{\lambda_o}\right) + k\frac{\lambda}{\lambda_o}MD = \mathcal{P} = cst.$$

There is stigmatism for $\boxed{\lambda = 2m\frac{\lambda_o}{k}}$ $k > o$

<u>To summarize</u> : when the photosensitive surface is a sphere and when one of the recording points is at its centre, then, there is, in general, three points stigmatic rigorously :

- the two points of recording,
- the harmonic conjuguate of the recording point that is not generally at the centre of the grating with regard to the diameter of the grating,
- The following schemes summarize the properties of these gratings :

$\boxed{\frac{MH}{MC} = m}$ $\boxed{DC = \frac{R}{m}}$ $\boxed{DH = m.R.}$

We have : $\frac{MH}{MC} = m$ $DC = \frac{R}{m}$ $DH = m.R$

a) If $m > 1$

- necessarily we have $m < \frac{1}{n.\lambda_o}$ n=number of grooves per mm of the grating, in the middle.
- C is in the circle of the grating.

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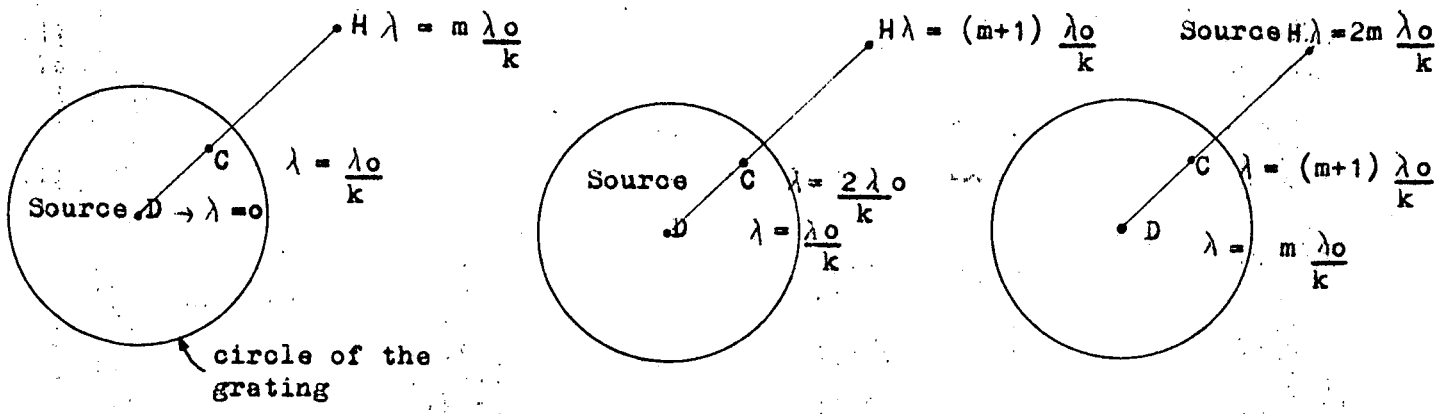

Fig. 6

b) If m < 1

- No limitations for m.
- C is out of the circle of the grating.

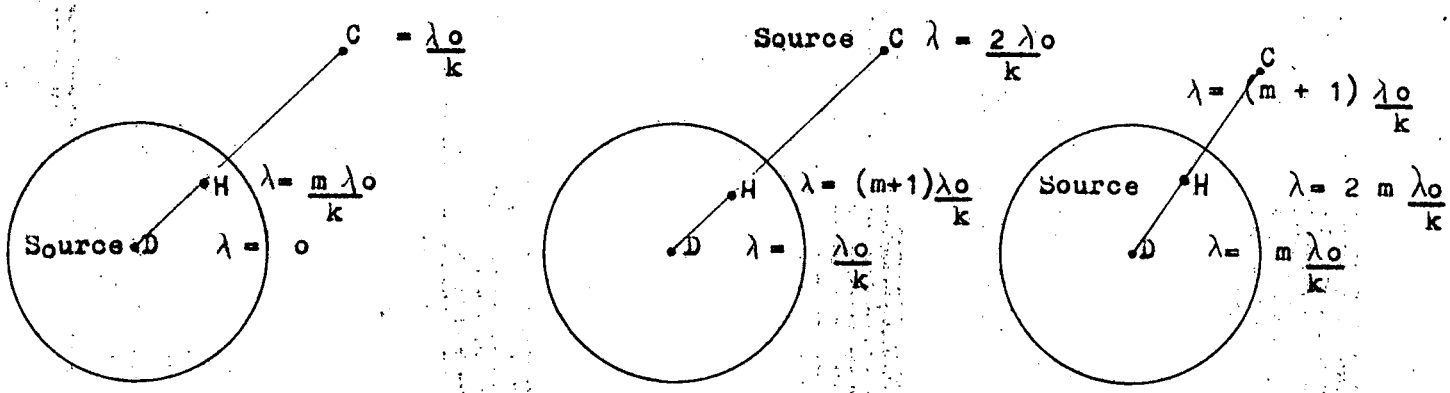

Fig. 7

- The Fig. 6 and 7 indicate the zones of the plane in which is the point C with the different values of m.
- If C comes in the Rowland circle, the third point of stigmatism becomes virtual. This case is not so attractive.

.../..

Likewise, we may reach a rigorous stigmatism when the recording points C and D are the point C, on one hand, and the point H harmonic conjugate of C with respect to the circle of the grating, on the other hand.

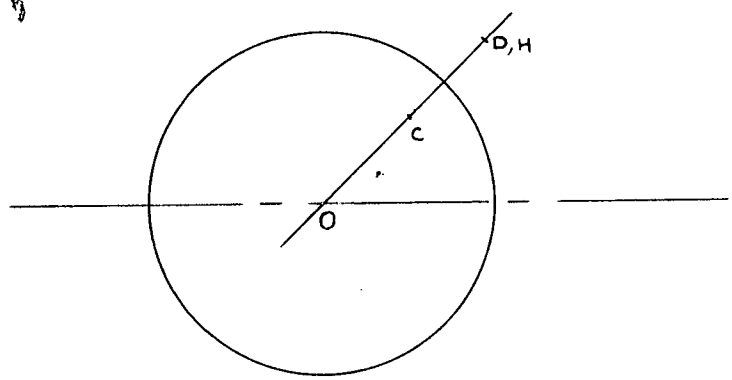

Fig. 8

Identically with the previous case, (D at O), the wavelengths of stigmatism are depending on the locus of both the source point and the image point B and on the parameter m

$$m = \frac{MH}{MC} = \frac{MD}{MC} \qquad OD = mR$$

$$OC = \frac{R}{m}$$

Considering the case in which the source is located at C, the condition (6) is as follows :

$$MC + MB - \frac{k\lambda}{\lambda_0}\,(MC - m\,MC) = \text{cst.}$$

$$MB + MC\left[1 - \frac{\lambda}{\lambda_0}\,k\,(1 - m)\right] = \text{cst.}$$

.../..

.../..

When the image B is located at D

$$\longrightarrow \quad MB = MD = m\,MC$$

the above expression is :

$$MC\left[1 + m - \frac{k\lambda}{\lambda_0}(1 - m)\right] = cst.$$

We observe that the wavelength of stigmatism is :

$$\boxed{\lambda = \frac{\lambda_0}{k}\,\frac{1+m}{1-m}}$$

We may follow the same reasoning if the source is at O or at H and if the point B is at O or at H.

The Tables below indicate the wavelengths corresponding to the different configurations of stigmatism.

A) D is at O

Table I

source position / image position	D (O)	C	H
D (O)	$\lambda = 0$	$\lambda = \frac{\lambda_0}{k}$	$\lambda = \frac{m\,\lambda_0}{k}$
C	$\lambda = \frac{\lambda_0}{k}$	$\lambda = 2\,\frac{\lambda_0}{k}$	$\lambda = \frac{(m+1)\,\lambda_0}{k}$
H	$\lambda = \frac{m\,\lambda_0}{k}$	$\lambda = (m+1)\,\frac{\lambda_0}{k}$	$\lambda = 2\,m\,\frac{\lambda_0}{k}$

m : any value lower than $\frac{2000}{N}$

.../..

.../..

B) Recording with conjugate points.

D is at H (harmonic conjugate of C)

Table 2

source position / image position	O	C	H (D)
O	$\lambda = 0$	$\lambda = \frac{1}{1-m} \frac{\lambda_o}{k}$	$\lambda = \frac{m}{1-m} \frac{\lambda_o}{k}$
C	$\lambda = \frac{1}{1-m} \frac{\lambda_o}{k}$	$\lambda = \frac{2}{1-m} \frac{\lambda_o}{k}$	$\lambda = \frac{1+m}{1-m} \frac{\lambda_o}{k}$
H (D)	$\lambda = \frac{m}{1-m} \frac{\lambda_o}{k}$	$\lambda = \frac{1+m}{1-m} \frac{\lambda_o}{k}$	$\lambda = \frac{2\ m}{1-\cos} \frac{\lambda_o}{k}$

m : any value lower than sin γ

.../..

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III - GENERAL CALCULATION'S METHOD OF THE ABERRANT OPTICAL PATH.

Terms of 2nd and 3rd orders

We have seen that the general equation of holographic grating can be written :

$$\boxed{(MA + MB) - (IA + IB)\ \frac{R\lambda}{\lambda_0}\left[(MC - MD) - (IC - ID)\right]} = 0$$

In the chapter V, we have settle conditions to obtain for the point object A a perfectly stigmatic image.

But in the general case the image B will not be perfectly stigmatic. We can calculate the quality of the image by the study of the aberrant optical path.

(10) $\Delta = MA + MB - R\frac{\lambda}{\lambda_0}\ (MC - MD) = \mathcal{P}$ when M is going along the surface Σ.

Therefore $\mathcal{P} = \underline{I}\,A + I\,B - \frac{R\lambda}{\lambda_0}\ (I\,C - I\,D)$

To obtain Δ one must calculate the values of MA, MB, MC and MD. We are going to explain the principles of MA calculation.

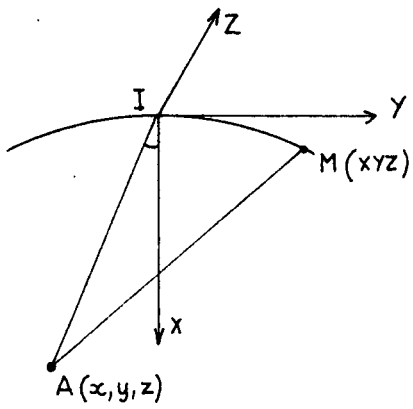

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Fig. 9

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In the I (X Y Z) coordinate system, (x y z) are the point A coordinates and (X Y Z) are the coordinates of M any point of Σ.
The relation between the coordinates of M is given by the equation of Σ. In this first report we suppose that Σ is a sphere.
In a next report we shall give the equations valid in all cases in which Σ is a paraboloid, an ellipsoid, a cylinder, a tore.

<u>In all the following sections we suppose that the dimensions of Y and Z are small with regard to the others dimensions as X, x, y, l.</u>
Also we suppose, when we study an object slit that z is also small.
When we add other hypothesis at these basic hypothesis, we shall express that obviously.

Therefore, the equation of the sphere in our coordinates system can be written :

$$X^2 + Y^2 + Z^2 - 2RX = 0$$

$$X = R \pm \left[R^2 - (Y^2 + Z^2)\right]^{\frac{1}{2}}$$

$$X = R \pm R\left[1 - \frac{Y^2 + Z^2}{R^2}\right]^{\frac{1}{2}}$$

The equation can be written in a series development as follows :

$$(11) \quad X = \frac{Y^2 + Z^2}{2R} + \frac{(Y^2 + Z^2)^2}{8R^3} + \varphi(Y^6 \; Z^6)$$

More, in the trihedron I (X Y Z) one can choose a system of spheric coordinate to express A

$$x = \ell_A \cos\alpha$$

$$y = \ell_A \cos\beta \qquad \ell = AI$$

$$z = \ell_A \cos\gamma$$

with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

In these conditions :

$$\overline{AM}^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2$$

$$= X^2 + Y^2 + Z^2 - 2Xx - 2Yy - 2Zz + x^2 + y^2 + z^2$$

.../..

.../..

We can substitute X and X^2 for their values

$$AM^2 = \ell^2 - 2 \times \left[\frac{Y^2 + Z^2}{2R} + \frac{(Y^2 + Z^2)^2}{8R^3} \right] - 2(yY + zZ) + Y^2 + Z^2 + \frac{(Y^2 + Z^2)^2}{4R^2}$$

that can be arranged in regular order with regard to Y and Z,

$$AM^2 = \ell^2 \left[1 - \frac{2(yY + zZ)}{\ell^2} + \frac{Y^2 + Z^2}{\ell^2} - \frac{x}{R\ell^2}(Y^2 + Z^2) + \frac{(Y^2 + Z^2)^2}{\ell^2}\left(\frac{1}{4R^2} - \frac{x}{4R^3}\right) \right]$$

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Let us put $AM^2 = \ell^2 (1 + \theta)$

so

$$\boxed{AM = \ell \left(1 + \frac{\theta}{2} - \frac{\theta^2}{8} + \frac{\theta^3}{16} - \frac{5\theta^4}{128}\right)} \quad (12)$$

$$\frac{\theta}{2} = \frac{1}{2\ell^2} \left[-2(yY + zZ) + Y^2 + Z^2 - \frac{x}{R}(Y^2 + Z^2) + (Y^2 + Z^2)^2 \left(\frac{1}{4R^2} - \frac{x}{4R^3}\right) \right]$$

$$-\frac{\theta^2}{8} = -\frac{1}{8\ell^4} \left[4(yY + zZ)^2 + (Y^2 + Z^2)^2 + \frac{x^2}{R^2}(Y^2 + Z^2)^2 - 4(yY + zZ)(Y^2 + Z^2) + \frac{4x}{R}(yY + zZ)(Y^2 + Z^2) - \frac{2x}{R}(Y^2 + Z^2)^2 \right]$$

$$+\frac{\theta^3}{16} = \frac{1}{16\ell^6} \left[-8(yY + zZ)^3 + 12(yY + zZ)^2(Y^2 + Z^2) - \frac{12x}{R}(yY + zZ)^2(Y^2 + Z^2) \right]$$

$$-\frac{5\theta^4}{128} = -\frac{5 \times 16}{128\ell^8}(yY + zZ)^4$$

.../..

.../..

If we arrange in regular order, it follows :

$$
\begin{aligned}
M\,A = {} & \ell \\
& - \frac{y\,Y + z\,Z}{\ell} \\
& + \frac{1}{2\ell}\left[Y^2 \left(1 - \frac{x}{R} - \frac{y^2}{\ell^2}\right) + Z^2 \left(1 - \frac{x}{R} - \frac{z^2}{\ell^2}\right) - 2\,\frac{y\,z\,Y\,Z}{\ell^2} \right] \\
& + \frac{y\,Y + z\,Z}{2\ell^3}\left[Y^2 \left(1 - \frac{x}{R} - \frac{y^2}{\ell^2}\right) + Z^2 \left(1 - \frac{x}{R} - \frac{z^2}{\ell^2}\right) - \frac{2\,y\,z\,Y\,Z}{\ell^2} \right] \\
& + \text{terms of forth order.}
\end{aligned}
\tag{13}
$$

We shall give the value of terms of fourth rank in the Chapter IV.-

---

Furthermore, in this Chapter, we consider points located in the plane I (X Y).

Therefore with z = o

In these conditions $\gamma = \frac{\pi}{2}$ and one can write

$x = \ell \cos \alpha$

$y = \ell \sin \alpha$

Then it follows :

$$
\begin{cases}
M\,A = & \ell \\
\text{term of first rank} & - Y \sin\alpha \\
\text{term of second rank} & + \frac{Y^2}{2} \cdot \left(\frac{\cos^2\alpha}{\ell} - \frac{\cos\alpha}{R}\right) + \frac{Z^2}{2}\left(1 - \frac{\cos\alpha}{R}\right) \\
\text{term of third rank} & + \frac{Y^3}{2}\,\frac{\sin\alpha}{\ell}\left(\frac{\cos^2\alpha}{\ell} - \frac{\cos\alpha}{R}\right) + \frac{Y\,Z^2}{2}\,\sin\alpha\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) \\
\text{term of forth order} & + Y^4 \ldots
\end{cases}
\tag{14}
$$

.../..

.../..

Obviously, one calculates MB, MC and MD in the same way.
We introduce the polar coordinates of points A B C D

Source point A $\alpha, \ell_A = IA$

Image point B $\beta \ \ell_B = IB$

First recording point C $\gamma \ \ell_C = IC$

Second recording point D $\delta \ \ell_D = ID$

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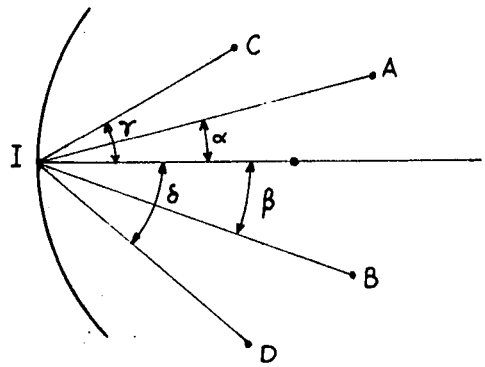

Fig. 10

Comment.

As a matter of fact, the point B is not necessarily the image but the point at which one decides to calculate the optical aberrant path. One can consequently vary the position of B until one obtains an optimum value of aberrations.
The point B determined in this way is then really image of A.
Substituting MA, MB, MC, MD by their developments in the following expression : (eq. 10)

$$\Delta = MA + MB - \frac{R\lambda}{\lambda_0}(-MC - MD) - \mathcal{P}$$

.../..

.../..

Therefore $\mathcal{P} = I\,A + I\,B - \frac{k\lambda}{\lambda_o}\,(I\,C - I\,D) =$

$$\ell_A + \ell_B - \frac{k\lambda}{\lambda_o}\,(\ell_C - \ell_D)$$

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In these conditions :

$$\Delta = -Y\left[\sin\alpha + \sin\beta - \frac{k\lambda}{\lambda_o}\,(\sin\gamma - \sin\delta)\right]$$

$$+\frac{Y^2}{2}\left[\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - \frac{k\lambda}{\lambda_o}\left[\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) - \left(\frac{\cos^2\gamma}{\ell_D} - \frac{\cos\delta}{R}\right)\right]\right]$$

$$+\frac{Z^2}{2}\left[\frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{1}{\ell_B} - \frac{\cos\beta}{R} - \frac{k\lambda}{\lambda_o}\left[\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)\right]\right] \qquad (15)$$

$$+\frac{Y^3}{2}\left[\frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{k\lambda}{\lambda_o}\left[\frac{\sin\gamma}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)\right]\right]$$

$$+\frac{Y\,Z^2}{2}\left[\frac{\sin\alpha}{\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{k\lambda}{\lambda_o}\left[\frac{\sin\gamma}{\ell_C}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)\right]\right]$$

To have signification, the series expansion must have at least its term of first order egal to zero. Therefore, we must

.../..

.../..

fulfil the following expression :

$$\sin\alpha + \sin\beta - \frac{k\lambda}{\lambda_o}(\sin\gamma - \sin\delta) = 0 \qquad (16)$$

If we write $d = \dfrac{\lambda_o}{\sin\gamma - \sin\delta}$ (17)

we obtain : $d(\sin\alpha + \sin\beta) = k\lambda$

that is the classical formula for the gratings.

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The rays IA and IB are corresponding by diffraction on the grating of grooves-spacing d in the order k

Now, precisely the quantity $d = \dfrac{\lambda_o}{\sin\gamma - \sin\delta}$ is the pitch of the interference fringes produced at the vicinity of the surface Σ top, by the two coherent points C and D emitting a radiation λ_o.

Then, the relation $\sin\alpha + \sin\beta - \dfrac{k\lambda}{\lambda_o}(\sin\gamma - \sin\delta) = 0$ is met.

In the next report we shall explain the consequences that can be deduced from the terms of second and third orders.

Relation between aberrant optical path and image quality.

At last, we must point out that it is possible to determine the widening dy_B and dz_B in a chosen plane from the aberrant optical path using the relation of Nijboer.

$$(18) \quad \begin{aligned} dy &= \frac{\cos\varphi}{\cos\omega}\frac{\partial\Delta}{\partial\omega} - \frac{\sin\varphi}{\sin}\frac{\partial\Delta}{\partial\varphi} \\ dz &= \frac{\sin\varphi}{\cos\omega}\frac{\partial\Delta}{\partial\omega} + \frac{\cos\varphi}{\sin\omega}\frac{\partial\Delta}{\partial\varphi} \end{aligned}$$

.../..

.../..

where φ is the azimuthal angle of the pupil

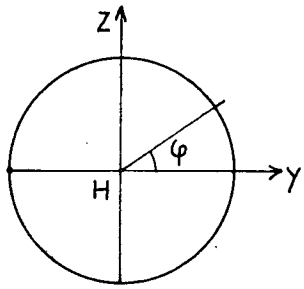


Fig.11

$Y = H \cos \varphi$

$Z = H \sin \varphi$

and ω is the aperture's angle of the beam.

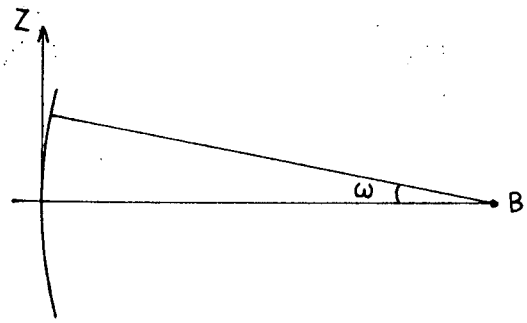


Fig.12

IV - SPHERICAL ABERRATION

STUDY OF THE ABERRANT PATH OF FOURTH DEGREE.

In the Chapter III, we have mentioned the value of the terms of second and third degrees of the aberrant optical path.

Now we are going to study the value of fourth order term.

Let us remind that $A\,M = \ell \left[1 + \frac{\theta}{2} - \frac{\theta^2}{8} + \frac{\theta^3}{16} - \frac{5\,\theta^4}{128} \right]$ (1)

(eq. 12)

So, the term of fourth degree may be written :

$$\ell \Bigg[\underbrace{\frac{1}{2\ell^2} (Y^2 + Z^2)^2 \left(\frac{1}{4R^2} - \frac{x}{4R^3} \right)}_{A} - \underbrace{\frac{1}{8\ell^4} (Y^2 + Z^2)^2}_{B} -$$

$$\underbrace{\frac{1}{8\ell^4} \frac{x^2}{R^2} (Y^2 + Z^2)^2}_{C} + \underbrace{\frac{x}{4R\ell^4} (Y^2 + Z^2)^2}_{D} + \underbrace{\frac{12}{16} \frac{(y\,Y + z\,Z)^2}{\ell^6} (Y^2 + Z^2)}_{E}$$

$$\underbrace{- \frac{12}{16} \frac{x}{R\ell^6} (y\,Y + z\,Z)^2 (Y^2 + Z^2)}_{F} \underbrace{- \frac{5 \times 16}{128\,\ell^8} (y\,Y + z\,Z)^4}_{G} \Bigg]$$

.../..

Let us group the terms A + C

$$A + C = \ell(Y^2 + Z^2)^2 \left[\frac{1}{2\ell^2} \left(\frac{1}{4R^2} - \frac{x}{4R^3} \right) - \frac{1}{8\ell^4} \frac{x^2}{R^2} \right] =$$

$$\frac{(Y^2 + Z^2)^2}{8R^2\ell} \left(1 - \frac{x}{R} - \frac{x^2}{\ell^2} \right)$$

Let us group the terms E + F + G :

$$E + F + G = \frac{(y\,Y + z\,Z)^2}{4\ell^5} \left[3\,(Y^2 + Z^2)\left(1 - \frac{x}{R}\right) - \frac{5}{2\ell^2}(Y\,y + Z\,z)^2 \right]$$

$$= \frac{3\,(y\,Y + z\,Z)^2}{4\ell^5} \left[Y^2 \left(1 - \frac{x}{R} - \frac{5}{6}\frac{y^2}{\ell^2}\right) + Z^2 \left(1 - \frac{x}{R} - \frac{5}{6\ell^6} Z^2\right) - \frac{5}{3}\frac{y^2}{\ell^2}\, Y\,Z \right]$$

Finally the value of the fourth order term, for A M, is :

$$AM^{(4)} = \frac{(Y^2 + Z^2)^2}{8\,R^2\ell}\left(1 - \frac{x}{R} - \frac{x^2}{\ell^2}\right) - \frac{1}{8\ell^3}(Y^2 + Z^2)^2\left(1 - \frac{2\,x}{R}\right) +$$

$$\frac{3}{4\ell^5}(yY + zZ)^2 \left[Y^2\left(1 - \frac{x}{R} - \frac{5}{6}\frac{y^2}{\ell^2}\right) + Z^2\left(1 - \frac{x}{R} - \frac{5}{6\ell^2} z^2\right) - \frac{5}{3}\frac{yz}{\ell^2}\, Y\,Z \right]$$

We use the same hypothesis as previously i.e.

$z = o$ so $x = \ell\cos\alpha$

$y = \ell\sin\alpha$

.../..

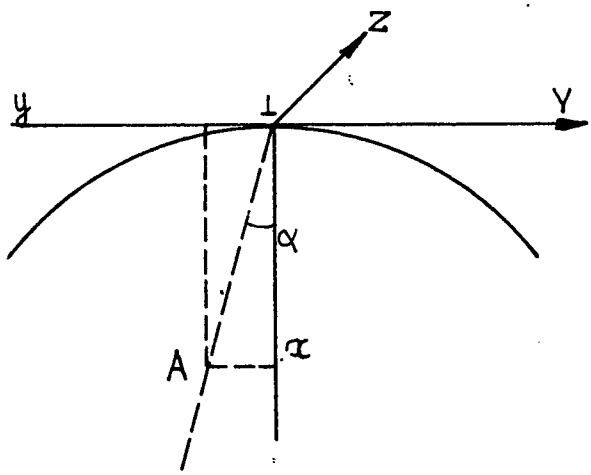


Fig. n° 13

Then one can write :

$$AM^{(4)} = \frac{(Y^2 + Z^2)^2}{8R^2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{\cos^2\alpha}{\ell}\right) - \frac{1}{8\ell^2}(Y^2 + Z^2)^2\left(\frac{1}{\ell} - \frac{2\cos\alpha}{R}\right)$$

$$+ \frac{3}{4\ell^4}(yY)^2\left[Y^2\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6}\frac{\sin^2\alpha}{\ell}\right) + Z^2\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right)\right]$$

We are going now to study successively the terms in Y^4, $Y^3 Z$, $Y^2 Z^2$ which may be written : $F(Y^4)$, $F(Y^3 Z)$ etc.

Term in Y^4

$$F(Y^4) = Y^4\left[\frac{1}{8R^2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{\cos^2\alpha}{\ell}\right) - \frac{1}{8\ell^2}\left(\frac{1}{\ell} - \frac{2\cos\alpha}{R}\right) + \frac{3}{4\ell^2}\sin^2\alpha\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6}\frac{\sin^2\alpha}{\ell}\right)\right]$$

$$= Y^4\left[\frac{1}{8R^2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) - \frac{\cos^2\alpha}{8R^2\ell} - \frac{1}{8\ell^3} + \frac{2\cos\alpha}{8R\ell^2} + \frac{3}{4\ell^2}\sin^2\alpha\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6}\frac{\sin^2\alpha}{\ell}\right)\right]$$

.../..

.../..

$$= Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{8\ell} \left(\frac{\cos\alpha}{R} - \frac{1}{\ell} \right)^2 + \frac{3}{4\ell^2} \sin^2\alpha \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell} \right) \right]$$

$$= Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{8\ell} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - 6 \frac{\sin^2\alpha}{\ell} \right) \right.$$

$$\left. - \frac{3}{4} \frac{\sin^2\alpha}{\ell^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) + \frac{3}{4} \frac{\sin^2\alpha}{\ell^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell} \right) \right]$$

$$= Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{8\ell} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - 6 \frac{\sin^2\alpha}{\ell} \right) - \frac{5}{8} \frac{\sin^4\alpha}{\ell^3} \right]$$

$$= Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{8\ell} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{\sin^2\alpha}{\ell} \right) \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5 \sin^2\alpha}{\ell} \right) \right]$$

(19) $$\boxed{F(Y^4) = Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{8\ell} \left(\frac{\cos^2\alpha}{\ell} - \frac{\cos\alpha}{R} \right) \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5 \sin^2\alpha}{\ell} \right) \right]}$$

( )

<u>Term in $Y^3 Z$</u>

$$F(Y^3 Z) = \frac{Y^3 Z}{\ell^4} \left\{ \frac{3}{2} y z \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell} \right) - \frac{5}{4} \frac{y^3 z}{\ell^3} \right\}$$

$$= \frac{Y^3 Z}{\ell^4} y z \left\{ \frac{3}{2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell} \right) - \frac{5}{4} \frac{\sin^2\alpha}{\ell} \right\}$$

.../.

.../..

$$(20) \quad \boxed{F(Y^3 Z) = \frac{Y^3 Z}{\ell^4} \; y \; z \; \frac{3}{2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{3} \frac{\sin^2\alpha}{\ell} \right)} \quad (\ldots)$$

This term is zero if $Z = o$.

<u>Term in $Y^2 Z^2$</u>

$$F(Y^2 Z^2) = Y^2 Z^2 \left\{ \frac{1}{4R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{\cos^2\alpha}{\ell} \right) - \frac{1}{4\ell^2} \left(\frac{1}{\ell} - \frac{2\cos\alpha}{R} \right) \right.$$

$$\left. + \frac{3 z^2}{4\ell^4} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell} \right) + \frac{3}{4} \frac{\sin^2\alpha}{\ell^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5}{6\ell^3} z^2 \right) \right\}$$

If $z = o$

$$F(Y^2 Z^2) = Y^2 Z^2 \left\{ \frac{1}{4R^2} \left(\frac{\sin^2\alpha}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{4\ell^2} \left(\frac{1}{\ell} - \frac{2\cos\alpha}{R} \right) + \frac{3}{4} \frac{\sin^2\alpha}{\ell^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) \right\}$$

$$= Y^2 Z^2 \left\{ \frac{1}{4R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{4\ell} \left(\frac{1}{\ell^2} - \frac{2\cos\alpha}{R\ell} - \frac{\cos^2\alpha}{R2} \right) + \frac{3}{4} \frac{\sin^2\alpha}{\ell^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) \right\}$$

$$= Y^2 Z^2 \left\{ \frac{1}{4R^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) - \frac{1}{4\ell} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right)^2 + \frac{3}{4} \frac{\sin^2\alpha}{\ell^2} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) \right\}$$

$$F(Y^2 Z^2) = Y^2 Z^2 \left\{ \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) \left(\frac{1}{4R^2} - \frac{1}{4\ell} \left(\frac{1}{\ell} - \frac{\cos\alpha}{R} \right) + \frac{3}{4} \frac{\sin^2\alpha}{\ell^2} \right) \right\}$$

.../..

Term in $Y\,Z^3$

$$F\,(Y\,Z^3) = Y\,Z^3 \left\{ \frac{6\,y\,z}{4\ell^5}\left(1 - \frac{x}{R} - \frac{5}{6\ell^2}\,z^2\right) - \frac{3\,z^3\,y}{4\,\ell^7}\,\frac{5}{3} \right.$$

$$= Y\,Z^3 \left(\frac{3}{2}\,\frac{y\,z}{\ell^4}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) - \frac{5}{4\ell^7}\,y\,z^3 - \frac{5}{4}\,\frac{y\,z^3}{\ell^7} \right)$$

$$(21) \quad \boxed{= Y\,Z^3\,\frac{y\,z}{\ell^4}\left(\frac{3}{2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) - \frac{5}{2\ell^3}\,z^2\right)} \quad ( )$$

This term is zero if $z = o$.

Term in Z^4

$$F\,(Z^4) = Z^4 \left(\frac{1}{8R^2\ell}\left(1 - \frac{x}{R} - \frac{x^2}{\ell^2}\right) - \frac{1}{8\ell^3}\left(1 - \frac{2\,x}{R}\right) \right.$$

$$\left. + \frac{3\,z^2}{4\ell^5}\left(1 - \frac{x}{R} - \frac{5}{6\ell^2}\,z^2\right) \right)$$

$$= Z^4 \left(\frac{1}{8R^2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{\cos^2\alpha}{\ell}\right) - \frac{1}{8\ell^2}\left(\frac{1}{\ell} - \frac{2\cos\alpha}{R}\right) \right.$$

$$\left. + \frac{3\,z^2}{4\ell^4}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5\,z^2}{6\ell^3}\right) \right)$$

.../..

$$= Z^4 \left| \frac{1}{8R^2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) - \frac{1}{8\ell}\left(\frac{\cos^2\alpha}{R^2} + \frac{I}{\ell 2} - \frac{2\cos\alpha}{R}\right) \right.$$

$$\left. + \frac{3\,z^2}{4\ell^4}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5\,z^2}{6\ell^3}\right) \right|$$

$$= Z^4 \left| \frac{1}{8R^2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) - \frac{1}{8\ell}\left(\frac{\cos\alpha}{R} - \frac{I}{\ell}\right)^2 + \right.$$

$$\left. \frac{3\,z^2}{4\ell^4}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5\,z^2}{6\ell^3}\right) \right|$$

So the term independant of z may be written :

(22) $$\boxed{F(Z^4) = Z^4 \left| \frac{1}{8}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\left|\frac{1}{R^2} - \frac{1}{\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\right| \right|}$$

Therefore, in the hypothesis z = o one can sum up the fourth order term of expansion of AM :

(23) $$\boxed{\begin{aligned} A\,M^{(4)} = Y^4 & \left[\frac{1}{8R^2}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) - \frac{1}{8\ell}\left(\frac{\cos^2\alpha}{\ell} - \frac{\cos\alpha}{R}\right)\left(\frac{1}{\ell} - \frac{\cos\alpha}{R} - \frac{5\sin^2\alpha}{\ell}\right) \right] \\ & + Y^2 Z^2 \left[\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right)\left[\frac{1}{4R^2} - \frac{1}{4\ell}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right) + \frac{3}{4}\,\frac{\sin^2\alpha}{\ell^2}\right] \right] \\ & + Z^4 \left[\frac{1}{8}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right)\left[\frac{1}{R^2} - \frac{1}{\ell}\left(\frac{1}{\ell} - \frac{\cos\alpha}{R}\right)\right] \right] \end{aligned}}$$

.../..

Obviously, one calculates MB, MC and MD in the same way.

One substitutes MA MB MC and MD by their development in the following expression :

$$\Delta = MA + MB - k \frac{\lambda}{\lambda_0} (MC - MD) - \mathcal{P}$$

V - ANALYTICAL EXPRESSION OF THE STIGMATIC POINTS

We have seen in Chapter II - 2 (page n° 4 and following), that a group of three points (O, C and H) had stigmatic properties when firstly the source A and secondly the image B were placed at these points.

In that case we know that :

a) the construction points C and D are :

D is at O so $\ell_D = R$ and $\delta = o$

C is anywhere

b) H is the harmonic conjuguate of C

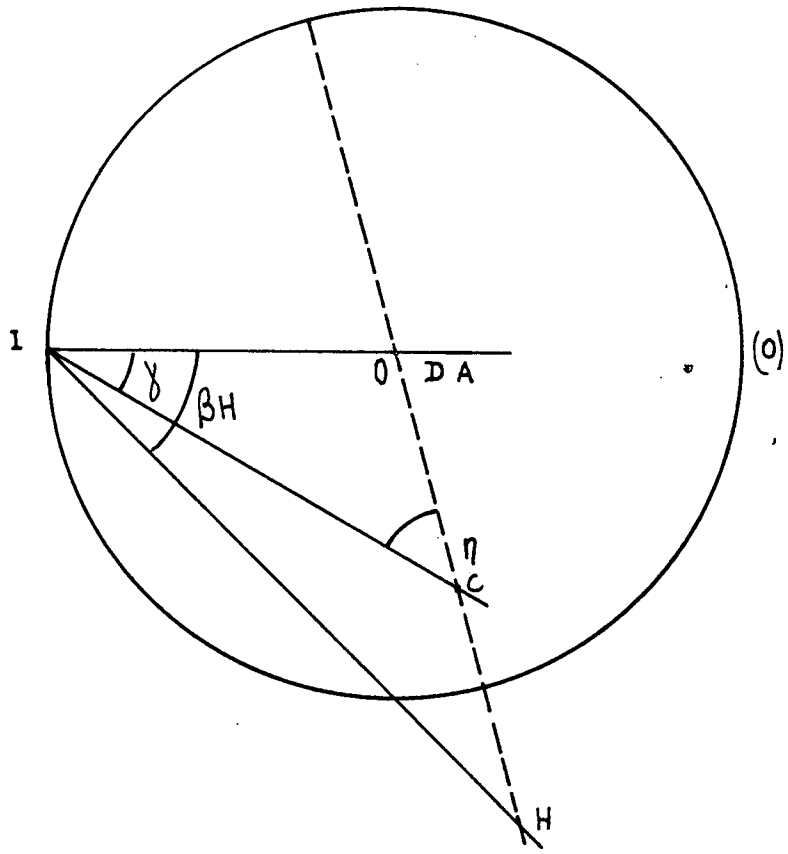


Fig. n° 14

Let us consider the Fig. n° 5 of the **Chapter II, or the Fig. n° 14.** We know that, if C and H are harmonic conjuguates with regards to the diameter of the circle (O), H is the stigmatic image of a point source A located at O for the wavelength λ_H.

Let us apply the relation

$$\sin\alpha + \sin\beta = k \frac{\lambda}{\lambda_0} (\sin\gamma - \sin\delta)$$

$$\delta = 0$$

$$\alpha = 0 \longrightarrow \sin\beta_H = \frac{k\lambda_H}{\lambda_0} \sin\gamma$$

we have $\frac{k\lambda_H}{\lambda_0} = m$

$$\sin\beta_H = m \sin\gamma$$

$$CD = \frac{R}{m} \quad DH = Rm \quad (\text{eq. } 9)$$

In the triangle I O C $\frac{\sin\eta}{R} = \frac{\sin\gamma}{\frac{R}{m}}$

$$\sin\eta = m\sin\gamma = \sin\beta_H$$

$$\boxed{\begin{aligned} \ell_C &= \frac{R}{m} \frac{\sin(\eta+\gamma)}{\sin\gamma} = \frac{R}{m}\left(m\cos\gamma + \sqrt{1-m^2\sin^2\gamma}\right) \\ \ell_H &= m\,\ell_C \end{aligned}} \qquad (24)$$

<u>Particular case of grazing incidence.</u>

We suppose that $\beta_H = \frac{\pi}{2}$ $\alpha = 0$

$\longrightarrow$ (in fact we use the grating under the converse condition $\alpha = \frac{\pi}{2}$ $\beta = 0$ but the equations are identical.)

.../..

So $m \sin \gamma = 1$

$$\ell_C = R \cos \gamma$$

$$\ell_H = m R \cos \gamma = \frac{R}{tg\,\gamma} \qquad (25)$$

VII - STUDY OF THE ASTIGMATISM -

VI - 1 - Study of no-astigmatism conditions at a given point.

Study in the general case.

VI-1-1- We have seen that the term of second degree may be written :

$$MA^{(2)} = \frac{1}{2\ell}\left[Y^2\left(1 - \frac{x}{R} - \frac{y^2}{\ell^2}\right) + Z^2\left(1 - \frac{x}{R} - \frac{z^2}{\ell^2}\right) - \frac{2\,y\,z\,Y\,Z}{\ell^2}\right]$$

expression that leads for z = o to the value of the aberrant optical path as follows :

$$\Delta^{(2)} = \frac{Y^2}{2}\left[\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - k\frac{\lambda}{\lambda_o}\left[\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)\right]\right]$$

(26)

$$+ \frac{Z^2}{2}\left[\frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{1}{\ell_B} - \frac{\cos\beta}{R} - \frac{k\lambda}{\lambda_o}\left[\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)\right]\right]$$

For a given point source A (ℓ_A, α) the coefficients of Y^2 and Z^2 are respectively the locus of sagittal and tangential focals.

$$T = 0 \quad \frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_{TB}} - \frac{\cos\beta}{R} - \frac{k\lambda}{\lambda_o}\left[\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)\right] = 0$$

(27)

.../..

$$S = 0 \quad \frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{1}{\ell_{SB}} - \frac{\cos\beta}{R} - \frac{k\lambda}{\lambda_o} \left[\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R} \right) - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R} \right) \right] = 0 \tag{28}$$

Obviously, as a rule, the curves $\ell_{TB} = f\,1\,(\beta)$ and $\ell_{SB} = f\,2\,(\beta)$ are distinct.

Of course, for a spectrograph, we shall consider the tangential focal surface as being the image surface.
Under such conditions, the height of the tangential focal, for a point object, is given by

$$h_T = Y_M = \frac{\ell_{SB} - \ell_{TB}}{\ell_{SB}}$$

In this formula Y_M is the height of the pupil. Moreover, we have the relation

$$\sin\alpha + \sin\beta = \frac{k\lambda}{\lambda_o} \cdot (\sin\gamma - \sin\delta)$$

So, T = o may be written :

$$\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_{TB}} - \frac{\cos\beta}{R} - \frac{\sin\alpha + \sin\beta}{\sin\gamma - \sin\delta} \times \left[\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} \right) - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R} \right) \right] = 0$$

We can write :

(29)

$$K_o = \sin\gamma - \sin\delta$$

$$K_1 = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R} \right)$$

$$K_3 = \frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R} \right)$$

By proceeding in the same way for the equation S= o and by writing →

.../..

(30) T = 0 becomes

$$T = \frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_{TB}} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\ \frac{KI}{Ko} = 0$$

S = o becomes

(31)
$$S = \frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{1}{\ell_{SB}} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\ \frac{K3}{Ko} = o$$

From the equations T = o and S = o we obtain :

(32)
$$\ell_{TB} = \frac{R\,\ell_A \cos^2\beta}{-R\cos^2\alpha + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\ \frac{KI}{Ko}}$$

$$\ell_{SB} = \frac{R\,\ell_A}{-R + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin^2\beta)\ \frac{K3}{Ko}}$$

so $h_T = Y_M \left(1 - \frac{\ell_{TB}}{\ell_{SB}}\right)$

(33)
$$h_T = Y_M \left[1 - \frac{\cos^2\beta\left[-R + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\ \frac{K3}{Ko}\right]}{-R\cos^2\alpha + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\ \frac{KI}{Ko}}\right]$$

In order to obtain the astigmatism equal to zero, we must have $\ell_{SB} = \ell_{TB}$. This condition may be written :

$$\cos^2\beta\left[-R + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\ \frac{K3}{Ko}\right]$$

$$= -R\cos^2\alpha + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\ \frac{KI}{Ko}$$

.../..

$$R\,(\cos^2\alpha - \cos^2\beta) = \ell_A\,(\cos\alpha + \cos\beta)\,(1 - \cos^2\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)$$

i.e.

$$(34)\qquad \boxed{\frac{\cos\alpha - \cos\beta}{\ell_A} = \frac{\sin^2\beta}{R} + \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)}$$

This is the necessary condition for having the astigmatism equal to zero.

VI-1-2- CASE OF CONVENTIONAL GRATINGS

In the case of a conventional grating KI = K3 = o
the above condition becomes :

$$(35)\qquad \frac{\cos\alpha - \cos\beta}{\ell_A} = \frac{\sin^2\beta}{R}$$

Let us carry forward this condition in the equation S = o (28) which is valid for conventional gratings i.e. with KI = K3 = o

$$\frac{\sin^2\beta}{R(\cos\alpha - \cos\beta)} - \frac{(\cos\alpha + \cos\beta)}{R} + \frac{1}{\ell_{SB}} = o$$

$$\ell_{BS} = R\,\frac{\cos\alpha - \cos\beta}{\cos^2\alpha - 1}$$

.../..

Placing side by side

(36) $$\ell_A = R\,\frac{\cos\alpha - \cos\beta}{\sin^2\beta} \quad \text{et} \quad \ell_B = -R\,\frac{\cos\alpha - \cos\beta}{\sin^2\alpha}$$

we observe that, in general, either ℓ_A or ℓ_B must be negative in order that the two relations may be simultaneously satisfied.

Therefore, generally, there is no solution to avoid the astigmatism with a conventional concave spherical grating.

WADSWORTH MOUNTING.

However, there is a special solution
if $\beta = 0$ In this case $\ell_A = \infty$

(37) and $\ell_B = \dfrac{R}{1 + \cos\alpha}$ It is the Wadsworth mounting.

VI-1-2-2- ASTIGMATISM OF THE CONVENTIONAL GRATING ON THE ROWLAND CIRCLE.

In the case of conventional grating, the focal's height may be written in the general case :

(38) $$\boxed{h_T = Y_M \; \frac{\sin^2\beta - \dfrac{R}{\ell_A}(\cos\alpha - \cos\beta)}{1 - \dfrac{R}{\ell_A}\,\dfrac{\cos^2\alpha}{\cos\alpha + \cos\beta}}}$$

In the Rowland case $\ell_A = R\cos\alpha$

$$h_T = Y_M \; \frac{\sin^2\beta - \cos\alpha(\cos\alpha + \cos\beta) - \cos^2\alpha + \cos^2\beta}{\cos\alpha(\cos\alpha + \cos\beta) - \cos^2\alpha}$$

(39) $$\boxed{h_T = Y_M\,(\sin^2\beta + \sin\alpha\, tg\,\alpha \cos\beta)}$$

that is the classical formula.

.../..

VI - 2 - DETERMINATION OF THE RELATIONS OF NO-ASTIGMATISM IN THE HOLOGRAPHIC GRATING CASE. -

If the condition to avoid the astigmatism is satisfied (eq. 34) it follows that :

$$\frac{1}{\ell_A} = \frac{\sin^2\beta}{\cos\alpha - \cos\beta} + \frac{\sin\alpha + \sin\beta}{\cos^2\alpha - \cos^2\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)$$

Let us carry forward in the equation S = o (eq. 31)

$$\frac{\sin^2\beta}{\cos\alpha - \cos\beta} - \frac{\cos\alpha + \cos\beta}{R} + \frac{\sin\alpha + \sin\beta}{\cos^2\alpha - \cos^2\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)$$

$$+ \frac{1}{\ell_B} - (\sin\alpha + \sin\beta)\frac{K3}{Ko} = o$$

so :

$$\frac{1}{\ell_B} = \frac{-\sin^2\alpha}{R(\cos\alpha - \cos\beta)} + \frac{\sin\alpha + \sin\beta}{\cos^2\alpha - \cos^2\beta}\left[\frac{K3}{Ko}\cos^2\alpha - \frac{KI}{Ko}\right]$$

so, for obtaining in Bo an image of the source point A without astigmatism, for the wavelength λBo, it is necessary that the following four equations may be solved :

.../..

$$\frac{1}{\ell_A} = \frac{\sin^2\beta}{R(\cos\alpha - \cos\beta)} + \frac{\sin\alpha + \sin\beta}{\cos^2\alpha - \cos^2\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)$$

$$\frac{1}{\ell_B} = \frac{-\sin^2\alpha}{R(\cos\alpha - \cos\beta)} + \frac{\sin\alpha + \sin\beta}{\cos^2\alpha - \cos^2\beta}\left(\frac{K3}{Ko}\cos^2\alpha - \frac{KI}{Ko}\right)$$

(40) $$KI = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} - \frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R})$$

$$K3 = \frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

with $Ko = \sin\gamma - \sin\delta$

and ℓ_A, ℓ_B, ℓ_C and ℓ_D essentially positive terms.

<u>Comment</u> : We know that one can gather the conventional grating's equations from the holographic grating's equations when writing $KI = K3 = o$.

The previous equations applied to the conventional grating lead to

$$\frac{1}{\ell_A} = \frac{\sin^2\beta}{R(\cos\alpha - \cos\beta)}$$

$$\frac{1}{\ell_B} = \frac{-\sin^2\alpha}{(\cos\alpha - \cos\beta)}$$

We point out that there is no real solution to that problem since, if $\ell_A > o$ it follows inevitably $\ell_B < o$.

.../..

VI - 3 - CONDITION ALLOWING THE EXTENSION OF NO-ASTIGMATISM PROPERTIES TO THE VICINITY OF THE CORRECTING POINT.

We are going to establish the relations allowing the astigmatism $\mathcal{A}$ to be equal to zero as well as $\frac{\partial \mathcal{A}}{\partial \beta}$ and $\frac{\partial^2 \mathcal{A}}{\partial \beta^2}$.

We are going to determine rapidly these relations in the general case i.e. $\alpha \neq \frac{\pi}{2}$ so as to apply them to the configurations in which A and Bo are linked by the relations of stigmatism.
We know that the relation ruling the no-astigmatism between a object point A and an image point Bo (for the wavelength λ Bo), is :

$$\frac{\cos\alpha - \cos\beta}{\ell_A} = \frac{\sin^2\beta}{R} + \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\,\frac{KI}{Ko}$$

(equation 34)

$$- \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\,\cos^2\beta\,\frac{K3}{Ko}$$

that may be written :

$$\frac{\cos\alpha - \cos\beta}{\ell_A} = \frac{\sin^2\beta}{R} + \frac{KI}{Ko}\,tg\frac{\alpha+\beta}{2} - \frac{K3}{Ko}\,tg\frac{\alpha+\beta}{2}\,\cos^2\beta$$

Deriving

$$\frac{\sin\beta}{\ell_A} = \frac{2\sin\beta\cos\beta}{R} + \frac{1}{2}\,\frac{KI}{Ko}\left(1 + tg^2\frac{\alpha+\beta}{2}\right) -$$

$$\frac{K3}{Ko}\left[\frac{1}{2}\left(1 + tg^2\frac{\alpha+\beta}{2}\right)\cos^2\beta - 2\cos\beta\sin\beta\,tg\frac{\alpha+\beta}{2}\right]$$

Deriving once again :

$$\frac{\cos\beta}{\ell_A} = \frac{2\cos 2\beta}{R} + \frac{1}{4}\times 2\,\frac{KI}{Ko}\,tg\frac{\alpha+\beta}{2}\,\frac{1}{\cos^2\frac{\alpha+\beta}{2}} -$$

.../..

$$\frac{K3}{Ko}\left[-\cos\beta\sin\beta\left(1+tg^2\frac{\alpha+\beta}{2}\right)+\frac{\cos^2\beta}{2}\times 2\times\frac{1}{2}\,tg\frac{\alpha+\beta}{2}\,\frac{1}{\cos^2\frac{\alpha+\beta}{2}}\right.$$

$$\left.-2\cos2\beta\,tg\frac{\alpha+\beta}{2}-\frac{\sin2\beta}{2\cos^2\frac{\alpha+\beta}{2}}\right]$$

This group of three equations is written as follows :

(41) $$\frac{\cos\alpha-\cos\beta}{\ell_A}=\frac{\sin^2\beta}{R}+\frac{KI}{Ko}\,tg\frac{\alpha+\beta}{2}-\frac{K3}{Ko}\,tg\frac{\alpha+\beta}{2}\,\cos^2\beta$$

(42) $$\frac{\sin\beta}{\ell_A}-\frac{\sin2\beta}{R}+\frac{1}{2}\frac{KI}{Ko}\,\frac{1}{\cos^2\frac{\alpha+\beta}{2}}-\frac{K3}{Ko}\left(\frac{\cos^2\beta}{2\cos^2\frac{\alpha+\beta}{2}}-\sin2\beta\,tg\frac{\alpha+\beta}{2}\right)$$

(43) $$\frac{\cos\beta}{\ell_A}-\frac{2\cos2\beta}{R}+\frac{KI}{Ko}\,\frac{\sin\frac{\alpha+\beta}{2}}{2\cos^3\frac{\alpha+\beta}{2}}-\frac{K3}{Ko}\left[-\frac{\sin2\beta}{\cos^2\frac{\alpha+\beta}{2}}+\frac{\cos^2\beta}{2\cos^3\frac{\alpha+\beta}{2}}\sin\frac{\alpha+\beta}{2}\right.$$

$$\left.-2\cos2\beta\,tg\frac{\alpha+\beta}{2}\right]$$

We may determine the unknown quantities ℓ_A , $\frac{KI}{Ko}$ and $\frac{K3}{Ko}$ from these three equations.

From the equation $T = 0$ or $S = 0$ ℓ_B is determined.

For example : $$\frac{1}{\ell_A}+\frac{1}{\ell_B}-\frac{\cos\alpha+\cos\beta}{R}-\frac{K3}{Ko}\left(\sin\alpha+\sin\beta\right)=0$$

.../..

VI - 4 - FOCALS' HEIGHT

We have established the relations conditioning the no-astigmatism at a determined point Bo and then for a point B closed to Bo, 1°) in the first order and 2°) in the second order.

If these relations are verified, we may now determine what is the value of the slit height.

We know that $h_T = Z\,m\left(1 - \frac{\ell_{TB}}{\ell_{SB}}\right) = Z\,m\,\frac{\ell_{SB} - \ell_{TB}}{\ell_{SB}}$

and we have determined the general equation of h_T from the equation giving the values ℓ_{TB} and ℓ_{SB} in the general case (Equation 32) ——

$$h_T = Zm\left[1 - \frac{-\cos^2\beta\left[-R + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\,\frac{K3}{Ko}\right]}{-R\cos^2\alpha + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\,\frac{KI}{Ko}}\right]$$

(Equation 33) ——

We may write $h_T = Zm \cdot \frac{U}{V}$

with $U = -R\cos^2\alpha + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\,\frac{KI}{Ko}$

$$-\cos^2\beta\left[-R + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\,\frac{K3}{Ko}\right]$$

$$V = -R\cos^2\alpha + \ell_A(\cos\alpha + \cos\beta) + R\,\ell_A(\sin\alpha + \sin\beta)\,\frac{KI}{Ko}$$

.../..

.../..

We may write :

$$h_T = Zm \cdot \frac{R(\cos^2\beta - \cos^2\alpha) + \ell_A(\cos\alpha + \cos\beta)(1-\cos^2\beta) + R\ell_A(\sin\alpha + \sin\beta)\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)}{-R\cos^2\alpha + \ell_A(\cos\alpha + \cos\beta) + R\ell_A(\sin\alpha + \sin\beta)\frac{KI}{Ko}}$$

$$h_T = Zm \cdot \frac{\ell_A R(\cos\alpha + \cos\beta)\left[\frac{\cos\beta - \cos\alpha}{\ell_A} + \frac{\sin^2\beta}{R} + \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)\right]}{\ell_A R(\cos\alpha + \cos\beta)\left[\frac{-\cos^2\alpha}{(\cos\alpha + \cos\beta)\ell_A} + \frac{1}{R} + \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\frac{KI}{Ko}\right]}$$

If we suppose $\cos\alpha + \cos\beta \neq 0$

$$\boxed{h_T = \frac{\frac{\cos\beta - \cos\alpha}{\ell_A} + \frac{\sin^2\beta}{R} + \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)}{-\frac{\cos^2\alpha}{(\cos\alpha + \cos\beta)\ell_A} + \frac{1}{R} + \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\frac{KI}{Ko}}} \qquad (44)$$

We identify at the numerator of the formula (44) providing h_T, the condition of no-astigmatism already determined. (34).

Now, let us suppose that the condition of no-astigmatism is really fulfiled. The focal's height, for a point B close to Bo will be obtained by developing the function $h_T(\beta)$ according to the Taylor's formula:

$$f(\beta_0 + \theta) = f(\beta_0) + \theta f'(\beta_0) + \frac{\theta^2}{2!} f''(\beta_0) + \frac{\theta^3}{3!} f'''(\beta_0)$$

in which the function $f(\beta)$ is the function $h_T(\beta)$

Therefore, if $h_T(\beta_0) = 0$, the focal's height, for the point B corresponding to the angle $\beta_0 + \theta$, will be given by

.../..

.../..

$$h_T\ (\beta_0 + \theta) = \theta\ h'_T\ (\beta_0)$$

Then, we may write $h_T\ (\beta) = \dfrac{U}{V}$

So $h'_T\ (\beta) = U' \cdot \dfrac{1}{V} - \dfrac{U' \cdot U}{V^2}$

However, we have just supposed that $h_T\ (\beta_0) = 0$ i.e. $U = 0$

Under such conditions $h'_T\ (\beta)$ is reduced to $U' \cdot \dfrac{1}{V}$

On the other hand, U' is the derived of the numerator i.e. of the equation (34)

We may then deduce that :

$$h'_T\ (\theta) = Zm.\theta \frac{-\dfrac{\sin\beta}{\ell_A} + \dfrac{\sin 2\beta}{R} + \dfrac{1}{2}\ \dfrac{KI}{Ko}\ \dfrac{1}{\cos^2\frac{\alpha+\beta}{2}} - \dfrac{K3}{Ko}\left(\dfrac{\cos^2\beta}{\cos^2\frac{\alpha+\beta}{2}} - \sin 2\beta\ tg\frac{\alpha+\beta}{2}\right)}{-\dfrac{\cos^2\alpha}{(\cos\alpha + \cos\beta)\ell_A} + \dfrac{1}{R} + tg\dfrac{\alpha+\beta}{2}\ \dfrac{KI}{Ko}} \tag{45}$$

If h'_T is zero, we may reason identically and we observe that :

$$h''_T\ (\theta^2) = \frac{Zm}{2}\ \theta^2 \frac{-\dfrac{\cos\beta}{\ell_A} + \dfrac{2\cos 2\beta}{R} + \dfrac{KI}{Ko}\ \dfrac{\sin\frac{\alpha+\beta}{2}}{2\cos^3\frac{\alpha+\beta}{2}} - \dfrac{K3}{Ko}\left[\dfrac{-\sin 2\beta}{\cos^2\frac{\alpha+\beta}{2}} + \dfrac{\cos^2\beta\ \sin\frac{\alpha+\beta}{2}}{2\cos^3\frac{\alpha+\beta}{2}} - 2\cos 2\beta\ tg\frac{\alpha+\beta}{2}\right]}{-\dfrac{\cos^2\alpha}{(\cos\alpha + \cos\beta)\ell_A} + \dfrac{1}{R} + tg\dfrac{\alpha+\beta}{2}\ \dfrac{KI}{Ko}} \tag{46}$$

One may go on with the reasoning and calculate h'''_T in the event of $h''_T = 0$. However, the calculation becomes complex and it is advisable to refer to the general formula of h_T.

VI - 5 STUDY OF THE ASTIGMATISM AT TANGENTIAL INCIDENCE.

Now we suppose that $\alpha \neq \frac{\pi}{2}$

i.e. $\sin\alpha = 1$ $\cos\alpha = 0$.

In this case

$$(47)\quad \left|\begin{array}{l} \dfrac{1}{\ell_A} = -\dfrac{\sin^2\beta}{R\cos\beta} - \dfrac{1+\sin\beta}{\cos^2\beta}\left(\dfrac{KI}{Ko} - \dfrac{K3}{Ko}\cos^2\beta\right) \\[2ex] \dfrac{1}{\ell_B} = +\dfrac{1}{R\cos\beta} + \dfrac{1+\sin\beta}{\cos^2\beta}\,\dfrac{KI}{Ko} \end{array}\right.$$

Particular case.

Let us write that the diffraction angle $\beta = 0$

$$\left|\begin{array}{l} \dfrac{1}{\ell_A} = -\dfrac{KI}{Ko} + \dfrac{K3}{Ko} \\[2ex] \dfrac{1}{\ell_B} = +\dfrac{I}{R} + \dfrac{KI}{Ko} \end{array}\right.$$

When writing ℓ_A and ℓ_B have to be positive, obviously we obtain the condition :

$$\boxed{\frac{K3}{Ko} > \frac{KI}{Ko} > -\frac{I}{R}}$$

EXTENSION OF THE ASTIGMATISM'S REDUCTION.

We wrote that the condition

$$\frac{\cos\alpha - \cos\beta}{\ell_A} = \frac{\sin^2\beta}{R} + \frac{\sin\alpha + \sin^2\beta}{\cos\alpha + \cos\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)$$

was valid for the pair A and Bo.

.../..

Now, let us write that this condition is not only valid for the Bo point but also at the vicinity of Bo.

Let us consider the particular concerned case $\alpha = \frac{\pi}{2}$

So, the condition may be written :

$$-\frac{\cos\beta}{\ell_A} = \frac{\sin^2\beta}{R} + \frac{1+\sin\beta}{\cos\beta}\,\frac{KI}{Ko} - \frac{K3}{Ko}\cos\beta\,(1+\sin\beta)$$

We have $\beta = \beta_0 + \theta$ with θ small. Let us develop, taking no account of the terms in θ^3 or terms higher order in θ.

$$-\frac{\cos(\beta_0+\theta)}{\ell_A} = \frac{-\cos\beta_0\cos\theta + \sin\beta_0\sin\theta}{\ell_A}$$

$$= -\frac{\cos\beta_0}{\ell_A}\left(1 - \theta\, tg\beta_0 - \frac{\theta^2}{2}\right)$$

$$\frac{\sin^2(\beta_0+\theta)}{R} = \frac{(\sin\beta_0\cos\theta + \sin\theta\cos\beta_0)^2}{R} =$$

$$\sin^2\beta_0\left(1 + \frac{\theta}{tg\beta_0} - \frac{\theta^2}{2}\right)^2$$

$$\frac{KI}{Ko}\,\frac{1+\sin\beta}{\cos\beta} = \frac{KI}{Ko\cos\beta_0}\left[1 + \sin\beta_0\left(1 + \frac{\theta}{tg\beta_0} - \frac{\theta^2}{2}\right)\right]$$

$$\left[1 + \theta\, tg\beta_0 + \left(\frac{1}{2} + tg^2\beta_0\right)\theta^2\right]$$

$$-\frac{K3}{Ko}\cos\beta\,(1+\sin\beta) = -\frac{K3}{Ko}\cos\beta_0\left[1 + \sin\beta_0\left(1 + \frac{\theta}{tg\beta_0} - \frac{\theta^2}{2}\right)\right]$$

$$\left[1 - \theta\, tg\beta_0 - \frac{\theta^2}{2}\right]$$

.../..

We may group the terms in θ and in θ^2

Terms in θ

$$\frac{\sin\beta_o}{\ell_A} = \frac{2\sin^2\beta_o}{R\,tg\beta_o} + \frac{KI}{Ko}\left[\frac{\sin\beta_o}{\cos\beta_o}\,\frac{1}{tg\beta_o} + \frac{tg\beta_o}{\cos\beta_o}\,(1+\sin\beta_o)\right]$$

$$-\frac{K3}{Ko}\left[\frac{\sin\beta_o\cos\beta_o}{tg\beta_o} - tg\beta_o\,\cos\beta_o\,(1+\sin\beta_o)\right]$$

$$\frac{\sin\beta_o}{\ell_A} = \frac{2\sin\beta_o\cos\beta_o}{R} + \frac{KI}{Ko}\left[1 + \frac{tg\beta_o}{\cos\beta_o}\,(1+\sin\beta_o)\right]$$

$$-\frac{K3}{Ko}\left[\cos\beta_o - tg\beta_o\,(1+\sin\beta_o)\right]\cos\beta_o$$

Terms in θ^2

$$\frac{\cos\beta_o}{2\,\ell_A} = \frac{\sin^2\beta_o}{R}\left(\frac{1}{tg^2\beta_o} - 1\right) + \frac{KI}{Ko}\,\frac{I}{\cos\beta_o}\left[\left(\frac{1}{2} + tg^2\beta_o\right)(1+\sin^2\beta_o)\right.$$

$$\left.-\frac{\sin\beta_o}{2} + \sin\beta_o\right] - \frac{K3}{Ko}\cos\beta_o\left[-\frac{\sin\beta_o}{2} - \left(\frac{1+\sin\beta_o}{2}\right) - \sin\beta_o\right]$$

$$\frac{\cos\beta_o}{2\,\ell_A} = \frac{\cos 2\beta_o}{R} + \frac{KI}{Ko}\,\frac{2\sin\beta_o + \sin^2\beta_o}{2\cos^3\beta_o} + 1 + \frac{K3}{Ko}\cos\beta_o\,\frac{1+4\sin\beta_o}{2}$$

.../..

So we have the equations :

$$(48)\quad \boxed{\begin{aligned}
-\frac{\cos\beta_o}{\ell_A} &= \frac{\sin^2\beta_o}{R} + \frac{1+\sin\beta_o}{\cos\beta_o}\frac{KI}{Ko} - \frac{K3}{Ko}\cos\beta_o(1+\sin\beta_o)\\
\frac{\sin\beta_o}{\ell_A} &= \frac{\sin 2\beta_o}{R} + \frac{1+\sin\beta_o}{\cos^2\beta_o}\frac{KI}{Ko} - \frac{K3}{Ko}(\cos 2\beta_o - \sin\beta_o)\\
\frac{\cos\beta_o}{\ell_A} &= \frac{2\cos 2\beta_o}{R} + \frac{(1+\sin\beta_o)^2}{\cos^3\beta_o}\frac{KI}{Ko} + \frac{K3}{Ko}(\cos\beta_o + 2\sin 2\beta_o)
\end{aligned}}$$

Let us resolve the equations :

$$① \quad -\frac{\cos\beta_o}{\ell_A} = \frac{\sin^2\beta_o}{R} + \frac{1+\sin\beta_o}{\cos\beta_o}\frac{KI}{Ko} - \frac{K3}{Ko}\cos\beta_o(1+\sin\beta_o)$$

$$② \quad \frac{\sin\beta_o}{\ell_A} = \frac{\sin 2\beta_o}{R} + \frac{1+\sin\beta_o}{\cos^2\beta_o}\frac{KI}{Ko} - \frac{K3}{Ko}(\cos 2\beta_o - \sin\beta_o)$$

$$③ \quad \frac{\cos\beta_o}{\ell_A} = \frac{2\cos 2\beta_o}{R} + \frac{(1+\sin\beta_o)^2}{\cos^3\beta_o}\frac{KI}{Ko} + \frac{K3}{Ko}(\cos\beta_o + 2\sin 2\beta_o)$$

Let us add ① + ③ :

$$④ \quad \frac{2\cos^2\beta_o - \sin^2\beta_o}{R} + \frac{2+3\sin\beta_o - \sin^3\beta_o}{\cos^3\beta_o}\frac{KI}{Ko} + \frac{3}{2}\sin^2\beta_o\frac{K3}{Ko} = 0$$

.../..

Let us add ① x $\sin\beta$ + ② x $\cos\beta$:

$$⑤ \quad \frac{2\sin\beta_0 - \sin^3\beta_0}{R} + \frac{(1\ \sin\beta_0)^2\cos\beta_0}{\cos^2\beta_0}\ \frac{KI}{Ko} - \cos^3\beta_0\ \frac{K3}{Ko} = 0$$

Let us write now

① x $\cos^2\beta_0$ + ② x 3 $\sin\beta_0$

$$\frac{2\cos^4\beta_0 - \sin^2\beta_0\cos^2\beta_0 + 6\sin^2\beta_0 - 3\sin^4\beta_0}{R} +$$

$$\frac{2 + 3\sin\beta_0 - \sin^3\beta_0 + 3\sin\beta_0 + 6\sin^2\beta_0 + 3\sin^3\beta_0}{\cos\beta_0}\ \frac{KI}{Ko} = 0$$

Finally we obtain :

$$\frac{KI}{Ko} = -\frac{I}{2R}\ \frac{2 + \sin^2\beta}{(1 + \sin\beta)^3}\cos\beta$$

by replacing $\frac{KI}{Ko}$ by its value in the equation

⑤ we have

$$\frac{K3}{Ko} = \frac{1}{R\cos^3\beta}\left[\sin\beta\ (2 - \sin^2\beta) - \frac{2 + \sin^2\beta}{2\ (1 + \sin\beta)}\right]$$

For example, by carrying forward the $\frac{KI}{Ko}$ and $\frac{K3}{Ko}$ values in the equation ③ we have

$$(49) \quad \frac{\cos\beta_0}{\ell_A} = \frac{2\cos^2\beta_0}{R} - \frac{2 + \sin^2\beta_0}{2R\ (1 + \sin\beta_0)\cos^2\beta_0} + \frac{1 + 4\sin\beta_0}{R\cos^2\beta_0}\left[\sin\beta_0(2 - \sin^2\beta_0) - \frac{2 + \sin^2\beta_0}{2\ (1 + \sin\beta_0)}\right]$$

.../..

VI-5-1- BEING CONDITIONS OF SOLUTIONS

We are going to see whether, in the case of the holographic grating, it is possible to get values of ℓ_C ℓ_D γ and δ in such a way that the Bo point (ℓ_B, βo) may be without astigmatism.

We have seen that when $\alpha \neq \frac{\pi}{2}$

$$\frac{1}{\ell_A} = \frac{-\sin^2\beta}{R\cos\beta} - \frac{1+\sin\beta}{\cos^2\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right)$$

$$\frac{1}{\ell_B} = \frac{1}{R\cos\beta} + \frac{1+\sin\beta}{\cos^2\beta}\,\frac{KI}{Ko}$$

Let us write that ℓ_A and ℓ_B are positive.

We have the unequalities

$$\frac{1}{R\cos\beta} + \frac{1+\sin\beta}{\cos^2\beta}\,\frac{KI}{Ko} > 0$$

$$\frac{\sin^2\beta}{R\cos\beta} + \frac{1+\sin\beta}{\cos^2\beta}\left(\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta\right) < 0$$

or $$\frac{KI}{Ko} > -\frac{\cos\beta}{R(1+\sin\beta)}$$

$$\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta < -\frac{\cos\beta\,\sin^2\beta}{R(1+\sin\beta)}$$

One can write the following system of unequality :

.../..

$$\frac{KI}{Ko} > \frac{-\cos\beta}{R\,(1+\sin\beta)}$$

$$-\frac{KI}{Ko} + \frac{K3}{Ko}\cos^2\beta > \frac{\cos\beta\sin^2\beta}{R\,(1+\sin\beta)}$$

which gives the conditions as follows :

$$(50)\qquad \boxed{\frac{K3}{Ko}\cos^2\beta - \frac{\cos\beta\sin^2\beta}{R\,(1+\sin\beta)} > \frac{KI}{Ko} > \frac{-\cos\beta}{R\,(1+\sin\beta)}}$$

with

$$\frac{KI}{Ko} = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos^2\delta}{\ell_D} - \frac{\cos\gamma - \cos\delta}{R}$$

$$\frac{K3}{Ko} = \frac{1}{\ell_C} - \frac{1}{\ell_D} - \frac{\cos\gamma - \cos\delta}{R}$$

We can replace $\frac{KI}{Ko}$ and $\frac{K3}{Ko}$ by their value.

First, let us consider the unequality

$$\frac{KI}{Ko} > \frac{-\cos\beta}{R\,(1+\sin\beta)} \qquad \text{with} \qquad \frac{KI}{Ko} = -\frac{I}{2R}\,\frac{2+\sin^2\beta}{(1+\sin\beta)^3}\cos\beta$$

$$-\frac{I}{2R}\,\frac{2+\sin^2\beta}{(1+\sin\beta)^3}\cos\beta > \frac{-\cos\beta}{(1+\sin\beta)\,R}$$

or $\dfrac{2+\sin^2\beta}{(1+\sin\beta)^2} < 2$

and finally

$$(\rightarrow)\qquad \boxed{\sin\beta\,(\sin\beta + 1) > 0}$$

.../..

.../..

Let us use the same method as regards the second unequality

$$\frac{K3}{Ko} \cos^2\beta \; \frac{-\cos\beta \sin^2\beta}{R\,(1+\sin\beta)} > -\frac{I}{2R} \; \frac{2+\sin^2\beta}{(1+\sin\beta)^3} \cos\beta$$

$$\text{with } \frac{K3}{Ko} = \frac{1}{R\cos^3\beta} \left[\sin\beta\,(2-\sin^2\beta) - \frac{2+\sin^2\beta}{2\,(1+\sin\beta)} \right]$$

So, we obtain :

$$\frac{1}{\cos\beta} \left[\sin\beta\,(2-\sin^2\beta) - \frac{2+\sin^2\beta}{2(1+\sin\beta)} \right] \frac{-\cos\beta \sin^2\beta}{1+\sin\beta} > -\frac{2+\sin^2\beta}{2\,(1+\sin\beta)^3} \cos\beta$$

$$\sin\beta\,(2-\sin^2\beta) - \frac{2+\sin^2\beta}{2\,(1+\sin\beta)} - \frac{\cos^2\beta \sin^2\beta}{1+\sin\beta} > -\frac{I}{2} \frac{2+\sin^2\beta}{(1+\sin\beta)^3} \cos^2\beta$$

$$2\sin\beta\,(1+\sin\beta)(2-\sin^2\beta) - 2 - 3\sin^2\beta + 2\sin^4\beta > \frac{-2+\sin^2\beta}{(1+\sin\beta)^2}\,(1-\sin^2\beta)$$

$$-2\sin^3\beta + \sin^2\beta + 4\sin\beta - 2 > -\frac{(2+\sin^2\beta)\,(1-\sin\beta)}{1+\sin\beta}$$

Simplifying the previous expression :

$$-\sin^2\beta - \sin\beta + 3 > 0$$

that is always right.

So, the only condition for β is $\sin\beta > 0$

VI-5-2-ASTIGMATISM IN THE VICINITY OF THE STIGMATIC POINTS

We have seen that there was stigmatic correspondence between a point H defined by

$$\left| \begin{array}{l} \ell_H = m\ell_C \\ \sin\beta H = m \sin\gamma \qquad \text{with } m = \dfrac{\lambda_H}{\lambda_o} \end{array} \right.$$

and the middle of the grating (D is at the centre of curvature).

So, the coordinates of A will be :

$$\ell_A = m\,\ell_C$$

$$\sin\alpha = m \sin\gamma$$

and the coordinates of B and D :

$$\left| \begin{array}{ll} \ell_B = R & \ell_D = R \\ \beta = o & \delta = o \end{array} \right.$$

More we suppose $\alpha = \dfrac{\pi}{2}$

Under such conditions, the equations of no-astigmatism can be written :

$$(51) \quad \left| \begin{array}{l} -\dfrac{I}{\ell_A} = \dfrac{KI}{Ko} - \dfrac{K3}{Ko} \\ 0 = \dfrac{KI}{Ko} - \dfrac{K3}{Ko} \\ \dfrac{1}{\ell_A} = \dfrac{2}{R} + \dfrac{KI}{Ko} + \dfrac{K3}{Ko} \end{array} \right.$$

.../..

We had the same equations already in the solution with $\ell_A = \infty$

$$\frac{KI}{Ko} = \frac{K3}{Ko} = -\frac{I}{R}$$

So $\ell_C = \infty$

and
$$\left| \begin{array}{l} KI = -\dfrac{\cos\gamma}{R} \\ \\ K3 = -\dfrac{\cos\gamma}{R} \end{array} \right.$$

Then it is necessary that $\gamma \backsim 0$

There is no solution.

However, if we consider $\frac{\partial \mathcal{A}}{\partial \beta} = o$ as sufficient, we just need that $\frac{KI}{Ko} = \frac{K3}{Ko}$ and it is possible to make the grating with any γ angle.

VI-5-3- FEASIBILITY OF THE GRATING WITHOUT ASTIGMATISM.

By writing that $\frac{\partial \mathcal{A}}{\partial \beta} = o$ and $\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = o$ we have seen that some values of $\frac{KI}{Ko}$ and of $\frac{K3}{Ko}$ allowed to obtain in Bo an image of the point A without astigmatism, for the wavelength λBo, and more to reduce that astigmatism around Bo.

The matter is to know if such a grating is feasible.

$$\frac{\cos^2\gamma}{\ell_C} - \frac{\cos^2\delta}{\ell_D} - \frac{\cos\gamma - \cos\delta}{R} = KI$$

$$\frac{1}{\ell_C} - \frac{1}{\ell_D} - \frac{\cos\gamma - \cos\delta}{R} = K3$$

.../..

We can obtain $\frac{1}{\ell_C}$ and $\frac{1}{\ell_D}$ from those equations

$$(52)\quad \boxed{\begin{aligned} &\frac{\cos^2\gamma - \cos^2\delta}{\ell_C} - \frac{\sin^2\gamma}{R}(\cos\gamma - \cos\delta) = KI - K3\cos^2\gamma \\ &\frac{\cos^2\gamma - \cos^2\delta}{\ell_D} - \frac{\sin^2\delta}{R}(\cos\gamma - \cos\delta) = KI - K3\cos^2\delta \end{aligned}}$$

The condition to fulfil is $\ell_C > 0$

$\ell_D > 0$

Let us suppose $\alpha = \frac{\pi}{2}$ and $\beta = 0$.

In this case the equations $T = 0$ $\frac{\partial \mathcal{A}}{\partial \beta} = 0$ and $\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0$ may be written :

$$-\frac{I}{\ell_A} = \frac{KI}{Ko} - \frac{K3}{Ko}$$

$$0 = \frac{KI}{Ko} - \frac{K3}{Ko}$$

$$\frac{1}{\ell_A} = \frac{2}{R} + \frac{KI}{Ko} + \frac{K3}{Ko}$$

The evident solutions are $\ell_A = \infty$

$$\frac{KI}{Ko} = \frac{K3}{Ko} = \frac{I}{R}$$

.../..

C.2

In that case, we have :

$$\frac{1}{\ell_C} = \frac{I}{R} \; \frac{\cos\gamma - \cos\delta - I}{\cos^2\gamma - \cos^2\delta} \; \sin^2\gamma$$

(53)

$$\frac{1}{\ell_D} = \frac{1}{R} \; \frac{\cos\gamma - \cos\delta - I}{\cos^2\gamma - \cos^2\delta} \; \sin^2\delta$$

It is necessary that $\cos\gamma - \cos\delta < 1$

and $\cos\gamma < \cos\delta$

for obtaining a real solution.

VI-6 STUDY OF THE ASTIGMATISM ON THE ROWLAND CIRCLE.

VI-6-1- General Study.

It is a particular and very important case.

Let us consider the general expression of the aberrations :

$$\Delta = \frac{Y^2}{2}\left[\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\,\frac{KI}{Ko}\right]$$

$$+ \frac{Z^2}{2}\left[\frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{1}{\ell_B} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\,\frac{K3}{Ko}\right]$$

$$+ \frac{Y^3}{2}\left[\frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right)\right.$$

$$\left. - (\sin\alpha + \sin\beta)\,\frac{K2}{Ko}\right]$$

.../..

$$+ \frac{Y\,Z^2}{2} \left[\frac{\sin\alpha}{\ell_A} \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} \right) + \frac{\sin\beta}{\ell_B} \left(\frac{1}{\ell_B} - \frac{\cos\beta}{R} \right) - (\sin\alpha + \sin\beta) \frac{K4}{Ko} \right]$$

$$+ Y^4 \ldots$$

The notations are :

$$Ko = \sin\gamma - \sin\delta$$

$$KI = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R} \right)$$

$$K2 = \frac{\sin\gamma}{\ell_C} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} \right) - \frac{\sin\delta}{\ell_D} \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R} \right)$$

$$K3 = \frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R} \right)$$

$$K4 = \frac{\sin\gamma}{\ell_C} \left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R} \right) - \frac{\sin\delta}{\ell_D} \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R} \right)$$

We suppose that the point source A and the images B will be on the Rowland circle, i.e. that the relations

$\ell_A = R\cos\alpha \quad \ell_B = R\cos\beta$ are verified.

Then the aberration equations can be written :

$$\Delta = \frac{Y^2}{2} \left[- (\sin\alpha + \sin\beta) \frac{KI}{Ko} \right] \tag{54}$$

.../..

$$+ \frac{Z^2}{2} \left[\frac{\sin^2\alpha}{R\cos\alpha} + \frac{\sin^2\beta}{R\cos\beta} - (\sin\alpha + \sin\beta)\frac{K3}{Ko} \right]$$

$$+ \frac{Y^3}{2} \left[- (\sin\alpha + \sin\beta)\frac{K2}{Ko} \right]$$

$$+ \frac{Y Z^2}{2} \left[\frac{\sin^3\alpha}{R^2\cos^2\alpha} + \frac{\sin^3\beta}{R^2\cos^2\beta} - (\sin\alpha + \sin\beta)\frac{K4}{Ko} \right]$$

We know that for a conventional concave grating (ruled grating)
$KI = K2 = K3 = K4 = o$

If we want 1) to have the locus of the tangential focus on the Rowland circle,

2) to have the coma (term in Y^3) null on the tangential focus,

it is necessary and sufficient that the terms in Y^2 and Y^3 should be zero whatever β is, i.e. that $KI = K2 = o$

Under these conditions the holographic grating has the same properties as those of the conventional grating, i.e. tangential's locus : Rowland circle, coma equal to zero on the tangential curve.

For a conventional grating, we see immediately that the residual aberrations are :

$$\Delta = \frac{Z^2}{2} \left[\frac{\sin^2\alpha}{R\cos\alpha} + \frac{\sin^2\beta}{R\cos\beta} \right]$$

(55)
$$+ \frac{YZ^2}{2} \left[\frac{\sin^3\alpha}{R^2\cos^2\alpha} + \frac{\sin^3\beta}{R^2\cos^2\beta} \right]$$

whereas in the case of holographic grating, they will be as follows :

.../..

.../..

$$\Delta = \frac{Z^2}{2}\left[\frac{\sin^2\alpha}{R\cos\alpha} + \frac{\sin^2\beta}{R\cos\beta} - (\sin\alpha + \sin\beta)\frac{K3}{Ko}\right]$$

(56)

$$+ \frac{Y\,Z^2}{2}\left[\frac{\sin^3\alpha}{-R^2\cos^2\alpha} + \frac{\sin^3\beta}{R^2\cos^2\beta} - (\sin\alpha + \sin\beta)\frac{K4}{Ko}\right]$$

So, we can resolve the system

$$(57)\quad \left|\begin{array}{l} KI = o \\ K2 = o \\ \dfrac{K3}{Ko} = \left[\dfrac{\sin^2\alpha}{\cos\alpha} + \dfrac{\sin^2\beta}{\cos\beta}\right]\dfrac{1}{R(\sin\alpha + \sin\beta)} \\ \dfrac{K4}{Ko} = \left[\dfrac{\sin^3\alpha}{\cos^2\alpha} + \dfrac{\sin^3\beta}{\cos^2\beta}\right]\dfrac{1}{R^2(\sin\alpha + \sin\beta)} \end{array}\right.$$

in such a way to avoid the aberrations in Z^2 (astigmatism) and $Y\,Z^2$.

Let us consider the equations

$KI = o \quad K2 = o$

$$(58)\quad \left\{\begin{array}{l} \dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R} - \dfrac{\cos^2\delta}{\ell_D} + \dfrac{\cos\delta}{R} = o \\[2ex] \dfrac{\sin\gamma}{\ell_C}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) - \dfrac{\sin\delta}{\ell_D}\left(\dfrac{\cos^2\delta}{\ell_D} - \dfrac{\cos\delta}{R}\right) = o \end{array}\right.$$

.../..

We have :

$$\left|\begin{array}{l} \dfrac{\cos\gamma}{\ell_C} - \dfrac{1}{2R} = X \\[2ex] \dfrac{\cos\delta}{\ell_D} - \dfrac{1}{2R} = Y \end{array}\right.$$

The system may be written :

$$(59) \quad \left|\begin{array}{l} X\cos\gamma - Y\cos\delta - \dfrac{\cos\gamma - \cos\delta}{2R} = 0 \\[2ex] X^2\sin\gamma - Y^2\sin\delta - \dfrac{\sin\gamma - \sin\delta}{4R^2} = 0 \end{array}\right.$$

Those two equations have an evident solution

$$X = \frac{1}{2R}$$

$$Y = \frac{1}{2R}$$

This solution consists in locating the C and D points on the Rowland circle.

Effectively in that case $\ell_C = R\cos\gamma$ and $y = \dfrac{I}{2R}$

$$\ell_D = R\cos\delta \quad \text{and} \quad Y = \frac{I}{2R}$$

But, obviously, that system may be solved in another way.

We may obtain $X = \dfrac{Y\cos\delta}{\cos\gamma} + \dfrac{1}{\cos\gamma}\left(\dfrac{\cos\gamma - \cos\delta}{2R}\right)$

.../..

By carrying forward in the second equation

$$\left[\frac{Y \cos\delta}{\cos\gamma} + \frac{1}{\cos\gamma} \frac{(\cos\gamma - \cos\delta)}{2R} \right]^2 \sin\gamma - Y^2 \sin\delta$$

$$- \frac{\sin\gamma - \sin\delta}{4R^2} = 0$$

It is an equation of second degree in Y for which the sum of the roots is $-\frac{b}{a}$ i.e.

$$- \frac{\cos\delta\,(\cos\gamma - \cos\delta)}{R\cos^2\gamma} \sin\gamma \times \frac{1}{\frac{\cos^2\delta}{\cos^2\gamma} \sin^2\gamma - \sin\delta} = - \frac{b}{a}$$

So we can find out the second Y_2 root by writing

$$- \frac{\cos\delta\,(\cos\gamma - \cos\delta)\sin\gamma}{R\,(\cos^2\delta \sin\gamma - \cos^2\gamma \sin\delta)} - \frac{I}{2R} = Y2 = \frac{\cos\delta}{\ell_D} - \frac{I}{2R}$$

$$\boxed{\frac{1}{\ell_D} = \frac{-(\cos\gamma - \cos\delta)\sin\gamma}{R\,(\cos^2\delta \sin\gamma - \cos^2\gamma \sin\delta)}}$$

and, by changing (using the symmetry)

$$\cos\gamma \longrightarrow -\cos\delta$$

$$\sin\gamma \longrightarrow -\sin\delta$$

$$\frac{1}{\ell_C} = + \frac{(\cos\delta - \cos\gamma)\sin\delta}{R\,(-\cos^2\gamma \sin\delta + \cos^2\delta \sin\gamma)}$$

.../..

(60)
$$\boxed{\begin{aligned} \frac{1}{\ell_D} &= \frac{-(\cos\gamma - \cos\delta)\sin\gamma}{R\,(\cos^2\delta\,\sin\gamma - \cos^2\gamma\,\sin\delta)} \\ \frac{1}{\ell_C} &= \frac{-(\cos\gamma - \cos\delta)\sin\delta}{R\,(\cos^2\delta\,\sin\gamma - \cos^2\gamma\,\sin\delta)} \end{aligned}}$$

Now it is possible to calculate the coefficients K3 and K4 :

$$K3 = \frac{1}{\ell_C} - \frac{1}{\ell_D} - \frac{\cos\gamma - \cos\delta}{R}$$

$$= \frac{(\cos\gamma - \cos\delta)(\sin\gamma - \sin\delta)}{R\,(\cos^2\delta\,\sin\gamma - \cos^2\gamma\,\sin\delta)} - \frac{\cos\gamma - \cos\delta}{R}$$

$$= \frac{(\cos\gamma - \cos\delta)(\sin\gamma - \sin\delta)}{R\,(\sin\gamma - \sin\delta - \sin^2\delta\,\sin\gamma + \sin^2\gamma\,\sin\delta)} - \frac{\cos\gamma - \cos\delta}{R}$$

$$= \frac{\cos\gamma - \cos\delta}{R\,(1 + \sin\delta\,\sin\gamma)} - \frac{\cos\gamma - \cos\delta}{R}$$

Finally (61)
$$\boxed{K3 = \frac{-(\cos\gamma - \cos\delta)\sin\gamma\,\sin\delta}{R\,(1 + \sin\gamma\,\sin\delta)}}$$

$$K4 = \frac{\sin\gamma}{\ell_C}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$= \frac{-\sin\gamma\,\sin\delta\,(\cos\gamma - \cos\delta)}{R\,(\cos^2\delta\,\sin\gamma - \cos^2\gamma\,\sin\delta)}\left(\frac{1}{\ell_C} - \frac{\cos\delta}{R}\right) + \frac{\sin\gamma\,\sin\delta\,(\cos\gamma - \cos\delta)}{R\,(\cos^2\delta\,\sin\gamma - \cos^2\gamma\,\sin\delta)}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

.../..

$$= \frac{\sin\gamma \sin\delta (\cos\gamma - \cos\delta)}{R\,(\cos^2\delta \sin\gamma - \cos^2\gamma \sin\delta)} \left| \frac{1}{\ell_D} - \frac{\cos\delta}{R} - \frac{1}{\ell_C} + \frac{\cos\gamma}{R} \right|$$

We identify K3.

So :

$$(62) \qquad K4 = + \frac{\sin^2\gamma \sin^2\delta (\cos\gamma - \cos\delta)^2}{R^2\,(\cos^2\delta \sin\gamma - \cos^2\gamma \sin\delta)\,(1 + \sin\gamma \sin\delta)}$$

For resolving our problem : to avoid the astigmatism on the Rowland circle, we have just to write :

$$(63) \qquad \boxed{\left[\frac{\sin^2\alpha}{\cos\alpha} + \frac{\sin^2\beta}{\cos\beta} \right] \frac{1}{\sin\alpha + \sin\beta} = \frac{1}{\sin\gamma - \sin\delta} \frac{(\cos\gamma - \cos\delta) \sin\gamma \sin\delta}{1 + \sin\gamma \sin\delta}}$$

More, if we remove the aberration in $Y\,Z^2$, we have then

$$\left[\frac{\sin^3\alpha}{\cos^2\alpha} + \frac{\sin^3\beta}{\cos^2\beta} \right] \frac{1}{(\sin\alpha + \sin\beta)} = - \frac{1}{(\sin\gamma - \sin\delta)} \frac{\sin^2\gamma \sin^2\delta (\cos\gamma - \cos\delta)^2}{(\cos^2\delta \sin\gamma - \cos^2\gamma \sin\delta)(1 + \sin\gamma \sin\delta)}$$

VI - 6 - 2 COMPARISON WITH TWO SOLUTIONS CORRESPONDING TO THE ROWLAND CIRCLE.

We have seen that if a grating is working on the Rowland circle and if we want the coma to be null, it results that

$$KI = K2 = 0$$

Under such conditions the equation of the aberrant optical path is :

$$\Delta = \frac{Z^2}{2}\left[\frac{\sin^2\alpha}{R\cos\alpha} + \frac{\sin^2\beta}{R\cos\beta} - (\sin\alpha + \sin\beta)\frac{K3}{Ko}\right]$$

$$+ Y Z^2\left[\frac{\sin^3\alpha}{R^2\cos^2\alpha} + \frac{\sin^3\beta}{R^2\cos^2\beta} - (\sin\alpha + \sin\beta)\frac{K4}{Ko}\right]$$

We noticed that the two equations $KI = 0$

$K2 = 0$

led to two groups of solution :

1st group $\ell_C = R\cos\gamma$

$\ell_D = R\cos\delta$

2nd group $\ell_C = \dfrac{R\cos^2\delta\sin\gamma - \cos^2\gamma\sin\delta}{-(\cos\gamma - \cos\delta)\sin\delta}$

$\ell_D = R\dfrac{\cos^2\delta\sin\gamma - \cos^2\gamma\sin\delta}{-(\cos\gamma - \cos\delta)\sin\gamma}$

.../..

.../..

One may write ℓ_C and ℓ_D under another form

$$\ell_C = R \; \frac{\sin\gamma(1 - \sin^2\delta) - \sin\delta(1 - \sin^2\gamma)}{-(\cos\gamma - \cos\delta)\sin\delta}$$

$$= R \left(- \frac{\sin\gamma - \sin\delta}{\cos\gamma - \cos\delta} \quad \frac{1 + \sin\gamma\sin\delta}{\sin\delta} \right)$$

(64)
$$\boxed{\begin{aligned} \ell_C &= \frac{R}{tg\,\frac{\gamma+\delta}{2}} \cdot \frac{1 + \sin\gamma\sin\delta}{\sin\delta} \\ \ell_D &= \frac{R}{tg\,\frac{\gamma+\delta}{2}} \cdot \frac{1 + \sin\gamma\sin\delta}{\sin\gamma} \end{aligned}}$$

Let us calculate $\frac{K3}{Ko}$ and $\frac{K4}{Ko}$ in these two groups of solution

$$K3 = \frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$K4 = \frac{\sin\gamma}{\ell_C}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

a) solution $\ell_C = R\cos\gamma$

$\ell_D = R\cos\delta$

$$K3 = \frac{1}{R\cos\gamma} - \frac{\cos\gamma}{R} - \left(\frac{1}{R\cos\delta} - \frac{\cos\delta}{R}\right) = \frac{\sin^2\gamma}{R\cos\gamma} - \frac{\sin^2\delta}{R\cos\delta}$$

(65)
$$\boxed{K3 = -(\cos\gamma - \cos\delta)\left(\frac{1 + \cos\delta\cos\gamma}{R\cos\gamma\cos\delta}\right)}$$

.../..

.../..

$$K4 = \frac{\sin\gamma}{R\cos\gamma}\left(\frac{1}{R\cos\gamma} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{R\cos\delta}\left(\frac{1}{R\cos\delta} - \frac{\cos\delta}{R}\right)$$

(66) $$\boxed{K4 = \frac{\sin^3\gamma}{R^2\cos^2\gamma} - \frac{\sin^3\delta}{R^2\cos^2\delta}}$$

b) Solution with $\ell_C = \dfrac{R}{tg\,\dfrac{\gamma+\delta}{2}} \cdot \dfrac{1 + \sin\gamma\sin\delta}{\sin\delta}$

$$\ell_D = \frac{R}{tg\,\dfrac{\gamma+\delta}{2}} \quad \frac{1 + \sin\gamma\sin\delta}{\sin\delta}$$

K3 and K4 have already been calculated in those cases (equations 61 and 62 , ———————)

(67) $$K3 = \frac{-(\cos\gamma - \cos\delta)\sin\gamma\sin\delta}{R\,(1 + \sin\gamma\sin\delta)}$$

(68) $$K4 = \frac{\sin^2\gamma\sin^2\delta\,(\cos\gamma - \cos\delta)^2}{R^2\,(1 + \sin\gamma\sin\delta)^2\,(\sin\gamma - \sin\delta)}$$

.../..

.../..

We have the two following possibilities :

I	II
$\ell_C = R \cos\gamma$	$\ell_C = \dfrac{R}{tg\,\frac{\gamma+\delta}{2}} \cdot \dfrac{1+\sin\gamma\sin\delta}{\sin\delta}$
$\ell_D = R \cos\delta$	$\ell_D = \dfrac{R}{tg\,\frac{\gamma+\delta}{2}} \cdot \dfrac{1+\sin\gamma\sin\delta}{\sin\gamma}$
$KI = K2 = 0$	$KI = K2 = 0$
$K3 = \dfrac{-(\cos\gamma - \cos\delta)\,(1+\cos\delta\cos\gamma)}{R\cos\gamma\cos\delta}$	$K4 = \dfrac{-(\cos\gamma - \cos\delta)\sin\gamma\sin\delta}{R\,(1+\sin\gamma\sin\delta)}$
$K4 = \left(\dfrac{\sin^3\gamma}{\cos^2\gamma} - \dfrac{\sin^3\delta}{\cos^2\delta}\right)\dfrac{1}{R^2}$	$K4 = \dfrac{\sin^2\gamma\sin^2\delta\,(\cos\gamma - \cos\delta)^2}{R^2\,(1+\sin\gamma\sin\delta)^2\,(\sin\gamma - \sin\delta)}$

<u>Comment</u> : Considering the solution type II

$$\frac{K3}{Ko} = -\frac{(\cos\gamma - \cos\delta)\sin\gamma\sin\delta}{R\,(1+\sin\gamma\sin\delta)(\sin\gamma - \sin\delta)} \qquad \frac{K4}{Ko} = \frac{\sin^2\gamma\sin^2\delta(\cos\gamma - \cos\delta)^2}{R^2\,(1+\sin\gamma\sin\delta)^2\,(\sin\gamma - \sin\delta)^2}$$

we observe that $\dfrac{K4}{Ko} = \left(\dfrac{K3}{Ko}\right)^2$

.../..

.../..

Under the above mentioned conditions, the focal's heights may be calculated.

Using the formula (44)

$$h_T = Zm \times \frac{\dfrac{\cos\beta - \cos\alpha}{\ell_A} + \dfrac{\sin^2\beta}{R} + \dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\left(\dfrac{KI}{Ko} - \dfrac{K3}{Ko}\cos^2\beta\right)}{-\dfrac{1}{(\cos\alpha + \cos\beta)}\dfrac{\cos^2\alpha}{\ell_A} + \dfrac{1}{R} + \dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\dfrac{KI}{Ko}}$$

Under the Rowland conditions $\ell_A = R\cos\alpha$

$$\ell_B = R\cos\beta$$

$$KI = K2 = 0$$

$$h_T = Zm \times \frac{\dfrac{\cos\beta - \cos\alpha}{R\cos\alpha} + \dfrac{\sin^2\beta}{R} - \dfrac{K3}{Ko}\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\cos^2\beta}{-\dfrac{1}{\cos\alpha + \cos\beta)}\dfrac{\cos^2\alpha}{R} + \dfrac{1}{R}}$$

$$h_T = Zm \times \frac{\dfrac{\cos\beta}{\cos\alpha} - \cos^2\beta - \dfrac{K3}{Ko}R\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}\cos^2\beta}{1 - \dfrac{\cos\alpha}{\cos\alpha + \cos\beta}}$$

.../..

.../..

$$h_T = Zm \frac{\frac{\cos\beta + \cos\alpha}{\cos\alpha \cos\beta} - (\cos\alpha + \cos\beta) - \frac{K3}{Ko} R (\sin\alpha + \sin\beta)}{\frac{\cos\beta}{\cos\alpha + \cos\beta} \times \frac{\cos\alpha + \cos\beta}{\cos^2\beta}}$$

(69) $$\boxed{h_T = Z\,m \times \left[\left(\frac{\sin^2\alpha}{\cos\alpha} + \frac{\sin^2\beta}{\cos\beta}\right) - \frac{K3}{Ko} R (\sin\alpha + \sin\beta) \right] \cos\beta}$$

In the same way, $h_T(\theta)$ may be calculated in the hypothesis $h_T = 0$ i.e. if $\frac{K3}{Ko} = \frac{1}{R(\sin\alpha + \sin\beta)} \left(\frac{\sin^2\alpha}{\cos\alpha} + \frac{\sin^2\beta}{\cos\beta}\right)$

Deriving the condition $\mathcal{A} = 0$ with respect to β :

$$2\frac{\sin\beta\cos\beta}{\cos\beta} + \frac{\sin^3\beta}{\cos^2\beta} - \frac{K3}{Ko} R \cos\beta$$

One concludes that :

(70) $$\boxed{h_T(\theta) = Z\,m \times \left(2\sin\beta + \frac{\sin^3\beta}{\cos^2\beta} - \frac{K3}{Ko} R \cos\beta\right) \cos\beta \cdot \theta}$$

with $\frac{K3}{Ko} = \frac{1}{R(\sin\alpha + \sin\beta)} \left(\frac{\sin^2\alpha}{\cos\alpha} + \frac{\sin^2\beta}{\cos\beta}\right)$

VII - STUDY OF THE COMA IN A TYPICALLY SPECTROGRAPH MOUNTING.

VII - 1 - GENERAL CASE

I - Initially the equations are :

$$T = o \quad \frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\ \frac{KI}{Ko} = o$$

$$C = o \quad \frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - (\sin\alpha + \sin\beta)\ \frac{K2}{Ko} = o$$

The point A (ℓ_A, α) is the source point to be considered as fixed with respect to the grating while the position of the image point B (ℓ_B, β) depends on the wavelength.

If we choose any condition the equation T = o makes possible to define the locus of the tangential but, generally, the equation C = o will not be satisfied.

We designate Bo the image of A for the wavelength λ_{Bo} and we suppose that Bo is chosen in such a way that the equations T = o and C = o are simultaneously satisfied.

Then we shall have to determine :

A, Bo, $\frac{KI}{Ko}$ and $\frac{K2}{Ko}$ so that not only the equations T = o and C = o may be satisfied but also

$$\frac{\partial \Delta c}{\partial \beta} = o \qquad \frac{\partial^2 \Delta c}{\partial \beta^2} = o$$

.../..

Let us consider the équations T = o and C = o

C may be written :

$$\sin\alpha\left(\frac{\cos\alpha}{\ell_A} - \frac{1}{2R}\right)^2 + \sin\beta\left(\frac{\cos\beta}{\ell_B} - \frac{1}{2R}\right)^2 - \frac{K2}{Ko}\left(\sin\alpha + \sin\beta\right)$$

$$- \frac{1}{4R^2}\left(\sin\alpha + \sin\beta\right) = o$$

With the same change of variable

T may be written :

$$\cos\alpha\left(\frac{\cos\alpha}{\ell_A} - \frac{1}{2R}\right) + \cos\beta\left(\frac{\cos\beta}{\ell_B} - \frac{1}{2R}\right) - \frac{KI}{Ko}\left(\sin\alpha + \sin\beta\right)$$

$$- \frac{\cos\alpha + \cos\beta}{2\,R} = o$$

We have : $X = \frac{\cos\alpha}{\ell_A} - \frac{1}{2R}$

$$Y = \frac{\cos\beta}{\ell_B} - \frac{1}{2R}$$

We obtain the two equations :

(71)

T = o	$X\cos\alpha + Y\cos\beta - \frac{KI}{Ko}\left(\sin\alpha + \sin\beta\right) - \frac{\cos\alpha + \cos\beta}{2\,R} = o$
C = o	$X^2\sin\alpha + Y^2\sin\beta - \left(\sin\alpha + \sin\beta\right)\left(\frac{K2}{Ko} + \frac{1}{4R^2}\right) = o$

.../..

II - We may try to see whether it is possible that the two equations $T = o$ and $C = o$ may be simultaneously satisfied, whatever β is.

$C = o$ can be written :

$$X^2 \sin\alpha - \sin\alpha \left(\frac{K2}{Ko} + \frac{1}{4R^2}\right) + \sin\beta \left(Y^2 - \frac{K2}{Ko} - \frac{1}{4R^2}\right) = o$$

If this equation is zero, whatever β is, that involves that simultaneously :

$$(72) \quad \begin{cases} X^2 \sin\alpha - \sin\alpha \left(\frac{K2}{Ko} + \frac{1}{4R^2}\right) = o \\ Y^2 - \frac{K2}{Ko} - \frac{1}{4R^2} = o \end{cases}$$

with $\frac{K2}{Ko} > -\frac{1}{4R^2}$ so

$$(41) \quad \begin{cases} X = \pm \sqrt{\frac{K2}{Ko} + \frac{1}{4R^2}} \\ Y = \pm \sqrt{\frac{K2}{Ko} + \frac{1}{4R^2}} \end{cases}$$

For the time being we do not discuss the signs' meaning.
We notice that those relations imply $X = \pm Y$.
Let us carry forward the values of X and Y in the equation $T = o$

$$\pm \sqrt{\frac{K2}{Ko} + \frac{1}{4R^2}} \cos\alpha \pm \sqrt{\frac{K2}{Ko} + \frac{1}{4R^2}} \cos\beta - \frac{KI}{Ko} (\sin\alpha + \sin\beta)$$

$$- \frac{\cos\alpha + \cos\beta}{2R} = o$$

This equation is to be valid whatever β is.
Let us remark that the choice of the sign + or - before the first term don't lead to the choice of the sign before the second term.

.../..

We have $\cos\beta = \dfrac{1 - t^2}{1 + t^2}$

$$t = \operatorname{tg} \frac{\beta}{2}$$

$$\sin\beta = \frac{2\,t}{1 + t^2}$$

The coefficients of t^2, t and the independant term of t must be equal to zero.

The only term's coefficient in t is $\dfrac{KI}{Ko} \longrightarrow \dfrac{KI}{Ko} = 0$

Then, the equation can be written :

$$\cos\alpha\left(\pm\sqrt{\frac{K2}{Ko} + \frac{I}{4R^2}} - \frac{1}{2R}\right) + \cos\beta\left(\pm\sqrt{\frac{K2}{Ko} + \frac{I}{4R^2}} - \frac{1}{2R}\right) = o$$

The only solution is $\dfrac{K2}{Ko} + \dfrac{I}{4R^2} = \dfrac{1}{4R^2}$

The equations T = o and C = o become then the equations of a conventional grating and we obtain <u>the Rowland circle</u> solution.

Yet, it does not mean that it is necessarily a conventional grating, i.e. that $\ell_C = \ell_D = \infty$ when writing KI = K2 = o.

We shall demonstrate that a holographic grating used under such conditions may have a lower astigmatism than a conventional grating.

<u>Conclusion</u> : The only solution to have T = o and C = o satisfied is to obtain simultaneously

$$\boxed{\begin{aligned} KI &= o \\ K2 &= o \end{aligned}}$$

whatever β is.

.../..

III - Let us consider the equations T = o and C = o as follows :

$$\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\ \frac{KI}{Ko} = o$$

$$\frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B} - \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\Big) - (\sin\alpha + \sin\beta)\ \frac{K2}{Ko} = o$$

Then we are going to determine the parameters α β_o ℓ_A ℓ_{Bo} K1 and K2 in such a way that, these two equations being satisfied, we might have in addition :

$$\left(\frac{\partial C}{\partial \beta}\right)_{\beta=\beta_o} = o \qquad \left(\frac{\partial^2 C}{\partial \beta^2}\right)_{\beta=\beta_o} = o$$

In order to simplify we shall write :

$$P = \frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} - \sin\alpha\ \frac{KI}{Ko}$$

$$Q = \frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2}{A} - \frac{\cos}{R}\right) - \sin\ \frac{K2}{Ko}$$

Then the equations can be written as follows :

$$T = \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - \frac{KI}{Ko}\ \sin\beta + P = o$$

(73)

$$C = \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{K2}{Ko}\ \sin\beta + Q = o$$

System equivalent to :

$$\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - \frac{KI}{Ko}\ \sin\beta + P = o$$

(74)

$$\frac{\sin\beta}{\ell_B}\left(\frac{KI}{Ko}\ \sin\beta - P\right) - \frac{K2}{Ko}\ \sin\beta + Q = o$$

.../..

The essential fact is to keep thoroughly the right position on the tangential.

Therefore, from T = o we are going to obtain $\frac{1}{\ell_B}$ with $\beta = \beta_0 + \theta$

and the value will be carried forward in the equation C = o.

From T = o we have $\frac{1}{\ell_B} = \frac{1}{R\cos\beta} + \frac{\sin\beta}{\cos^2\beta}\,\frac{KI}{Ko} - \frac{P}{\cos^2\beta}$

Let us develop with $\beta = \beta_0 + \theta$

$$\cos(\beta_0+\theta) = \cos\beta_0\cos\theta - \sin\beta_0\sin\theta$$

$$\cos(\beta_0+\theta) = \cos\beta_0 - \theta\sin\beta_0 - \frac{\theta^2}{2}\cos\beta_0$$

$$= \cos\beta_0\left(1 - \theta\, tg\,\beta_0 - \frac{\theta^2}{2}\right)$$

$$\frac{1}{\cos(\beta_0+\theta)} = \frac{1}{\cos\beta_0}\left[1 + \theta\, tg\,\beta_0 + \left(\frac{I}{2} + tg^2\beta_0\right)\theta^2\right]$$

$$\frac{1}{\cos^2(\beta_0+\theta)} = \frac{1}{\cos^2\beta_0}\left[1 + 2\theta\, tg\,\beta_0 + (1+3tg^2\beta_0)\theta^2\right]$$

$$\sin(\beta_0+\theta) = \sin\beta_0\cos\theta + \sin\theta\cos\beta_0 =$$

$$\sin\beta_0\left(1 + \frac{\theta}{tg\,\beta_0} - \frac{\theta^2}{2}\right.$$

$\cos\theta \sim 1 - \frac{\theta^2}{2}$

$\sin\theta \sim \theta$

by keeping within bounds of second degree.

$\frac{1}{1-x} = 1 + x + x^2 + \dots$

$x = \theta\, tg\,\beta_0 + \frac{\theta^2}{2}$

$x^2 = \theta^2\, tg^2\beta_0$

So :

$$\frac{1}{\ell_B} = \frac{1}{R\cos\beta_0}\left[1 + \theta\, tg\,\beta_0 + \left(\frac{1}{2} + tg^2\beta_0\right)\theta^2\right]$$

$$+ \frac{KI}{Ko}\,\frac{\sin\beta_0}{\cos^2\beta_0}\left[1 + \frac{\theta}{tg\,\beta_0} - \frac{\theta^2}{2}\right]\left[1 + 2\theta\, tg\,\beta_0 + (1+3\, tg^2\beta_0)\theta^2\right]$$

.../..

$$- \frac{P}{\cos^2\beta_0}\left[1 + 2\theta\, tg\,\beta_0 + (1 + 3\, tg^2\beta_0)\,\theta^2\right]$$

$$\frac{1}{\ell_B} = \frac{1}{R\cos\beta_0}\left[1 + \theta\, tg\,\beta_0 + \left(\frac{1}{2} + tg^2\beta_0\right)\theta^2\right]$$

$$+ \frac{KI}{Ko}\,\frac{\sin\beta_0}{\cos^2\beta_0}\left[1 + \left(\frac{1}{tg\,\beta_0} + 2\, tg\,\beta_0\right)\theta + \left(\frac{5}{2} + 3\, tg^2\beta_0\right)\theta^2\right]$$

$$- \frac{P}{\cos^2\beta_0}\left[1 + 2\theta\, tg\,\beta_0 + (1 + 3\, tg^2\beta_0)\,\theta^2\right]$$

Let us consider now the coma's expression :

$$\Delta\mathcal{C} = \frac{KI}{Ko}\,\frac{\sin^2\beta}{\ell_B} - \frac{P\sin\beta}{\ell_B} - \frac{K2}{Ko}\sin\beta + Q$$

$$\sin^2(\beta_0 + \theta) = \sin^2\beta_0\left[1 + \frac{2\theta}{tg\,\beta_0} + \left(\frac{1}{tg^2\beta_0} - 1\right)\theta^2\right]$$

$$(75)\quad \begin{cases} \dfrac{KI}{Ko}\,\dfrac{\sin^2\beta}{\ell_B} \longrightarrow \dfrac{KI}{Ko}\,\dfrac{\sin^2\beta}{R\cos\beta} + \left(\dfrac{KI}{Ko}\right)^2 \dfrac{\sin^3\beta}{\cos^2\beta} - \dfrac{P\,KI}{Ko}\,\dfrac{\sin^2\beta}{\cos^2\beta} \\[2ex] -\dfrac{P\sin\beta}{\ell_B} \longrightarrow \dfrac{P\sin\beta}{R\cos\beta} - \dfrac{P\,KI}{Ko}\,\dfrac{\sin^2\beta}{\cos^2\beta} + P^2\,\dfrac{\sin\beta}{\cos^2\beta} \\[2ex] -\dfrac{K2}{Ko}\sin\beta \\[2ex] +Q \end{cases}$$

Let us consider the terms one by one :

.../..

.../..

$$\frac{KI}{Ko}\ \frac{\sin^2\beta}{R\cos\beta} = \frac{KI}{R\,Ko}\ \frac{\sin^2\beta_0}{\cos\beta_0}\left[1+\theta\left(tg\,\beta_0+\frac{2}{tg\,\beta_0}\right)+\left(\frac{1}{2}+tg^2\beta_0-1+\frac{1}{tg^2\beta_0}+2\right)\theta^2\right]$$

(76)
$$\boxed{=\frac{KI}{R\,Ko}\ \frac{\sin^2\beta_0}{\cos\beta_0}\ 1+\left(tg\,\beta_0+\frac{2}{tg\,\beta_0}\right)\theta+\left(\frac{3}{2}+tg^2\beta_0+\frac{1}{tg^2\beta_0}\right)\theta^2}$$

$$-P\,\frac{KI}{Ko}\ \frac{\sin^2\beta}{\cos^2\beta} = -\frac{P\,KI}{Ko}\ \frac{\sin^2\beta_0}{\cos^2\beta_0}\left[1+\left(\frac{1}{tg\,\beta_0}+2\,tg\,\beta_0\right)\theta+\left(\frac{5}{2}+3\,tg^2\beta_0\right)\theta^2\right]\left[1+\frac{\theta}{tg\,\beta_0}-\frac{\theta^2}{2}\right]$$

$$=-\frac{P\,KI}{Ko}\ \frac{\sin^2\beta_0}{\cos^2\beta_0}\left[1+\left(\frac{1}{tg\,\beta_0}+2\,tg\,\beta_0+\frac{1}{tg\,\beta_0}\right)\theta+\left(\frac{5}{2}+3\,tg\,\beta_0-\frac{I}{2}+\frac{1}{tg^2\beta_0}+2\right)\theta^2\right]$$

(77)
$$\boxed{=-P\,\frac{KI}{Ko}\ \frac{\sin^2\beta_0}{\cos^2\beta_0}\left[1+\left(2\,tg\,\beta_0+\frac{2}{tg\,\beta_0}\right)\theta+\left(4+3\,tg^2\beta_0+\frac{1}{tg^2\beta_0}\right)\theta^2\right]}$$

$$\left(\frac{KI}{Ko}\right)^2\frac{\sin^3\beta}{\cos^2\beta} = \left(\frac{KI}{Ko}\right)^2\frac{\sin^3\beta_0}{\cos^2\beta_0}\left[1+\left(2\,tg\,\beta_0+\frac{2}{tg\,\beta_0}\right)\theta+\left(4+3\,tg^2\beta_0+\frac{I}{tg^2\beta_0}\right)\theta^2\right]\left[1+\frac{\theta}{tg\,\beta_0}-\frac{\theta^2}{2}\right]$$

.../..

$$= \left(\frac{KI}{K2}\right)^2 \frac{\sin^3\beta_0}{\cos^2\beta_0}\left[1 + \left(2tg\beta_0 + \frac{2}{tg\beta_0} + \frac{1}{tg\beta_0}\right)\theta + \left(4+3tg^2\beta_0 + \frac{1}{tg^2\beta_0} - \frac{1}{2} + 2 + \frac{2}{tg^2\beta_0}\right)\theta^2\right]$$

(78) $$\boxed{= \left(\frac{KI}{Ko}\right)^2 \frac{\sin^3\beta_0}{\cos^2\beta_0}\; 1 + \left(2tg\beta_0 + \frac{3}{tg\beta_0}\right)\theta + \left(\frac{11}{2} + 3tg^2\beta_0 + \frac{3}{tg^2\beta_0}\right)\theta^2}$$

$$- \frac{P\sin\beta}{R\cos\beta} = -\frac{P}{R}\frac{\sin\beta}{\cos\beta_0}\left(1 + \frac{\theta}{tg\beta_0} - \frac{\theta^2}{2}\right)\left[1 + \theta tg\beta_0 + \left(\frac{1}{2} + tg^2\beta_0\right)\theta^2\right]$$

$$= -\frac{P}{R}\frac{\sin\beta_0}{\cos\beta_0}\left[1 + \left(\frac{1}{tg\beta_0} + tg\beta_0\right)\theta + \left(\frac{1}{2} + tg^2\beta_0 - \frac{I}{2} + 1\right)\theta^2\right]$$

(79) $$\boxed{= -\frac{P}{R}\frac{\sin\beta_0}{\cos\beta_0}\left[1 + \left(\frac{1}{tg\beta_0} + tg\beta_0\right)\theta + \left(1 + tg^2\beta_0\right)\theta^2\right]}$$

$- P\frac{KI}{Ko}\frac{\sin^2\beta}{\cos^2\beta}$ term identical with the above calculated one.

$$+ P^2\frac{\sin\beta}{\cos^2\beta} = P^2\frac{\sin\beta_0}{\cos^2\beta_0}\left[1 + 2\theta tg\beta_0 + \left(1 + 3tg^2\beta_0\right)\theta^2\right]\left(1 + \frac{\theta}{tg\beta_0} - \frac{\theta^2}{2}\right)$$

$$= P^2\frac{\sin\beta_0}{\cos^2\beta_0}\left[1 + \left(2tg\beta_0 + \frac{1}{tg\beta_0}\right)\theta + \left(1 + 3tg^2\beta_0 + 2 - \frac{I}{2}\right)\theta^2\right]$$

.../..

$$(80)\quad \boxed{= P^2 \frac{\sin\beta_0}{\cos^2\beta_0} \quad 1 + \left(2\,tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right)\theta + \left(\frac{5}{2} + 3\,tg^2\beta_0\right)\theta^2}$$

$$-\frac{K2}{Ko}\sin\beta = \boxed{-\frac{K2}{Ko}\sin\beta_0\left(1 + \frac{\theta}{tg\,\beta_0} - \frac{\theta^2}{2}\right)} \quad (81)$$

$$+ Q$$

Classifying, we obtain as follows :

$$C = \frac{KI}{Ko}\frac{\sin^2\beta_0}{R\cos\beta_0}\left[1 + \left(tg\,\beta_0 + \frac{2}{tg\,\beta_0}\right)\theta + \left(\frac{3}{2} + tg^2\beta_0 + \frac{1}{tg^2\beta_0}\right)\theta^2\right]$$

$$+ \left(\frac{KI}{Ko}\right)^2 \frac{\sin^3\beta_0}{\cos^2\beta_0}\left[1 + \left(2\,tg\,\beta_0 + \frac{3}{tg\,\beta_0}\right)\theta + \left(\frac{11}{2} + 3\,tg^2\beta_0 + \frac{3}{tg^2\beta_0}\right)\theta^2\right]$$

$$- 2P\frac{KI}{Ko}\frac{\sin^2\beta_0}{\cos^2\beta_0}\left[1 + \left(2\,tg\,\beta_0 + \frac{2}{tg\,\beta_0}\right)\theta + \left(4 + 3\,tg^2\beta_0 + \frac{1}{tg^2\beta_0}\right)\theta^2\right]$$

$$(82)\quad - \frac{P}{R}\frac{\sin\beta_0}{\cos\beta_0}\left[1 + \left(tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right)\theta + \left(1 + tg^2\beta_0\right)\theta^2\right]$$

$$+ \frac{P^2}{\cos^2\beta_0}\sin\beta_0\left[1 + \left(2\,tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right)\theta + \left(\frac{5}{2} + 3\,tg^2\beta_0\right)\theta^2\right]$$

$$- \frac{K2}{Ko}\sin\beta_0\left[1 + \frac{\theta}{tg\,\beta_0} - \frac{\theta^2}{2}\right]$$

$$+ Q.$$

.../..

So, this is the expansion of

$$\Delta C = \frac{KI}{Ko}\,\frac{\sin^2\beta}{\ell_B} - \frac{P\sin\beta}{\ell_B} - \frac{K2}{Ko}\sin\beta + Q$$

with $P = \frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} - \sin\alpha\frac{KI}{Ko} = -\left[\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R} - \sin\beta_o\frac{KI}{Ko}\right]$

$$Q = \frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) - \sin\alpha\frac{K2}{Ko} = -\left[\frac{\sin\beta_o}{\ell_{Bo}}\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right) - \sin\beta_o\frac{K2}{Ko}\right]$$

As it is supposed the equations T = o and C = o are satisfied for the pair (A Bo), obviously, if we substitute P and Q for their βo value into the equation $\Delta C = \ldots$, the independant terms of θ have to disappear.

We can check it :

$$\Delta C = \frac{KI}{R\,Ko}\,\frac{\sin^2\beta_o}{\cos\beta_o} + \left(\frac{KI}{Ko}\right)^2\frac{\sin^3\beta_o}{\sin^2\beta_o} - 2\,P\,\frac{KI}{Ko}\,\frac{\sin^2\beta_o}{\cos^2\beta_o} -$$

$$\frac{P\sin\beta_o}{R\cos\beta_o} + P^2\,\frac{\sin\beta_o}{\cos^2\beta_o} - \frac{K2}{Ko}\sin\beta_o + Q$$

$$= \frac{KI}{KoR}\,\frac{\sin^2\beta_o}{\cos\beta_o} + \left(\frac{KI}{Ko}\right)^2\frac{\sin^3\beta_o}{\cos^2\beta_o} - 2\,\frac{KI}{Ko}\,\frac{\sin^2\beta_o}{\cos^2\beta_o}\left(\frac{\cos^2\beta_o}{\ell_{Bo}} -\right.$$

$$\left.\frac{\cos\beta_o}{R} - \frac{KI}{Ko}\sin\beta_o\right) + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R} - \frac{KI}{Ko}\sin\beta_o\right)\frac{\sin\beta_o}{R\cos\beta_o}$$

$$+\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R} - \frac{KI}{Ko}\sin\beta_o\right)^2\frac{\sin\beta_o}{\cos^2\beta_o} - \frac{K2}{Ko}\sin\beta_o$$

.../..

$$- \frac{\sin \beta_o}{\ell_{Bo}} \left(\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} \right) + \frac{K2}{Ko} \sin \beta_o =$$

$$\left(\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} \right) \left[\left(\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} \right) \frac{\sin \beta_o}{\cos^2 \beta_o} + \frac{\sin \beta_o}{R \cos \beta_o} - \frac{\sin \beta_o}{\ell_{Bo}} \right] = 0$$

Let us determine now the terms in θ and θ^2 by substituting into (6) P and Q for their value.

a) Term in $\frac{KI}{R\ Ko} \frac{\sin^2 \beta_o}{\cos \beta_o}$

$$\frac{KI}{R\ Ko} \frac{\sin^2 \beta_o}{\cos \beta_o} \left[1 + \left(tg\, \beta_o + \frac{2}{tg\, \beta_o} \right) \theta + \left(\frac{3}{2} + tg^2 \beta_o + \frac{1}{tg^2 \beta_o} \right) \theta^2 \right]$$

$$- \frac{KI}{R\ Ko} \frac{\sin^2 \beta_o}{\cos \beta_o} \left[1 + \left(tg\, \beta_o + \frac{1}{tg\, \beta_o} \right) \theta + \left(1 + tg^2 \beta_o \right) \theta^2 \right]$$

(83) $$\boxed{= \frac{KI}{R\ Ko} \frac{\sin^2 \beta_o}{\cos \beta_o} \frac{\theta}{tg\, \beta_o} + \left(\frac{1}{2} + \frac{1}{tg^2 \beta_o} \right) \theta^2}$$

b) Term in $\left(\frac{KI}{K2} \right)^2 \frac{\sin^3 \beta_o}{\cos^2 \beta_o}$

$$\left(\frac{KI}{Ko} \right)^2 \frac{\sin^3 \beta_o}{\cos^2 \beta_o} \left[1 + \left(2\, tg\, \beta_o + \frac{3}{tg\, \beta_o} \right) \theta + \left(\frac{11}{2} + 3\, tg^2 \beta_o + \frac{3}{tg^2 \beta_o} \right) \theta^2 \right]$$

$$- 2 \left(\frac{KI}{Ko} \right)^2 \frac{\sin^3 \beta_o}{\cos^2 \beta_o} \left[1 + \left(2\, tg\, \beta_o + \frac{2}{tg\, \beta_o} \right) \theta + \left(4 + 3\, tg^2 \beta_o + \frac{1}{tg^2 \beta_o} \right) \theta^2 \right]$$

.../..

$$+ \left(\frac{KI}{K2}\right)^2 \frac{\sin^3 \beta_o}{\cos^2 \beta_o} \left[1 + \left(2 tg\beta_o + \frac{1}{tg\beta_o}\right) + \left(\frac{5}{2} + 3 tg^2\beta_o\right)\theta^2 \right]$$

(84)
$$\boxed{= \left(\frac{KI}{Ko}\right)^2 \frac{\sin^3 \beta_o}{\cos^2 \beta_o} \frac{1}{tg^2\beta_o} \theta^2}$$

c) Term in $2 \frac{KI}{Ko} \frac{\sin^2 \beta_o}{\cos^2 \beta_o} \left(\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} \right)$

$$= 2 \frac{KI}{Ko} \frac{\sin^2 \beta_o}{\cos^2 \beta_o} \left(\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} \right) \left[1 + \left(2 tg\beta_o + \frac{2}{tg\beta_o}\right)\theta + \left(4 + 3 tg^2\beta_o + \frac{I}{tg^2\beta_o}\right)\theta^2 \right]$$

$$- 2 \frac{KI}{Ko} \frac{\sin^2 \beta_o}{\cos^2 \beta_o} \left(\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} \right) \left[1 + \left(2 tg\beta_o + \frac{I}{tg\beta_o}\right)\theta + \left(\frac{5}{2} + 3 tg^2\beta_o\right)\theta^2 \right]$$

(85)
$$\boxed{= 2 \frac{KI}{Ko} \frac{\sin^2 \beta_o}{\cos^2 \beta_o} \left(\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} \right) \left[\frac{\theta}{tg\beta_o} + \left(\frac{3}{2} + \frac{1}{tg^2\beta_o}\right)\theta^2 \right]}$$

d) Term in $\frac{K2}{Ko} \sin \beta_o$

$$- \frac{K2}{Ko} \sin \beta_o \left[1 + \frac{\theta}{tg\beta_o} - \frac{\theta^2}{2} \right] + \frac{K2}{Ko} \sin \beta_o$$

(86)
$$\boxed{= - \frac{K2}{Ko} \sin \beta_o \left[\frac{\theta}{tg\beta_o} - \frac{\theta^2}{2} \right]}$$

.../..

e) Left terms.

$$\frac{(\cos^2\beta_0 - \cos\beta_0)}{\ell_{Bo}} \frac{}{R} \left[\frac{(\cos^2\beta_0 - \cos\beta_0)}{\ell_{Bo}} \frac{}{R} \frac{\sin\beta_0}{\cos^2\beta_0} \left[1 + \left(2\,tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right)\theta + \left(\frac{5}{2} + 3\,tg^2\beta_0\right)\theta^2 \right] + \frac{\sin\beta_0}{R\cos\beta_0} \left[1 + \left(tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right)\theta + \left(1 + tg\,\beta_0\right)\theta^2 \right] - \frac{\sin\beta_0}{\ell_{Bo}} \right]$$

$$= \frac{(\cos^2\beta_0 - \cos\beta_0)}{\ell_{Bo}} \frac{}{R} \left[\frac{\sin\beta_0}{\ell_{Bo}} \left[\left(2\,tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right)\theta + \left(\frac{5}{2} + 3\,tg^2\beta_0\right)\theta^2 \right] + \frac{\sin\beta_0}{R\cos\beta_0} \left[-tg\,\beta_0\,\theta - \left(\frac{3}{2} + 2\,tg^2\beta_0\right)\theta^2 \right] \right] \tag{87}$$

Classifying we obtain as follows :

$$\Delta C = \frac{KI}{R\,Ko} \frac{\sin^2\beta_0}{\cos\beta_0} \left[\frac{\theta}{tg\,\beta_0} + \left(\frac{1}{2} + \frac{1}{tg^2\beta_0}\right)\theta^2 \right]$$

$$+ \left(\frac{KI}{Ko}\right)^2 \frac{\sin^3\beta_0}{\cos^2\beta_0} \frac{1}{tg^2\beta_0} \theta^2$$

$$+ 2\,\frac{KI}{Ko} \frac{\sin^2\beta_0}{\cos^2\beta_0} \left(\frac{\cos^2\beta_0}{\ell_{Bo}} - \frac{\cos\beta_0}{R} \right) \left[\frac{\theta}{tg\,\beta_0} + \left(\frac{3}{2} + \frac{1}{tg^2\beta_0}\right)\theta^2 \right]$$

$$- \frac{K2}{Ko} \sin\beta_0 \left[\frac{\theta}{tg\,\beta_0} - \frac{\theta^2}{2} \right]$$

.../..

$$+ \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left[\frac{\sin\beta_o}{\ell_{Bo}}\left[\left(2\,tg\,\beta_o + \frac{1}{tg\,\beta_o}\right)\theta + \left(\frac{5}{2} + 3\,tg^2\beta_o\right)\theta^2\right] - \frac{tg\,\beta_o}{R}\left[tg\,\beta_o\,\theta + \left(\frac{3}{2} + 2\,tg^2\beta_o\right)\theta^2\right]\right]$$

Term of first order in θ.

$$\frac{KI}{R\,Ko}\sin\beta_o - \frac{K2}{Ko}\cos\beta_o + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left[2\frac{KI}{Ko}\frac{\sin^2\beta_o}{\cos^2\beta_o}\frac{1}{tg\,\beta_o} + \frac{\sin\beta_o}{\ell_{Bo}}\left(2\,tg\,\beta_o + \frac{1}{tg\,\beta_o}\right) - \frac{tg^2\beta_o}{R}\right]$$

(88)
$$\boxed{\frac{KI}{R\,Ko}\sin\beta_o - \frac{K2}{Ko}\cos\beta_o + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left[2\frac{KI}{Ko}\,tg\,\beta_o + \frac{\sin\beta_o}{\ell_{Bo}}\left(2\,tg\,\beta_o + \frac{1}{tg\,\beta_o}\right) - \frac{tg^2\beta_o}{R}\right]}\ \theta$$

Term of second order in θ.

(89)
$$\boxed{\begin{aligned}&\frac{KI}{R\,Ko}\frac{\sin^2\beta_o}{\cos\beta_o}\left(\frac{1}{2} + \frac{1}{tg^2\beta_o}\right) + \left(\frac{KI}{Ko}\right)^2\sin\beta_o \\ &+ 2\frac{KI}{Ko}tg^2\beta_o\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left(\frac{3}{2} + \frac{1}{tg^2\beta_o}\right) + \frac{K2}{Ko}\frac{\sin\beta_o}{2} \\ &+ \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\frac{\sin\beta_o}{\ell_{Bo}}\left(\frac{5}{2} + 3\,tg^2\beta_o\right) - \frac{tg\,\beta_o}{R}\left(\frac{3}{2} + 2\,tg^2\beta_o\right)\end{aligned}}\ \theta^2$$

We must add to those equations $T = o$

$C = o$

VI-2- STUDY OF THE PARTICULAR SOLUTION : Bo ON THE ROWLAND CIRCLE.

Let us consider the equations $T = o$ and $C = o$ written as follows (74)

$$\begin{cases} \dfrac{\cos^2 \beta_o}{\ell_{Bo}} - \dfrac{\cos \beta_o}{R} - \dfrac{KI}{Ko} \sin \beta_o + P = o \\[2ex] \dfrac{\sin \beta_o}{\ell_{Bo}} \left(\dfrac{KI}{Ko} \sin \beta_o - P \right) - \dfrac{K2}{Ko} \sin \beta_o + Q = o \end{cases}$$

An evident solution of $C = o$ is $\begin{cases} P = \dfrac{KI}{Ko} \sin \beta_o \\[2ex] Q = \dfrac{K2}{Ko} \sin \beta_o \end{cases}$

If we carry forward P in $T = o$ it results that

$$\frac{\cos^2 \beta_o}{\ell_{Bo}} - \frac{\cos \beta_o}{R} = 0$$

$\longrightarrow$ Bo is over the Rowland circle; yet it is not proved that A is over the Rowland circle, namely that the spectrum is not necessarily spreading over the Rowland circle.

.../..

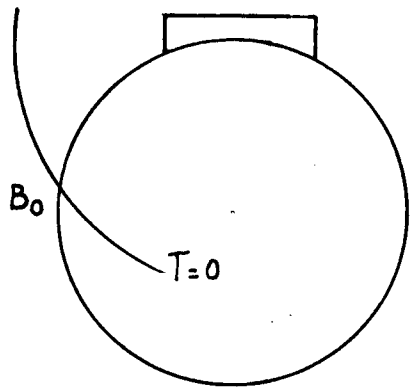


Fig. n° 15

So, we have :

$$(90)\quad \begin{cases} \dfrac{KI}{Ko}\sin\beta_0 = P = \dfrac{\cos^2\alpha}{\ell_A} - \dfrac{\cos\alpha}{R} - \sin\alpha\,\dfrac{KI}{Ko} \\[2ex] \dfrac{K2}{Ko}\sin\beta_0 = Q = \dfrac{\sin\alpha}{\ell_A}\left(\dfrac{\cos^2\alpha}{\ell_A} - \dfrac{\cos\alpha}{R}\right) - \sin\alpha\,\dfrac{K2}{Ko} \\[2ex] \ell_{B_0} = R\cos\beta_0 \end{cases}$$

Let us use the equations (88) and (89) for that specific case

$$\begin{cases} \dfrac{KI}{R\,Ko}\sin\beta_0 - \dfrac{K2}{Ko}\cos\beta_0 = 0 & (91) \\[2ex] \dfrac{KI}{R\,Ko}\,\dfrac{\sin^2\beta_0}{\cos\beta_0}\left(\dfrac{1}{2} + \dfrac{1}{tg^2\beta_0}\right) + \left(\dfrac{KI}{Ko}\right)^2\sin\beta_0 + \dfrac{K2}{Ko}\,\dfrac{\sin\beta_0}{2} = 0 & (92) \end{cases}$$

.../..

We can write (92) taking account of (91) :

$$\frac{KI}{R\,Ko}\,\frac{\sin^2\beta_0}{\cos\beta_0}\left(\frac{1}{2}+\frac{1}{tg^2\beta_0}\right)+\left(\frac{KI}{Ko}\right)^2\sin\beta_0+\frac{KI}{R\,Ko}\,\frac{I}{2}\,\frac{\sin^2\beta_0}{\cos\beta_0}=0$$

We may divide by $\frac{KI}{Ko} \longrightarrow \frac{KI}{Ko} \neq 0$

$$\frac{1}{R}\,\frac{\sin^2\beta_0}{\cos\beta_0}\left(1+\frac{1}{tg^2\beta_0}\right)+\frac{KI}{Ko}\sin\beta_0=0$$

$$\sin\beta_0 \neq 0$$

$$\longrightarrow \frac{KI}{Ko}=-\frac{1}{R}\left(tg\,\beta_0+\frac{1}{tg\,\beta_0}\right)$$

$$\frac{K2}{Ko}=\frac{I}{R}\,\frac{KI}{Ko}\,tg\beta_0=-\frac{I}{R^2}\left(1+tg^2\beta_0\right)$$

Finally

$$\left\{\begin{array}{l} KI=-\dfrac{1}{R}\left(tg\,\beta_0+\dfrac{1}{tg\,\beta_0}\right)=-\dfrac{I}{R}\,\dfrac{1}{\sin\beta_0\cos\beta_0} \\[2ex] \dfrac{K2}{Ko}=-\dfrac{1}{R^2}\,\dfrac{1}{\cos^2\beta_0} \end{array}\right. \qquad (93)$$

Let us introduce into the tangential :

$$\frac{\cos^2\alpha}{\ell_A}-\frac{\cos\alpha}{R}+\frac{\cos^2\beta}{\ell_B}-\frac{\cos\beta}{R}-(\sin\alpha+\sin\beta)\,\frac{KI}{Ko}=0$$

$$\longrightarrow \left\{\begin{array}{l} \dfrac{\cos^2\alpha}{\ell_A}-\dfrac{\cos\alpha}{R}+(\sin\alpha+\sin\beta_0)\,\dfrac{1}{R\sin\beta_0\cos\beta_0}=0 \\ \text{to which we must add :} \\ \sin\alpha+\sin\beta=\dfrac{\lambda}{\lambda_0}\,(\sin\gamma-\sin\delta) \end{array}\right. \qquad (94)$$

$$Ko=\sin\gamma-\sin\delta$$

.../..

Let us introduce into the coma :

$$\frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta_o}{\ell_{Bo}}\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right) - (\sin\alpha + \sin\beta)\ \frac{K2}{Ko} = o$$

$$(95) \qquad \frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{(\sin\alpha + \sin\beta_o)}{R^2\cos^2\beta_o} = o$$

VII-3- DETERMINATION OF USE CONDITIONS.

From the above mentioned equations, we have to find out whether it may be effectively possible to determine the ℓ_A ℓ_B α and β_o values.

From the equations (94) and (95) we have :

$$\frac{\sin\alpha}{\ell_A} - \frac{(\sin\alpha + \sin\beta_o)}{R\sin\beta_o\cos\beta_o} + \frac{\sin\alpha + \sin\beta_o}{R^2\cos^2\beta_o} = o$$

We divide by $\dfrac{\sin\alpha + \sin\beta_o}{\cos\beta_o}$

$$\longrightarrow \boxed{\ell_A = \frac{R\sin\alpha}{tg\,\beta_o}} \qquad (96)$$

<u>Conclusion</u> : <u>α and β_o have to be of same sign.</u>

More, we know that $\ell_{Bo} = R\cos\beta_o$

so $$\left\{\frac{\ell_A}{\sin\alpha} = \frac{\ell_{Bo}}{\sin\beta_o}\right.$$

By carrying forward (96) in (94) or in (95), we have :

$$\frac{\cos^2\alpha}{\frac{R\sin\alpha}{tg\,\beta_o}} - \frac{\cos\alpha}{R} + \frac{\sin\alpha + \sin\beta_o}{R\sin\beta_o\cos\beta_o} = o$$

.../..

$$\frac{\cos^2\alpha}{\sin\alpha}\, tg\,\beta_0 - \cos\alpha + \frac{\sin\alpha + \sin\beta_0}{\sin\beta_0 \cos\beta_0} = 0$$

$$(97) \qquad \boxed{\cos\alpha\left(\frac{tg\,\beta_0}{tg\,\alpha} - 1\right) + \frac{\sin\alpha + \sin\beta_0}{\sin\beta_0 \cos\beta_0} = 0}$$

This is the relation that must link α and β so as to be able to use the grating, providing that it might be possible to do it.

We may try to transform the relation (97) by using the following notations

$$\begin{cases} \omega = \dfrac{\alpha + \beta}{2} \\ \varphi = \dfrac{\alpha - \beta}{2} \end{cases}$$

(97) may be written :

$$\cos\alpha\left(\frac{\sin\beta_0 \cos\alpha}{\cos\beta_0 \sin\alpha} - I\right) + \frac{\sin\alpha + \sin\beta_0}{\sin\beta_0 \cos\beta_0} = 0$$

$$\cos\alpha\,\frac{(\sin\beta_0 \cos\alpha - \cos\beta_0 \sin\alpha)}{\sin\alpha} + \frac{\sin\alpha + \sin\beta_0}{\sin\beta_0} = 0$$

$$\cos\alpha \sin(\beta - \alpha) + 2 \sin\omega \cos\varphi = 0$$

$$-2 \cos\alpha \sin\frac{\alpha - \beta}{2} \cos\frac{\alpha - \beta}{2} + 2 \sin\omega \cos\varphi = 0$$

$$\boxed{\cos\alpha = \frac{\sin\omega}{\sin\varphi}} \qquad (98)$$

VII-4- DETERMINATION OF MANUFACTURING CONDITIONS.

The question is to determine γl_C δl_D in such a way that :

.../..

$$\begin{cases} KI = \dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R} - \left(\dfrac{\cos^2\delta}{\ell_D} - \dfrac{\cos\delta}{R}\right) \\[2ex] K2 = \dfrac{\sin\gamma}{\ell_C}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) - \dfrac{\sin\delta}{\ell_D}\left(\dfrac{\cos^2\delta}{\ell_D} - \dfrac{\cos\delta}{R}\right) \\[2ex] Ko = \sin\gamma - \sin\delta \end{cases}$$

Simplifying hypothesis :

We suppose $\ell_D = R\cos\delta$

i.e. that D point on the Rowland circle.

$$\underline{KI} = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} = \frac{-Ko}{R\sin\beta_0\cos\beta_0}$$

$$K2 = \frac{\sin\gamma}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) = \frac{-Ko}{R^2\cos\beta_0}$$

so $\dfrac{K2}{KI} = \dfrac{\sin\gamma}{\ell_C}$

$$\frac{1}{\ell_C} = \frac{K2}{KI\sin\gamma} = \frac{tg\,\beta}{R\sin\gamma}$$

$\ell_C > 0$ <u>Conclusion</u> : <u>β and γ have to be of same sign.</u>

By carrying forward in KI :

$$\frac{\cos^2\gamma + tg\,\beta}{R\sin\gamma} - \frac{\cos\gamma}{R} = -\frac{\sin\gamma - \sin\delta}{R\sin\beta\cos\beta}$$

.../..

$$\cos^2\gamma\, tg\beta - \cos\gamma \sin\gamma = -\frac{\sin^2\gamma}{\sin\beta\cos\beta} + \frac{\sin\delta}{\sin\beta\cos\beta}\sin\gamma$$

$$tg\beta - tg\gamma = -\frac{tg^2\gamma}{\sin\beta\cos\beta} + \frac{\sin\delta}{\sin\beta\cos\beta}\,\frac{\sin\gamma}{\cos^2\gamma}$$

that may be also written :

$$\frac{\cos^2\gamma\, tg\beta}{R\sin\gamma} - \frac{\cos\gamma}{R} = \frac{-Ko}{R\sin\beta\cos\beta}$$

or $$\boxed{tg\beta - tg\gamma = -\frac{KI\ \sin\gamma}{\sin\beta\cos\beta\cos^2\gamma}} \qquad (99)$$

We may transform :

$$\frac{\sin\beta}{\cos\beta} - \frac{\sin\gamma}{\cos\gamma} = -\frac{tg^2\gamma}{\sin\beta\cos\beta} + \frac{\sin\delta}{\sin\beta\cos\beta}\,\frac{\sin\gamma}{\cos^2\gamma}$$

$$\sin(\beta-\gamma) = -\frac{\sin^2\gamma}{\cos\gamma\sin\beta} + \frac{\sin\delta\ \sin\gamma}{\sin\beta\cos\gamma}$$

$$\sin(\beta-\gamma) = \frac{\sin\gamma\,(-\sin\gamma + \sin\delta)}{\sin\beta\cos\gamma}$$

$$\sin(\beta-\gamma) = \frac{-Ko\ \sin\gamma}{\sin\beta\cos\gamma} \qquad (100)$$

<u>Conclusion</u> : As β and γ are of same sign and that $Ko > o$, γ must be $> \beta$.

.../..

Let us take a simple numerical example :

$\gamma = 60°$

$\beta = 30°$

$$-\frac{I}{2} = KO \frac{\sqrt{3}}{2} \frac{1}{\frac{1}{2} \times \frac{1}{2}}$$

$$Ko = \frac{1}{4\sqrt{3}} = 0,1443$$

$$\sin\delta = \sin\gamma - Ko = 0,7217$$

$$\delta = 46° \; 193'$$

$$\ell_C = \frac{R \sin\gamma}{tg\,\beta} = 1,5 \; R$$

$$\sin\alpha + \sin\beta = \frac{\lambda}{\lambda_0} \; Ko$$

Let us choose arbitrarily $\alpha = 45°$

$$\longrightarrow \frac{\sqrt{2}}{2} + \frac{1}{2} = \frac{\lambda}{4880} \times 0,1443$$

$$\lambda = \frac{1+\sqrt{2}}{2} \times \frac{4\,880}{0.1443} = 4,19 \; \mu$$

$$\ell_A = R \times \frac{0,707}{0,5773} = 1,2247 \; R$$

$$\ell_B = R \cos\beta = 0,886 \; R$$

$$\ell_D = R \cos\delta = 0,6922 \; R$$

One could check that the equations T = o, C = o $\frac{\partial c}{\partial \beta}$, ($\theta$ = o) and $\frac{\partial^2 c}{\partial \beta^2}$ (θ^2 = o) are effectively zero with those parameters.

.../..

The choice is not attractive as it leads to $\lambda_{Bo} = 4,19\ \mu$. However, it proves the feasibility of the system in spite of conditions arbitrarily given ($\ell_B = R\cos\beta - \ell_D = R\cos\delta$).

<u>Comment</u> : For having a large number of grooves, which seems interesting for the far U V region, Ko must be large; so, if possible, γ and δ must be of opposite sign.

Let us consider the equation (89) under the following form :

$$\frac{tg^2\gamma}{\sin\beta\cos\beta} - tg\gamma + tg\beta = \frac{\sin\delta\ \sin\gamma}{\sin\beta\cos\beta\cos^2\gamma}$$

Since γ and β are of same sign

Let us suppose γ and $\beta > o$ it needs $\delta < o$

$$\longrightarrow \frac{tg^2\gamma}{\sin\beta\cos\beta} - tg\gamma + tg\beta < o$$

The minimum of the function is achieved,

$$\text{for } tg\gamma = -\frac{b}{2a} = \frac{\sin\beta\cos\beta}{2}$$

$$\longrightarrow \frac{\sin^2\beta\ \cos^2\beta}{4\sin\beta\cos\beta} - \frac{\sin\beta\cos\beta}{2} + tg\beta = tg\beta - \frac{\sin\beta\cos\beta}{2}$$

$$\longrightarrow \sin\beta\left(\frac{1}{\cos\beta} - \frac{\cos\beta}{4}\right) < o$$

Since $\beta > o$ $\quad \frac{1}{\cos\beta} < \frac{\cos\beta}{4}$

$$\longrightarrow \cos^2\beta > 4 \text{ impossible.}$$

We can choose the sign for γ and β arbitrarily. If we choose it negative, the same reasoning leads to the same impossibility.

.../..

$\longrightarrow$ In this simplifying hypothesis

$$\begin{cases} \ell_B = R \cos \beta \\ \ell_D = R \cos \delta \end{cases}$$

it is impossible to obtain γ and δ with opposite sign.
The minimum value is reached for $tg\,\gamma = \dfrac{\sin\beta \cos\beta}{2}$

$$\longrightarrow \frac{\sin\delta \sin\gamma}{\sin\beta \cos\beta \cos^2\gamma} = \sin\beta \left(\frac{1}{\cos\beta} - \frac{\cos\beta}{4} \right)$$

$$\frac{\sin\delta}{\cos\gamma} = \frac{\sin\beta}{\cos\beta} \left(4 - \cos^2\beta \right)$$

$$\underline{\sin\delta = 2 \cos\gamma\, tg\beta \left(4 - \cos^2\beta \right)}$$

In this case $\sin(\beta - \gamma) = -\dfrac{Ko \sin\gamma}{\sin\beta \cos\gamma}$ may be written

$$\sin(\beta - \gamma) = - Ko \frac{\cos\beta}{2}$$

$$Ko = -2 \frac{\sin(\beta - \gamma)}{\cos\beta}$$

Let us resume the study in the general case.

<u>GENERAL CASE:</u>

Let us consider the equation (88) and (89)

$$\frac{K2}{Ko} = \frac{KI}{R\,Ko}\, tg\beta_o + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R} \right) \left[2 \frac{KI}{Ko} \frac{\sin\beta_o}{\cos^2\beta_o} + \frac{tg\beta_o}{\ell_{Bo}} \left(\frac{2\,tg^2\beta_o + 1}{tg\beta_o} \right) - \frac{tg^2\beta_o}{R \cos\beta_o} \right]$$

.../..

Under those conditions, the equation (57) may be written :

$$\left(\frac{KI}{Ko}\right)^2 \sin\beta_o + \frac{KI}{R\,Ko}\left(\frac{\sin^2\beta_o}{2\cos\beta_o} + \cos\beta_o + tg\beta_o\,\frac{\sin\beta_o}{2}\right)$$

$$+ 2\,\frac{KI}{Ko}\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left(\frac{3\,tg^2\beta_o}{2} + 1 + \frac{\sin\beta_o}{\cos^2\beta_o}\,\frac{\sin\beta_o}{2}\right)$$

$$+\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left(\frac{\sin\beta_o}{\ell_{Bo}}\left(\frac{5}{2} + 3\,tg^2\beta_o\right) - \frac{tg\beta_o}{R}(3+2\,tg^2\beta_o) + \frac{\sin\beta_o}{2}(2tg^2\beta_o+1) - \frac{\sin\beta_o}{2R\cos\beta_o}\,tg^2\beta_o\right)$$

We can write :

$$\boxed{\begin{aligned}&\left(\frac{KI}{Ko}\right)^2 \sin\beta_o + \frac{KI}{Ko}\,\frac{1}{R\cos\beta_o} + 2\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)(2\,tg^2\beta_o + 1)\\ &+\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\frac{\sin\beta_o}{\ell_{Bo}}(3 + 4\,tg^2\beta_o) - \frac{tg\beta_o}{R}\left(\frac{3}{2} + \frac{5}{2}\,tg^2\beta_o\right)\end{aligned}} \qquad (101)$$

So, we have an equation of second degree in $\left(\frac{KI}{Ko}\right)$.

We are going to study the discriminant of that equation so as to determine what are the being conditions of roots.

$$\Delta = b'^2 - ac$$

First, let us calculate b' by developing

$$b' = \frac{1}{2\,R\cos\beta_o} + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)(2\,tg^2\beta_o + 1) =$$

.../..

$$= \frac{1 + \sin^2\beta_0}{\ell_{Bo}} + \frac{1}{R}\left(\frac{1}{2\cos\beta_0} - \frac{2\sin^2\beta_0\cos\beta_0}{\cos^2\beta_0} - \cos\beta_0\right)$$

$$= \frac{1 + \sin^2\beta_0}{\ell_{Bo}} + \frac{1}{R}\,\frac{(1 - 4\sin^2\beta_0 - 2\cos^2\beta_0)}{2\cos\beta_0}$$

$$= \frac{1 + \sin^2\beta_0}{\ell_{Bo}} + \frac{1}{R}\left(\frac{-1 + 2\sin^2\beta_0}{2\cos\beta_0}\right)$$

$$b' = \frac{1 + \sin^2\beta_0}{\ell_{Bo}} - \frac{1}{R}\,\frac{1 + 2\sin^2\beta_0}{2\cos\beta_0} =$$

$$b'^2 = \frac{(1 + \sin^2\beta_0)^2}{\ell_{B\,o}^2} + \frac{1}{4R^2}\,\frac{1 + 4\sin^4\beta_0 + 4\sin^2\beta_0}{\cos^2\beta_0}$$

$$- \frac{1 + 3\sin^2\beta_0 + 2\sin^4\beta_0}{R\,\ell_{Bo}\cos\beta_0}$$

$-ac =$

$$\left(\frac{\cos^2\beta_0}{\ell_{Bo}} - \frac{\cos\beta_0}{R}\right)\left[-\frac{\sin^2\beta_0}{\ell_{Bo}}(3 + 4\,tg^2\beta_0) + \frac{\sin^2\beta_0}{R\cos\beta_0}\left(\frac{3}{2} + \frac{5}{2}\,tg^2\beta_0\right)\right] =$$

$$\frac{1}{\ell_{B^2o}}\left(-3\sin^2\beta_0\cos^2\beta_0 - 4\sin^4\beta_0\right) - \frac{\sin^2\beta_0}{R^2}\left(\frac{3}{2} + \frac{5}{2}\,\frac{\sin^2\beta_0}{\cos^2\beta_0}\right)$$

.../..

$$+ \frac{\sin^2\beta_0 \cos\beta_0}{\ell_{Bo}\ R} \left[\left(\frac{3}{2} + \frac{5}{2} tg^2\beta_0 \right) + \left(3 + 4\, tg^2\beta_0 \right) \right]$$

$b'^2 - ac =$ a) Term in $\frac{1}{\ell_{B^2o}}$

$$\frac{1}{\ell_{B^2o}} \left[1 + 2\sin^2\beta_0 + \sin^4\beta_0 - 3\sin^2\beta_0 \cos^2\beta_0 - 4\sin^4\beta_0 \right] =$$

$$\left[1 + 2\sin^2\beta_0 - 3\sin^2\beta_0 \left(\cos^2\beta_0 + \sin^2\beta\right) \right] =$$

$$\left(1 - \sin^2\beta_0 \right) = \cos^2\beta_0$$

$$\longrightarrow \text{Term in } \frac{1}{\ell_{B^2o}} = \frac{\cos^2\beta_0}{\ell_{B^2o}}$$

b) Term in $\frac{1}{R^2}$

$$\frac{1}{4R^2\cos^2\beta_0} \left[1 + 4\sin^4\beta_0 + 4\sin^2\beta_0 - 6\sin^2\beta_0 \cos^2\beta_0 - 10\sin^4\beta_0 \right]$$

$$= \frac{1}{4R^2\cos^2\beta_0} \left[1 - 6\sin^4\beta_0 + 4\sin^2\beta_0 - 6\sin^2\beta_0 \cos^2\beta_0 \right] =$$

$$\frac{1}{4R^2\cos^2\beta_0} \left[1 - 6\sin^2\beta_0 \left(\sin^2\beta_0 + \cos^2\beta_0\right) + 4\sin^2\beta_0 \right] = \frac{1 - 2\sin^2\beta_0}{4R^2\cos^2\beta_0}$$

$$\longrightarrow \text{Term in } \frac{1}{R^2} : \frac{1 - 2\sin^2\beta_0}{4R^2\cos^2\beta_0}$$

.../..

c) Term in $\frac{1}{\ell_{Bo} R}$

$$\frac{1}{\ell_{Bo} R \cos\beta_0}\left[9 \sin^2\beta_0 \cos^2\beta_0 + 13 \sin^4\beta_0 - 2 - 6 \sin^2\beta_0 - 4 \sin^4\beta_0\right]$$

$$= \frac{1}{2 \ell_{Bo} R \cos\beta_0}\left[9 \sin^2\beta_0 \cos^2\beta_0 + 9 \sin^4\beta_0 - 2 - 6 \sin^2\beta_0\right]$$

$$= \frac{1}{2 \ell_{Bo} R \cos\beta_0}\left[9 \sin^2\beta_0 - 6 \sin^2\beta_0 - 2\right]$$

$\longrightarrow$ Term $\frac{1}{\ell_{Bo} R}$

$$\frac{1}{2 \ell_{Bo} R \cos\beta}\left[3 \sin^2\beta_0 - 2\right]$$

$\longrightarrow$ The discriminant

$$\boxed{\Delta = \frac{\cos^2\beta_0}{\ell_{Bo}^2} + \frac{1 - 2 \sin^2\beta_0}{4 R^2 \cos^2\beta_0} + \frac{3 \sin^2\beta_0 - 2}{2 \ell_{Bo} R \cos\beta_0}} \qquad (102)$$

Δ must be $>$ o.

So, let us investigate if the trinome in $\frac{1}{\ell_{Bo}}$ has real roots.

.../..

$$\frac{\cos^2\beta_o}{\ell_{Bo}^2} + \frac{3\sin^2\beta_o - 2}{2\,\ell_{Bo} R\cos\beta_o} + \frac{1 - 2\sin^2\beta_o}{4R^2\cos^2\beta_o}$$

The roots will be :

$$\frac{1}{\ell_{Bo}} = \frac{1}{2\cos^2\beta_o}\left[\frac{2 - 3\sin^2\beta_o}{2\cos\beta_o} \pm \sqrt{\frac{(3\sin^2\beta_o - 2)^2}{4\cos^2\beta_o} - \frac{4\cos^2\beta_o(1 - 2\sin^2\beta_o)}{4\cos^2\beta_o}}\right]$$

$$= \frac{1}{2\cos^2\beta_o}\left[\frac{2 - 3\sin^2\beta_o}{2\cos\beta_o} \pm \frac{1}{2\cos\beta_o}\sqrt{(9\sin^2\beta_o - 2)^2 - 4\cos^2\beta_o(1 - 2\sin^2\beta_o)}\right]$$

Let us explain the term under $\sqrt{}$

$$9\sin^4\beta_o - 12\sin^2\beta_o + 4 - 4\cos^2\beta_o + 8\sin^2\beta_o\cos^2\beta_o =$$

$$\sin^4\beta_o + 8\sin^2\beta_o(\sin^2\beta_o + \cos^2\beta_o) - 12\sin^2\beta_o - 4\cos^2\beta_o + 4 =$$

$$\sin^4\beta_o - 4\sin^2\beta_o - 4\cos^2\beta_o + 4 = \sin^4\beta_o$$

so we have the following roots :

$$\frac{1}{\ell_{Bo}} = \frac{1}{4\cos^3\beta_o}(2 - 3\sin^2\beta_o \pm \sin^2\beta_o)$$

$$= \frac{1}{\ell_{Bo}} = \frac{1}{4\cos^2\beta_o}(2 - 4\sin^2\beta_o) = \frac{1 - 2\sin^2\beta_o}{2\cos^3\beta_o}$$

$$\frac{1}{\ell_{Bo}} = \frac{1}{4\cos^3\beta_o}(2 - 2\sin^2\beta_o) = \frac{1}{2\cos\beta_o}$$

so

$$\boxed{\begin{aligned} \ell_{Bo} &= \frac{2R\cos^3\beta_o}{1 - 2\sin^2\beta_o} \\ \ell_{Bo} &= 2R\cos\beta_o \end{aligned}} \qquad (103)$$

.../..

Let us see under such condition

$$\frac{\cos^3 \beta_0}{1 - 2 \sin^2 \beta_0} > 0$$

as $\cos \beta_0$ is > 0 it needs $1 - 2 \sin^2 \beta_0 > 0$

$$\sin^2 \beta_0 < \frac{1}{2}$$

$$\longrightarrow \beta_0 < \frac{\pi}{4}$$

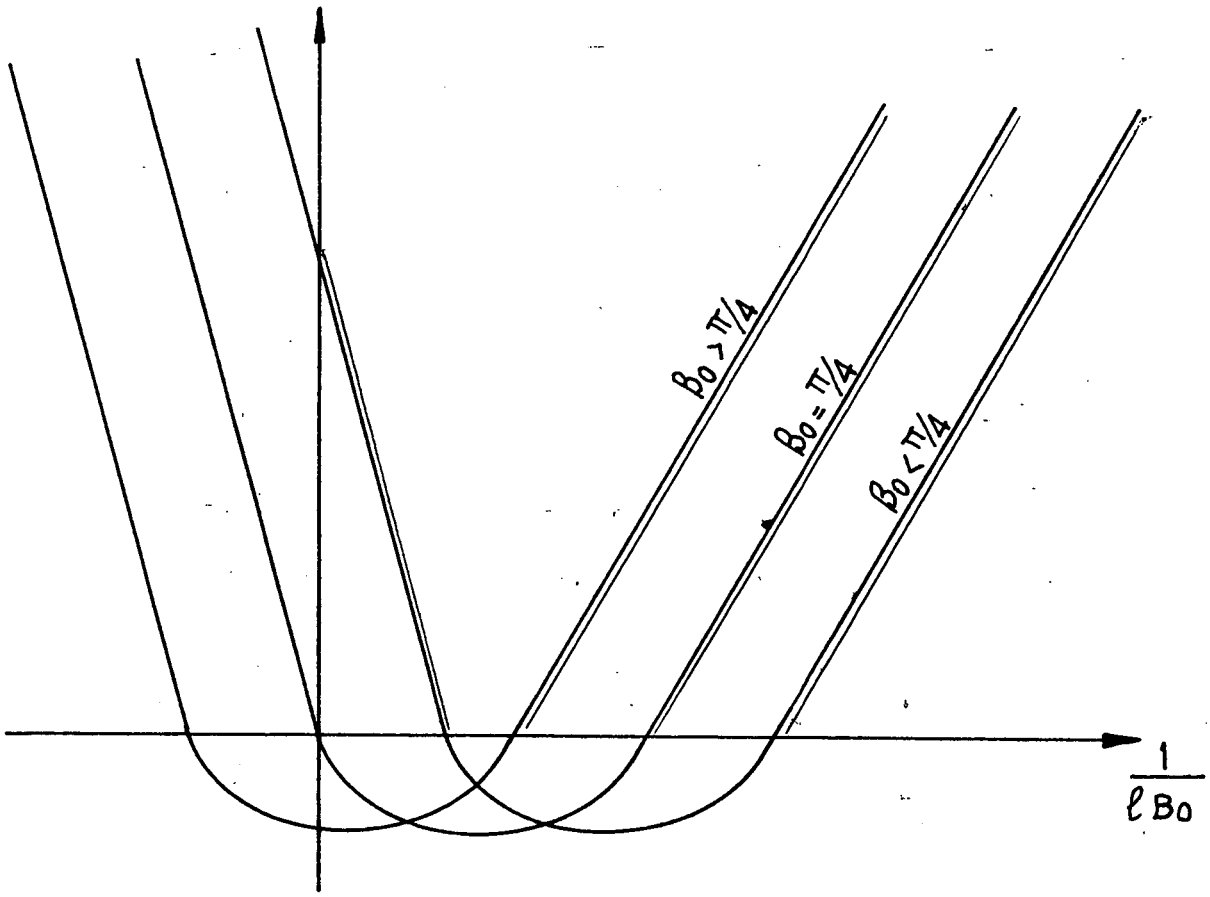


Fig. n° 16

.../..

The red zones are the only ones for which we get a real value of KI.

We may easily demonstrate that, in any case,

$$\frac{1 - 2\sin^2\beta_0}{2R\cos^3\beta} < \frac{1}{2R\cos\beta}$$

INTERPRETATION

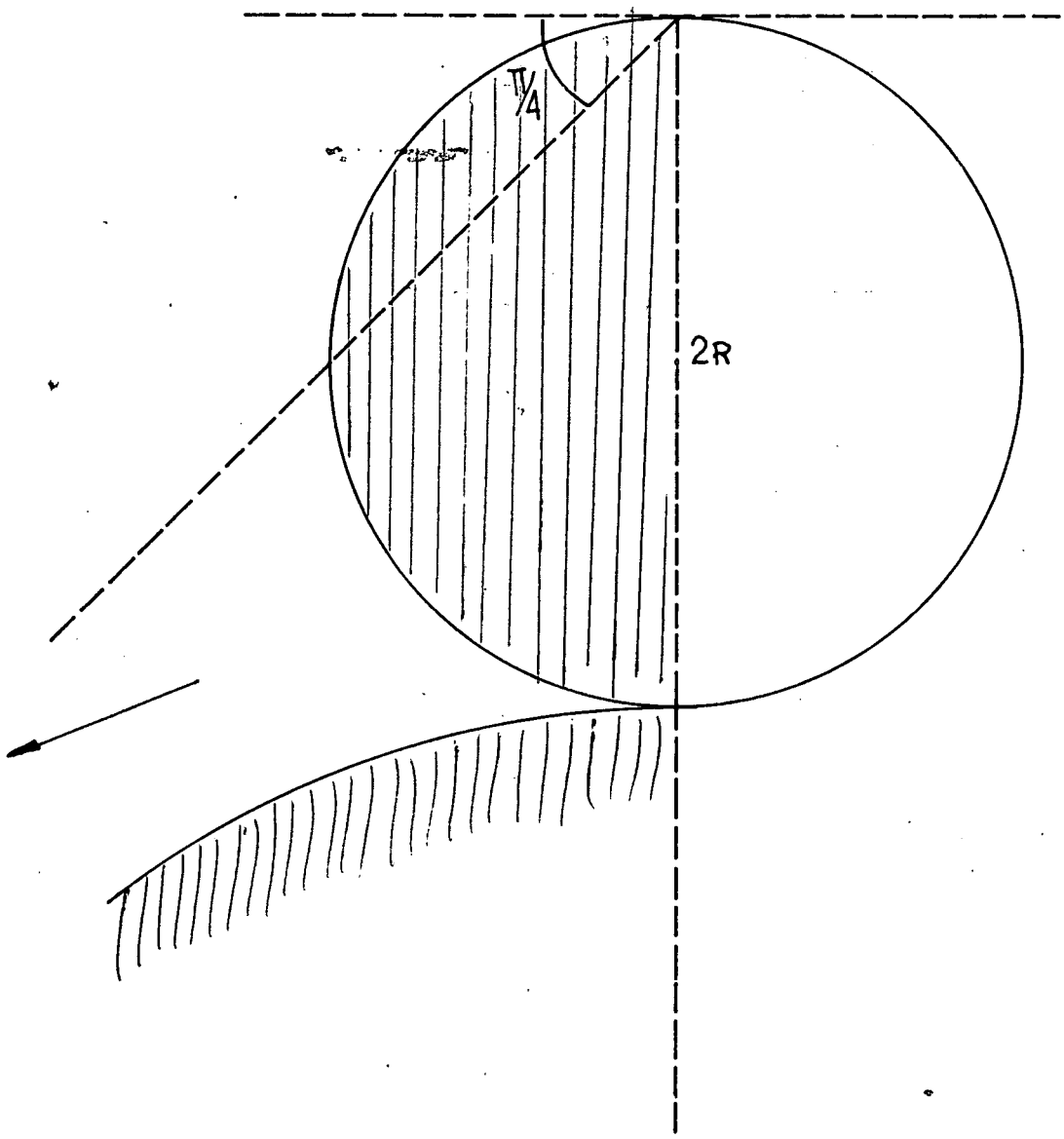


Fig. n° 17

= allowed zones.

VII-5- STUDY OF THE LIMITATIONS

We can use the following equations :

$$T = o \quad \frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\frac{KI}{Ko} = o$$

$$C = o \quad \frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - (\sin\alpha + \sin\beta)\frac{K2}{Ko} = o$$

$$\frac{\partial C}{\partial \beta} = o \longrightarrow \frac{KI}{R\,Ko}\sin\beta_o - \frac{K2}{Ko}\cos\beta_o + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left(2\frac{KI}{Ko}\,tg\,\beta_o + \frac{\sin\beta_o}{\ell_{Bo}}\left(2\,tg\,\beta_o + \frac{1}{tg\,\beta_o}\right) - \frac{tg^2\beta_o}{R}\right) = o$$

$$\frac{\partial^2 C}{\partial \beta^2} = o \longrightarrow \frac{KI}{R\,Ko}\frac{\sin^2\beta_o}{\cos\beta_o}\left(\frac{1}{2} + \frac{1}{tg^2\beta_o}\right) + \left(\frac{KI}{K2}\right)^2 \sin\beta_o$$

$$+ 2\frac{KI}{Ko}\,tg^2\beta_o\left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left(\frac{3}{2} + \frac{1}{tg^2\beta_o}\right) + \frac{K2}{Ko}\frac{\sin\beta_o}{2}$$

$$+ \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left(\frac{\sin\beta_o}{\ell_{Bo}}\left(\frac{5}{2} + 3\,tg^2\beta_o\right) - \frac{tg\,\beta_o}{R}\left(\frac{3}{2} + 2\,tg^2\beta_o\right)\right) = o$$

First, let us transform the equations $\frac{\partial C}{\partial \beta}$ and $\frac{\partial^2 C}{\partial \beta^2}$:

$$\frac{\partial C}{\partial \beta} \longrightarrow$$

$$\frac{KI}{R\,Ko}\sin\beta_o - \frac{K2}{Ko}\cos\beta_o + \frac{2\,KI}{Ko\,\ell_{Bo}}\frac{\cos^2\beta_o\sin\beta_o}{\cos\beta_o} + \frac{\cos^2\beta_o\sin\beta_o}{\ell_{B^2o}}\frac{2\,tg^2\beta_o + 1}{tg\,\beta_o}$$

$$- \frac{\sin^2\beta_o}{R\,\ell_{Bo}} - \frac{2\,KI}{R\,Ko}\sin\beta_o - \frac{\cos\beta_o\sin\beta_o}{R\,\ell_{Bo}}\frac{2\,tg^2\beta_o + 1}{tg\,\beta_o} + \frac{\sin^2\beta_o}{\cos\beta_o R^2}$$

.../..

.../..

$$-\frac{KI}{R\,Ko}\sin\beta_0-\frac{K2}{Ko}\cos\beta_0+\frac{2\,KI}{Ko\,\ell_{Bo}}\sin\beta_0\cos\beta_0+\frac{\cos^3\beta_0}{\ell_{B^2o}}\;\frac{2\sin^2\beta_0+\cos^2\beta_0}{\cos^3\beta_0}$$

$$-\frac{\sin^2\beta_0}{R\,\ell_{Bo}}-\frac{\cos^2\beta_0}{R\,\ell_{Bo}}\left(\frac{2\sin^2\beta_0+\cos^2\beta_0}{\cos^2\beta_0}\right)+\frac{\sin^2\beta_0}{R^2\cos\beta_0}=0$$

Finally :

$$\frac{\partial c}{\partial\beta}=0$$

$$\boxed{\begin{aligned}&-\frac{KI}{R\,Ko}\sin\beta_0-\frac{K2}{Ko}\cos\beta_0+\frac{2\,KI}{Ko\,\ell_{Bo}}\sin\beta_0\cos\beta_0+\frac{\cos\beta_0(1+\sin^2\beta_0)}{\ell_{B^2o}}\\&-\frac{1+2\sin^2\beta_0}{R\,\ell_{Bo}}+\frac{\sin^2\beta_0}{R^2\cos\beta_0}=0\end{aligned}}\quad(104)$$

Equation $\dfrac{\partial^2 c}{\partial\beta^2}$

$$\frac{KI}{R\,Ko}\;\frac{\sin^2\beta_0}{2\cos\beta_0}\left(1+\frac{2\cos^2\beta_0}{\sin^2\beta_0}\right)+\left(\frac{KI}{Ko}\right)^2\sin\beta_0+\frac{KI}{Ko}\;\frac{\sin^2\beta_0}{\ell_{Bo}}\;\frac{(3\,tg^2\beta_0+2)}{tg^2\beta_0}$$

$$-\frac{KI}{Ko\,R}\;\frac{\sin^2\beta_0}{\cos\beta_0}\;\frac{3\,tg^2\beta_0+2}{tg^2\beta_0}+\frac{K2}{Ko}\;\frac{\sin\beta_0}{2}+\frac{\cos^2\beta_0\sin\beta_0}{\ell_{B^2o}}\;\frac{5+6\,tg^2\beta_0}{2}$$

$$+\frac{\cos\beta_0}{R^2}\;\frac{\sin\beta_0}{\cos\beta_0}\;\frac{3+4\,tg^2\beta_0}{2}-\frac{\cos\beta_0\sin\beta_0}{R\,\ell_{Bo}}\;\frac{5+6\,tg^2\beta_0}{2}-\frac{\sin\beta_0\cos\beta_0}{R\,\ell_{Bo}}\;\frac{(3+4\,tg^2\beta_0)}{2}$$

$$=\frac{KI}{R\,Ko}\;\frac{1+\cos^2\beta_0}{2\cos\beta_0}+\left(\frac{KI}{Ko}\right)^2\sin\beta_0+(2+\sin^2\beta_0)\frac{KI}{Ko\,\ell_{Bo}}-\frac{2+\sin^2\beta_0}{\cos\beta_0}\;\frac{KI}{R\,Ko}$$

.../..

.../..

$$+ \frac{K2}{Ko} \frac{\sin\beta_o}{2} + (5 + \sin^2\beta_o) \frac{\sin\beta_o}{2\ell_B{}^2{}_o} + \frac{(3 + \sin^2\beta_o)}{2R^2} \frac{\sin\beta_o}{\cos^2\beta_o} - \frac{(5+\sin^2\beta_o)}{2R\ell_{Bo}} \frac{\sin\beta_o}{\cos\beta_o}$$

$$- \frac{\sin\beta_o}{\cos\beta_o} \frac{3 + \sin^2\beta_o}{2R\ell_{Bo}}$$

$$\doteq \frac{KI}{R\,Ko} \frac{(1+\cos^2\beta_o - 4 - 2\sin^2\beta_o)}{2\cos\beta_o} + \left(\frac{KI}{Ko}\right)^2 \sin\beta_o + (2 + \sin^2\beta_o) \frac{KI}{Ko\ell_{Bo}} + \frac{K2}{Ko} \frac{\sin\beta_o}{2}$$

$$+ (5 + \sin^2\beta_o) \frac{\sin\beta_o}{2\ell_B{}^2{}_o} + \frac{\sin\beta_o}{2R^2\cos^2\beta_o} (3 + \sin^2\beta_o) - \frac{4 + \sin^3\beta_o}{R\ell_{Bo}} \frac{\sin\beta_o}{\cos\beta_o}$$

Finally

$$\frac{\partial^2 C}{\partial\beta^2} \longrightarrow$$

(10.5)

$$\boxed{\begin{aligned} &- \frac{KI}{R\,Ko} \frac{2 + 3\sin^2\beta_o}{2\cos\beta_o} + \left(\frac{KI}{Ko}\right)^2 \sin\beta_o + (2 + \sin^2\beta_o) \frac{KI}{Ko\ell_{Bo}} + \frac{K2}{Ko} \frac{\sin\beta_o}{2} \\ &+ (5 + \sin^2\beta_o) \frac{\sin\beta_o}{2\ell_B{}^2{}_o} - \frac{4 + \sin^2\beta_o}{R\ell_{Bo}} \frac{\sin\beta_o}{\cos\beta_o} + \frac{\sin\beta_o(3 + \sin^2\beta_o)}{2R^2\cos^2\beta_o} \end{aligned}}$$

Let us replace $\frac{K2}{Ko}$ by its value

$$\frac{K2}{Ko} \frac{\sin\beta_o}{2} = - \frac{KI}{R\,Ko} \frac{\sin^2\beta_o}{2\cos\beta_o} + \frac{KI}{Ko\ell_{Bo}} \sin^2\beta_o + \frac{\sin\beta_o}{2\ell_B{}^2{}_o} (1 + \sin^2\beta_o)$$

$$- \frac{1 + 2\sin^2\beta_o}{R\ell_{Bo}} \frac{\sin\beta_o}{2\cos\beta_o} + \frac{\sin^3\beta_o}{2R^2\cos^2\beta_o} = 0$$

.../..

By grouping

$$\frac{\partial^2 C}{\partial \beta^2}$$

$$(106)\quad \boxed{\begin{aligned} &-\frac{KI}{R\,Ko}\,\frac{1+2\sin^2\beta_0}{\cos\beta_0}+\left(\frac{KI}{K2}\right)^2\sin\beta_0+2\,(1+\sin^2\beta_0)\,\frac{KI}{\ell_{Bo}\,Ko}+\frac{(3+\sin^2\beta_0)\,\sin\beta_0}{\ell_B{}^2{}_0}\\ &-(9+4\sin^2\beta_0)\,\frac{\sin\beta_0}{2\,R\ell_{Bo}\cos\beta_0}+\frac{\sin\beta_0(3+2\sin 2\beta_0)}{2\,R^2\cos^2\beta_0}=0\end{aligned}}$$

We have to obtain the same result if the equation (69) is developed.

$$\left(\frac{KI}{K2}\right)^2\sin\beta_0+\frac{KI}{RKo}\left[\frac{1}{\cos\beta_0}-2\cos\beta_0\frac{(2\sin^2\beta_0+\cos^2\beta_0)}{\cos^2\beta_0}\right]+\frac{2KI}{\ell_{Bo}\,Ko}\cos^2\beta_0\frac{(2\sin^2\beta_0+\cos\beta_0)}{\cos^2\beta_0}$$

$$+\cos^2\beta_0\,\frac{\sin\beta_0(3\cos^2\beta_0+4\sin^2\beta_0)}{\cos^2\beta_0\,\ell_B{}^2{}_0}\,\frac{1}{R\ell_{Bo}}\left[\cos^2\beta_0\,\frac{\sin^2\beta_0}{2\cos\beta_0}(3+5tg^2\beta_0)+\sin\beta_0\cos\beta_0(3+4tg^2\beta_0)\right]$$

$$+\frac{\sin\beta_0}{R^2}\,\frac{(3+5tg^2\beta_0)}{2}$$

that is right.

Let us consider the equation in $\frac{1}{\ell_{Bo}}$

$$\frac{(3+\sin^2\beta_0)\,\sin\beta_0}{\ell_B{}^2{}_0}+\frac{1}{\ell_{Bo}}\left(2\,(1+\sin^2\beta_0)\,\frac{KI}{Ko}-\frac{(9+4\sin^2\beta_0)}{2R}\,\frac{\sin\beta_0}{\cos\beta_0}\right)\quad(75)$$

$$+\left(\frac{KI}{Ko}\right)^2\sin\beta_0-\frac{KI}{R\,Ko}\,\frac{1+2\sin^2\beta_0}{\cos\beta_0}+\frac{\sin\beta_0(3+2\sin^2\beta_0)}{2R^2\cos^2\beta_0}=0$$

The equation (75) should have a real and positive root in $\frac{1}{\ell_{Bo}}$

.../..

.../..

Let us calculate the determinant

$$\left[2\,\frac{KI}{Ko}\,(1+\sin^2\beta_0) - \frac{(9+4\sin^2\beta_0)}{2R}\,\frac{\sin\beta_0}{\cos\beta_0} \right]^2 - 4\,(3+\sin^2\beta_0)\,\sin\beta_0$$

$$\left[\left(\frac{KI}{Ko}\right)^2 \sin\beta_0 - \frac{KI}{R\,Ko}\,\frac{1+2\sin^2\beta_0}{\cos\beta_0} + \frac{\sin\beta_0(3+2\sin^2\beta_0)}{2\,R^2\cos^2\beta_0} \right]$$

Term in $\left(\frac{KI}{Ko}\right)^2$

$$\left(\frac{KI}{Ko}\right)^2 \left[4\,(1+\sin^4\beta_0\,2\sin^2\beta_0) - 4\,(3+\sin^2\beta_0)\,\sin^2\beta_0 = 4 - 4\sin^2\beta_0 \right.$$

$$\left. = 4\cos^2\beta_0 \left(\frac{KI}{Ko}\right)^2 \right]$$

Term in $\frac{KI}{Ko}$

$$\frac{2}{R}\left(\frac{KI}{Ko}\right) \left[-(1+\sin^2\beta_0)\,(9+4\sin^2\beta_0)\frac{\sin\beta_0}{\cos\beta_0} + 2(3+\sin^2\beta_0)\,\frac{\sin\beta_0}{\cos\beta_0}(1+2\sin^2\beta_0) \right]$$

$$= \frac{2}{R}\left(\frac{KI}{Ko}\right)\frac{\sin\beta_0}{\cos\beta_0} \left[6 + 14\sin^2\beta_0 + 4\sin^4\beta_0 - 9 - 13\sin^2\beta_0 - 4\sin^4\beta_0 \right]$$

$$= \frac{2}{R}\,\frac{KI}{Ko}\,(-3+\sin^2\beta_0)\,\frac{\sin\beta_0}{\cos\beta_0}$$

Term independant of $\frac{KI}{Ko}$

$$\frac{(9+4\sin^2\beta_0)^2}{4R^2}\,\frac{\sin^2\beta_0}{\cos^2\beta_0} - 4\,\frac{(3+\sin^2\beta_0)\,(3+2\sin^2\beta_0)\,\sin^2\beta_0}{2\,R^2\cos^2\beta_0}$$

$$= \frac{\sin^2\beta_0}{4\cos^2\beta_0\,R^2} \left[81 + 16\sin^4\beta_0 + 72\sin^2\beta_0 - 8\times 9 - 8\times 9\sin^2\beta_0 - 16\sin^4\beta_0 \right]$$

$$= \frac{9\sin^2\beta_0}{4\,R^2\cos^2\beta_0}$$

.../..

⟶ Finally

$$(107) \qquad \boxed{\Delta = 4\cos^2\beta_0\left(\frac{KI}{Ko}\right)^2 + \frac{2\,KI}{R\,Ko}\,\frac{\sin\beta_0}{\cos\beta_0}\,(\sin^2\beta_0 - 3) + \frac{9\sin^2\beta_0}{4\,R^2\cos^2\beta_0}}$$

Δ has the same sign as its 1st coefficient i.e. $>$ o out of roots if any.

Research of Δ roots.

The determinant may be written :

$$\frac{\sin^2\beta_0}{R^2\cos^2\beta_0}(\sin^2\beta_0 - 3)^2 - \frac{9 \times 4\sin^2\beta_0\cos^2\beta_0}{4\cos^2\beta_0 R^2} =$$

$$\frac{\sin^2\beta_0}{\cos^2\beta_0}\left[\sin^4\beta_0 - 6\sin^2\beta_0 + 9 - 9\cos^2\beta_0\right] =$$

$$\frac{\sin^2\beta_0}{\cos^2\beta_0}\left[\sin^4\beta_0 + 3\sin^3\beta_0\right]$$ always positive.

$$\longrightarrow \boxed{\Delta = \frac{\sin^4\beta_0}{\cos^2\beta_0}\left[\sin^2\beta_0 + 3\right]}$$

The roots are :

$$\frac{(3 - \sin^2\beta_0)\,\dfrac{\sin\beta_0}{\cos\beta_0} \pm \sqrt{\dfrac{\sin^4\beta_0}{\cos^2\beta_0}\,(3\sin^2\beta_0)}}{4\cos^2\beta_0}$$

We have said that we don't limit the generality when saying that β_0 is $>$ o.

Let us see, $\dfrac{KI}{Ko}$. It is necessary that

.../..

$$(108)\quad \boxed{\begin{aligned} \frac{KI}{Ko} &< \frac{(3-\sin^2\beta_o)\sin\beta_o}{4\cos^3\beta_o} - \frac{\sin^2\beta_o}{\cos\beta_o}\sqrt{3+\sin^2\beta_o} \\ \frac{KI}{Ko} &> \frac{(3-\sin^2\beta_o)\sin\beta_o}{4\cos^3\beta_o} + \frac{\sin^2\beta_o}{\cos^2\beta_o}\sqrt{3+\sin^2\beta_o} \end{aligned}}$$

$$\beta o = \frac{\pi}{4}\ \frac{KI}{Ko} < \frac{(3-\frac{1}{2}) - \frac{\sqrt{2}}{2}\sqrt{3+\frac{1}{2}}}{4\times\frac{1}{2}} = \frac{5}{4} - \frac{\sqrt{2}}{2}\sqrt{\frac{1}{2}}$$

βo					
10°	0,134986	± 0,0533	→	$\frac{KI}{Ko}$	> 0,188286
					< 0,081686
20°	0,297086	± 0,21978	→	$\frac{KI}{Ko}$	< 0,077306
					> 0,51686
40°	0,924725	± 0,99646	→	$\frac{KI}{Ko}$	< − 0,071735
					> 1,921185
60°	3,897114	± 2,904738		$\frac{KI}{Ko}$	< 0,992376
					> 6,801852
80°	95,45738	± 11,12806		$\frac{KI}{Ko}$	> 106,58544
					< 84,32878

But, more it is necessary that $\frac{1}{\ell_{Bo}} > o$.

.../..

.../..

$$\frac{C}{d} = \left[\left(\frac{KI}{Ko}\right)^2 \sin\beta_0 - \frac{KI}{R\,Ko}\,\frac{1 + 2\sin^2\beta_0}{\cos\beta_0} + \frac{\sin\beta_0(3 + 2\sin^2\beta_0)}{2\,R^2\cos^2\beta_0}\right] \frac{1}{\sin\beta_0(3+\sin^2\beta_0)}$$

Let us suppose $\sin\beta_0 > o$. So, it is sufficient that C may be $> o$ i.e we have the following relation

$$b^2 - 4\,ac = \left(\frac{1 + 2\sin^2\beta_0}{\cos^2\beta_0}\right)^2 - \frac{4}{2}\,\frac{\sin^2\beta_0(3 + 2\sin^2\beta_0)}{\cos^2\beta_0} > o$$

$$\frac{1}{\cos^2\beta_0}\left[1 + 4\sin^2\beta_0 + 4\sin^4\beta_0 - 6\sin^2\beta_0 - 4\sin^4\beta_0\right]$$

$$b^2 - 4\,ac = \frac{1 - 2\sin^2\beta_0}{\cos^2\beta_0}$$

$$\longrightarrow \sin\beta_0 < \frac{\sqrt{2}}{2} \quad \text{or} \quad \beta_0 < \frac{\pi}{4}$$

The product will be positive if $\frac{KI}{Ko}$ is out of roots i.e.

$$\frac{\dfrac{1 + 2\sin^2\beta_0}{\cos\beta_0} + \dfrac{I}{\cos\beta_0}\sqrt{1 - 2\sin^2\beta_0}}{2\sin\beta_0}$$

$$\frac{KI}{Ko} \quad \text{out of} \quad \frac{1 + 2\sin^2\beta_0 + \sqrt{1 - 2\sin^2\beta_0}}{2\sin\beta_0\cos\beta_0}$$

Let us see $-\frac{b}{a}$

$$\frac{1}{\sin\beta_0(3+\sin^2\beta_0)}\left[2\,(1 + \sin^2\beta_0)\,\frac{KI}{Ko} - \frac{9 + 4\sin^2\beta_0}{2R}\,\frac{\sin\beta_0}{\cos\beta_0}\right] > o$$

i.e. $$\frac{KI}{Ko} > \frac{9 + 4\sin^2\beta_0}{4\,(1 + \sin^2\beta_0)}\,\frac{\sin\beta_0}{\cos\beta_0}$$

.../..

Value of $\dfrac{1 + 2\sin^2\beta_0}{\sin^2\beta_0} \pm \sqrt{\dfrac{1 - 2\sin^2\beta_0}{\sin^2\beta_0}}$

β			
10°	3,100131	± 2,83427	↗ 5,9344
			↘ 0,26586
20°	2.35485	± 1.36163	↗ 3,71648
			↘ 0,99322
40°	1,85453	± 0,42314	↗ 2,27767
			↘ 1,43139

Value $\dfrac{9 + 4\sin^2\beta_0}{4\,(1 + \sin^2\beta_0)\cos\beta_0}\,\sin\beta_0$

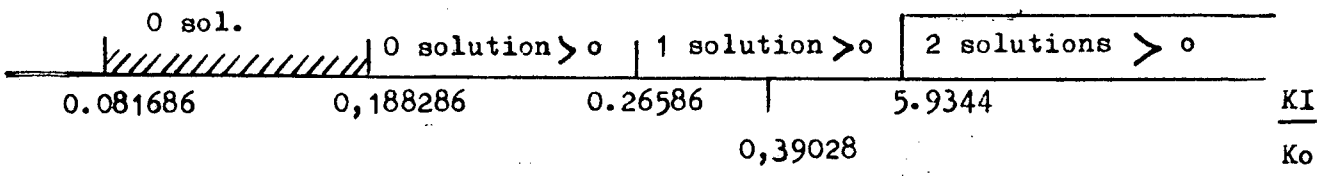
10°	0,39028
20°	0,77129
40°	1,58131

---→ If we have two roots they are both > 0

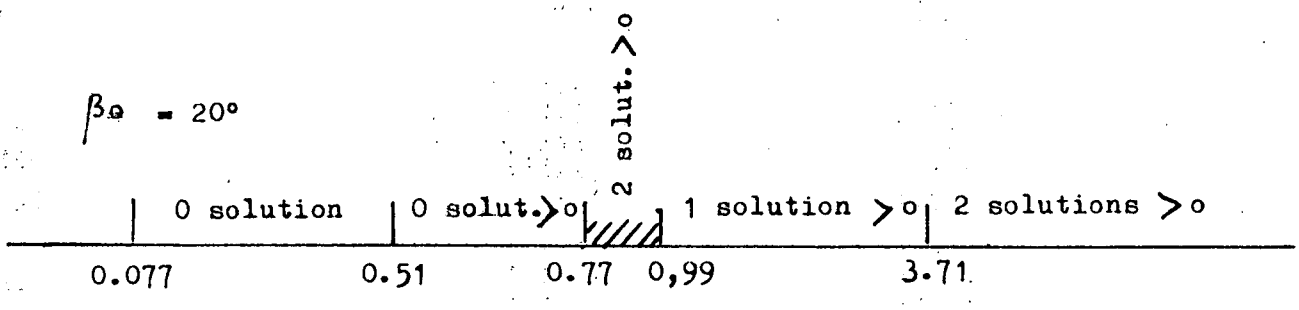
.../..

To summarize :

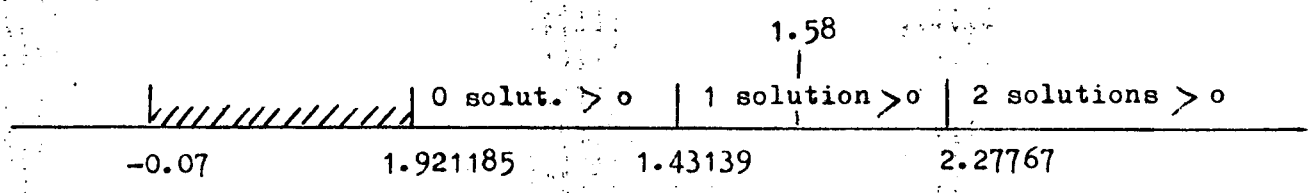
$\beta o = 10°$



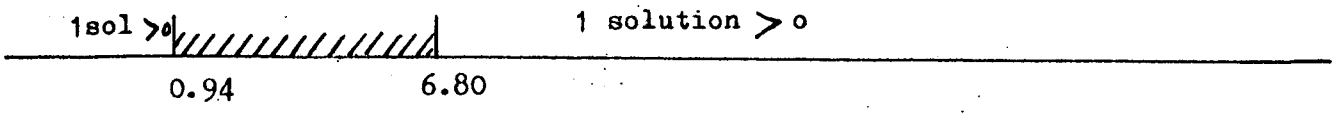
$\beta o = 20°$



$\beta = 40°$



$\beta o = 60°$



Let us consider again the equation (101) in $\left(\frac{KI}{Ko}\right)$. If any solution exists

$$\frac{KI}{Ko} = \frac{\left[- \frac{I}{R \cos\beta_o} + 2 \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right) \left(2 tg^2\beta_o + I\right)\right] \pm \sqrt{\Delta}}{2 \sin\beta_o}$$

$$\frac{KI}{Ko} = - \frac{1}{2 \sin\beta_o} \left[\left[\frac{1}{R \cos\beta_o} + \frac{4 \sin^2\beta_o}{\ell_{Bo}} + \frac{2\cos^2\beta_o}{\ell_{Bo}} - \frac{4\sin^2\beta_o}{\cos\beta_o R} - \frac{2\cos\beta_o}{R} \right] \pm \sqrt{\Delta} \right]$$

$$\frac{KI}{Ko} = - \frac{I}{2 \sin\beta_o} \left[\frac{2 + 2 \sin^2\beta_o}{\ell_{Bo}} + \frac{1}{R \cos\beta_o} \left[1 - 4 \sin^2\beta_o - 2 \cos^2\beta_o \right] \pm \sqrt{\Delta} \right]$$

$$= - \frac{I}{2 \sin\beta_o} \left[\frac{2 + 2 \sin^2\beta_o}{\ell_{Bo}} + \frac{1}{R \cos\beta_o} \left[- I - 2 \sin^2\beta_o \right] \pm \sqrt{\Delta} \right]$$

(109) $$\boxed{\frac{KI}{Ko} = \frac{1}{\sin\beta_o} \left[- \frac{1 + \sin^2\beta_o}{\ell_{Bo}} + \frac{1 + 2 \sin^2\beta_o}{2 R \cos\beta_o} \pm \sqrt{\Delta} \right]}$$

(110) with $$\boxed{\Delta = \frac{\cos^2\beta_o}{\ell_{B^2o}} + \frac{1 + 2 \sin^2\beta_o}{4 R^2 \cos^2\beta_o} + \frac{3 \sin^2\beta_o - 2}{2 \ell_{Bo} R \cos\beta_o}}$$

We may deduct $\frac{K2}{Ko}$

$$\frac{K2}{Ko} = - \frac{KI}{R Ko} \frac{\sin\beta_o}{\cos\beta_o} + \frac{2 KI}{\ell_{Bo} Ko} \sin\beta_o + \frac{1 + \sin^2\beta_o}{\ell_{B^2o}} - \frac{1 + 2 \sin^2\beta_o}{R \ell_{Bo} \cos\beta_o}$$

$$+ \frac{\sin^2\beta_o}{\cos^2\beta_o R^2}$$

.../..

so

$$\frac{K2}{Ko} = \frac{+1 + \sin^2\beta_o}{R\,\ell_{Bo}\,\cos\beta_o} - \frac{1 + 2\sin^2\beta_o}{2R^2\cos^2\beta_o} \pm \frac{\sqrt{\Delta}}{R\cos\beta_o}$$

$$- \frac{2}{\ell_B{}^2{}_o}(1 + \sin^2\beta_o) + \frac{(1 + 2\sin^2\beta_o)}{R\,\ell_{Bo}\cos\beta_o} \pm \frac{2\sqrt{\Delta}}{\ell_{Bo}} + \frac{1 + \sin^2\beta_o}{\ell_B{}^2{}_o}$$

$$- \frac{I + 2\sin^2\beta_o}{R\,\ell_{Bo}\cos\beta_o} + \frac{\sin^2\beta_o}{\cos^2\beta_o\,R^2}$$

$$\frac{K2}{Ko} = \frac{+1 + \sin^2\beta_o}{R\,\ell_{Bo}\cos\beta_o} - \frac{1}{\ell_B{}^2{}_o}(1 + \sin^2\beta_o) - \frac{1}{2R^2\cos^2\beta_o} \pm\sqrt{\Delta}\left(-\frac{1}{R\cos\beta_o} + \frac{2}{\ell_{Bo}}\right)$$

(111)
$$\boxed{\frac{K2}{Ko} = \frac{1 + \sin^2\beta_o}{R\,\ell_{Bo}\cos\beta_o} - \frac{1 + \sin^2\beta_o}{\ell_B{}^2{}_o} - \frac{1}{2R^2\cos\beta_o} \pm \sqrt{\Delta}\left(\frac{2}{\ell_{Bo}} - \frac{1}{R\cos\beta_o}\right)}$$

Δ has the value indicated in the equation (80)

Likewise, when considering the equation in $\frac{1}{\ell_{Bo}}$

(112)
$$\boxed{\frac{1}{\ell_{Bo}} = \frac{\frac{9 + 4\sin 2\beta_o}{R}\,\frac{\sin\beta_o}{2\cos\beta_o} - 2\,\frac{KI}{Ko}(1 + \sin^2\beta_o) \pm \sqrt{\Delta}}{2\sin\beta_o(3 + \sin^2\beta_o)}}$$

with $\Delta = 4\cos^2\beta_o\left(\frac{KI}{Ko}\right)^2 + 2\,\frac{KI}{R\,Ko}\,\frac{\sin\beta_o}{\cos\beta_o}(\sin^2\beta_o - 3) + \frac{9\sin^2\beta_o}{4R^2\cos^2\beta_o}$

.../..

General remark about the feasibility of gratings without coma.

We know that we may write :

$$\frac{KI}{Ko} = X \cos\gamma - Y \cos\delta - \frac{\cos\gamma - \cos\delta}{2R}$$

$$\frac{K2}{Ko} = X^2 \sin\gamma - Y^2 \sin\delta - \frac{\sin\gamma - \sin\delta}{4R^2}$$

by writing $X = \frac{\cos\gamma}{\ell_C} - \frac{1}{2R}$

$$Y = \frac{\cos\delta}{\ell_D} - \frac{1}{2R}$$

The first equation shows a straight line and the second a conic. So, the C and D points are located at the intersection of the straight line and of the conic.

Let us write that the conic is real.
Suppose $\gamma > o$ and consider the two cases $\delta > o$ and $\delta < o$

If δ is positive, the conic is an hyperbola and it is always real.
If δ is negative, the conic is an ellipse and it is necessary that

$$\boxed{\frac{K2}{Ko} + \frac{\sin\gamma - \sin\delta}{4R^2} > o}$$

C.3

VIII - STUDY OF THE ABERRATIONS AT THE VICINITY OF STIGMATIC POINTS

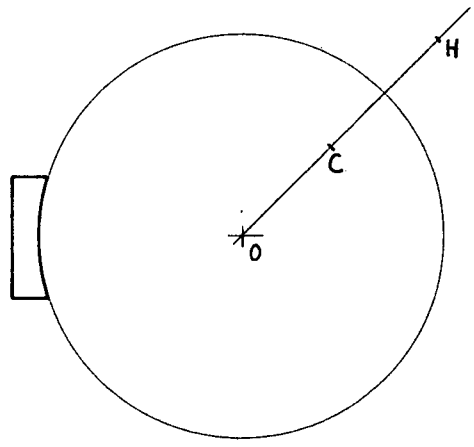
VIII - 1 - GENERAL CONSIDERATIONS

Let us write the general expression of the aberrant optical path of the holographic gratings :

$$\Delta = (MA + MB) - (IA + IB) - \frac{k\lambda}{\lambda_o}\left[(MC - MD) - (IC - ID)\right]$$

to which we add the relation

$$\sin\alpha + \sin\beta = \frac{\lambda}{\lambda_o}(\sin\gamma - \sin\delta)$$



- Fig. n° 18 -

O = the centre of the grating,

C = an arbitrary point and H its harmonic conjuguate with respect to the circle (O).

.../..

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- We know that if M is at any point of the circle (O)

$$\frac{MH}{MC} = m \quad \text{and} \quad \sin \eta = m \sin \gamma$$

- We know that the image B of the point A will be perfectly stigmatic, for some determined wavelength , provided that simultaneously :

  D is at O or H
  A is at O, C or H
  B is at O, C or H

  (there is no interest when it occurs that A and B are simultaneously at O).

- We know that if the point B is the image of the point A, the general equation is :

$$MA + MB - \frac{\lambda}{\lambda_0}(MC - MD) = (IA + IB) - \frac{\lambda}{\lambda_0}(IC - ID) = \text{cste}$$

This equation is verified, whatever M is,

. strictly - if there is a perfect stigmatic correspondence between A and B,

. approachingly - if the image B is with aberration.

I - Let us consider <u>the point D at the centre O of the circle</u>.
It follows that $MD = ID = R$

$$\delta = 0$$

1° Considering A at O

We have $MA = IA = R$

$$\alpha = 0$$

The general equation is :

$$\Delta = R + MB - \frac{\lambda}{\lambda_0}(MC - R) - \left[R + IB - \frac{\lambda}{\lambda_0}(IC - R) \right]$$

.../..

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or $\Delta = MB - \frac{\lambda}{\lambda_0} MC - (IB - \frac{\lambda}{\lambda_0} IC)$

In this case $\sin\beta = \frac{\lambda}{\lambda_0} \sin\gamma$

Finally, we have

$$\Delta = MB - \frac{\sin\beta}{\sin\gamma} M C - (IB - \frac{\sin\beta}{\sin\gamma} IC)$$

2° We have A at C

$MA = MC \quad \alpha = \gamma$

$IA = IC$

$$\Delta = MB + MC - \frac{\lambda}{\lambda_0} (MC - R) - \left[IB + IC - \frac{\lambda}{\lambda_0} (IC - R)\right]$$

$$\Delta = MB + MC (1 - \frac{\lambda}{\lambda_0}) - \left[IB + IC (1 - \frac{\lambda}{\lambda_0})\right]$$

$$\sin\beta = (\frac{\lambda}{\lambda_0} - 1) \sin\gamma$$

$$\Delta = MB - \frac{\sin\beta}{\sin\gamma} MC - (IB - IC \frac{\sin\beta}{\sin\gamma})$$

3° We have A at H

$MA = MH = m M C \quad \alpha = \gamma$

$IA = IH = m I C$

$$\Delta = m MC + MB - \frac{\lambda}{\lambda_0} (MC - R) - \left[m IC + IB - \frac{\lambda}{\lambda_0} (IC - R)\right]$$

$$\Delta = (m - \frac{\lambda}{\lambda_0}) MC + MB - \left[(m - \frac{\lambda}{\lambda_0}) IC + IB\right]$$

$$m \sin\gamma + \sin\beta = \frac{\lambda}{\lambda_0} \sin\gamma$$

$$\Delta = MB - \frac{\sin\beta}{\sin\gamma} MC - (IB - IC \frac{\sin\beta}{\sin\gamma})$$

.../..

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II - Let us consider <u>the point D at H harmonic conjuguate of C</u>

$$MD = MH = m\ MC$$

$$\sin\delta = m \sin\gamma$$

1° Considering the point A at O

$$MA = IA = R$$

$$\alpha = 0$$

$$\Delta = R + MB - \frac{\lambda}{\lambda_0}(MC - m\ MC) - \left[R + IB - \frac{\lambda}{\lambda_0}(IC - m\ IC)\right]$$

$$\sin\beta = \frac{\lambda}{\lambda_0}(\sin\gamma - m \sin\gamma)$$

$$\frac{\lambda}{\lambda_0} = \frac{\sin\beta}{\sin\gamma}\ \frac{1}{1-m}$$

$$\Delta = MB - \frac{\sin\beta}{\sin\gamma}\ \frac{1-m}{1-m}\ MC - \left(IB - \frac{\sin\beta}{\sin\gamma}\ IC\right)$$

$$\Delta = MB - \frac{\sin\beta}{\sin\gamma}\ MC\left(IB - \frac{\sin\beta}{\sin\gamma}\ IC\right)$$

2° Considering the point A at C

$$MA = MC$$

$$\alpha = \gamma$$

$$\Delta = MC + MB - \frac{\lambda}{\lambda_0}(MC - m\ MC) - \left[IC + IB - \frac{\lambda}{\lambda_0}(IC - m\ IC)\right]$$

$$\sin\gamma + \sin\beta = \frac{\lambda}{\lambda_0}(\sin\gamma - m \sin\gamma)$$

$$\frac{\lambda}{\lambda_0} = \frac{1}{1-m} + \frac{\sin\beta}{(1-m)\sin\gamma}$$

.../..

.../..

$$\Delta = MC + MB - \left[\frac{1}{1-m} + \frac{\sin\beta}{(1-m)\sin\gamma}\right](1-m)\,MC - \left[IC + IB - \frac{\lambda}{\lambda_0}(IC - m\,IC)\right]$$

$$\Delta = MC + MB - MC - \frac{\sin\beta}{\sin\gamma}MC - \left[IC + IB - \frac{\lambda}{\lambda_0}(IC - m\,IC)\right]$$

$$\Delta = MB - \frac{\sin\beta}{\sin\gamma}MC - \left(IB - IC\,\frac{\sin\beta}{\sin\gamma_0}\right)$$

3° Considering the point A at H

$$MA = MH = MD = m\,MC$$

$$\alpha = \delta$$

$$\sin\alpha = m\sin\gamma$$

$$\Delta = m\,MC + MB - \frac{\lambda}{\lambda_0}(MC - m\,MC) - \left[m\,ID + IB - \frac{\lambda}{\lambda_0}(IC - m\,IC)\right]$$

$$m\sin\gamma + \sin\beta = \frac{\lambda}{\lambda_0}\sin\gamma(1-m)$$

$$\frac{\lambda}{\lambda_0} = \frac{m}{1-m} + \frac{\sin\beta}{\sin\gamma}\,\frac{1}{1-m}$$

$$\Delta = m\,MC + MB - \left(\frac{m}{1-m} + \frac{\sin\beta}{\sin\gamma}\,\frac{1}{1-m}\right)(1-m)\,MC - \left[IB - IC\,\frac{\sin\beta}{\sin\gamma}\right]$$

$$\Delta = m\,MC + MB - m\,MC - \frac{\sin\beta}{\sin\gamma}MC - \left(IB - IC\,\frac{\sin\beta}{\sin\gamma}\right)$$

$$\Delta = MB - \frac{\sin\beta}{\sin\gamma}MC - \left(IB - IC\,\frac{\sin\beta}{\sin\gamma}\right)$$

.../..

.../..

So, we may notice that, if D is one of the points O or H and A one of the points O, C or H, the aberrant optical path corresponding to the point image B is given by

$$\boxed{\Delta = MB - \frac{\sin\beta}{\sin\gamma}\, MC - \left(IB - IC\,\frac{\sin\beta}{\sin\gamma}\right)} \tag{113}$$

This equation is independant of the locus of D and A.

VIII - 2 - STUDY OF THE FOCAL CURVES

We are going to define the properties of the focal curves in the specific case of perfect stigmatism for a determined wavelength, i.e. simultaneously :

- D is one of the points O or H
- A is one of the points O, C or H

We know that if B is, moreover, one of the points O, C or H, it is the stigmatic image of the point A for a determined wavelength.

Our purpose is to study the locus of the point B out of those specific points.

We shall consider the special case D and A at the centre of the grating ; Then we have seen that, in fact, the results to be performed will be valid for every locus of D and of A, - providing being one of the privileged loci.

Therefore, in our hypothesis $\ell_A = R \qquad \alpha = 0$

$\ell_D = R \qquad \delta = 0$

(114) and $$\boxed{\sin\beta = \frac{k\lambda}{\lambda_0}\sin\gamma}$$

Sagittal focal equation :

(115) $$\boxed{\frac{1}{\ell_B} = \frac{\cos\beta}{R} + \frac{\sin\beta}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)}$$

We identify the straight line 's equation. It passes through the points

.../..

.../..

$\beta = 0 \quad \ell_B = R$ i.e. D

$\beta = \gamma \quad \ell_B = \ell_C$ i.e. C

<u>Therefore, the locus of the sagittal focal is the straight line C D.</u>

Equation of the tangential focal length :

$$\boxed{\frac{1}{\ell_T} = \frac{\frac{\cos\beta}{R} + \frac{\sin\beta}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)}{\cos^2\beta}} \qquad (116)$$

that may be written :

$$\ell_T = \frac{R\cos\beta}{1 + \frac{tg\,\beta}{tg\,\gamma}\left(\frac{R\cos\gamma}{\ell_C} - 1\right)}$$

We have $\quad tg\,\beta_a = \dfrac{tg\,\gamma}{1 - \frac{R\cos\gamma}{\ell_C}}$

so

$$\boxed{\ell_T = \frac{R\cos\beta}{1 - \frac{tg\,\beta}{tg\,\beta_a}}} \qquad (117)$$

Therefore, the curve shows an asymptotic direction for the polar angle β_a defined by :

$$tg\,\beta_a = \frac{tg\,\gamma}{1 - \frac{R}{\ell_C}\cos\gamma} \qquad (118)$$

.../..

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Conventionally, we shall have $0 < \gamma < \frac{\Omega}{2}$

If $\ell_C > R \cos\gamma$, i.e. the point C out of the Rowland circle, $\beta_a > 0$.

If $\ell_C < R \cos\gamma$, i.e. the point C inside the Rowland circle, $\beta_a < 0$.

The asymptote 's locus is defined by the quantity

$$p = I\ H = \text{limit of } \ell_T \cdot \sin(\beta - \beta_a) \text{ when } \beta \to \beta_a.$$

Then ℓ_T may be written :

$$\ell_T = \frac{R \cos^2\beta \sin\beta_a}{\sin(\beta_a - \beta)}$$

and we obtain

(119) $$\boxed{p = -R(\sin\beta_a - \sin^3\beta_a)}$$

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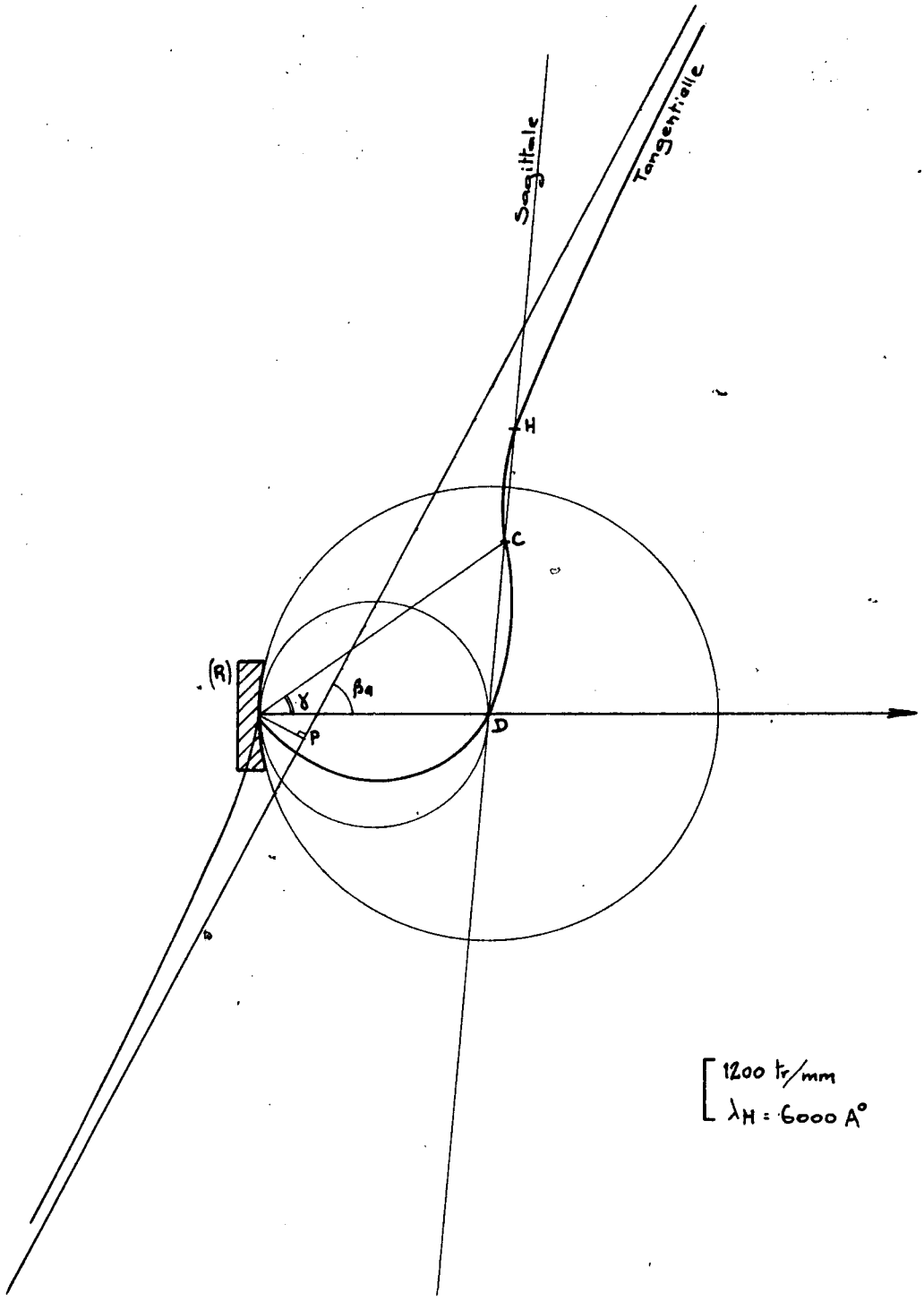

- Fig. n°19 -

Focal curves for C out of the Rowland Circle.

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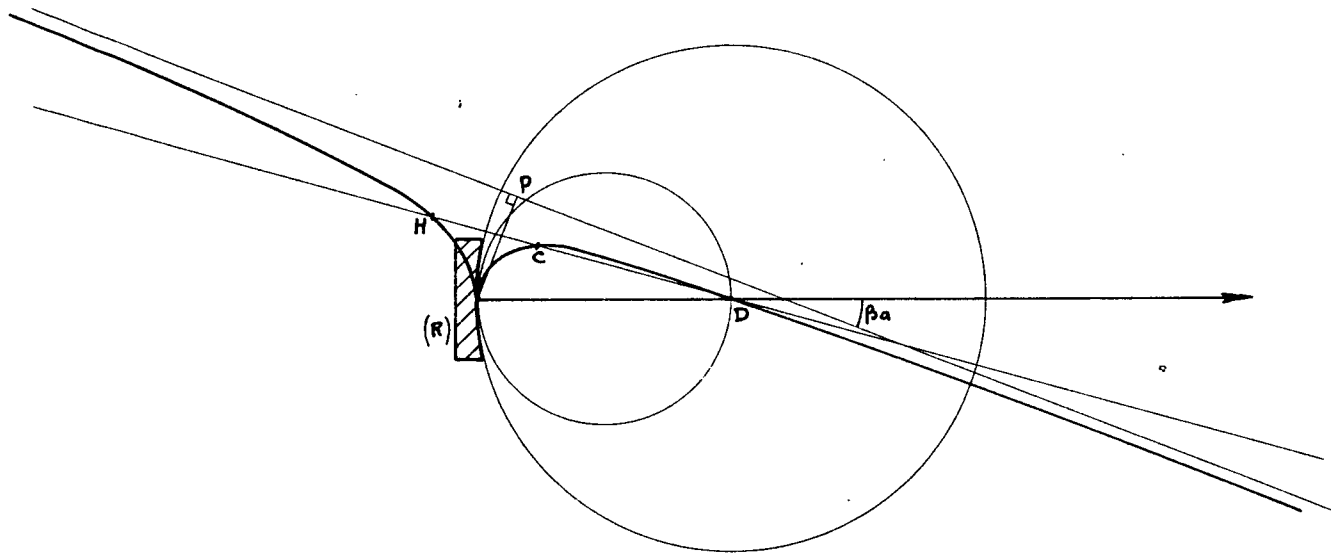

[1200 tr/mm
λH = 6000 A°

- Fig. n°20 -

Focal curves for C in the Rowland Circle.
We observe that H is virtual.

.../..

.../..

For $\beta = \frac{\pi}{2}$ we obtain $\ell_T = 0$

Then, the curve of the tangential focal length passes through the top of the grating with a vertical tangent.

The angle $\hat{V}$ between the tangent located at a point of the curve of tangential focal length and the radius vector is given by :

$$tg\,V = \frac{\ell_T}{\frac{d\ell_T}{d\beta}}$$

Then $$\frac{d\ell_T}{\partial\beta} = R \times \frac{1 - \sin\beta\cos\beta\,(tg\,\beta_a - tg\,\beta)}{\cos\beta \cdot tg\,\beta_a \left(1 - \frac{tg\,\beta}{tg\,\beta_a}\right)^2}$$

We have $$\boxed{tg\,V = \frac{\cos\beta\,(tg\,\beta_a - tg\,\beta)}{1 - \sin^2\beta\,\cos\beta\,(tg\,\beta_a - tg\,\beta)}}$$

(120)

STUDY OF THE ABERRATIONS AT THE VICINITY OF STIGMATIC POINTS

VIII - 3 - STUDY OF THE ASTIGMATISM

We have seen that the aberration of the optical path corresponding to a point image B was depending neither on the locus of the point A (source) nor on the position of the recording point D, providing that the points D and A be at one of these privileged loci compatible with a rigorous stigmatism, but just on the locus of points C and B.

Now, we may ask - the points A and D being at determined privileged loci - Is it possible to obtain an image B without astigmatism, namely out of the privileged points ?

As the answer is not depending on the exact locus of points A and D providing that they are at one of the privileged loci, our reasoning is bearing upon a particular case.
For this case we choose to locate D at O (centre of the circle) and A at C.

Then we have :

$$\ell_D = R \qquad \ell_A = \ell_C$$

$$\delta = 0 \qquad \alpha = \gamma$$

In the general case, we know that the condition to avoid astigmatism, is :

$$\frac{\cos\alpha - \cos\beta}{\ell_A} = \frac{\sin 2\beta}{R} + \frac{KI}{Ko}\,\frac{tg\,\frac{\alpha+\beta}{2}}{} - \frac{K3}{Ko}\,\frac{tg\,\frac{\alpha+\beta}{2}}{}\,\cos^2\beta$$

If we introduce the initial conditions

$$\frac{KI}{Ko} = \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\frac{K3}{Ko} = \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

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The condition may be written :

$$\frac{\cos\gamma - \cos\beta}{\ell_C} = \frac{\sin^2\beta}{R} + \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{\sin\gamma + \sin\beta}{\cos\gamma + \cos\beta}$$

$$- \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{\sin\gamma + \sin\beta}{\cos\gamma + \cos\beta}\cos^2\beta$$

$$\frac{\cos\gamma - \cos\beta}{\ell_C} = \frac{\sin^2\beta}{R} - \frac{\cos\gamma}{R\sin\gamma}\,\frac{\sin\gamma + \sin\beta}{\cos\gamma + \cos\beta}\left(1 - \cos^2\beta\right)$$

$$+ \frac{1}{\ell_C}\cdot\frac{1}{\sin\gamma}\,\frac{\sin\gamma + \sin\beta}{\cos\gamma + \cos\beta}\left(\cos^2\gamma - \cos^2\beta\right)$$

$$- \frac{\cos\gamma - \cos\beta}{\ell_C} + \frac{\sin^2\beta}{R}\left(1 - \frac{\cos\gamma}{\sin\gamma}\,\frac{\sin\gamma + \sin\beta}{\cos\gamma + \cos\beta}\right)$$

$$+ \frac{1}{\ell_C}\,\frac{\sin\gamma + \sin\beta}{\sin\gamma}\left(\cos\gamma - \cos\beta\right) = 0$$

$$\frac{\cos\gamma - \cos\beta}{\ell_C}\left(-1 + \frac{\sin\gamma + \sin\beta}{\sin\gamma}\right) + \frac{\sin^2\beta}{R}\,\frac{\sin\gamma\cos\gamma + \sin\gamma\cos\beta - \sin\gamma\cos\gamma - \cos\gamma\sin\beta}{\sin\gamma\left(\cos\gamma + \cos\beta\right)} = 0$$

$$\frac{\cos\gamma - \cos\beta}{\ell_C}\,\frac{\sin\beta}{\sin\gamma} + \frac{\sin^2\beta}{R}\,\frac{\sin\left(\gamma - \beta\right)}{\sin\gamma\left(\cos\gamma + \cos\beta\right)} = 0$$

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$$\frac{\sin\beta}{\sin\gamma}\,\frac{1}{\cos\gamma+\cos\beta}\left[\frac{\cos^2\gamma-\cos^2\beta}{\ell_C}+\frac{\sin\beta\,.\,\sin(\gamma-\beta)}{R}\right]=0$$

$$\frac{\sin\beta}{\sin\gamma}\,\frac{1}{(\cos\gamma+\cos\beta)}\left[\frac{(\cos\gamma+\cos\beta)\,(\cos\gamma-\cos\beta)}{\ell_C}+\frac{\sin\beta\sin(\gamma-\beta)}{R}\right]=0$$

$$\frac{\sin\beta}{\sin\gamma\,(\cos\gamma+\cos\beta)}\left[\frac{-2\cos\frac{\gamma+\beta}{2}\cos\frac{\gamma-\beta}{2}\,2\sin\frac{\gamma+\beta}{2}\sin\frac{\gamma-\beta}{2}}{\ell_C}+\frac{\sin\beta\sin(\gamma-\beta)}{R}\right]=0$$

$$\frac{\sin\beta}{\sin\gamma}\,\frac{1}{(\cos\gamma+\cos\beta)}\left[\frac{-\sin(\gamma+\beta)\,\sin(\gamma-\beta)}{\ell_C}+\frac{\sin\beta\sin(\gamma-\beta)}{R}\right]=0$$

Finally :

$$(121)\qquad \boxed{\frac{\sin\beta\sin(\gamma-\beta)}{\sin\gamma\,(\cos\gamma+\cos\beta)}\left(\frac{\sin\beta}{R}-\frac{\sin(\beta+\gamma)}{\ell_C}\right)=0}$$

Then this expression may be cancelled if :

<u>1st solution</u>

1) $\sin\beta = 0 \longrightarrow \beta = 0$

B is at O. It is one of the stigmatic points.

<u>2nd solution</u>

2) $\sin(\gamma-\beta) = 0 \longrightarrow \gamma = \beta$

B is at C. It is the second stigmatic point.

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3rd solution

3) $$\frac{\sin\beta}{R} = \frac{\sin(\gamma+\beta)}{\ell_C}$$

We have seen (equation 24 ————)
that the 3rd stigmatic point corresponded to $\beta = \eta$ and we had the two following relations :

$$\ell_C = \frac{R}{m}\,\frac{\sin(\eta+\gamma)}{\sin\gamma} \quad \text{and} \quad \sin\eta = m\sin\gamma$$

from which we may write $$\frac{\sin(\eta+\gamma)}{\ell_C} = \frac{m\sin\gamma}{R} = \frac{\sin\eta}{R}$$

The relations are well identical if $\beta = \eta$

The third solution corresponds to the third stigmatic point.

Conclusion : The points A and D being at the determined privileged loci, the astigmatism is zero for the images B located at the privileged points and at these points only.

Focal's height

Let us consider the formula (equation 44 ————)
which becomes in the specific case :

$$h_T = \frac{\dfrac{\sin\beta\,\sin(\gamma-\beta)}{\sin\gamma\,(\cos\gamma+\cos\beta)}\left[\dfrac{\sin\beta}{R} - \dfrac{\sin(\beta+\gamma)}{\ell_C}\right]}{-\dfrac{1}{\cos\gamma+\cos\beta}\,\dfrac{\cos^2\gamma}{\ell_C} + \dfrac{1}{R} + \dfrac{\sin\gamma+\sin\beta}{\cos\gamma+\cos\beta}\,\dfrac{1}{\sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right)}$$

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That may be written :

$$h_T = \frac{\sin\beta \ \sin(\gamma-\beta) \ \frac{\sin\beta}{R} - \frac{\sin(\beta+\gamma)}{C}}{\frac{1}{R}\sin(\gamma-\beta) + \frac{\cos^2\gamma}{\ell_C}\sin\beta} \qquad (122)$$

Under the above defined conditions, i.e. D at O or at H and A at O, C or H, there is no possibility to avoid the astigmatism but to choose, for point Bo, one of the points O, C or H.

Then we may ask the question : <u>Bo being one of these points, is it possible to cancel</u> $\frac{\partial \mathcal{A}}{\partial \beta}$?

We keep the specific chosen conditions as above, i.e. D at O and A at C and we choose for B the three privileged loci : B at O, C and H.

1) <u>B at O</u>

Under the chosen conditions

$$\ell_D = R \qquad \ell_A = \ell_C$$
$$\delta = 0 \qquad \alpha = \gamma$$

General conditions (resulting from the fact that a grating is used in such conditions that it is possible to obtain stigmatic points).

$$\frac{KI}{Ko} = \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\frac{K3}{Ko} = \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

Specific conditions linked to the fact that B is at O

$$\beta = 0$$
$$\ell_B = R \qquad \lambda = \lambda_o$$

.../..

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Let us remind that the general equation corresponding to $\frac{\partial \mathcal{A}}{\partial \beta} = 0$ is :

$$\frac{\sin\beta}{\ell_A} - \frac{\sin 2\beta}{R} + \frac{1}{2}\frac{KI}{K\hat{o}}\frac{1}{\cos^2\frac{\alpha+\beta}{2}} - \frac{K3}{K\hat{o}}\left(\frac{\cos^2\beta}{2\cos^2\frac{\alpha+\beta}{2}} - \sin 2\beta\, tg\frac{\alpha+\beta}{2}\right)$$

Applying the hypothesis we have the relation :

$$\frac{1}{2\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{1}{\cos^2\frac{\gamma}{2}} - \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{1}{2\cos^2\frac{\gamma}{2}} = 0$$

that may be written :

$$\frac{1}{2\sin\gamma\cos^2\frac{\gamma}{2}}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{1}{\ell_C}\right) = -\frac{\sin^2\gamma}{2\sin\gamma\cos^2\frac{\gamma}{2}\,\ell_C} = 0$$

According to definition $\gamma \neq 0$

So $\frac{\partial \mathcal{A}}{\partial \beta}$ may be cancelled only if $\ell_C = \infty$

In that case $\frac{KI}{K\hat{o}} = -\frac{\cos\gamma}{\sin\gamma}\cdot\frac{1}{R}$ $\quad \frac{K3}{K\hat{o}} = -\frac{\cos\gamma}{\sin\gamma}\cdot\frac{1}{R}$

Now, under the same conditions, let us study $\frac{\partial^2 \mathcal{A}}{\partial \beta^2}$

The general condition to cancel the above term is :

$$\frac{\cos\beta}{\ell_A} - \frac{2\cos 2\beta}{R} + \frac{KI}{K\hat{o}}\frac{\sin\frac{\alpha+\beta}{2}}{2\cos^3\frac{\alpha+\beta}{2}} - \frac{K3}{K\hat{o}}\left[\frac{-\sin 2\beta}{\cos^2\frac{\alpha+\beta}{2}} + \frac{\cos^2\beta\,\sin\frac{\alpha+\beta}{2}}{2\cos^3\frac{\alpha+\beta}{2}} - 2\cos 2\beta\, tg\frac{\alpha+\beta}{2}\right]$$

Under this specific case, the equation is :

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.../..

$$\frac{2}{R} - \frac{\cos\gamma}{R\sin\gamma}\,\frac{\sin\frac{\gamma}{2}}{2\cos^3\frac{\gamma}{2}} + \frac{\cos\gamma}{R\sin\gamma}\left[\frac{\sin\frac{\gamma}{2}}{2\cos^3\frac{\gamma}{2}} - 2\,tg\,\frac{\gamma}{2}\right] = 0$$

$$\text{or}\quad \frac{2}{R}\left(1 - \frac{tg\,\frac{\gamma}{2}}{tg\,\gamma}\right) = 0$$

$\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0$ means that $tg\,\gamma = tg\,\frac{\gamma}{2}$

That is equivalent to $2\cos^2\frac{\gamma}{2} = \cos\gamma$

or $\cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2} = 0$

that is impossible.

So, we cannot obtain $\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0$

Let us calculate <u>the focal's height</u> in this case

a) If $\ell_C \neq \infty$ i.e. if $\frac{\partial \mathcal{A}}{\partial \beta} \neq 0$

from the equations (122) and (45)

$$h_T(\theta) = Z\,m\,x\,\frac{\dfrac{-\sin^2\gamma}{2\sin\gamma\,\cos^2\frac{\gamma}{2}\cdot\ell_C}}{-\dfrac{-1}{(\cos\alpha+1)}\,\dfrac{\cos^2\alpha}{\ell_A} + \dfrac{1}{R} + tg\,\dfrac{\alpha+\beta}{2}\,\dfrac{KI}{Ko}}\,x\,\theta$$

$$h_T(\theta) = Z\,m\,x\,\frac{\dfrac{-\sin\gamma}{2\cos^2\frac{\gamma}{2}\,\ell_C}}{\dfrac{\cos^2\gamma}{(1+\cos\gamma)\,\ell_C} + \dfrac{1}{R} + tg\,\dfrac{\gamma}{2}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right)\dfrac{1}{\sin\gamma}}\,x\,\theta$$

.../.

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Let us remark that $tg\,\frac{\gamma}{2} = \frac{\sin\gamma}{1+\cos\gamma}$

$$h_T(\theta) = Zm \times \frac{\dfrac{-\sin\gamma}{2\cos^2\frac{\gamma}{2}\cdot \ell_C}}{\dfrac{\cos^2\gamma}{\ell_C}\left[\dfrac{\sin\gamma}{(1+\cos\gamma)\sin\gamma} - \dfrac{1}{1+\cos\gamma}\right] + \dfrac{1}{R}\left(1 - \dfrac{\cos\gamma}{1+\cos\gamma}\right)}$$

As $1 + \cos\gamma = 2\cos^2\frac{\gamma}{2}$ we obtain finally :

$$h_T(\theta) = Z\,m \frac{-\dfrac{\sin\gamma}{\ell_C(1+\cos\gamma)}}{\dfrac{1}{R} \times \dfrac{1}{1+\cos\gamma}} \times \theta$$

(123) $$\boxed{h_T(\theta) = -Z\,m\,\frac{R\sin\gamma}{\ell_C}\,\theta}$$

b) If $\ell_C = \ell_A = \infty$ i.e. if $\frac{\partial\mathcal{A}}{\partial\beta} = 0$

we may calculate $h_T(\theta^2)$ from the formula (46)

In this specific case $\frac{KI}{Ko} = \frac{K3}{Ko} = -\frac{\cos\gamma}{\sin\gamma}\cdot\frac{1}{R}$

$$h_T(\theta^2) = \frac{Z\,m}{2}\cdot\frac{\dfrac{2}{R}\left(1 - \dfrac{tg\frac{\gamma}{2}}{tg\,\gamma}\right)}{\dfrac{1}{R} - tg\frac{\gamma}{2}\cdot\dfrac{\cos\gamma}{\sin\gamma}\cdot\dfrac{1}{R}}\cdot\theta^2$$

$$= \frac{Z\,m}{2}\,\frac{2\left(tg\,\gamma - tg\frac{\gamma}{2}\right)}{tg\,\gamma - tg\frac{\gamma}{2}}\cdot\theta^2$$

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(124) $$\boxed{h_T(\theta^2) = Z\,m\,.\,\theta^2}$$

Besides, it is easy to find out the rigorous value of h_T under those conditions

$$\boxed{h_T = Z\,m\,.\,\sin^2\beta} \qquad (125)$$

2) <u>B at C</u>

Under the chosen conditions $\begin{cases} \ell_D = R \quad \ell_A = \ell_C \\ \delta = 0 \quad \alpha = \gamma \end{cases}$

General conditions for $\mathcal{A} = 0$ $\begin{cases} \dfrac{KI}{Ko} = \dfrac{1}{\sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \\ \dfrac{K3}{Ko} = \dfrac{1}{\sin\gamma}\left(\dfrac{1}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \end{cases}$

Specific conditions linked to the fact that B is at C $\begin{cases} \beta = \gamma \quad \lambda = 2\lambda_0 \\ \ell_B = \ell_C = \ell_A \end{cases}$

Let us recall that the general equation corresponding to $\dfrac{\partial \mathcal{A}}{\partial \beta} = 0$ is :

$$\frac{\sin\beta}{\ell_A} = \frac{\sin 2\beta}{R} + \frac{1}{2}\,\frac{KI}{Ko}\,\frac{1}{\cos^2\frac{\alpha+\beta}{2}} - \frac{K3}{Ko}\left[\frac{\cos^2\beta}{2\cos^2\frac{\alpha+\beta}{2}} - \sin 2\beta\, tg\frac{\alpha+\beta}{2}\right]$$

Applying the hypothesis, this relation is :

$$\frac{\sin\gamma}{\ell_C} = \frac{\sin 2\gamma}{R} + \frac{1}{2\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{1}{\cos^2\gamma}$$

$$- \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)\left(\frac{\cos^2\gamma}{2\cos^2\gamma} - \sin 2\gamma\, tg\gamma\right)$$

.../..

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or

$$\frac{1}{\ell_C}\left(\sin\gamma - \frac{1}{2\sin\gamma} + \frac{1}{2\sin\gamma} - \frac{\sin 2\gamma\, tg\gamma}{\sin\gamma}\right) =$$

$$\frac{1}{R}\left(\sin 2\gamma - \frac{1}{2\sin\gamma\cos\gamma} + \frac{\cos\gamma}{2\sin\gamma} - \frac{\cos\gamma\sin 2\gamma\, tg\gamma}{\sin\gamma}\right)$$

$$\frac{1}{\ell_C}\left(\sin\gamma - 2\sin\gamma\right) = \frac{1}{2R}\left(\frac{\cos\gamma}{\sin\gamma} - \frac{1}{\sin\gamma\cos\gamma}\right)$$

$$\frac{\sin\gamma}{\ell_C} = \frac{\sin^2\gamma}{2\sin\gamma\cos\gamma}\cdot\frac{1}{R}$$

$$\sin\gamma\left(\frac{1}{\ell_C} - \frac{1}{2R\cos\gamma}\right) = 0$$

$$\frac{\partial A}{\partial\beta} = 0 \text{ if } \boxed{\ell_C = 2R\cos\gamma} \qquad (126)$$

If we retain the hypothesis $\ell_C = 2R\cos\gamma$

$$\frac{KI}{Ko} = -\frac{\cos\gamma}{2\sin\gamma}\cdot\frac{1}{R}$$

$$\frac{K3}{Ko} = -\frac{\cos 2\gamma}{\sin 2\gamma}\cdot\frac{1}{R}$$

Let us study now $\frac{\partial^2 A}{\partial\beta^2}$ under the same conditions.

The condition $\frac{\partial^2 A}{\partial\beta^2} = 0$ is :

$$\frac{\cos\beta}{\ell_A} = \frac{2\cos^2\beta}{R} + \frac{KI}{Ko}\,\frac{\sin\frac{\alpha+\beta}{2}}{2\cos^3\frac{\alpha+\beta}{2}} - \frac{K3}{Ko}\left[\frac{-\sin 2\beta}{\cos^2\frac{\alpha+\beta}{2}}\right.$$

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$$+ \frac{\cos^2\beta \quad \sin\frac{\alpha+\beta}{2}}{2\cos^3\frac{\alpha+\beta}{2}} - 2\cos 2\beta \; tg\frac{\alpha+\beta}{2} \Bigg]$$

To make easier the calculations we shall keep temporarily

$\frac{KI}{Ko}$ and $\frac{K3}{Ko}$ under the form

$$\frac{KI}{Ko} = \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \qquad \frac{K3}{Ko} = \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\frac{\cos\gamma}{\ell_C} - \frac{2\cos 2\gamma}{R} + \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{\sin\gamma}{2\cos^3\gamma}$$

$$- \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[-\frac{\sin 2\gamma}{\cos^2\gamma} + \frac{\cos^2\gamma\sin\gamma}{2\cos^3\gamma} - 2\cos 2\gamma \; tg\gamma\right]$$

$$\frac{\cos\gamma}{\ell_C} = \frac{2\cos 2\gamma}{R} + \frac{1}{2\ell_C}\cos\gamma - \frac{1}{2R\cos\gamma} -$$

$$\frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[-\frac{2\sin\gamma}{\cos\gamma} + \frac{\sin\gamma}{2\cos\gamma} - 2\cos 2\gamma\frac{\sin\gamma}{\cos\gamma}\right]$$

$$\frac{\cos\gamma}{\ell_C} = \frac{2\cos 2\gamma}{R} + \frac{1}{\ell_C\cos\gamma} - \frac{1}{2R\cos^2\gamma} - \frac{1}{\sin\gamma}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\left(-\frac{3}{2}\frac{\sin\gamma}{\cos\gamma} - 2\cos 2\gamma\frac{\sin\gamma}{\cos\gamma}\right)$$

$$\frac{\cos\gamma}{\ell_C} = \frac{2\cos 2\gamma}{R} + \frac{1}{2\ell_C\cos\gamma} - \frac{1}{2R\cos^2\gamma} + \frac{1}{\ell_C\cos\gamma}\left(\frac{3}{2} + 2\cos 2\gamma\right)$$

$$- \frac{1}{R}\left(\frac{3}{2} + 2\cos 2\gamma\right)$$

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$\ell_C = 2\,R\cos\gamma$

$$\frac{1}{2R} = \frac{2\cos 2\gamma}{R} + \frac{1}{4\,R\cos^2\gamma} - \frac{1}{2\,R\cos^2\gamma} + \frac{1}{2\,R\cos^2\gamma}\left(\frac{3}{2} + 2\cos 2\gamma\right)$$

$$- \frac{1}{R}\left(\frac{3}{2} + 2\cos 2\gamma\right)$$

$$\frac{2}{R} = \frac{1}{2\,R\cos^2\gamma} + \frac{\cos 2\gamma}{R\cos\gamma}$$

The condition should be $\dfrac{2}{R} = \dfrac{1 + 2\cos 2\gamma}{2\,R\cos^2\gamma}$

that is impossible.

<u>Let us calculate the focal's height</u> at the vicinity of C

a) If $\ell_C \neq 2\,R\cos\gamma$ i.e if $\dfrac{\partial\alpha}{\partial\beta} \neq 0$

From the equations (5) and (23)

$$h_T(\theta) = Z\,m\,x\,\frac{\sin\gamma\left(\dfrac{1}{\ell_C} - \dfrac{1}{2\,R\cos\gamma}\right)}{-\dfrac{1}{2\cos\gamma}\cdot\dfrac{\cos^2\gamma}{\ell_C} + \dfrac{1}{R} + tg\,\gamma\,x\,\dfrac{1}{\sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right)}\,x\,\theta$$

$$h_T(\theta) = Z\,m\,x\,\frac{\sin\gamma\left(\dfrac{1}{\ell_C} - \dfrac{1}{2\,R\cos\gamma}\right)}{\dfrac{\cos^2\gamma}{2\ell_C}}\,\theta$$

or

$$\boxed{h_T(\theta) = Zm\,x\,\frac{1}{R}\,\frac{\sin\gamma}{\cos^2\gamma}\left(\ell_C - 2\,R\cos\gamma\right)\,\theta} \qquad (127)$$

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.../..

b) If $\ell_C = 2\,R\cos\gamma$ i.e. $\dfrac{\partial \mathcal{A}}{\partial \beta} = 0$ and $h_T(\theta) = 0$

In that case we may calculate $h_T(\theta^2)$ from the equations (127) and (46).

Then we know that $\dfrac{KI}{Ko} = -\dfrac{\cos\gamma}{2\sin\gamma}\cdot\dfrac{1}{R}$

$$\frac{K3}{Ko} = -\frac{\cos 2\gamma}{\sin 2\gamma}\cdot\frac{1}{R}$$

$$h_T(\theta^2) = Z\,m \times \frac{\dfrac{1}{2R\cos^2\gamma}}{-\dfrac{\cos\gamma}{2\,\ell_C} + \dfrac{1}{R} - tg\,\gamma\,\dfrac{\cos\gamma}{2\sin\gamma}\cdot\dfrac{1}{R}}\cdot\theta^2$$

$$h_T(\theta^2) = Z\,m \times \frac{\dfrac{1}{2\,R\cos^2\gamma}}{\dfrac{1}{4\,R}}\,\theta^2$$

$$\boxed{h_T(\theta^2) = Z\,m \times \frac{2}{\cos^2\gamma}\,\theta^2} \qquad (128)$$

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3) <u>B at H</u>

General conditions for $\mathcal{A} = 0$

$$\begin{cases} \ell_D = R \quad \ell_A = \ell_C \\ \delta = 0 \quad \alpha = \gamma \\ \dfrac{KI}{Ko} = \dfrac{1}{\sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \\ \dfrac{K3}{Ko} = \dfrac{1}{\sin\gamma}\left(\dfrac{1}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \end{cases}$$

Specific conditions linked to the fact that B is at H

$$\begin{cases} \sin\beta = \sin\eta = m \sin\gamma \\ \ell_B = \ell_H = m\,\ell_C \\ \dfrac{\lambda}{\lambda_0} = 1 + m \end{cases}$$

Now we have to use a dissimilar method (with regards to the one used previously for B at O and B at C) ; effectively the introduction of the specific conditions linked to the fact that B is at H into the general equation $\dfrac{\partial \mathcal{A}}{\partial \beta} = 0$, would lead to very complicated calculations, without interest.

Let us consider the equation (121) which gives the general condition for avoiding the astigmatism when A and D are at one of the privileged points, i.e.

$$\mathcal{A} = \frac{\sin\beta \,.\, \sin(\gamma-\beta)}{\sin\gamma(\cos\gamma + \cos\beta)} \cdot \left(\frac{\sin\beta}{R} - \frac{\sin(\beta+\gamma)}{\ell_C}\right) = 0$$

Then, we have a product that we may write as follows :

$$\mathcal{A} = U \times V \quad \text{with } U = \frac{\sin\beta \,.\, \sin(\gamma-\beta)}{\sin\gamma(\cos\gamma + \cos\beta)}$$

$$V = \frac{\sin\beta}{R} - \frac{\sin(\beta+\gamma)}{\ell_C}$$

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.../..

So $$\frac{\partial \mathcal{A}}{\partial \beta} = U \times \frac{\partial V}{\partial \beta} + V . \frac{\partial U}{\partial \beta}$$

Then we have B at H i.e. $V = 0$

So $$\frac{\partial \mathcal{A}}{\partial \beta} = U \times \frac{\partial V}{\partial \beta}$$

and

$$\frac{\partial \mathcal{A}}{\partial \beta} = \frac{\sin \beta \, . \, \sin (\gamma - \beta)}{\sin \gamma \, (\cos \gamma + \cos \beta)} \left[\frac{\cos \beta}{R} - \frac{\cos (\beta + \gamma)}{\ell_C} \right]$$

If we want $\frac{\partial \mathcal{A}}{\partial \beta} = 0$ it is necessary that

$$\ell_C = \frac{R \cos (\beta + \gamma)}{\cos \beta}$$

Then, we know that if B is at H $\ell_C = \frac{\sin (\beta + \gamma)}{\sin \beta}$

Therefore, it would be necessary that

$$\sin (\beta + \gamma) \cos \beta - \cos (\beta + \gamma) \sin \beta = 0$$

Then this equation is reduced to $\gamma = 0$

Consequently, it is impossible in this configuration, to obtain $\frac{\partial \mathcal{A}}{\partial \beta} = 0$

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Calculation of the focal's height at the vicinity of H

$$h_T(\theta)\ Z.m \times \frac{\dfrac{\sin\beta \sin(\gamma-\beta)}{\sin\gamma(\cos\gamma+\cos\beta)}\left(\dfrac{\cos\beta}{R} - \dfrac{\cos(\beta+\gamma)}{\ell_C}\right)}{-\dfrac{1}{(\cos\gamma+\cos\beta)}\dfrac{\cos^2\gamma}{\ell_C} + \dfrac{1}{R} + \dfrac{\sin\gamma+\sin\beta}{\cos\gamma+\cos\beta}\dfrac{1}{\sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right)} \cdot \theta$$

with $\ell_C = R\dfrac{\sin(\beta+\gamma)}{\sin\beta}$

that may be written :

$$h_T(\theta) = Zm \times \frac{\dfrac{\sin\beta \sin(\gamma-\beta)}{\sin\gamma(\cos\gamma+\cos\beta)}\left(\dfrac{\cos\beta}{R} - \dfrac{\cos(\beta+\gamma)}{\sin(\beta+\gamma)}\cdot\dfrac{\sin\beta}{R}\right)}{-\dfrac{\cos^2\gamma}{(\cos\gamma+\cos\beta)}\dfrac{1}{\ell_C} + \dfrac{1}{R} + \dfrac{\sin\gamma+\sin\beta}{\cos\gamma+\cos\beta} \times \dfrac{1}{R\sin\gamma}\dfrac{\sin\beta}{\sin(\beta+\gamma)}\cos^2\gamma - \dfrac{\cos\gamma}{\sin\gamma} \times \dfrac{\sin\gamma+\sin\beta}{\cos\gamma+\cos\beta}\cdot\dfrac{1}{R}}$$

$$h_T(\theta) = Zm \frac{\dfrac{\sin\beta \sin(\gamma-\beta)}{\sin\gamma(\cos\gamma+\cos\beta)}\left[\dfrac{\cos\beta}{R} - \dfrac{\cos(\beta+\gamma)}{\sin(\beta+\gamma)}\cdot\dfrac{\sin\beta}{R}\right]}{\dfrac{1}{R}\left(1 - \dfrac{\cos\gamma}{\sin\gamma}\dfrac{\sin\gamma+\sin\beta}{\cos\gamma+\cos\beta}\right) - \dfrac{\cos^2\gamma \times \sin\beta}{R(\cos\beta+\cos\gamma)(\sin(\beta+\gamma)} + \dfrac{1}{R}\dfrac{\sin\gamma+\sin\beta}{\cos\gamma+\cos\beta}\dfrac{\sin\beta\cos^2\gamma}{\sin\gamma\sin(\beta+\gamma)}} \times \theta$$

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$$h_T(\theta) = Zm \times \frac{\dfrac{\sin\beta \sin(\gamma-\beta)}{\sin\gamma(\cos\gamma + \cos\beta)} \left[\dfrac{\cos\beta}{R} - \dfrac{\cos(\beta+\gamma)\sin\beta}{\sin(\beta+\gamma)} \dfrac{1}{R} \right]}{\dfrac{1}{R} \dfrac{\sin(\beta-\gamma)}{\sin\gamma(\cos\gamma + \cos\beta)} - \dfrac{\cos^2\gamma \sin\beta}{R \sin(\beta+\gamma)(\cos\beta + \cos\gamma)} \dfrac{(\sin\beta + \sin\gamma}{\sin\gamma} - 1)} \times \theta$$

$$h_T(\theta) = Z\ m \frac{\sin\beta \sin(\gamma-\beta) \left[\dfrac{\cos\beta}{} - \dfrac{\cos(\beta+\gamma)\sin\beta}{\sin(\beta+\gamma)} \right]}{\sin(\beta-\gamma) - \dfrac{\cos^2\gamma \sin^2\beta}{\sin(\beta+\gamma)}} \times \theta$$

$$h_T(\theta) = Z\ m \times \frac{\sin\beta \sin(\gamma-\beta) \sin(\beta+\gamma-\beta)}{\sin(\beta-\gamma)\sin(\beta+\gamma) - \cos^2\gamma \sin^2\beta} \times \theta$$

$$= Z\ m \times \frac{\sin\beta \sin(\gamma-\beta) \sin\gamma}{\sin^2\beta \cos^2\gamma - \sin^2\gamma \cos^2\beta - \cos^2\gamma \sin^2\beta} \times \theta$$

Finally :

(129) $$\boxed{h_T(\theta) = Z\ m \frac{\sin\beta \sin(\gamma-\beta)}{\sin\gamma \cos^2\beta} \times \theta}$$

STUDY OF THE ABERRATIONS AT THE VICINITY OF THE STIGMATIC POINTS

VIII - 4 STUDY OF THE COMA

We are going to use a method identical with the one applied to the case of the astigmatism.

First, we may put in practice the conclusions of our general study and say that the coma located at any point B - in the case of the points A and D being one of the privileged points - is not depending on the special locus of A and D among these privileged points.

We choose, as special case, to place D at O (centre of the circle) and A at C

So, it results $\ell_D = R$ $\qquad \ell_A = \ell_C$

$\delta = 0$ $\qquad \alpha = \gamma$

We know that for having the coma equal to zero while remaining on the locus of the tangential focal length, the two equations $T = 0$ and $C = 0$ have to be resolved simultaneously :

$$T = \frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\ \frac{KI}{Ko} = 0$$

$$C = \frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - (\sin\alpha + \sin\beta)\ \frac{K2}{Ko} = 0$$

.../..

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Introducing the initial conditions

$$\begin{cases} \dfrac{KI}{Ko} = \dfrac{1}{\sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \\[2ex] \dfrac{K2}{Ko} = \dfrac{1}{\ell_C}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \end{cases}$$

The equations may be written as follows :

$$T = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - \frac{(\sin\gamma + \sin\beta)}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) = 0$$

$$C = \frac{\sin\gamma}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{(\sin\gamma + \sin\beta)}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) = 0$$

From $T = 0$ we have :

$$\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left(\frac{\sin\gamma + \sin\beta}{\sin\gamma} - 1\right)$$

$$\boxed{\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{\sin\beta}{\sin\gamma}}$$

$$C = 0\ \frac{\sin\gamma}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{\sin\beta}{\sin\gamma} - (\sin\gamma + \sin\beta)\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) = 0$$

$$C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[\frac{\sin\gamma}{\ell_C} + \frac{\sin^2\beta}{\ell_B \sin\gamma} - \frac{\sin\gamma + \sin\beta}{\ell_C}\right]$$

$$\boxed{C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\sin\beta\left(\frac{\sin\beta}{\ell_B \sin\gamma} - \frac{1}{\ell_C}\right)} \qquad (130)$$

We have $\dfrac{1}{\ell_B} = \dfrac{1}{R\cos\beta} + \dfrac{\sin\beta}{\cos^2\beta\ \sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right)$

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By carrying forward in the formula (130) we have :

$$C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \sin\beta \left[\frac{\sin\beta}{\sin\gamma}\,\frac{1}{R\cos\beta} + \frac{\sin^2\beta}{\cos^2\beta\,\sin^2\gamma}\left[\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right] - \frac{1}{\ell_C}\right]$$

$$C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \sin\beta \left[\frac{\sin\beta}{R}\left(\frac{1}{\sin\gamma\cos\beta} - \frac{\cos\gamma\sin\beta}{\cos^2\beta\,\sin^2\gamma}\right) + \frac{1}{\ell_C}\left(\frac{\cos^2\gamma\sin^2\beta}{\cos^2\beta\,\sin^2\gamma} - 1\right)\right]$$

$$C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \sin\beta \left[\frac{\sin\beta}{R}\,\frac{(\cos\beta\sin\gamma - \cos\gamma\sin\beta)}{\cos^2\beta\,\sin^2\gamma} + \frac{1}{\ell_C}\,\frac{\cos^2\gamma\sin^2\beta - \cos^2\beta\,\sin^2\gamma}{\cos^2\beta\,\sin^2\gamma}\right]$$

$$C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \frac{\sin\beta}{\cos^2\beta\,\sin^2\gamma}\left[\frac{\sin\beta}{R}\sin(\gamma-\beta) + (\cos\gamma\sin\beta + \cos\beta\sin\gamma)(\cos\gamma\sin\beta - \cos\beta\sin\gamma)\right]$$

$$C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \frac{\sin\beta}{\cos^2\beta\,\sin^2\gamma}\left[\frac{\sin\beta\,\sin(\gamma-\beta)}{R} + \frac{\sin(\beta+\gamma)\,\sin(\beta-\gamma)}{\ell_C}\right]$$

Finally :

$$(131) \quad \boxed{C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \frac{\sin\beta\,\sin(\gamma-\beta)}{\cos^2\beta\,\sin^2\gamma}\left[\frac{\sin\beta}{R} - \frac{\sin(\beta+\gamma)}{\ell_C}\right]}$$

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That expression may be equal to zero

1) if $\frac{\cos^2\gamma}{\ell} - \frac{\cos\gamma}{R} = 0$ i.e. if $\underline{KI = K2 = 0}$

we have studied that case page 54 and following. It is the Rowland circle solution.

2) if $\sin\beta = 0$ $\beta = 0$

B is at O : it is one of the stigmatic points

3) if $\sin(\gamma - \beta) = 0$

$\gamma = \beta$

B is at C : it is the second stigmatic point

4) $\frac{\sin\beta}{R} = \frac{\sin(\beta+\gamma)}{\ell_C}$

We have seen, in Chapter VIII-3 called "Astigmatism at the vicinity of stigmatic points" that this relation defined the third stigmatic point.

Conclusion

Except the solution $\ell_C = R\cos\gamma$ which corresponds to the conditions of the Rowland circle, and under the following conditions : D at O or at H, A at O, C or H, there is no other possibility to have the coma zero, than to choose one of the points O, C or H as point Bo.

The question we ask now is : Bo being one of these points, is it possible to have $\frac{\partial c}{\partial \beta}$ equal to zero ?

We keep the special conditions chosen as above, i.e. D at O and A at C which have - as demonstrated - a general value.
We choose for B the 3 privileged loci : B at O - B at C - B at H.

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Let us reconsider the equation (88) which gives the value of $\frac{\partial c}{\partial \beta}$ and the equation (89) providing the value of $\frac{\partial^2 c}{\partial \beta^2}$

$$\frac{\partial c}{\partial \beta} = \frac{KI}{R\,Ko} \sin\beta_o - \frac{K2}{Ko} \cos\beta_o + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right) \left[2\,\frac{KI}{Ko}\, tg\,\beta_o + \frac{\sin\beta_o}{\ell_{Bo}} \left(2\, tg\,\beta_o + \frac{1}{tg\,\beta_o} \right) - \frac{tg^2\beta_o}{R} \right]$$

$$\frac{\partial^2 c}{\partial \beta^2} = \frac{KI}{R\,Ko} \frac{\sin^2\beta_o}{\cos\beta_o} \left(\frac{1}{2} + \frac{1}{tg^2\beta_o} \right) + \left(\frac{KI}{Ko}\right)^2 \sin\beta_o$$

$$+ 2\,\frac{KI}{Ko}\, tg^2\beta_o \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right) \left(\frac{3}{2} + \frac{1}{tg^2\beta_o} \right) + \frac{K2}{Ko} \frac{\sin\beta_o}{2}$$

$$+ \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right) \left[\frac{\sin\beta_o}{\ell_{Bo}} \left(\frac{5}{2} + 3\,tg^2\beta_o \right) - \frac{tg\,\beta_o}{R} \left(\frac{3}{2} + 2\,tg^2\beta_o \right) \right]$$

1) <u>We have Bo at O</u>

In that case

$$\begin{cases} \ell_D = R \qquad \ell_A = \ell_C \\ \delta = 0 \quad \alpha = \gamma \\ \frac{KI}{Ko} = \frac{1}{\sin\gamma} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \\ \frac{K2}{Ko} = \frac{1}{\ell_C} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \end{cases}$$

$$\beta = 0$$

$$\ell_B = R \qquad \lambda = \lambda_o$$

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$$\frac{\partial c}{\partial \beta} = -\frac{K2}{Ko}$$

The only solution is $\frac{K2}{Ko} = 0$

So, we may write, in the general case

(132) $$\boxed{\frac{\partial c}{\partial \beta} = -\frac{K2}{Ko}\,\theta}$$

Therefore, one has the following possibilities for obtaining the coma equal to zero :

$\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} = 0 \longrightarrow$ We find again the Rowland circle solution (2nd report, page 33).

The other solution is $\ell_C = \infty$

In this case $\frac{K2}{Ko} = 0$ $\quad \frac{KI}{Ko} = -\frac{\cos\gamma}{R\sin\gamma}$

Let us study $\frac{\partial^2 c}{\partial \beta^2}$ under the conditions : $\sin\beta_0 = 0$ $\ell_{B_0} = R$

(133) $$\boxed{\frac{\partial^2 c}{\partial \beta^2} = \frac{1}{R}\,\frac{KI}{Ko}\,\theta^2 = -\frac{\cos\gamma}{R^2\sin\gamma}\,\theta^2}$$

In fact, that is a particular case which will be examined in chapter "Wadsworth".

.../..

2) We have Bo at C

In that case

$$\ell_D = R \qquad \ell_A = \ell_C$$

$$\delta = 0 \qquad \alpha = \gamma$$

General conditions

$$\frac{KI}{Ko} = \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\frac{K2}{Ko} = \frac{1}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

Specific conditions linked to the fact that Bo is at C

$$\beta = \gamma \qquad \ell_B = \ell_C$$

$$\lambda = 2\,\lambda_0$$

$$\frac{\partial c}{\partial \beta} = \frac{KI}{R\,Ko}\sin\gamma - \frac{K2}{Ko}\cos\gamma + \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[\frac{2\,KI}{Ko}\,tg\gamma + \frac{\sin\gamma}{\ell_C}\left(2\,tg\gamma + \frac{1}{tg\gamma}\right) - \frac{tg^2\gamma}{R}\right]$$

$$\frac{\partial c}{\partial \beta} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[\frac{1}{R} - \frac{\cos\gamma}{\ell_C} + \frac{2}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)tg\,\gamma + \frac{\sin\gamma}{\ell_C}\left(2\,tg\gamma + \frac{1}{tg\gamma}\right) - \frac{tg^2\gamma}{R}\right]$$

$$\frac{\partial c}{\partial \beta} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[\frac{1}{R}\left(1 - \frac{2\cos\gamma}{\sin\gamma}tg\gamma - tg^2\gamma\right) + \frac{1}{\ell_C}\left[-\cos\gamma + 2\,\frac{\cos^2\gamma}{\sin\gamma}tg\,\gamma + \left(2\,tg\gamma + \frac{1}{tg\gamma}\right)\sin\gamma\right]\right]$$

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$$\frac{\partial c}{\partial \beta} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[-\frac{1}{R\cos^2\gamma} + \frac{1}{\ell_C}\left(\cos\gamma + \frac{2\sin^2\gamma}{\cos\gamma} + \cos\gamma\right)\right]$$

(134) $$\boxed{\frac{\partial c}{\partial \beta} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[-\frac{1}{R\cos^2\gamma} + \frac{1}{\ell_C}\left(\frac{2}{\cos\gamma}\right)\right]\theta}$$

$\frac{\partial c}{\partial \beta}$ may be zero if

$$\boxed{\ell_C = 2\,R\cos\gamma}$$

Under these conditions

$$\frac{KI}{Ko} = \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) = -\frac{\cos\gamma}{2\sin\gamma}\cdot\frac{1}{R}$$

$$\frac{K2}{Ko} = \frac{1}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) = -\frac{1}{4R^2}$$

Let us study $\frac{\partial^2 c}{\partial \beta^2}$ under these conditions.

To make easier the calculations, we shall keep $\frac{KI}{Ko}$ and $\frac{K2}{Ko}$ under their general form

$$\frac{\partial^2 c}{\partial \beta^2} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{1}{\sin\gamma}\,\frac{\sin^2\gamma}{R\cos\gamma}\left(\frac{1}{2} + \frac{1}{tg^2\gamma}\right)$$

$$+\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)^2\frac{\sin\gamma}{\sin^2\gamma} + 2\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)^2\frac{tg^2\gamma}{\sin\gamma}\left(\frac{3}{2} + \frac{1}{tg^2\gamma}\right)$$

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$$+ \frac{1}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{\sin\gamma}{2} + \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[\frac{\sin\gamma}{\ell_C}\left(\frac{5}{2} + 3\,tg^2\gamma\right)\right.$$

$$\left. - \frac{tg\,\gamma}{R}\left(\frac{3}{2} + 2\,tg^2\gamma\right)\right]$$

that may be written :

$$\frac{\partial^2 C}{\partial \beta^2} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[\frac{tg\,\gamma}{R}\left(\frac{1}{2} + \frac{1}{tg^2\gamma}\right) + \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{1}{\sin\gamma}\right.$$

$$+ 2\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left(\frac{3\,tg^2\gamma}{2\sin\gamma} + \frac{1}{\sin\gamma}\right) + \frac{1}{\ell_C}\,\frac{\sin\gamma}{2} +$$

$$\left.\frac{\sin\gamma}{\ell_C}\left(\frac{5}{2} + 3\,tg^2\gamma\right) - \frac{tg\,\gamma}{R}\left(\frac{3}{2} + 2\,tg^2\gamma\right)\right]$$

$$\frac{\partial^2 C}{\partial \beta^2} = -\frac{\cos\gamma}{2R^2}\left[\frac{tg\,\gamma}{2} + \frac{1}{tg\,\gamma} - \frac{\cos\gamma}{2\sin\gamma} - \cos\gamma \times \frac{3}{2}\,\frac{tg^2\gamma}{\sin\gamma}\right.$$

$$- \frac{\cos\gamma}{\sin\gamma} + \frac{\sin\gamma}{4\cos\gamma} + \frac{5}{4}\,\frac{\sin\gamma}{\cos\gamma} + \frac{3\,tg^2\gamma\sin\gamma}{2\cos\gamma}$$

$$\left. - \frac{3\sin\gamma}{2\cos\gamma} - 2\,tg^3\gamma\right]$$

$$\frac{\partial^2 C}{\partial \beta^2} = -\frac{\cos\gamma}{2R^2}\left[-\frac{tg^3\gamma}{2} - tg\,\gamma - \frac{1}{2\,tg\,\gamma}\right] =$$

$$= \frac{\cos^2\gamma}{4R^2\sin\gamma}\left(tg^4\gamma + 2\,tg^2\gamma + 1\right) = \frac{\cos^2\gamma}{4R^2\sin\gamma}\left(1 + tg^2\gamma\right)^2$$

Finally :

$$\boxed{\frac{\partial^2 C}{\partial \beta^2} = \frac{1}{4R^2\sin\gamma\cos^2\gamma}\,\theta^2} \qquad (135)$$

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3) We have B at H

We know that in this case :

General conditions

$$\begin{cases} \ell_D = R \quad \ell_A = \ell_C \\ \delta = 0 \quad \alpha = \gamma \\ \dfrac{KI}{Ko} = \dfrac{1}{\sin\gamma}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \\ \dfrac{K2}{Ko} = \dfrac{1}{\ell_C}\left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right) \end{cases}$$

Specific conditions linked to the fact that B is at H

$$\ell_C = R\,\frac{\sin(\beta+\gamma)}{\sin\beta}$$

For that case, we use another method and consider C under the form of the equation (131)

$$C = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\frac{\sin\beta\sin(\gamma-\beta)}{\cos^2\beta\,\sin^2\gamma}\left[\frac{\sin\beta}{R} - \frac{\sin(\beta+\gamma)}{\ell_C}\right]$$

We know that effectively this function is zero as regards the initial chosen conditions.

We have $C = U \times V$

with $U = \left(\dfrac{\cos^2\gamma}{\ell_C} - \dfrac{\cos\gamma}{R}\right)\dfrac{\sin\beta\,\sin(\beta-\gamma)}{\cos^2\beta\,\sin^2\gamma}$

$$V = \frac{\sin\beta}{R} - \frac{\sin(\beta+\gamma)}{\ell_C}$$

$$\frac{\partial C}{\partial\beta} = V\frac{\partial U}{\partial\beta} + U\frac{\partial V}{\partial\beta}$$

Then we know that $V = 0$

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So : $$\frac{\partial c}{\partial \beta} = U \times \frac{\partial V}{\partial \beta}$$

and $$\frac{\partial c}{\partial \beta} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \frac{\sin\beta \sin(\beta-\gamma)}{\cos^2\beta \sin^2\gamma} \left[\frac{\cos\beta}{R} - \frac{\cos(\beta+\gamma)}{\ell_C}\right]$$

Under the conditions we have chosen, there is no other way to get that function equal to zero than to have

$$\ell_C = R \frac{\cos(\beta+\gamma)}{\cos\beta}$$

Then, this condition is incompatible with the condition $C = 0$

$$\ell_C = R \frac{\sin(\beta+\gamma)}{\sin\beta}$$

Therefore, in this case, it is impossible to obtain $\frac{\partial c}{\partial \beta} = 0$ and the value of the coma is

$$\frac{\partial c}{\partial \beta} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \frac{\sin\beta \sin(\gamma-\beta)}{\cos^2\beta \sin^2\gamma} \left[\frac{\cos\beta}{R} - \frac{\cos(\beta+\gamma)}{\ell_C}\right] \theta$$

$$\text{With} \quad \ell_C = R \frac{\sin(\beta+\gamma)}{\sin\beta}$$

and we write :

$$\frac{\partial c}{\partial \beta} = \frac{1}{R}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \frac{\sin\beta \sin(\gamma-\beta)}{\cos^2\beta \sin^2\gamma} \times \frac{\sin\gamma}{\sin(\gamma+\beta)} \theta$$

(136) $$\boxed{\frac{\partial c}{\partial \beta} = \frac{1}{R}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) \frac{\sin\beta \sin(\gamma-\beta)}{\cos^2\beta \sin\gamma \sin(\gamma+\beta)} \theta}$$

.../..

VIII-5- STUDY OF THE COMBINATION COMA-ASTIGMATISM AT THE VICINITY OF STIGMATIC POINTS.

The study of the astigmatism on one hand, of the coma on the other hand, has led to the following results :

B at O $\qquad K_o = \sin\gamma$

$$h_T(\theta) = Z\,m \frac{R \sin\gamma}{\ell_C}\,\theta$$

$$\frac{\partial c}{\partial \beta} = -\frac{K2}{K_o}\,\theta$$

Then $$\frac{K2}{K_o} = \frac{1}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

Therefore there are two available solutions for having $\frac{K2}{K_o}$ equal to zero :

a) $\ell_C = R\cos\gamma$

which involves $K1 = K2 = 0$

and $K3 = \frac{\sin^2\gamma}{R\cos\gamma}$

In that case $h_T(\theta) = Zm \frac{R\sin\gamma}{R\cos\gamma}\,\theta$

(137) $$\boxed{h_T(\theta) = Zm\,tg\,\gamma\,.\,\theta}$$

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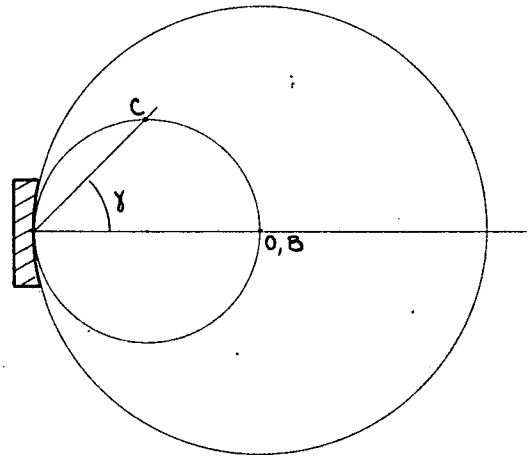
We easily observe that, in this case, $\frac{\partial^2 C}{\partial \beta^2} = 0$



- Fig. n° 21 -

So, A may be located either at O or at C or at H. Obviously, there is no interest in having A located at O.

The only interesting solution is A at C or at H.

If A is at C, it is a configuration in which A is on the Rowland circle and, obviously, the locus of the tangential focal length is that same Rowland circle (as C and D are on the Rowland circle as well.).

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From the general properties of the configurations of that type, it results that the coma is null for all the wavelengths.

The correcting wavelength which corresponds to B located at O is λ^* so that

$$\sin\alpha + \sin\beta = \frac{\lambda^*}{\lambda_0}(\sin\gamma - \sin\delta)$$

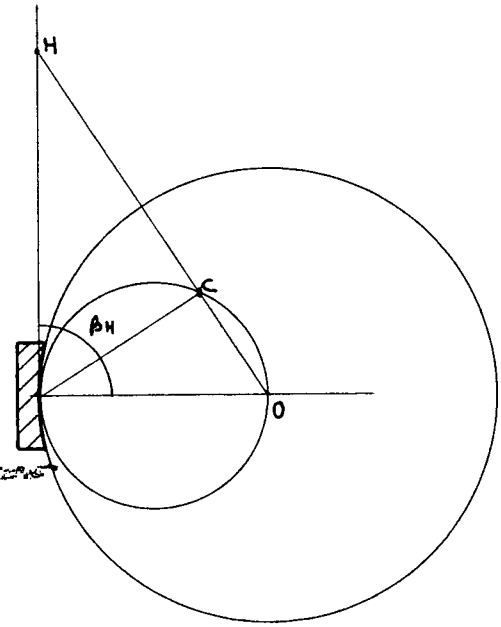
with $\delta = 0$

$$\beta = 0 \qquad \alpha = \gamma$$

Therefore $\lambda^* = \lambda_0$

If A is at H we know that $\lambda^* = m\,\lambda_0$

In this case, m may be easily determined



- Fig.n° 22 -

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$$m = \frac{R}{OC} = \frac{1}{\sin\gamma} \qquad \ell_H = m\,\ell_C = \frac{R}{tg\,\gamma}$$

It results $\sin\beta_H = 1$ so $\beta_H = \frac{\pi}{2}$

So
$$\boxed{\lambda^* = \frac{\lambda_0}{\sin\gamma}} \qquad (138)$$

Le point H is a singular point of the spectrum.

Let us remind that, if the conventional grating were concerned in the same conditions i.e. the source point A on the Rowland circle and the point image B located at the centre of curvature of the grating corresponding to the wavelength λ_0, the focal's height would be :

$$h_T = Z\,m\,(\sin^2\beta + \sin\alpha\, tg\,\alpha \cos\beta)$$

(Equation 39) ———)

with $\beta = 0$ therefore
$$\boxed{h_T = Z\,m\,\frac{\sin^2\alpha}{\cos\alpha}} \qquad (139)$$

Likewise, in the holographic grating case, we observe that, for that configuration, the coefficient of Y^4 is null (spherical aberration).

As regards the conventional grating with the same parameters, the

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coefficient of Y^4 is

$$\boxed{\Delta^{(4)} = \frac{Y^4}{8R^3} \frac{\sin^2\alpha}{\cos\alpha}} \qquad (140)$$

b) $\ell_C = \infty$ which involves $K2 = 0$

<u>Wadsworth mounting.</u>

$$KI = -\frac{\cos\gamma}{R}$$

$$K3 = -\frac{\cos\gamma}{R}$$

We observe in that case (equation 133) $\dfrac{\partial^2 C}{\partial\beta^2} = -\dfrac{\cos\gamma}{R^2\sin\gamma}\,\theta^2$

and $h_T(\theta^2) = Z\,m\,x\,\theta^2$

More generally, it is easy to see that

$$h_T = Z\,m\,\sin^2\beta$$

The equation $T = 0$ is written as follows :

$$-\frac{\cos\gamma}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} + (\sin\beta + \sin\gamma)\frac{\cos\gamma}{R\sin\gamma} = 0$$

$$\frac{\cos^2\beta}{\ell_B} + \frac{\sin(\beta-\gamma)}{R\sin\gamma} = 0$$

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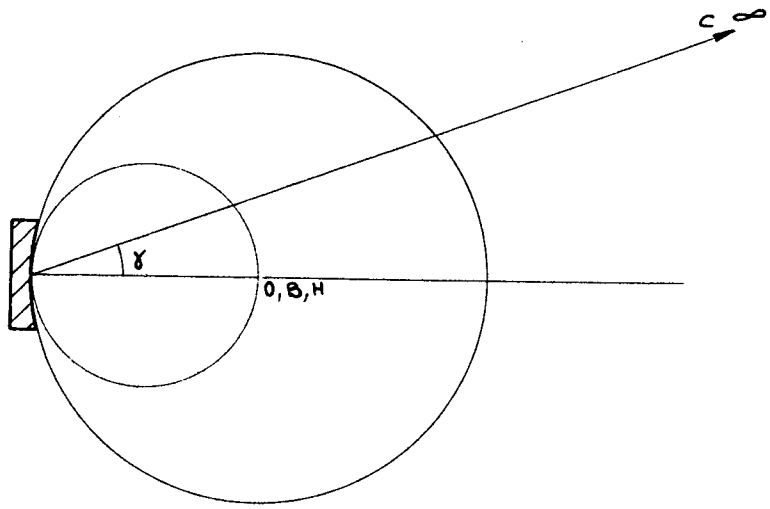

- Fig. n° 23 -

If C is located at infinity H is at O; The locus of A at O is without interest. A may be located only at C. So it results, as previously, that the correcting wavelength $\lambda^* = \lambda_0$

Reconsidering the formula (38) giving the focal's height of the conventional grating in the general case :

$$h_T = Zm \frac{\sin^2\beta - \frac{R}{\ell_A}(\cos\alpha - \cos\beta)}{1 - \frac{R}{\ell_A}\frac{\cos^2\alpha}{\cos\alpha + \cos\beta}}$$

In the Wadsworth mounting case, $\ell_A = \infty$

and the formula is reduced to $h_T = Z\,m \times \sin^2\beta$ which is precisely the same one obtainable in the holographic grating case.

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Therefore, the use of a holographic grating brings no improvement to the classical Wadsworth mounting as regards the astigmatism.

<u>Coma</u> : we may easily observe that the coma is null at B when B corresponds to $\beta = 0$ in the classical Wadsworth mounting case:

$$C = \frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) = 0$$

if $\ell_A = \infty$ and $\beta = 0$ $C = 0$

Let us calculate $\frac{\partial C}{\partial \beta}$: Reconsidering the equation (88) in the classical Wadsworth mounting conditions i.e.

$KI = K2 = 0$ $\beta = 0$

$$\frac{1}{\ell_B} = \frac{1 + \cos\alpha}{R}$$

$$\frac{\partial C}{\partial \beta} = \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right)\left[\frac{\cos\beta_o}{\ell_{Bo}}\right]\theta$$

$$= \left(\frac{1 + \cos\alpha}{R} - \frac{1}{R}\right)\left(\frac{1 + \cos\alpha}{R}\right)\theta$$

$$(141) \qquad \boxed{\frac{\partial C}{\partial \beta} = \cos\alpha \frac{1 + \cos\alpha}{R^2}\theta}$$

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Then, it has been observed that in the holographic Wadsworth mounting

$$\frac{\partial c}{\partial \beta} = 0 \quad \frac{\partial^2 c}{\partial \beta^2} = -\frac{\cos\gamma}{R^2 \sin\gamma}\,\theta^2$$

We know that, as B located at O is a stigmatic point, the coefficient of the Term in Y^4 is null in the holographic grating case.

One may calculate it in the conventional grating case :

$$\Delta^{(4)} = Y^4 \left[-\frac{\cos\alpha}{8R^3} + \frac{1}{8R^2}\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{1}{8\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right)\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R} - \frac{5\sin^2\beta}{\ell_B}\right)\right]$$

with $\frac{1}{\ell_B} = \frac{1+\cos\alpha}{R}$

$$\Delta^{(4)} = Y^4 \left[-\frac{\cos\alpha}{8R^3} + \frac{1}{8R^2}\left(\frac{1+\cos\alpha}{R} - \frac{1}{R}\right) - \frac{1+\cos\alpha}{8R}\left(\frac{1+\cos\alpha}{R} - \frac{1}{R}\right)\left(\frac{1+\cos\alpha}{R} - \frac{1}{R}\right)\right]$$

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$$\Delta^{(4)} = Y^4 \left[-\frac{\cos\alpha}{8R^3} + \frac{1}{8R^2} \frac{\cos\alpha}{R} - \frac{1+\cos\alpha}{8R} \times \frac{\cos^2\alpha}{R^2} \right]$$

$$(142) \quad \Delta^{(4)} = -\frac{Y^4}{8R^3} (1+\cos\alpha) \cos^2\alpha$$

<u>B at C</u>

$$Ko = \sin\gamma$$

$$\frac{KI}{Ko} = \frac{1}{\sin\gamma} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\frac{K3}{Ko} = \frac{1}{\sin\gamma} \left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\frac{K2}{Ko} = \frac{1}{\ell_C} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$h_T(\theta) = Zm \frac{\sin\gamma\left(\frac{1}{\ell_C} - \frac{1}{2R\cos\gamma}\right)}{\frac{\cos\gamma}{2\ell_C}} \times \theta$$

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$$\frac{\partial c}{\partial \beta} = \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left[-\frac{1}{R\cos^2\gamma} + \frac{1}{\ell_C} \times \frac{2}{\cos\gamma}\right] \times \theta.$$

So, there are two available solutions for having $\frac{\partial c}{\partial \beta}$ equal to zero.

1) $\ell_C = R\cos\gamma$ which involves $\begin{cases} K1 = 0 \\ K2 = 0 \\ K3 = \dfrac{\sin^2\gamma}{R\cos\gamma} \end{cases}$

In that case :

$$h_T(\theta) = Zm \cdot \frac{\sin\gamma\left(\frac{1}{\cos\gamma} - \frac{1}{2\cos\gamma}\right)}{\frac{\cos\gamma}{2\cos\gamma}} \times \theta$$

(143) $$\boxed{h_T(\theta) = Zm \times \frac{\sin\gamma}{\cos\gamma} \cdot \theta}$$

$$\frac{\partial^2 c}{\partial \beta^2} = 0$$

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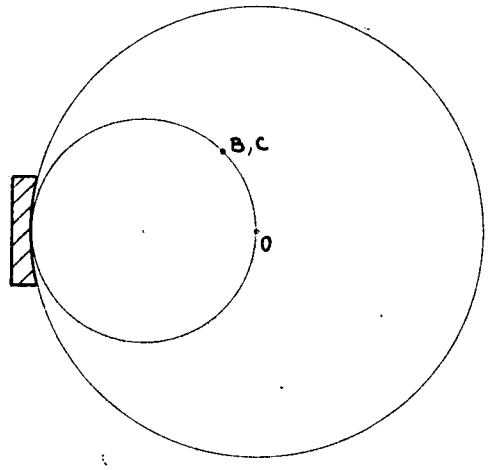

- Fig. n° 24 -

Therefore, one may locate A either at O or at C (autocollimation) or at H.

If A is at O :

$\sin\alpha + \sin\beta = \frac{\lambda^*}{\lambda_0}(\sin\gamma - \sin\delta)$ is written as follows :

$\sin\gamma = \frac{\lambda^*}{\lambda_0}\sin\gamma \longrightarrow$ so $\lambda^* = \lambda_0$

If A is at C :

$\sin\gamma + \sin\gamma = \frac{\lambda^*}{\lambda_0}\sin\gamma \longrightarrow$ so $\lambda^* = 2\lambda_0$

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2) $\ell_C = 2\,R\cos\gamma$

which involves
$$\begin{cases} KI = -\dfrac{\cos\gamma}{R} \\ K2 = -\dfrac{\sin\gamma}{4R^2} \\ K3 = -\dfrac{\cos^2\gamma}{2\cos\gamma}\cdot\dfrac{1}{R} \end{cases}$$

Under those conditions :

$$h_T(\theta) = Zm\,\frac{\sin\gamma\left(\dfrac{1}{2\cos\gamma} - \dfrac{1}{2\cos\gamma}\right)}{\dfrac{\cos\gamma}{2\cos\gamma}}\times\theta = 0$$

One may calculate the term in $h_T(\theta^2)$

(144) $$h_T(\theta^2) = Z\,m \times \frac{2}{\cos^2\gamma} \times \theta^2$$

(145) $$\frac{\partial^2 c}{\partial p^2} = \frac{1}{4\,R^2\sin\gamma\cos^2\gamma}\cdot\theta^2$$

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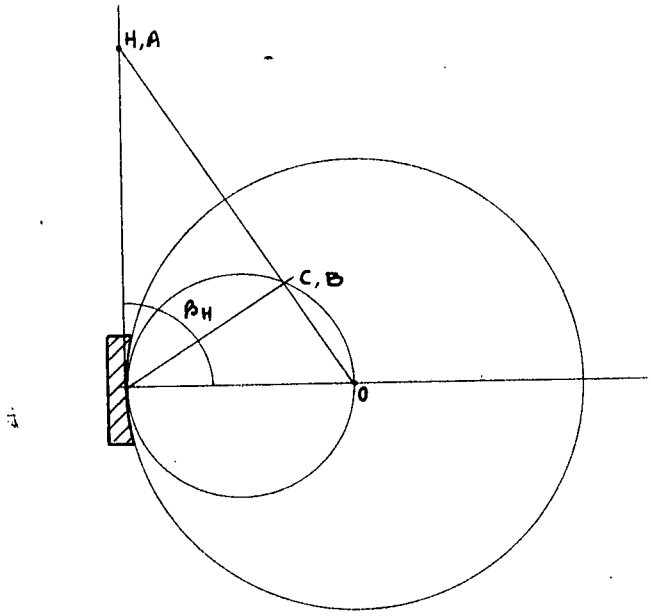

- Fig. n° 25 -

If A is at H :

$$\lambda^* = (m + 1)\lambda_o \text{ with } \ell_C = R \cos \gamma$$

$$OC = R \sin \gamma$$

$$m = \frac{R}{OC} = \frac{R}{R \sin \gamma}$$

$$m = \frac{1}{\sin \gamma}$$

So

(146) $$\boxed{\lambda^* = \frac{1 + \sin \gamma}{\sin \gamma} \lambda_o}$$ with $\sin \beta_H = 1$ $\beta_H = \frac{\pi}{2}$

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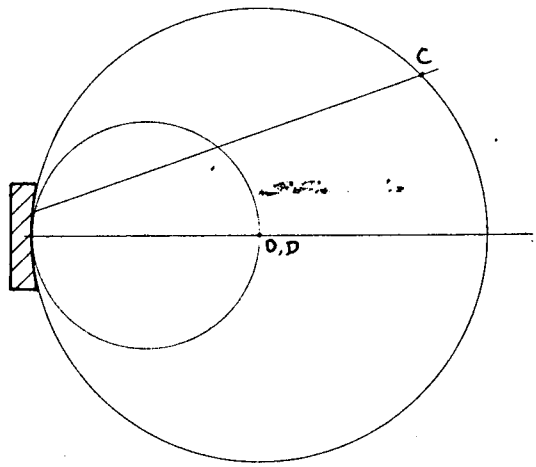

- Fig. n° 26 -

In that case, C and H run together.

A may be, therefore, located either at O or at C (autocollimation).

The wavelenths λ^* corresponding to the perfect stigmatism, are respectively :

A at O $\lambda^* = \lambda_0$

A at C $\lambda^* = 2\lambda_0$

B at H

$$\frac{KI}{Ko} = \frac{1}{\sin\gamma}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\frac{K2}{Ko} = \frac{1}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\ell_C = R\,\frac{\sin(\beta+\gamma)}{\sin\beta}$$

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$$\frac{\partial c}{\partial \beta} = \frac{1}{R} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} \right) \frac{\sin\beta \sin(\gamma-\beta)}{\cos^2\beta \sin\gamma \sin(\gamma+\beta)}$$

(147) $$h_T(\theta) = Z\,m \frac{\sin\beta \sin(\gamma-\beta)}{\sin\gamma \cos^2\beta} \times \theta$$

The only solution for having $\frac{\partial c}{\partial \beta}$ equal to zero consists in choosing $\ell_C = R \cos\gamma$

We may observe that, under those conditions $\frac{\partial^2 c}{\partial \beta^2}$ and the other derived will be null ; effectively :

$$\frac{\partial c}{\partial \beta} = \frac{1}{R} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} \right) F(\beta)$$

The successive derived are :

$$\frac{\partial^n c}{\partial \beta^n} = \frac{1}{R} \left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} \right) \frac{\partial^n F^n}{\partial F^n}(\beta)$$

and $\frac{\partial^n c}{\partial \beta^n} = 0$ if $\ell_C = R \cos\gamma$

A may be located either at O, or at C or at H

A at O $\lambda^* = m\,\lambda_o$

A at C $\lambda^* = (m+1)\lambda_o$

A at H $\lambda^* = 2m\,\lambda_o$

.../..

The Table below summarizes the characteristics of the gratings with stigmatic points, with respect the aberrations, according to the different cases concerned and in connection with the locus of the point B :

ℓ_C \ locus B	O	C	H
General case	$h_T = -Zm\,\frac{R\sin\gamma}{\ell_C}\cdot\theta$ $\frac{\partial c}{\partial\beta} = \frac{-K2}{K_\odot} = -\frac{1}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C}-\frac{\cos\gamma}{R}\right)$	$h_T = -Zm\,\frac{\sin\gamma}{\cos^2\gamma}\left(\frac{2R\cos\gamma-\ell_C}{R}\right)\theta$ $\frac{\partial c}{\partial\beta} = \left(\frac{\cos^2\gamma}{\ell_C}-\frac{\cos\gamma}{R}\right)-\left(\frac{1}{R\cos^2\gamma}+\frac{2}{\ell_C\cos\gamma}\right)$	$h_T = -Zm\,\frac{\sin\beta\sin(\gamma-\beta)}{\sin\gamma\cos^2\beta}$ $\frac{\partial c}{\partial\beta} = \frac{1}{R}\left(\frac{\cos^2\gamma}{\ell_C}-\frac{\cos\gamma}{R}\right)\frac{\sin\beta\sin(\gamma-\beta)}{\cos^2\beta\sin\gamma\sin(\gamma+\beta)}$
$\ell_C = R\cos\gamma$	$h_T = Zm.\,tg\gamma\,.\,\theta$ $\frac{\partial c}{\partial\beta} = \frac{\partial^2 c}{\partial\beta^2} = \dots = 0$	$h_T = Zm.\,tg\gamma\,.\,\theta$ $\frac{\partial c}{\partial\beta} = \frac{\partial^2 c}{\partial\beta^2} = \dots = 0$	
$\ell_C = 2R\cos\gamma$		$h_T = Zm\,.\,\frac{2}{\cos^2\gamma}\,.\,\theta^2$ $\frac{\partial^2 c}{\partial\beta^2} = \frac{1}{4R^2\sin\gamma\cos^2\gamma}$	
$\ell_C = \infty$	$h_T = Zm\,.\,\theta^2 = Zm\sin^2\beta$ $\frac{\partial^2 c}{\partial\beta^2} = \frac{1}{R^2}\,\frac{\cos\gamma}{\sin\gamma}$		

IX - ABERRATIONS AT GRAZING INCIDENCE.

IX-1- STUDY OF THE COMA AT GRAZING INCIDENCE

First, it is necessary to keep locus on the tangential i.e the equation $T = 0$ is verified.

$$\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - (\sin\alpha + \sin\beta)\ \frac{KI}{Ko} = 0$$

If $\alpha \simeq \frac{\pi}{2}$ this equation becomes

$$(148)\quad \frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R} - (1 + \sin\beta)\ \frac{KI}{Ko} = 0 \qquad (48)$$

The general equation of the coma is :

$$C = \frac{Y^3}{2}\left[\frac{\sin\alpha}{\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - (\sin\alpha + \sin\beta)\ \frac{K2}{Ko}\right]$$

If $\alpha = \frac{\pi}{2}$ C is written as follows :

$$C = \frac{Y^3}{2}\left[\frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - (1 + \sin\beta)\ \frac{K2}{Ko}\right]$$

We have :

$$C = \frac{\sin\beta}{\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right) - (1 + \sin\beta)\ \frac{K2}{Ko}$$

$$\Delta C = \frac{Y^3}{2} \cdot C$$

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Taking account of (148) C may be written :

$$C = (1 + \sin\beta)\left[\frac{\sin\beta}{\ell_B}\,\frac{KI}{Ko} - \frac{K2}{Ko}\right] \qquad (149)$$

We notice that the coma will be 0 ($C = 0$) for the point Bo

if $\dfrac{\sin\beta}{\ell_B}\,\dfrac{KI}{Ko} = \dfrac{K2}{Ko}$ i.e. $\boxed{\dfrac{\sin\beta_o}{\ell_{Bo}} = \dfrac{K2}{KI}}$ (150)

That relation may be written also under the form :

$$\frac{1}{\ell_B} = \frac{1}{R\cos\beta} + \frac{1 + \sin\beta}{\cos^2\beta}\,\frac{KI}{Ko}$$

$$(151)\quad \boxed{C = (1 + \sin\beta)\left[\frac{\sin\beta}{R\cos\beta}\,\frac{1}{R}\,\frac{KI}{Ko} + \frac{(1 + \sin\beta)\sin\beta}{\cos^2\beta}\left(\frac{KI}{Ko}\right)^2 - \frac{K2}{Ko}\right]}$$

We have now to find out the conditions allowing the enlargement of that propertie on points B close to Bo, the equation (151) being satisfied i.e. the coma being equal to 0 for the point Bo.

It means that $\dfrac{\partial c'}{\partial\beta} = 0$

However, it will be also possible to calculate the value of C through the Taylor's formula.

$$C(\beta) = C(\beta_o) + \theta\, C'(\beta_o) + \frac{\theta^2}{2}\, C''(\beta_o) + \ldots$$

θ being the increment imposed to β_o

$$\beta = \beta_o + \theta$$

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Calculation of $\frac{\partial c}{\partial \beta}$

Let us write C as follows :

$$C = \frac{1 + \sin\beta}{\cos^2\beta} \cdot \sin\beta \left[\frac{KI}{R\,Ko} \cos\beta + (1 + \sin\beta) \left(\frac{KI}{Ko}\right)^2 - \frac{K2}{Ko} \frac{\cos^2\beta}{\sin\beta} \right]$$

In fact, we take an interest in the value of C' as far as $C = 0$
Under such conditions, if we have :

$$\frac{1 + \sin\beta}{\cos^2\beta} \sin\beta = U$$

$$\frac{KI}{R\,Ko} \cos\beta + (1 + \sin\beta) \left(\frac{KI}{Ko}\right)^2 - \frac{K2}{Ko} \frac{\cos^2\beta}{\sin\beta} = V$$

$$C = U \cdot V \quad \text{So} \quad C' = U'V + UV'$$

but $\frac{1 + \sin\beta}{\cos^2\beta} \sin\beta \neq 0$ excepted for $\beta = 0$

Thus, it is necessary that $U = 0$ for obtaining $C = 0$
Then we may write :

$$C' = U \cdot V'$$

Calculation of V'

The question is to derive the expression

$$\frac{KI}{R\,Ko} \cos\beta + (1 + \sin\beta) \left(\frac{KI}{Ko}\right)^2 - \frac{K2}{Ko} \frac{\cos^2\beta}{\sin\beta}$$

We have the derived :

$$-\frac{KI}{R\,Ko} \sin\beta + \cos\beta \left(\frac{KI}{Ko}\right)^2 - \frac{K2}{Ko} \left[-\frac{2\cos\beta\sin\beta}{\sin\beta} - \frac{\cos^3\beta}{\sin^2\beta} \right] =$$

.../..

C.4.

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However, as we consider the case $C = 0$ therefore $V = 0$

$$\frac{K2}{K\delta} = \frac{KI}{R\,Ko}\, tg\,\beta_o + \left(\frac{KI}{Ko}\right)^2 (1 + \sin\beta_o)\,\frac{\sin\beta_o}{\cos^2\beta_o}$$

The expression becomes :

$$-\frac{KI}{R\,Ko}\sin\beta_o + \cos\beta_o\left(\frac{KI}{Ko}\right)^2 + \left[\frac{KI}{R\,Ko}\, tg\,\beta_o + \left(\frac{KI}{Ko}\right)^2 \frac{\sin\beta_o}{\cos^2\beta_o}(1 + \sin\beta_o)\right]$$

$$\left[2\cos\beta_o + \frac{\cos^3\beta_o}{\sin^2\beta_o}\right] =$$

$$\frac{KI}{RKo}\left[-\sin\beta_o + 2\sin\beta_o + \frac{\cos^2\beta_o}{\sin\beta_o}\right] + \left(\frac{KI}{Ko}\right)^2 \left[\cos\beta_o + \right.$$

$$\left.\frac{\sin\beta_o}{\cos\beta_o}(1 + \sin\beta_o)\left(2 + \frac{\cos^2\beta_o}{\sin^2\beta_o}\right)\right] =$$

$$\frac{KI}{R\,Ko}\left(\sin\beta_o + \frac{\cos^2\beta_o}{\sin\beta_o}\right) + \left(\frac{KI}{Ko}\right)^2\left[\cos\beta_o + \frac{\sin\beta_o}{\cos\beta_o}\frac{(1+\sin\beta_o)\,(1 + \sin^2\beta_o)}{\sin^2\beta_o}\right]$$

$$= \frac{KI}{R\,Ko}\left(\frac{1}{\sin\beta_o}\right) + \left(\frac{KI}{Ko}\right)^2 \frac{1}{\sin\beta_o\cos\beta_o}\left(\sin\beta_o\cos^2\beta_o + 1 + \sin\beta_o + \sin^2\beta_o + \sin^3\beta_o\right)$$

$$= \frac{KI}{R\,Ko}\,\frac{1}{\sin\beta_o} + \left(\frac{KI}{Ko}\right)^2 \frac{1}{\sin\beta_o\cos\beta_o}\left[\sin\beta_o(1 - \sin^2\beta_o) + 1 + \sin\beta + \right.$$

$$\left.\sin^2\beta + \sin^3\beta\right] =$$

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$$\frac{KI}{R\,Ko}\;\frac{1}{\sin\beta_0} + \left(\frac{KI}{Ko}\right)^2 \frac{1}{\sin\beta_0\cos\beta_0}\left(\sin\beta_0 + 1 + \sin\beta_0 + \sin^2\beta_0\right) =$$

$$\frac{KI}{R\,Ko}\;\frac{1}{\sin\beta_0} + \left(\frac{KI}{Ko}\right)^2 \frac{(1+\sin\beta_0)^2}{\sin\beta_0\cos\beta_0}$$

So, we have $e' = \dfrac{1+\sin\beta}{\cos^2\beta}\,\sin\beta\left[\dfrac{KI}{R\,Ko}\;\dfrac{1}{\sin\beta_0} + \left(\dfrac{KI}{Ko}\right)^2 \dfrac{(1+\sin\beta)^2}{\sin\beta\cos\beta}\right]$

that may be written :

$$e' = (1+\sin\beta)\left[\frac{KI}{R\,Ko}\;\frac{1}{\cos^2\beta} + \left(\frac{KI}{Ko}\right)^2 \frac{(1+\sin\beta)^2}{\cos^3\beta}\right]$$

So :

$$\boxed{e' = \frac{1+\sin\beta}{\cos^2\beta}\left[\frac{KI}{R\,Ko} + \left(\frac{KI}{Ko}\right)^2 \frac{(1+\sin\beta)^2}{\cos\beta}\right]} \qquad (152)$$

Or :

$$e' = (1+\sin\beta)\left[\frac{KI}{R\,Ko}\;\frac{1}{\cos^2\beta} + \left(\frac{KI}{Ko}\right)^2 \frac{1+\sin\beta}{\cos\beta\,(1-\sin\beta)}\right]$$

$$\boxed{e' = \frac{(1+\sin\beta)}{\cos\beta}\left[\frac{KI}{R\,Ko}\;\frac{1}{\cos\beta} + \left(\frac{KI}{Ko}\right)^2 \frac{1+\sin\beta}{1-\sin\beta}\right]} \qquad (153)$$

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Therefore one can cancel C' if

$$\boxed{\frac{KI}{Ko} = \frac{1}{R}\,\frac{\sin\beta - 1}{(1+\sin\beta)\cos\beta}} \qquad (154)$$

or, under another form

$$\boxed{\frac{KI}{Ko} = \frac{-\cos\beta}{R\,(1+\sin\beta)^2}} \qquad (155)$$

We may write : $C' = U.V$

with $U = \dfrac{1+\sin\beta_o}{\cos^2\beta_o\,(1-\sin\beta_o)}$

$$V = \frac{KI}{R\,Ko}\,(1-\sin\beta_o) + \left(\frac{KI}{Ko}\right)^2 (1+\sin\beta)\cos\beta$$

$$V' = -\frac{KI}{R\,Ko}\cos\beta_o + \left(\frac{KI}{Ko}\right)^2 \left(-\sin\beta_o + \cos^2\beta_o - \sin^2\beta_o\right)$$

but $\dfrac{KI}{Ko} = \dfrac{1-\sin\beta_o}{R\cos\beta_o(1+\sin\beta_o)} = \dfrac{-\cos\beta_o}{R\,(1+\sin\beta_o)^2}$

Therefore :

$$V' = \frac{1}{R^2}\,\frac{\cos^2\beta_o}{(1+\sin\beta_o)^2} + \frac{\cos^2\beta_o}{R^2\,(1+\sin\beta_o)^4}\left(-\sin\beta_o + \cos^2\beta_o - \sin^2\beta_o\right)$$

$$= \frac{1}{R^2}\,\frac{\cos^2\beta_o}{(1+\sin\beta)^4}\left[(1+\sin\beta)^2 + \cos^2\beta - \sin^2\beta_o - \sin\beta_o\right]$$

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$$= \frac{1}{R^2} \frac{\cos^2 \beta_o}{(1+ \sin \beta)^4} (1 + \sin\beta_o + \cos^2\beta_o)$$

$$= \frac{1}{R^2} \frac{\cos^2 \beta_o}{(1 + \sin \beta)^4} (2 + \sin\beta_o - \sin^2\beta_o)$$

$$= - \frac{\cos^2\beta_o (1 + \sin\beta_o) (\sin\beta_o - 2)}{R^2 (1 + \sin\beta_o)^4}$$

From which we have :

$$C'' = \frac{1 + \sin \beta_o}{R^2 \cos^2\beta_o (1 - \sin\beta_o)} \times \frac{\cos^2\beta_o (1 + \sin\beta_o) (2 - \sin\beta_o)}{(1 + \sin \beta_o)^4}$$

$$\boxed{C'' = \frac{2 - \sin \beta}{R^2 (1 - \sin\beta) (1 + \sin\beta)^2}} \quad (156)$$

In that case, we may calculate $\frac{K2}{Ko}$ We know that :

$$\frac{K2}{Ko} = \frac{KI}{R\ Ko} tg\ \beta_o + \left(\frac{KI}{Ko}\right)^2 (1 + \sin\beta) \frac{\sin \beta_o}{\cos^2\beta_o}$$

with $\frac{KI}{Ko} = \frac{-\cos\beta}{R (1+\sin\beta)^2}$

$$\frac{K2}{Ko} = \frac{-\cos\beta}{R^2 (1+\sin\beta_o)^2} \frac{\sin\beta}{\cos\beta} + \frac{\cos^2\beta_o}{R^2 (1+\sin\beta_o)^4} (1+\sin\beta_o) \frac{\sin\beta_o}{\cos^2\beta_o}$$

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$$= \frac{1}{R^2\,(1+\sin\beta)^2}\left[\frac{\sin\beta}{1+\sin\beta} - \sin\beta\right]$$

$$\boxed{\frac{K2}{Ko} = -\frac{\sin^2\beta}{R^2\,(1+\sin\beta)^3}} \qquad (157)$$

So, we have the coma equations as follows :

$$C = (1+\sin\beta)\left[\frac{\sin\beta}{R\cos\beta}\,\frac{1}{R}\,\frac{KI}{Ko} + \frac{(1+\sin\beta)\sin\beta}{\cos^2\beta}\left(\frac{KI}{Ko}\right)^2 - \frac{K2}{Ko}\right]$$

$$\frac{\partial C}{\partial\beta} = \frac{(1+\sin\beta)}{\cos^2\beta}\left[\frac{KI}{R\,Ko} + \left(\frac{KI}{Ko}\right)^2\frac{(1+\sin\beta)^2}{\cos\beta}\right]$$

with $\frac{K2}{Ko} = \frac{KI}{Ko}\,tg\,\beta_o + \left(\frac{KI}{Ko}\right)^2(1+\sin\beta)$

$$\frac{\partial^2 C}{\partial\beta^2} = \frac{2-\sin\beta}{R^2\,(1-\sin\beta)(1+\sin\beta)^2}$$

with $\frac{KI}{Ko} = -\frac{\cos\beta_o}{R\,(1+\sin\beta)^2}$

$$\frac{K2}{Ko} = -\frac{\sin^2\beta}{R^2\,(1+\sin\beta)^3}$$

<u>Comment</u> : It can be easily demonstrated that if one introduces the grazing incidence conditions into the equations (88) and (89) ——— ———, we rightly find again the above mentioned equations.

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a) $$\frac{\partial c}{\partial \beta} = \frac{KI}{R\,Ko} \sin\beta_o - \frac{K2}{Ko} \cos\beta_o + \left(\frac{\cos^2\beta_o}{\ell_{Bo}} - \frac{\cos\beta_o}{R}\right) \left[2\,\frac{KI}{Ko}\, tg\,\beta_o \right.$$

$$\left. + \frac{\sin\beta_o}{\ell_{Bo}} \left(2\,tg\,\beta_o + \frac{1}{tg\,\beta_o} \right) - \frac{tg^2\beta_o}{R} \right] = 0$$

The tangential incidence conditions are :

$$\left| \begin{array}{l} \dfrac{\cos^2\beta}{\ell_B} - \dfrac{\cos\beta}{R} - (1 + \sin\beta)\,\dfrac{KI}{Ko} = 0 \\[2ex] \dfrac{1}{\ell_B} = \dfrac{1}{R\cos\beta} + \dfrac{1+\sin\beta}{\cos^2\beta}\,\dfrac{KI}{Ko} \\[2ex] \dfrac{\sin\beta}{\ell_B}\,\dfrac{KI}{Ko} = \dfrac{K2}{Ko} \qquad (\text{coma} = 0) \end{array} \right.$$

The general equation $\dfrac{\partial c}{\partial \beta}$ is :

$$\frac{\partial c}{\partial \beta} = \frac{KI}{R\,Ko} \sin\beta_o - \frac{\sin\beta}{\ell_B}\,\frac{KI}{Ko} \cos\beta_o + (1+\sin\beta)\,\frac{KI}{Ko} \left[2\,\frac{KI}{Ko}\, tg\,\beta_o \right.$$

$$\left. + \frac{\sin\beta_o}{\ell_{Bo}} \left(2\,tg\,\beta_o + \frac{1}{tg\,\beta_o} \right) - \frac{tg^2\beta_o}{R} \right]$$

$$= \frac{KI}{R\,Ko} \sin\beta_o - \frac{KI}{Ko} \sin\beta_o \cos\beta_o \left(\frac{1}{R\cos\beta} + \frac{1+\sin\beta}{\cos^2\beta_o}\,\frac{KI}{Ko} \right) +$$

$$2\left(\frac{KI}{Ko}\right)^2 (1+\sin\beta_o)\,tg\,\beta_o + (1+\sin\beta)\,\frac{KI}{Ko} \sin\beta_o \left(2\,tg\,\beta_o + \frac{1}{tg\,\beta_o}\right) \left(\frac{1}{R\cos\beta_o} + \frac{1+\sin\beta_o}{\cos^2\beta_o}\,\frac{KI}{Ko} \right)$$

$$- (1 + \sin\beta)\,\frac{KI}{R\,Ko}\, tg^2\beta_o$$

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$$= \frac{KI}{R\,Ko}\left[\sin\beta_0 - \sin\beta_0 + (1+\sin\beta)\,tg\,\beta_0\left(2\,tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right) - (1+\sin\beta)\,tg^2\beta_0\right]$$

$$+ \left(\frac{KI}{Ko}\right)^2\left[-(1+\sin\beta)\,tg\,\beta + 2\,(1+\sin\beta)\,tg\,\beta + (1+\sin\beta)^2\,\frac{\sin\beta_0}{\cos^2\beta_0}\left(2\,tg\,\beta_0 + \frac{1}{tg\,\beta_0}\right)\right]$$

$$= (1+\sin\beta)\left[\frac{KI}{R\,Ko}\,(tg^2\beta_0+1) + \left(\frac{KI}{Ko}\right)^2\left[tg\,\beta + \frac{1+\sin\beta}{\cos\beta}\,(2\,tg^2\beta_0 + 1)\right]\right]$$

$$= (1+\sin\beta)\left[\frac{KI}{R\,Ko}\,\frac{1}{\cos^2\beta} + \left(\frac{KI}{Ko}\right)^2\left[\frac{\sin\beta\,(1-\sin^2\beta+2\sin^2\beta+\cos^2\beta+2\sin^3\beta + \sin\beta\cos^2\beta}{\cos^3\beta}\right]\right]$$

$$= \frac{1+\sin\beta}{\cos^2\beta}\left[\frac{KI}{R\,Ko}\,\frac{1}{\cos^2\beta} + \left(\frac{KI}{Ko}\right)^2\,\frac{(1+\sin\beta)^2}{\cos\beta}\right]$$

b) $$\frac{\partial^2 C}{\partial\beta^2} = \frac{KI}{R\,Ko}\,\frac{\sin^2\beta}{\cos\beta}\left(\frac{1}{2} + \frac{1}{tg^2\beta}\right) + \left(\frac{KI}{Ko}\right)^2\sin\beta + \frac{K2}{Ko}\,\frac{\sin\beta}{2}$$

$$+ \frac{2\,KI}{R\,Ko}\,tg^2\beta\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right)\left(\frac{3}{2} + \frac{1}{tg^2\beta}\right) +$$

$$\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right)\left[\frac{\sin\beta}{\ell_B}\left(\frac{5}{2} + 3\,tg^2\beta\right) - \frac{tg\,\beta}{R}\left(\frac{3}{2} + 2\,tg^2\beta\right)\right]$$

$$= \frac{KI}{R\,Ko}\left\{\frac{\sin^2\beta}{\cos\beta}\left(\frac{1}{2} + \frac{1}{tg^2\beta}\right) + \frac{KI}{Ko}\sin\beta + \frac{1}{\ell_B}\,\frac{\sin^2\beta}{2} + 2\,\frac{KI}{Ko}\,tg^2\beta\,(1+\sin\beta)\right.$$

$$\left.\left(\frac{3}{2} + \frac{1}{tg^2\beta}\right) + (1+\sin\beta)\left[\frac{\sin\beta}{\ell_B}\left(\frac{5}{2} + 3\,tg^2\beta\right) - tg\,\beta\left(\frac{3}{2} + 2\,tg^2\beta\right)\right]\right\}$$

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$$= \frac{KI}{R\,Ko}\left\{ \frac{\sin^2\beta}{\cos\beta}\left(\frac{1}{2}+\frac{1}{tg^2\beta}\right) - \frac{\sin\beta\cos\beta}{(1+\sin\beta)^2} + \frac{\sin^3\beta}{2\cos\beta\,(1+\sin\beta)} - \frac{2\cos\beta\, tg^2\beta}{1+\sin\beta}\left(\frac{3}{2}+\frac{1}{tg^2\beta}\right) \right.$$

$$\left. + (1+\sin\beta)\left[\frac{\sin^2\beta}{\cos\beta(1+\sin\beta)}\left(\frac{5}{2}+3tg^2\beta\right) - tg\beta\left(\frac{3}{2}+2tg^2\beta\right)\right]\right\}$$

Or

$$\frac{KI}{R\,Ko}\left\{ \frac{\sin^2\beta}{\cos\beta}\left(\frac{1}{2}+\frac{1}{tg^2\beta}\right) + \frac{\sin^3\beta}{2\cos\beta\,(1+\sin\beta)} - \frac{\sin\beta\cos\beta}{(1+\sin\beta)^2} + \frac{\sin^2\beta}{\cos\beta}\left(\frac{5}{2}+3tg^2\beta\right) \right.$$

$$\left. - \frac{2\cos\beta}{1+\sin\beta}tg^2\beta\left(\frac{3}{2}+\frac{1}{tg^2\beta}\right) - (1+\sin\beta)\,tg\beta\left(\frac{3}{2}+2tg^2\beta\right)\right\}$$

$$\frac{KI}{RKo}\left\{ \frac{\sin^2\beta}{\cos\beta}\left(\frac{1}{2}+\frac{1}{tg^2\beta}+\frac{5}{2}+3tg^2\beta-\frac{3}{2}-2tg^2\beta\right) + \frac{\sin^3\beta}{2\cos\beta\,(1+\sin\beta)} - \frac{\sin\beta\cos\beta}{(1+\sin\beta)^2} \right.$$

$$\left. - \frac{\sin\beta}{\cos\beta}\left(\frac{3}{2}+2tg^2\beta\right) - \frac{3\sin^2\beta+2\cos^2\beta}{\cos\beta\,(1+\sin\beta)}\right\}$$

$$\frac{KI}{RKo}\left\{ \frac{\sin^2\beta}{\cos\beta}\left(\frac{3}{2}+tg^2\beta+\frac{1}{tg^2\beta}\right) + \frac{\sin^3\beta-6\sin^2\beta-4\cos^2\beta}{2\cos\beta\,(1+\sin\beta)} - \frac{2\sin\beta\cos^2\beta}{2\cos\beta\,(1+\sin\beta)^2} \right.$$

$$\left. - \frac{\sin\beta}{\cos\beta}\left(\frac{3}{2}+2tg^2\beta\right)\right\}$$

$$\frac{KI}{RKo}\left\{ \frac{\sin^2\beta}{\cos\beta}\left(\frac{3}{2}+\frac{\sin^4\beta+\cos^4\beta}{\sin^2\beta\cos^2\beta}\right) - \frac{2\sin\beta\cos^2\beta}{2\cos^3\beta}\left(\frac{3\cos^2\beta+4\sin^2\beta}{2\cos^2\beta}\right) \right.$$

$$\left. + \frac{(\sin^3\beta-6\sin^2\beta-4\cos^2\beta)(1+\sin\beta)-2\sin\beta\cos^2\beta}{2\cos\beta\,(1+\sin\beta)^2}\right\}$$

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$$\frac{KI}{RKo}\left\{\frac{\sin^2\beta}{\cos\beta}\left(\frac{2-\sin^2\beta\cos^2\beta}{2\cos^2\beta\sin^2\beta}\right)-\frac{6\sin\beta\cos^2\beta-8\sin^3\beta}{4\cos^3\beta}\right.$$

$$\left.+\frac{\sin^3\beta-6\sin^2\beta-4\cos^2\beta+\sin^4\beta-6\sin^3\beta-4\sin\beta\cos^2\beta-2\sin\beta\cos^2\beta}{2\cos\beta(1+\sin\beta)^2}\right\}$$

$$\frac{KI}{RKo}\left\{\frac{2-\sin^2\beta\cos^2\beta-3\sin\beta\cos^2\beta-4\sin^3\beta}{2\cos^3\beta}+\frac{\sin^4\beta-5\sin^3\beta-2\sin^2\beta-4-6\sin\beta\cos^2\beta}{2\cos\beta(1+\sin\beta)^2}\right\}$$

$$\frac{KI}{RKo}\left\{\frac{2-3\sin\beta(\sin^2\beta+\cos^2\beta)-\sin^2\beta\cos^2\beta-\sin^3\beta}{2\cos^3\beta}+\frac{\sin^4\beta-4-5\sin\beta(\sin^2\beta+\cos^2\beta)-\sin\beta\cos^2\beta-2\sin^2\beta}{2\cos\beta(1+\sin\beta)^2}\right\}$$

$$\frac{KI}{RKo}\left\{\frac{2-3\sin\beta-\sin^3\beta-\sin^2\beta\cos^2\beta}{2\cos^3\beta}+\frac{\sin^4\beta-4-5\sin\beta-\sin\beta(1-\sin^2\beta)-2\sin^2\beta}{2\cos\beta(1+\sin\beta)^2}\right\}$$

$$\frac{KI}{RKo}\left\{\frac{2-3\sin\beta-\sin^3\beta-\sin^2\beta\cos^2\beta}{2\cos\beta(1+\sin\beta)(1-\sin\beta)}+\frac{(\sin^3\beta-2\sin\beta-4)(1+\sin\beta)}{2\cos\beta(1+\sin\beta)^2}\right\}$$

$$\frac{KI}{RKo}\left\{\frac{2-3\sin\beta-\sin^3\beta-\sin^2\beta\cos^2\beta+(\sin^3\beta-2\sin\beta-4)(1-\sin\beta)}{2\cos\beta(1+\sin\beta)(1-\sin\beta)}\right\}$$

$$\frac{KI}{RKo}\left\{\frac{2-3\sin\beta-\sin^3\beta-\sin^2\beta\cos^2\beta+\sin^3\beta-2\sin\beta-4-\sin^4\beta+2\sin^2\beta+4\sin\beta}{2\cos\beta(1+\sin\beta)(1-\sin\beta)}\right\}$$

$$=\frac{KI}{RKo}\left\{\frac{-2+\sin^2\beta-\sin\beta}{2\cos\beta(1+\sin\beta)(1-\sin\beta)}\right\}$$

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$$= \frac{1}{R} \cdot \frac{(\sin^2\beta - \sin\beta - 2)\cos\beta}{2\cos\beta\,(1 + \sin\beta)^3\,(1 - \sin\beta)}$$

The numerator is divisible by $(1 + \sin\beta)$ There is left :

$$\frac{\partial^2 C}{\partial \beta^2} = \frac{-\sin\beta + 2}{2\,R\,(1 - \sin\beta)\,(1 + \sin\beta)^2}$$

IX - 2 - STUDY OF THE COMBINATION COMA - ASTIGMATISM AT GRAZING INCIDENCE.

We are going to reconsider the relations established previously for the astigmatism and the coma at grazing incidence.

First, we notice that the equation $T = 0$ i.e.

$$\frac{\cos^2\alpha}{\ell_A} + \frac{\cos^2\beta}{\ell_B} - \frac{\cos\alpha + \cos\beta}{R} - (\sin\alpha + \sin\beta)\frac{KI}{Ko} = 0$$

is, with $\alpha = \frac{\pi}{2}$:

$$\boxed{\frac{\cos^2\beta}{\ell_B} = \frac{\cos\beta}{R} + (1 + \sin\beta)\frac{KI}{Ko}} \qquad (158)$$

Therefore, we see that the locus of the tangential focal length at grazing incidence is independant of the distance ℓ_A . If $KI = 0$ this locus is the Rowland circle (Let us remind that $KI = 0$ as regards the classical gratings).

The equation $S = 0$ i.e

$$\frac{1}{\ell_A} + \frac{1}{\ell_B} - \frac{\cos\alpha + \cos\beta}{R} - (\sin\alpha + \sin\beta)\frac{K3}{Ko} = 0$$

is, with $\alpha = \frac{\pi}{2}$:

$$\boxed{\frac{1}{\ell_A} + \frac{1}{\ell_B} - \frac{\cos\beta}{R} - (1 + \sin\beta)\frac{K3}{Ko} = 0} \qquad (159)$$

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Precisely, the choice of ℓ_A will allow the astigmatism to be reduced. It is the combination of the equations (158) and (159) which makes possible obtaining the relation of the astigmatism reduction $\mathcal{A} = 0$

Let us consider the equations $\mathcal{A} = 0 \quad \frac{\partial \mathcal{A}}{\partial \beta} = 0 \quad \frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0$

(equation 22,2nd report)

$$\mathcal{A} = 0 \longrightarrow -\frac{\cos\beta_0}{\ell_A} = \frac{\sin^2\beta_0}{R} + \frac{1+\sin\beta_0}{\cos\beta_0}\frac{KI}{Ko} - \frac{K3}{Ko}\cos\beta_0\,(1+\sin\beta_0)$$

$$\frac{\partial \mathcal{A}}{\partial \beta} = 0 \longrightarrow \frac{\sin\beta_0}{\ell_A} = \frac{\sin 2\beta_0}{R} + \frac{1+\sin\beta_0}{\cos^2\beta_0}\frac{KI}{Ko} - \frac{K3}{Ko}\,(\cos 2\beta_0 - \sin\beta_0)$$

$$\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0 \longrightarrow \frac{\cos\beta_0}{\ell_A} = \frac{2\cos 2\beta_0}{R} + \frac{(1+\sin\beta_0)^2}{\cos^3\beta_0}\frac{KI}{Ko} + \frac{K3}{Ko}\,(\cos\beta_0 + 2\sin 2\beta_0)$$

$$C = 1 + \sin\beta\left[\frac{\sin\beta}{R\cos\beta_0}\,\frac{1}{R}\,\frac{KI}{Ko} + \frac{(1+\sin\beta)\sin\beta}{\cos^2\beta}\left(\frac{KI}{Ko}\right)^2 - \frac{K2}{Ko}\right]$$

$$\frac{\partial C}{\partial \beta} = \frac{(1+\sin\beta)}{\cos^2\beta}\left[\frac{KI}{R\,Ko} + \left(\frac{KI}{Ko}\right)^2\frac{(1+\sin\beta)^2}{\cos\beta}\right]$$

with $\frac{K2}{Ko} = \frac{KI}{R\,Ko}\,tg\,\beta + \left(\frac{KI}{Ko}\right)^2(1+\sin\beta)\frac{\sin\beta}{\cos^2\beta}$

$$\frac{\partial^2 C}{\partial \beta^2} = \frac{2 - \sin\beta}{R^2\,(1-\sin\beta)\,(1+\sin\beta)^2}$$

with $\frac{KI}{Ko} = \frac{-\cos\beta_0}{R\,(1+\sin\beta)^2}$

$\frac{K2}{Ko} = \frac{-\sin^2\beta_0}{R^2\,(1+\sin\beta)^3}$

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We notice that the relation $\frac{\partial \mathcal{A}}{\partial \beta} = 0$ is significant only if $\mathcal{A} = 0$

Likewise with the relation $\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0$ if $\mathcal{A} = 0$ and $\frac{\partial \mathcal{A}}{\partial \beta} = 0$

We are going to modify the form of the relations $\mathcal{A}$, $\frac{\partial \mathcal{A}}{\partial \beta}$ and $\frac{\partial^2 \mathcal{A}}{\partial \beta^2}$

by using the relation $S = 0$ $\quad \frac{1}{\ell_A} = -\frac{1}{\ell_B} + \frac{\cos\beta}{R} + (1 + \sin\beta)\frac{K3}{Ko}$

$\mathcal{A} = 0$ is written as follows :

$$-\cos\beta_o\left(-\frac{1}{\ell_B} + \frac{\cos\beta}{R} + (1+\sin\beta)\frac{K3}{Ko}\right) = \frac{\sin^2\beta}{R} + \frac{1+\sin\beta}{\cos\beta}\frac{KI}{Ko} - \cos\beta_o(1+\sin\beta_o)\frac{K3}{Ko}$$

$$\frac{\cos\beta}{\ell_B} = \frac{1}{R} + \frac{1+\sin\beta}{\cos\beta}\frac{KI}{Ko}$$

However, that equation is not sufficient: It is just the equation $T = 0$

The equation $S = 0$ must be added for obtaining $\mathcal{A} = 0$.

$\frac{\partial \mathcal{A}}{\partial \beta} = 0$ is written :

$$\sin\beta_o\left(-\frac{1}{\ell_B} + \frac{\cos\beta_o}{R} + (1+\sin\beta_o)\frac{K3}{Ko}\right) = \frac{\sin 2\beta_o}{R} + \frac{1+\sin\beta_o}{\cos^2\beta_o}\frac{KI}{Ko} - (\cos 2\beta_o - \sin\beta_o)\frac{K3}{Ko}$$

$$-\frac{\sin\beta_o}{\ell_B} - \frac{\sin\beta_o\cos\beta_o}{R} + \frac{1+\sin\beta_o}{\cos^2\beta_o}\frac{KI}{Ko} - \frac{K3}{Ko}\left(\cos^2\beta_o - \sin^2\beta_o - \sin\beta_o + \sin\beta_o + \sin^2\beta_o\right)$$

$$-\frac{\sin\beta_o}{\ell_B} - \frac{\sin\beta_o\cos\beta_o}{R} + \frac{1+\sin\beta_o}{\cos^2\beta_o}\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta_o$$

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$\frac{\partial^2 \mathcal{A}}{\partial \beta^2}$ is written as follows :

$$\cos\beta_0 \left[-\frac{1}{\ell_B} + \frac{\cos\beta_0}{R} + (1+\sin\beta_0)\frac{K3}{Ko} \right] = \frac{2\cos 2\beta_0}{R} + \frac{(1+\sin\beta_0)^2}{\cos^3\beta_0}\frac{KI}{Ko}$$

$$+ \frac{K3}{Ko}(\cos\beta_0 + 2\sin 2\beta_0)$$

$$-\frac{\cos\beta_0}{\ell_B} = \frac{\cos^2\beta_0 - 2\sin^2\beta_0}{R} + \frac{(1+\sin\beta_0)^2}{\cos^3\beta_0}\frac{KI}{Ko} + \frac{K3}{Ko}(\cos\beta_0 + 4\sin\beta_0\cos\beta_0$$

$$-\cos\beta_0 - \cos\beta_0\sin\beta_0)$$

$$-\frac{\cos\beta_0}{\ell_B} = \frac{1-3\sin^2\beta_0}{R} + \frac{(1+\sin\beta_0)^2}{\cos^3\beta_0}\frac{KI}{Ko} + \frac{K3}{Ko}\, 3\sin\beta_0\cos\beta_0$$

So, in $\frac{1}{\ell_B}$, the equations of no-astigmatism are as follows :

(160)

$$\boxed{\begin{aligned}
&T = 0 \longrightarrow \frac{\cos\beta}{\ell_B} = \frac{1}{R} + \frac{1+\sin\beta}{\cos\beta}\frac{KI}{Ko} \\
&S = 0 \longrightarrow \frac{1}{\ell_A} + \frac{1}{\ell_B} = \frac{\cos\beta}{R} + (1+\sin\beta)\frac{K3}{Ko} \\
&\frac{\partial\mathcal{A}}{\partial\beta} = 0 \longrightarrow -\frac{\sin\beta_0}{\ell_B} = \frac{\sin\beta_0\cos\beta_0}{R} + \frac{1+\sin\beta_0}{\cos^2\beta_0}\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta_0 \\
&\frac{\partial^2\mathcal{A}}{\partial\beta^2} = 0 \longrightarrow -\frac{\cos\beta_0}{\ell_B} = \frac{1-3\sin^2\beta_0}{R} + \frac{(1+\sin\beta_0)^2}{\cos^3\beta_0}\frac{KI}{Ko} + 3\sin\beta_0\cos\beta_0\frac{K3}{Ko}
\end{aligned}}$$

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We notice that the first of these equations is not $\mathcal{A} = 0$ but just $T = 0$ For having effectively $\mathcal{A} = 0$ it is necessary to add $S = 0$ i.e. we are placed under the condition in $\frac{1}{\ell_A}$

$$-\frac{\cos\beta_0}{\ell_A} = \frac{\sin^2\beta_0}{R} + \frac{1+\sin\beta_0}{\cos\beta_0}\,\frac{KI}{Ko} - \frac{K3}{Ko}\cos\beta_0(1+\sin\beta_0)$$

From pages n° 43 and n° 44, –) we have seen that the solution of the three equations system $\mathcal{A} = 0$ $\frac{\partial\mathcal{A}}{\partial\beta} = 0$ $\frac{\partial^2\mathcal{A}}{\partial\beta^2} = 0$ led to

$$\boxed{\frac{KI}{Ko} = -\frac{1}{2R}\,\frac{2+\sin^2\beta}{(1+\sin\beta)^3}\cdot\cos\beta} \qquad (161)$$

One may deduce that :

$$\frac{\cos\beta}{\ell_B} = \frac{1}{R} - \frac{1}{2R}\,\frac{2+\sin^2\beta}{(1+\sin\beta)^3}\cdot\frac{1+\sin\beta}{\cos\beta}\cdot\cos\beta$$

$$\frac{\cos\beta}{\ell_B} = \frac{1}{R}\left[1 - \frac{2+\sin^2\beta}{2(1+\sin\beta)^2}\right]$$

$$(162) \qquad \boxed{\frac{\cos\beta}{\ell_B} = \frac{\sin\beta(\sin\beta + 4)}{(1+\sin\beta)^2}}$$

Let us remind that our comments about the being conditions (page n° 50 ---------- and following) had led to a discussion regarding the product's sign $\sin\beta$ ~~(———)~~ i.e. we finally wrote that ℓ_A should be positive.

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It had been established :

$$\boxed{\frac{K3}{Ko} = \frac{1}{R\cos^3\beta_o}\left[\sin\beta_o(2-\sin^2\beta_o) - \frac{2+\sin^2\beta_o}{2(1+\sin\beta_o)}\right]}$$

and

$$\boxed{\frac{\cos\beta_o}{\ell_A} = \frac{2\cos 2\beta_o}{R} = \frac{2+\sin^2\beta_o}{2R(1+\sin\beta_o)\cos^2\beta_o} + \frac{1+4\sin\beta_o}{R\cos^2\beta_o}\left[\sin\beta_o(2-\sin^2\beta_o) - \frac{2+\sin^2\beta_o}{2(1+\sin\beta_o)}\right]}$$

(equation 49)

First, we know that it is impossible to obtain $\frac{\partial^2 C}{\partial\beta^2} = 0$.

Let us see if there is a minimum value of that function.
We know that if C and $\frac{\partial C}{\partial\beta}$ are null

$$\frac{\partial^2 C}{\partial\beta^2} = \frac{2-\sin\beta}{R^2(1-\sin\beta)(1+\sin\beta)^2}$$

β	$\frac{\partial^2 C}{\partial\beta^2}$	β	$\frac{\partial^2 C}{\partial\beta^2}$
0°	2	- 10°	2,7122
10°	1,6045		
15°	1,4825	- 15°	3,2664
20°	1,3991	- 20°	4,0309
25°	1,3499	- 25°	5,10823
30°	1,3333	- 30°	6,6667

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β	$\frac{\partial^2 c}{\partial \beta^2}$	β	$\frac{\partial^2 c}{\partial \beta^2}$
35°	1,3509	- 35°	8,9943
40°	1,4079	- 40°	12,6075
50°	1,6911	- 50°	28,6148
60°	2,4308	- 60°	85,5692
70°	4,6730	- 70°	416,7044
80°	16,9625	- 80°	6515,59
85°	66,1995		

At this stage the whole conditions leading to $\mathcal{A} = 0$, $\frac{\partial \mathcal{A}}{\partial \beta} = 0$ and $\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0$ have been determined and we know that we can determine $\frac{1}{\ell_C}$ and $\frac{1}{\ell_D}$ through the relations (52) ———————

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Now we are trying to determine whether the conditions of coma null and of no-astigmatism are compatible.

I) If $\mathcal{A} = 0$, $\frac{\partial \mathcal{A}}{\partial \beta} = 0$, $\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = 0$ $\longrightarrow$ $\frac{KI}{Ko} = - \frac{1}{2R} \frac{2 + \sin^2 \beta}{(1 + \sin \beta)^3} \cos \beta$

if $c = 0$, $\frac{\partial c}{\partial \beta} = 0$ $\longrightarrow$ $\frac{KI}{Ko} = - \frac{\cos \beta_o}{R (1 + \sin \beta)^2}$

$\frac{K2}{Ko} = - \frac{\sin^2 \beta_o}{R^2 (1 + \sin \beta)^3}$

<u>Is it compatible</u> ?

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$$\frac{-\cos\beta_0}{(1+\sin\beta)^2} = -\frac{1}{2}\,\frac{2+\sin^2\beta}{(1+\sin\beta)^3}\cos\beta$$

$$2\,(1+\sin\beta) = 2+\sin^2\beta$$

$$\sin\beta\,(2-\sin\beta) = 0$$

The only solution is $\beta_0 = 0$

Observing the solution $\beta = 0$

$$\frac{KI}{Ko} = -\frac{1}{R} \qquad \frac{K2}{Ko} = 0$$

The equation (158) leads to $\frac{1}{\ell_B} = 0$ So $\ell_B = \infty$

The equation (159) leads to $\frac{1}{\ell_A} = \frac{2}{R} - \frac{1}{R} + \frac{1}{R}\,(-1)$

so $\ell_A = \infty$

It does not seem that solution being of a great practical interest. Therefore we may conclude that it is impossible to obtain simultaneously

$$\begin{cases} \mathcal{A} = 0 & \frac{\partial \mathcal{A}}{\partial\beta} = 0 \quad \frac{\partial^2 \mathcal{A}}{\partial\beta^2} = 0 \\ C = 0 & \frac{\partial C}{\partial\beta} = 0 \end{cases}$$

II) Therefore, one condition is to be left.
Let us leave $\frac{\partial^2 \mathcal{A}}{\partial\beta^2} = 0$

$$\frac{\cos\beta_0}{\ell_B} = \frac{1}{R} + \frac{1+\sin\beta_0}{\cos\beta_0}\,\frac{KI}{Ko}$$

$$-\frac{\sin\beta_0}{\ell_B} = \frac{\sin\beta_0\cos\beta_0}{R} + \frac{1+\sin\beta_0}{\cos^2\beta_0}\,\frac{KI}{Ko} - \frac{K3}{Ko}\cos^2\beta_0$$

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$$\frac{1}{\ell_A} + \frac{1}{\ell_B} - \frac{\cos\beta_o}{R} - (1 + \sin\beta_o)\ \frac{K3}{Ko} = 0$$

$$\frac{KI}{Ko} = -\frac{\cos\beta_o}{R(1+\sin\beta)^2} \qquad \frac{K2}{Ko} = \frac{-\sin^2\beta_o}{R^2\ (1 + \sin\beta)^3}$$

So we have :

$$\frac{\cos\beta_o}{\ell_B} = \frac{1}{R} - \frac{\cos\beta_o}{(1+\sin\beta)^2} \times \frac{1+\sin\beta_o}{R\cos\beta_o} = \frac{1}{R} - \frac{1}{R\ (1+\sin\beta_o)}$$

$$\frac{1}{\ell_B} = \frac{\sin\beta_o}{R\cos\beta_o(1+\sin\beta_o)}$$

(163) $$\boxed{\ell_B = R\frac{\cos\beta_o}{\sin\beta_o}\ (1 + \sin\beta_o)}$$

$$\frac{-\sin\beta_o \times \sin\beta_o}{R\cos\beta_o(1+\sin\beta_o)} = \frac{\sin\beta_o\cos\beta_o}{R} - \frac{1+\sin\beta_o}{R\cos^2\beta_o} \times \frac{\cos\beta_o}{(1+\sin\beta)^2} - \frac{K3}{Ko}\ \cos^2\beta_o$$

$$\frac{1}{R(1+\sin\beta_o)} = \frac{\sin\beta_o}{R} - \frac{K3}{Ko}\ \cos\beta_o$$

$$\frac{K3}{Ko}\ \cos\beta_o = \frac{1}{R}\ \left(\sin\beta_o - \frac{1}{1+\sin\beta_o}\right)$$

(164) $$\boxed{\frac{K3}{Ko} = \frac{\sin^2\beta_o + \sin\beta_o - 1}{R\cos\beta_o(1+\sin\beta_o)}}$$

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$$\frac{1}{\ell_A} + \frac{\sin\beta_0}{R\cos\beta_0(1+\sin\beta_0)} - \frac{\cos\beta_0}{R} - \frac{(1+\sin\beta_0)}{R\cos\beta_0}\,\frac{\sin^2\beta_0 + \sin\beta_0 - 1}{1+\sin\beta_0} = 0$$

$$\frac{1}{\ell_A} + \frac{\sin\beta_0}{R\cos\beta_0(1+\sin\beta_0)} - \frac{\cos^2\beta_0 + \sin^2\beta_0 + \sin\beta_0 - 1}{R\cos\beta_0} = 0$$

$$\frac{1}{\ell_A} = -\frac{\sin\beta_0}{R\cos\beta_0(1+\sin\beta_0)} + \frac{\sin\beta_0}{R\cos\beta_0} = \frac{\sin\beta_0}{R\cos\beta_0}\left(1 - \frac{1}{1+\sin\beta_0}\right)$$

$$\frac{1}{\ell_A} = \frac{\sin^2\beta_0}{R\cos\beta_0(1+\sin\beta_0)}$$

$$\boxed{\ell_A = \frac{R\cos\beta_0(1+\sin\beta_0)}{\sin^2\beta_0}}$$

(165)

If the following conditions are satisfied :

$$\ell_A = \frac{R\cos\beta_0(1+\sin\beta_0)}{\sin^2\beta_0} \qquad \ell_B = \frac{R\cos\beta_0}{\sin\beta_0}(1+\sin\beta_0)$$

$$\frac{K1}{Ko} = \frac{-\cos\beta_0}{R(1+\sin\beta)^2} \qquad \frac{K2}{Ko} = \frac{-\sin^2\beta_0}{R^2(1+\sin\beta)^3}$$

$$\frac{K3}{Ko} = \frac{\sin^2\beta_0 + \sin\beta_0 - 1}{R\cos\beta_0(1+\sin\beta_0)}$$

we have :

$$\begin{cases} A = 0 & \dfrac{\partial A}{\partial\beta} = 0 \\ C = 0 & \dfrac{\partial C}{\partial\beta} = 0 \end{cases}$$

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Let us study the value of $\dfrac{\partial^2 \mathcal{A}}{\partial \beta^2}$

$$\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = \frac{\cos\beta_0}{\ell_B} + \frac{1 - 3\sin^2\beta_0}{R} + \frac{(1+\sin\beta_0)^2}{\cos^3\beta_0}\,\frac{KI}{Ko} + 3\sin\beta_0\cos\beta_0\,\frac{K3}{Ko}$$

$$= \frac{\cos\beta_0 \sin\beta_0}{R\cos\beta_0(1+\sin\beta_0)} + \frac{1 - 3\sin^2\beta_0}{R} - \frac{(1+\sin\beta_0)^2}{\cos^3\beta_0}\,\frac{\cos\beta_0}{R\,(1+\mathrm{Sin}\beta_0)^2}$$

$$+ \frac{3\sin\beta_0\cos\beta_0(\sin^2\beta_0 + \sin\beta_0 - 1)}{R\cos\beta_0(1+\sin\beta_0)}$$

$$= \frac{\sin\beta_0 + (1-3\sin^2\beta_0)(1+\sin\beta_0) + 3\sin^3\beta_0 + 3\sin^2\beta_0 - 3\sin\beta_0}{R\,(1+\sin\beta_0)}$$

$$- \frac{1}{R\cos^2\beta_0}$$

$$= \frac{1-\sin\beta_0}{R\,(1+\sin\beta_0)} - \frac{1}{R\cos^2\beta_0} = \frac{(1-\sin\beta_0)^2 - 1}{R\cos^2\beta_0}$$

(166) $$\frac{\partial^2 \mathcal{A}}{\partial \beta^2} = \sin\beta_0\,\frac{\sin\beta_0 - 2}{R\cos^2\beta_0}$$

From the formula (46) the focal's height may be obtained

$$h''_T(\theta^2) = \frac{Z\,m}{2} \times \frac{\dfrac{\sin\beta_0(\sin\beta_0 - 2)}{R\cos^2\beta_0}}{\dfrac{1}{R} - \dfrac{1+\sin\beta_0}{\cos\beta_0} \times \dfrac{\cos\beta_0}{(1+\sin\beta)^2}} \times \theta^2$$

$$h_T(\theta^2) = \frac{Z\,m}{2} \times \frac{\dfrac{\sin\beta_0(\sin\beta_0 - 2)}{\cos^2\beta_0}}{1 - \dfrac{1}{1+\sin\beta}} \times \theta^2$$

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$$h_T(\theta^2) = \frac{Zm}{2} \times \frac{\sin\beta_0(\sin\beta_0 - 2)}{\cos^2\beta_0} \times \frac{1 + \sin\beta_0}{\sin\beta_0} \times \theta^2$$

(167)
$$\boxed{h_T(\theta^2) = \frac{Z\,m}{2} \frac{(\sin\beta_0 - 2)(1 + \sin\beta_0)}{\cos^2\beta_0}\,\theta^2}$$

Particular case $\beta = 30°$. This case is interesting as it corresponds to the minimum of $\frac{\partial^2 C}{\partial \beta^2}$

$$\cos\beta = \frac{\sqrt{3}}{2} \qquad \sin\beta = \frac{1}{2}$$

$$\ell_B = R \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \times \frac{3}{2} = \frac{3\sqrt{3}}{2} R$$

$$\ell_A = R \frac{\frac{\sqrt{3}}{2}}{\frac{1}{4}} \times \frac{3}{2} = 3\sqrt{3}\, R$$

$$\frac{KI}{Ko} = -\frac{\sqrt{3}}{2R} \times \frac{1}{\left(\frac{3}{2}\right)^2} = -\frac{2\sqrt{3}}{9R}$$

$$\frac{K2}{Ko} = -\frac{1}{4R^2} \times \frac{1}{\left(\frac{3}{2}\right)^3} = -\frac{2}{27R^2}$$

$$\frac{K3}{Ko} = \frac{\frac{1}{4} + \frac{1}{2} - 1}{R\frac{\sqrt{3}}{2} \times \frac{3}{2}} = -\frac{1}{3\sqrt{3}\,R}$$

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III) $KI = K2 = 0$ at grazing incidence.

We have seen that the equation of the coma at grazing incidence is : (149)

$$C = \frac{Y^3}{2} \left(1 + \sin\beta \right) \left[\frac{\sin\beta}{\ell_B} \frac{KI}{Ko} - \frac{K2}{Ko} \right]$$

Obviously the condition $\frac{KI}{Ko} = \frac{K2}{Ko} = 0$ is sufficient for having the coma null whatever the point B is.

We have seen, page 54, that the conditions $KI = K2 = 0$ led to two groups of solution :

1) $\ell_C = R \cos\gamma$

$\ell_D = R \cos\delta$

2) $\ell_C = - R \dfrac{\cos^2\delta \sin\gamma - \cos^2\gamma \sin\delta}{\sin\gamma(\cos\gamma - \cos\delta}$

$\ell_D = - R \dfrac{\cos^2\delta \sin\gamma - \cos^2\gamma \sin\delta}{\sin\gamma(\cos\gamma - \cos\delta)}$

The equation of the tangential focal length is :

$$\frac{\cos\beta}{\ell_B} = \frac{1}{R} + \frac{1 + \sin\beta}{\cos\beta} \frac{KI}{Ko}$$

If $\frac{KI}{Ko} = 0$ $\ell_B = R \cos\beta$

The locus of the tangential focal length <u>is the Rowland circle</u>.

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The equation $S = 0$ is :

$$\frac{1}{\ell_A} + \frac{1}{\ell_B} - \frac{\cos\beta}{R} - (1 + \sin\beta)\ \frac{K3}{Ko} = 0$$

or

$$\boxed{-\frac{\cos\beta_o}{\ell_A} = \frac{\sin^2\beta_o}{R} - \frac{K3}{Ko}\ \cos\beta_o\,(1 + \sin\beta_o)} \qquad (168)$$

The equation (168) shows clearly that - for some wavelength corresponding to β_o,-it is possible to avoid the astigmatism by choosing ℓ_A in such a way that the equation (168) may be resolved.

So, the method consists in :

- choosing one of the solutions in ℓ_C and ℓ_D allowing KI and K2 to be zero,
- calculating $\frac{K3}{Ko}$ and deducing ℓ_A from it.

We may calculate the value of K4 that would allow the term $Y\,Z^2$ to be zero:

$$\frac{Y\,Z^2}{2}\left[\frac{\sin\alpha}{\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{\ell_B}\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{K4}{Ko}\,(\sin\alpha + \sin\beta)\right]$$

$$\boxed{\frac{Y\,Z^2}{2}\left[\frac{1}{\ell_A^2} + \frac{\sin^3\beta}{R^2\cos^2\beta} - \frac{K4}{Ko}\,(1 + \sin\beta)\right]} \qquad (169)$$

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By applying the general formula (45) we may calculate the focal's height in the case of the equation (168) being satisfied.

$$h'_T(\theta) = Zm \frac{-\frac{\sin\beta}{\ell_A} + \frac{\sin 2\beta}{R} + \frac{1}{2}\frac{KI}{Ko}\frac{1}{\cos^2\frac{\alpha+\beta}{2}} - \frac{K3}{Ko}\left[\frac{\cos^2\beta}{2\cos^2\frac{\alpha+\beta}{2}} - \frac{\sin 2\beta}{tg\frac{\alpha+\beta}{2}}\right]}{-\frac{1}{\cos\alpha + \cos\beta}\frac{\cos^2\alpha}{\ell_A} + \frac{1}{R} + tg\frac{\alpha+\beta}{2}\frac{KI}{Ko}}$$

$$\frac{KI}{Ko} = 0 \qquad \alpha = \frac{\pi}{2}$$

$$h'_T(\theta) = Zm \times \frac{-\frac{\sin\beta}{\ell_A} + \frac{\sin 2\beta}{R} - \frac{K3}{Ko}\left[\cos^2\beta\left(\frac{1+\sin\beta}{\cos\beta}\right)^2 - \sin 2\beta\left(\frac{1+\sin\beta}{\cos\beta}\right)\right]}{\frac{1}{R}}$$

$$h'_T(\theta) = Zm \times \frac{-\frac{\sin\beta}{\ell_A} + \frac{\sin 2\beta}{R} - \frac{K3}{Ko}\left[(2+2\sin\beta) - 2\sin\beta_o(1+\sin\beta_o)\right]}{\frac{1}{R}},$$

(170) $$\boxed{h'_T(\theta) = Zm \times \left[\frac{-\sin\beta}{\ell_A} + \frac{\sin 2\beta}{R} - 2\frac{K3}{Ko}\cos^2\beta\right] R.\theta}$$ (—)

X - SPHERICAL ABERRATION AND LIMITATION OF THE GRATING'S WIDTH

We have established that the fourth order term of the optical path

$$\Delta = AM + MB - \frac{k\lambda}{\lambda_0}(MC - MD) - \mathcal{P}$$

might be determined from the fourth order term of expansion of A M by calculating M B M C M D as previously made for A M and leading to the equation 23.

$$AM^{(4)} = Y^4 \left\{ \frac{1}{8R^2}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) - \frac{1}{8\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right)\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5\sin^2\alpha}{\ell_A}\right)\right\}$$

$$+ Y^2 Z^2 \left\{ \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\left[\frac{1}{4R^2} - \frac{1}{4\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{3}{4}\frac{\sin^2\alpha}{\ell^2_A}\right]\right\}$$

$$+ Z^4 \left\{ \frac{1}{8}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\left[\frac{1}{R^2} - \frac{1}{\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\right]\right\}$$

We are going to see how the term in Y^4 is generally linked to the grating's width. Effectively, let us suppose we are on the Rowland circle

$$AM^{(4)} = \frac{Y^4}{8R^3}\frac{\sin^2\alpha}{\cos\alpha} + (\qquad) Y^2 Z^2 + (\qquad) Z^4$$

Page n° 37 ———, we have seen that the height of the tangential focal length, in the case of a classical grating, used on the Rowland circle, is :

$$h_T = Z\,m\,(\sin^2\beta + \sin\alpha\, tg\alpha \cos^2\beta)$$

(equation 39).

We observe that if $\alpha \longrightarrow \frac{\pi}{2}$ the coefficient of Z m becomes very high and h_T as well.

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Then, we are obliged to limit the value of Zm if we want h_T to be limited, that is necessary if we use a photoelectric receiver : the cathodes height of the P M is limited.

If we use a photographic film the whole flux is divided by the focal's height, so the illumination decreases while the focal's height increases.

Those remarks lead to the limitation of Zm due to focal's height considerations, at least, as regards the conventional concave gratings.

On the other hand, Ym is not generally limited to the fourth order stage if we work on the Rowland circle ; effectively, we have seen that, in this case, the coma i.e. the term in Y^3 is zero.

First, let us study the example of the conventional grating operating on the Rowland circle:

$$\boxed{\Delta^{(4)} = A M^{(4)} + M B^{(4)} = \frac{Y^4}{8 R^3} \left(\frac{\sin^2 \alpha}{\cos \alpha} + \frac{\sin^2 \beta}{\cos \beta} \right)} \qquad (171)$$

We observe that it is the formula given by J.E. Mack - J.R. Stehn and Bengt Edlen. J.O.S.A. May 1932 under the form

$$A P + P_0 A' = \ldots + \frac{y^4}{8 e^3} \left(tg\,\theta \sin\theta + tg\,\theta' \sin\theta' \right)$$

Placing side by side the Mack's notations and ours $\begin{cases} \theta = \alpha & e = R. \\ \theta' = \beta & y = Y. \end{cases}$

The method of calculation of Ym consists in saying that it is not hoped that Δ be greater than $\frac{\lambda}{4}$, for example, which corresponds to 80% of the theoretical resolving power of the grating.

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One also may calculate the widening of the image for a given value of Ym by using the relation of Nijboer given in the 1st report,

$$dy = \frac{-\cos\varphi}{\cos\omega}\,\frac{\partial\Delta}{\partial\omega} - \frac{\sin\varphi}{\sin\omega}\,\frac{\partial\Delta}{\partial\varphi}$$

φ is the azimuthel angle of the pupil

ω is the aperture's angle of the beam

$$Y = Y\,m\cos\varphi$$

$$\omega = \frac{Y}{R}$$

However, the relation of Nijboer is rigorous only if the pupil is circular, that is not the case generally for the conventional gratings due to precisely the above mentioned considerations about the influence of Zm over the slit's height.

Regarding the holographic gratings, the term in Y^4 of expansion of Δ is written :

$$\Delta^{(4)} = Y^4 \Bigg\{ \frac{1}{8R^2}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) - \frac{1}{8\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right)\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - 5\,\frac{\sin^2\alpha}{\ell_A}\right)$$

$$+ \frac{1}{8R^2}\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{1}{8\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right)\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R} - 5\,\frac{\sin^2\beta}{\ell_B}\right)$$

(172)

$$- \frac{(\sin\alpha + \sin\beta)}{K_o}\Bigg[\frac{1}{8R^2}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{1}{8\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R} - 5\,\frac{\sin^2\gamma}{\ell_C}\right)$$

$$- \frac{1}{8R^2}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right) + \frac{1}{8\ell_D}\left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R} - 5\,\frac{\sin^2\delta}{\ell_D}\right)\Bigg]\Bigg\}$$

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That equation is rather complex and is numerically calculated in a specific given case only.
However, we notice that if we are on the Rowland circle and use the group of solutions

$$\begin{cases} \ell_C = R \cos \gamma \\ \ell_D = R \cos \delta \end{cases}$$

and if, on the other hand, $\ell_A = R \cos \alpha$

$$\ell_B = R \cos \beta$$

$\Delta^{(4)}$ is reduced to :

$$\boxed{\Delta^{(4)} = \frac{Y^4}{8R^2} \left[\frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{1}{\ell_B} - \frac{\cos\beta}{R} - \frac{(\sin\alpha + \sin\beta)}{Ko} \left[\frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R} \right) \right] \right]} \quad (173)$$

We identify the term in Z^2 of second order development.

$$\Delta^{(4)} = \frac{Y^4}{8R^2} \left[\frac{\sin^2\alpha}{R \cos\alpha} + \frac{\sin^2\beta}{R \cos\beta} - (\sin\alpha + \sin\beta) \frac{K3}{Ko} \right]$$

Let us remind the formula (169)

$$h_T = Z\, m \left[\frac{\sin^2\alpha}{\cos\alpha} + \frac{\sin^2\beta}{\cos\beta} - \frac{K3}{Ko} R \left(\sin\alpha + \sin\beta \right) \right] \cos\beta$$

So, we may say that the spherical aberration is in direct ratio to the

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astigmatism.

Specially if we want obtaining $\frac{K3}{Ko} = \frac{1}{R(\sin\alpha + \sin\beta)}\left[\frac{\sin^2\alpha}{\cos\alpha} + \frac{\sin^2\beta}{\cos\beta}\right]$

<u>the astigmatism and the spherical aberration are simultaneously null.</u> (In fact, as regards the spherical aberration, the widening due to the term in Y^4 is more precisely concerned).

Let us calculate, in this case, the coefficient of the term in Z^4

$$\frac{Z^4}{8}\left[\frac{1}{R^2}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) - \frac{1}{\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)^2 + \right.$$

$$\frac{1}{R^2}\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R}\right) - \frac{1}{\ell_B}\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R}\right)^2 -$$

$$\frac{(\sin\alpha + \sin\beta)}{Ko}\left[\frac{1}{R^2}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{1}{\ell_C}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)^2\right.$$

$$\left.\left. - \left[\frac{1}{R^2}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right) - \frac{1}{\ell_D}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)^2\right]\right]\right]$$

that may be written :

$$\frac{Z^4}{8}\frac{1}{R^2}\left[\frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{1}{\ell_B} - \frac{\cos\beta}{R} - \frac{(\sin\alpha + \sin\beta)}{Ko}\left[\frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)\right]\right.$$

$$- \frac{1}{\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)^2 - \frac{1}{\ell_B}\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R}\right)^2 - \frac{(\sin\alpha + \sin\beta)}{Ko}\left[-\frac{1}{\ell_C}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right)^2\right.$$

$$\left.\left. + \frac{1}{\ell_D}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)^2\right]\right]$$

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From the first part of this expression, we identify the coefficient of the term in Y^4 which is zero according to our hypothesis. The term in Z^4 is therefore reduced to :

$$\boxed{\Delta^{(4)} = -\frac{Z^4}{8R^3}\left[\frac{\sin^4\alpha}{\cos^3\alpha} + \frac{\sin^4\beta}{\cos^3\beta} - \frac{(\sin\alpha + \sin\beta)}{Ko}\left[\frac{\sin^4\gamma}{\cos^3\gamma} - \frac{\sin^4\delta}{\cos^3\delta}\right]\right]} \quad (174)$$

In the same way and under identical conditions we may calculate the term in $Y^2\ Z^2$

First, let us consider the term in ℓ_A , α

$$AM^{(4)} = Y^2\ Z^2\left[\frac{1}{4\ R^2}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) + \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\left[-\frac{1}{4\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{3}{4}\ \frac{\sin^2\alpha}{\ell_A}\right]\right]$$

We are on the Rowland circle ; the astigmatism is supposed to be equal to zero and we attempt to calculate the coefficient of the term in $Y^2\ Z^2$ corresponding to $\Delta(4)$.

For reasons identical with those indicated above, the sum of the terms in $\frac{1}{4R^2}\left(\frac{1}{\ell} - \frac{\cos}{R}\right)$ will be null.

Then, we shall have to add the terms such as :

$$A\ M^{(4)} = +\ Y^2\ Z^2\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\left[-\frac{1}{4\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right) + \frac{3}{4}\ \frac{\sin^2\alpha}{\ell_A}\right]$$

with : $\ell_A = R\cos\alpha$

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$$AM^{(4)} = + Y^2 Z^2 \times \frac{\sin^2\alpha}{R\cos\alpha} \left[- \frac{1}{4 R\cos\alpha} \times \frac{\sin^2\alpha}{R\cos\alpha} + \frac{3}{4} \frac{\sin^2\alpha}{\ell_A^2} \right]$$

$$AM^{(4)} = Y^2 Z^2 \frac{\sin^2\alpha}{R\cos\alpha} \times \frac{1}{2} \frac{\sin^2\alpha}{R^2 \cos^2\alpha}$$

$$\boxed{A M^{(4)} = \frac{Y^2 Z^2}{2 R^3} \frac{\sin^4\alpha}{\cos^3\alpha}} \quad (175)$$

So, under those conditions, the whole terms of $\Delta^{(4)}$ may be written :

$$(176) \quad \boxed{\Delta^{(4)} = \left[\frac{\sin^4\alpha}{\cos^3\alpha} + \frac{\sin^4\beta}{\cos^3\beta} - \frac{(\sin\alpha + \sin\beta)}{Ko} \left(\frac{\sin^4\gamma}{\cos^3\gamma} - \frac{\sin^4\delta}{\cos^3\delta} \right) \right] \left(\frac{Y^2 Z^2}{2 R^3} - \frac{Z^4}{8R^3} \right)}$$

We notice that $\Delta^{(4)}$ is zero if $Y^2 = \frac{Z^2}{4}$

$$(177) \quad \text{or} \quad \boxed{Y = \pm \frac{1}{2} Z}$$

.../..

.../..

Now, we are going to study the problem relating to the spherical aberration at grazing incidence.

The problem regarding the coma and the astigmatism both equal to zero at grazing incidence under the condition $\frac{KI}{Ko} = \frac{K2}{Ko} = 0$ has been examined (page 191.).

That led us to say that the astigmatism was null if the condition

$$-\frac{\cos\beta_o}{\ell_A} = \frac{\sin^2\beta_o}{R} - \frac{K3}{Ko}\cos\beta_o(1+\sin\beta_o)$$

was satisfied.

However, in grazing incidence conditions, the formula (172) providing the value of the term Y^4, of $\Delta^{(4)}$, is :

$$\Delta^{(4)} = Y^4\left[\frac{1}{8R^2\ell_A} + \frac{1}{8R^3}\frac{\sin^2\beta}{\cos\beta} - \frac{(1+\sin\beta)}{8R^2}\frac{K3}{Ko}\right]$$

(Let us remind that $\ell_C = R\cos\gamma$

$\ell_D = R\cos\delta$

$\ell_B = R\cos\beta$ $\longrightarrow$ but to avoid the astigmatism, the relation ( ) is to be satisfied,

$$-\frac{\cos\beta}{\ell_A} = \frac{\sin^2\beta}{R} - \frac{K3}{Ko}\cos\beta_o(1+\sin\beta_o)$$

By substituting $\frac{1}{\ell_A}$ for its value, we have :

$$\Delta^{(4)} = \frac{Y^4}{8R^2}\left[-\frac{\sin^2\beta}{R\cos\beta} + \frac{K3}{Ko}(1+\sin\beta) + \frac{\sin^2\beta}{R\cos\beta} - (1+\sin^2\beta)\frac{K3}{Ko}\right]$$

.../..

Therefore, the term Y^4 is zero if the grating operates at grazing incidence with C D and B on the Rowland circle and A chosen in such a way that the astigmatism is avoided.

So, the spherical aberration and the astigmatism are simultaneously avoided in this case too.

XI - STUDY OF THE SURFACES TYPE { ELLIPSOID, HYPERBOLOID, PARABOLOID } of revolution

used near the pole corresponding to the axis of revolution.

Let us write the equation of a quadric of revolution under the form :

$$(178) \quad Y^2 + Z^2 + (1 - e^2) X^2 - 2 R X = 0$$

1) e is defined as the eccentricity of the conic section

2) the term containing R is the axis of revolution

Condition	conic
$1 - e^2 = 0$	parabola
$1 - e^2 = 1$	sphere
$1 - e^2 < 0$	hyperbola
$0 < 1 - e^2 < 1$	ellipsoid of revolution about major axis
$1 - e^2 < 1$ $1 - e^2 > 0$	ellipsoid of revolution about minor axis

.../..

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From the equation (178) we are going to use the same method as previously followed concerning the spherical mirror :

$$X = \frac{R \pm \left[R^2 - (1 - e^2)\,(Y^2 + Z^2) \right]^{\frac{1}{2}}}{1 - e^2}$$

$$X = \frac{R \pm R\left[1 - \frac{1 - e^2}{R^2}\,(Y^2 + Z^2) \right]^{\frac{1}{2}}}{1 - e^2}$$

The equation of the quadric may be written :

$$(179) \qquad X = \frac{Y^2 + Z^2}{2R} + \frac{1 - e^2}{8\,R^3}\,(Y^2 + Z^2)^2 + \varphi(Y^6 , Z^6)$$

We choose a source point A located by three coordinates x_A, y_A, z_A with $\ell_A^2 = x_A^2 + y_A^2 + z_A^2$

and if M is a point of the quadric surface,

$$\overline{AM}^2 = (X - x_A)^2 + (Y - y_A)^2 + (Z - z_A)^2$$

$$= X^2 + Y^2 + Z^2 - 2\,X\,x_A - 2\,Y\,y_A - 2\,Z\,z_A + x_A^2 + y_A^2 + z_A^2$$

By substituting X and X^2 for their value

$$\overline{AM}^2 = \ell_A^2 - 2\,x_A \left[\frac{Y^2 + Z^2}{2R} + \frac{1 - e^2}{8\,R^3}\,(Y^2 + Z^2)^2 \right] - 2\,(y_A\,Y + z_A\,Z)$$

$$+ Y^2 + Z^2 + \frac{(Y^2 + Z^2)^2}{4\,R^2}$$

.../..

.../..

that may be written by arranging in Y and Z

$$\overline{AM}^2 = \ell_A^2 \left[1 - 2 \left(\frac{y_A Y + z_A Z}{\ell_A^2} \right) + \frac{Y^2 + Z^2}{\ell_A^2} - \frac{x_A}{R \ell_A^2} (\lambda Y^2 + \mu Z^2) \right.$$

$$\left. + \frac{(Y^2 + Z^2)^2}{4R^2} \left[1 - \frac{x_A (1 - e^2)}{R} \right] \right]$$

We have $\overline{AM}^2 = \ell_A^2 (1 + \theta)$

So $AM = \ell_A \left(1 + \frac{\theta}{2} - \frac{\theta^2}{8} + \frac{\theta^3}{16} - \frac{5\theta^4}{128} \right)$

$$\frac{\theta}{2} = \frac{1}{2\ell_A^2} \left[-2(y_A Y + z_A Z) + Y^2 + Z^2 - \frac{x}{R} (Y^2 + Z^2) \right.$$

$$\left. + \frac{(Y^2 + Z^2)^2}{4R^2} \left[1 - \frac{x}{R} (1 - e^2) \right] \right]$$

$$\frac{\theta^2}{8} = -\frac{1}{8\ell_A^4} \left[4(y_A Y + z_A Z)^2 + (Y^2 + Z^2) + \frac{x^2}{R^2} (Y^2 + Z^2)^2 \right.$$

$$-4 (y_A Y + z_A Z)(Y^2 + Z^2) + \frac{4 x_A}{R} (y_A Y + z_A Z)(Y^2 + Z^2)$$

$$\left. - \frac{2 x_A}{R} (Y^2 + Z^2)^2 \right]$$

.../..

.../..

$$+\frac{\theta^3}{16} = \frac{1}{16\,\ell^6}\left[-8(y_A Y + z_A Z)^3 + 12\,(y_A Y + z_A Z)^2\,(Y^2 + Z^2)\right.$$

$$\left. - \frac{12\,x_A}{R}\,(y_A Y + z_A Z)^2\,(Y^2 + Z^2)\right]$$

$$-\frac{5\theta^4}{128} = -\frac{5 \times 16}{128\,\ell^8}\,(y_A Y + z_A Z)^4$$

Arranging we have

$$M\,A = \ell_A$$

$$-\frac{y_A Y + z_A Z}{\ell_1}$$

$$+\frac{1}{2\,\ell_A}\left[Y^2\left(1 - \frac{x_A}{R} - \frac{y_A^2}{\ell_A^2}\right) + Z^2\left(1 - \frac{x_A}{R} - \frac{\alpha_A^2}{\ell_A^2}\right) - 2\,\frac{y_A z_A\,Y Z}{\ell_A^2}\right]$$

$$+\frac{y_A Y + z_A Z}{2\,\ell_A^3}\left[Y^2\left(1 - \frac{x_A}{R} - \frac{y_A^2}{\ell_A^2}\right) + Z^2\left(1 - \frac{x_A}{R} - \frac{z_A^2}{\ell_A^2}\right) - 2\,\frac{y_A z_A\,Y Z}{\ell_A^2}\right]$$

$$+\frac{1}{2\,\ell_A}\,\frac{(Y^2 + Z^2)^2}{4\,R^2}\left[1 - \frac{x_A}{R}\,(1 - e^2) - \frac{1}{8\,\ell_A^3}\,(Y^2 + Z^2)^2\right.$$

A B

.../..

.../..

$$- \frac{1}{8\,\ell_A^3} \quad \frac{x_A^2}{R^2} \quad (Y^2 + Z^2)^2 + \frac{x_A}{4R\,\ell_A^3} (Y^2 + Z^2)^2 + \frac{12}{16\,\ell_A^5} (y_A Z + z_A Z)^2 (Y^2 + Z^2)$$

C D E

$$- \frac{12}{16} \frac{x_A}{R\,\ell_A^5} (yY + zZ)^2 (Y^2 + Z^2) - \frac{5 \times 16}{128\,\ell^7} (y\,Y + z\,Z)^4$$

F G

At this stage we notice that all terms of expansion of MA - in the case of a quadric operating near the poles- are strictly identical with the terms of MA previously obtained for the sphere (page n° 16 ------), term (A) excepted, which is contributing to the expansion of fourth order that is :

$$\frac{1}{2\,\ell_A} \frac{(Y^2 + Z^2)}{4R^2} \left[\left(1 - \frac{x_A}{R} \right) (1 - e^2) \right]$$

So, we are going to study the fourth order terms specifically by using the method followed - page n° 22 and following.

Grouping the terms A + C

$$A + C = \frac{1}{2\ell_A} \frac{(Y^2 + Z^2)^2}{4R^2} \left[1 - \frac{x_A (1 - e^2)}{R} - \frac{x_A^2}{\ell_A^2} \right]$$

.../..

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The classifying of the terms E F G includes the same elements as the sphere's ones. -

Finally, the value of the fourth order term of AM is :

$$AM^{(4)} = \frac{(Y^2+Z^2)^2}{8R^2\,\ell_A}\left(1 - \frac{x_A\cdot(1-e^2)}{R} - \frac{x_A^2}{\ell_A^2}\right) - \frac{1}{8\,\ell_A^3}(Y^2+Z^2)^2\left(1 - \frac{2x_A}{R}\right)$$

$$(180)\qquad + \frac{3}{4\,\ell_A^5}(y_A Y + z_A Z)^2\left[Y^2\left(1 - \frac{x_A}{R} - \frac{5}{6}\frac{y_A^2}{\ell_A^2}\right) + Z^2\left(1 - \frac{x_A}{R} - \frac{5}{6}\frac{z_A^2}{\ell_A^2}\right) - \frac{5}{3}\frac{y_A z_A}{\ell_A^2}\,Y Z\right]$$

We use the same hypothesis as previously i.e. $z = 0$ so

$$x = \ell\cos\alpha$$
$$y = \ell\sin\alpha$$

We may then write :

$$A\,M^{(4)} = \frac{(Y^2+Z^2)^2}{8\,R^2}\left[\frac{1}{\ell_A} - (1-e^2)\frac{\cos\alpha}{R} - \frac{\cos^2\alpha}{\ell_A}\right] -$$

$$\frac{1}{8\ell_A^2}(Y^2+Z^2)^2\left(\frac{1}{\ell_A} - \frac{2\cos\alpha}{R}\right) +$$

$$\frac{3}{4\,\ell_A^4}(y_A Y)^2\left[Y^2\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5}{6}\frac{\sin^2\alpha}{\ell_A}\right) + Z^2\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\right]$$

.../..

.../..

Let us study the term in Y^4

<u>Term of $A\,M^{(4)}$ in Y^4</u>

$$F(Y^4) = Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R} - \frac{\cos^2\alpha}{\ell_A} \right) - \right.$$

$$\left. \frac{1}{8\ell_A^2} \left(\frac{1}{\ell_A} - \frac{2\cos\alpha}{R} \right) + \frac{3}{4\ell_A^2} \sin^2\alpha \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell_A} \right) \right]$$

$$= Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R} \right) - \frac{\cos^2\alpha}{\ell_A} - \frac{1}{8\ell_A^3} + \frac{2\cos\alpha}{8R\ell_A^2} \right.$$

$$\left. + \frac{3}{4\ell_A} \sin^2\alpha \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell_A} \right) \right]$$

The whole terms of the above expression (the first one excepted) are exactly the same as those obtained in the case of the sphere (Equation 19, page 25). Therefore we may calculate the terms in the case of the quadric exactly as we did in the case of the sphere ; the evident result is :

(181)
$$\boxed{F(Y^4) = Y^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R} \right) - \frac{1}{8\ell_A} \left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R} \right) \right.}$$

$$\left. \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5\sin^2\alpha}{\ell_A} \right) \right]$$

.../..

.../..

Considering the equation (180) from which we get the value of $AM^{(4)}$, it is easy to see that the value of the F term (Y^3 Z) will differ in no way from the value of the term in F (Y^3 Z) obtained in the case of the sphere (equation 20).

<u>Term in Y^2 Z^2</u>

$$F(Y^2 Z^2) = Y^2 Z^2 \left\{ \frac{1}{4R^2} \left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R} - \frac{\cos^2\alpha}{\ell_A} \right) \right.$$

$$- \frac{1}{4\ell_A^2} \left(\frac{1}{\ell_A} - \frac{2\cos\alpha}{R} \right) + \frac{3 z^2}{4\ell_A^4} \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5}{6} \frac{\sin^2\alpha}{\ell_A} \right)$$

$$\left. + \frac{3}{4} \frac{\sin^2\alpha}{\ell_A^2} \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5}{6\ell_A^3} z^2 \right) \right.$$

$$\boxed{F(Y^2 Z^2) = Y^2 Z^2 \left\{ \frac{1}{4R^2} \left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R} \right) - \frac{1}{4\ell_A} \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} \right) \left[\frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{3}{4} \frac{\sin^2\alpha}{\ell_A^2} \right] \right\}}$$

(182)

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Term in $Y Z^3$

Considering the general equation of AM [4] we notice that when the quadric is used near the poles, the term in $Y Z^3$ does not differ from the term in $Y Z^3$ obtained in the case of the sphere (equation n° 21 -).

Term in Z^4

$$F(Z^4) = Z^4 \left(\frac{1}{8 R^2 \ell_A} \left(1 - \frac{x_A (1-e^2)}{R} - \frac{x_A^2}{\ell_A^2}\right) - \right.$$

$$\left. \frac{1}{8 \ell_A^3} \left(1 - \frac{2 x_A}{R}\right) + \frac{3 z^2}{4 \ell^5} \left(1 - \frac{x_A}{R} - \frac{5}{6 \ell_A^2} z^2\right) \right)$$

Following the same method as the one used in the sphere' case, we have

$$Z \quad F(Z^4) = Z^4 \left[\frac{1}{8R^2} \left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R} \right) - \right.$$

$$\left. \frac{1}{8 \ell_A} \left(\frac{\cos^2\alpha}{R^2} + \frac{1}{\ell_A^2} - \frac{2\cos\alpha}{R \ell_A} \right) + \frac{3 z^2}{4 \ell_A^7} \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5}{6 \ell_A^2} \right. \right]$$

The independant term of z is :

$$\boxed{F(Z^4) = \left[\frac{1}{8R^2} \left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R} \right) - \frac{1}{8 \ell_A} \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} \right)^2 \right]}$$

(183)

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So, the fourth order term of expansion of AM may be written as follows in the hypothesis $z = 0$

$$
\begin{aligned}
AM^{(4)} = Y^4 \Bigg\{ & \frac{1}{8R^2}\left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R}\right) - \frac{1}{8\,\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right) \\
& \left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R} - \frac{5\sin^2\alpha}{\ell_A}\right)\Bigg\} + Y^2 Z^2 \Bigg\{ \frac{1}{4R^2}\left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R}\right) \\
& - \frac{1}{4\,\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)\left[\frac{1}{\ell_A} - \frac{\cos\alpha}{R} + \frac{3}{4}\,\frac{\sin^2\alpha}{\ell_A}\right]\Bigg\} \\
& + Z^4 \Bigg\{ \frac{1}{8R^2}\left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R}\right) - \frac{1}{8\,\ell_A}\left(\frac{1}{\ell_A} - \frac{\cos\alpha}{R}\right)^2\Bigg\}
\end{aligned}
\tag{184}
$$

Conclusion

We have seen that the whole properties studied(up to fourth order) in the case of the sphere are fully valid as regards a quadric of revolution used near its poles.

In other words, the use of a quadric does not bring any new possibility as to the correction of the aberrations of second and third orders.

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The terms of fourth order are different according to the type of the quadric and are effectively dependent on e.

Let us consider the term in Y^4 of expansion of $\Delta^{(4)}$

$$\Delta^{(4)} = Y^4 \left\{ \frac{1}{8R^2}\left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R}\right) - \right.$$

$$- \frac{1}{8\ell_A}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\cos\alpha}{R}\right)\left(\frac{1}{\ell_A} - \frac{\cos\gamma}{R} - \frac{5\sin^2\alpha}{\ell_A}\right) +$$

$$\frac{1}{8R^2}\left(\frac{1}{\ell_B} - \frac{(1-e^2)\cos\beta}{R}\right) - \frac{1}{8\ell_B}\left(\frac{\cos^2\beta}{\ell_B} - \frac{\cos\beta}{R}\right)$$

$$\left(\frac{1}{\ell_B} - \frac{\cos\beta}{R} - \frac{5\sin^2\beta}{\ell_B}\right) - \frac{\sin\alpha + \sin\beta}{Ko}\left[\right.$$

$$\frac{1}{8R^2}\left[\frac{1}{\ell_C} - \frac{(1-e^2)\cos\gamma}{R}\right] - \frac{1}{8\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right)$$

$$\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \frac{5\sin^2\gamma}{\ell_C}\right) - \frac{1}{8R^2}\left(\frac{1}{\ell_D} - \frac{(1-e^2)\cos\delta}{R}\right)$$

$$\left. + \frac{1}{8\ell_D}\left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R} - \frac{5\sin^2\delta}{\ell_D}\right]\right\}$$

(185)

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Obviously, according to the particular chosen conditions, it will be possible to calculate the term in Y^4 and in some cases this term will prove to be more favourable in the case of a quadric ($e \neq 0$) than in the sphere's one ($e = 0$).

However, the quadric operating near its poles may be coupled with the same Rowland circle as the one coupled with the sphere that results from the quadric with $e = 0$.

The Rowland circle coupled with the quadric will have similar properties with the one coupled with the sphere as regards the second order terms (astigmatism) and the third order ones (the coma will be null on the Rowland circle coupled with the quadric).

Let us suppose we are under the Rowland conditions with the group of solutions

$$\begin{cases} \ell_C = R \cos\gamma \\ \ell_D = R \cos\delta \end{cases}$$

and more

$$\ell_A = R \cos\alpha$$

$$\ell_B = R \cos\beta$$

$\Delta^{(4)}$ is reduced to :

$$\boxed{\Delta^{(4)} = \frac{Y^4}{8R^2}\left[\left(\frac{1}{\ell_A} - \frac{(1-e^2)\cos\alpha}{R}\right) + \left(\frac{1}{\ell_B} - \frac{(1-e^2)\cos\beta}{R}\right) - \frac{(\sin\alpha + \sin\beta)}{K_o}\left[\left(\frac{1}{\ell_C} - \frac{(1-e^2)\cos\gamma}{R}\right) - \left(\frac{1}{\ell_D} - \frac{(1-e^2)\cos\delta}{R}\right)\right]\right]}$$

.../..

.../..

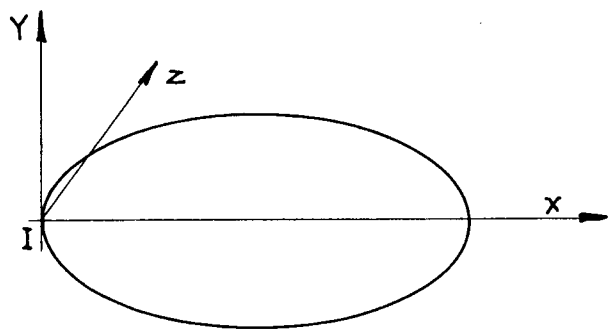
that may be written :

$$\boxed{\Delta^{(4)} = \frac{Y^4}{8R^2}\left[\frac{\sin^2\alpha}{R\cos\alpha} + \frac{\sin^2\beta}{R\cos\beta} - (\sin\alpha + \sin\beta)\frac{K3}{Ko}\right] + \frac{Y^4}{8R^2}\, e^2\left[\frac{\cos\alpha}{R} + \frac{\cos\beta}{R} - \frac{(\sin\alpha + \sin\beta)}{Ko}\left(\frac{\cos\gamma}{R} - \frac{\cos\delta}{R}\right)\right]}$$

(186)

If $\frac{K3}{Ko} = \frac{1}{(\sin\alpha + \sin\beta)}\left(\frac{\sin^2\alpha}{R\cos\alpha} + \frac{\sin^2\beta}{R\cos\beta}\right)$ that is the condition to avoid the astigmatism on the Rowland circle, $\Delta^{(4)}$ is null only if $\underline{e = 0}$ i.e. in the sphere's case.

Therefore we may say that, in this case precisely, the quadrics' properties are less attractive than the sphere's ones.



- Fig. n° 27 -

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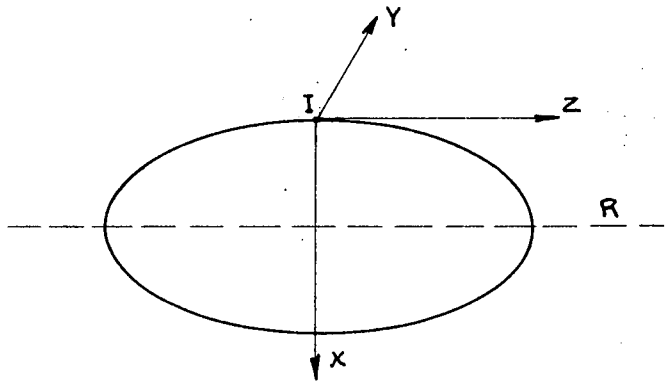
.../..

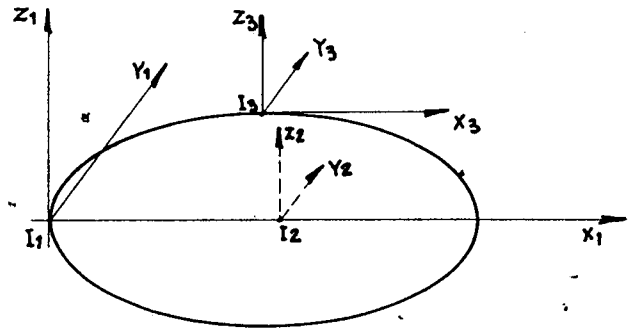
The above study is relating to the quadrics used near the poles i.e. close to the intersection between the quadric and the axis of revolution which is IX in this case.

Let us consider what would be the properties of the quadric used with an axis perpendicular to the axis of revolution :



- Fig. 28 -

We know that, when applied to the axis system I_1 (X_1 Y_1 Z_1), the quadric's equation is : $Y^2 + Z^2 + (1 - e^2) X^2 - 2 R X = 0$



- Fig. 29 -

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The intersections between the quadric and the axis of revolution are $X_1 = 0$ and $X_2 = \frac{2R}{1-e^2}$

So, when changing the variables $x = X - \frac{R}{1-e^2}$

we shall obtain the quadric equation applied to its axis of symmetry I_2 (X_2 Y_2 Z_2) , the axis I_2 X_2 being colinear with the axis I_1 X_1

(See Fig. n° 29.)

We have :

$$\frac{Z^2 + Y^2}{\frac{R^2}{1-e^2}} + \frac{X^2}{\frac{R^2}{(1-e^2)^2}} - 1 = 0$$

the major semi-axis' length is $\frac{R}{1-e^2}$

the minor semi-axis' length is $\frac{R}{\sqrt{1-e^2}}$

Now we are proceeding to a new change of axis which will apply the equation to the trihedral I_3 (X_3 Y_3 Z_3). The axis I_3 Y_3 and the I_2 Y_2 one, are colinear :

.../..

.../..

Under such conditions, the equation is :

$$\frac{Z^2}{R^2} + \frac{Y^2}{\frac{R^2}{1-e^2}} + \frac{2\,Y\,\sqrt{1-e^2}}{R} + \frac{X^2}{\frac{R^2}{(1-e^2)^2}} = 0$$

Obviously, from the Fig. n° 29 , we observe that the system $I_3(X_3\ Y_3\ Z_3)$ is not the one generally used. For obtaining the equation under our usual conditions, it is necessary to change $X \rightarrow Y$ and $Y \rightarrow -X$

So that leads finally to the equation :

$$(187)\quad \frac{Z^2}{R^2} + \frac{X^2}{\frac{R^2}{1-e^2}} - \frac{2\,X\,\sqrt{1-e^2}}{R} + \frac{Y^2}{\frac{R^2}{(1-e^2)^2}} = 0$$

We may write that equation under the form

$$\frac{Z^2}{\sqrt{1-e^2}} + X^2\,\sqrt{1-e^2} - 2\,R\,X + Y^2\,\frac{(1-e^2)^2}{\sqrt{1-e^2}} = 0$$

In order to simplify, we write :

$$\lambda = \frac{(1-e^2)^2}{\sqrt{1-e^2}} \qquad \mu = \frac{1}{\sqrt{1-e^2}} \qquad \nu = \sqrt{1-e^2}$$

So it results the following equation :

$$\nu\,X^2 + \lambda\,Y^2 + \mu\,Z^2 - 2\,R\,X = 0$$

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We may write :

$$X = \frac{1}{\nu}\left[R \pm \left[R^2 - \nu(\lambda Y^2 + \mu Z^2)\right]^{\frac{1}{2}}\right]$$

$$X = \frac{R}{\nu} \pm \frac{R}{\nu}\left[1 - \frac{\nu}{R^2}(\lambda Y^2 + \mu Z^2)\right]^{\frac{1}{2}}$$

$$X = \frac{R}{\nu} \pm \frac{R}{\nu}\left[1 - \frac{\nu}{2R^2}(\lambda Y^2 + \mu Z^2) - \frac{\nu^2}{8R^4}(\lambda Y^2 + \mu Z^2)^2 - \frac{\nu^3}{16R^6}(\lambda Y^2 + \mu Z^2)^3 + \dots\right]$$

(188) So $$X = \frac{\lambda Y^2 + \mu Z^2}{2R} + \frac{\nu(\lambda Y^2 + \mu Z^2)^2}{8R^3} + \varphi(Y^6 \quad Z^6)$$

and $$X^2 = \frac{(\lambda Y^2 + \mu Z^2)^2}{4R^2} + \varphi(Y^6 \quad Z^6)$$

We are going to calculate A M by using the same method as previously followed $A \mid x_A \; y_A \; z_A$

with $\ell_A^2 = x_A^2 + y_A^2 + z_A^2$

$$\overline{AM}^2 = (X - x_A)^2 + (Y - y_A)^2 + (Z - z_A)^2$$

.../..

$$= X^2 + Y^2 + Z^2 - 2\,X\,x_A - 2\,Y\,y_A - 2Z\,z_A + x_A^2 + y_A^2 + z_A^2$$

Substituting X and X^2 for their value

$$A\,M^2 = \ell_A^2 - 2\,x_A \left[\frac{\lambda Y^2 + \mu Z^2}{2R} + \frac{\nu(\lambda Y^2 + \mu Z^2)^2}{8R^3}\right] - 2\,(y_A\,Y + z_A\,Z)$$

$$+ Y^2 + Z^2 + \frac{(\lambda Y^2 + \mu Z^2)^2}{4R^2}$$

By classifying in Y and Z we may write :

$$\overline{A\,M}^2 = \ell_A^2 \left[1 - 2\,\frac{(y_A\,Y + z_A\,Z)}{\ell_A^2} + \frac{Y^2 + Z^2}{\ell_A^2} - \frac{x_A}{R\ell_A^2}(\lambda Y^2 + \mu Z^2)\right.$$

$$\left. + \frac{(\lambda Y^2 + \mu Z^2)^2}{4R^2}\left(1 - \frac{\nu x_A}{R}\right)\right]$$

We have $\overline{A\,M}^2 = \ell_A^2\,(1 + \theta)$

So $A\,M = \ell_A\left(1 + \frac{\theta}{2} - \frac{\theta^2}{8} + \frac{\theta^3}{16} - \frac{5\theta^4}{128}\right)$

We shall take an interest in <u>the second order terms only</u>

$$\Delta^{(2)} = \frac{1}{2\ell_A}\left[Y^2 + Z^2 - \frac{x_A}{R}(\lambda Y^2 + \mu Z^2)\right] - \frac{1}{2\ell_A^3}(y_A\,Y + z_A\,Z)^2$$

(189)
$$= \frac{1}{2\ell_A}\left[Y^2\left(1 - \frac{\lambda x_A}{R} - \frac{y_A^2}{\ell_A^2}\right) + Z^2\left(1 - \frac{\mu x_A}{R} - \frac{Z_A^2}{\ell_A^2}\right) - \frac{2y_A\,z_A\,YZ}{\ell_A^2}\right]$$

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At this stage, we notice that the second term is <u>different</u> from the one obtained in the case of the sphere while it was identical when using the quadric near its pole.

Let us write $z = 0$ $\quad x_A = \ell_A \cos\alpha$

$$y_A = \ell_A \sin\alpha$$

$$\Delta^{(2)} = \frac{Y^2}{2}\left(\frac{\cos^2\alpha}{\ell_A} - \frac{\lambda\cos\alpha}{R}\right) + \frac{Z^2}{2}\left(\frac{1}{\ell_A} - \frac{\mu\cos\alpha}{R}\right) \tag{190}$$

If we calculate M B, M C and M D, in the same way, we shall have finally :

$$\boxed{\begin{aligned}\Delta^{(2)} = \frac{Y^2}{2}\Bigg[&\frac{\cos^2\alpha}{\ell_A} - \frac{\lambda\cos\alpha}{R} + \frac{\cos^2\beta}{\ell_B} - \frac{\lambda\cos\beta}{R} - \\ &\frac{(\sin\alpha + \sin\beta)}{Ko}\left[\frac{\cos^2\gamma}{\ell_C} - \frac{\lambda\cos\gamma}{R} - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\lambda\cos\delta}{R}\right)\right]\Bigg] \\ + \frac{Z^2}{2}\Bigg[&\frac{1}{\ell_A} - \frac{\mu\cos\alpha}{R} + \frac{1}{\ell_B} - \frac{\mu\cos\beta}{R} - \\ &\frac{(\sin\alpha + \sin\beta)}{Ko}\left[\frac{1}{\ell_C} - \frac{\mu\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\mu\cos\delta}{R}\right)\right]\Bigg]\end{aligned}} \tag{191}$$

We notice that two coefficients may be defined :

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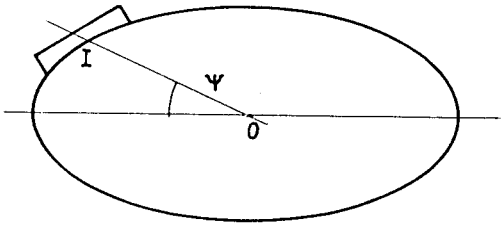
$$(192)\quad \boxed{\begin{aligned} KI' &= \frac{\cos^2\gamma}{\ell_C} - \frac{\lambda\cos\gamma}{R} - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\lambda\cos\delta}{R}\right) \\ K3' &= \frac{1}{\ell_C} - \frac{\mu\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\mu\cos\delta}{R}\right) \end{aligned}}$$

those coefficients KI' and $K3'$ are different from those reached with the sphere (or the quadric near its poles).

To sum up it appears that the fact of using a quadric located at $\underline{\cap}$ from the poles allows using a new parameter, as soon as the second order terms, whereas this new parameter appears only with the fourth order terms if the quadric is used near its poles.

It seems that if one uses the quadric under general conditions, one benefits by a new correction parameter which is the angle Ψ between the straight line I O and the axis of revolution.



- Fig. n° 28 -

XII - DETERMINATION OF CONSTRUCTION PARAMETERS

Construction parameters are the focal distances ℓ_C , ℓ_D and the angles γ and δ .

A first relation is given by

$$Ko = \sin\gamma - \sin\delta \text{ (eq. 11- 2nd report)}$$

Then, we have seen (page n° 16 of 1st report) that we have the relation

$$\sin\alpha + \sin\beta - k\frac{\lambda}{\lambda_0}(\sin\gamma - \sin\delta) = 0$$

i.e.

$$\sin\alpha + \sin\beta = \frac{Ko}{\lambda_0} \cdot k \cdot \lambda$$

So we remark that $\frac{\lambda_0}{Ko}$ is the groove spacing if compared to the classical gratings formula

$$\sin\alpha + \sin\beta = \frac{k \cdot \lambda}{d}$$

Our studies relating to the properties of spectrographs lead to the determination of the coefficients :

$\frac{KI}{Ko}$ $\frac{K2}{Ko}$ $\frac{K3}{Ko}$ and $\frac{K4}{Ko}$

According to the problem concerned, either the whole coefficients or some of them only have been determined.
Therefore, those coefficients $\frac{Ki}{Ko}$ have been obtained from the use conditions required i.e. $\frac{Ki}{Ko}$ is, then, expressed in ℓ_A, ℓ_B, α and β.

More, the grooves number, i.e. Ko, is known.

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From the values $\frac{Ki}{Ko}$ we may then obtain the necessary Ki values so as to meet the requirements. We have now to know about the feasibility of the grating i.e. determine the ℓ_C ℓ_D γ and δ parameters from the Ki values obtained as said above.

We have to solve the following system:

$$Ko = \sin\gamma - \sin\delta$$

$$KI = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$K2 = \frac{\sin\gamma}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$K3 = \frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$K4 = \frac{\sin\gamma}{\ell_C}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

In this system, Ko, KI, K2, K3 and K4 are numerical values. Then, we have 4 unknown quantities : ℓ_C ℓ_D γ and δ. Therefore, - in principle - <u>there will be no solution for the equations system</u>.

Two cases may be considered :

- First, one decide to work with a strictly fixed number of grooves and then one must give up one of the four other equations,
- Second, one decide that the number of grooves may be variable and then the solution is feasible as we introduce a new unknown quantity $d\,Ko$ which is the variation of the groove spacing.

The Table (II) (separated from this report) indicates - for some combinations of γ, δ, the numbers of grooves per mm - therefore the Ko - as $d = \frac{\lambda o}{Ko}$ d being the groove spacing. So, if n is the number of grooves per mm $n = \frac{1}{d}$ d being expressed in mm. We have

$$n = \frac{Ko}{\lambda o} \quad \lambda o \quad " \quad " \quad " \quad .$$

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The Table (III) (same remark as above.) indicates the values of the various Ki for some values of ℓ_C ℓ_D γ and δ. We shall identify the Ki of the Table : Ki^* .

Now, let us suppose that the study of the use requirements leads to draw up the Table of the necessary Ki which we shall identify : $\overline{Ki}$.

The best conditions would be to obtain, from the Ki^* Table, a group of values coinciding with the $\overline{Ki}$, precisely.

Generally, that will not occur but one may hope to find out a group of Ki^* approaching the $\overline{Ki}$.

A series of given values of ℓ_C ℓ_D γ and δ corresponds to those Ki^* .

Now, if the parameters ℓ_C ℓ_D γ and δ are respectively given the increments $d\ell_C$, $d\ell_D$, $d\gamma$, $d\delta$ one may write :

$$Ki\,(\ell_C + d\ell_C\,,\ \ell_D + d\ell_D\,,\ \gamma + d\gamma\,,\ \delta + d\delta) =$$

$$Ki\,(\ell_C\,,\ell_D,\ \gamma\,,\ \delta) + \frac{\partial Ki}{\partial \ell_C}\,d\ell_C + \frac{\partial Ki}{\partial \ell_D}\,d\ell_D$$

$$+ \frac{\partial Ki}{\partial \gamma}\,d\gamma + \frac{\partial Ki}{\partial \delta}\,d\delta$$

So, we have four relations as follows :

$$\overline{Ki} = Ki^* + \frac{\partial Ki}{\partial \ell_C}\,d\ell_C + \frac{\partial Ki}{\partial \ell_D}\,d\ell_D + \frac{\partial Ki}{\partial \gamma}\,d\gamma$$

(193)

$$+ \frac{\partial Ki}{\partial \delta}\,d\delta$$

equations in which $\overline{Ki}$, Ki^*, $\frac{\partial Ki}{\partial \ell_C}$, $\frac{\partial Ki}{\partial \ell_D}$, $\frac{\partial Ki}{\partial \gamma}$ and $\frac{\partial Ki}{\partial \delta}$ are coefficients

and $d\ell_C$, $d\ell_D$, $d\gamma$, $d\delta$ and possibly Ko, are unknown quantities.

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CALCULATION of $\frac{\partial Ki}{\partial}$

It is easy to observe that :

- First $\frac{\partial Ki}{\partial \ell_C}$ and $\frac{\partial Ki}{\partial \ell_D}$

- Second $\frac{\partial Ki}{\partial \gamma}$ and $\frac{\partial Ki}{\partial \delta}$

are expressed in the same way, except for sign

$$KI = \frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)$$

In fact, we choose $\frac{1}{\ell_C}$ as variable, preferably to ℓ_C

(194) $$\frac{\partial KI}{\partial\left(\frac{1}{\ell_C}\right)} = \cos^2\gamma \qquad \frac{\partial KI}{\partial\left(\frac{1}{\ell_D}\right)} = -\cos^2\delta \qquad (195)$$

$$\frac{\partial KI}{\partial \gamma} = -\frac{2\cos\gamma\,\sin\gamma}{\ell_C} + \frac{\sin\gamma}{R} = \sin\gamma\left(\frac{1}{R} - \frac{2\cos\gamma}{\ell_C}\right) \qquad (196)$$

$$\frac{\partial KI}{\partial \delta} = -\sin\delta\left(\frac{1}{R} - 2\frac{\cos\delta}{\ell_D}\right) \qquad (197)$$

$$K2 = \frac{\sin\gamma}{\ell_C}\left(\frac{\cos^2\gamma}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{\cos^2\delta}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$\frac{\partial K2}{\delta\left(\frac{1}{\ell_C}\right)} = \frac{2\sin\gamma\cos^2\gamma}{\ell_C} - \frac{\sin\gamma\cos\gamma}{R} = \sin\gamma\cos\gamma\left(\frac{2\cos\gamma}{\ell_C} - \frac{1}{R}\right) \qquad (198)$$

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$$\frac{\partial K2}{\partial\left(\frac{1}{\ell_D}\right)} = -\sin\delta\cos\delta\left(\frac{2\cos\delta}{\ell_D} - \frac{1}{R}\right) \qquad (199)$$

$$\frac{\partial K2}{\partial\gamma} = \frac{\cos^3\gamma - 2\cos\gamma\sin^2\gamma}{\ell^2_C} - \frac{\cos^2\gamma - \sin^2\gamma}{R\,\ell_C} \qquad (200)$$

$$\frac{\partial K2}{\partial\delta} = -\frac{\cos^3\delta - 2\cos\delta\sin^2\delta}{\ell^2_D} + \frac{\cos^2\delta - \sin^2\delta}{R\,\ell_D} \qquad (201)$$

$$K3 = \frac{1}{\ell_C} - \frac{\cos\gamma}{R} - \left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$\frac{\partial K3}{\partial\left(\frac{1}{\ell_C}\right)} = 1 \quad (202) \qquad \frac{\partial K3}{\partial\left(\frac{1}{\ell_D}\right)} = -1 \qquad (203)$$

$$(204)\quad \frac{\partial K3}{\partial\gamma} = \frac{\sin\gamma}{R} \qquad \frac{\partial K3}{\partial\delta} = \frac{-\sin\delta}{R} \qquad (205)$$

$$K4 = \frac{\sin\gamma}{\ell_C}\left(\frac{1}{\ell_C} - \frac{\cos\gamma}{R}\right) - \frac{\sin\delta}{\ell_D}\left(\frac{1}{\ell_D} - \frac{\cos\delta}{R}\right)$$

$$\frac{\partial K4}{\partial\left(\frac{1}{\ell_C}\right)} = \frac{2\sin\gamma}{\ell_C} - \frac{\sin\gamma\cos\gamma}{R} = \sin\gamma\left(\frac{2}{\ell_C} - \frac{\cos\gamma}{R}\right) \qquad (206)$$

$$\frac{\partial K4}{\partial\left(\frac{1}{\ell_D}\right)} = -\sin\delta\left(\frac{2}{\ell_C} - \frac{\cos\delta}{R}\right) \qquad (207)$$

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$$\frac{\partial K4}{\partial \gamma} = \frac{\cos\gamma}{\ell_C^2} - \frac{\cos^2\gamma - \sin^2\gamma}{R\,\ell_C} = \frac{1}{\ell_C}\left(\frac{\cos\gamma}{\ell_C} - \frac{\cos^2\gamma - \sin^2\gamma}{R}\right) \quad (208)$$

$$\frac{\partial K4}{\partial \delta} = -\frac{1}{\ell_D}\left(\frac{\cos\delta}{\ell_D} - \frac{\cos^2\delta - \sin^2\delta}{R}\right) \quad (209)$$

So, the following system has to be solved :

$$\cos^2\gamma\, d\left(\frac{1}{\ell_C}\right) - \cos^2\delta\, d\left(\frac{1}{\ell_D}\right) + \sin\gamma_0\left(\frac{1}{R} - \frac{2\cos\gamma}{\ell_C}\right) d\gamma$$

$$- \sin\delta\left(\frac{1}{R} - \frac{2\cos\delta}{\ell_D}\right) d\delta = \overline{Ki} - Ki^*$$

$$\sin\gamma\cos\gamma\left(\frac{2\cos\gamma}{\ell_C} - \frac{1}{R}\right) d\left(\frac{1}{\ell_C}\right) - \sin\delta\cos\delta\left(\frac{2\cos\delta}{\ell_D} - \frac{1}{R}\right) d\left(\frac{1}{\ell_D}\right)$$

$$+ \left(\frac{\cos^3\gamma - 2\cos\gamma\sin^2\gamma}{\ell_C^2} - \frac{\cos^2\gamma - \sin^2\gamma}{R\,\ell_C}\right) d\gamma$$

$$- \left(\frac{\cos^3\delta - 2\cos\delta\sin^2\delta}{\ell_D^2} - \frac{\cos^2\delta - \sin^2\delta}{R\,\ell_D}\right) d\delta = \overline{K2} - K2^*$$

$$d\left(\frac{1}{\ell_C}\right) - d\left(\frac{1}{\ell_D}\right) + \frac{\sin\gamma}{R}\, d\gamma - \frac{\sin\delta}{R}\, d\delta = \overline{K3} - K3^*$$

$$\sin\gamma\left(\frac{2}{\ell_C} - \frac{\cos\gamma}{R}\right) d\left(\frac{1}{\ell_C}\right) - \sin\delta\left(\frac{2}{\ell_D} - \frac{\cos\delta}{R}\right) d\left(\frac{1}{\ell_D}\right)$$

$$+ \frac{1}{\ell_C}\left(\frac{\cos\gamma}{\ell_C} - \frac{\cos^2\gamma - \sin^2\gamma}{R}\right) d\gamma - \frac{1}{\ell_D}\left(\frac{\cos\delta}{\ell_D} - \frac{\cos^2\delta - \sin^2\delta}{R}\right) d\delta = \overline{K4} - K4^*$$

$$d\,Ko = \cos\gamma\, d\gamma - \cos\delta\, d\delta \quad (210)$$

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Obviously, we may proceed by successive approximations.
Let us suppose we have solved the above system. We may introduce into the functions Ki the values of $d\left(\frac{1}{\ell_C}\right)$, $d\left(\frac{1}{\ell_D}\right)$, $d\gamma$ and $d\delta$, thus determined.

By this way we obtain a new series of Ki.*

If that series differs yet from the $\overline{Ki}$, one calculate the new coefficients $\frac{\partial Ki}{\partial}$ which we introduce into the new system of linear equations from which we obtain the new values for $d\left(\frac{1}{\ell_C}\right)$ $d\left(\frac{1}{\ell_D}\right)$ $d\gamma$ and $d\delta$.