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APOLLO GUIDANCE AND NAVIGATION NOTES

These notes are working papers generated by Bissett-Berman personnel working on NASA contracts. Notes 1 through 171 were written under NASA Headquarters direction. The following notes, starting with Note 172, are under the direction of NASA, Manned Spacecraft Center.

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APOLLO NOTE NO. 172

H. Dale/C. Siska
17 February 1964

A PRELIMINARY ANALYSIS OF THE ABILITY OF THE MSFN TO TRACK THE LEM DURING THE BOOST PORTION OF THE LUNAR ASCENT

The present computer program, described in Apollo Note No. 164, is not capable of handling thrusting trajectories. However, a continuous thrust trajectory may be approximated by a series of short coasting segments, the endpoints of which have been properly adjusted. By this technique it is possible to estimate the covariance matrix as a function of time up to and including burnout.

This note considers the case of three doppler range stations tracking the ascending LEM. The LEM is assumed to be following a thrust magnitude and body angular rate program. The computing center on Earth is assumed to be receiving the LEM's body-measured longitudinal acceleration and body rates through a data link. What the Earth does not know are the errors in the LEM's initial attitude and the error in the longitudinal accelerometer. As the Earth-based computing center increases its knowledge of the actual ascent, it is assumed to be able to update the LEM's program of body rates and cut-off time. In this preliminary analysis, the tracking stations are assumed to measure range ($\sigma R = 0.5$ meters for 1 second samples) with a bias which is known to within 10 meters. The initial position of the LEM on the surface of the Moon is assumed to be known to within 1 meter. The one sigma accelerometer error is assumed to be one per cent of the measured acceleration, and the initial attitude errors are: sigma pitch = sigma yaw = sigma roll = 2°.

Detailed Method of Analysis

Table 1 contains points on a reference trajectory from lunar ignition to burnout. x is defined along the local radius and y is the ground trace. Now it is possible to fit a parabolic trajectory, thrust free, to the points on the reference trajectory.

Time (sec.) t	Altitude (m.) x	Range (m.) y	Impulsive Velocity Components Necessary to Match Ref. Traj. with Coasting Segments (m/sec.) $\dot{x}_i$	$\dot{y}_i$	
0	0	0	59.36	69.72	
15	707.96	1045.85	96.90	201.66	
30	1979.07	4070.77	124.79	351.83	
45	3668.52	9348.29	141.95	518.10	
60	5615.41	17119.80	148.39	698.30	
75	7658.84	27594.40	141.95	891.38	
90	9605.73	40965.10	122.65	1097.33	
105	11263.00	57425.20	89.39	1315.08	
120	12421.50	77151.60	42.20	1545.70	
135	12872.00	100377.00			

Table 1

A Reference Trajectory

Thus

$$\dot{y}_i(t_k) = \frac{y(t_{k+1}) - y(t_k)}{t_{k+1} - t_k}$$

$$\dot{x}_i(t_k) = \frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k} + g\,\frac{t_{k+1} - t_k}{2}$$

describe the incremental velocity components at x_k, y_k, and t_k which will allow the LEM to coast under the influence of lunar gravity to x_{k+1}, y_{k+1} in time t_{k+1}. These numbers are also shown in Table 1.

Now it is necessary to develop the information matrix for the first segment of the thrusting portion of the ascent. The input conditions for the three stations are as follows:

Madrid Station

A1	$= .17525 \times 10^4$ m		initial orbit radius
A4	$= .59359 \times 10^2$ m/s		initial radial velocity
A5	$= .697229 \times 10^2$ m/s		initial tangential velocity
GAM ID	$= 1.0$		
BETA	$= .18308 \times 10^2$	deg.	inclination of Earth-Moon Orbit to Equator
XI	$= 0$	deg.	Euler angles relating x' y' z' coordinates to $\hat{x}\ \hat{y}\ \hat{z}$ coordinates (see Note No. 82)
ETA	$= .180 \times 10^3$	deg.	
ZETA	$= .180 \times 10^3$	deg.	
LAMBA	$= .402 \times 10^2$	deg.	station latitude
ALPHA	$= -.370 \times 10^1$	deg.	station longitude
OMEGA E	$= .729116 \times 10^{-4}$	rad/sec	earth angular rate
OMEGA M	$= .4236 \times 10^{-6}$	rad/sec	earth-moon angular rate
RHO E	$= .63781 \times 10^7$	m.	radius of earth
RHO M	$= .385 \times 10^9$		earth-moon distance
MU	$= .4896 \times 10^{13}$	m^3/sec^2	gravitational constant
BIG L	$= .40 \times 10^1$	deg.	initial latitude of sub-lunar point
SMA L	$= .60 \times 10^2$	deg.	initial longitude of sub-lunar point
SD IND	$= 0$		
S IND	$= 1$		
PS 1	$= 0$		
PS 2	$= 0$		
PVW IND	= ——		

TJ	= .25 min.	maximum time of observation
Q IND	= 1	
Q F	= ——	
TI (initial)	= .016666 min.	time of first observation
TI (interval)	= .016666 min.	interval between observation

Ascension Station		Carnarvon Station	
same except:		same except:	
LAMBA	$= -.79 \times 10^{1}$ deg.	LAMBA	$= -.2487 \times 10^{2}$ deg.
ALPHA	$= -.143 \times 10^{2}$ deg.	ALPHA	$= .1137 \times 10^{3}$ deg.

The above three sets of input conditions are processed using range as a measurable ($\sigma R = 0.5$ meters) with the assumption that a bias exists in the measurement.

In addition to the information derived by the above three radars, there exists some a priori information regarding the orbital segment. It is assumed that the initial position is known to about one meter in each of three orthogonal directions. The velocity components are assumed to be quite well known ($\sqrt{10^{-3}}$ m/s not including platform or accelerometer errors). The range measurement biases are assumed to be known to within 10 meters for each station. Thus an a priori information matrix may be generated, the diagonal terms of which are the reciprocals of the squares of the assumed mean deviations. The off-diagonal terms are zero.

	x	y	z	$\dot{x}$	$\dot{y}$	$\dot{z}$	R_1	R_2	R_3
x	1	0	0	0	0	0	0	0	0
y	0	1	0	0	0	0	0	0	0
z	0	0	1	0	0	0	0	0	0
$\dot{x}$	0	0	0	1000	0	0	0	0	0
$\dot{y}$	0	0	0	0	1000	0	0	0	0
$\dot{z}$	0	0	0	0	0	1000	0	0	0
R_1	0	0	0	0	0	0	.01	0	0
R_2	0	0	0	0	0	0	0	.01	0
R_3	0	0	0	0	0	0	0	0	.01

In the computer program the above information is combined with the tracking information to yield a total information matrix. By inverting this and through the use of a Jacobian transformation, the covariance matrix at the end of the orbital segment is obtained. Up to this point no allowance has been made for unknown errors due to the initial errors in the LEM attitude and due to an unknown error in the longitudinal accelerometer. Thus the covariance matrix mentioned above is not yet complete.

Assume a fixed reference system with x, y, and z defined as altitude, surface range, and out-of-plane coordinate respectively. The initial errors in the LEM attitude rotate this frame by α_x, αy, and αz successively giving the body coordinates $\overline{x}$, $\overline{y}$, and $\overline{z}$. The programmed thrust vector will have components $\overline{T}_x$, $\overline{T}_y$, and $\overline{T}_z$ (which is defined as zero). Thus in the fixed inertial frame the thrust components to first orders are:

$$
\begin{aligned}
T_x &= \overline{T}_x + \alpha_z \overline{T}_y \\
T_y &= \overline{T}_y - \alpha_z \overline{T}_x \qquad (1) \\
T_z &= \alpha_x \overline{T}_y + \alpha_y \overline{T}_x
\end{aligned}
$$

Assuming a constant g field, the inertial accelerations are:

$$\ddot{x} = \frac{T_x}{m} - g$$

$$\ddot{y} = \frac{T_y}{m} \qquad (2)$$

$$\ddot{z} = \frac{T_z}{m}$$

If the LEM is correctly aligned to the inertial frame ($\alpha_x = \alpha_y = \alpha_z = 0$), then the programmed thrust will generate the desired ascent trajectory. Thus at any time, t_k

$$\dot{\bar{x}}_{t_k} = \int_0^{t_k} \frac{\overline{T}_x}{m}\, dt - g\, t_k$$

$$\dot{\bar{y}}_{t_k} = \int_0^{t_k} \frac{\overline{T}_y}{m}\, dt \qquad (3)$$

$$\dot{\bar{z}}_{t_k} = 0$$

Defining the velocity component errors as $\Delta\dot{x} = \int_0^{t_k} \ddot{x}\, dt - \dot{\bar{x}}_{t_k}$, etc. for each time, t_k, results in

$$\Delta\dot{x} = \alpha_z\, \dot{\bar{y}}_{t_k}$$

$$\Delta\dot{y} = -\alpha_z \left(\dot{\bar{x}}_{t_k} + g\, t_{k+1} \right) \qquad (4)$$

$$\Delta\dot{z} = \alpha_x\, \dot{\bar{y}}_{t_k} + \alpha_y \left(\dot{\bar{x}}_{t_k} + g\, t_k \right)$$

Now if α_x, α_y, and α_z are constant with time but are random variables with variances $\overline{\alpha_x^2}$, $\overline{\alpha_y^2}$, $\overline{\alpha_z^2}$, then the variances of the velocity errors are:

$$\sigma \dot{x}^2 = \overline{\alpha_z^2} \; \dot{\bar{y}}_{t_k}^2$$

$$\sigma \dot{y}^2 = \overline{\alpha_z^2} \left(\dot{\bar{x}}_{t_k} + g\, t_k \right)^2 \tag{5}$$

$$\sigma \dot{z}^2 = \overline{\alpha_x^2} \; \dot{\bar{y}}_{t_k}^2 + \overline{\alpha_y^2} \left(\dot{\bar{x}}_{t_k} + g\, t_k \right)^2$$

However, this does not include errors in the accelerometer. Since the accelerometer is aligned with the longitudinal or thrust axis, any error in the accelerometer will add a first order error to the horizontal in-plane and vertical components of velocity. Assuming that the accelerometer error is some constant fractional part of the actual acceleration, then equation (5) may be rewritten as:

$$\sigma \dot{x}^2 = \overline{\alpha_z^2} \; \dot{\bar{y}}_{t_k}^2 + e^2 \left(\dot{\bar{x}}_{t_k} + g\, t_k \right)^2$$

$$\sigma \dot{y}^2 = \overline{\alpha_z^2} \left(\dot{\bar{x}}_{t_k} + g\, t_k \right)^2 + e^2 \; \dot{\bar{y}}_{t_k}^2 \tag{6}$$

$$\sigma \dot{z}^2 = \overline{\alpha_x^2} \; \dot{\bar{y}}_{t_k}^2 + \overline{\alpha_y^2} \left(\dot{\bar{x}}_{t_k} + g\, t_k \right)^2$$

where e is the fractional expected random error in the accelerometer ($e = .01$ for the numerical cases studied herein).

In a similar fashion equations for the variance of position may be written:

$$\sigma x^2 = \overline{\alpha_z^2}\, \bar{y}_{t_k}^{\,2} + e^2 \left(\bar{x}_{t_k} + \frac{g}{z} t_k^{\,2} \right)^2$$

$$\sigma y^2 = \overline{\alpha_z^2} \left(\bar{x}_{t_k} + \frac{g}{2} t_k^{\,2} \right)^2 + e^2\, \bar{y}_{t_k}^{\,2} \qquad (7)$$

$$\sigma z^2 = \overline{\alpha_x^2}\, \bar{y}_{t_k}^{\,2} + \overline{\alpha_y^2} \left(\bar{x}_{t_k} + \frac{g}{z} t_k^{\,2} \right)^2$$

Equations (6) and (7) express the variances in position and velocity for an ascent which is never corrected or updated during the thrusting period. Thus in order to find the additional uncertainty which should be added to the previously derived covariance matrix between times t_k and t_{k+1}, differences in the above equations must be taken as a function of time:

$$\Delta \sigma x^2 = \sigma x^2 (t_{k+1}) - \sigma x^2 (t_k)$$

$$\Delta \sigma y^2 = \sigma y^2 (t_{k+1}) - \sigma y^2 (t_k) \qquad (8)$$

$$\vdots$$

$$\Delta \sigma \dot{z}^2 = \sigma \dot{z}^2 (t_{k+1}) - \sigma \dot{z}^2 (t_k)$$

These numbers may be used to generate an a priori matrix which expresses the additional uncertainties in the desired parameters caused by platform misalignment.

$$\begin{bmatrix} \Delta\sigma x^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta\sigma y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta\sigma z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta\sigma \dot{x}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta\sigma \dot{y}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta\sigma \dot{z}^2 \end{bmatrix}$$

When the above covariance matrix is added to the previously derived covariance matrix (which used tracking and old a priori data), a new and total estimate of the expected errors is obtained at the end of the boost increment. (In this example it is the first boost increment.)

Now by inverting this total covariance matrix a new information matrix is obtained which is used as a priori information at the start of the next boost segment. This process is continued until ascent engine burnout.

Numerical Results

First of all, the numerical results will be presented for the case of no tracking. This is a check or worst case which may be used to show what improvement can be made through the use of

MSFN tracking. As pointed out previously, with no tracking the uncertainty in position and velocity is given by equations (6) and (7) above. Figure 1 presents the uncertainty in positional elements as a function of time (measured from lift-off) for an assumed a priori knowledge of initial alignment (2° about each axis: pitch, yaw, and roll; with a 1% longitudinal accelerometer). Figure 2 presents corresponding velocity uncertainties. Also included in Figure 2 is the RMS error in total velocity which is simply

$$\sigma V = \sqrt{\sigma \dot{x}^2 + \sigma \dot{y}^2 + \sigma \dot{z}^2}$$

In the computer program used for the enclosed example calculation, the burning period was split into nine steps. The computer print-out can thus be tabulated at the end of each of the nine segments.

Table 2

Example Output of Computer Program
Velocity Variances vs. Time

Computer Output Before Adding A Priori Errors $(m/sec.)^2$

t (sec.)	$\sigma \dot{x}^2$	$\sigma \dot{y}^2$	$\sigma \dot{z}^2$	σV^2	σV (m/sec)
0	0	0	0	0	0
15	$.249 \times 10^{-3}$	$.999 \times 10^{-3}$	$.100 \times 10^{-2}$	$.225 \times 10^{-2}$	0.0
30	$.396 \times 10^{-3}$	$.144 \times 10^{1}$	$.612 \times 10^{1}$	$.756 \times 10^{1}$	2.8
45	$.463 \times 10^{-3}$	$.179 \times 10^{1}$	$.913 \times 10^{1}$	$.109 \times 10^{2}$	3.3
60	$.577 \times 10^{-3}$	$.185 \times 10^{1}$	$.978 \times 10^{1}$	$.116 \times 10^{2}$	3.4
75	$.806 \times 10^{-3}$	$.188 \times 10^{1}$	$.100 \times 10^{2}$	$.119 \times 10^{2}$	3.4
90	$.123 \times 10^{-2}$	$.189 \times 10^{1}$	$.102 \times 10^{2}$	$.121 \times 10^{2}$	3.5
105	$.199 \times 10^{-2}$	$.189 \times 10^{1}$	$.102 \times 10^{2}$	$.121 \times 10^{2}$	3.5
120	$.323 \times 10^{-2}$	$.190 \times 10^{1}$	$.103 \times 10^{2}$	$.122 \times 10^{2}$	3.5
135	$.518 \times 10^{-2}$	$.190 \times 10^{1}$	$.103 \times 10^{2}$	$.122 \times 10^{2}$	3.5

(Table 2 continued)

A Priori Additional Errors to be Added After Each Segment $(m/sec)^2$ (See Equation 8)

t(sec)	$\Delta\sigma \dot{x}^2$	$\Delta\sigma \dot{y}^2$	$\Delta\sigma \dot{z}^2$	$\sum\Delta\sigma \dot{x}^2$	$\sum\Delta\sigma \dot{y}^2$	$\sum\Delta\sigma \dot{z}^2$	σV^2
0	0	0	0	0	0	0	0
15	23.1	10.7	31.2	23.1	10.7	31.2	65
30	78.6	24.5	95.	101.7	35.2	126.	263
45	144.3	31.8	164.	246.	67.	290.	603
60	238.	38.	254.	484.	105.	544.	1133
75	336.	42.	350.	820.	147.	894.	1861
90	449.	45.	456.	1269.	192.	1350.	2811
105	622.	48.	620.	1891.	240.	1970.	4101
120	780.	53.	770.	2671.	293.	2741.	5704
135	957.	62.	942.	3628.	335.	3682.	7665

$\sqrt{}$ Plotted in Figure 2

Computer Output Including A Priori Addition $(m/sec)^2$

t(sec)	$\sigma \dot{x}^2$	$\sigma \dot{y}^2$	$\sigma \dot{z}^2$	σV^2	σV (m/sec)
0	0	0	0	0	0
15	$.231 \times 10^2$	$.107 \times 10^2$	$.310 \times 10^2$	$.648 \times 10^2$	8.0
30	$.787 \times 10^2$	$.259 \times 10^2$	$.101 \times 10^3$	$.206 \times 10^3$	13.4
45	$.144 \times 10^3$	$.338 \times 10^2$	$.173 \times 10^3$	$.351 \times 10^3$	18.7
60	$.238 \times 10^3$	$.399 \times 10^2$	$.264 \times 10^3$	$.542 \times 10^3$	23.3
75	$.336 \times 10^3$	$.439 \times 10^2$	$.360 \times 10^3$	$.740 \times 10^3$	27.2
90	$.449 \times 10^3$	$.469 \times 10^2$	$.466 \times 10^3$	$.962 \times 10^3$	31.0
105	$.622 \times 10^3$	$.499 \times 10^2$	$.630 \times 10^3$	$.130 \times 10^4$	36.1
120	$.780 \times 10^3$	$.549 \times 10^2$	$.780 \times 10^3$	$.162 \times 10^4$	40.2
135	$.957 \times 10^3$	$.639 \times 10^2$	$.952 \times 10^3$	$.197 \times 10^4$	44.4

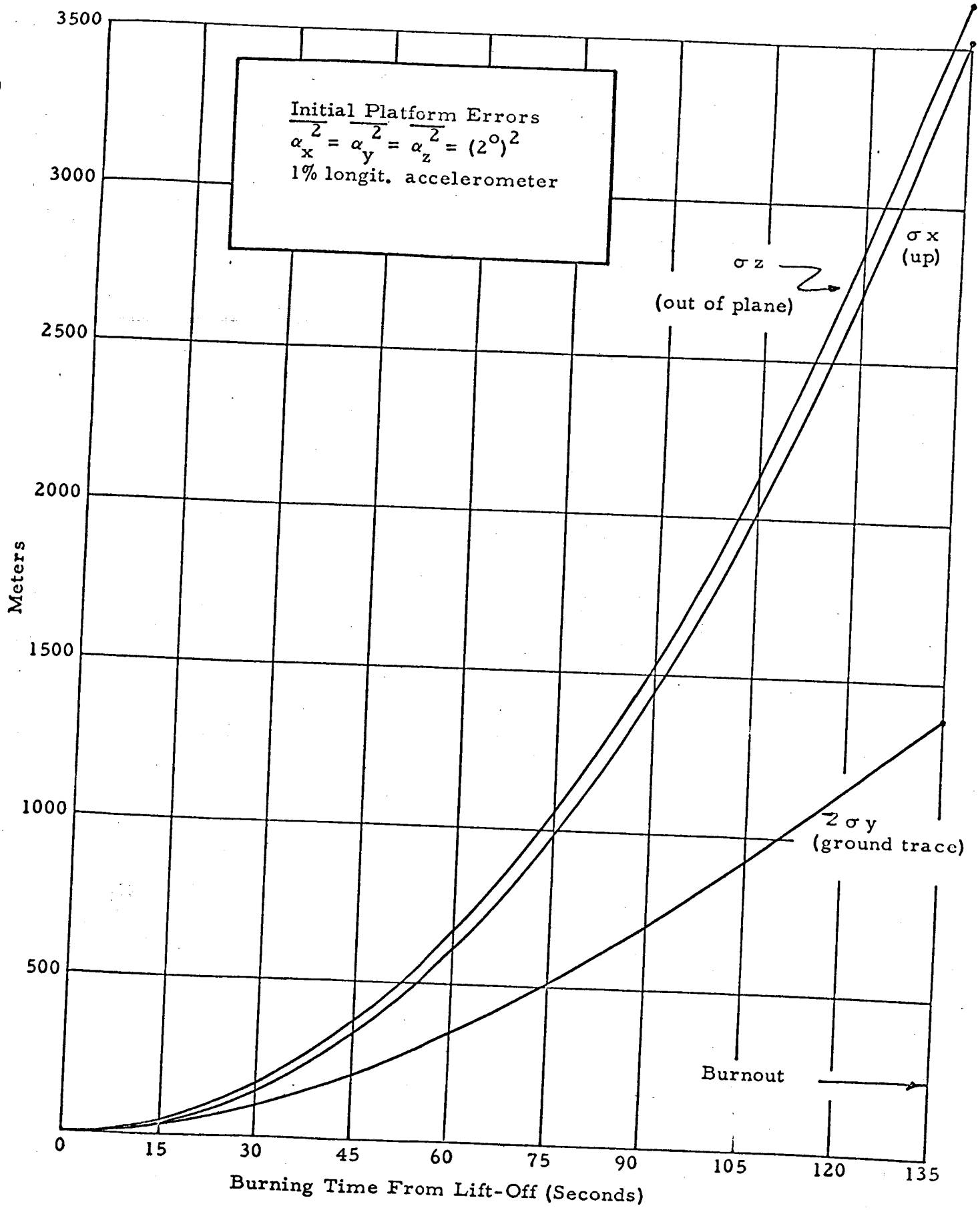

Figure 1 Positional Errors for no Tracking

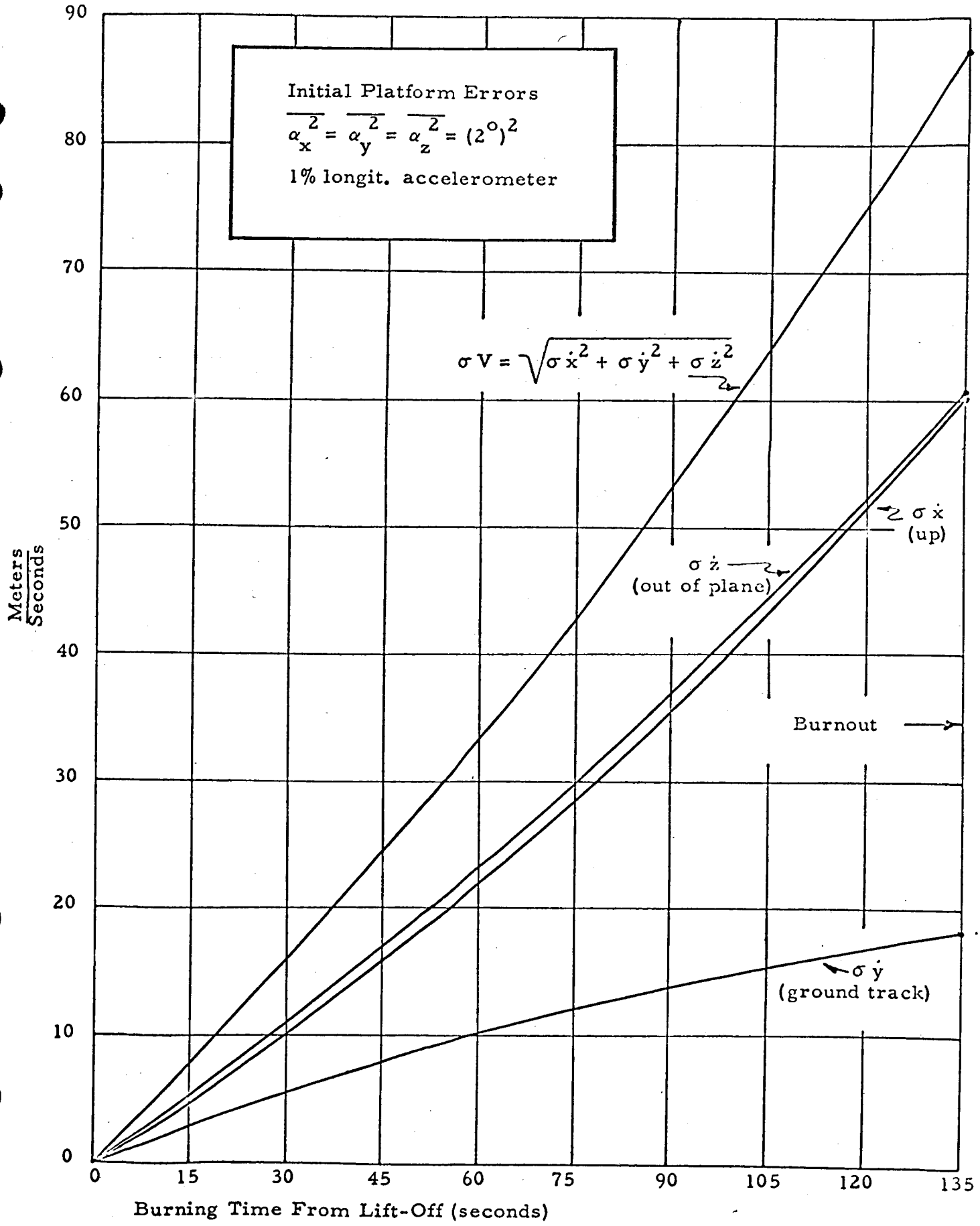

Figure 2 Velocity Errors for no Tracking

As it turns out, the time increments chosen (15 sec.) were too large to yield meaningful answers. This can be seen by looking at the $\dot{x}$ terms in Table 2. The total variance after an increment is almost exactly equal to the errors added during the increment, implying that the initial errors have been completely smoothed out by the tracking information during that period. However, if this is true, then somewhere in the middle of the increment the errors built up in this shorter time should start to be acted upon by the tracking data, reducing greatly the final error at the end of the 15-second segment. To better explain what is happening, the last segment of the $\sigma \dot{x}$ component will be analyzed.

From Table 2 at 120 seconds $\sigma \dot{x}^2$ is found to be equal to 780 m^2/sec^2. After 15 seconds of smoothing and without including platform errors, the MSFN is able to reduce $\sigma \dot{x}^2$ to .00518 m^2/sec^2. From Apollo Note 4, the relationship between a measurable (range) and the estimated slope (velocity term) of a linear fit is given. Combining this with the initial a priori variance gives a relationship of the form:

$$\frac{1}{\sigma \dot{x}^2_{\text{final}}} = \frac{1}{\sigma \dot{x}^2_{\text{a priori}}} + \frac{t^3}{k} \quad ; \quad t = 15 \text{ seconds} \tag{9}$$

where for our case $\sigma \dot{x}^2_{\text{final}} = 780\ m^2/sec^2$, $\sigma \dot{x}^2_{\text{a priori}} = .00518$ and k is a constant dependent upon the variance of the measurable and the geometry. t is time measured from the start of the increment of tracking (120 sec) or where $\sigma \dot{x}_{\text{a priori}}$ is given. Solving this equation for k with t equal 15 sec. gives $k = 17.47144\ m^2$ sec. This allows equation (9) to be plotted for any tracking time

$$\sigma x(t) = \sqrt{\frac{1}{\frac{1}{\sigma \dot{x}^2_{\text{a priori}}} + \frac{t^3}{k}}} \tag{10}$$

This is shown in Figure 3.

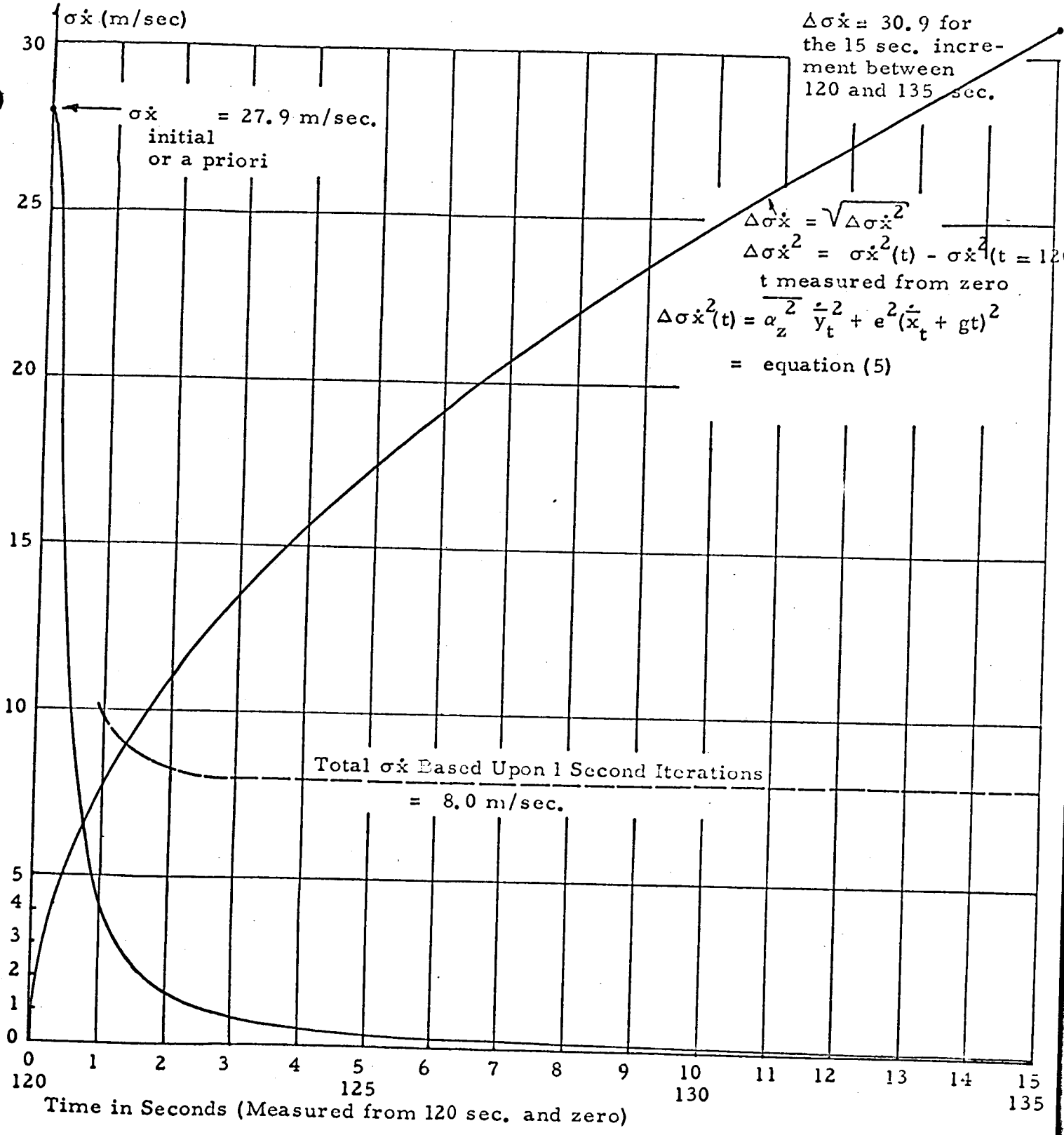

Figure 3

Effect on $\sigma\dot{x}$ Estimate of Using One Second Iteration

Also plotted in Figure 3 is the additive error due to the 2° platform, $\Delta\sigma\dot{x}$. Now the total error at the end of an increment (15 seconds in the example computed case) is simply equal to the square root of the sum of $\sigma\dot{x}(t)^2$ from equation (10) and $\Delta\sigma\dot{x}^2$ from equation (5) or as shown on the figure. This is true for incremental times other than 15 seconds. In particular it has been done for 1-second increments and is shown on the figure. The resulting error in $\sigma\dot{x}$ of 8 m/sec. is closer to reality than the 30.9 m/sec. given by the computer. In $\dot{y}$ and $\dot{z}$, the relative improvement through the use of shorter increments is not quite as great since the k of equation (9) is much larger. This follows from geometrical reasoning: $\dot{x}$ is essentially along the radar measurable and thus a good handle on x and $\dot{x}$ is quickly obtained; but $\dot{y}$ and $\dot{z}$ are perpendicular to the Earth-Moon line and thus longer measurements are required to improve a priori estimates of $\dot{y}$ and $\dot{z}$.

Thus reasonable estimates of the actual uncertainty in $\dot{z}$ and $\dot{y}$, at the end of the last increments, are 16 and 4 meters per second respectively. The computed output was 31 and 8 meters per second for these two quantities.

Figure 4 presents the computer print-out of the estimated total velocity error as function of time for the ascending LEM.

$$\sigma V = \sqrt{\sigma\dot{x}^2 + \sigma\dot{y}^2 + \sigma\dot{z}^2}$$

Also shown is a more reasonable estimate of the actual ability of three radar stations based upon the above reasoning. It should be remembered that this is a preliminary effort and not an exact statement of the utility of the MSFN during lunar ascent.

A refinement, which has not been encorporated in this analysis, can further improve the estimates shown in Figure 4. This refinement consists of assuming some knowledge of the actual platform alignment as the MSFN builds confidence in the computed velocity. For example,

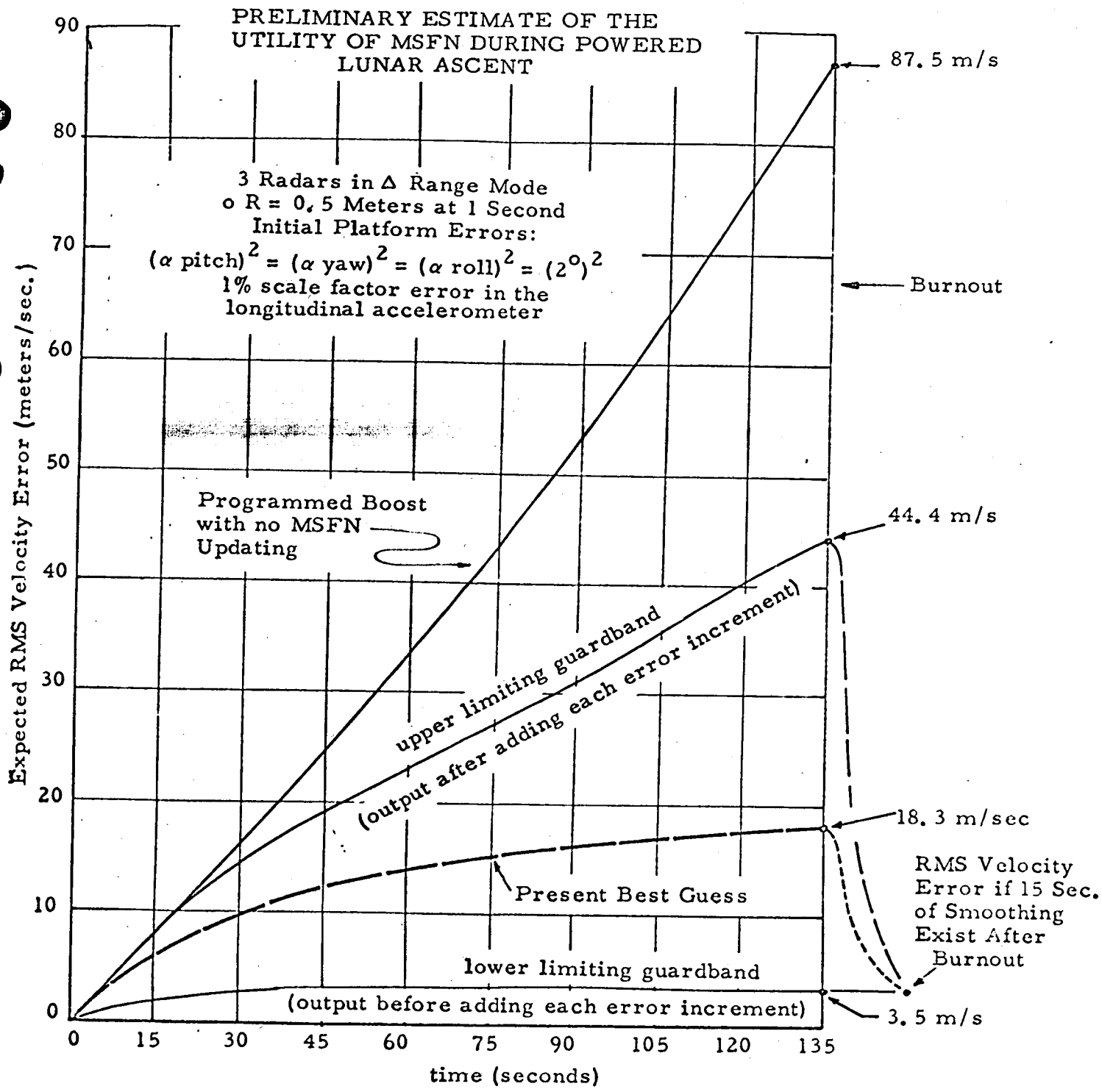

Figure 4

Velocity Uncertainty with Tracking

assume that the MSFN computes at close to real time. Then from Figure 4 at 45 seconds, the total uncertainty in the velocity is about 12 meters per second. This is a factor of two better than the error which would exist for a 2° platform error. Thus the MSFN could make a second guess at the platform alignment and thus decrease the additional uncertainty occurring after 45 seconds.

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APOLLO NOTE NO. 173

H. Dale/H. Engel
3 March 1964

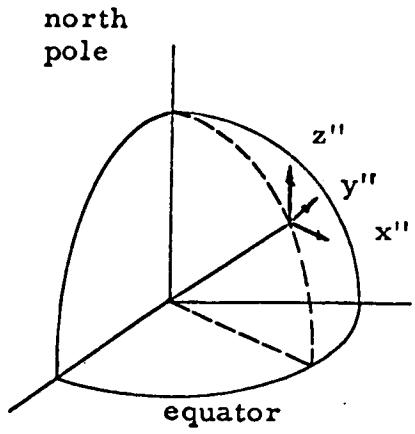
EARTH ORBIT INJECTION AND TRACKING ERRORS INCLUDING STATION LOCATION UNCERTAINTY

This study was initiated with Apollo Note No. 144 which analyzed the Apollo Earth Orbit injection from various station combinations. Two reasons make this note necessary. First, it has been shown that range and range-rate are not really independent, uncorrelated measurables. Second, the computer program has been modified to include station location errors (see Apollo Note No. 108).

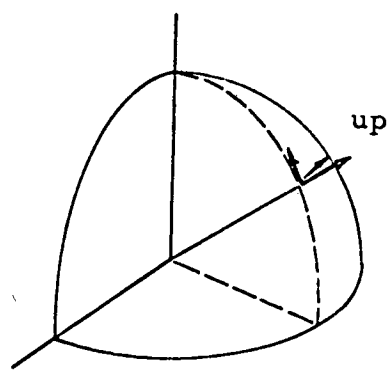
The location of a station, for the purpose of describing station location errors, is as follows:

x''	measured perpendicular to the axis of the Earth's rotation
z''	measured parallel to the axis of rotation
y''	following a right-hand rule

These coordinates differ from those specified by Dr. F. O. Vonbun and described in Apollo Note No. 144. This is the case



Station Reference Used



Vonbun's Reference
(see Apollo Note No. 144)

because our previous discussions with JPL have indicated that station coordinates are known best in the x'' direction and known to lesser but approximately equal accuracies in the y'' and z'' directions. We could have found the covariance matrix of the x''y''z'' components of a priori station location by a matrix rotation, but for the purposes of this note we have simply applied Dr. Vonbun's numbers to the x'', y'' and z'' axes.

It is of interest that as the tracking time increases not only does the effect of station location errors decrease, but the actual knowledge of the station location improves also. This occurs simply because each station location component is no more than another parameter of the problem. Thus with one station (3 components of location) the six-by-six covariance matrix increases to a nine-by-nine equivalent. Similarly, two stations produce twelve-by-twelves, and three stations produce fifteen-by-fifteen covariance matrices. This presents a problem if bias errors in the measurables are to be included. To keep from exceeding the available program format, biases were not included in the cases reported in this note. However, an estimate of the degradations due to the presence of biases in the range and angle measurables may be obtained from Apollo Note No. 144.

The results of the comparison with and without station location errors are presented in Table 1. Here only the RMS position and velocity uncertainties are shown. As a generality one may conclude that station location errors do not seriously degrade orbit determination. As one would expect, the major degradation occurs in position rather than velocity.

Table 2 shows how the station location uncertainties decrease with tracking.

A short and simple program has been written which will diagonalize the position or velocity (3 x 3) covariance matrices and print out the main diagonals and their direction cosines in an appropriate space. This can be explained through the use of the following sketch:

Table 1

EFFECT OF STATION LOCATION UNCERTAINTY ON
RMS POSITION AND VELOCITY ERRORS FOR VARIOUS RADARS

Stations	Radar/Mode	No Station Location Errors: RMS V	No Station Location Errors: RMS Position	With Station Location Errors: RMS V	With Station Location Errors: RMS Position
		m/sec.	meters	m/sec.	meters
Bermuda alone	UFS/R, ϕ, ϕ	3.33	139.	3.33	167.
	FPS-16/R, ϕ, ϕ	.928	36.3	.928	99.8
	UFS/$\dot{R}$, ϕ, ϕ	3.03	28.3		
	FPS-16/$\dot{R}$, ϕ, ϕ	7.71	1425.		
Carnarvon alone	UFS/R, ϕ, ϕ	.071	9.5	.075	90.6
	FPS-16/R, ϕ, ϕ	.019	2.5	.028	81.9
	UFS/$\dot{R}$, ϕ, ϕ	.058	8.1		
	FPS-16/$\dot{R}$, ϕ, ϕ	.255	39.4		
Bermuda and Carnarvon	UFS/R, ϕ, ϕ	.035	6.5	.068	62.7
	FPS-16/R, ϕ, ϕ	.011	1.8	.022	43.4
	UFS/$\dot{R}$, ϕ, ϕ	.039	5.3		
	FPS-16/R, ϕ, ϕ	.045	7.9		
Bermuda and Ship No. 1	UFS/R, ϕ, ϕ	.088	23.9	.160	105.
				.161	110.
	FPS-16/R, ϕ, ϕ	.022	6.4	.054	79.5
				.054	82.7
	UFS/$\dot{R}$, ϕ, ϕ	.056	15.6		
	FPS-16/$\dot{R}$, ϕ, ϕ	.180	3.7		
Bermuda, Carnarvon, and Guaymas	UFS/R, ϕ, ϕ	.0060	2.42	.037	37.8
	FPS-16/R, ϕ, ϕ	.0017	.78	.012	18.2
	UFS/$\dot{R}$, ϕ, ϕ	.0051	1.74	.036	36.7
	FPS-16/$\dot{R}$, ϕ, ϕ	.0053	5.14	.025	27.3

Notes

1 σ Errors in the measurables are (for 1 sec. Δt)

UFS:
σR = 15 m
$\sigma \dot{R}$ = .1 m/s
$\sigma \phi$ = .8 mils

FPS-16:
σR = 6 m
$\sigma \dot{R}$ = 6 m/s
$\sigma \phi$ = .2 mils

A priori sta. loc. errors are for each sta. (except ship):
x'' = 10 m.
y'' = z'' = 65 m.

A priori Ship No. 1 Sta. loc. errors
x'' = 20 m.
y'' = z'' M 200 m
x'' = 20 m.
y'' = z'' = 1000 m

UFS

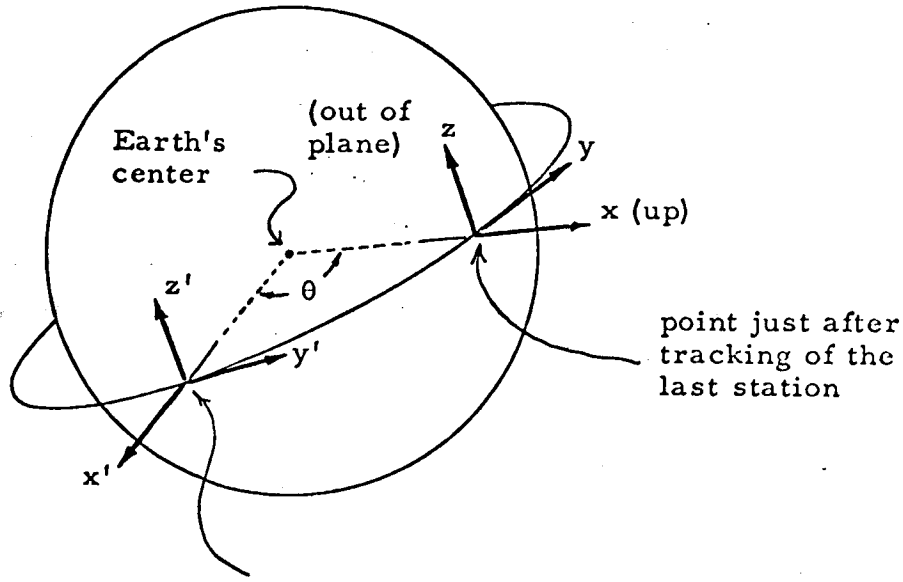
Tracking Stations	Bermuda $\sigma_{x''}$	Bermuda $\sigma_{y''}$	Bermuda $\sigma_{z''}$	Carnarvon $\sigma_{x''}$	Carnarvon $\sigma_{y''}$	Carnarvon $\sigma_{z''}$	Ship No. 1 $\sigma_{x''}$	Ship No. 1 $\sigma_{y''}$	Ship No. 1 $\sigma_{z''}$	Guaymas $\sigma_{x''}$	Guaymas $\sigma_{y''}$	Guaymas $\sigma_{z''}$
A priori	10.0	65.4	65.4	10.0	65.4	65.4	20.0	200.	200.	10.0	65.4	65.4
Bermuda	10.0	65.4	65.4									
Carnarvon				10.0	62.2	65.1						
Bermuda and Ship No. 1	9.8	61.0	65.1				18.1	64.9	75.4			
Bermuda and Carnarvon	9.8	63.0	43.3	9.0	50.5	35.4						
Bermuda, Carnarvon, and Guaymas	9.5	61.2	32.5	7.9	30.3	17.2				8.0	30.1	17.4

FPS-16

Tracking Stations	Bermuda $\sigma_{x''}$	Bermuda $\sigma_{y''}$	Bermuda $\sigma_{z''}$	Carnarvon $\sigma_{x''}$	Carnarvon $\sigma_{y''}$	Carnarvon $\sigma_{z''}$	Ship No. 1 $\sigma_{x''}$	Ship No. 1 $\sigma_{y''}$	Ship No. 1 $\sigma_{z''}$	Guaymas $\sigma_{x''}$	Guaymas $\sigma_{y''}$	Guaymas $\sigma_{z''}$
A priori	10.0	65.4	65.4	10.0	65.4	65.4	20.0	200.	200.	10.0	65.4	65.4
Bermuda	10.0	65.4	65.4									
Carnarvon				10.0	50.6	62.9						
Bermuda and Ship No. 1	9.3	47.2	61.0				13.3	47.3	62.3			
Bermuda and Carnarvon	9.4	59.7	34.0	7.9	33.2	27.1						
Bermuda, Carnarvon, and Guaymas	7.7	53.0	15.2	5.0	13.9	10.1				5.4	13.5	9.9

Table 2

Station Location, Standard Deviation (meters)



The total covariance matrix is presented as an output of the old computer program in the x', y', z' system which is defined with x' along the radial direction at t = 0. z' is normal to the orbit (see Apollo Note No. 82). If θ is the central angle traveled between t = 0 and the last point of tracking, then a local coordinate system may be defined with x up or radially away from the Earth's center and z parallel to the old z'.

Now the 3 x 3 set of covariance components corresponding to either position or velocity and in the x', y', z' system may be written:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Using the above defined angle, θ, the following quantities may be defined:

$$N_{11} = M_{11} \cos^2\theta + 2 M_{12} \sin\theta \cos\theta + M_{22} \sin^2\theta$$

$$N_{12} = N_{21} = M_{12} (\cos^2\theta - \sin^2\theta) + (M_{22} - M_{11}) \sin\theta \cos\theta$$

$$N_{13} = N_{31} = M_{13} \cos\theta - M_{23} \sin\theta$$

$$N_{22} = M_{11} \sin^2\theta - 2 M_{12} \sin\theta \cos\theta + M_{22} \cos^2\theta$$

$$N_{23} = N_{32} = - M_{13} \sin\theta + M_{23} \cos\theta$$

$$N_{33} = M_{33}$$

which simply describe the rotation of the matrix about the z' axis by an angle 0. Now the main diagonals may be found by solving the following equation for λ_1, λ_2, and λ_3:

$$\lambda^3 + B_2\lambda^2 + B_1\lambda + B_o = 0$$

where

$$B_2 = - N_{11} - N_{22} - N_{33}$$

$$B_1 = N_{11}N_{33} + N_{33}N_{22} + N_{22}N_{11} - N_{23}^2 - N_{12}^2 - N_{13}^2$$

$$B_o = - N_{11}N_{22}N_{33} + N_{11}N_{23}^2 + N_{22}N_{13}^2 + N_{33}N_{12}^2 - 2 N_{12}N_{23}N_{13}$$

Then for each λ

$$P_2 = \frac{N_{11}N_{23} - N_{13}N_{12} - N_{23}\lambda}{N_{22}N_{13} - N_{12}N_{23} - N_{13}\lambda}$$

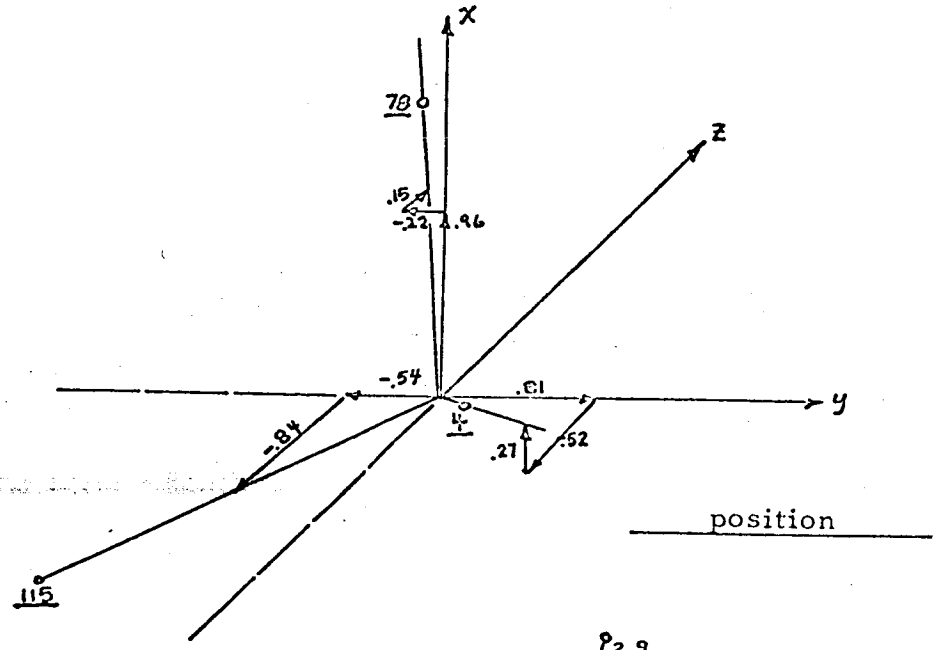
$$P_3 = \frac{\lambda - N_{12}P_2 - N_{11}}{N_{13}}$$

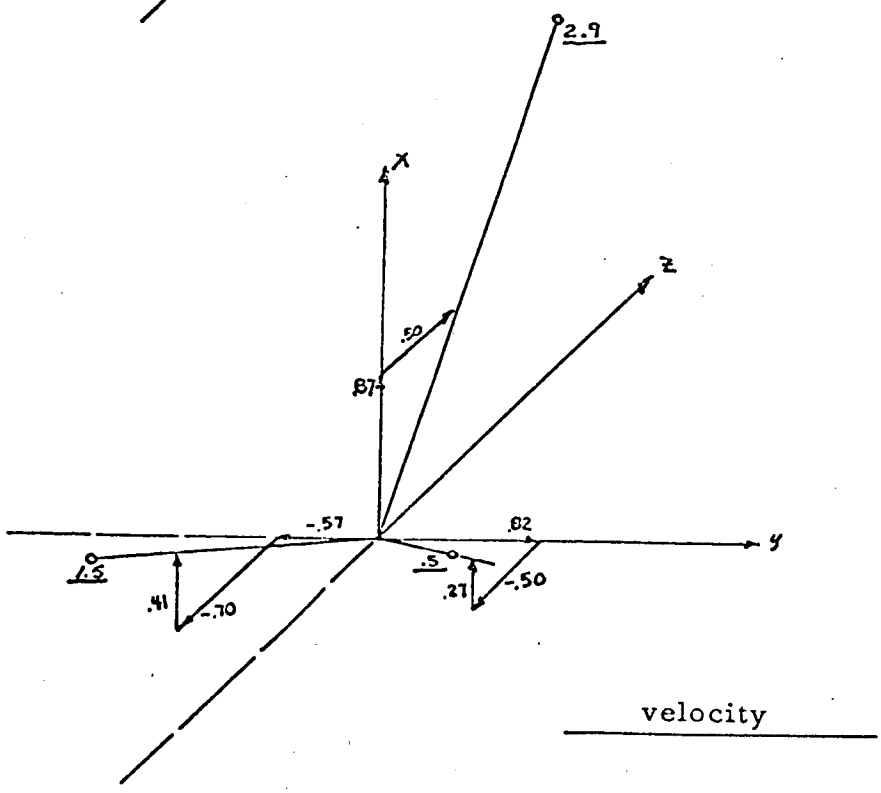
$$P = \sqrt{1 + P_2^2 + P_3^2}$$

Using these three quantities, the direction cosines to the x, y, z axes may be defined:

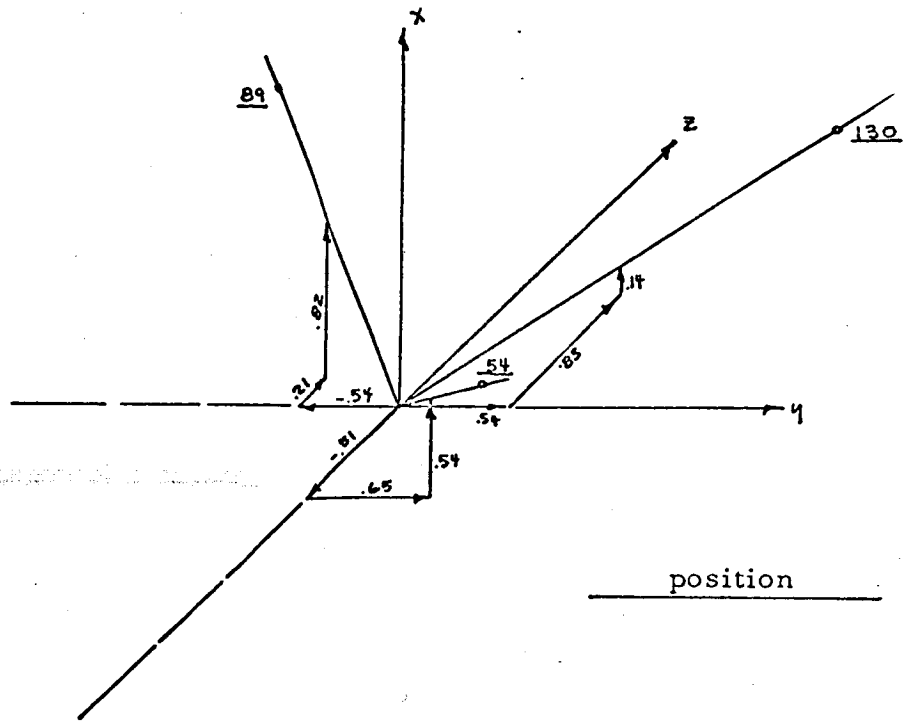
$$\left.\begin{aligned} \text{x direction cosine} &= 1/P \\ \text{y direction cosine} &= P_2/P \\ \text{z direction cosine} &= P_3/P \end{aligned}\right\} \text{ for each } \lambda$$

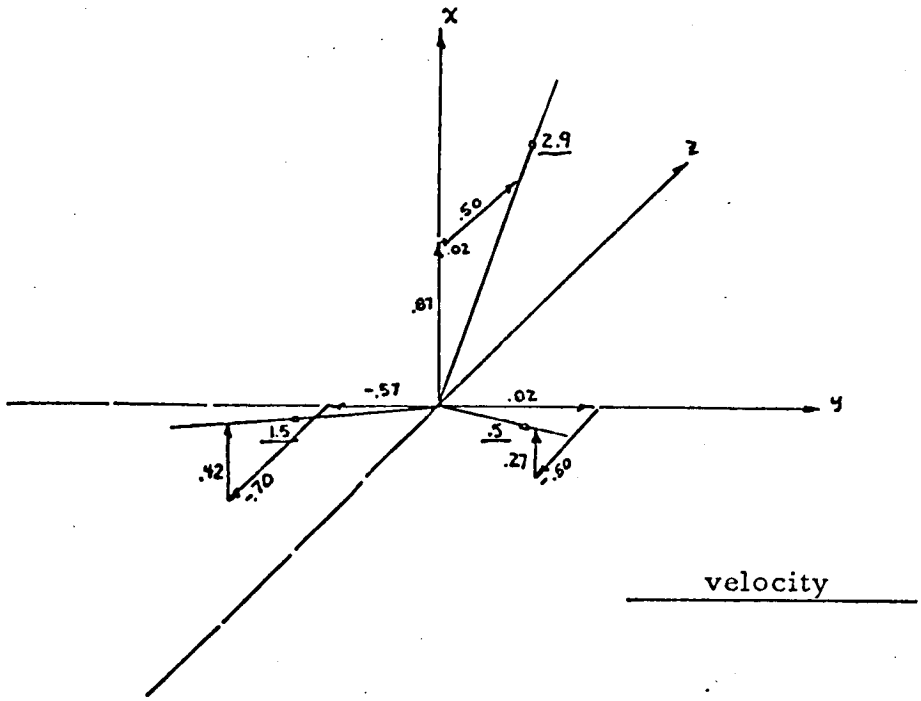
Use of such a computer program allowed the author to sketch the main diagonals of the covariance matrix for all of the (R, ϕ, ϕ) cases shown in Table 1. These follow and end this note. The main diagonals are in meters and meters per second.



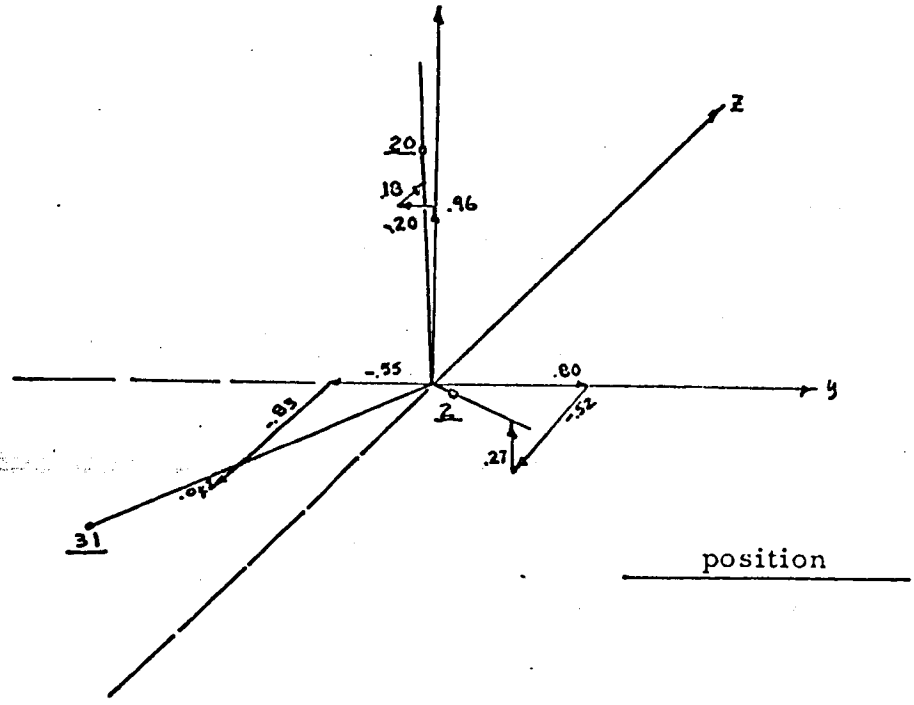


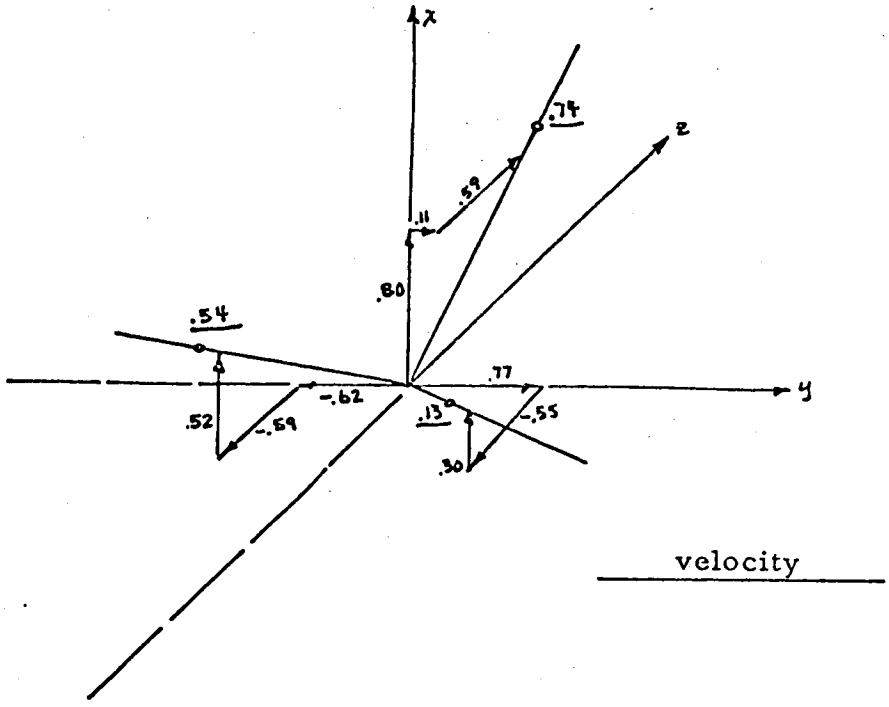
Bermuda (R, ϕ, ϕ) UFS
with no Station Location Errors



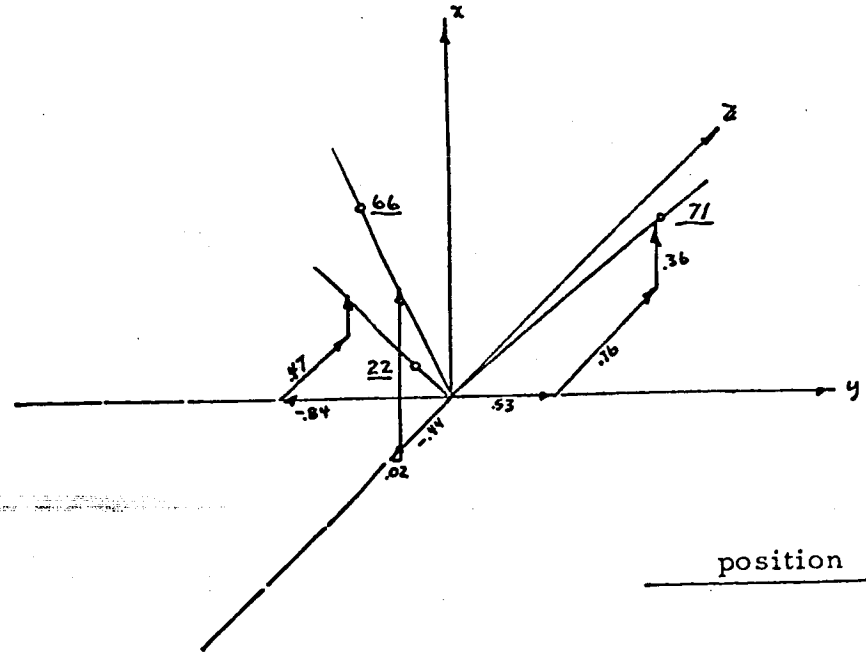


Bermuda (R, ϕ, ϕ) UFS
with Station Location Errors

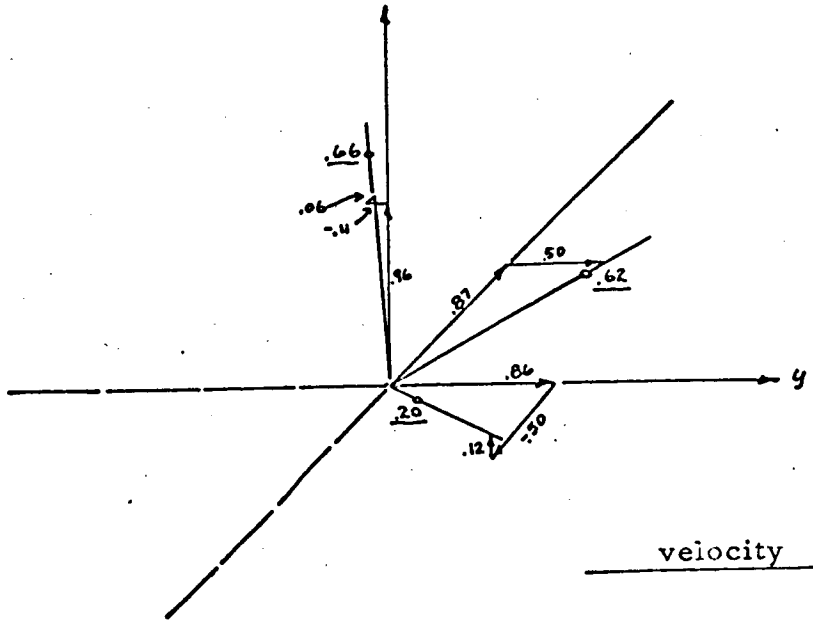




Bermuda (R, ϕ, ϕ) FPS-16
with no Station Location Errors

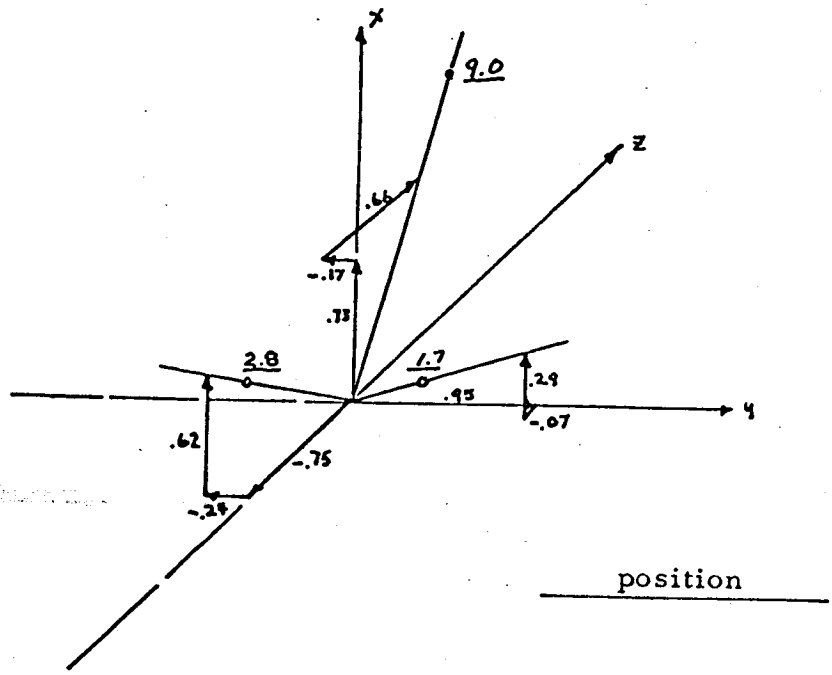


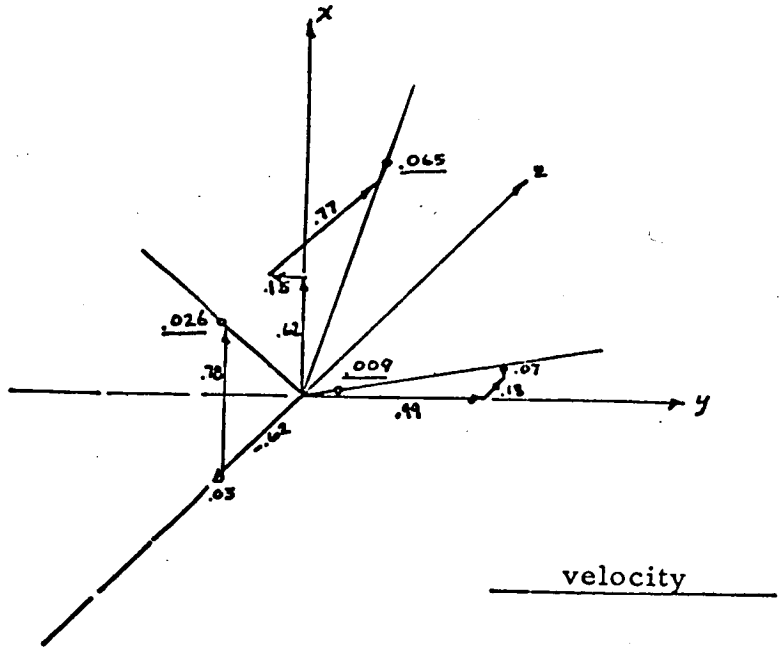
position



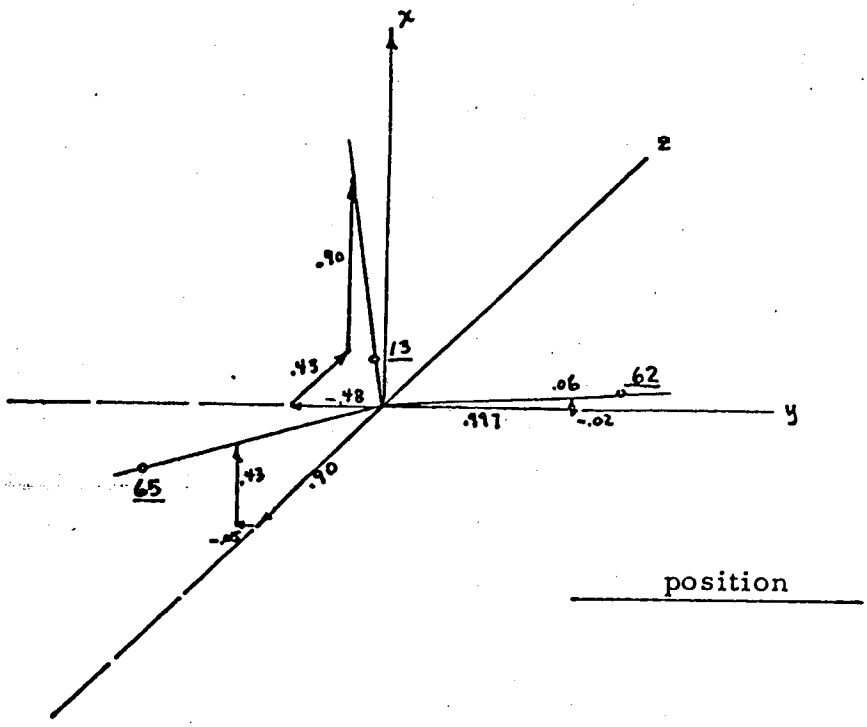
velocity

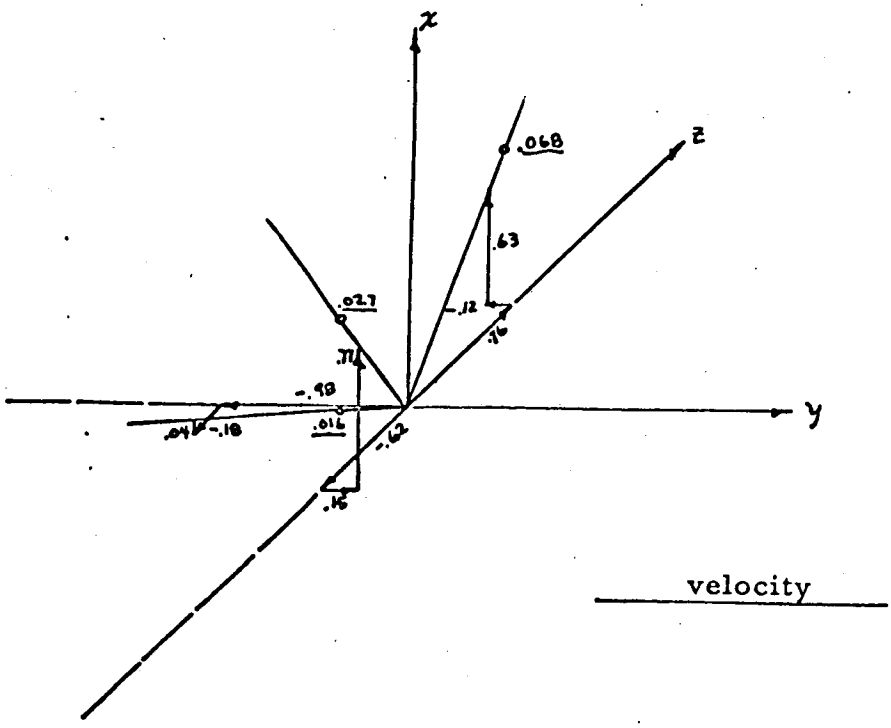
Bermuda (R, ϕ, ϕ) FPS-16
with Station Location Errors



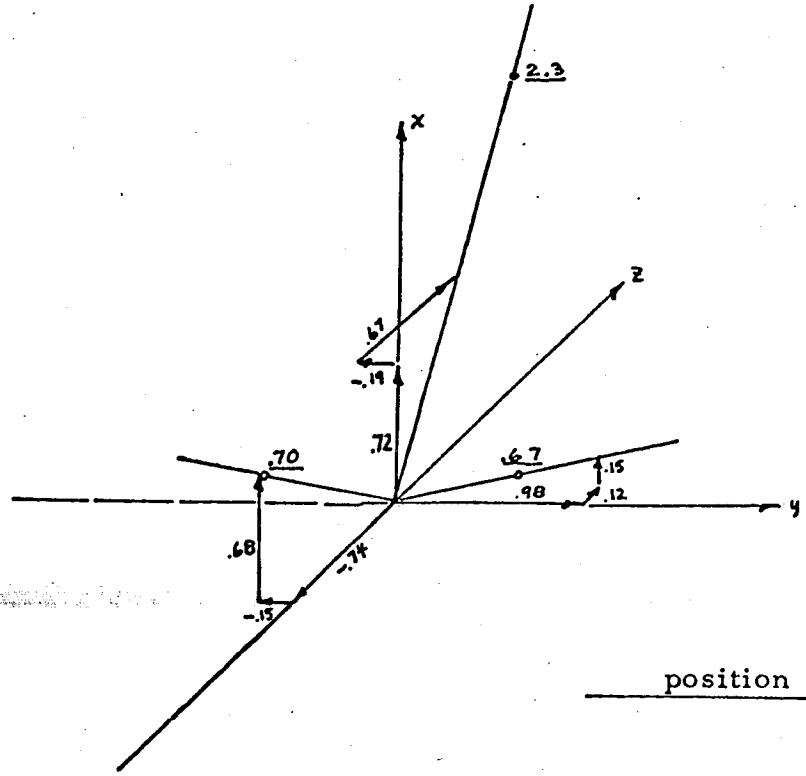


Carnarvon (R, ϕ, ϕ) UFS
with no Station Location Errors

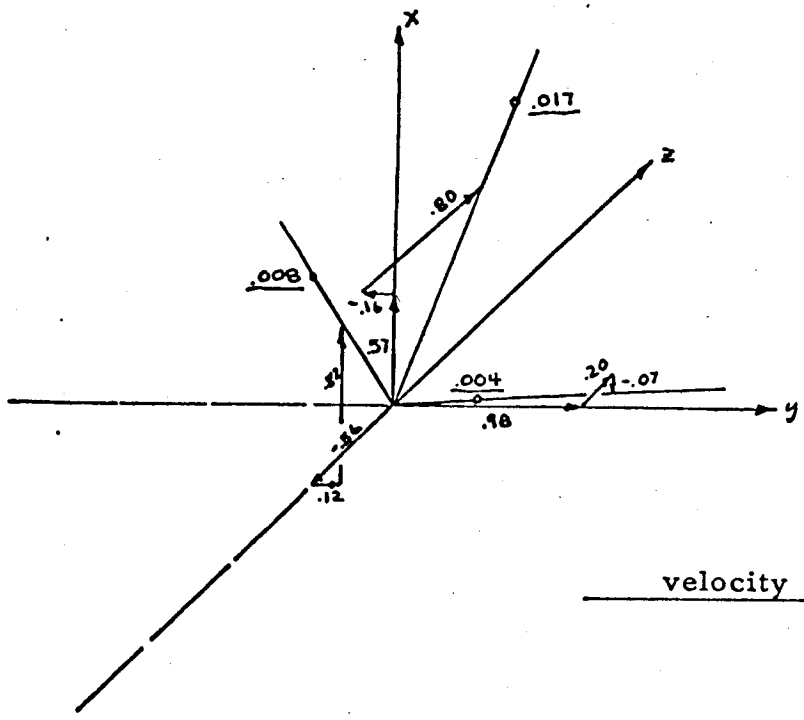




Carnarvon (R, ϕ, ϕ) UFS
with Station Location Errors

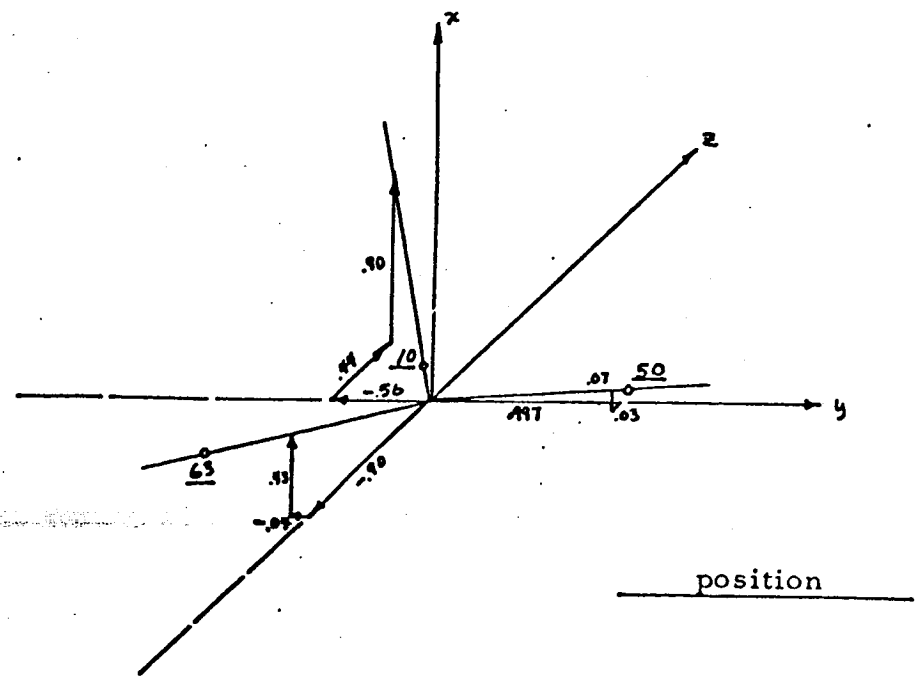


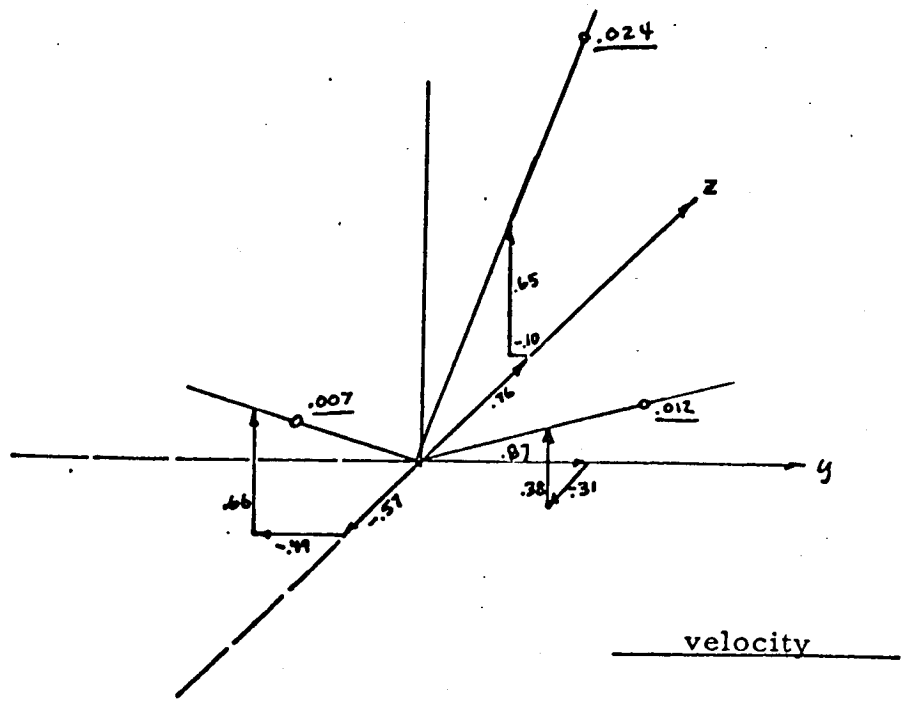
position



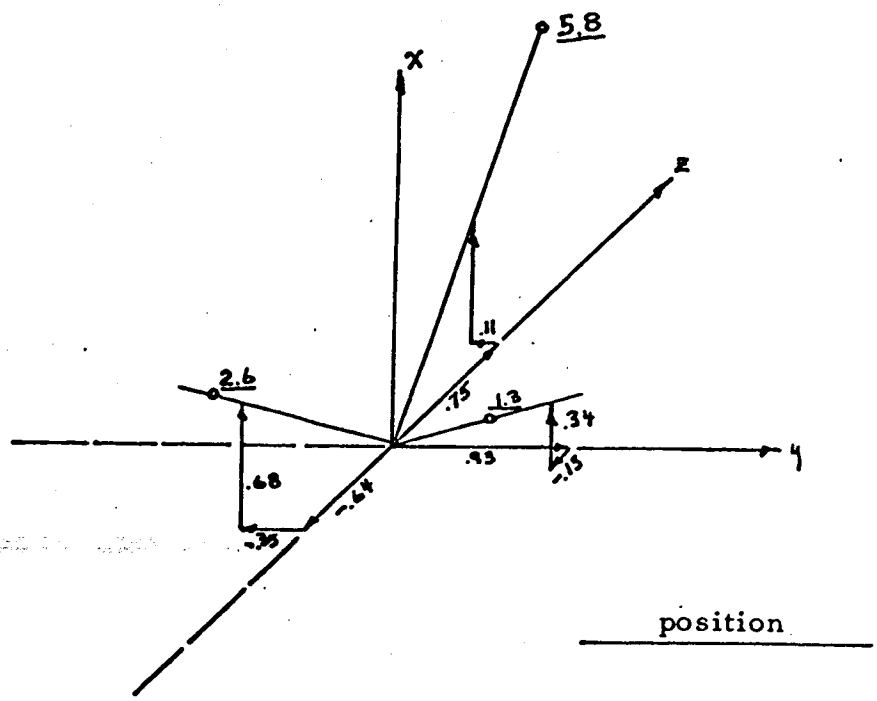
velocity

Carnarvon (R, ϕ, ϕ) FPS-16
with no Station Location Errors

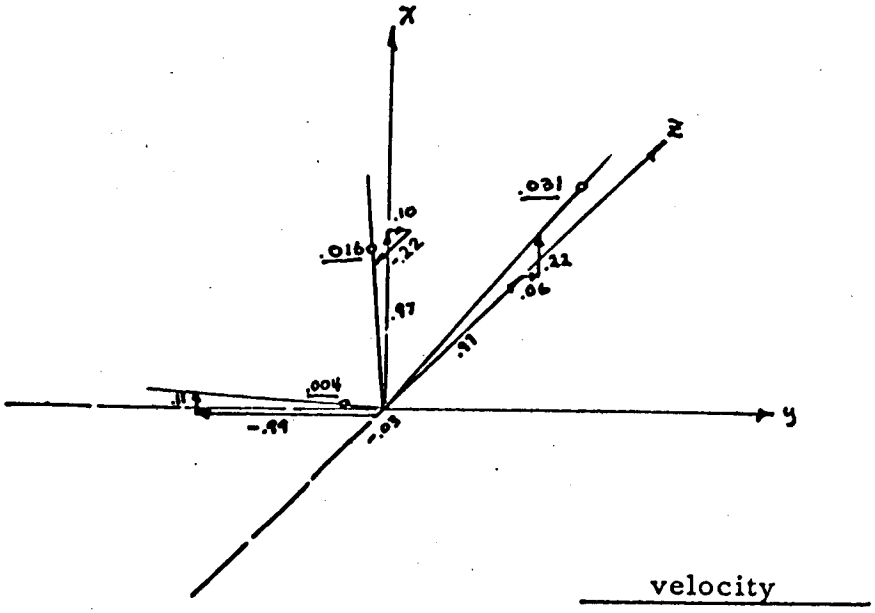




Carnarvon (R, ϕ, ϕ) FPS-16
with Station Location Errors

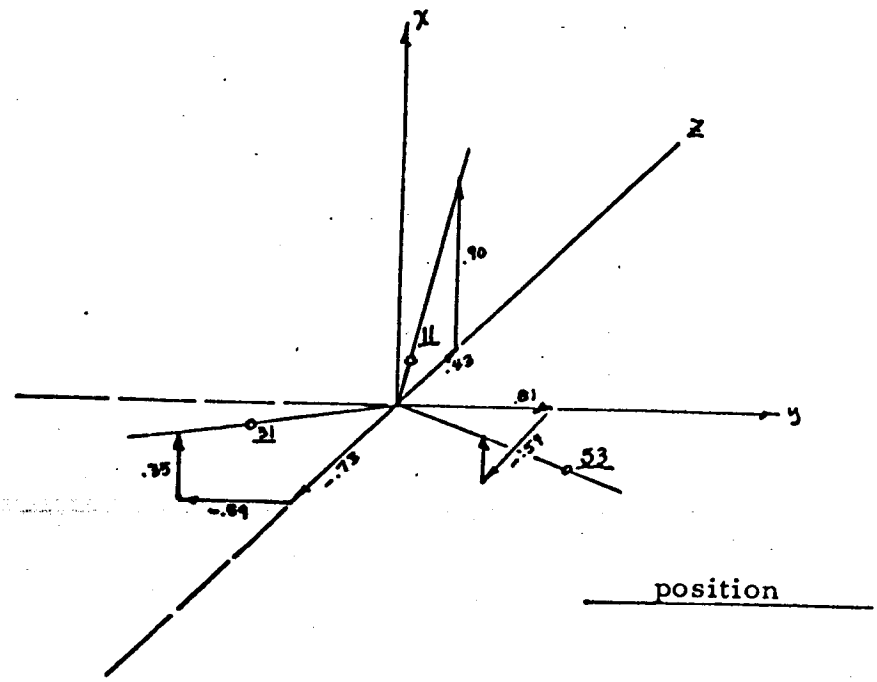


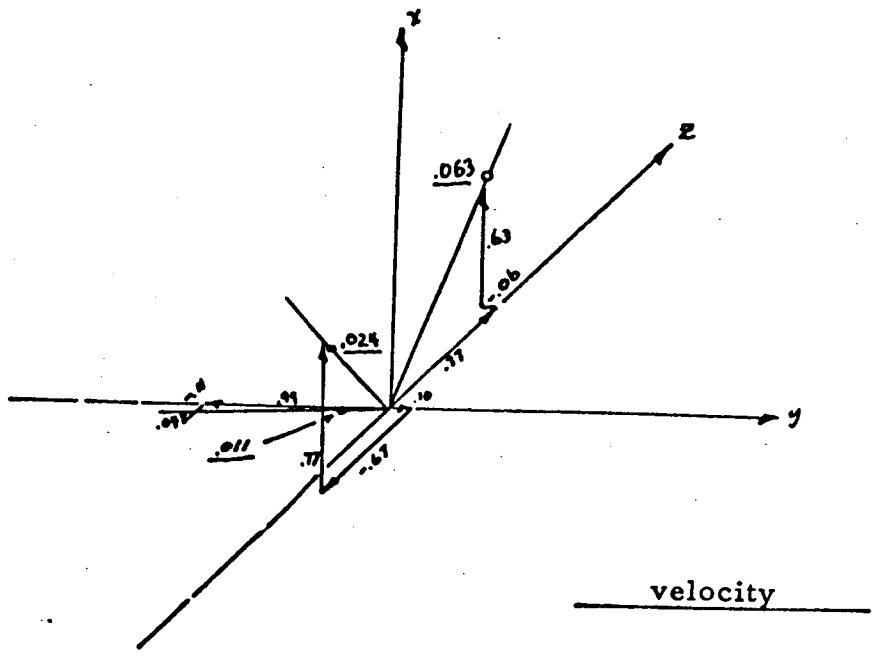
position



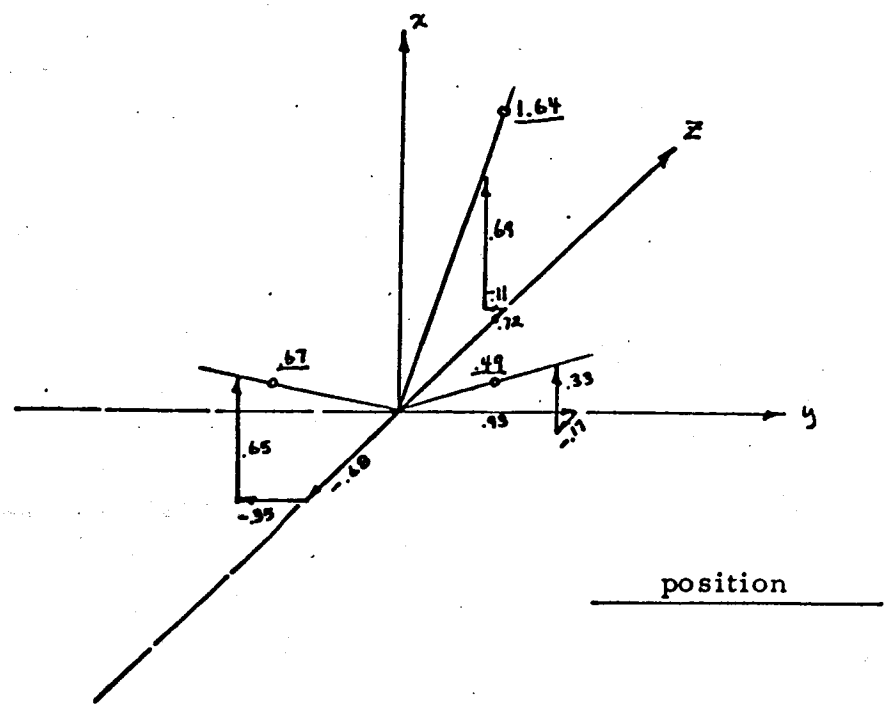
velocity

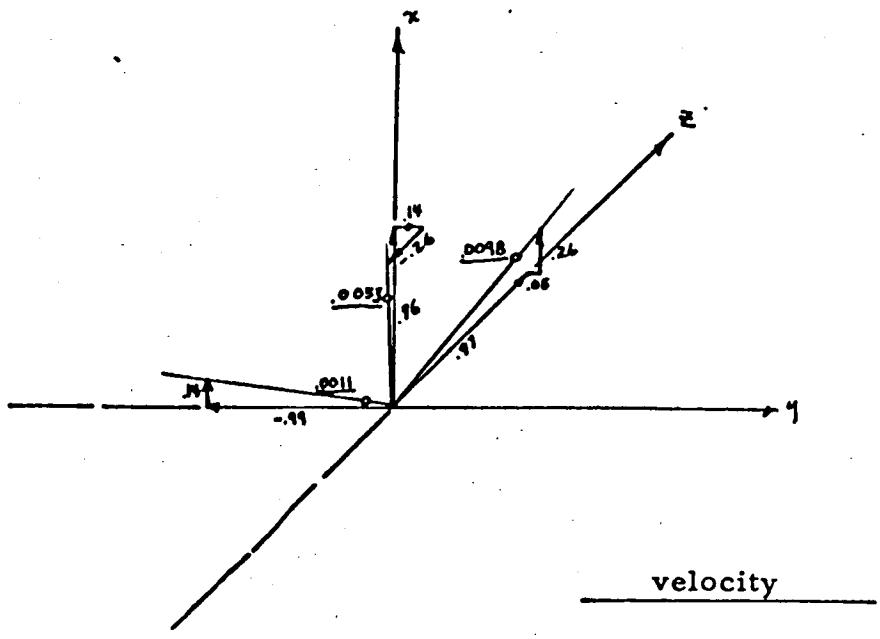
Bermuda and Carnarvon (R, ϕ, ϕ) UFS
with no Station Location Errors



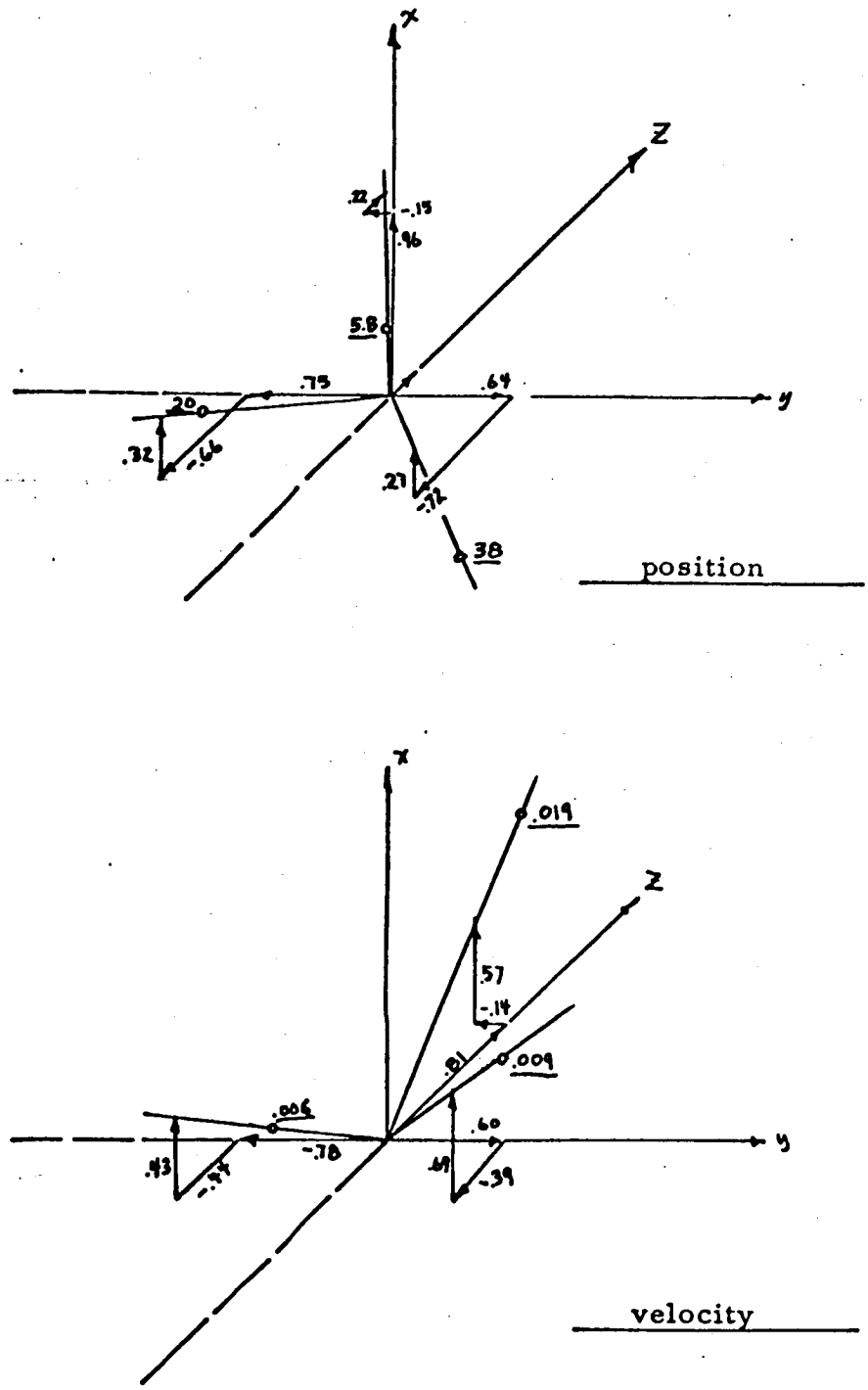


Bermuda and Carnarvon (R, ϕ, ϕ) UFS
with Station Location Errors

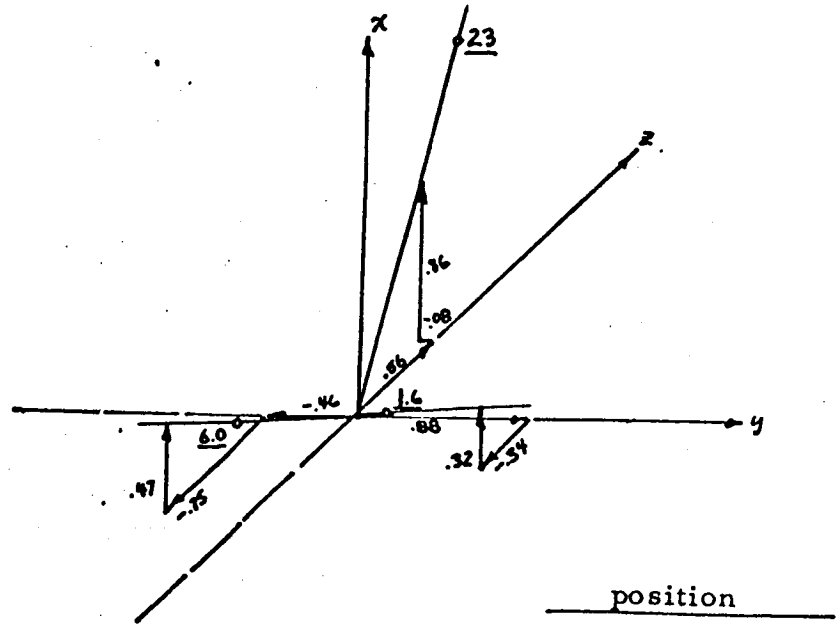


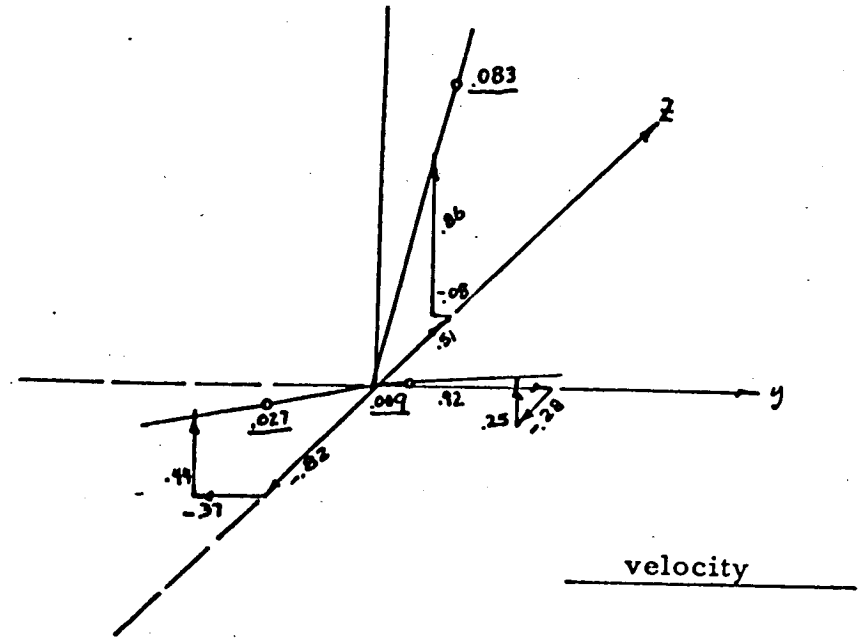


Bermuda and Carnarvon (R, ϕ, ϕ) FPS-16
with no Station Location Errors

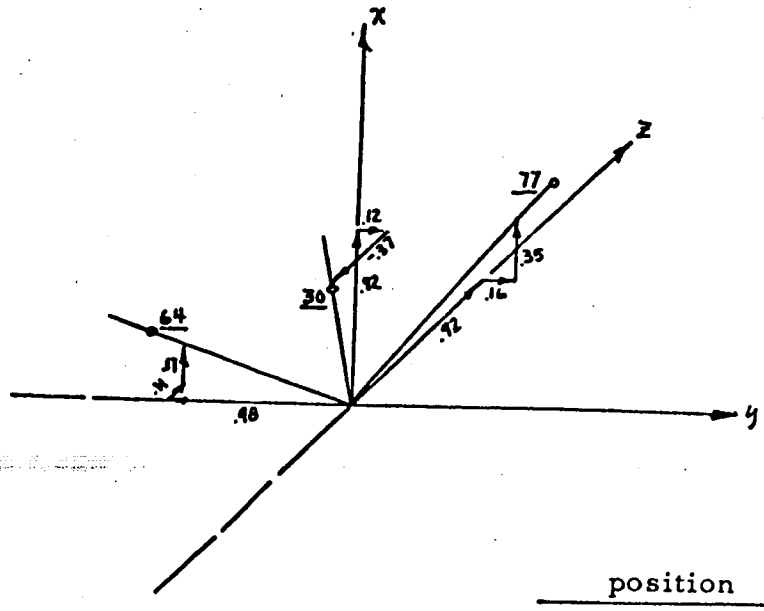


Bermuda and Carnarvon (R, ϕ, ϕ) FPS-16
with Station Location Errors

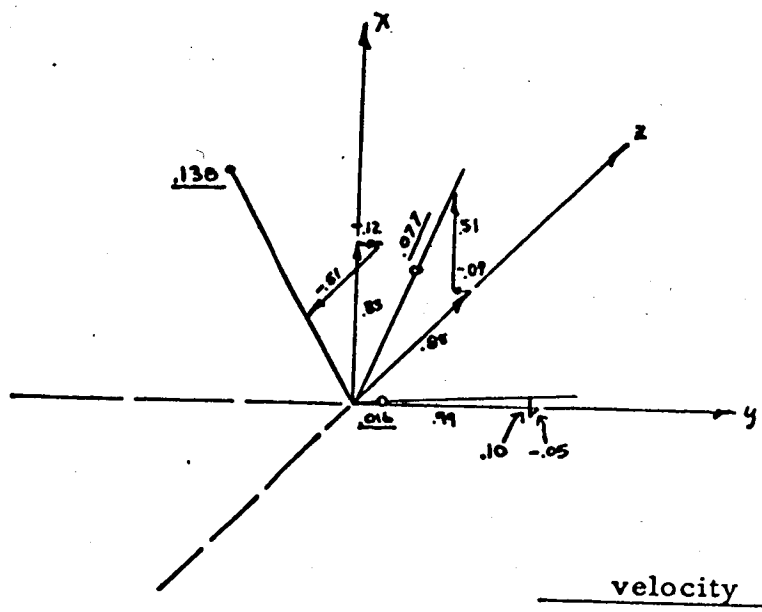




Bermuda and Ship No. 1 (R, ϕ, ϕ) UFS
with no Station Location Errors



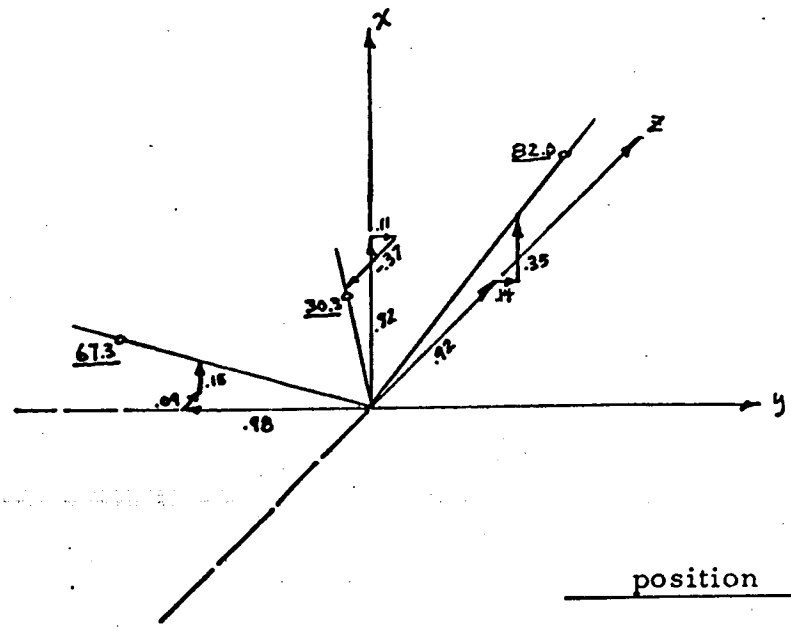
position



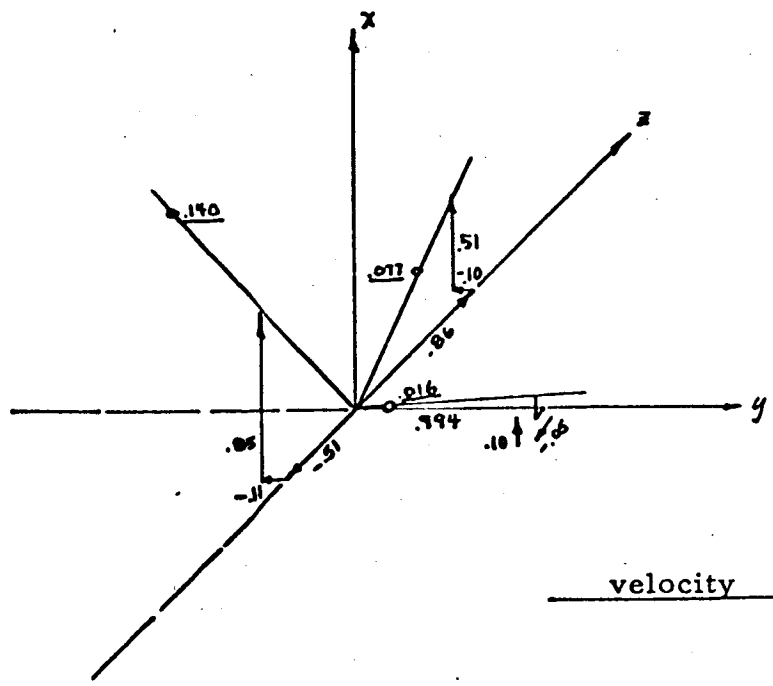
velocity

Bermuda and Ship No. 1 (R, ϕ, ϕ) UFS
with Station Location Errors

A priori Ship Location: x'' = 20 m., y'' = z'' = 200 m.



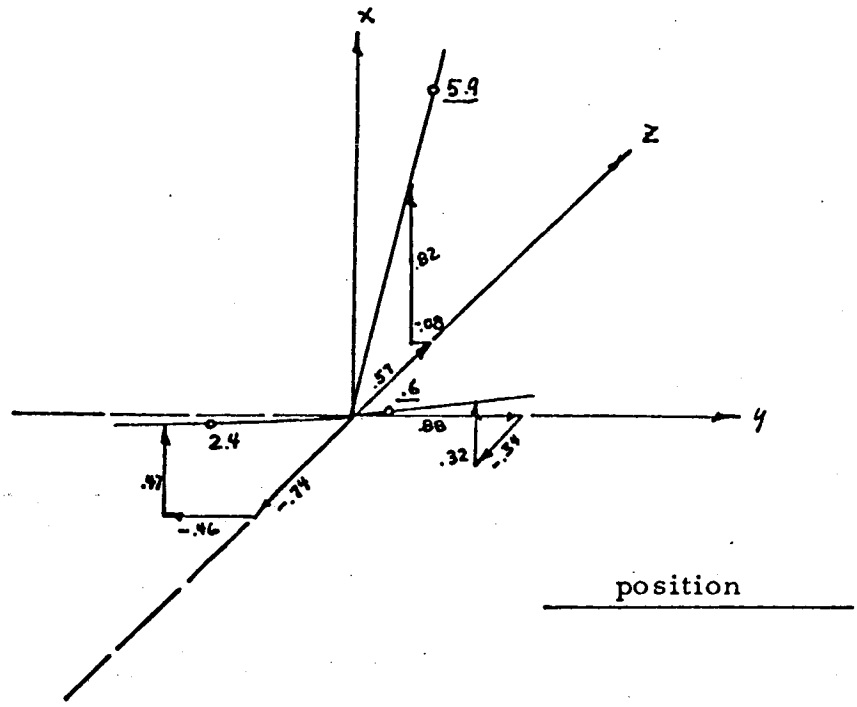
position

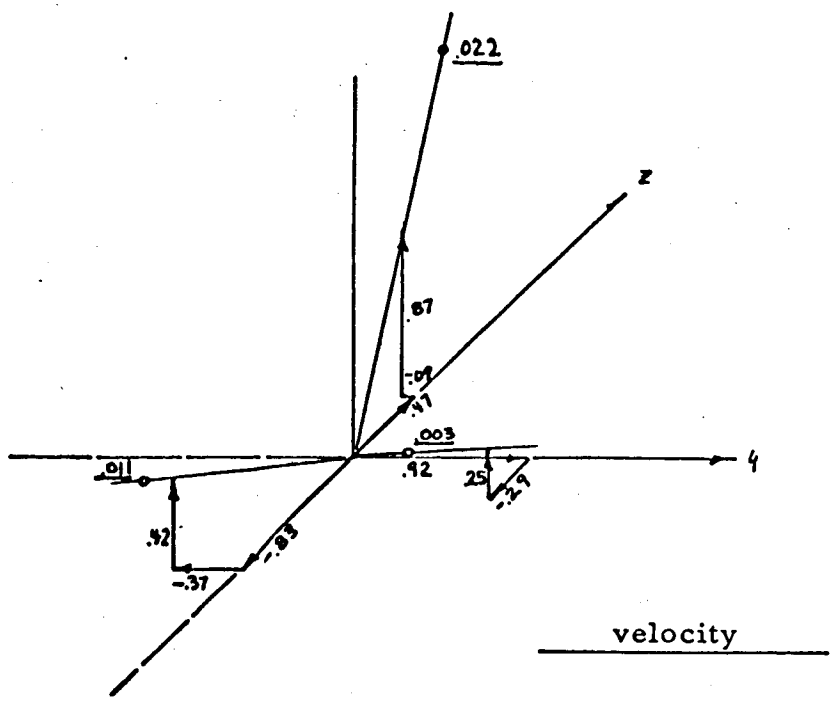


velocity

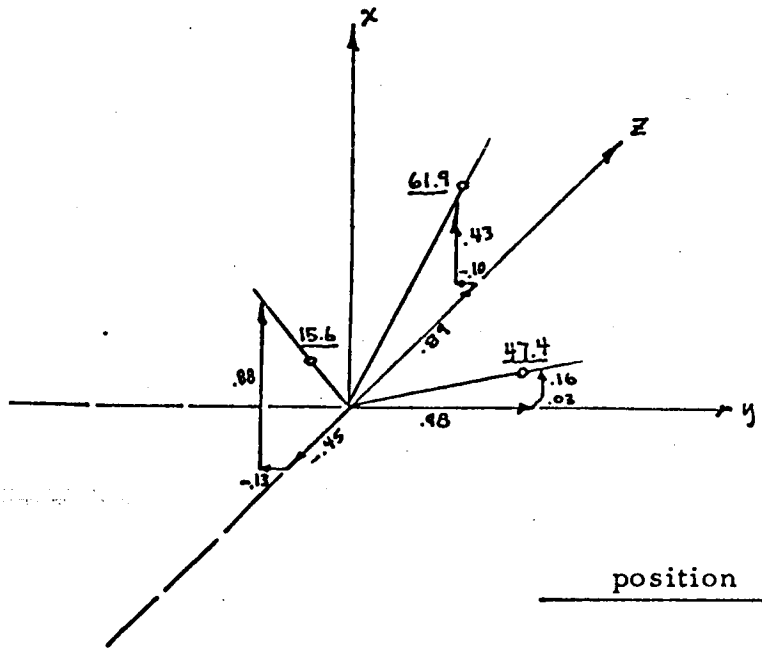
Bermuda and Ship No. 1 (R, ϕ, ϕ) UFS
with Station Location Errors

A priori Ship Location: x"= 20 m., y"= z"= 1000 m

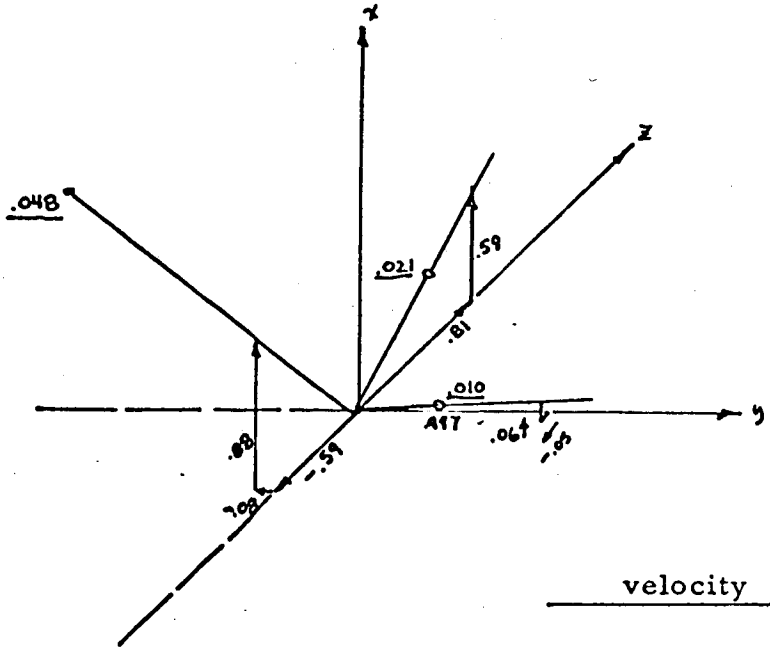




Bermuda and Ship No. 1 (R, ϕ, ϕ) FPS-16
with no Station Location Errors



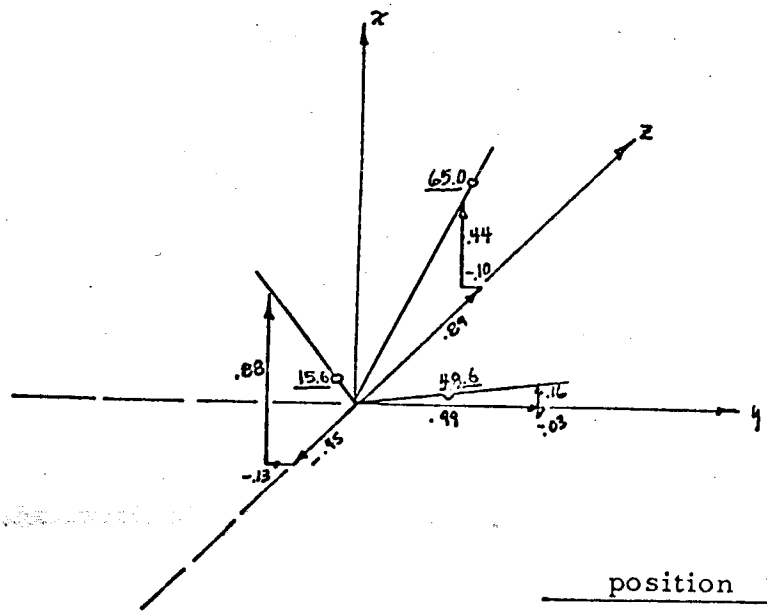
position



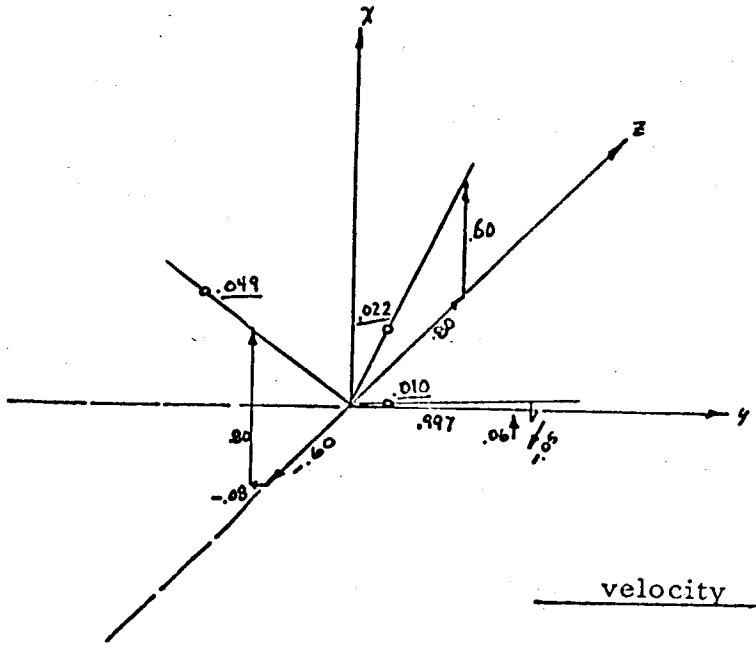
velocity

Bermuda and Ship No. 1 (R, ϕ, ϕ) FPS-16
with Station Location Errors

A priori Ship Location: $x'' = 20$ m., $y'' = z'' = 200$ m.



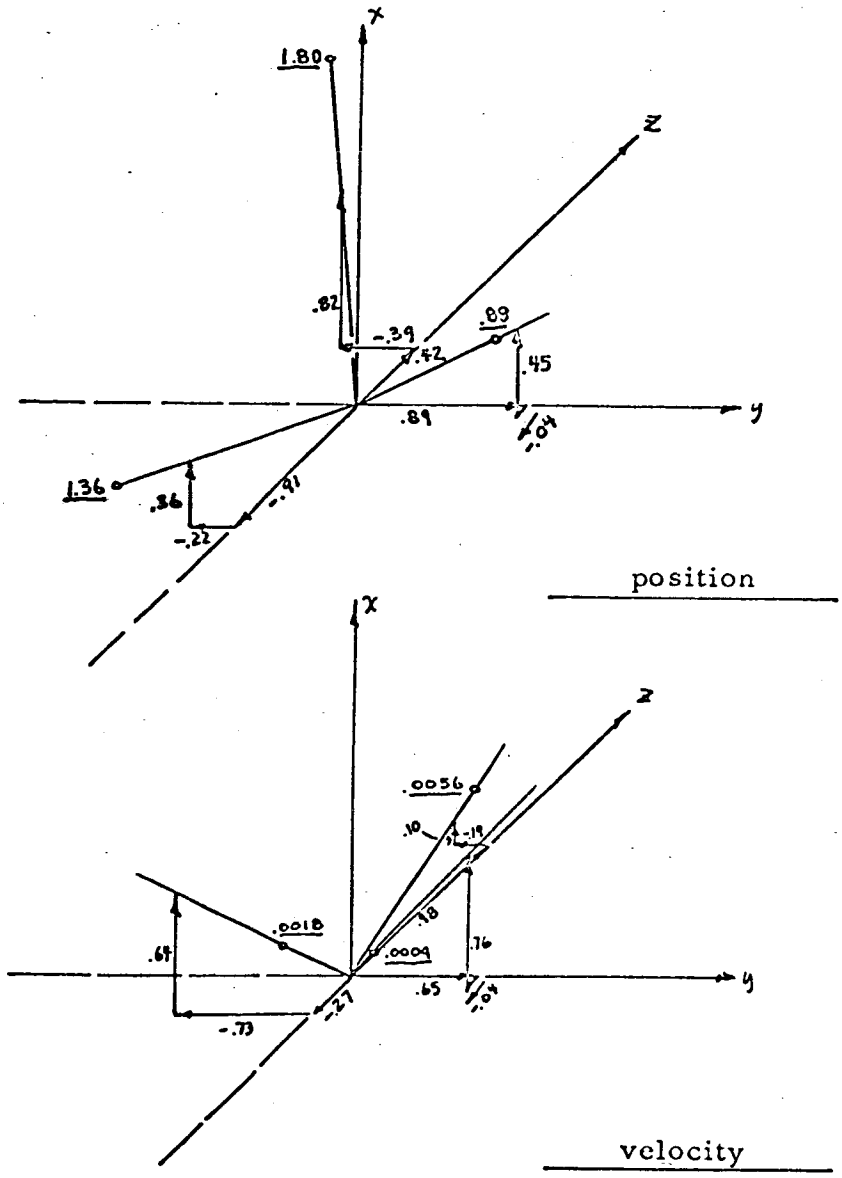
position



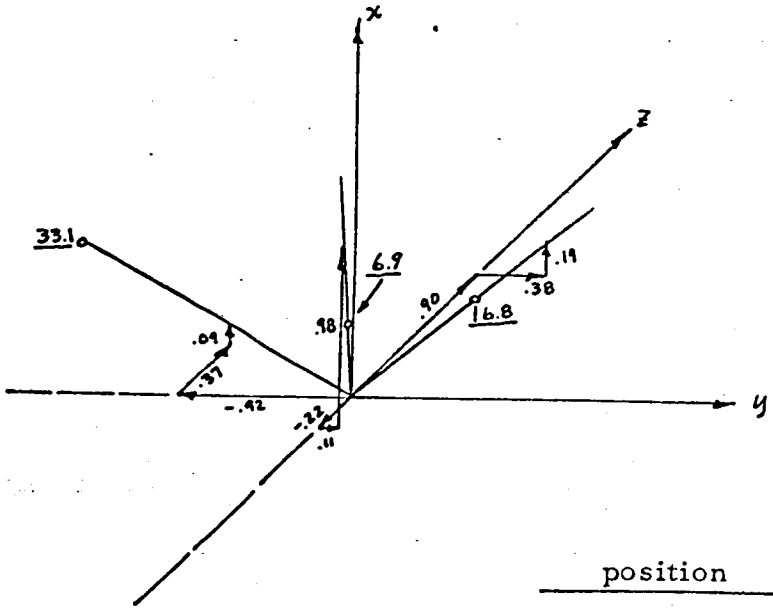
velocity

Bermuda and Ship No. 1 (R, ϕ, ϕ) FPS-16
with Station Location Errors

A priori Ship Location: x"= 20 m., y"= z"= 1000 m.



Bermuda, Carnarvon, and Guaymas (R, ϕ, ϕ) UFS
no Station Location Errors



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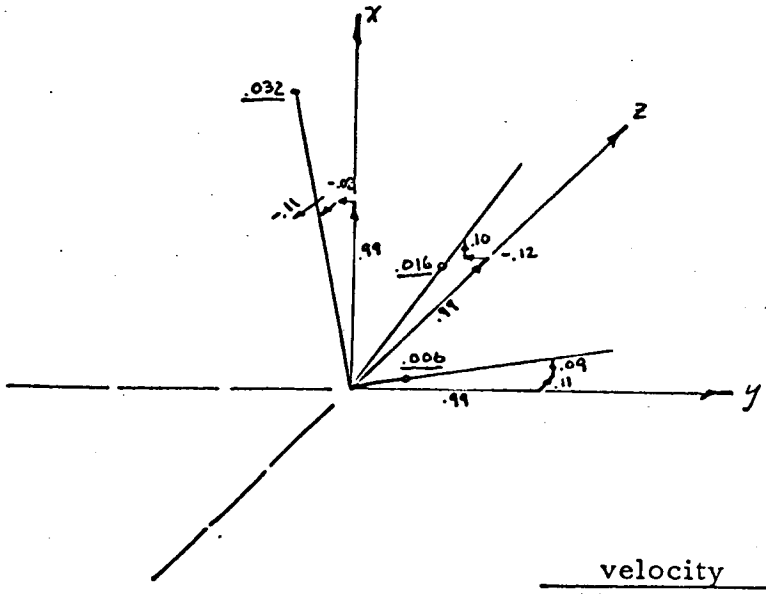
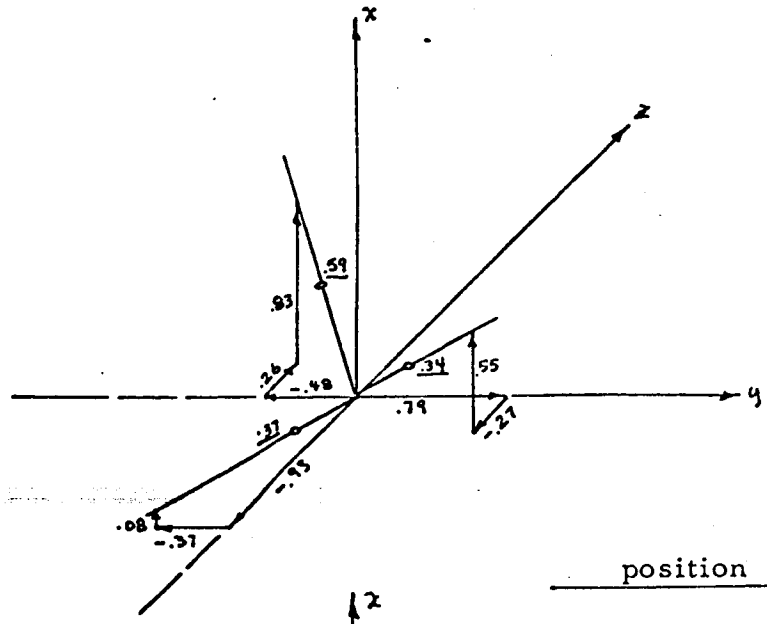


velocity

Bermuda, Carnarvon, and Guaymas (R, ϕ, ϕ) UFS
with Station Location Errors



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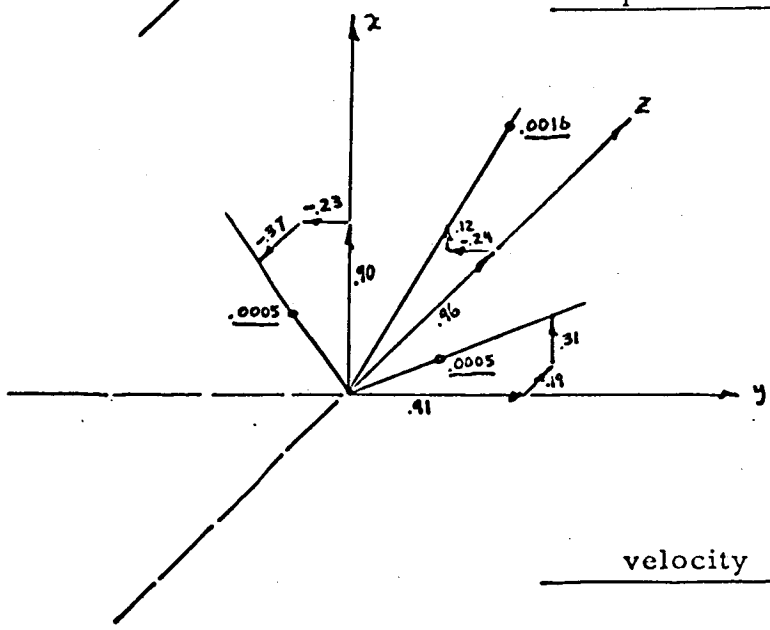
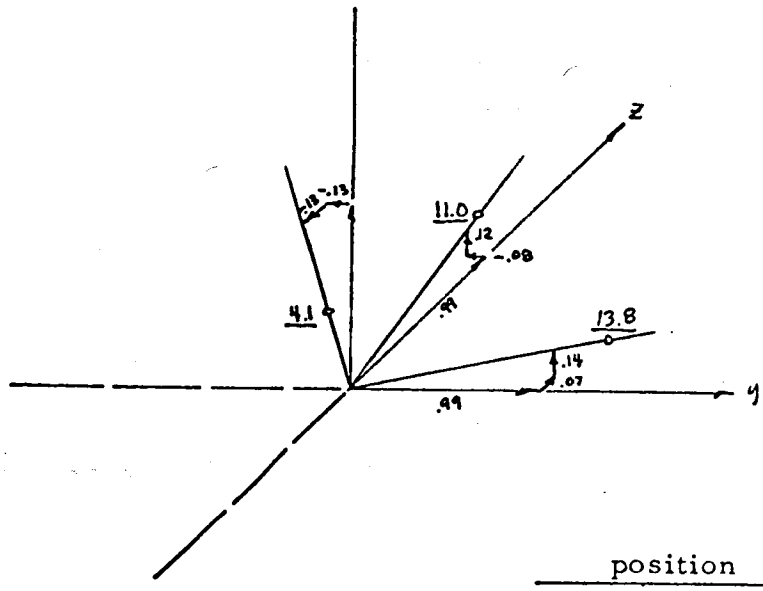
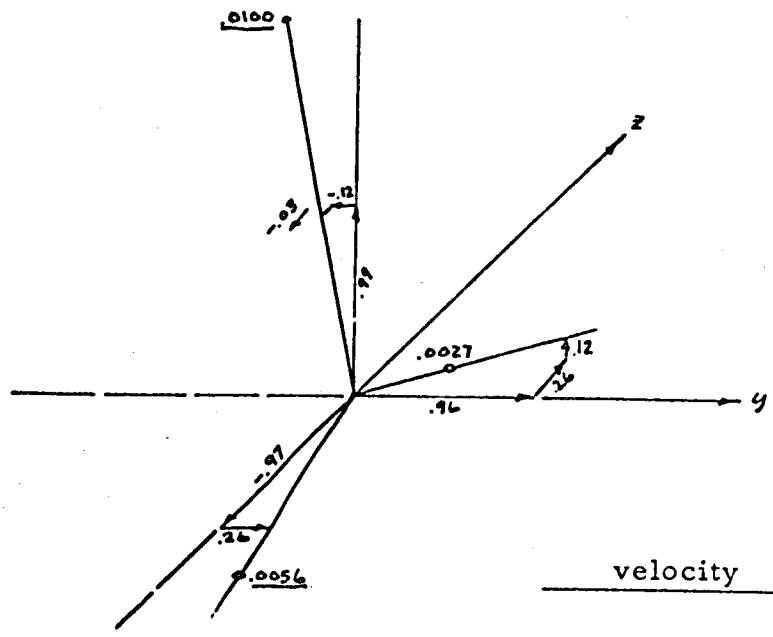


velocity

Bermuda, Carnarvon, and Guaymas (R, ϕ, ϕ) FPS-16
no Station Location Errors



position



velocity

Bermuda, Carnarvon, and Guaymas (R, ϕ, ϕ) FPS-16
with Station Location Errors

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 174

H. Engel/L. Lustick
27 February 1964

THE EFFECT OF THE UNCERTAINTY IN NUISANCE PARAMETERS ON THE ACCURACY OF ORBIT DETERMINATIONS

Purpose

The purpose of this report is to indicate the effect of uncertainties in the following quantities on the accuracy of orbit determination.

1. Station location errors
2. Errors in velocity of light
3. Errors in the clock
4. Error in the gravitational constant

Introduction

The measurables depend on quantities in addition to the parameters which determine the orbit. If these quantities are not known exactly, their inaccuracy is reflected in additional uncertainty in the orbit parameters desired.

It is the purpose of this report to indicate to what extent these nuisance parameters result in uncertainty in the desired parameters.

Method

1. The results were obtained with the Bissett-Berman error analysis program for determination of orbit parameters (see Note No. 82).
2. A typical lunar rendezvous trajectory was used.

3. The change in the uncertainty of the miss $\left(\sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}\right)$ at rendezvous was used as an indication of the effect of the nuisance parameter.

Conditions Investigated

1. Lunar rendezvous trajectory (perilune at 8 n. mi.; apolune at 80 n. mi.).
2. Stations (2): Madrid; Ascension.
3. Measurable: Range
4. Accuracy of measurable: 0.6 m (1 σ) at one minute intervals.
5. Observations times of: 5, 10, 15, 20, 25, 30 mins.
6. Hohman transfer rendezvous.
7. The a priori knowledge of the unknown parameters is listed below. These values represent approximately the published uncertainty in the knowledge of the nuisance parameters.

Parameter	(1σ) Value of Uncertainty
Position components of LEM	1 km } Consistent with Accuracy of I. M. U.
Velocity components of LEM	3 m/sec. } Consistent with Accuracy of I. M. U.
Station location components	25 m (x, y, z)
Velocity of light	1 part in 10^7
Gravitational constant of Moon	1 part in 10^3
Apparent velocity due to clock rate difference at Stations (see Note 157)	1 part in 10^{12}; and 10^{10}

Results

Figure 1 compares the knowledge of the miss at rendezvous for uncertainties in the nuisance parameters with the miss at rendezvous for perfect knowledge of the nuisance parameters. The effect of the uncertainty in the velocity of light could not be determined because for the case considered the uncertainty in velocity of light was apparently too much like a range bias resulting in an ill-conditioned matrix. Over the range of errors investigated, only the uncertainty in the gravitational constant and the error in clock rate (at least for 10^{-10} parts/part) affected the uncertainty in the miss at rendezvous. The effect of station location errors was negligible. The clock with 10^{-10} parts/part error was the only variable to have an appreciable effect after 30 minutes of observation. The miss at rendezvous with no errors in the nuisance parameters was 330 meters and this was increased to 550 meters due to an error in the clock of 10^{-10} parts/part. For errors in the clock of 10^{-12} parts/part, there was no effect in the uncertainty in the miss at rendezvous.

The effect of the uncertainty in the gravitational constant, μ, shown in Figure 1, can perhaps best be understood by examining Figure 2, which shows the improvement in the nuisance parameters as a function of observation time. The improvement in μ is small until after 20 minutes of observation, whereafter a marked improvement is observed (2 order of magnitude at 30 minutes). The effect of μ on the uncertainty in the miss at rendezvous (Figure 1) emulates, as to be expected, the effect shown in Figure 2. The only other parameter which was improved during the 30 minute observation interval was the clock rate with an a priori error of 10^{-10} parts/part.

Conclusions

After 30 minutes of observation, the only nuisance parameter which significantly effects the uncertainty in the miss at rendezvous is the clock error. Even for pessimistic assumption as to clock accuracy (1×10^{-10} parts/part), the uncertainty in the miss at rendezvous is small (550 meters) compared to the allowable rendezvous window.

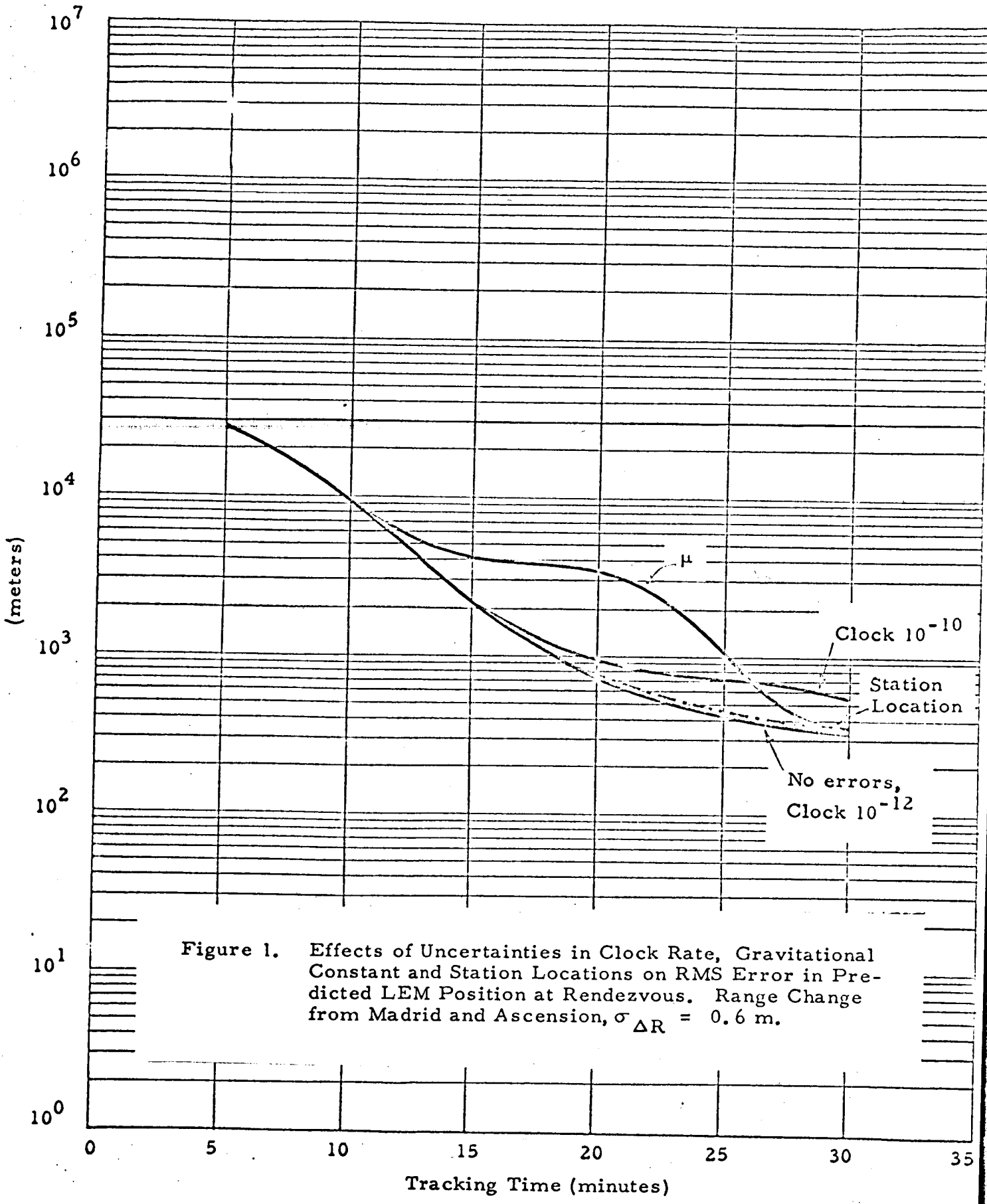


Figure 1. Effects of Uncertainties in Clock Rate, Gravitational Constant and Station Locations on RMS Error in Predicted LEM Position at Rendezvous. Range Change from Madrid and Ascension, $\sigma_{\Delta R}$ = 0.6 m.

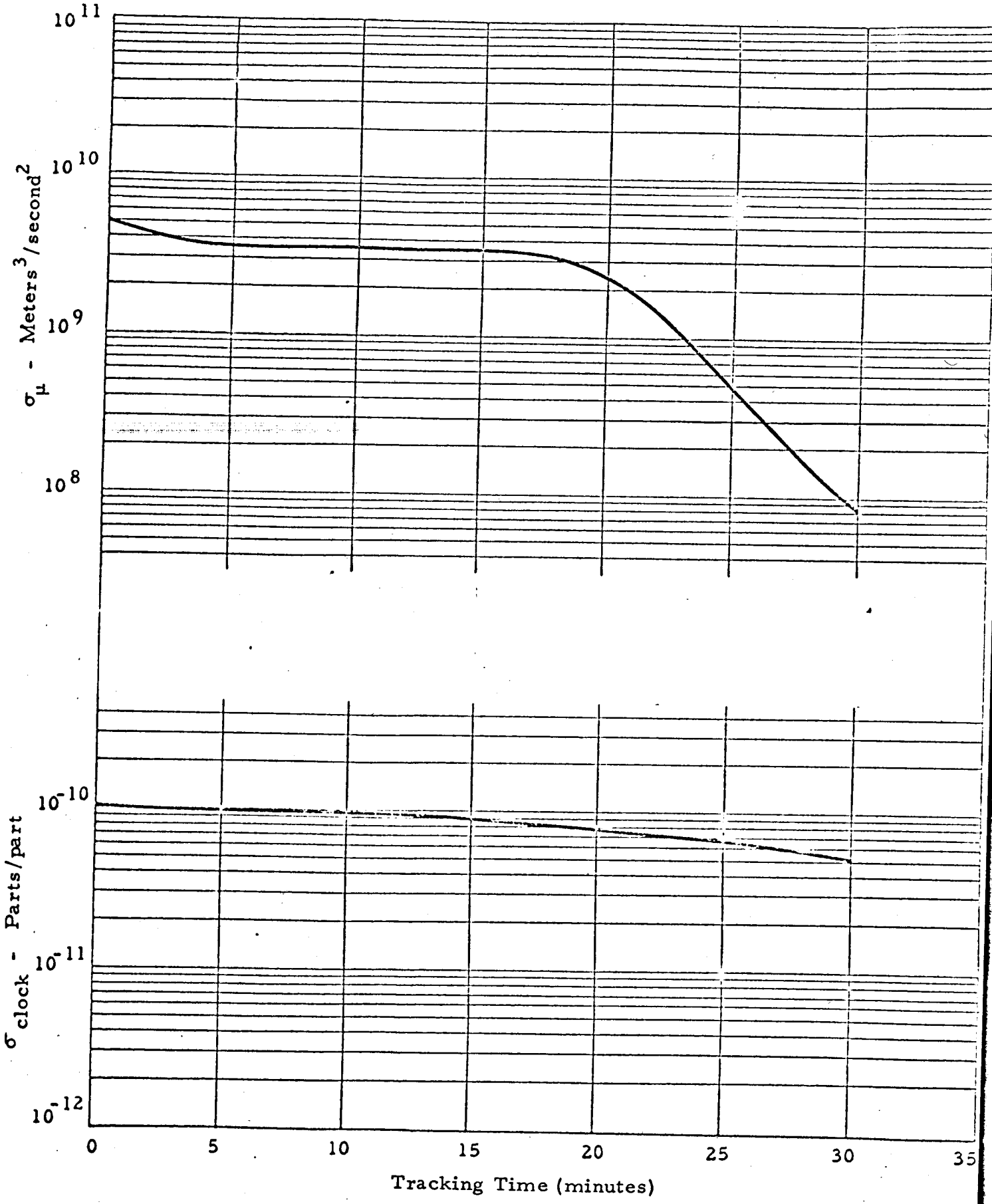


Figure 2. Reduction of Uncertainties in Clock Rate and Gravitational Constant with Tracking Time. Range Change from Madrid and Ascension. $\sigma_{\Delta R} = 0.6$ m.

BISSETT BERMAN

APOLLO NOTE NO. 175

L. Horowitz
March 6, 1964

OPTIMUM GUIDANCE THROUGH LUNAR DESCENT PHASE

Introduction: Apollo Note 46 describes optimum lunar ascent trajectories based on minimum fuel considerations. This note provides developments for lunar descent trajectories, optimizing exchanges between fuel displaced by more accurate on-board guidance systems and fuel economies afforded thereby.

Formulation: We suppose a flat lunar surface and a constant gravitational field "g":

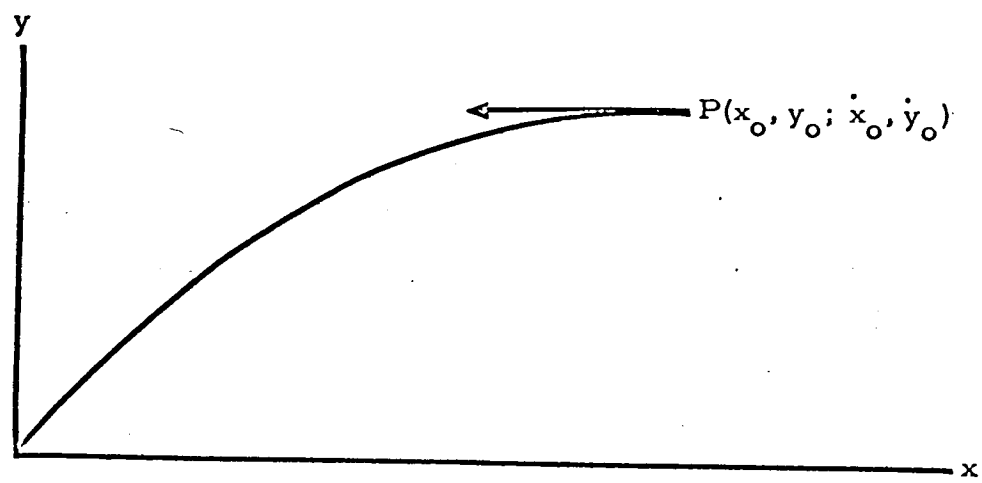


Figure 1

For a nominal case, we take y_o = 8 n. mi.; $\dot{x}_o$ = -5000 ft/sec; $\dot{y}_o$ = 0; and x_o to be determined subject to the requirement that the origin of phase space (0, 0, 0, 0) be acquired with minimum fuel expenditure. Taking the rocket engine to be operating at full thrust over the entire descent trajectory, it is clear that the minimum fuel requirement can be replaced by the requirement of minimum transfer time. As an approximation, we take the thrust to mass ratio of the vehicle to be 6g, where g is the lunar

gravitational constant. Then, letting θ be the thrust angle (measured clockwise from the positive y axis as in Note 46), T the transfer time and "a" the thrust magnitude, we have

$$\ddot{x} = a \sin \theta$$
$$\ddot{y} = a \cos \theta - g \tag{1}$$

subject to the indicated initial and final conditions. Then, by (28) of Note 46, we have

$$\theta(t) = \tan^{-1}\left[\frac{1}{\cot \theta_T - \frac{a \ln \gamma}{\dot{x}_o}(T - t)}\right], \tag{2}$$

where we have "reversed time" by replacing t with T - t. Here, $\theta_o = \theta(0)$; $\theta_T = \theta(T)$, and $\gamma = \frac{\csc \theta_o - \cot \theta_o}{\csc \theta_T - \cot \theta_T}$.

Rewriting (2) as

$$\theta(t) = \tan^{-1}\left(\frac{1}{\alpha + \beta t}\right), \tag{3}$$

we obtain

$$\cos \theta = \frac{\alpha + \beta t}{\sqrt{1 + (\alpha + \beta t)^2}}$$
$$\sin \theta = \frac{1}{\sqrt{1 + (\alpha + \beta t)^2}} \tag{4}$$

Then, by (1)

$$\dot{x}(t)\Big|_0^T = a\int_0^T \frac{dt}{\sqrt{1+(\alpha+\beta t)^2}}$$

$$\dot{y}(t)\Big|_0^T = a\int_0^T \frac{(\alpha+\beta t)\,dt}{\sqrt{1+(\alpha+\beta t)^2}} - gT \tag{5}$$

so that

$$T = \frac{\dot{x}_o}{a\,\ell n\,\gamma}\left[\cot\theta_T - \cot\theta_o\right]. \tag{6}$$

From (2),

$$\theta_o = \tan^{-1}\left[\frac{1}{\theta_T - \frac{aT\ell n\gamma}{\dot{x}_0}}\right], \tag{7}$$

so that θ_T is specified by θ_o. Now, if we start with $\theta_o = 3\pi/2$ (to provide maximum diminution in initial kinetic energy $\frac{1}{2}m\dot{x}_o^2$), we have

$$\cot\theta_T = \theta_T \tag{8}$$

with solutions .860, 3.43, 6.44, 9.53, 12.7, 15.8, ... , $n\pi$ (asympototically).

Now it is clear on physical grounds that the horizontal component of thrust will always oppose the horizontal component of velocity (since minimum transfer time is desired). Hence, for $\dot{x}_o$ negative, the "principal values" of $\theta(t)$ as given by (2) must be

$$(2n-1)\pi \le \theta(t) \le 2n\pi\,; \quad n = 1, 2, \ldots \tag{9}$$

But note that if it were actually possible to have $n > 1$ for this optimal trajectory, the horizontal component of velocity would have to reverse

itself, contradicting the assumed optimality. Hence, the principal values (9) for $\theta(t)$ must be taken for $n = 1$:

$$\pi \leq \theta(t) \leq 2\pi \ . \tag{10}$$

The required zero of (8) is therefore $\theta_T = 3.43$ rad $\approx 197^\circ$. The quantity γ is then .281; $\ln\gamma = -1.27$, so that $T = 382$ sec. for this optimal transfer.

The horizontal range is

$$x_o = -\int_0^T \dot{x}(t)\, dt \ , \tag{11}$$

where $\dot{x}(t)$ is given by (5). Effecting this computation,

$$x_o = \left[\frac{a \sinh^{-1}\alpha}{\beta} - \dot{x}_o\right] T - \frac{a}{\beta^2}\left[u \sinh^{-1} u - \sqrt{1+u^2}\right]_{u=\alpha}^{u=\alpha+\beta T} \tag{12}$$

since $x(T) = 0$. The numerical value of x_o is 192 n. mi. for this case. The development (12) is given in Appendix I.

Appendix II provides an analogous development for optimum guidance with horizontal range specified. The result is

$$\theta(t) = \tan^{-1}\left[\frac{1 + C_1 t}{C_2 + C_3 t}\right] \ . \tag{13}$$

We see that $C_1 = 0$ for the optimum range case.

<u>Error Study I, Single Terminal Correction</u>: Using on-board altimetry as the guidance and navigation observable, we begin by assuming errors in estimated position and velocity to be themselves proportional to position and velocity respectively, and to be of the order of 1%. For simplicity, we first consider position errors and velocity errors separately, and take the extreme case of a single terminal correction under which no correctional measures are taken until the total apparent transfer time T has elapsed:

1. Positional Errors: For an initial error ϵx_o in the initial horizontal coordinate x_o, we have

$$x_\epsilon(T) = a\int_0^T (T-t)\sin\theta(t)\,dt + \dot{x}_o T + (1+\epsilon)\,x_o$$

where $x_\epsilon(T)$ is the horizontal coordinate at $t = T$ corresponding to an initial error $\epsilon - x_o$. But from (1), $x(T) = a\int_0^T (T-t)\sin\theta\,dt + \dot{x}_o T + x_o = 0$ so that

$$x_\epsilon(T) = \epsilon x_o . \tag{14}$$

Now at full thrust ($a = 6g$), the time required to travel from $(x, y; \dot{x}, \dot{y}) = (\epsilon x_o, 0; 0, 0)$ to $(0, 0; 0, 0)$ is $T_{penalty} = \sqrt{\frac{2\epsilon x_o}{3g}}$, which (for $\epsilon = .01$) is 38 sec. or about 10% for the extreme case of no correctional measures until the (apparent) optimal trajectory has been completely negotiated.

2. Velocity Errors: For an initial error $\epsilon\dot{x}_o$ in the initial horizontal component of velocity $\dot{x}_o$, we have

$$\dot{x}_\epsilon(T) = \int_0^T a\sin\theta\,dt + (1+\epsilon)\,\dot{x}_o ,$$

where $\dot{x}_\epsilon(T)$ is the horizontal component of velocity at $t = T$ corresponding to an initial error $\epsilon\dot{x}_o$. But from (1), $\dot{x}(T) = a\int_0^T \sin\theta\,dt + \dot{x}_o = 0$ so that

$$\dot{x}_\epsilon(T) = \epsilon\dot{x}_o . \tag{15}$$

Integrating again, we have

$$x_\epsilon(T) = a\int_0^T (T-t)\sin\theta\,dt + (1+\epsilon)\,\dot{x}_o T + x_o ,$$

where

$$x(T) = a\int_0^T (T-t)\sin\theta\,dt + \dot{x}_o T + x_o = 0.$$

Hence,

$$x_e(T) = e\dot{x}_o T \tag{16}$$

for this special case. By symmetry, the optimum thrust program is to apply full thrust in the negative x direction until $x = \frac{e\dot{x}_o}{24g}\left[e\dot{x}_o + 12gT\right]$, then full thrust in the positive x direction until the origin of phase space (0, 0; 0, 0) is acquired (see Appendix III). The time penalty for this transfer is $\frac{1}{6g}\left(e\dot{x}_o + \sqrt{2e^2\dot{x}_o^2 + 24ge\dot{x}_o T}\right)$ as is also shown in Appendix III, which (for $\epsilon = .01$, $T = 382$ sec. as before) is 49 seconds or about 13% for this extreme case of a single terminal correction on an initial velocity error of 1%. Note that penalties for position errors and penalties for velocity errors are about the same for this situation.

Error Study II: Marginal Utility of Additional Guidance and Navigation Units: To a first approximation, doubling the number of on-board guidance and navigation units should result in reducing the (1σ) error in estimated initial position and velocity by $\frac{1}{\sqrt{2}}$. Relating fuel displaced by this additional equipment to fuel saved in the required terminal correction should define the optimum exchange between the two. Note that there is an additional complication introduced by reliability considerations, however; should one of the units fail, there is less fuel available (hence diminished reliability) than before—we ignore this problem for the moment, however.

Suppose then that the (1σ) guidance and navigation accuracy is improved by a factor $\frac{1}{\sqrt{m}}$ by adding "m" additional G and N units. Then, from Error Study I (above), the consequent time penalties resulting from terminal corrections (expressed as fractions of the initial penalties) become:

Fractional penalty, seconds

Position Error $\frac{\epsilon}{\sqrt{m}}$: $\frac{1}{\sqrt[4]{m}}$

$$\text{Velocity Error } \frac{\epsilon}{\sqrt{m}}: \begin{cases} \frac{1}{\sqrt[4]{m}}, & \frac{12gT}{e\dot{x}_o} >> 1 \\ \frac{1}{\sqrt{m}}, & \frac{12gT}{e\dot{x}_o} << 1 \end{cases} \tag{17}$$

where T is the (apparent) optimal transfer time (382 seconds for the case under consideration).

Figure 2 is a plot of total transfer time versus percent error in initial position/velocity for optimum descent from 8 n. mi. orbit. A single terminal correction at T = 382 seconds is made to acquire correct terminal conditions.

Figure 3 shows net fuel penalty versus pounds of added guidance and navigation units for the same optimum descent problem. By negative fuel penalty, we mean net fuel savings. Net fuel penalty F(m) due to adding "m" additional guidance and navigation units is:

$$F(m) = F_d(m) - F_s(m) \tag{18}$$

where F_d is the fuel displaced by m additional units and $F_s(m)$ is the fuel saved by improved accuracy provided with m additional units. One additional guidance and navigation unit is taken to provide a (1 σ) probable error in estimated position/velocity of 1%.

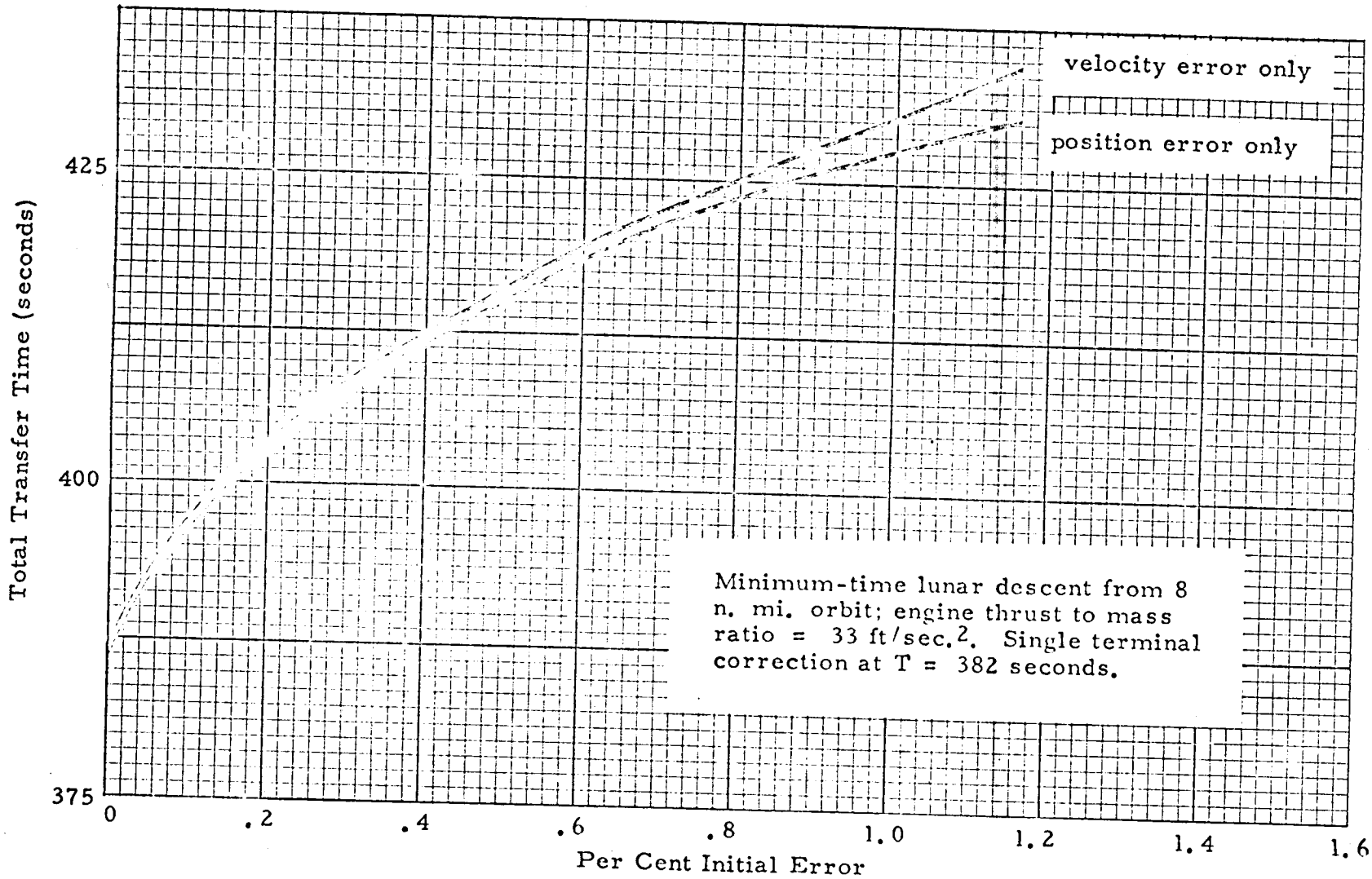


Figure 2.

Transfer Time vs. Initial Error (%)

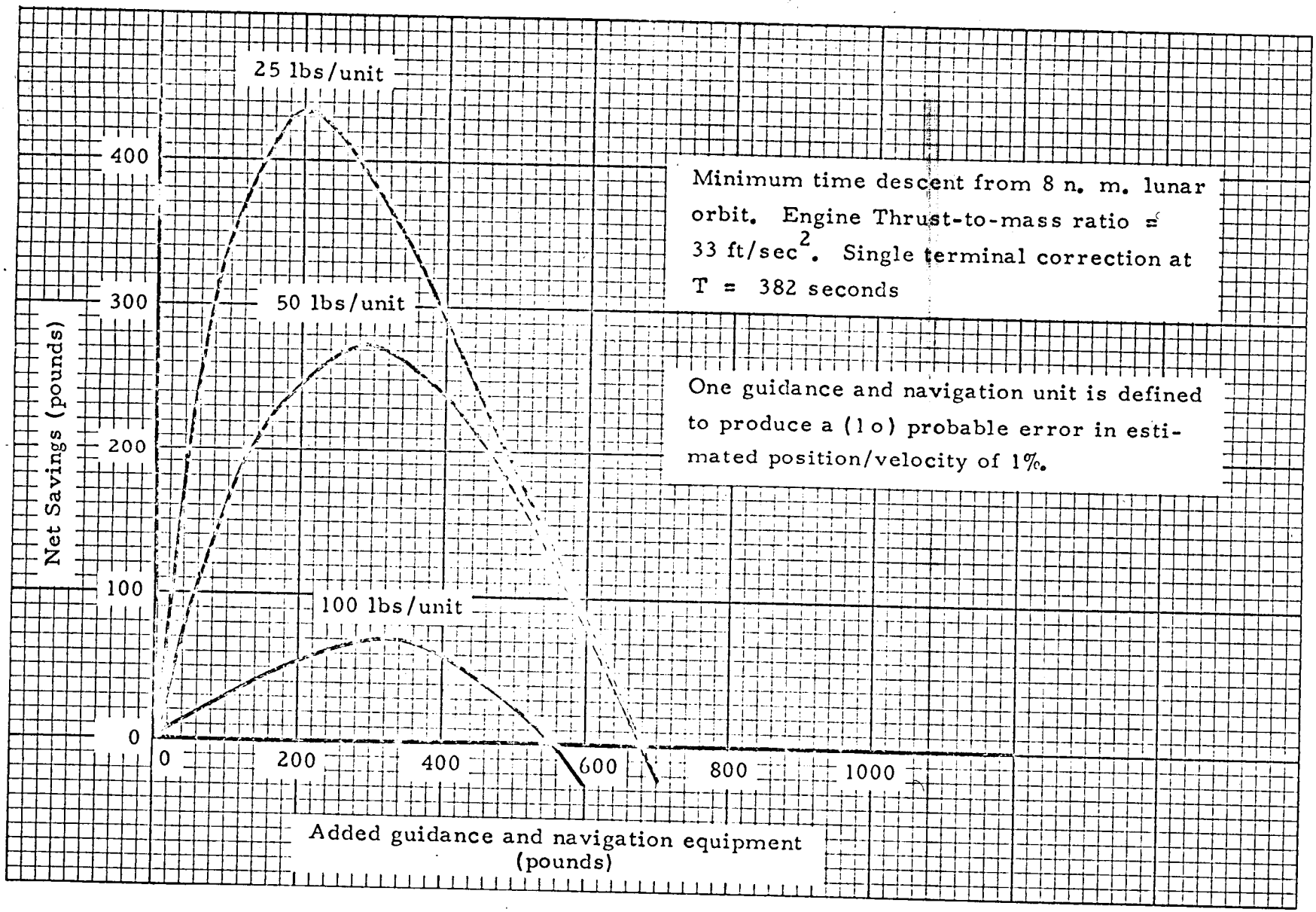


Figure 3

Net Fuel Savings vs. Added Guidance and Navigation Equipment

APPENDIX I

INTEGRALS OF MOTION

$$\tan\theta = \frac{1}{\alpha + \beta t} \quad \text{from (3),}$$

$$\ddot{x} = a \sin\theta\,, \quad \text{from (1),}$$

$$\sin\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}\,, \quad \text{so that}$$

$$\ddot{x} = \frac{a}{\sqrt{1 + (\alpha + \beta t)^2}}\,,$$

$$\dot{x}(t) - \dot{x}(0) = \frac{\alpha}{\beta} \int_{\alpha}^{\alpha + \beta t} \frac{du}{\sqrt{1 + u^2}}\,, \quad \text{where } u = \alpha + \beta t.$$

$$\dot{x}(t) = \dot{x}_o + \frac{\alpha}{\beta}\left[\sinh^{-1}(\alpha + \beta t) - \sinh^{-1}\alpha\right].$$

$$x(t) - x(0) = \dot{x}_o t + \frac{\alpha}{\beta}\int_o^t \sinh^{-1}(\alpha + \beta t)\,dt + \left(\frac{a \sinh^{-1}\alpha}{\beta}\right) t,$$

so that

$$x_o = \left[\frac{a \sinh^{-1}\alpha}{\beta} - \dot{x}_o\right] T - \frac{a}{\beta^2}\left[u \sinh^{-1} u - \sqrt{1 + u^2}\right]_{\alpha}^{\alpha + \beta T}.$$

APPENDIX II

VARIATIONAL SOLUTION FOR OPTIMUM DESCENT

The solution $\theta = \tan^{-1}\left(\frac{1}{\alpha + \beta t}\right)$ provides optimum thrust control for the descent phase described in this note, leaving optimum horizontal range open for variational specification. For study of small excursions about the optimum trajectory, however, optimization must be conducted with horizontal range specified.

To formulate this problem variationally, we write

$$\begin{aligned} \ddot{x} &= a \sin \theta \\ \ddot{y} &= a \cos \theta - g \end{aligned} \tag{II-1}$$

with $(x, y; \dot{x}, \dot{y})\big|_{t=0} = (x_o, y_o; \dot{x}_o, \dot{y}_o)$ and $(x, y; \dot{x}, \dot{y})\big|_{t=T} = (0, 0, 0, 0)$ for the as yet unspecified total transfer time T. Hence,

$$\begin{aligned} &\int_0^T \left(a \sin \theta + \frac{\dot{x}_o}{T}\right) dt = 0 \\ &\int_0^T \left(a \cos \theta - g + \frac{\dot{y}_o}{T}\right) dt = 0 \\ &\int_0^T \left[(T - t)\, a \sin \theta + \frac{x_o}{T} + \dot{x}_o\right] dt = 0 \\ &\int_0^T \left[(T - t)\,(a \cos \theta - g) + \frac{y_o}{T}\right] dt = 0 \qquad \text{as before.} \end{aligned} \tag{II-2}$$

Again, the functional on θ to be optimized is

$$T = \int_0^T dt\,, \tag{II-3}$$

for which the Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\theta}}\right) - \frac{\partial F}{\partial \theta} = 0 , \tag{II-4}$$

subject to the integral constraints (II-2). Using Lagrange multipliers λ_i, the equivalent optimization may be conducted on

$$F = 1 + \lambda_1 (a \sin\theta + \frac{\dot{x}_o}{T}) + \lambda_2 \ (a \cos\theta - g + \frac{\dot{y}_o}{T})$$
$$+ \lambda_3\left[(T-t)\ a \sin\theta + \frac{\dot{x}_o}{T} + \dot{x}_o\right] + \lambda_4\left[(T-t)(a \cos\theta - g) + \frac{y_o}{T}\right],$$

so that

$$\lambda_1 \cos\theta - \lambda_2 \sin\theta + \lambda_3(T - t) \cos\theta - \lambda_4(T - t) \sin\theta = 0, \quad \text{and}$$

$$\theta(t) = \tan^{-1}\left[\frac{1 + C_1 t}{C_2 + C_3 t}\right] . \tag{II-5}$$

Note that for the unspecified - range problem, $C_1 = 0$ as in Apollo Note 46.

APPENDIX III

OPTIMUM ONE-DIMENSIONAL TRANSFER

This note has shown that initial error in a single velocity component results in both position and velocity errors at termination of an optimum transfer. To determine the optimum thrust program for effecting terminal correction, we are led to the one-dimensional optimization problem illustrated in Figure III-1:

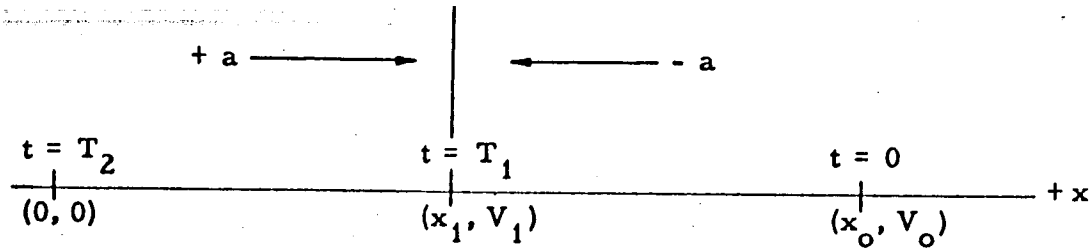


Figure III-1

Here we know only x_o and V_o; we wish to acquire (0, 0) in minimum time. T_1, T_2, x_1 and V_1 are as yet unspecified. Full thrust in the negative x direction is applied until (x_1, V_1) is acquired at time T_1; full thrust in the positive x direction is then applied until the origin of phase space is acquired at time $t = T_2$. The unknown quantities T_1, T_2, x_1, V_1 are obtained in the following fashion:

$0 \leq t \leq T_1$:

$$\ddot{x} = -a$$

$$\dot{x} = -at + V_o$$

$$x = -\frac{1}{2}at^2 + V_o t + x_o$$

$t = T_1$:

$$V_1 = -aT_1 + V_o$$

$$x_1 = -\frac{1}{2}aT_1^2 + V_o T_1 + x_o$$

$T_1 \leq t \leq T_2$:

$$\ddot{x} = +a$$

$$\dot{x} = at + k_1$$

$$x = \frac{1}{2}at^2 + k_1 t + k_2$$

$t = T_1$:

$$V_1 = aT_1 + k_1$$

$$x_1 = \frac{1}{2}aT_1^2 + k_1 T_1 + k_2$$

(Solving for k_1, k_2)

$t = T_1$

$$V_1 = -aT_1 + V_o$$

$$x_1 = -\frac{1}{2}aT_1^2 + T_1 V_o + x_o$$

$T_1 \leq t \leq T_2$:

$$\ddot{x} = +a$$

$$\dot{x} = at - 2aT_1 + V_o$$

$$x = \frac{1}{2}at^2 + (V_o - 2aT_1)\,t + aT_1^2 + x_o$$

$t = T_2$:

$$0 = aT_2 - 2aT_1 + V_o$$

$$0 = \frac{1}{2}aT_2^2 + (V_o - 2aT_1)\,T_2 + aT_1^2 + x_o$$

Solving simultaneously:

$$T_1 = \frac{1}{a}\left(V_o + \sqrt{\frac{V_o^2}{2} + ax_o}\right)$$

$$T_2 = \frac{1}{a}\left(V_o + 2\sqrt{\frac{V_o^2}{2} + ax_o}\right)$$

Remaining unknowns:

$$V_1 = -\sqrt{\frac{V_o^2}{2} + ax_o}$$

$$x_1 = \frac{V_o^2}{4a} + \frac{x_o}{2}$$

The positive sign for the square root was selected in solving the quadratic for T_1 to provide physically meaningful answers.

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APOLLO NOTE NO. 176

C. Siska
9 March 1964

CHANGE IN ΔV REQUIRED TO ESTABLISH A LUNAR POLAR ORBIT MEASURED RELATIVE TO A FREE RETURN ORBIT IN EARTH-MOON PLANE

1. APPROACH TO PROBLEM

1.1 Principal Ground Rules

a. Launch from 28.5° latitude (Cape Kennedy) into E-M plane. This implies launch in the 1968-1969 period when the Moon's orbit inclination is 28.5°.

b. Lunar polar orbit will have same shape as lunar phase of free return orbit; only the orientation differs.

c. Injection radius (and therefore parking orbit altitude) will be the same in each case.

d. No orbit plane change is considered after launch.

1.2 ΔV Required

From 1.1(b), lunar deboost will be the same for both free return and lunar polar orbits, therefore the only change in velocity required here is at injection in order to establish the proper Earth phase orbit which yields a lunar polar orbit.

From 1.1(c), the only velocity change required is at ascent burnout due to different launch azimuth (or equivalently, different orbit inclination to Equator). This ΔV has to compensate for the effect of Earth's rotation speed at different launch azimuths, since the inertial velocity in the parking orbit remains the same in each case.

2. REFERENCE FREE-RETURN ORBIT

2.1 Earth Phase

Manipulating graphical results in NASA TR R-122 yields,

V_o = 35830 fps - injection velocity

γ_o = 20° - injection angle to local horizontal

r_o = 4100 st. mi. - injection radius

$\tau \approx 90^o$ - lead angle of Moon relative to injection position

72 hours flight time from injection to perilune (approximately 60 hours to LSOI plus 12 hours to perilune).

2.2 Lunar Phase

The following values are used,

V_S = 3920 fps - selenographic velocity at LSOI

β = 4° - flight path angle measured from inward direction of radius vector (i.e., $\beta = 0$ directed toward Moon's center)

$\overline{\xi}$ = 51° - West longitude (E-M plane) of LSOI entry

$\overline{\eta}$ = 0° - latitude (E-M plane) of LSOI entry

100 st. mi. is perilune altitude.

3. GEOCENTRIC ORBIT ELEMENTS AT LSOI

3.1 Method

The equations used to obtain the elements are described in NASA TN D-1780, quantities being referenced to the E-M plane (x, y, plane) with x axis as the E-M line at LSOI entry (see Figure 1 for notation).

The situation defined in Section 1.1(b) requires the geometry of Figure 2. For the polar orbit going northward, the variable $\theta = \beta = \eta$ (latitude) and the longitude is $\xi = \bar{\xi} - \beta$, where ξ, η is the LSOI entry position.

3.2 Geometric Elements, etc.

Quantities were computed to the first order in β.

Free-Return Orbit		Perturbed Orbit
$\bar{x}/R_{LS} = 6.040$	Position Components at LSOI	$x/R_{LS} = 5.989$
$\bar{y}/R_{LS} = 0.777$		$y/R_{LS} = 0.731$
$\bar{z}/R_{LS} = 0$		$z/R_{LS} = 0.0698$
$\dot{\bar{x}}/R_{LS} = 0.681$	Velocity Components at LSOI	$\dot{x}/R_{LS} = 0.681$
$\dot{\bar{y}}/R_{LS} = 0.252$		$\dot{y}/R_{LS} = 0.252$
$\dot{\bar{z}}/R_{LS} = 0$		$\dot{z}/R_{LS} = 0$

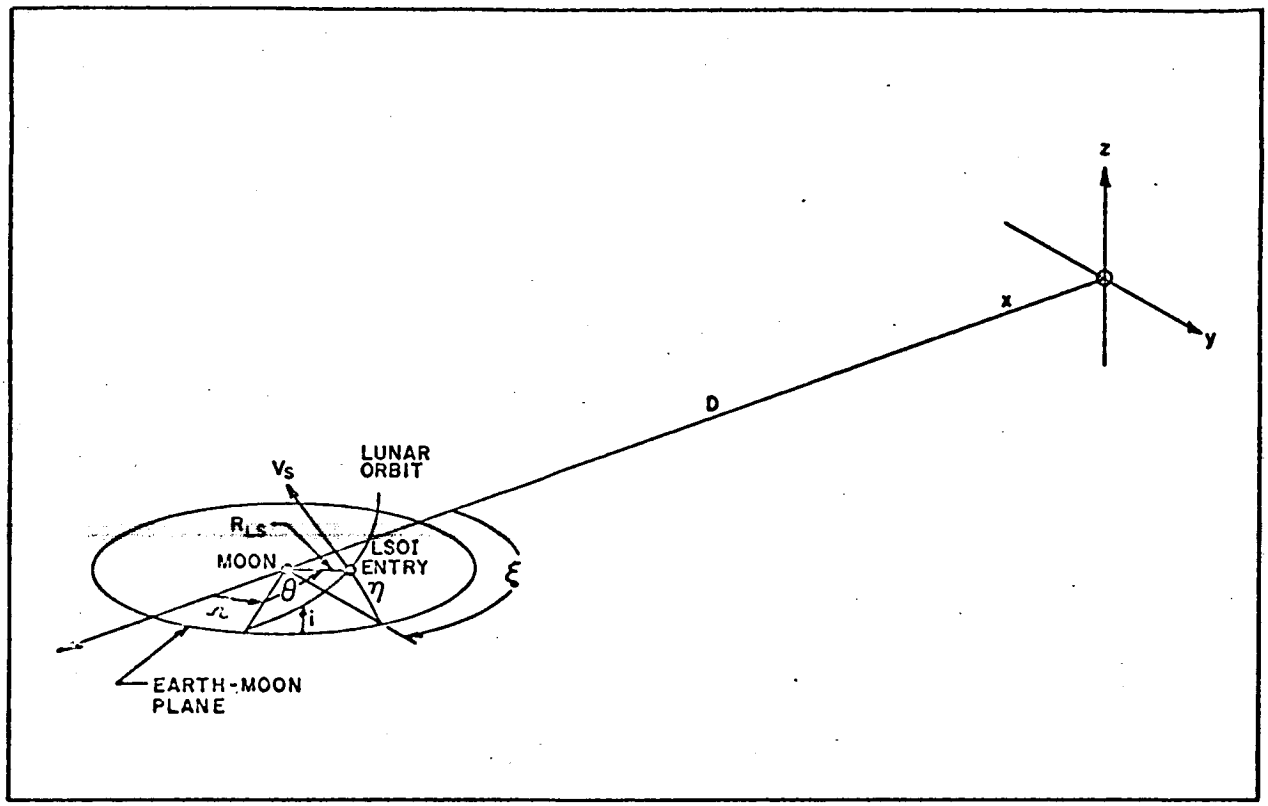


Figure 1.
Notation

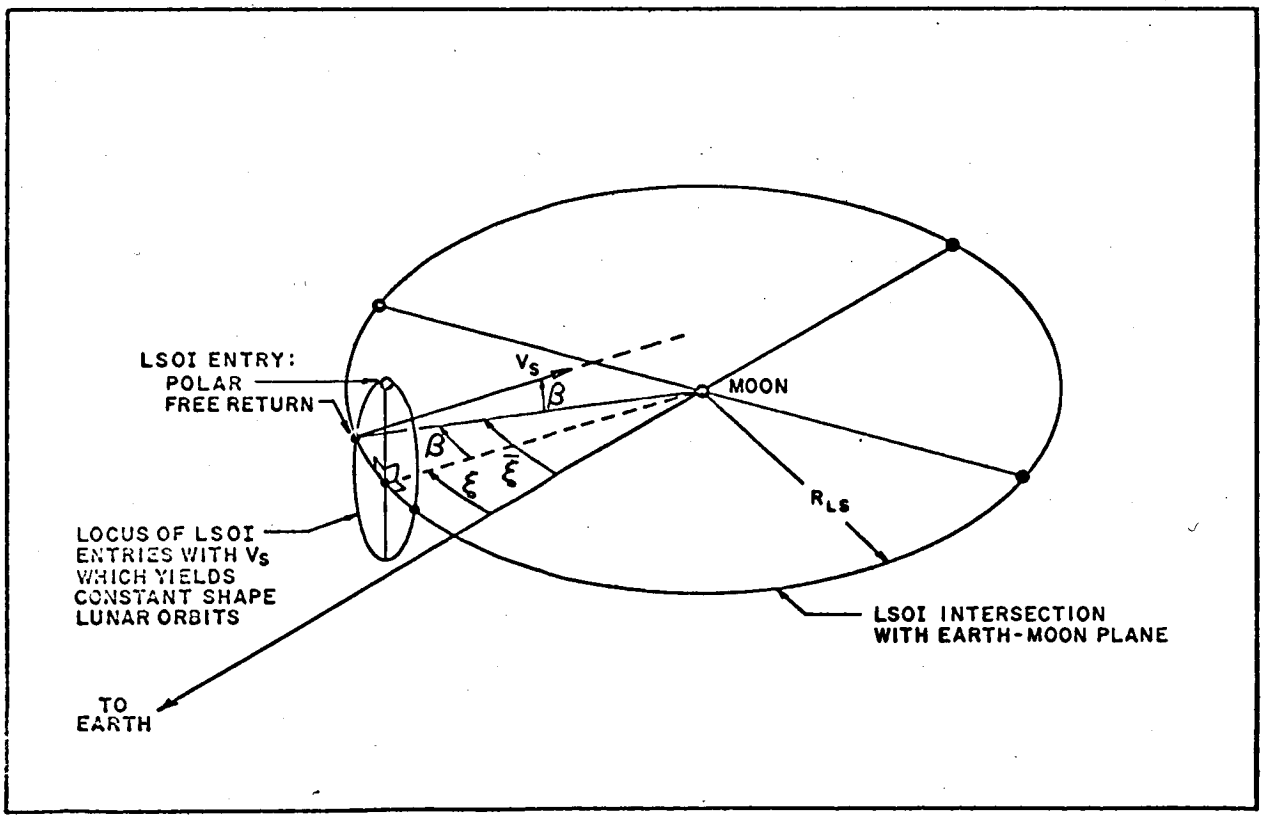


Figure 2.
Relative LSOI Position for Polar Orbit

Free-Return Orbit			Perturbed Orbit	
$\frac{\bar{h}_x}{R_{LS}V_S}$	= 0		$\frac{h_x}{R_{LS}V_S}$	= -.0176
$\frac{\bar{h}_y}{R_{LS}V_S}$	= 0	Momentum Components	$\frac{h_y}{R_{LS}V_S}$	= .0475
$\frac{\bar{h}_z}{R_{LS}V_S}$	= 0.995		$\frac{h_z}{R_{LS}V_S}$	= 1.013
$\frac{\bar{h}}{R_{LS}V_S}$	= 0.995	Momentum	$\frac{h}{R_{LS}V_S}$	= 1.0143
$\frac{\bar{r}}{R_{LS}}$	= 6.0672	Radius to LSOI Entry Position From Earth	$\frac{r}{R_{LS}}$	= 6.0366

From the momentum components we obtain the inclination $i_o = 2.94^o$ relative to the E-M plane and the descending node, $\Omega_o = 21.8^o$ relative to E-M line at LSOI entry for the perturbed orbit.

4. CHANGE IN INJECTION CONDITIONS

Note in Section 3.2 that to 1st order in β, the velocity components do not change, the momentum increases for the perturbed orbit while the radius to LSOI entry decreases (with consequent decrease in energy since the velocities are the same). Thus we have

$$\frac{\Delta h}{R_{LS}V_S} = \frac{h-\bar{h}}{R_{LS}V_S} = .0193 \quad \text{(more momentum for perturbed orbit)}$$

$$R_{LS}\Delta E = \frac{\mu_E}{\left(\bar{r}/R_{LS}\right)^2}\left(\frac{\Delta r}{R_{LS}}\right) = -1.17 \times 10^{13}\ ft^3/sec^2 \quad \text{(less energy for perturbed orbit)}$$

At the injection position, since $\overline{r}_o = r_o$, we have

$$\Delta E = \overline{V}_o \, \Delta V_o$$

$$\Delta h = r_o \left(\Delta V_o \cos \overline{\gamma}_o - \overline{V}_o \sin \overline{\gamma}_o \, \Delta \gamma_o \right)$$

Evaluating these expressions, we obtain

$$\boxed{\begin{aligned} \Delta V_o &= -17.25 \text{ fps (less injection velocity)} \\ \Delta \gamma_o &= -3.12^\circ \text{ (smaller injection angle)} \end{aligned}}$$

5. OTHER CHANGES FOR PERTURBED ORBIT

Δp	=	$.1495 \times 10^7$ ft.	semi parameter
Δe	=	1.072×10^{-3}	eccentricity
Δv_o	=	-6.79°	true anomaly at injection
Δv_1	=	-0.962°	true anomaly at LSOI intercept
δv	=	$\Delta v_1 - \Delta v_o = 5.82^\circ$	central angle change, injection to LSOI entry
ΔM	=	$-.157$ rad.	mean anomaly change, injection to LSOI entry
Δt_F	=	-9.91 hours	flight time change, injection to LSOI entry

6. TIMING FOR REFERENCE ORBIT

Each day Cape Kennedy (28.5° latitude) touches the E-M plane (inclination 28.5°) once, and can therefore launch into the E-M plane at time $\overline{t}_{Li}$ on the (i)th day. At this time the Moon's position (relative) is an angle U_i (see Figure 3).

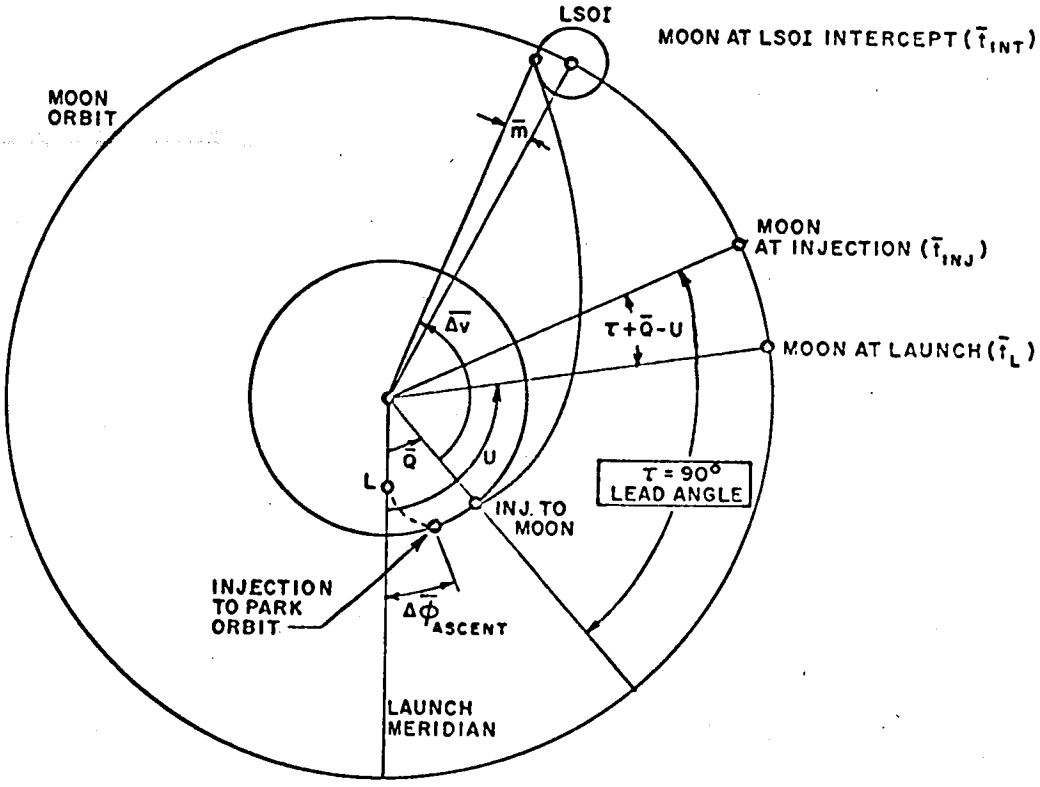


Figure 3.
Timing for Reference Orbit

At injection, the Moon's lead angle τ is already known for the reference orbit (Section 2.1). The possible launch times are known also from the lunar empheris (right ascension of max. Moon declination and associated meridian passage by Cape Kennedy). Additionally the angular separation U_i is known for the launch times t_{Li}. It remains to fix the proper relative injection position, angle $\overline{Q_i}$.

The time between launch and injection for the lunar vehicle encompasses the time to ascend to the parking orbit, time to traverse the angular difference between the injection positions (parking orbit and translunar) and the time in parking orbit for n complete revolutions.

Thus,

$$\overline{t}_{inj} - \overline{t}_L = \overline{\Delta t}_{ascent} + \frac{\overline{Q_i} - \overline{\Delta\phi}_{ascent}}{\omega_{co}} + \frac{2\overline{n}\pi}{\omega_{co}}$$

where

$$\omega_{co} = \frac{V_{co}}{\overline{r}_o} = \text{circular orbit angular velocity.}$$

Simultaneously (Figure 3) the Moon must travel,

$$\omega_M (\overline{t}_{inj} - \overline{t}_L) = \tau + \overline{Q_i} - U_i$$

Thus, acceptable angles $\overline{Q_i}$ are from the above conditions,

$$\overline{Q_i} = \left(\frac{1}{\omega_M} - \frac{1}{\omega_{co}}\right)^{-1} \left[\overline{\Delta t}_{ascent} + \frac{2\overline{n}\pi - \Delta\phi_{ascent}}{\omega_{co}} + \frac{U_i - \tau}{\omega_M}\right]$$

Each $\overline{Q_i}$ leads to a different Moon position at LSOI intercept and we will look at the effect of this on launch azimuth.

7. TIMING FOR PERTURBED ORBIT

Using the relation for $\overline{t}_{inj} - \overline{t}_L$ in Section 6 and the fact that the time from injection to the Moon to LSOI is the flight time, $\overline{t}_F$, we can write for the time from launch to LSOI as

$$\bar{t}_{int} - \bar{t}_L = \bar{t}_F + \left[\overline{\Delta t}_{ascent} + \frac{\bar{Q} - \overline{\Delta\phi}_{ascent}}{\omega_{co}} + \frac{2\,\bar{n}\pi}{\omega_{co}} \right]$$

In a similar vein for the perturbed orbit

$$t_{int} - t_L = t_F + \left[\Delta t_{ascent} + \frac{Q - \Delta\phi_{ascent}}{\omega_{co}} + \frac{2\,n\pi}{\omega_{co}} \right]$$

Assuming that in each case the same number of revs. in the parking orbit prevail, we have $n = \bar{n}$. Further assume that the possible changes in the ascent path are negligible so that $\Delta t_{ascent} = \overline{\Delta t}_{ascent}$ and $\overline{\Delta\phi}_{ascent} = \overline{\Delta\phi}_{ascent}$. Thus subtracting the above equations results in

$$(t_{int} - \bar{t}_{int}) - (t_L - \bar{t}_L) = (t_F - \bar{t}_F) + \frac{Q - \bar{Q}}{\omega_{co}}$$

Now the difference in absolute intercept time of the LSOI is reflected in the change of the E-M line at intercept (See Figure 4). Thus

$$(t_{int} - \bar{t}_{int}) = \frac{z}{\omega_M} \quad (\omega_M = \text{angular velocity of Moon})$$

The difference in launch times (Figure 4) can be expressed as

$$(t_L - \bar{t}_L) = \frac{\pi/2 - W}{\omega_E} \quad (\omega_E = \text{angular velocity of Earch})$$

Consequently the required injection position for the perturbed orbit relative to the launch position Q can be written

$$Q = \omega_{co} \left[\frac{\bar{Q}}{\omega_{co}} - \Delta t_F + \frac{z}{\omega_M} - \frac{\pi/2 - W}{\omega_E} \right]$$

The value Δt_F is already known (Section 5, $\Delta t_F = -9.91$ hours). From the spherical triangles in Figure 4 we can obtain the values

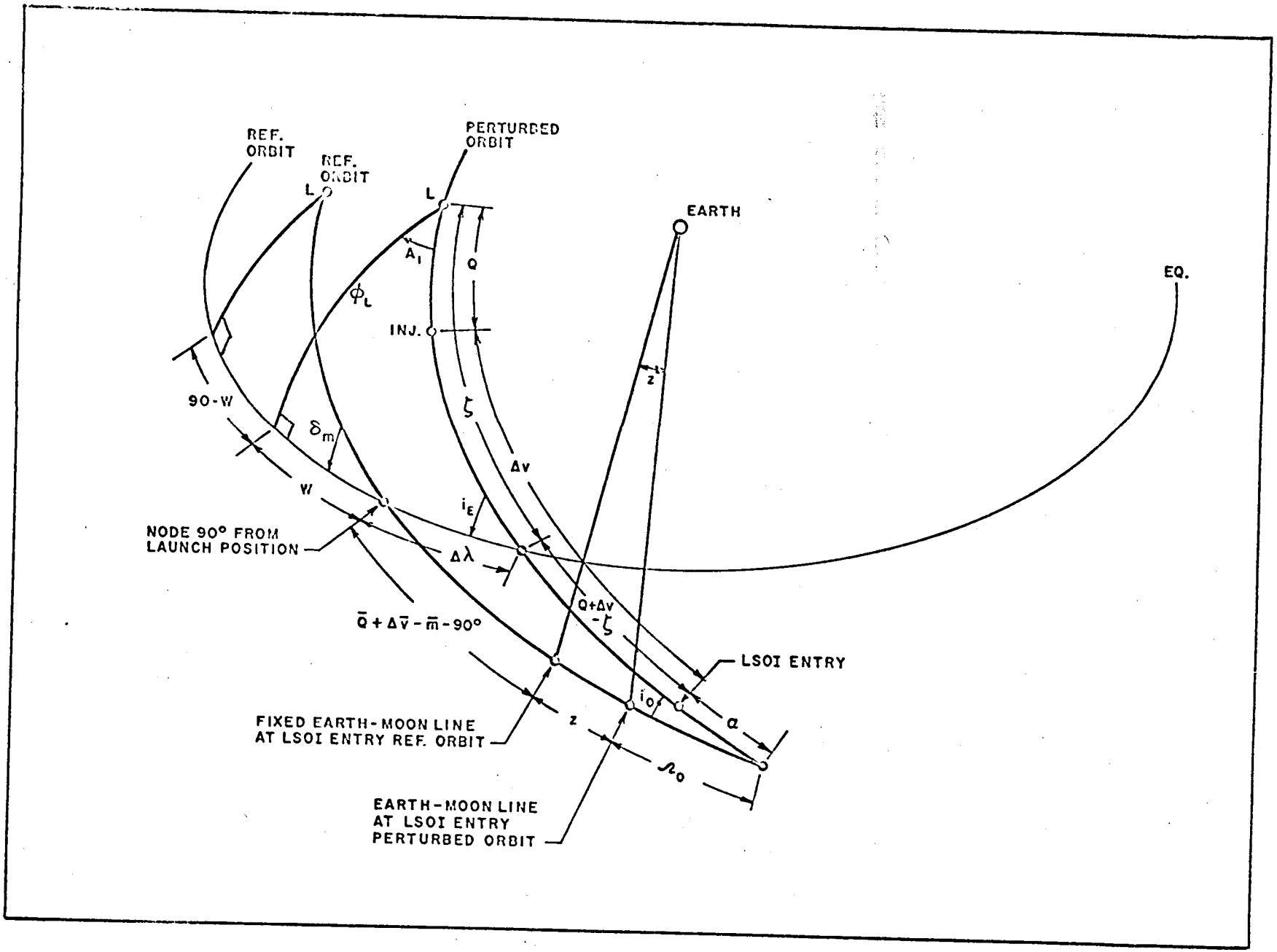


Figure 4.
Notation

Q, W, $\Delta\lambda$, i_E in terms of $\overline{Q}$ and z. Thus for each $\overline{Q_i}$ associated with $\overline{t}_{Li}$ there is a distinctive value of z required for proper LSOI intercept to ensue for the perturbed orbit.

In Figure 4, reference orbit quantities $\overline{\Delta v}$, $\overline{t}_F$ (central angle and flight time from Moon injection to LSOI) are 124° and 60 hours, respectively. Values i_o and r_o are already known. Quantities $\overline{m}$, α can be obtained using Figure 5 where r $(\overline{r})$, ξ $(\overline{\xi})$, z, $(\overline{z})$ are known.

Note that as z changes in Figure 4 that the inclination i_E changes and so does the angle A_1, which determines how much of the Earth's rotation speed is lost in developing the inertial velocity of the parking orbit.

8. ΔV REQUIRED IN ASCENT

Referring to Figure 6, the required boost burnout velocity is

$$V_B^2 = (V_\omega + \overline{V}_B)^2 + V_\omega^2 - 2V_\omega (V_\omega + \overline{V}_B) \cos \Delta A$$

where

$$V_\omega = R_E \omega_E \cos \phi_L$$

Considering that $\Delta V_B = V_B - \overline{V}_B$, we obtain to the second order in ΔA,

$$\Delta V_B \simeq \frac{1}{2} (\Delta A)^2 \left(\frac{V_\omega}{1 - \frac{V_\omega}{V_{co}}} \right)$$

where $\Delta A = \pi/2 - A_1$ which is a function of the value $\overline{Q}$.

9. RESULTS OF CALCULATION

Results are shown in Figure 7. Different values of $\overline{Q}$ correspond to different Moon positions at intercept. The 90°-W curve shows that launch is delayed up to 1.7 hours (15° longitude change per hour).

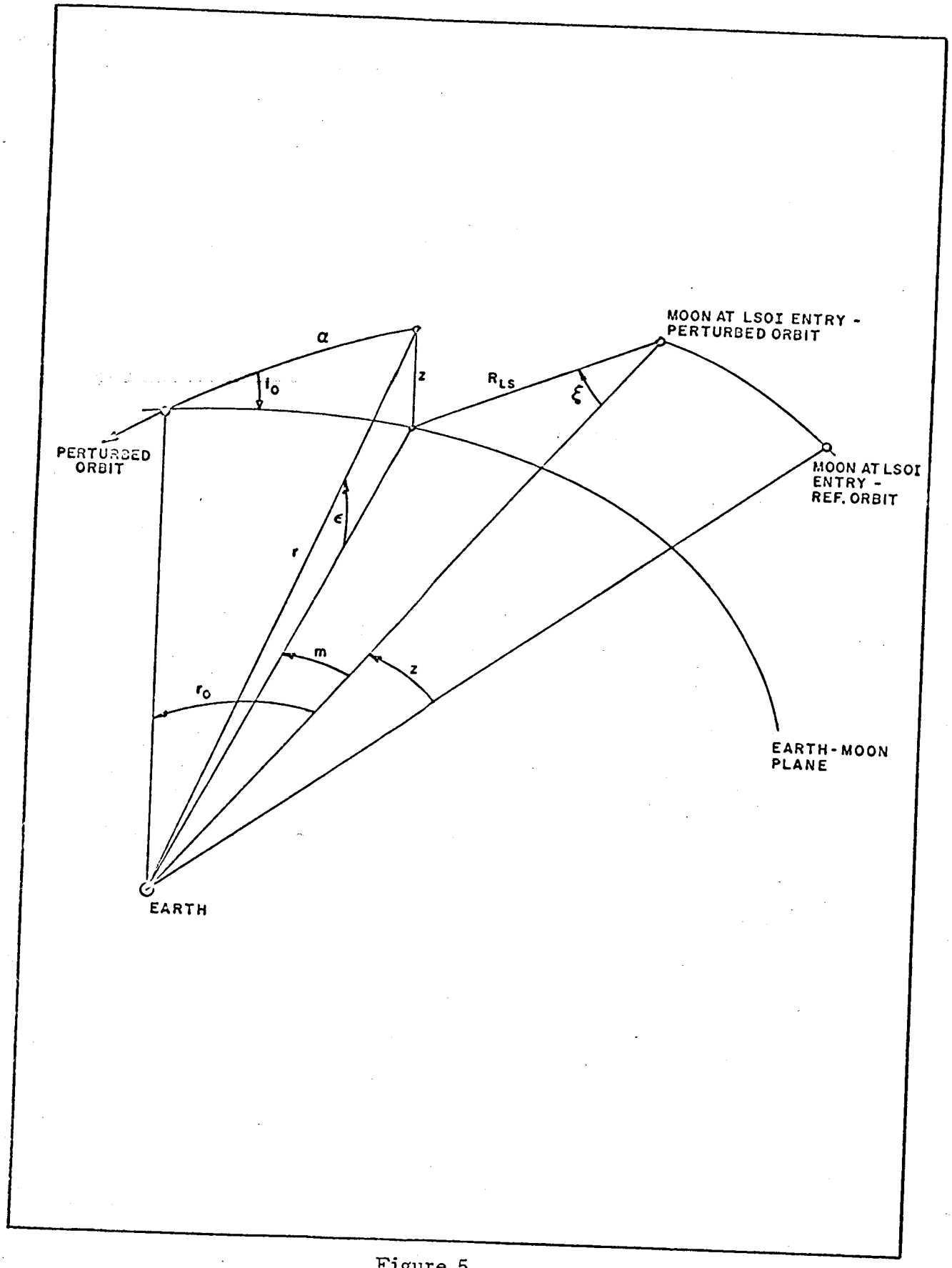


Figure 5.
More Notation

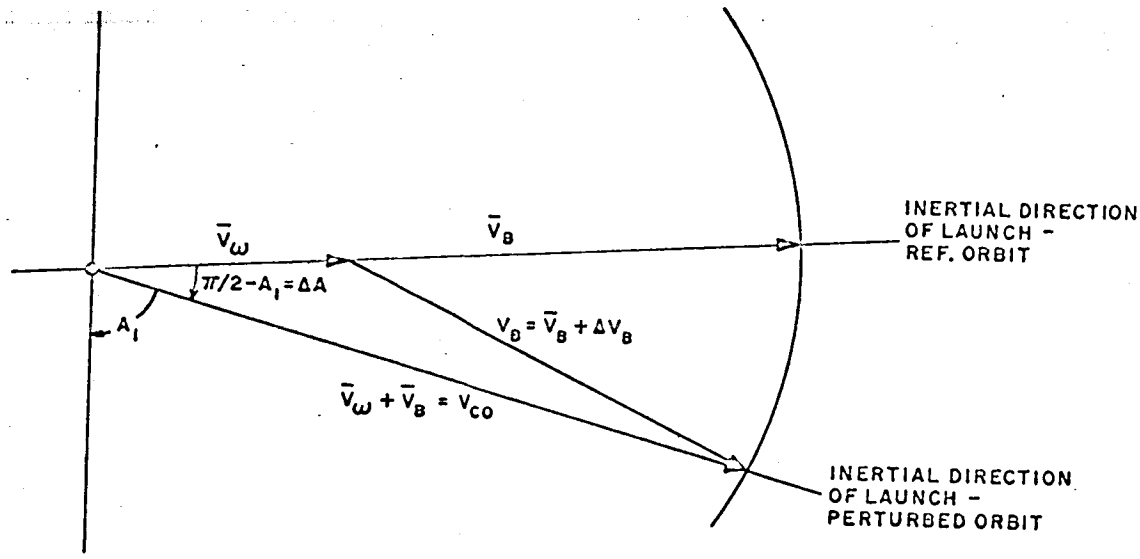
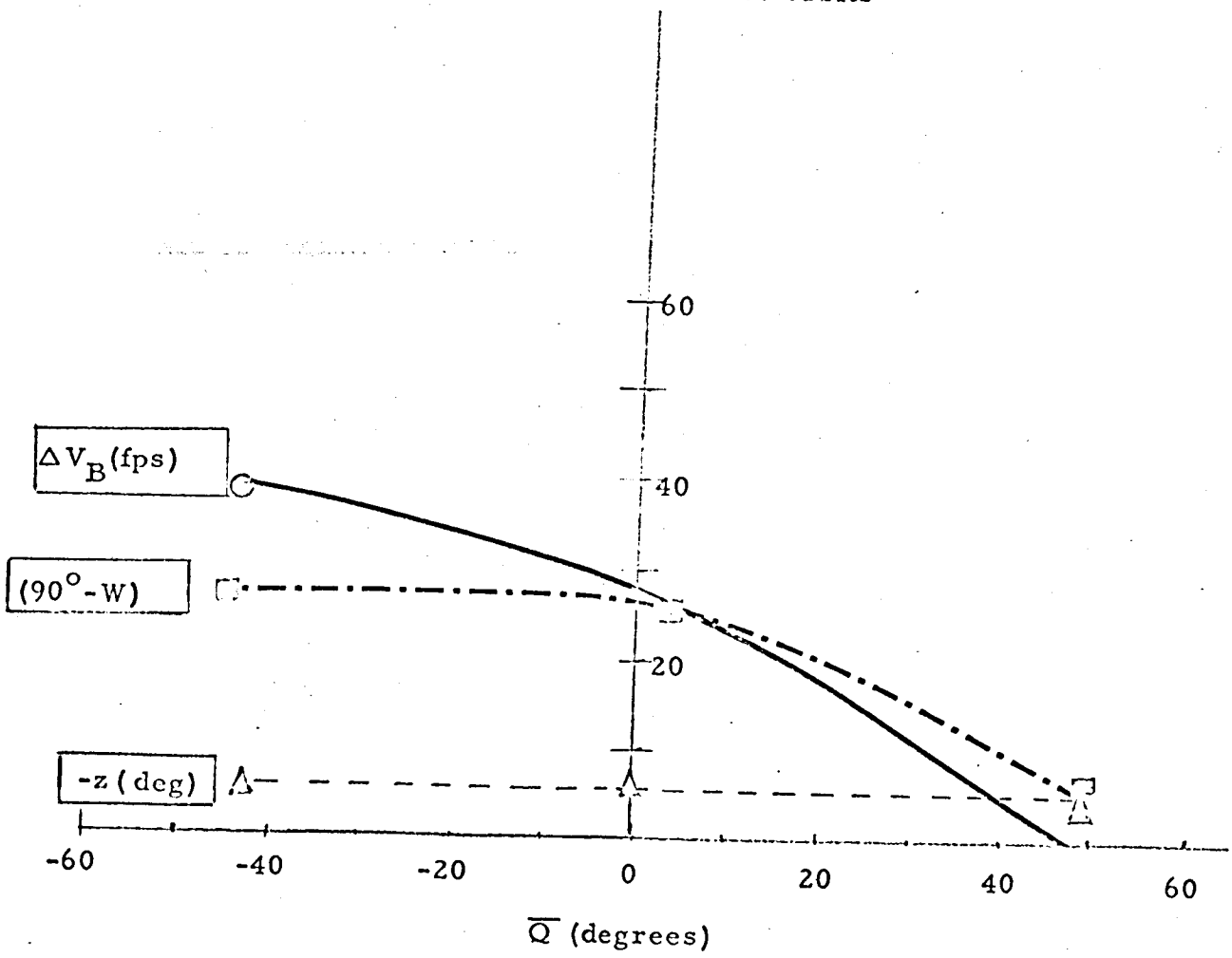

$\bar{V}_\omega$
$\bar{V}_B$
INERTIAL DIRECTION OF LAUNCH - REF. ORBIT
$\pi/2 - A_1 = \Delta A$
A_1
$V_D = \bar{V}_B + \Delta V_B$
$\bar{V}_\omega + \bar{V}_B = V_{CO}$
INERTIAL DIRECTION OF LAUNCH - PERTURBED ORBIT

Figure 6.

ΔV_B = Change ascent boost velocity for parking orbit

$90^\circ - W$ = Change in longitude of launch position

z = Difference in Moon position at LSOI entry for perturbed and reference orbits



Moon Injection Position Relative to Launch

Figure 7.
Changes Required For Perturbed Orbit

The perturbed orbit flight time is roughly 9.9 hours shorter and therefore the value z shows this, since the Moon moves at about $1/2^{\circ}$ per hour.

ΔV requirements over-all are small since the Moon injection is approximately 17 fps smaller while the ascent boost is up to 40 fps higher. If no jettison occurs from injection into parking orbit until injection to Moon, the weight landed on the Moon doesn't change appreciably.

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APOLLO NOTE NO. 177

H. Engel
10 March 1964

ACCURACY OF POSITION AND VELOCITY DETERMINATION DURING BOOST

The Kalman-Schmidt method of determining orbit parameters has been shown to provide a best linear estimate of these parameters. In spite of the warnings given about the use of the Kalman-Schmidt method to determine actual orbit parameters, it still provides a useful tool in determining the error covariance matrix of the maximum likelihood estimated trajectory parameters during boost.

The Kalman-Schmidt method is not restricted to free-fall orbit determination but may be used for any kind of motion for which the necessary measurements can be made and for which certain necessary matrices are known.

Before proceeding further with this discussion, let us once more define the quantities used in the Kalman-Schmidt method, with special attention to this particular application. We have

t_n Time of n-th set of observations.

x_n State (column) vector at t_n. The elements of x_n in this problem are the components of position, velocity and acceleration at t_n, and any other quantities to be estimated at t_n, such as measurement biases.

y_n Measurement (column) vector at t_n. Each element corresponds to one measurable, whether or not that quantity is used at t_n.

y_n^* Measured value of y_n.

u_n Zero mean random noise vector added to y_n that results in y_n^*. $y_n^* = y_n + u_n$. This noise has zero mean since all biases appear as part of the state vector.

Q_n Covariance of the noise u_n. Q_n is assumed known a priori, and the noise at t_n is assumed to be uncorrelated with the noise at any different time t_m.

$\phi_{n+1,n}$ Square matrix that transforms orbit parameters on a trajectory near the reference trajectory at time t_n into orbit parameters on the same trajectory at t_{n+1}. We discuss $\phi_{n+1,n}$ below.

M_n Rectangular matrix relating y_n and x_n. $y_n = M_n x_n$. Its elements are the first partial derivatives of the measurables with respect to the trajectory parameters.

x_n^* Best estimate of x_n based on all y_i^* for $i \prec n$. $E(x_n^* - x_n) = 0$.

P_n^* Covariance matrix of errors in estimated state vector at t_n. $P_n^* = E\left[(x_n^* - x_n)(x_n^* - x_n)^T\right]$

In addition we must introduce the following quantities for powered flight,

v_n Zero mean random noise added to x_n, corresponding to an uncontrolled change in acceleration between the interval from t_{n-1} to t_n and the interval t_n to t_{n+1}.

W_n Covariance matrix of v_n. W_n is assumed known a priori and the noise at t_n is assumed to be uncorrelated with the noise at any different time t_m.

From Appendix A of the Bissett-Berman Final Report on Contract NASw-688, Amendment 1, "Capabilities of MSFN for Apollo Guidance and Navigation," we find successive values of x^*_n and P^*_n, given values of $\phi_{n+1,n}$, M_n, Q_n, x^*_o and P^*_o, from iteration of the following set of four equations.

$$\Delta^*_n = \phi_{n+1,n} P^*_n M^T_n (M_n P^*_n M^T_n + Q_n)^{-1}$$

$$\phi^*_{n+1,n} = \phi_{n+1,n} - \Delta^*_n M_n$$

$$P^*_{n+1} = \phi^*_{n+1,n} P^*_n \phi^T_{n+1,n}$$

$$x^*_{n+1} = \phi^*_{n+1,n} x^*_n + \Delta^*_n y^*_n$$

in which Δ^*_n and $\phi^*_{n+1,n}$ are defined only by these equations.

In that report, however, there is no noise v_n, since the vehicle is in free-fall. We must correct the expression for P^*_{n+1} to include the effects of v_n. From that report we have

$$P^*_{n+1} = E\left[(x^*_{n+1} - x_{n+1}) (x^*_{n+1} - x_{n+1})^T \right]$$

and in this note we must use

$$x_{n+1} = \phi_{n+1,n} (x_n + v_n)$$

instead of

$$x_{n+1} = \phi_{n+1,n}\, x_n$$

Since we assume v_n to be uncorrelated with any other quantity we find

$$P^*_{n+1} = \phi^*_{n+1,n}\, P^*_n\, \phi^T_{n+1,n} + \phi_{n+1,n}\, W_n\, \phi^T_{n+1,n}$$

Now, we are not interested in finding x^*_{n+1}, so we need not evaluate Δ^*_n and $\phi^*_{n+1,n}$ separately. Instead, we may combine the above equations. We find by substitution that

$$\phi^*_{n+1,n} = \phi_{n+1,n} \left[I - P^*_n M^T_n (M_n P^*_n M^T_n + Q_n)^{-1} M_n \right]$$

and then

$$P^*_{n+1} = \phi_{n+1,n} \left[P^*_n - P^*_n M^T_n (M_n P^*_n M^T_n + Q_n)^{-1} M_n P^*_n + W_n \right] \phi^T_{n+1,n}$$

The number of matrix multiplications can be minimized if we write this as

$$P^*_{n+1} = \phi_{n+1,n} \left[P^*_n - (M_n P^*_n)^T \left\{ (M_n P^*_n) M^T_n + Q_n \right\}^{-1} (M_n P^*_n) + W_n \right] \phi^T_{n+1,n}$$

This one equation provides a means for iteratively finding P^*_n if $\phi_{n+1,n}$, M_n, Q_n, W_n, and P^*_o are known.

Let us next find an expression for $\phi_{n+1,n}$. Initially we shall restrict ourselves to the case in which there are no biases, drift rates or scale factors to be determined. We shall assume that the time τ between t_n and t_{n+1} is independent of n, and that the acceleration due to thrust is constant within each interval. We denote the components

of thrust acceleration within the interval t_n to t_{n+1} by a_{xn}, a_{yn}, a_{zn}. Then for any trajectory near the actual trajectory

$$
\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \\ \tilde{z}_{n+1} \\ \dot{\tilde{x}}_{n+1} \\ \dot{\tilde{y}}_{n+1} \\ \dot{\tilde{z}}_{n+1} \\ \ddot{\tilde{x}}_{n+1} \\ \ddot{\tilde{y}}_{n+1} \\ \ddot{\tilde{z}}_{n+1} \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & \tau & 0 & 0 & \frac{\tau^2}{2} & 0 & 0 \\
0 & 1 & 0 & 0 & \tau & 0 & 0 & \frac{\tau^2}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & \tau & 0 & 0 & \frac{\tau^2}{2} \\
0 & 0 & 0 & 1 & 0 & 0 & \tau & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \tau & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tau \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \\ \tilde{z}_n \\ \dot{\tilde{x}}_n \\ \dot{\tilde{y}}_n \\ \dot{\tilde{z}}_n \\ \ddot{\tilde{x}}_n \\ \ddot{\tilde{y}}_n \\ \ddot{\tilde{z}}_n \end{bmatrix}
= \phi_{n+1,n}
\begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \\ \tilde{z}_n \\ \dot{\tilde{x}}_n \\ \dot{\tilde{y}}_n \\ \dot{\tilde{z}}_n \\ \ddot{\tilde{x}}_n \\ \ddot{\tilde{y}}_n \\ \ddot{\tilde{z}}_n \end{bmatrix}
$$

where $\tilde{x}_n$, $\tilde{y}_n$, $\tilde{z}_n$ are displacements from the reference trajectory, and ignoring any difference in gravity between points on the reference trajectory and the neighboring trajectory. Let us determine the effects of these gravity differences separately since we intend to show that they may be neglected.

We let $\bar{r}$ be the vector from the center of the attracting body to the vehicle. The gravitational acceleration is $-\mu\bar{r}/r^3$. For a vehicle at a slightly different position $\bar{r} + \Delta\bar{r}$, the difference in gravitational acceleration is

$$\Delta \bar{a}_g = - \frac{\mu(\bar{r} + \Delta\bar{r})}{(r + \Delta r)^3} + \frac{\mu \bar{r}}{r^3}$$

$$= - \frac{\mu}{r^3} (\Delta\bar{r} - 3 \Delta r \hat{r}) + \text{higher order terms}$$

in which $\hat{r}$ is a unit vector in the direction of $\bar{r}$.

Then,

$$|\Delta \bar{a}_g| = \frac{\mu}{r^3} \left[(2 \Delta r)^2 + (\Delta r_T)^2 + (\Delta r_p)^2 \right]^{1/2}$$

$$\leq \frac{2\mu}{r^3} |\Delta\bar{r}|$$

in which Δr, Δr_T and Δr_p are the three orthogonal components of $\Delta\bar{r}$. For a LEM in the vicinity of the Moon, r is of the order of 1.7×10^6 meters and μ is 4.896×10^{12} meters^3/sec.^{2} Assuming that the magnitude of $\Delta\bar{r}$ is less than 500 meters, we find that $|\Delta\bar{a}_g|$ is less than 2×10^{-3} meters/sec.^{2} The ascent burn time is about 140 seconds. In this time the difference in gravitational acceleration due to a constant position difference of 500 meters results in a velocity error of 0.15 meters/sec. and a position error of 5.0 meters. These errors are negligible compared to those that will result from thrust magnitude and direction uncertainties and hence we are justified in neglecting them in the analysis of LEM ascent. A similar analysis must be performed to justify this assumption in other operational situations, but we shall leave that for other Apollo Notes.

In the case of a LEM ascent observed from the Earth the geometry of the Earth-Moon-vehicle system does not change greatly during the 140 seconds of boost, so in the error analysis we can use a fixed matrix M instead of M_n. It has already been shown that for fixed measurement interval $\phi_{n+1,n}$ is independent of n, so we

may write it simply as ϕ. Also, during LEM ascent boost Q_n is independent of n, and we assume W_n is independent of n. The iterative equation for the error covariance matrix then becomes

$$P^*_{n+1} = \phi \left[P^*_n - (MP^*_n)^T \left\{ (MP^*_n) M^T + Q \right\}^{-1} (MP^*_n) + W \right] \phi$$

and may easily be evaluated using a digital computer.

If there are additional parameters in the state vector, such as IMU orientation, range biases; and so forth, then the ϕ and M matrices must be augmented, but this is not difficult.

It should be borne in mind that this note indicates only how we can find out how well the error covariance matrix can be determined during boost. It provides no clues as to how to actually determine the trajectory parameters. That is for a later note.

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APOLLO NOTE NO. 178

G. F. Floyd
10 March 1964

POWERED FLIGHT TRACKING

1. RELATIVE IMPORTANCE OF POSITION, VELOCITY AND ACCELERATION

The problem of radio guiding the LEM during the lift off from the Moon is the same as that of the radio-guided Atlas in the sense we are adjusting the position and velocity of LEM so that after LEM cut-off, it will be on an impact trajectory with CSM. Computationally this is done continuously towards the end of the boost by computing the position miss which will occur assuming the boost is stopped now. The vector miss equation involved is of the form,

$$\overline{M}(T,t) = \left[T_p(T,t)\right] \cdot \overline{R}(t) + \left[T_V(T,t)\right] \cdot V(t) \qquad (1)$$

where the bracketed expressions are transformation matrices which relate components of present position and velocity to conditions at rendezvous. Terms in the position transformation are non-dimensional while those in the velocity transformation have the units of time. Considering the leading terms, for 180° transfer orbit we see that position magnitudes translate about 1 to 1 while for the most sensitive velocity component, (the magnitude at injection) the scale factor for the rendezvous is about 4150 sec. That is, 72 n. mi. altitude change for a 106 ft/sec. speed change. Thus, from a miss point of view we have,

$$\Delta M = \Delta R_f + t_{coast}\ \Delta V_f, \qquad (2)$$

with $t_{coast} \approx 4150$ sec.

Now the LEM starts off from the lunar surface where it has been sitting for some period of time. Therefore, the MSFN has had

plenty of time to determine the position (hence the velocity of the landing site due to Moon spin), and so if the MSFN can determine the velocity of the LEM during ascent every (t_s) sec. with a random error (dv) then the position error is a random walk of magnitude ((dv) t_s) and this will be repeated (t_b/t_s) times where t_b is the burn time. Thus,

$$\Delta R_f = (dv\ t_s)\sqrt{\frac{t_b}{t_s}} = (dv)\ \sqrt{t_b\ t_s} \tag{3}$$

so if (dv) is of the same size as Δv_f we get

$$\Delta M = \left[\sqrt{t_b\ t_s} + 4150 \text{ sec}\right] \Delta v_f \tag{4}$$

Now for the LEM $t_b \approx 135$ sec., so that with 1 sec. samples $\sqrt{t_b\ t_s} \approx 12$ sec. which is completely negligible in comparison with the velocity scale factor of 4150 sec.

Since the coast trajectory is completely determined by the position and velocity at cut-off, acceleration of the vehicle at cut-off doesn't effect the miss at all, so like position, is unimportant.

Now for a 180° transfer, the azimuth component of velocity doesn't affect the miss at all; we get to the same point at the same time in half an orbit regardless of inclination angle. However, any azimuth error at injection must be corrected by an equal component of plane change in addition to circularization at rendezvous. Thus to keep the required boost velocity within the available fuel pad we must keep the azimuth velocity error small. Then since the magnitude and initial altitude rate directly affects the miss, we see that all three components of velocity are important.

Finally, because of fuel pad limitations we must keep the velocity near the programmed velocity schedule throughout the entire boost phase. That is, if we boost in the wrong direction early, we don't have the fuel to cancel the error and then add the right value. Thus we must keep the velocity nearly correct throughout

the burn phase and the Moon's position will stay near the planned on value and so the actual Moon's gravitational acceleration will stay very near to the planned on value and for error studies the gravitational differences can be neglected.

Thus, we conclude that during the ascent phase, we simply want the MSFN to keep accurate track of the LEM velocity vector.

2. VELOCITY DETERMINATION WITH UNKNOWN RANDOM BOOSTS

In this section we consider the following operational situation. The LEM starts from rest from a known position on the lunar surface hence its initial velocity (due to the Moon's slow spin), is also known. On the ground we assume a three station MSFN system operating in the doppler mode so that we can make a sequence of measurements of the vector velocity of the spacecraft. These samples of measured velocity are in error and we assume the sample errors to be uncorrelated with a normal distribution and zero mean.

In addition to the velocity measurements, the MSFN will be assumed to have data on the velocity changes which are applied to the spacecraft between the times the velocity measurements are made. In the following analysis the MSFN knowledge of these changes will be assumed to be in error with these errors uncorrelated, normally distributed and of zero mean. This assumption then corresponds to either of the following operational situations.

a. LEM under manual control of the astronauts who are trying to follow a boost program which has been or is being sent to them from the ground. There is no operational IMU or strapped-down inertial system on board or if there is, the outputs are not being telemetered to the ground.

b. LEM under either manual or automatic control with a "noisy" inertial system which is making random errors in its measurement of the velocity changes which are actually applied.

Note that under this assumption as far as the error which the MSFN makes in estimating the velocity of the LEM, it makes no difference whether the velocity changes actually have random errors or actually have no errors but with the inertial system telemetering unknown random errors which are then used by the ground system.

We now consider each of the three components of the LEM velocity separately and calculate the most likely value of the component (V_N) after a set of (N) measurements have been made. The ground has two sets of data to use for estimating V_N. First the set of measured values ($V_{m1}, V_{m2}, \ldots V_{mN}$)

where

$$V_{mi} = V_i + e_{Vmi} \tag{5}$$

and secondly has the set of measured or assumed changes ($\Delta V_{a1}, \Delta V_{a2}, \Delta V_{aN}$)

where

$$V_{ai} = V_i - V_{i-1} + e_{\Delta V_{ai}} \tag{6}$$

with $e_{V_{mi}}$ and $e_{\Delta V_{ai}}$ normally distributed, uncorrelated (with themselves and each other), zero mean, and variance σ_V^2 and $\sigma_{\Delta V}^2$ respectively.

Now given the set of (N) measurements and assumed changes we have,

$$\mathcal{L}(V_N, V_{N-1}, \ldots V_1) = \frac{1}{\sigma_V^N \; \sigma_{\Delta V}^N \; (2\pi)^N} \exp - \frac{1}{2} \left[\frac{1}{\sigma_V^2} \sum_{i=1}^{N} (V_i - V_{mi})^2 + \frac{1}{\sigma_{\Delta V}^2} \sum_{i=1}^{N} (V_i - V_{i-1} - \Delta V_{ai})^2 \right]$$

with $V_o = 0$ from the assumed known initial condition.

Denoting the maximum likelihood estimators with a hat ($\wedge$) we then have the set of simultaneous equations after the N measurements,

$$\left.\begin{aligned}
\frac{\partial L}{\partial \hat{V}_N} &= 0 = \frac{1}{\sigma_V^2}(\hat{V}_N - V_{mN}) + \frac{1}{\sigma_{\Delta V}^2}(\hat{V}_N - \hat{V}_{N-1} - \Delta V_{aN}) \\
\frac{\partial L}{\partial \hat{V}_{N-1}} &= 0 = \frac{1}{\sigma_V^2}(\hat{V}_{N-1} - V_{m(N-1)}) - \frac{1}{\sigma_{\Delta V}^2}(\hat{V}_N - \hat{V}_{N-1} - \Delta V_{aN}) \\
&\quad + \frac{1}{\sigma_{\Delta V}^2}(\hat{V}_{N-1} - \hat{V}_{N-2} - \Delta V_{a\,(N-1)})
\end{aligned}\right\} \quad (7)$$

and so forth down to $\hat{V}_1$. The analysis difficulty now is that after each new measurement, we should revise our estimates of the velocity at all of the past points. This difficulty is however easily side-stepped by first considering the best estimate of V_N just after the (N-1) measurement hence <u>before</u> the N th measurement. Then on the basis of the (N-1) measurements (V_m) and the (<u>N</u>) assumed changes (ΔV_a) we have,

$$\left.\begin{aligned}
\mathcal{L}(V_N, V_{n-1}, \ldots V_1) = \frac{1}{\sigma_V^{N-1}\,\sigma_{\Delta V}^{N-1}\,(2\pi)^{N-1}} \exp - \frac{1}{2}\Bigg[\frac{1}{\sigma_V^2}\sum_1^{N-1}(V_i - V_{mi})^2 \\
+ \frac{1}{\sigma_{\Delta V}^2}\sum_1^{N}(V_i - V_{i-1} - \Delta V_{ai})^2 \Bigg]
\end{aligned}\right\} \quad (8)$$

Denoting the maximum likelihood estimate of V_N based on the (N-1) velocity measurements by $\hat{V}_N$ (N-1) we then have

$$\frac{\partial \mathcal{L}}{\partial \hat{V}_N(N-1)} = 0 = \frac{1}{\sigma^2_{\Delta V}} \left(\hat{V}_N(N-1) - \hat{V}_{N-1}(N-1) - \Delta V_{aN}\right)$$

$$\text{so (8)} \quad \hat{V}_N(N-1) = \hat{V}_{N-1}(N-1) + \Delta V_{aN}.$$

Now from the form of the equation (7) we see that the estimators for V_N etc., depend <u>linearly</u> on the measurements and from (8) we see that $\hat{V}_N(N-1)$ depends linearly on $\hat{V}_{N-1}(N-1)$ and ΔV_{aN}. Thus taking expected values, the linearity dependence plus the zero mean assumption means the expected error in $\hat{V}_N(N-1)$ will be zero so it will be unbiased. Next the linearity dependence plus the normal distribution of the measurement errors means that $\hat{V}_N(N-1)$ will be normally distributed, with a variance which is easily calculated from (8) as

$$\begin{aligned}
\sigma^2_{\hat{V}_N(N-1)} &= E\left[\left(\hat{V}_N(N-1) - V_N\right)^2\right] \\
&= E\left[\left(\hat{V}_{N-1}(N-1) + \Delta V_{aN} - V_N\right)^2\right] \\
&= E\left[\left(\hat{V}_{N-1}(N-1) + V_N - V_{N-1} + e_{\Delta V_{aN}} - V_N\right)^2\right] \qquad (9) \\
&= \sigma^2_{\hat{V}_{N-1}(N-1)} + \sigma^2_{\Delta V}
\end{aligned}$$

since ΔV_{aN} is not related to $\hat{V}_{N-1}(N-1)$.

Thus we can write the conditional probability of V_N given $\hat{V}_{N-1}(N-1)$ and ΔV_{aN} as

$$P\left(V_N \,\middle|\, \hat{V}_{N-1}, \Delta V_{aN}\right) = K \exp - \frac{\left(V_N - \hat{V}_{N-1}(N-1) - \Delta V_{aN}\right)^2}{2\left(\sigma^2_{\hat{V}_{N-1}}(N-1) + \sigma^2_{\Delta N}\right)} \qquad (10)$$

Now because of the uncorrelated nature of the measurements and because V_{mN} cannot be used before it is measured, it is not involved in $\hat{V}_{N-1}(N-1)$, the likelihood of $\hat{V}_N$ <u>after</u> the measurement V_{mN} can be written in the form

$$L(V_N) = K \; P(V_N \mid \hat{V}_{N-1}, V_{aN}) \; \exp - \frac{(V_N - V_{mN})^2}{2\,\sigma_V^2} \tag{11}$$

Using (10) and (11) and solving for the maximum likelihood estimate $\hat{V}_N$, we get

$$\frac{\partial L}{\partial \hat{V}_N} = 0 = \frac{(\hat{V}_N - \hat{V}_{N-1}(N-1) - V_{aN})}{\sigma^2_{\hat{V}_{N-1}}(N-1) + \sigma^2_{\Delta V}} + \frac{(\hat{V}_N - V_{mN})}{\sigma_V^2}$$

so after the N-th measurement we have

$$\hat{V}_N = \lambda_N \; V_{mN} + (1 - \lambda_N)\;(\hat{V}_{N-1}(N-1) + \Delta V_{aN}) \tag{12}$$

where

$$\lambda_N \triangleq \frac{\dfrac{1}{\sigma_V^2}}{\dfrac{1}{\sigma_V^2} + \dfrac{1}{\sigma^2_{\hat{V}_{N-1}(N-1)} + \sigma^2_{\Delta V}}} = \frac{1}{1 + \dfrac{\sigma_V^2}{\sigma^2_{\hat{V}_{N-1}(N-1)} + \sigma^2_{\Delta V}}} \tag{13}$$

No confusion will now result if we use the notation

$$\sigma^2_{\hat{V}_{N-1}(N-1)} = \sigma^2_{\hat{V}_{N-1}} \tag{14}$$

and since $\hat{V}_N$ is now a linear combination of terms, each of zero mean error, the expected value of $\hat{V}_N$ will be V_N and so from (12) the variance of $\hat{V}_N$ will be

$$\sigma^2_{\hat{V}_N} = \lambda_N^2 \; \sigma_V^2 + (1-\lambda_N)^2 \, (\sigma^2_{\hat{V}_{N-1}} + \sigma^2_{\Delta V})$$

and using (13) we find after simple arithmetic

$$\frac{1}{\sigma^2_{\hat{V}_N}} = \frac{1}{\sigma_V^2} + \frac{1}{\sigma^2_{\hat{V}_{N-1}} + \sigma^2_{\Delta V}} \tag{15}$$

and since $\sigma^2_{V_0} = 0$,

$$\frac{1}{\sigma^2_{V_1}} = \frac{1}{\sigma_V^2} + \frac{1}{\sigma^2_{\Delta V}}$$

or

$$\boxed{\frac{\sigma_{V_1}}{\sigma_V} = \frac{\sigma_{\Delta V}/\sigma_V}{\sqrt{1 + \left(\sigma_{\Delta V}/\sigma_V\right)^2}}} \tag{16}$$

also letting N become large so

$$\sigma^2_{\hat{V}_N} \longrightarrow \sigma^2_{\hat{V}_{N-1}}$$

from (15) we get for $\sigma_{\hat{V}_\infty}$

$$\boxed{\frac{\sigma_{\hat{V}_\infty}}{\sigma_V} = \frac{\sigma_{\Delta V}}{\sqrt{2}\,\sigma_V}\sqrt{\sqrt{1 + 4\left(\frac{\sigma_V}{\sigma_{\Delta V}}\right)^2} - 1}} \tag{17}$$

Equations 16 and 17 are plotted in Figure 1 versus the ratio $\sigma_{\Delta V} / \sigma_V$.

Now from (12) if we made $\lambda_N = 1$ we would simply be making the estimate $(\tilde{V}_N)$,

$$\tilde{V}_N = V_{mN}$$

with

$$\boxed{\frac{\sigma_{\tilde{V}_N}}{\sigma_V} = 1} \tag{18}$$

Finally if we let

$$V^*_N = \lambda V_{mN} + (1-\lambda)(V_{m(N-1)} + V_{aN})$$

and we choose λ to minimize $E(V^*_N - V_N)$ we find easily

$$\lambda = \frac{1 + \left(\frac{\sigma_{\Delta V}}{\sigma_V}\right)^2}{2 + \left(\frac{\sigma_{\Delta V}}{\sigma_V}\right)^2}$$

and

$$\boxed{\frac{\sigma_{V_N^*}}{\sigma_V} = \sqrt{\frac{1 + \left(\frac{\sigma_{\Delta V}}{\sigma_V}\right)^2}{2 + \left(\frac{\sigma_{\Delta V}}{\sigma_V}\right)^2}}} \tag{19}$$

and these results are also shown in Figure 1.

This result is then interesting since the figure shows that unless $\sigma_{\Delta V}/\sigma_V$ is well less than one, both the old measured data (V_{mi}) as well as the assumed or measured velocity change data (ΔV_{ai}) improve the ground system's knowledge of LEM velocity very little, and the better estimate V_N^*, based on a single old piece of data is very nearly as good as using all the data. Also, the figure shows that even when ($\sigma_{\Delta V}/\sigma_V$) is small, the uncertainty in LEM velocity increases only a small amount with (N), (i.e., as time progresses during the ascent).

3. MSFN VELOCITY MEASUREMENT ACCURACY

First consider the MSFN stations operating in the Doppler mode. Then the measured $\dot{R}_m$ is the scalar range change (ΔR) divided by the sampling time (t_s). So without noise,

$$\dot{R}_m = \frac{\Delta R + e_{\Delta R}(t_s)}{t_s} \tag{20}$$

Now if over a measurement interval R is quadratic in form so

$$R(t) = a + bt + ct^2 \tag{21}$$

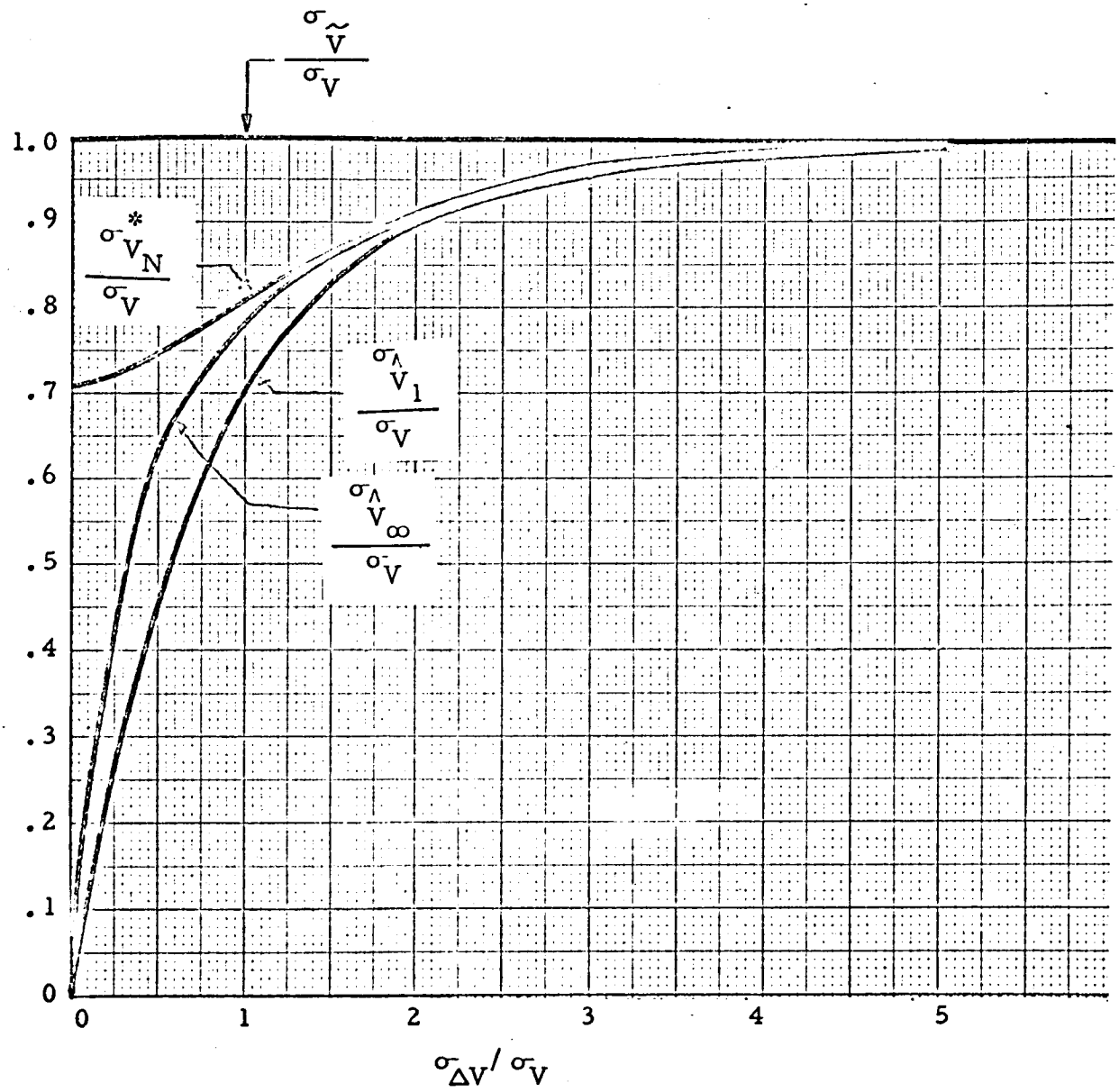

Figure 1.
Standard Deviation of Various Predictors

$$\dot{R}_m = \frac{a + bt + ct^2 + e_{\Delta R} - a}{t} = b + ct + \frac{e_{\Delta R}}{t_s} \tag{22}$$

and from (21)

$$\dot{R}\,(t/2) = b + 2\,c\,(t/2) = b + ct \tag{23}$$

Thus if R is varying quadratically the measured doppler value ($\dot{R}_m$) is related to the instantaneous range at t/2 as,

$$\dot{R}_m = \dot{R}\,(t/2) + \frac{e_{\Delta R}}{t_s} \tag{24}$$

Now due to the small angle (θ) between the line-of-sights from the different MSFN stations to the spacecraft, the errors in the components of LEM velocity in the plane normal to the Earth-Moon line get amplified by the factor ($1/\theta$).

Thus if the landing site is in the center of the Moon and the MSFN stations are separated by 5000 n.mi. (83°) so

$$(1/\theta) = \frac{250,000}{8,000 \sin (83/2)} = 48,$$ then the measured spacecraft altitude rate will have a σ equal to the σ of measured doppler while the other two components will have σ's which are the measured doppler σ multiplied by 48.

Using a rubidium frequency standard and a 50 sec. sampling time, JPL on the Mariner II flight found 17 cm as the maximum value for the random value of the 50 sec. measured doppler range change. If the source of this final error was uncorrelated at 1 sec. sampling times, this would indicate that for 1 sec. sample times,

$$\sigma_{\dot{R}} = \frac{17}{\sqrt{50}} = 2.4 \text{ cm/sec.} \tag{25}$$

so with the "wind up" factor of 48 for two components,

$$\sigma_V = 115,\ 115,\ 2.4\ (\text{cm/sec.})$$
$$= 3.8,\ 3.8,\ 01\ \text{ft/sec.} \tag{26}$$

with the smallest error in altitude rate.

Now assume that near the end of ascent the LEM has an acceleration of 20 ft/sec.^{2}, and the astronaut does 6° steering. Then for 1 sec. samples,

$$\sigma_{\Delta V} = (20)\ (1)\ \left(\frac{6}{57}\right) = 2\ \text{ft/sec.} \tag{27}$$

Thus

$$\frac{\sigma_{\Delta V}}{\sigma_V} \approx .5,\ .5,\ 20 \tag{28}$$

and from Figure 1, the use of the old data would reduce the σ of raw measurement velocity by only a factor of .65 in two components and not at all for altitude rate yielding:

$$\sigma_{\hat{V}} = 2.5,\ 2.5,\ .1\ \text{ft/sec.} \tag{29}$$

and so the old data and assumed velocity changes do not improve the accuracy very much.

A more important problem is due to the transit time delays which makes the MSFN orders about 3 seconds behind and this requires extrapolation through the assumed (ΔV) changes. This problem will be treated in a later note.

It appears that even with station location, speed-of-light, instrumentation errors and assumed biases, and transmission delays, a 3 station MSFN operating in the precision doppler mode may be able to track the LEM velocity to 10 ft/sec. during the ascent phase.

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APOLLO NOTE NO. 179

C. Siska
27 March 1964

ANALYSIS FOR A ONE BOOST MIDCOURSE CORRECTION TO THE RETURN TO EARTH TRAJECTORY

Introduction

As the return vehicle leaves the lunar sphere of influence in returning to Earth, it will generally be on a trajectory which misses the re-entry dive angle "corridor" as a result of transearth injection errors.

The present report develops equations regarding the questions of how much boost velocity is required in making a single midcourse correction to nullify the initial dive angle error, and how accurately the eventual re-entry conditions are established. Subsequent studies will examine the more complicated situation of multiple boost corrections.

Discussion of Orbits Involved

In order to define the various orbits which must be considered and some of the symbology used, a brief discussion of the correction process follows with the aid of the schematic diagram in Figure 1.

Generally three different orbits are involved, all of which are fairly close to one another. First is the reference orbit which represents the desired objective for the return mission. Properties of this orbit will be labeled with a subscript zero. The second orbit (subscript A), is the one which the vehicle is actually traveling and this orbit is assumed to contain an unacceptable dive angle error which is to be removed via a boost correction. The third orbit is the one which is predicted from observational data, and properties of this orbit will be signified by circumflexed parameters (e.g., $\hat{\delta}_F$).

When a boost correction is applied, the latter two orbits will be changed and therefore additional subscripts 1 and 2 are used to

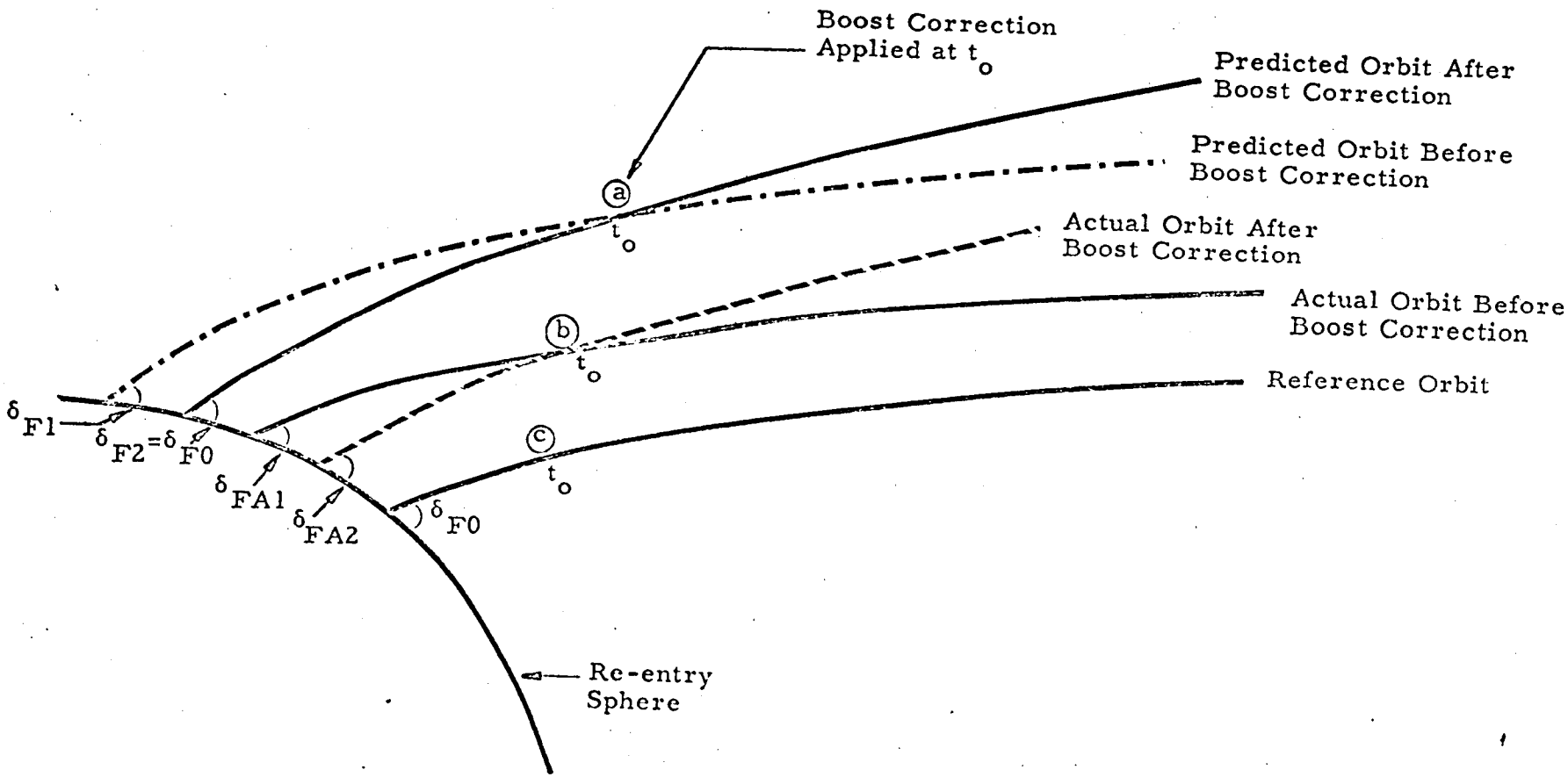

Figure 1. Schematic Diagram of Return-to-Earth Orbit Situation

signify the orbits before and after correction.

At some time t_o (position (a) along the predicted orbit), a boost correction is computed so as to nullify the predicted dive angle error $\hat{\delta}_{F1} - \delta_{F0}$, and therefore the predicted orbit after boost will have an estimated dive angle $\hat{\delta}_{F2} = \delta_{F0}$. In actuality the executed boost at t_o is applied at position (b) on the actual orbit and results in changing the re-entry dive angle δ_{FA1} to δ_{FA2} with an error $\delta_{FA2} - \delta_{F0}$.

Using only a single boost correction, the standard deviation of the value $(\delta_{FA2} - \delta_{F0})$ must be of such magnitude that the safe re-entry criterion is satisfied. For our purposes this implies a re-entry dive angle $\sigma = 0.1^{\circ}$ associated with a zero lift atmospheric trajectory to landing.

Corrective Boost Components

As observational data is gathered on the actual return orbit, a predicted orbit can be determined. At any given time, the predicted re-entry dive angle $\hat{\sigma}_{F1}$ can be obtained from the predicted orbit parameters, and generally, the greater the number of data the closer the value $\hat{\delta}_{F1}$ approaches the actual value δ_{FA1}. The desired objective is to have an orbit with a re-entry dive angle δ_{F0}, the same as the reference orbit. To accomplish this end, a single boost correction is made, based on the predicted orbit parameters, which changes the value $\hat{\delta}_{F1}$ to δ_{F0}.

Using a local x, y, z, earth-centered coordinate system with x along the local vertical, y perpendicular to x and in the direction of motion, z out of plane, and corresponding velocity components $\dot{x}$, $\dot{y}$, $\dot{z}$, the required condition (linearized) can be written,

$$\left(\frac{\partial \hat{\delta}_F}{\partial \dot{x}}\right)_1 \Delta\hat{\dot{x}} + \left(\frac{\partial \hat{\delta}_F}{\partial \dot{y}}\right)_1 \Delta\hat{\dot{y}} + (\hat{\delta}_{F1} - \delta_{F0}) = 0 \qquad (1)$$

Here $\widehat{\Delta\dot{x}}$ and $\widehat{\Delta\dot{y}}$ are the estimated boost components and we wish to select these so as to minimize the resultant boost velocity $\widehat{\Delta V}$. Equation (1) is a straight line in the $\widehat{\Delta\dot{x}}$, $\widehat{\Delta\dot{y}}$ plane and therefore the minimum $\widehat{\Delta V}$ is a vector normal to this line from the origin. Labeling the components $\widehat{\Delta\dot{x}}$, $\widehat{\Delta\dot{y}}$ of the minimum boost as $\hat{u}$, $\hat{v}$, we have the following relations,

$$\left.\begin{aligned}
\widehat{\Delta V}^2 &= \hat{u}^2 + \hat{v}^2 = (\widehat{\delta_{F1}} - \delta_{F0})^2 \left[\left(\frac{\partial \widehat{\delta_F}}{\partial \dot{x}}\right)_1^2 + \left(\frac{\partial \widehat{\delta_F}}{\partial \dot{y}}\right)_1^2\right]^{-1} \\
\hat{u} &= -(\widehat{\delta_{F1}} - \delta_{F0}) \left(\frac{\partial \widehat{\delta_F}}{\partial \dot{x}}\right)_1 \left[\left(\frac{\partial \widehat{\delta_F}}{\partial \dot{x}}\right)_1^2 + \left(\frac{\partial \widehat{\delta_F}}{\partial \dot{y}}\right)_1^2\right]^{-1} \\
\hat{v} &= -(\widehat{\delta_{F1}} - \delta_{F0}) \left(\frac{\partial \widehat{\delta_F}}{\partial \dot{y}}\right)_1 \left[\left(\frac{\partial \widehat{\delta_F}}{\partial \dot{x}}\right)_1^2 + \left(\frac{\partial \widehat{\delta_F}}{\partial \dot{y}}\right)_1^2\right]^{-1}
\end{aligned}\right\} \quad (2)$$

To change the orbit plane requires a boost component of velocity in the z direction--this consideration is neglected in this analysis.

Boost Execution Errors

The consideration of boost execution errors will involve the following simplifying assumptions,

a. The boost velocity error V_e, with components u_e, v_e, w_e will be pointed equally likely in all directions.

b. The variance in V_e will be proportional to the variance in the delivered velocity ΔV_D, i.e., $\overline{V_e^2} = k^2 \overline{\Delta V_D^2}$

As a consequence of (a), the velocity error components satisfy the following relations,

$$\overline{u_e^2} = \overline{v_e^2} = \overline{w_e^2} = \frac{1}{3}\,\overline{V_e^2}$$

$$\overline{u}_e = \overline{v}_e = \overline{w}_e = \overline{u_e v_e} = \overline{u_e w_e} = \overline{v_e w_e} = 0 \tag{3}$$

The delivered velocity in terms of components associated with the predicted orbit can be written,

$$\Delta V_D^2 = (\hat{u} + u_e)^2 + (\hat{v} = v_e)^2 + w_e^2 \tag{4}$$

from which, using Equation (3) and assumption (b) above, we obtain

$$\overline{V_e^2} = \frac{k^2}{1-k^2}\,\overline{\Delta\hat{V}^2}$$

$$\overline{u_e^2} = \overline{v_e^2} = \overline{w_e^2} = \frac{1}{3}\left(\frac{k^2}{1-k^2}\right)\overline{\Delta\hat{V}^2} \tag{5}$$

Re-entry Dive Angle Error

Writing the delivered boost components in the local coordinate system of the actual orbit as u_{A1}, v_{A1}, w_{A1}, the resulting re-entry dive angle is

$$\delta_{FA2} = \delta_{FA1} + \left(\frac{\partial\delta_F}{\partial\dot{x}}\right)_{A1} u_{A1} + \left(\frac{\partial\delta_F}{\partial\dot{y}}\right)_{A1} v_{A1} \tag{6}$$

The difference between u_{A1}, v_{A1} and $\hat{u} + u_e$, $\hat{v} + v_e$ results from the difference in the local coordinate systems associated with positions on the predicted and actual orbits. This difference is a second order quantity and can therefore be neglected in a linearized analysis, so that Equation (6) can be written,

$$\left.\begin{aligned}(\delta_{FA2} - \delta_{F0}) = (\delta_{FA1} - \delta_{F0}) &+ \left[\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)_{A1} \hat{u} + \left(\frac{\partial \delta_F}{\partial \dot{y}}\right)_{A1} \hat{v}\right] \\ &+ \left[\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)_{A1} u_e + \left(\frac{\partial \delta_F}{\partial \dot{y}}\right)_{A1} v_e\right]\end{aligned}\right\} \quad (7)$$

In view of Equation (2) and since $\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)_{A1} \hat{u} \approx \left(\frac{\partial \hat{\delta}_F}{\partial \dot{x}}\right)_1 \hat{u}$ to first order, the second term on the right of Equation (6) is just $-(\hat{\delta}_{F1} - \delta_{F0})$. Therefore, Equation (7) becomes,

$$(\delta_{FA2} - \delta_{F0}) = (\delta_{FA1} - \hat{\delta}_{F1}) + \left[\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)_{A1} u_e + \left(\frac{\partial \delta_F}{\partial \dot{y}}\right)_{A1} v_e\right] \quad (8)$$

Equation (8) shows that the final dive angle error due to the error in predicting the actual orbit and the error in boost execution. Since these errors are independent and in view of Equations (3) and (5), the variance in the error is

$$\overline{(\delta_{FA} - \delta_{F0})^2} = \overline{(\delta_{FA1} - \hat{\delta}_{F1})^2} + \frac{1}{3}\left(\frac{k^2}{1-k^2}\right)\left[\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)^2_{A1} + \left(\frac{\partial \delta_F}{\partial \dot{y}}\right)^2_{A1}\right] \overline{\hat{\Delta V}^2} \quad (9)$$

Replacing $\left(\frac{\partial \hat{\delta}_F}{\partial \dot{x}}\right)_1$ with $\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)_{A1}$ etc., in Equation (2), according to preceding arguments, the variance $\overline{\hat{\Delta V}^2}$ becomes

$$\overline{\hat{\Delta V}^2} = \left[\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)^2_{A1} + \left(\frac{\partial \delta_F}{\partial \dot{y}}\right)^2_{A1}\right] \overline{(\hat{\delta}_{F1} - \delta_{F0})^2} \quad (10)$$

However, the variance $\overline{(\hat{\delta}_{F1} - \delta_{F0})^2}$ can be written

$$\overline{(\hat{\delta}_{F1} - \delta_{F0})^2} = \overline{\left[(\hat{\delta}_{F1} - \delta_{FA1}) + (\delta_{FA1} - \delta_{F0})\right]^2} = \overline{(\hat{\delta}_{F1} - \delta_{FA1})^2} + (\delta_{FA1} - \delta_{F0})^2 \tag{11}$$

since $(\delta_{FA1} - \delta_{F0})$ is a constant and the mean value of $(\hat{\delta}_{F1} - \delta_{FA1})$ equals zero.

Combining Equations (9), (10), and (11) yields the final form,

$$\overline{\delta_{FA2} - \delta_{F0})^2} = \left[1 + \frac{1}{3}\frac{k^2}{1-k^2}\right]\overline{(\hat{\delta}_{F1} - \delta_{FA1})^2} + \frac{1}{3}\left(\frac{k^2}{1-k^2}\right)(\delta_{FA1} - \delta_{F0})^2 \tag{12}$$

Since the reference and actual orbits are not too dissimilar, the error $(\hat{\delta}_{F1} - \delta_{FA1})$ in predicting the actual orbit is equivalent to the error in predicting the reference orbit. Consequently only the second term on the right of Equation (12) can be considered to vary if the actual orbit (initial error) is changed for a given observation data sample.

Using Equations (3), (4), (5), (10), and (11), the variance in the delivered boost velocity becomes,

$$\overline{\Delta V_D^2} = \left(\frac{1}{1-k^2}\right)\left[\left(\frac{\partial \delta_F}{\partial \dot{x}}\right)_{A1}^2 + \left(\frac{\partial \delta_F}{\partial \dot{y}}\right)_{A1}^2\right]^{-1}\left[\overline{(\hat{\delta}_{F1} - \delta_{FA1})^2} + (\delta_{FA1} - \delta_{F0})^2\right] \tag{13}$$

Here the above argument can apply also to the partial derivatives, i.e., for a given time along the reference orbit, assume the partials for the actual orbit at the same instant of time are equivalent to those of the reference orbit.

Miss Distance in Reference Plane (Miss Along)

The re-entry miss distance in the reference plane after the boost correction is

$$M_{A2} = M_{A1} + \left[\left(\frac{\partial M_A}{\partial \dot{x}} \right)_{A1} u_{A1} + \left(\frac{\partial M_A}{\partial \dot{y}} \right)_{A1} v_{A1} \right] \tag{14}$$

where M_{A1} is the initial miss along associated with the actual return orbit.

In a vein similar to that leading to Equation (8), we obtain

$$M_{A2} = M_{A1} + F(\hat{\delta}_{F1} - \delta_{F0}) + \left[\left(\frac{\partial M_A}{\partial \dot{x}} \right)_{A1} u_e + \left(\frac{\partial M_A}{\partial \dot{y}} \right)_{A1} v_e \right] \tag{15}$$

where

$$F = \left[\left(\frac{\partial M_A}{\partial \dot{x}} \right) \left(\frac{\partial \delta_F}{\partial \dot{x}} \right) + \left(\frac{\partial M_A}{\partial \dot{y}} \right) \left(\frac{\partial \delta_F}{\partial \dot{y}} \right) \right]_{A1} \left[\left(\frac{\partial \delta_F}{\partial \dot{x}} \right)^2 + \left(\frac{\partial \delta_F}{\partial \dot{y}} \right)^2 \right]_{A1}^{-1} \tag{16}$$

The variance for Equation (15) can be written as follows,

$$\overline{M_{A2}^2} = \left[M_{A1} + F(\delta_{FA1} - \delta_{F0}) \right]^2 + (F^2 + G) \overline{(\hat{\delta}_{F1} - \delta_{FA1})^2} + G(\delta_{FA1} - \delta_{F0})^2 \tag{17}$$

where

$$G = \frac{1}{3} \left(\frac{k^2}{1-k^2} \right) \left[\left(\frac{\partial M_A}{\partial \dot{x}} \right)^2 + \left(\frac{\partial M_A}{\partial \dot{y}} \right)^2 \right]_{A1} \left[\left(\frac{\partial \delta_F}{\partial \dot{x}} \right)^2 + \left(\frac{\partial \delta_F}{\partial \dot{y}} \right)^2 \right]_{A1}^{-1} \tag{18}$$

In general, the class of orbits typified by the initial error $(\delta_{FA1} - \delta_{F0})$ can generate a variety of miss along errors M_{A1}.

Miss Distance Normal to Reference Plane (Miss Across)

Here the miss distance after boost correction can be written

$$M_{C2} = M_{C1} + \left(\frac{\partial M_C}{\partial \dot{z}} \right)_{A1} w_{A1} \tag{19}$$

Since consideration of a plane change by a boost correction has been ignored, we have $w_{A1} = w_e$.

Continuing as before, we obtain the variance

$$\overline{M_{C2}^2} = M_{C1}^2 + H \left[\overline{(\hat{\delta}_{F1} - \delta_{FA1})^2} + (\delta_{FA1} - \delta_{F0})^2 \right] \tag{20}$$

where

$$H = \frac{1}{3} \left(\frac{k^2}{1-k^2} \right) \left(\frac{\partial M_C}{\partial \dot{z}} \right)_{A1}^2 \left[\left(\frac{\partial \delta_F}{\partial \dot{x}} \right)^2 + \left(\frac{\partial \delta_F}{\partial \dot{y}} \right)^2 \right]_{A1}^{-1} \tag{21}$$

Evaluation Procedure

The immediate objective is to determine an upper limit to the boost velocity requirement for a single midcourse correction which permits a safe re-entry.

Since the mean value of the delivered boost $\overline{\Delta V_D}$ is equal to $\overline{\hat{\Delta V}}$, we obtain from Equation (2),

$$\overline{\Delta V_D} = (\delta_{FA1} - \delta_{F0}) \left[\left(\frac{\partial \delta_F}{\partial \dot{x}} \right)^2 + \left(\frac{\partial \delta_F}{\partial \dot{y}} \right)^2 \right]_{A1}^{-1} \tag{22}$$

Defining an upper limit to the boost velocity as the sum of the mean value plus n standard deviations, we can write

$$\Delta V_{max.} = \overline{\Delta V_D} + n \sqrt{\overline{\Delta V_D^2}} \tag{23}$$

where ΔV_{max} can be evaluated by means of Equations (13) and (22).

Adopting a zero lift dive angle re-entry "corridor" requires that the error $(\delta_{FA2} - \delta_{F0})$ should have a standard deviation of 0.1° for safe re-entry. For a given boost execution parameter k, Equation (12) then relates the required dive angle prediction accuracy $(\hat{\delta}_{F1} - \delta_{FA1})$ to the initial error $(\delta_{FA1} - \delta_{F0})$.

From Apollo Notes 151 and 159 we can obtain the duration of observation (or time to re-entry) which yields the determined accuracy $(\hat{\delta}_{F1} - \delta_{FA1})$ for a variety of tracking situations. The time to re-entry, or equivalently, a position on the return orbit, enables one to determine the partial derivatives $\frac{\partial \delta_F}{\partial \dot{x}}$ etc., so that Equation (23) can then be evaluated for a given n. Generally the value of n will be about 3 in order to ensure a high probability (99% plus) that the computed value of ΔV_{max} is not exceeded.

Concluding Remarks

The foregoing procedure permits a determination of the boost velocity requirement for a given initial dive angle error. Thus ΔV_{max} can be graphed as a function of the initial dive angle error at the LSOI, $(\delta_{FA1} - \delta_{F0})$, for the parameters k (boost execution) and tracking situation. However, the latter value is also a statistical quantity arising from Moon-Earth injection uncertainties. A knowledge of these uncertainties permits one to establish an upper limit to the error $(\delta_{FA1} - \delta_{F0})$, say a 3 sigma value. This in turn, used with the above-mentioned graph will permit a determination of (1), the over-all boost limit for a single correction mode of operation and (2), the manner in which this value is affected by various tracking situations.

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APOLLO NOTE NO. 180

L. S. Lustick
26 March 1964

SCHMIDT-KALMAN METHOD AND POWERED FLIGHT

Purpose

The purpose of this note is to identify the fundamental theorem of the Schmidt-Kalman method with the expression for the conditional means of a joint multivariate normal distribution and to illustrate the manner in which this can be applied to powered flight.

Lemma

Given a joint normal distribution of variables, $a_{i = 1, 2 \ldots n}$ with zero mean (the covariance matrix A_{ij} consistent with this distribution is shown below); the conditional means $C_{a_{i=1, 2 \ldots r}}$ given $a_{i= r+1, r+2 \ldots n}$ can be expressed as

$$C_{a_{i= 1, 2 \ldots r}} = A_{12} A_{22}^{-1} \begin{bmatrix} a_{r+1} \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} .$$

Let matrix A be a covariance matrix defining a joint normal distribution between the variables, $a_{i=1, 2 \ldots n}$ (the variables have zero mean)

$$A_{ij} = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} j=r \\ \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array} \begin{array}{c} \\ i=r \end{array}$$

$$A_{ij}^{-1} = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} j=r \\ \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right] \end{array} i=r$$

The conditional means of the variables $a_{i=1,2\ldots r}$ given the variables $a_{r+1,\, r+2\ldots\ldots n}$ can be obtained as shown below.

$$C_{a_{i=1,2\ldots r}} = - \left[C_{11}^{-1} \times C_{12} \times \begin{bmatrix} a_{r+1} \\ a_{r+2} \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \right]^{*} \qquad (1)$$

where C_{a_i} is a column vector defining the conditional means of the distribution $P(a_1, a_2 \ldots a_r / a_{r+1} \ldots a_n)$.

This can be put into a form considered to be the fundamental theorem of the Schmidt-Kalman method.

The element of the inverse of matrix A can be obtained in the partitioned form as indicated on the following page.

This effect is interesting in its own right as it indicates how to find the effect of estimating nuisance parameters on the errors in the desired parameters without inverting large order matrices.

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* Alexander M. Mood, Introduction to the Theory of Statistics, p. 183.

$$C_{11} = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \qquad C_{21} = -A_{22}^{-1} A_{21} C_{11}$$ *

$$C_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} \qquad C_{12} = -A_{11}^{-1} A_{12} C_{22}$$

Substituting the preceding expressions in Equation (1), the following is obtained

$$C_{a_{i=1,2\ldots r}} = \left[A_{11} - A_{12} A_{22}^{-1} A_{21}\right]\left[A_{11}^{-1} A_{12}\right]\left[(A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}\right]\begin{bmatrix} a_{r+1} \\ \vdots \\ a_n \end{bmatrix}$$

$$C_{a_{i=1,2\ldots r}} = \left[I A_{12} - A_{12} A_{22}^{-1} A_{21} A_{11}^{-1} A_{12}\right]\left[(A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}\right]\begin{bmatrix} a_{r+1} \\ \vdots \\ a_n \end{bmatrix}$$

$$I A_{12} = A_{12} I = A_{12} A_{22}^{-1} A_{22}$$

$$C_{a_{i=1,2\ldots r}} = \left[A_{12} A_{22}^{-1} A_{22} - A_{12} A_{22}^{-1} A_{21} A_{11}^{-1} A_{12}\right]\left[(A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}\right]\begin{bmatrix} a_{r+1} \\ \vdots \\ a_n \end{bmatrix}$$

$$\boxed{C_{a_{i=1,2\ldots r}} = A_{12} A_{22}^{-1} \begin{bmatrix} a_{r+1} \\ a_{r+2} \\ \vdots \\ a_n \end{bmatrix}} \qquad (2)$$

In order to see the significance of the conditional mean, consider the bivariate normal distribution pictured in Figure 1. The cross-hatched

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* Granino A. Korn, Theresa M. Korn, Mathematical Handbook for Scientists and Engineers, p. 640.

area is the probability density function for the variable x given that the variable $y = y_o$.

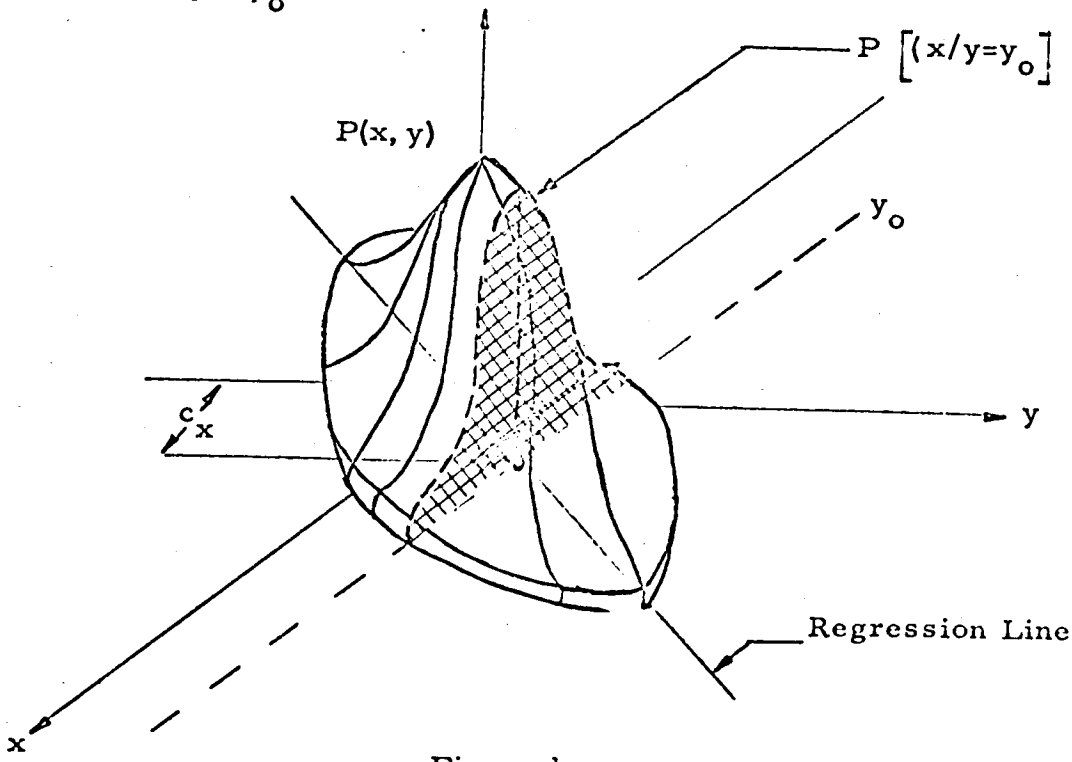

Figure 1.

The mean of this distribution, C_x, is termed the conditional mean, that is the mean of x on the condition $y = y_o$. The conditional mean is obviously located at the value of x where $P(x/y=y_o)$ is a maximum and the second moment of the distribution $P(x/y=y_o)$ is obviously a minimum taken about the conditional mean.

Error Equation with Schmidt-Kalman Approach

Let:

$m^a{}_{i=1,2...r}$ = column vector representing the errors in the state vector at time $t = t_m$ based on m pieces of data.

$_{m+1}\hat{a}_{i=1,2\ldots r}$ = column vector representing the errors in the state vector at time $t = t_{m+1}$ based on m pieces of data

$$_{m+1}\hat{a}_{i=1,2\ldots r} = (_m\phi_{m+1})(_m a_{i=1,2\ldots r})$$

$_{m=1}\hat{a}_{i=r+1,\ldots n}$ = column vector representing errors in the predicted values of the measurables prior to the measurements at $t = t_{m+1}$

$$_{m+1}\hat{a}_{i=r+1,r+2\ldots n} = M\,(_m a_{i=1,2\ldots r})$$

$_{m+1}a_{i=1,2\ldots r}$ = column vector representing the errors in the state vector at time $t = t_{m+1}$ based on (m+1) pieces of data (that is subsequent to measurement at $t = t_{m+1}$)

$$(_mP) = \overline{(_m a_{i=1,2\ldots r})\,(_m a_{i=1,2\ldots r})^T}$$

$$(_{m+1}P) = \overline{(_{m+1} a_{i=1,2,\ldots r})\,(_{m+1} a_{i=1,2\ldots r})^T}$$

$_{m+1}y_{i=r+1,\ldots n}$ = $_{m+1}\hat{a}_{i=r+1,\ldots n} - N_{i=r+1,\ldots n}$ =

$$M\,(_m a_{i=1,2\ldots r}) - N_{i=r+1,\ldots n}$$

where:

N_i = noise in the i-th measurable

$(_{m+1}y)$ = error in the difference between the predicted and measured quantity at time $t = t_{m+1}$

Using the above definitions, the elements of the A matrix can be expressed as shown below:

$$A_{11} = \overline{({}_{m+1}\hat{a}_{i=1,2,\ldots r})\,({}_{m+1}\hat{a}_{i=1,2\ldots r})^T} = {}_m\phi_{m+1}\ ({}_mP)\ {}_m\phi^T_{m+1}$$

$$A_{12} = ({}_m\hat{a}_{i=1,2\ldots r})\ (M\ {}_m a_{i=1,2\ldots r})^T = {}_m\phi_{m+1}\ ({}_mP)\ M^T$$

$$A_{22}^{-1} = \left[\ \overline{(y_i)\ (y^T)}\ \right]^{-1} = \left[M\ ({}_mP)\ M^T + \overline{N_i\ N_i^T}\ \right]^{-1}$$

Subsequent to obtaining the measurements at time $t = t_{m+1}$ we now have a posterori the value of y_i, we select as errors in the state vectors the conditional means of the state variables given the y_i. Our new estimate of the state variables after the measurement at $t = t_{m+1}$ is presented below.

(Estimate of state variables after measurements at time $t = t_{m+1}$) = (Estimates of state variables prior to measurements at $t = t_{m+1}$)

$$- \quad A_{12}\ A_{22}^{-1}\ y_i$$

The error in this expression $({}_{m+1}\ a_{i=1,2\ldots r})$ is presented below

$${}_{m+1}\ a_{i=1,2\ldots r} = {}_{m+1}\ \hat{a}_{i=1,2\ldots r} - A_{12}\ A_{22}^{-1}\ y_i$$

$$\left[{}_{m+1}P\right] = \left[({}_{m+1}\ \hat{a}_{i=1,2\ldots r}) - A_{12}\ A_{22}^{-1}\ y_i \right] \left[{}_{m+1}\ \hat{a}_{i=1,2\ldots r} - A_{12}\ A_{22}^{-1}\ y_i \right]^T$$

$${}_{m+1}P = A_{11} - A_{12}\ A_{22}^{-1}\ M\ ({}_mP)\ {}_m\phi^T_{m+1}$$

Hence: the recursion formula for the errors in the state variables is:

$$ {}_{m+1}P = {}_m\phi_{m+1} \left\{ {}_mP - {}_mP\, M^T \left[M({}_mP)\, M^T + \overline{N_i\, N_i^T} \right]^{-1} M\,({}_mP) \right\} {}_m\phi^T_{m+1} $$

Application to Powered Flight

This technique can be employed to determine errors during powered flight. Care must be exercised in defining (A_{11}, A_{12} and A_{22}^{-1}) for this application. The use of this technique is partially described in Apollo Note 177. A brief discussion defining A_{11}, A_{12} and A_{22}^{-1} for a restricted application is discussed below.

Consider a vehicle constrained to move along the coordinate x. The vehicle has a perfect pilot who flies the vehicle according to a predetermined acceleration versus time schedule. There is a random error in the acceleration during any particular time interval. The errors in acceleration during successive time intervals are uncorrelated.

Let

$$ \phi = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} $$

$$ \psi = \begin{pmatrix} t^2/2 \\ t \end{pmatrix} $$

$$ M = \begin{bmatrix} \dfrac{\partial M_1}{\partial x} & \dfrac{\partial M_1}{\partial \dot{x}} \\ \vdots & \vdots \\ \dfrac{\partial M_n}{\partial x} & \dfrac{\partial M_n}{\partial \dot{x}} \end{bmatrix} $$

$$\widehat{M} = \begin{pmatrix} \dfrac{\partial M_1}{\partial x} \\ \vdots \\ \dfrac{\partial M_n}{\partial \dot{x}} \end{pmatrix}$$

$$_mP = E\left[\begin{pmatrix} \delta x \\ \partial \dot{x} \end{pmatrix} (\delta x, \ \delta \dot{x})\right]_m = \text{covariance matrix in state vector}$$

W = variance in the acceleration

NN^T = covariance matrix of noise in measurables

then,

$$A_{11} = \phi \ {}_mP \ \phi^T + \psi \ W \ \psi^T$$

$$A_{12} = \phi \ {}_mP \ M^T + \psi \ W \ \widehat{M}^T$$

$$A_{22}^{-1} = \left[M \ {}_mP \ M^T + \widehat{M} \ W \ \widehat{M}^T + \overline{N N^T} \right]^{-1}$$

The recursion formula for the error in the state variables position and velocity can now easily be obtained as illustrated below.

$$_{m+1}P = A_{11} - A_{12} A_{22}^{-1} \left[M \ {}_mP \ \phi^T + \widehat{M} \ W \ \psi^T \right]$$

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APOLLO NOTE NO. 181

G. F. Floyd
25 March 1964

MSFN STATION LOCATION

At the present time the following stations are planned for the USF system:

DSIF	Goldstone Canberra Madrid
Phase I	Bermuda Carnarvon Guaymas Hawaii Cape Kennedy Corpus Christie Houston
Phase II	Guam Ascension Antigua
	3 ships

A plot of these stations on a globe shows that 11 of the 13 stations are essentially in the plane defined by an E-W launch from Cape Kennedy. Guam and Madrid are 30° out of this plane, and one of these can see the Moon 5° above the horizon when the sublunar point is on this plane except for 20° arc South of Goldstone and 10° arc in the Indian Ocean. Also the 3 DSIF sites can see the Moon at all times along this plane and there will be two transmitters

at each of the DSIF sites, so will be taken as the transmitting stations for the multiple station Doppler mode.

Now only one station can operate at a time in the range mode and one station is no good for short time operations. Also the range mode must allow for an unknown bias and then is no different than the range change mode except for a worse σ. Thus for these two reasons I see no point in further consideration of the ranging mode for short time lunar operations, and suggest we restrict our attention to the Doppler mode with one of the three DSIF sites transmitting and all the other MSFN sites that can see, receiving.

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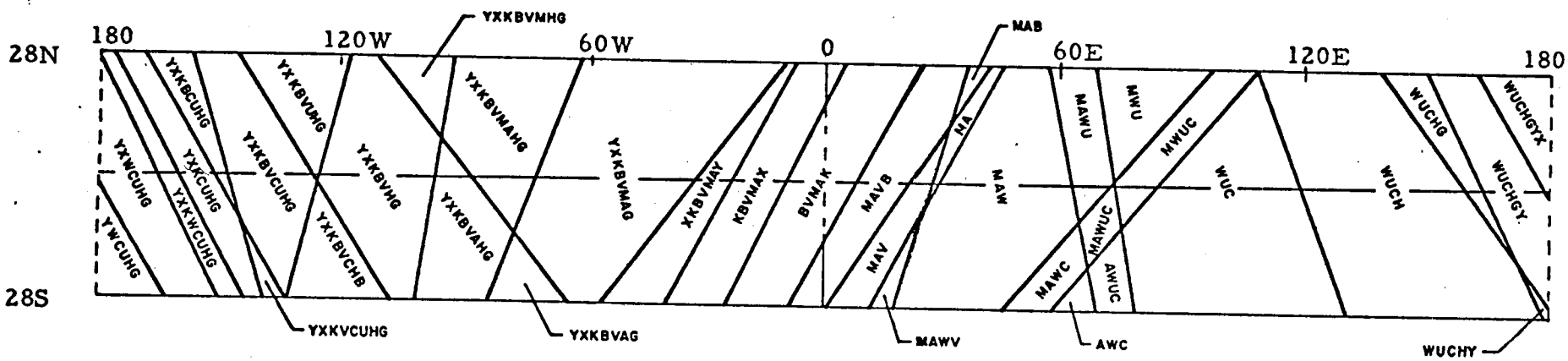
APOLLO NOTE NO. 182

H. Engel,
W. Iwata
30 March 1964

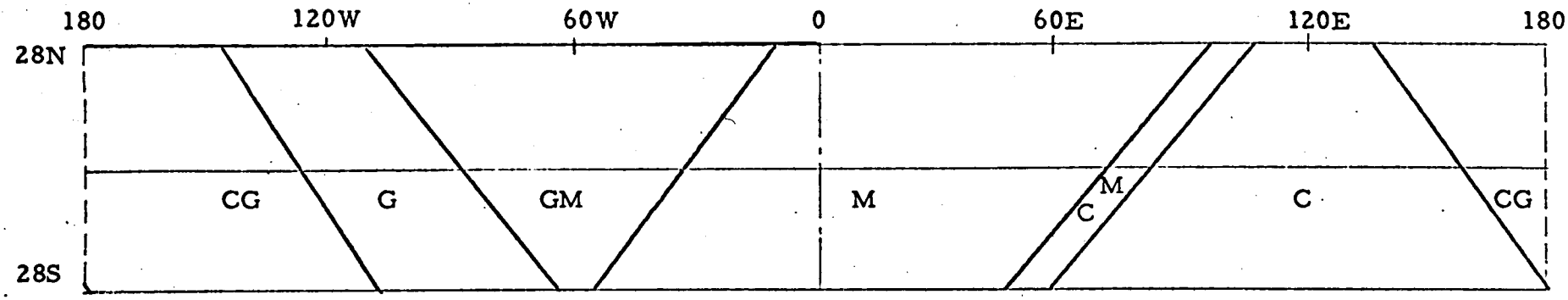
MSFN LUNAR COVERAGE, II

The attached figures show the number of stations from which the Moon is more than 5° above the horizon for any sub-lunar point.

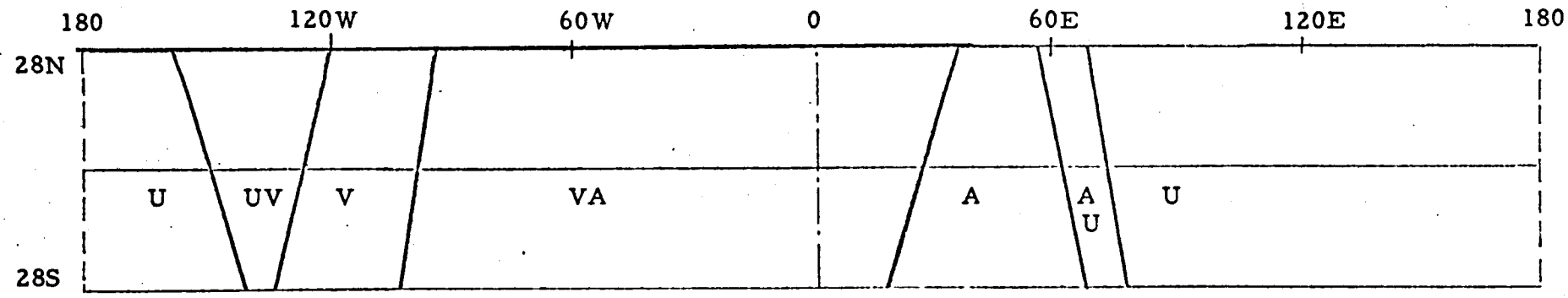
The stations used are those presently planned for the UFS system, and listed in Apollo Note No. 181.



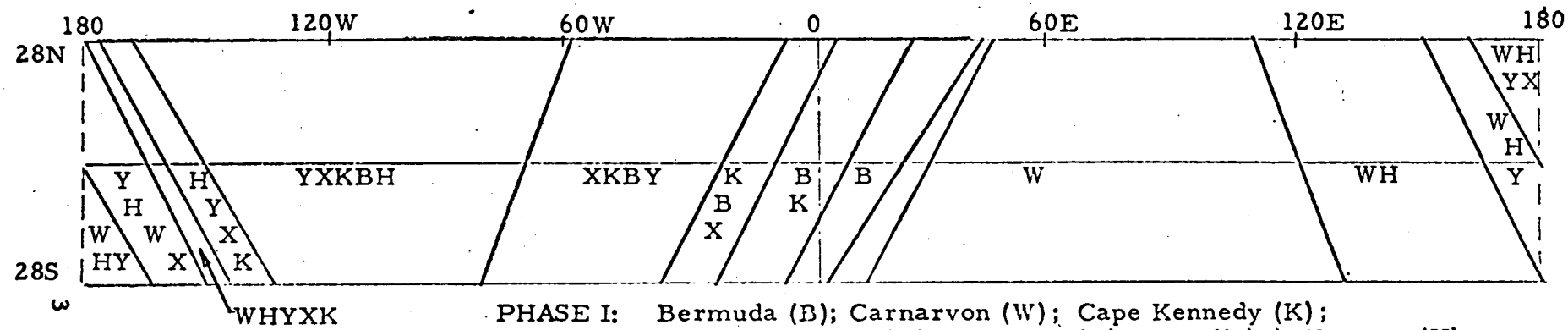
(A)	Ascension	(M)	Madrid
(B)	Bermuda	(U)	Guam
(C)	Canberra	(V)	Antigua
(G)	Goldstone	(W)	Carnarvon
(H)	Hawaii	(X)	Corpus Christie and Houston
(K)	Cape Kennedy	(Y)	Guaymas



DSIF: Goldstone (G); Canberra (C); Madrid (M)



PHASE II: Guam (U); Ascension (A); Antigua (V)



PHASE I: Bermuda (B); Carnarvon (W); Cape Kennedy (K); Corpus Christie (X); Guaymas (Y); Hawaii (H); Houston (X)

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APOLLO NOTE NO. 183

SOME COMMENTS ON THE IMPORTANCE OF AN APPROPRIATE MEASURE OF EFFECTIVENESS FOR THE CORRECTIVE BOOST PROBLEM

J. Holdsworth
30 March 1964

On the return-to-earth trajectory, a critical parameter is the re-entry dive angle. Current corrective boost philosophy seems to indicate that the boost should be such as to correct for the observed discrepancy between the estimated re-entry dive angle and the desired re-entry dive angle where there are random uncertainties both in the estimated position and the executed corrective boost. This philosophy is almost tantamount to assuming that one wishes to minimize the expected square discrepancy between the actual re-entry dive angle and the desired re-entry dive angle.

The above-mentioned criterion function, i.e., mean square discrepancy, is not the only reasonable criterion function. For example, another criterion function which might well be more realistic would be the probability that the actual re-entry dive angle lay within some permissible tolerance band about the nominally desired angle. In this latter situation, the optimal boosting program would be such as to maximize this probability that the actual re-entry dive angle lay within the permissible window.

Now there seems to be a fairly prevalent current fiction that a corrective boosting doctrine which is optimal for one criterion is also optimal for another. The purpose of this note is to dispel this notion by constructing a simple model which can be carried to completion analytically and to show by means of closed-form expressions that what is a good policy for one of the criteria just mentioned need not be good for another.

Also, we shall inquire in this overly-simplified case as to when the optimal policies are similar or identical. Even though the assumptions we shall make to enable us to carry out the analysis explicitly are perhaps unrealistic physically, it is none the less true that this simplified version does represent a limiting case and that if the dependence of the optimality of the correction program exists in our limiting case, then there is strong reason to expect that it exists also in the more realistic case. Hence, since human life may be involved, it behooves one to give some careful thought to the choice of a measure of success or effectiveness or equivalently to an intelligent pay-off of criterion function which we wish to maximize with respect to the various decision variables at our disposal.

Analytical Formulation:

For the purpose of our mathematical treatment, it is logically unnecessary to continue to use the physically suggestive and picturesque term re-entry dive angle, and since our comments apply to a somewhat wider class of decision problems, we shall give an abstract formulation of the problem which avoids the use of these specific terms.

Analysis Leading to Maximization of Probability of Mission Success:

We assume that we are initially in some state or position which may be characterized by some continuous real scalar quantity x which we assume to be known without error. We also assume that we would ideally prefer to be at some other state or position which we may assume to be zero without loss of generality. However, we shall be content and consider our mission a success if our terminal state, which we denote by a real number y, lies within some known prescribed tolerance band; i. e., if $-\tau \leq y \leq \tau$ where $\tau > 0$. We also assume that τ is known. Now if we are dissatisfied with our initial state or equivalently if $|x| > \tau$, then we may apply one single transformation upon x, which we call a corrective maneuver with the hope

that the resulting or terminal position y lies with the tolerance interval.

In performing our corrective maneuver, we assume that the actual maneuver performed deviates from the maneuver which we command in a random manner which is known only distributionally so that amongst other things, the terminal state y is a random quantity. The first case we shall consider is where the pay-off function G is the probability that $-\tau \leq y \leq \tau$. If we let C denote our desired or commanded change or maneuver, we may write the following expression for G which shows the explicit dependence of G upon the initial state x, tolerance half-width τ and commanded maneuver change C.

$$G = \text{Prob}\left\{-\tau \leq y \leq \tau \mid x, C\right\} \tag{1}$$

Thus succinctly, knowing x and τ perfectly, our problem is to choose an optimal C, say C_{opt} such that:

$$\text{Max } G = \text{Prob}\left\{|y| < \tau \mid x, C_{opt}\right\} \tag{2}$$

Now, as mentioned earlier, the actual change performed which we denote by δ will depart somewhat from the commanded change C in a random fashion. Furthermore, to closely parallel the physical situation involving auto-pilot errors, we assume that the root mean square error in δ is linearly proportional to the magnitude of the commanded change C. Also, we shall assume that the statistics of the change δ are such that the expected value of δ is the commanded change C. We further, for convenience, assume that the random departures of δ from C are distributed according to a gaussian probability law. Thus we may write the following equation for the actually performed maneuver δ:

$$\delta = (1 + \epsilon)C \tag{3}$$

where e is a dimensionless random variable whose probability density function is:

$$f(e) = \frac{1}{\sqrt{2\pi}\,\sigma_e} e^{-\frac{1}{2}\frac{e^2}{\sigma_e^2}} \tag{4}$$

Since the terminal position or configuration y is assumed to be the sum of the initial position x and the executed maneuver, we may clearly write:

$$y = x + \delta \tag{5}$$

$$y = x + (1 + e)C \tag{6}$$

From equation (6) it is easy to see that our assumptions imply that we are able to sustain a steady course or keep a constant position in spite of the randomness of e. We do this simply by commanding no change, i. e., by setting $C = 0$.

The last stated fact allows us to immediately solve the optimization problem for a certain singular situation. For example, suppose that x which we know with certainty already is such that $-\tau \leq x \leq \tau$. Then clearly we should command no change; i. e., we should take $C_{opt} = 0$ for in this case, we have $y = x$ and:

$$G = \text{Prob}\left\{-\tau \leq y \leq \tau \,|\, x,\ C = 0\right\} = \text{Prob}\left\{-\tau \leq x \leq \tau \,\middle|\, |x| < \tau\right\}$$
$$= 1 = \max G = \text{Prob}\left\{|y| < \tau \,|\, x,\ C_{opt}\right\} \tag{7}$$

It is perhaps not premature to say that it is in this almost trivial regime that the optimal correction policies corresponding to the two aforementioned pay-off functions will differ most drastically.

It is important to realize that our decision variable, i.e., the quantity under our control with which we hope to maximize G, is the commanded change C. Now since e has the distribution given by equation (4), it follows that the final position or state variable y has a gaussian probability distribution whose probability density function is given by:

$$f(y) = \frac{1}{\sqrt{2\pi}\ \sigma_e\ |C|} \exp - \frac{1}{2\sigma_e^2 C^2} (y - x - C)^2 \quad (8)$$

As defined earlier, we regard our mission as successful if the terminal position does not exceed τ in magnitude, thus from equation (8) and the definition of our pay-off function as being the probability of success, we may write:

$$\text{Prob}\left\{ |y| \leq \tau \mid x,\ C \right\} = \frac{1}{\sqrt{2\pi}\ \sigma_e\ |C|} \int_{-\tau}^{\tau} \exp - \frac{1}{2\sigma_e^2 C^2} (y-x-C)^2\, dy = G(C) \quad (9)$$

where we have written G(C) to emphasize the dependence of the pay-off function G on the decision variable C.

Now since we have disposed of the exceptional case where $-\tau \leq x \leq \tau$, there is no loss in generality in assuming $x > \tau$. Furthermore, if $x > \tau > 0$ and if G(C) is improved at all by choosing a $C \neq 0$, then all of the canons of both mathematical and physical intuition suggest that $C_{opt} < 0$. We shall take this to be a self-evident truth and not bother with a rigorous demonstration of such an obvious point.

With this last thought in mind, if we maximize G by setting:

$$\frac{dG}{dC} = 0 \quad (10)$$

we easily obtain by differentiation under the integral that:

$$\frac{\exp 2\tau(x - |C|)}{\sigma_e^2 C^2} = \frac{x + \tau}{x - \tau} \tag{11}$$

As we are now assuming that $x > \tau$, we may take logarithms of both sides of (11) to yield:

$$\frac{2\tau(x - |C|)}{\sigma_e^2 C^2} = \log \frac{x + \tau}{x - \tau} = k \tag{12}$$

where k depends only upon x and τ and not upon the decision variable C.

A slight rearrangement of (12) shows that the optimal C must satisfy:

$$k\sigma_e^2 C^2 + 2\tau |C| - 2\tau x = 0 \tag{13}$$

from which we easily obtain:

$$|C|_{opt} = \frac{-\tau + \sqrt{\tau^2 + 2k\sigma_e^2 \tau x}}{k\sigma_e^2} \tag{14}$$

whichgives the magnitude of the optimal command of change C. But since, as we noted earlier, $x > \tau > 0$ implies that the optimum commanded correction cannot be positive, we obtain:

$$C_{opt} = \frac{\tau - \sqrt{\tau^2 + 2k\sigma_e^2 \tau x}}{k\sigma_e^2} \tag{15}$$

where k is given by equation (12).

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correction

From equation (15) it is worthwhile to note that the optimum change for one criterion does not center the distribution of the terminal position at the most desired location, which seems to be a part of current dogma.

This is approximately the case, however, under certain limiting conditions. Thus consider the case where σ_e, the dimensionless auto-pilot error is very small; i. e.,

$$\sigma_e \ll 1. \tag{16}$$

Then we have:

$$\sqrt{\tau^2 + 2k\tau x\sigma_e^2} = \tau\left(1 + \frac{2kx}{\tau}\sigma_e^2\right)^{1/2} \approx \tau\left(1 + \frac{kx\sigma_e^2}{\tau}\right) \tag{17}$$

Substituting the approximate value of the radical as given by equation (17) into equation (15) we obtain:

$$C_{opt} = -x \tag{18}$$

which says that for limitingly small dimensionless auto-pilot error, to maximize G(C) we should choose C_{opt} in such a way that the center of the distribution of the final position is centered about zero.

However, to consider a contrary example, if

$$\sigma_e \gg \frac{\tau}{2kx} \tag{19}$$

that is the case where the auto-pilot erros are very large, then a similar limiting analysis shows that we should choose

$$C_{opt} = -\frac{1}{\sigma_e}\sqrt{\frac{2\tau x}{k}} \tag{20}$$

which is a markedly different mode of behavior than that advocated by equation (18).

Minimization of the Mean Square Distance Between the Actual and Preferred Terminal Positions

We now consider a similar situation with identical ground rules, and statistics for the auto-pilot error with one major exception. That is, instead of attempting to maximize the probability of achieving a terminal position within some prescribed tolerance band, we now attempt to modify our position by our corrective maneuver in such a manner that we minimize the mean square distance between our terminal configuration and the ideally desired zero configuration. In other words, our new pay-off, or perhaps more properly called a cost or penalty function, will be taken as $E y^2$ where E is the expected value operator. We shall denote this cost or criterion function by $\mathcal{C}(C)$ to exhibit its dependence upon the decision variable C; or more explicitly we may write:

$$\mathcal{C}(C) = E y^2 . \tag{21}$$

Now squaring equation (6), multiplying by equation (4) and performing an easy integration, it follows readily that $\mathcal{C}(C)$ may be more explicitly written as:

$$\mathcal{C}(C) = (x + C)^2 + C^2 \sigma_e^2 . \tag{22}$$

To compute the optimal commanded correction to minimize $\mathcal{C}(C)$ we note that:

$$\frac{\partial \mathcal{C}}{\partial C} = 2(x + C) + 2C\sigma_e^2 = 0 \tag{23}$$

$$\Longrightarrow \quad C^*_{opt} = -\frac{x}{1 + \sigma_e^2} \tag{24}$$

so that comparison with equation (15) clearly indicates that a commanded boost which is optimal for the first criterion function is clearly not optimal for the second.

General Comments

The rather remarkable result to be gleaned from both equations (15) and (24) is that for neither of the criterion functions considered in this simplified case is the optimal boost the one which on the average will reduce the undesired deviation to zero, although most papers treating the subject make the a-fortiori commitment that each of their corrective boosts will be such as to reduce the apparent discrepancy to zero.

As was initially admitted, the model we have considered is grossly oversimplified. Two of the most presumptious assumptions were that there was no uncertainty in positional knowledge and that it was possible to alter one's position by performing just one corrective maneuver. In addition, we have for simplicity's sake ignored the very real possibility that these efforts are coupled and that there actually exists a trade-off between the number of corrective maneuvers which may be performed and the certainty with which one knows his position at any given time.

However, there is no reason to believe that the inclusion of these other complexities will negate the validity of our basic point regarding the correct identification of a behavioral procedure with a given measure of effectiveness. Thus we reiterate our suggestion for as careful a formulation of the particular optimization problem as possible, together with a plea for a heightened awareness of the dependence of an optimal mode of procedure on the criterion function to be optimized.

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APOLLO NOTE NO. 184

L. S. Lustick
31 March 1964

COMPUTER PROGRAM FOR ESTIMATION OF EFFECT OF NUISANCE PARAMETERS

Purpose

The purpose of this report is to establish the basis for a computer program to evaluate the effect of nuisance parameters.

Introduction

There are many parameters such as the velocity of light, gravitational constants, etc., which are required in order to determine the parameters defining an orbit. These parameters are not known exactly. The purpose of this report is to describe the manner in which the effect of the uncertainty in these parameters on the desired parameters can be determined.

Method

Consider the estimation of a number of parameters $a_{i=1,2...n}$. Let us define arbitrarily the parameters $a_{i=1,2...r}$ as the parameters of interest and $a_{i=r+1,r+2...n}$ as the nuisance parameters. Further let the matrix A be the information matrix for the variables $a_{i=1,2...n}$. This matrix is shown below.

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cc} \phantom{A_{11}} & j=r \end{array} \\ \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array} \begin{array}{l} \\ \\ i=r \\ \\ \end{array}$$

Let the matrix C below be the covariance matrix for the variables $a_{i=1,2\ldots n}$.

$$C = A^{-1} = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]$$

then it can be shown that

$$C_{11} = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1}$$

$$C_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

$$C_{21} = -A_{22}^{-1} A_{21} C_{11}$$

$$C_{12} = - A_{11}^{-1} A_{12} C_{22}$$

If we are only interested in the elements of C_{11}, these can be obtained with less round-off error and in about 1/3 the machine time by using. the fact that $C_{11} = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1}$ instead of inverting the larger dimension matrix.

Alternately one might be interested in the effect of the uncertainty in the nuisance parameters when these are not updated with the current

measurements. The covariance matrix for this case can be expressed in the form below.[1/]

$$\mathrm{Cov}\,(a_{i=1,2\ldots r}) = A_{11}^{-1}\left[\, A_{11} + A_{12}\,\widehat{A}_{22}^{-1}\,A_{21}\,\right] A_{11}^{-1}$$

where $\widehat{A}_{22}^{-1}$ = original covariance matrix of nuisance parameters

Before summarizing, it is of interest to make the following observation. Consider the case when the knowledge in the nuisance parameters is being updated with the current observations. Then,

$$\mathrm{Cov}\,(a_{i=1,2\ldots r}) = (A_{11} - A_{12}\,A_{22}^{-1}\,A_{21})^{-1}$$

If the effect of the nuisance parameters is considered to be small then,

$$\left[\,\mathrm{Cov}\,(a_{i=1,2\ldots r}) \cong A_{11}^{-1}\,\right]$$

and the successive approximation to the inverse can be obtained as follows.[2/]

1/ Bissett-Berman Final Report, Appendix G, dated 2 March 1964, NASw-688, (C60-18).

2/ R. A. Frazer, W. J. Duncan, A. R. Collar, Elementary Matrices and Some Applications to Dynamics and Differential Equations, p. 120.

$$\mathrm{Cov}\,(a_{i=1,2\ldots r}) \cong A_{11}^{-1}\left[2I - (A_{11} - A_{12}A_{22}^{-1}A_{21})\,A_{11}^{-1}\right]$$

$$\cong A_{11}^{-1}\left[I + A_{11}A_{11}^{-1} - (A_{11} - A_{12}A_{22}^{-1}A_{21})A_{11}^{-1}\right]$$

$$\cong A_{11}^{-1}\left[I + A_{12}A_{22}^{-1}A_{21}A_{11}^{-1}\right]$$

$$\mathrm{Cov}\,(a_{i=1,2\ldots r}) \cong A_{11}^{-1}\left[A_{11} + A_{12}A_{22}^{-1}A_{21}\right]A_{11}^{-1}$$

The similarity to this last expression with the case when the knowledge of the nuisance parameters is not updated is shown below.

Nuisance Parameters Updated	Nuisance Parameters <u>Not</u> Updated
$\mathrm{Cov}\,(a_{i=1,2\ldots r}) \cong$ $A_{11}^{-1}\left[A_{11} + A_{12}A_{22}^{-1}A_{21}\right]A_{11}^{-1}$	$\mathrm{Cov}\,(a_{i=1,2\ldots r}) =$ $A_{11}^{-1}\left[A_{11} + A_{12}\widehat{A}_{22}^{-1}A_{21}\right]A_{11}^{-1}$

The difference in these expressions is the difference between $\widehat{A}_{22}^{-1}$ and A_{22}^{-1} where:

$\widehat{A}_{22}^{-1}$ = Covariance matrix of the nuisance parameter (not updated with current measurements)

A_{22}^{-1} = Covariance matrix of the nuisance parameters (updated with current measurements)

Summary

To summarize given A_{11}, A_{12}, A_{21}, and A_{22}, we are interested in the following quantities.

1. A_{11}^{-1} = covariance matrix in desired parameters assuming no error in the knowledge of the nuisance parameters.

2. $A_{11}^{-1} \left[A_{11} + A_{12} \hat{A}_{22}^{-1} A_{21}\right] A_{11}^{-1}$ = covariance matrix in desired parameters with no updating of knowledge in nuisance parameters.

   $\hat{A}_{22}^{-1}$ = covariance matrix of nuisance parameters previous to current data.

3. $(A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1}$ = covariance matrix in desired parameters with updating in nuisance parameters from current data.

4. $(A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$ = covariance matrix in estimates of nuisance parameters (updating with current data).

Computer Program

The purpose of this section of the note will be to describe the changes in our current program to calculate the quantities summarized previously. We will take for the starting point of this discussion the total information matrix just prior to obtaining the covariance matrix.

Let C_{ij} = Total Information Matrix

where

$$C_{ij} = \sum_{\ell=1}^{R} \frac{1}{\sigma_\ell^2} \sum_{k=1}^{N} \frac{\partial M_\ell(t_k)}{\partial a_i} \frac{\partial M_\ell(t_k)}{\partial a_j}$$

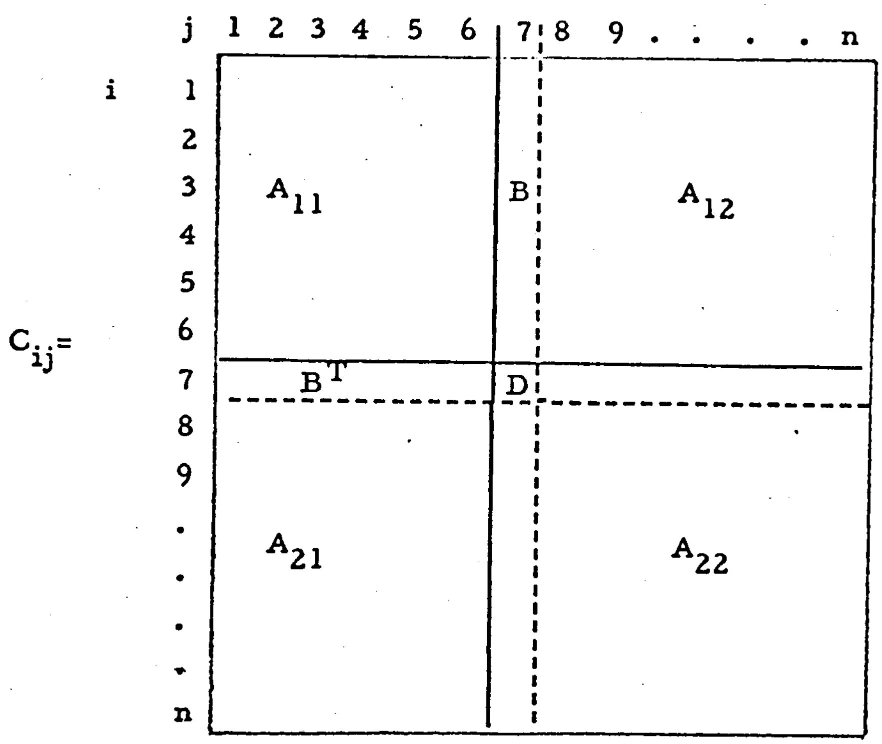
R = Number of measurables

σ_ℓ^2 = Variance in measurable

N = Number of observations

M_ℓ = Measurable

The C_{ij} matrix is indicated schematically below.



where:

$a_{i=1,2...6}$ = position and velocity components at time equal to zero

$a_{i=7}$ = gravitational attraction if errors in μ are to be considered

$a_{i>7}$ = other nuisance parameters

We wish to calculate the following items.

1. No error in the nuisance parameters

$$\text{Cov}\ (T=0) = A_{11}^{-1} \text{ is a } 6 \times 6 \text{ matrix} \tag{1}$$

$$\text{Cov}\ (T=t) = Q(t)\ \text{Cov}(T=0)\ Q^T(t) \tag{2}$$

where $Q(t)$ is a 6×6 matrix corresponding to elements $i = 1, 2 \ldots 6$

2. A priori data in nuisance parameters (no updating with current data).

A. No error in μ

$$\text{Cov}\ (T=0) = A_{11}^{-1} \left[A_{11} + A_{12} \widehat{A}_{22}^{-1} A_{21} \right] A_{11}^{-1} \tag{3}$$

where $\widehat{A}_{22}^{-1}$ is input data and is the covariance matrix corresponding to the a priori uncertainty in the nuisance parameters.

$$\text{Cov}\ (T=t) = Q(t)\ \text{Cov}\ (T=0)\ Q^T(t) \tag{4}$$

where $Q(t) = 6 \times 6$ matrix corresponding to elements $i = 1, 2 \ldots 6$

B. Error in μ to be considered

$$\mathrm{Cov}\,(T=0) = A_{11}^{-1} \left[A_{11} + A_{12}\, \widehat{A}_{22}^{-1}\, A_{21} \right] A_{11}^{-1} \qquad (5)$$

Since there is an error in μ we must augment this matrix before projecting the errors to future times.

$$C\Delta a_{i=1,2\ldots6} + B\Delta a_{i=7} + F\Delta a_{i=8,9\ldots n} = e_{i=1,2\ldots6}$$

$$\Delta a_{i=1,2\ldots6} = C^{-1}\left[e_{i=1,\ldots6} - B\Delta a_{i=7} - F\Delta a_{i=8,9\ldots n} \right]$$

The cross-correlation between $\Delta a_{i=7}$ and the vector $\Delta a_{i=1,2,\ldots6}$ can be obtained as shown below:

$$\overline{(\Delta a_{i=1,2\ldots6})\,(\Delta a_{i=7})} = -\,D\,C^{-1}\,B \qquad (6)$$

where $D = \sigma_{\mu}^{2}$ the variance in the a priori knowledge of μ

$$\mathrm{Cov}(T=0)_{\mathrm{Augmented}} = \begin{array}{c|c|c|}
 & j = 1\ 2\ 3\ 4\ 5\ 6 & 7 \\
\hline
i = 1\ 2\ 3\ 4\ 5\ 6 & \mathrm{Cov}\,(T=0) & L \\
\hline
7 & L^{T} & D \\
\hline
\end{array}$$

where $L = -\,D\,C^{-1}\,B$

$D = \sigma_{\mu}^{2}$

$$\mathrm{Cov}(T=t) = Q\ \mathrm{Cov}\ (T=0)_{\text{augmented}}\ Q^T \tag{7}$$

where $Q = 7 \times 7$ matrix

3. Error in nuisance parameters (updated with current data)

A. No error in μ

$$\mathrm{Cov}\ (T=0) = (A_{11} - A_{12}\ A_{22}^{-1}\ A_{21})^{-1} \tag{8}$$

$$\mathrm{Cov}(T=t) = Q\ \mathrm{Cov}\ (T=0)\ Q^T \tag{9}$$

where $Q = 6 \times 6$ matrix

B. Error in μ to be considered

$$\mathrm{Cov}\ (T=0) = (G_{11} - G_{12}\ G_{22}^{-1}\ G_{21})^{-1} \tag{10}$$

where:

$$C_{ij} = \begin{array}{c|c|c|}
 & j = 1\ 2\ 3\ 4\ 5\ 6\ 7 & 8\ 9\ \ldots\ n \\
\hline
i = 1\ 2\ 3\ 4\ 5\ 6\ 7 & G_{11} & G_{12} \\
\hline
8\ 9\ \ldots\ n & G_{21} & G_{22} \\
\hline
\end{array}$$

$$\text{Cov}\,(T=t) = Q\,\text{Cov}\,(T=0)\;Q^T \tag{11}$$

Q is 7 x 7 matrix

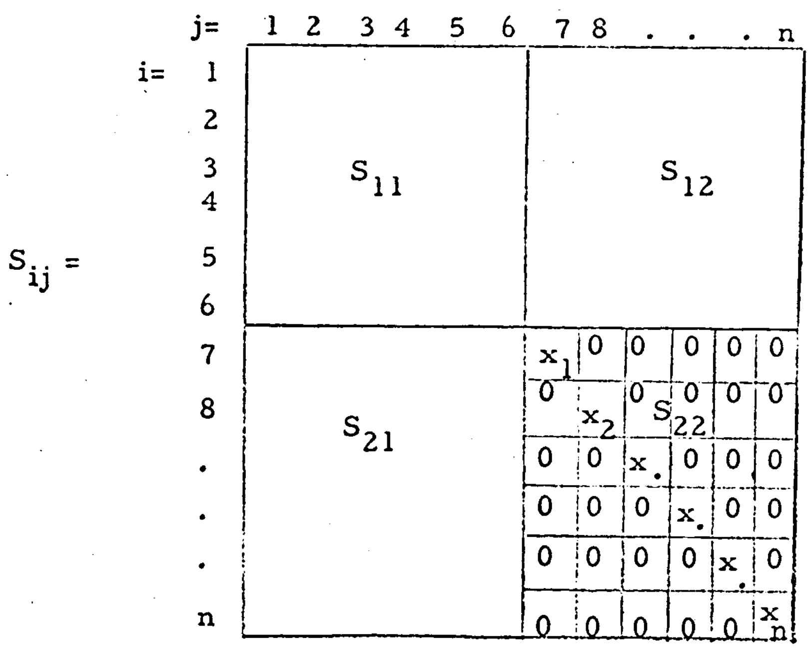
4. Ability to update the nuisance parameters

$$\text{Cov}\,(T=0) = (A_{22} - A_{21}\,A_{11}^{-1}\,A_{12})^{-1} \tag{12}$$

no operation wi th Q matrix.

<u>Input Data</u>

<u>A Priori Matrix</u>



S_{ij} is a matrix representing the a priori knowledge of both the orbit parameters and the nuisance parameters.

$$\widehat{A}_{22}^{-1} = S_{22}^{-1}$$

where S_{22} is a diagonal matrix and S_{22}^{-1} is a diagonal matrix with elements along the principal diagonal that are the reciprocal of those along the diagonal of matrix S_{22}.

Let C_{ij} be the total information matrix including the a priori matrix S_{ij} defined previously. This is the matrix defined in the current program 2 just prior to obtaining the inverse. The C_{ij} matrix has the following restrictions imposed on it.

1. The nuisance parameters are the elements $a_{i=7,8...n}$.

2. If the μ variable is considered, it corresponds to $a_{i=7}$ otherwise the order of elements for $i \geq 7$ is unimportant.

Calculation Procedure

1. From the a priori matrix calculate $\widehat{A}_{22}^{-1}$.

   The principal diagonal of $\widehat{A}_{22}^{-1}$ are the reciprocals of the elements on the principal diagonal of S_{22}.

2. Calculate A_{11}^{-1} (See C_{ij} matrix on page 6).

If there is a Q Matrix	If there is no Q Matrix
Calc: $Cov_{p=1}(T=t)$ from Equation (2)	$Cov_{p=2}(T=t) = A_{11}^{-1}$

3. Effect of μ to be considered

Calc: Cov(T=0) see Equation (5)

Is there a Q matrix?

Yes	No
Calc: $Cov(T=0)_{Aug.}$ see page 8 $Cov_{p=3}(T=t)$ Equation (7)	$Cov_{p=4}(T=t)$ = Cov(T=0)

Am I done?

No	Yes
Cov(T=0) Equation (10) Is there a Q matrix?	EXIT

Yes	No
$Cov_{p=7}(T=t)$ Equation (11)	$Cov_{p=8}(T=t)$= Cov(T=0)

Am I done?

No	Yes
$Cov_{p=11}(T=0)$ Equation (12) Go to EXIT	EXIT

μ Not to be considered

Calc: Cov (T=0) see Equation (3)

Is there a Q matrix?

Yes	No
$Cov_{p=5}(T=t)$ see Equation (4)	$Cov_{p=6}(T=t)$ = Cov(T=0)

Am I done?

No	Yes
Cov(T=0) Equation (8) Is there a Q Matrix?	EXIT

Yes	No
$Cov_{p=9}(T=t)$ Equation (9)	$Cov_{p=10}(T=t)$= Cov(T=0)

Am I done?

No	Yes
$Cov_{p=12}(T=0)$ Equation (12) Go to EXIT	EXIT

EXIT

Output

The output at most will be 4 matrices, $Cov_p(T=t)$, none of which is bigger than a 7 x 7.

$Cov_{p=1 \text{ or } 2}$	should be labeled as "zero errors in nuisance parameters"
$Cov_{p=3 \text{ or } 4}$	should be labeled as "no updating in nuisance parameters MU considered"
$Cov_{p=5 \text{ or } 6}$	should be labeled as "no updating in nuisance parameters MU not considered"
$Cov_{p=7 \text{ or } 8}$	"Nuisance parameters updated with current data MU considered"
$Cov_{p=9 \text{ or } 10}$	"Nuisance parameters updated with current data MU not considered"
$Cov_{p=11 \text{ or } 12}$	"Effect on the error in nuisance parameters"

If any of the output matrices are not printed out, an appropriate comment of this fact should be printed out.

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APOLLO NOTE NO. 185

H. Engel
6 April 1964

ACCURACY OF POSITION AND VELOCITY DETERMINATION DURING BOOST, II

Apollo Note No. 177 errs in using past estimates or measures of acceleration to determine current acceleration. We modify the results of Note No. 177 here to correct this.

It is convenient to treat the acceleration measurements separately from the remaining measurements. We let P_n be the error covariance matrix at time t_n, including all measurements up to that time except acceleration measurements at t_n. We denote the error covariance matrix at t_n after the acceleration measurements by V_n. Then, if Q_n is the covariance matrix of the noise in these measurements, and M_n is the matrix relating the state vector and the acceleration measurables, we have

$$V_n^{-1} = P_n^{-1} + M_n^T Q_n^{-1} M_n$$

according to the rules for combination of information matrices. Then

$$V_n = \left[P_n^{-1} + M_n^T Q_n^{-1} M_n\right]^{-1}$$

$$= \left[P_n^{-1} (M_n^T Q_n^{-1} M_n)^{-1} (M_n^T Q_n^{-1} M_n) + (M_n^T Q_n^{-1} M_n)\right]^{-1}$$

$$= (M_n^T Q_n^{-1} M_n)^{-1} \left[P_n^{-1} (M_n^T Q_n^{-1} M_n)^{-1} + I\right]^{-1}$$

Now the state vector, x_n, of position, velocity and acceleration at t_n and of accelerometer scale factor, range bias, etc., can be decomposed into two parts, x_{0n} and x_{1n},

$$x_n = \begin{bmatrix} x_{0n} \\ x_{1n} \end{bmatrix}$$

in which x_{1n} includes the three components of acceleration, and x_{0n} all the rest.

The M_n and P_n matrices may be partitioned in like fashion.

$$M_n = \begin{bmatrix} M_{0n} & M_{1n} \end{bmatrix}$$

$$P_n = \begin{bmatrix} P_{00n} & P_{01n} \\ P_{10n} & P_{11n} \end{bmatrix}$$

P_{11n}, however, is infinite since the acceleration error at time t_n is independent of that at t_{n-1}, and P_{01n} and P_{10n} are zero since the new acceleration error will not be correlated with the past errors in the other elements of the state vector. Still, we retain P_{11n}, even though it is infinite, considering it for the time being as though it were only very large. Then

$$P_n = \begin{bmatrix} P_{00n} & 0 \\ 0 & P_{11n} \end{bmatrix}$$

It follows that

$$V_n = (M_n^T Q_n^{-1} M_n)^{-1} \left[\begin{pmatrix} P_{00n}^{-1} & 0 \\ 0 & P_{11n}^{-1} \end{pmatrix} \begin{pmatrix} M_{0n}^T Q_n^{-1} M_{0n} & M_{0n}^T Q_n^{-1} M_{1n} \\ M_{1n}^T Q_n^{-1} M_{0n} & M_{1n}^T Q_n^{-1} M_{1n} \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right]^{-1}$$

$$= (M_n^T Q_n^{-1} M_n)^{-1} \begin{bmatrix} P_{00n}^{-1} M_{0n}^T Q_n^{-1} M_{0n} + I & P_{00n}^{-1} M_{0n}^T Q_n^{-1} M_{1n} \\ P_{11n}^{-1} M_{1n}^T Q_n^{-1} M_{0n} & P_{11n}^{-1} M_{1n}^T Q_n^{-1} M_{1n} + I \end{bmatrix}^{-1}$$

$$= (M_n^T Q_n^{-1} M_n)^{-1} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}$$

We find the inverse of the matrix on the right above by parts. We have

$$V_n = (M_n^T Q_n^{-1} M_n)^{-1} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1}$$

$$B_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

$$B_{12} = - A_{11}^{-1} A_{12} B_{22}$$

$$B_{21} = - A_{22}^{-1} A_{21} B_{11}$$

Now

$$A_{22} = I + P_{11n}^{-1} M_{1n}^T Q_n^{-1} M_{1n}$$

$$= I + O(\varepsilon)$$

so

$$A_{22}^{-1} \doteq I - P_{11n}^{-1} M_{1n}^T Q_n^{-1} M_{1n}$$

$$= I - O(\varepsilon)$$

Then

$$B_{11} = \left[A_{11} - P_{00n}^{-1} M_{0n}^T Q_n^{-1} M_{1n} (I - M_{1n}^T Q_n^{-1} M_{1n}) P_{11n}^{-1} M_{1n}^T Q_n^{-1} M_{0n}\right]^{-1}$$

$$= \left[A_{11} + O(\varepsilon)\right]^{-1} = A_{11}^{-1} - O(\varepsilon)$$

$$B_{22} = \left[(I + O(\varepsilon) - P_{11}^{-1} M_{1n}^T Q_n^{-1} M_{0n} A_{11}^{-1} P_{00n}^{-1} M_{0n}^T Q_n^{-1} M_{1n}\right]^{-1}$$

$$= \left[I + O(\varepsilon)\right]^{-1} = I - O(\varepsilon)$$

$$B_{12} = -A_{11}^{-1} A_{12} + O(\varepsilon)$$

$$B_{21} = -A_{21} A_{11}^{-1} + O(\varepsilon)$$

$$V_n = (M_n^T Q_n^{-1} M_n)^{-1} \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} \\ -A_{21} A_{11}^{-1} & I \end{bmatrix} + \left[O(\varepsilon)\right]$$

We find the inverse of $M_n^T \; Q_n^{-1} \; M_n$ in like fashion,

$$(M_n^T \; Q_n^{-1} \; M_n)^{-1} = \begin{bmatrix} M_{0n}^T \, Q_n^{-1} \; M_{0n} & M_{0n}^T \, Q_n^{-1} \, M_{1n} \\ M_{1n}^T \, Q_n^{-1} \; M_{0n} & M_{1n}^T \, Q_n^{-1} \; M_{1n} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

where

$$E_{11} = D_{11} - D_{12} \, D_{22}^{-1} \; D_{21}$$

$$E_{22} = D_{22} - D_{21} \, D_{11}^{-1} \; D_{12}$$

$$E_{12} = - D_{11}^{-1} \; D_{12} \, E_{22}$$

$$E_{21} = - D_{22}^{-1} \; D_{21} \, E_{11}$$

Then

$$V_n = \begin{bmatrix} E_{11} A_{11}^{-1} - E_{12} A_{21} A_{11}^{-1} & -E_{11} A_{11}^{-1} A_{12} + E_{12} \\ E_{21} A_{11}^{-1} - E_{22} A_{21} A_{11}^{-1} & -E_{21} A_{11}^{-1} A_{12} + E_{22} \end{bmatrix} + O(\mathcal{E})$$

$$= \begin{bmatrix} V_{00n} & V_{01n} \\ V_{10n} & V_{11n} \end{bmatrix}$$

but V_n must be symmetric, so

$$(E_{21} - E_{22} A_{21}) A_{11}^{-1} = - E_{11} A_{11}^{-1} A_{12} + E_{12}$$

On the other hand, A_{11} depends on P_{00n}, and E_{12} does not. Therefore,

$$E_{12} = 0$$

or

$$- D_{11}^{-1} D_{12} E_{22} = 0$$

and by further substitution

$$- D_{11}^{-1} D_{12} (D_{22} - D_{21} D_{11}^{-1} D_{21}) = 0.$$

Q_n, however, is an arbitrary matrix, so either D_{11}^{-1}, D_{12} or $(D_{22} - D_{21} D_{11}^{-1} D_{21})$ is zero. Again, since Q_n is arbitrary, these quantities may not be 0 unless M_{0n} or M_{1n} is zero. We know that M_{1n} is not zero, since the acceleration measurements are related to the acceleration components of the state vector. Therefore, we are led to the conclusion that

$$M_{0n} = 0$$

This, in turn, invalidates practically all of our derivation, which depends on M_{0n} being different from zero. Returning to the beginning, and letting M_{0n} be zero, we have

$$V_n^{-1} = \begin{bmatrix} P_{00n}^{-1} & 0 \\ 0 & P_{11n}^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M_{1n}^T Q_n^{-1} M_{1n} \end{bmatrix}$$

$$= \begin{bmatrix} P_{00n}^{-1} & 0 \\ 0 & (\infty)^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M_{1n}^T Q_n^{-1} M_{1n} \end{bmatrix}$$

$$= \begin{bmatrix} P_{00n}^{-1} & 0 \\ 0 & M_{1n}^T Q_n^{-1} M_{1n} \end{bmatrix}$$

and

$$V_n = \begin{bmatrix} P_{00n} & 0 \\ 0 & (M_{1n}^T Q_n^{-1} M_{1n})^{-1} \end{bmatrix}$$

Now if x_{1n} is the three components of acceleration and the measurables are these same three quantities, M_{1n} is an identity matrix and

$$V_n = \begin{bmatrix} P_{00n} & 0 \\ 0 & Q_n \end{bmatrix}$$

Thus we have gone all around Robin Hood's barn to arrive at a conclusion that seems immediately evident; that is, the measurement of acceleration can not improve the knowledge of position, velocity, platform tilt, and so forth, and can lead only to an estimate of acceleration.

Next, using the covariance V_n we can determine the covariance at t_{n+1} prior to any measurements at t_{n+1}. It is $\mathcal{P}_{n+1}$

$$\mathcal{P}_{n+1} = \phi_{n+1,n} V_n \phi^T_{n+1,n}$$

Now let us find P_{n+1}. The fundamental theorem of the Kalman-Schmidt method is the following:

<u>Theorem:</u>

Let α and β be two column vector random variables with 0 means and covariance matrices

$$\Lambda_\beta = F_{11} = E(\beta\,\beta^T)$$

$$\Lambda_\alpha = F_{22} = E(\alpha\,\alpha^T)$$

$$\Lambda_{\beta\alpha} = F_{12} = E(\beta\,\alpha^T)$$

$$\Lambda_{\alpha\beta} = F_{21} = E(\alpha\,\beta^T) = F_{12}^T$$

then the best estimate of β is $\tilde{\beta}$,

$$\tilde{\beta} = F_{12} F_{22}^{-1} \alpha$$

and

$$E(\tilde{\beta}\tilde{\beta}^T) = F_{12} F_{22}^{-1} E(\alpha\,\alpha^T) F_{22}^{-1} F_{12}^T$$

$$= F_{12} F_{22}^{-1} F_{22} F_{22}^{-1} F_{21}$$

$$= F_{12} F_{22}^{-1} F_{21}$$

Now we let the error in the best estimated state vector prior to measurements at t_{n+1} be β and the errors in the predicted measurements be α. Denoting the error in the estimated state vector by δx^*_{n+1} $(= x^*_{n+1} - x_{n+1})$, the error in the estimated measurements vector by δy^*_{n+1}, and the matrix relating the state vector and the measurables by $\mathcal{M}_{n+1}$,

$$y_{n+1} = \mathcal{M}_{n+1} \; x_{n+1},$$

$$y^*_{n+1} = \mathcal{M}_{n+1} \; x^*_{n+1} + n_{n+1}$$

we have

$$\alpha = \delta y^*_{n+1} = \mathcal{M}_{n+1} \, \delta x^*_{n+1} + n_{n+1}$$

in which n_{n+1} is the noise vector appearing in the measurements. Then

$$F_{22} = E\left[(\mathcal{M}_{n+1} \, \delta x^*_{n+1} + n_{n+1}) (\mathcal{M}_{n+1} \, \delta x^*_{n+1} + n_{n+1})^T \right]$$

$$= \mathcal{M}_{n+1} \, E\,(\delta x^*_{n+1} \, \delta x^{*\,T}_{n+1}) \, \mathcal{M}^T_{n+1} + E\,(n_{n+1} \, n^T_{n+1})$$

Denoting the covariance matrix of these measurements by $\mathcal{L}_{n+1}$, we have

$$F_{22} = \mathcal{M}_{n+1} \, \mathcal{P}_{n+1} \, \mathcal{M}^T_{n+1} + \mathcal{L}_{n+1}$$

Also,

$$F_{12} = E\,(\delta x^*_{n+1}\ \delta y^{*\,T}_{n+1})$$

$$= E\left(\delta x^*_{n+1}\left[\mathcal{H}_{n+1}\ \delta x^*_{n+1} + n_{n+1}\right]^T\right)$$

$$= E\,(\delta x^*_{n+1}\ \delta x^{*\,T}_{n+1})\ \mathcal{H}^T_{n+1} + 0$$

$$= \mathcal{P}_{n+1}\ \mathcal{M}^T_{n+1}$$

$$= \mathcal{M}_{n+1}\ \mathcal{P}_{n+1})^T$$

and

$$F_{21} = \mathcal{M}_{n+1}\ \mathcal{P}_{n+1}$$

Now the new best estimate of x_{n+1} is $\tilde{x}_{n+1}$

$$\tilde{x}_{n+1} - x_{n+1} = x^*_{n+1} - x_{n+1} - \text{Best estimate of } (x^*_{n+1} - x_{n+1})$$

$$= \delta x^*_{n+1} - \tilde{\beta}$$

$$= \delta x^*_{n+1} - F_{12}\ F_{22}^{-1}\ \alpha$$

and the expected value of $\tilde{x}_{n+1} - x_{n+1}$ is 0.

Then,

$$P_{n+1} = E\left[\ (\tilde{x}_{n+1} - x_{n+1})\ (\tilde{x}_{n+1} - x_{n+1})^T\ \right]$$

$$= E\left[\ (\delta x^*_{n+1} - F_{12}\ F_{22}^{-1}\ \alpha)\ (\delta x^*_{n+1} - F_{12}\ F_{22}^{-1}\ \alpha)^T\right]$$

$$= E\,(\delta x^*_{n+1}\ \delta x^{*\,T}_{n+1}) - E(\delta x^*_{n+1}\ \alpha^T)\ F_{22}^{-1}\ F_{12}^T$$

$$- F_{12}\ F_{22}^{-1}\ E\,(\alpha\ \delta x^{*\,T}_{n+1}) + F_{12}\ F_{22}^{-1}\ E\,(\alpha\ \alpha^T)\ F_{22}^{-1}\ F_{12}^T$$

$$P_{n+1} = \mathcal{P}_{n+1} - E\left[\delta x^*_{n+1}\,(\mathcal{M}_{n+1}\,\delta x^*_{n+1} + n_{n+1})^T\right] F_{22}^{-1}\,F_{12}^T$$

$$- F_{12}\,F_{22}^{-1}\,E\left[(\mathcal{M}_{n+1}\,\delta x^*_{n+1} + n_{n+1})\,\delta x^{*\,T}_{n+1}\right]$$

$$+ F_{12}\,F_{22}^{-1}\,F_{22}\,F_{22}^{-1}\,F_{12}^T$$

$$= \mathcal{P}_{n+1} - \mathcal{P}_{n+1}\,\mathcal{M}^T_{n+1}\,F_{22}^{-1}\,F_{12}^T - F_{12}\,F_{22}^{-1}\,\mathcal{M}_{n+1}\,\mathcal{P}_{n+1}$$

$$+ F_{12}\,F_{22}^{-1}\,F_{21}^T$$

$$= \mathcal{P}_{n+1} - F_{12}\,F_{22}^{-1}\,F_{12}^T - F_{12}\,F_{22}^{-1}\,F_{12}^T + F_{12}\,F_{22}^{-1}\,F_{12}^T$$

$$= \mathcal{P}_{n+1} - F_{12}\,F_{22}^{-1}\,F_{21}$$

or

$$P_{n+1} = \mathcal{P}_{n+1} - (\mathcal{M}_{n+1}\mathcal{P}_{n+1})^T\,(\mathcal{M}_{n+1}\mathcal{P}_{n+1}\mathcal{M}^T_{n+1} + \mathcal{R}_{n+1})^{-1}(\mathcal{M}_{n+1}\mathcal{P}_{n+1})$$

Since we only want $P_{00,n+1}$ we may decompose $\mathcal{M}_{n+1}$ according to

$$\mathcal{M}_{n+1} = \left[\mathcal{M}_{0,n+1} \quad \mathcal{M}_{1,n+1}\right]$$

and write

$$P_{00,n+1} = \mathcal{P}_{00,n+1} - (\mathcal{M}_{0,n+1}\,\mathcal{P}_{00,n+1})^T\,(\mathcal{M}_{n+1}\,\mathcal{P}_{n+1}\,\mathcal{M}^T_{n+1} + \mathcal{R}_{n+1})^{-1}$$

$$(\mathcal{M}_{0,n+1}\,\mathcal{P}_{00,n+1})$$

Since we use only measurements of distance, direction or rate of change of these in obtaining $P_{00,n+1}$ from $\mathcal{P}_{00,n+1}$, it follows that $M_{1,n+1}$ is zero and

$$P_{00,n+1} = \mathcal{P}_{00,n+1} - (\mathcal{M}_{0,n+1}\mathcal{P}_{00,n+1})^T (\mathcal{M}_{0,n+1}\mathcal{P}_{00,n+1}\mathcal{M}^T_{0,n+1} + \mathcal{L}_{n+1})^{-1}$$

$$(\mathcal{M}_{0,n+1}\mathcal{P}_{00,n+1})$$

in which

$$\mathcal{P}_{00,n+1} = \phi_{00,n} P_{00,n} \phi^T_{00,n} + \phi_{12,n} Q_n \phi^T_{12,n}$$

These results should now be consistent with those of Note 180, differing only in form because of differences in notation.

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APOLLO NOTE NO. 186

H. Dale/H. Engel/
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DIRECTIONS FOR USE OF ERROR ANALYSIS PROGRAM

Introduction

The Bissett-Berman Orbit Error Analysis Program provides a means for estimating the error covariance matrix of the estimates of position and velocity of a vehicle in orbit about the Earth or Moon on the basis of observations from a number of sources. The analysis upon which the program is based has been reported in previous Bissett-Berman Apollo Notes and in the Bissett-Berman Report, "Capabilities of MSFN for Apollo Guidance and Navigation."

The purpose of the present note is to explain how to use the program. Following this introduction, the capabilities of the program are described, and the data that must be provided as inputs are indicated. Finally, instructions for actual entry of data and running of the program are presented.

Program Capabilities

The program has a number of capabilities in addition to those sketched so very briefly in the introduction. These additional capabilities are described in this section. In addition, some further capabilities may be wrung from the existing program with a little ingenuity, and some of these are also indicated here.

The program is divided into two parts, called Program A and Program B. Program A computes the partials of the measurables with respect to the orbit parameters from a single tracking station on the basis of assumed values of physical constants and orbit parameters. Program B combines the results of Program A with data from other sources to determine the error covariance matrix of the orbit parameters and any other parameters to be

estimated on the basis of the measurements used.

Program A may be used for a vehicle in circular, elliptic or hyperbolic orbits about the Earth or Moon, with observations from a station on the rotating Earth, a vehicle in circular orbit about the Earth or Moon, or a station on the Moon. The measurables may be range, range rate or angles. The outputs of Program A are information matrices corresponding to measurables with a variance of unity. The information matrices may either be 7 x 7, corresponding to the six orbit parameters $a_1, \ldots, a_6$ and a fixed bias in the measurable, or else they may be 13 x 13, corresponding to the seven quantities already mentioned plus the following: three components of station location, speed of light, gravitational constant, and clock error rate. The 7 x 7 output should be called for if quantities in the larger matrix will not be used, since this will reduce computing time. Another output of Program A is Q matrices that are used in computing uncertainties in future values of position and velocity. The last output is the central angle about its orbital focus that the vehicle passes through from that time selected as time zero until the time corresponding to that of the corresponding Q matrix output.

Inputs for Program A

The inputs to Program A are the nominal values of the physical constants and orbit parameters. An input data sheet for program A is attached as Figure 1. The units employed are meters, seconds, and degrees, with the exception of time, which is specified in minutes for simpler comprehension.

We shall consider the entries on the input data sheet in order.

The data set number and station name are used to identify these computations and in obtaining the results of these computations from the magnetic tape on which they are stored for use in Program B. Data sets must be computed in the order in which they are numbered so that data sets which appear on the storage tape are in order of increasing data set numbers and hence can be accessed.

Station:

Data Set | | | | |

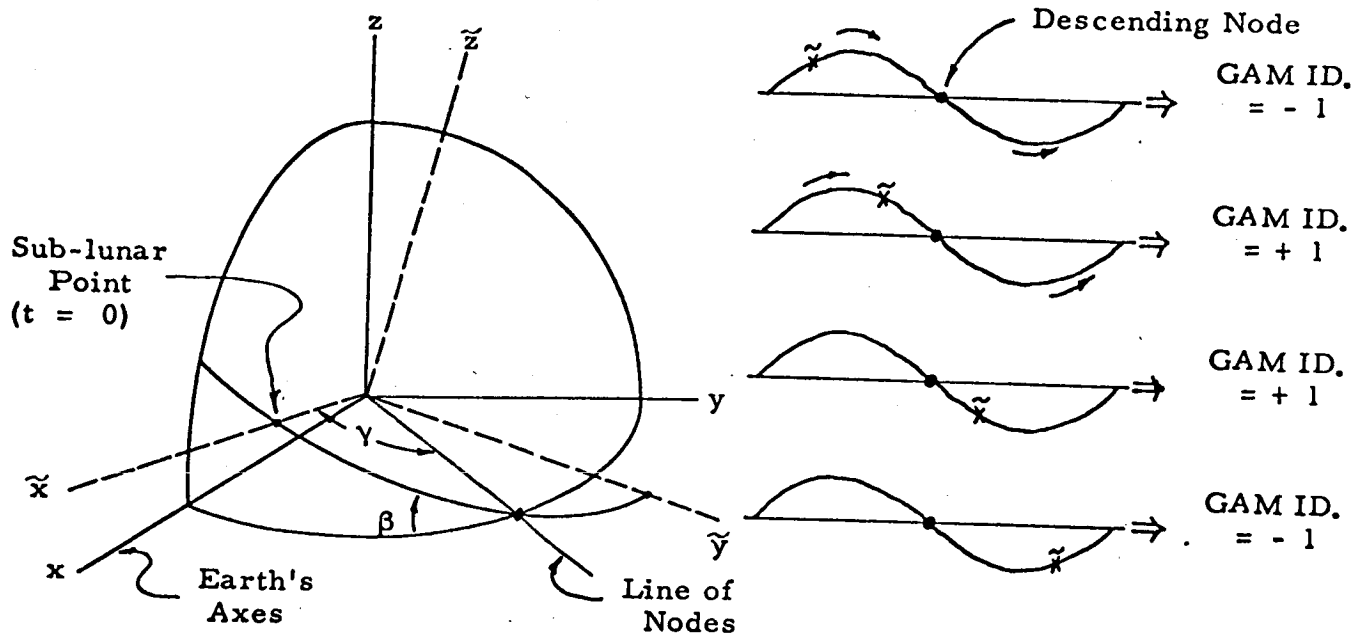
01	A 1	m	Initial orbit radius
02	A 4	m/sec	" " radial velocity
03	A 5	m/sec	" " tangent velocity
04	GAM ID		+ 1 if $\lvert\gamma\rvert < \frac{\pi}{2}$, - 1 otherwise
05	BETA	deg	Inclination of Earth-Moon orbit to Equator
06	XI	deg	Euler angles relating x' y' z'
07	ETA	deg	coordinates to $\tilde{x}\ \tilde{y}\ \tilde{z}$
08	ZETA	deg	coordinates
09	LAMBDA	deg	Station latitude
10	ALPHA	deg	" longitude
11	OMEGA E	rad/sec	Earth angular rate
12	OMEGA M	rad/sec	Earth-Moon angular rate
13	RHO E	m	Radius of Earth
14	RHO M	m	Earth-Moon distance
15	MU	m^3/sec^2	Gravitational constant
18	BIG L	deg	Latitude of sublunar point at T = 0
19	SMA L	deg	Longitude of sublunar point at T = 0
20	SD IND		If 1, calculate information for $\dot{R}$, otherwise 0
21	S IND		" " " R, " "
22	PS 1 IND		" " " ϕ_1, " "
25	PS 2 IND		" " " ϕ_2, " "
26	PVW IND		If 1, x-y mount
16	TJ	min	Maximum times of observation
23	Q IND		If 1, calculate Q for all TJ, otherwise 0
24	QF	min	Value of time for which other Q wanted
27	TI (initial)	min	Time of first observation
28	TI (interval)	min	Interval between observations
29	DIMENSIONS		Desired size of inf. matrix

REMARKS:

Figure 1. Example Input Data Sheet for Program A

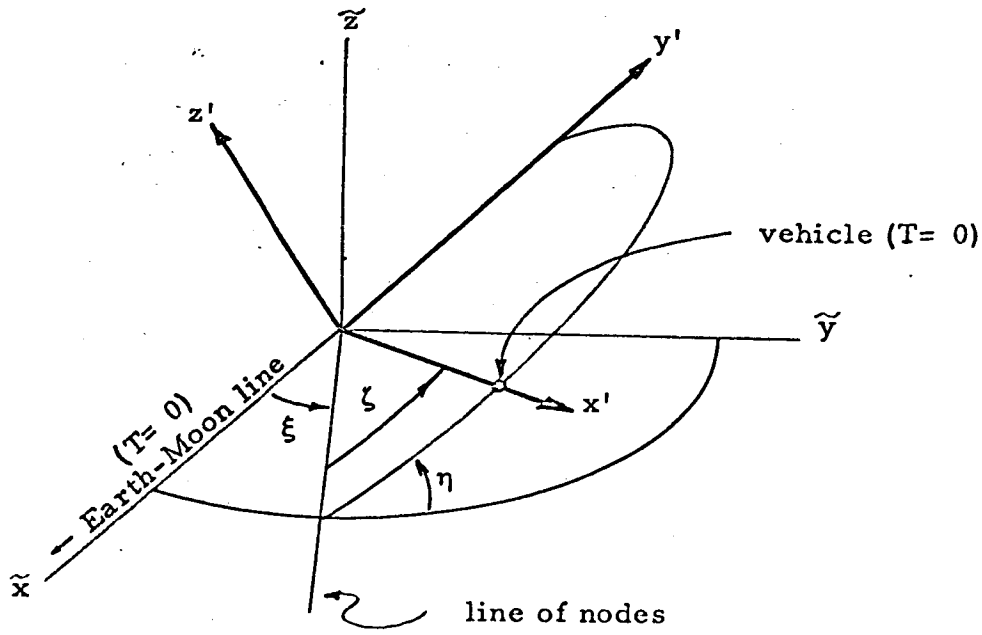
A_1 is the radial distance of the vehicle at time zero from the center of the body about which it is orbiting. A_4 is the radial component of velocity at time zero, and A_5 is the tangential component of velocity at time zero.

"Gamma Indicator" is an input which helps describe the quadrant of γ. Gamma Indicator should be entered as + 1 if $|\gamma|$ is less than 90°. The following sketch can be used to better visualize γ which is the angle measured by a right-hand rotation about $\tilde{z}$ from the sub-lunar point at time zero to the declining node of the Moon's orbit about the Earth. Also seen in the sketch is BETA, the input describing the inclination of the Earth-Moon orbit to the equator of the Earth.



Sketch Describing GAM ID. and BETA

XI, ETA, and ZETA are inputs which describe the orbital plane and initial position (T = 0) of the tracked body. They are Euler angles which are again best described by a sketch:



XI rotates the line of nodes out of the $\tilde{x}$ (Earth-Moon line at T = 0) axis. ETA is the inclination of the vehicular orbit plane to the $\tilde{x}$, $\tilde{y}$ plane, and ZETA describes the initial position of the vehicle and thus the set of coordinates: x', y', z'.

LAMBDA and ALPHA are the latitude and longitude of the tracking station in Earth coordinates. LAMBDA is entered as a positive quantity North of the Earth's Equator, while ALPHA is positive measured Eastward from the prime-meridian. It is of interest here to note that if a plurality of stations track the vehicle during the same time, then a Program A must be run for each station. However, the only difference between the inputs of each station may well be LAMBDA and ALPHA.

OMEGA E is the Earth's angular rate and is entered as $.729116 \times 10^{-4}$ rad./sec. This input allows the tracking station to be correctly described as the Earth rotates.

OMEGA M is the Earth-Moon angular rate and is entered as $.2669 \times 10^{-5}$ rad./sec.

RHO E is the radius of the Earth ($.63781 \times 10^7$ m) and is used to describe the position of the tracking station in x, y, z coordinates.

RHO M is the Earth-Moon distance which is entered as $.385 \times 10^9$ m.

MU is the gravitational constant of the body attracting the vehicle. For Lunar orbits the value entered is $.4896 \times 10^{13}$ m^3/sec^2, while for Earth orbits, which will be described later, the value entered is $.399689 \times 10^{15}$ m^3/sec^2.

BIG L and SMA L are the latitude and longitude, respectively, of the sub-lunar point at T = 0. In conjunction with GAM ID, BETA, and OMEGA M they describe the orbit of the Moon about the Earth. BIG L is measured as positive North from the Earth's equator and SMA L is measured positive East from the prime-meridian.

SD IND., S IND., PS 1 IND., and PS 2 IND. are all input indicators which command the type of information matrices to be calculated. If SD IND. is entered as + 1, an information matrix will be computed corresponding to range-rate measurements, while a zero input will delete this computation. S. IND. similarly commands a range information matrix. PS 1 and PS 2 are used to command the computation of information matrices corresponding to angular measurements. An entry of +1 for the PVW IND. indicates that the angular measurements are taken from an x-y mount; an entry of 0 indicates that the angle measurements are about two axes normal to the line of sight. A more detailed description of x-y mount measurements is given in Appendix E of The Bissett-Berman report, "Capabilities of MSFN for Apollo Guidance and Navigation."

TJ is a multiple entry which commands various maximum observation times. Thus an information matrix will be computed for each TJ entered. Since it is often of interest to see how the estimate of the vehicles orbit parameters changes with the observation time, space is provided for multiple TJ entries. The Q IND. is used (+ 1) to indicate

that Q matrices are to be calculated for each TJ. As an example, if we wish to know the estimates of orbit parameter errors at 5, 10, 20, and 40 minutes after the initiation of tracking, then these times should be entered as TJ's. Then if it is desired to have the covariance matrix of position and velocity errors at these times rather than the reference, T = 0, it is necessary to have Q matrices for each TJ. Thus a + 1 should be entered for Q IND. If it is not necessary to have covariance matrices expressed at the end of each TJ, then computer time will be saved by entering a zero for Q IND.

Two restrictions are placed on the TJ's entered. First, they must be equal to calculated observation times, as specified by TI (initial) and TI (interval). Second, they must be entered in the order of increasing time.

In many cases it will be of interest to compute the covariance matrix at some time different from the termination of tracking from a particular station. For example the covariance matrix at periapsis, apoapsis, rendezvous, etc., may be obtained through the use of a Q matrix. In this case the time of interest (measured from T = 0) should be entered at QF. If more than one station (Program A) is to be used with a particular vehicle, only one station need contain a QF entry since the Q matrix so calculated depends only on the vehicle orbit parameters and time and not on the station location.

TI (initial) is the entry which determines the first observation for a particular station while TI (interval) determines the time between observations. TI (initial) is seldom zero since, even if tracking starts at T = 0, the first observation is accomplished after one interval of tracking. In many cases stations start tracking much later than T = 0 because of visibility problems. This is particularly true of low Earth orbits.

At this point the trade off between the number of uncorrelated measurements and the variance in each measurement is pertinent. Since the program format has room for only 299 observations per station, it is often necessary to compress the actual measurements taken and to compensate for the reduced observations by decreasing the one-shot error in the measurable, σ, used in Program B. The following formula may be used.

$$\frac{\sigma \text{ used in Prog. B}}{\sigma \text{ actual}} = \sqrt{\frac{\text{N used in Prog. A}}{\text{N actual}}} \; ; \text{N = total number of observations}$$

Thus, if 10 minutes of tracking with one-second intervals are used, 600 observations are necessary. Since the above formula is good for more than about five observations (used), computer time may be saved by using only six observations without degrading the results appreciably. But then in Program B the σ of the observable should be multiplied by a factor of $\sqrt{6/600} = .1$ to compensate for the fewer observations. In order to decrease the number of observations from 600 to six, TI (interval) should be changed from .01666 min. to 1.666 min. and TI (initial) should also be changed from .01666 min. to 1.666 min.

The last input is the required dimension of the requested information matrices, either 7 x 7 or 13 x 13, as discussed previously.

Output of Program A

The first output of Program A is a list of the inputs. Following this are the various Q matrices each titled with the heading "Q Matrix" followed by the data set number and the time to which the Q matrix corresponds. The dimension of the Q matrix is the same as the input command; however, only significant terms are printed out. If at some future time an attempt is made at multiplying a 6 x 6 Q matrix by a larger information or covariance matrix, then "ones" are automatically assumed to be in the remaining diagonal elements and zeros are assumed elsewhere. With a 13 x 13 request as an input, the following output is printed:

$$
\left|\begin{array}{cccccc|ccccccc}
\frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} & \frac{\partial x_1}{\partial a_3} & \frac{\partial x_1}{\partial a_4} & \frac{\partial x_1}{\partial a_5} & \frac{\partial x_1}{\partial a_6} & 0 & 0 & 0 & 0 & 0 & \frac{\partial x_1}{\partial \mu} & 0 \\
\frac{\partial x_2}{\partial a_1} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & \frac{\partial x_2}{\partial \mu} & 0 \\
\frac{\partial x_3}{\partial a_1} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & \frac{\partial x_3}{\partial \mu} & 0 \\
\frac{\partial x_4}{\partial a_1} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & \frac{\partial x_4}{\partial \mu} & 0 \\
\frac{\partial x_5}{\partial a_1} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & \frac{\partial x_5}{\partial \mu} & 0 \\
\frac{\partial x_6}{\partial a_1} & \cdot & \cdot & \cdot & \cdot & \frac{\partial x_6}{\partial a_6} & 0 & 0 & 0 & 0 & 0 & \frac{\partial x_6}{\partial \mu} & 0 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & 0 & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right|
$$

where the x's are the orbit parameters at the future time (time of the Q matrix calculation) and the a's are the orbit parameters at T= 0. The last row and column is often not printed out since it would be filled in automatically as described above. The seventh column derivatives are with respect to the bias in the measurable. The eighth, ninth, and tenth columns are with respect to station location errors: x", y", and z". The eleventh column derivatives are with respect to clock rate errors. Non-zero derivatives off the main diagonal after column six appear in the twelfth column and are with respect to changes in the gravitational constant. This is true since the gravitational constant and the A1 through A6 are the only parameters which affect the dynamics of the orbit.

After each Q matrix there is a printout of SIN (THETA) and COS (THETA) where THETA is the central angle subtended by the vehicle from T = 0 to the time corresponding to the Q matrix.

The next printout is the information matrix corresponding to range-rate measurements. This matrix has the same dimensions as the input requirement. The title of the matrix is the name of the tracking station, the data set number, the term "R Rate" to indicate that it is an information matrix corresponding to range-rate, and finally the time at the termination of tracking (TJ). One of these matrices will exist for each TJ input, and if none exist, due to a zero input for S D IND., then a comment to that effect will appear.

Essentially the same sort of thing is repeated for Range, Angle 1, and Angle 2.

The terms of the information matrices are

$$\sum_{k=1}^{N} \frac{\partial M(t_k)}{\partial a_i} \cdot \frac{\partial M(t_k)}{\partial a_j}$$

where $M(t_k)$ is the value of the measurable at t_k and N is the total number of increments taken. a_i or a_j are the orbit parameters A1 through A6 followed in order by the bias, the station location errors (x'', y'', z''), the speed of light, the gravitational constant, and finally the error in the clock rate.

Inputs to Program B

The basic task of Program B is to add information matrices from one or more sources (Program A or a priori sources) weighted separately by known variances in each measurable, and finally to invert the total to form a covariance matrix. The resulting inverted matrix is an output referenced to T= 0. Also printed out is a similar output referenced to a later time. The reason for separate programs, A and B, is now clearly seen to be the fact that many outputs of Program A may be used in various combinations in Program B.

Figure 2 is a sample Input Data Sheet for Program B. The inputs are explained here in order.

The Data Set Number is similar to that of Program A. All Data Set numbers are consecutive and no distinction is made between Programs A and B. The results of Program B are logged on tape and the Data Set number is used for future retrieval. This will be discussed later in more detail. The next input is the Case Number. Case Numbers are consecutive starting with "one" for each Data Set. Each Case Number designates a covariance matrix. Within a Case Number might be many stations (results of various Program A's), but all of the input data within a Case Number is combined into a total matrix which is inverted and printed out.

In Figure 2 the first example shown (Case 1) uses three tracking stations. Each tracking station is retrieved from tape by calling out its Data Set number, the appropriate tracking duration (TJ) within

Station: Program B

Data Set | 0 | 1 | 0 | 1 |

Case Number	Data Set (4 digits)	Old Case No. or TJ	Measur-able	σ	Bias Indicator
1	0054	10	R	.6	B/7
	0054	58.05	Q		
	0055	10	R	.6	B/8 x''/9 y''/10 z''/11
	0056	10	R	.6	B/12 x''/13 y''/14 z''/15
	0093		A		
2	0054	20	R	.6	B/7
	0054	58.05	Q		
	0055	20	R	.6	B/8 x''/9 y''/10 z''/11
	0056	20	R	.6	B/12 x''/13 y''/14 z''/15
	0093		A		
3	0082	1	P		
	0085		A		
	0038	15.83	Q^T		

Remarks:

Figure 2. Example Input Data Sheet for Program B

that Data Set and the kind of measurable. Thus, in the example, the first tracking station (Data Set 0054) tracked until 10 minutes. The form of tracking is called out by the input, "Measurable." Since Program A handles range, range-rate, and two angles, these must be differentiated in the input to Program B. Thus under Measurable might appear: R, $\dot{R}$, ϕ_1 or ϕ_2. It goes without explanation that any combination of TJ and Measurable called out must exist in Program A.

The next input is the σ that should be associated with a particular measurable. Here the units are R in meters, $\dot{R}$ in meters/sec., and ϕ_1 or ϕ_2 in radians. In the example shown the first station (Data Set 0054) is measuring range with a standard deviation of 0.6 meters per sighting.

The Bias Indicator input handles bias errors in the measurable station location errors, etc. A separate row and column of the total information matrix must be used for each bias in a measurable from each station for each component of station location error from each station, for each clock rate error, and so on. In the example shown three stations are tracking in range, and a bias error is assumed to exist in range for each station. The first station location (Data Set 0054) is assumed to be known exactly, while the second and third are assumed to have station location errors in all three components (x'', y'', and z''). Thus the appropriate Bias Indicator input for the first station is B/7. This indicates that the seventh row and column of the total information matrix is confined to information regarding the bias error in the measurable (range) from the first station. It should be remembered that the first 6 x 6 elements of the total information matrix are taken up with the parameters of the orbit, and thus all other parameters to be estimated (biases and station location errors) must start with the seventh row and column. In the first example, the second station location is not known perfectly, and there exists a bias in the measurable; thus the Bias Indicator entry is B/8, x''/9, y''/10, z''/11. The size of the covariance matrix is determined by these entries. In Data Set 0101 Case 1, the covariance matrix is 15 x 15.

Two other inputs are shown in the first case. A Q matrix is called out as a measurable for the first station. Since more than one Q matrix might exist within Data Set 0054, the time of the Q matrix, in this case 58.05 minutes, is entered under TJ. This input allows the product $Q(T_1)Cov(T = 0)\ Q^T(T_1)$ to be formed where Cov(T = 0) is the inverse of the total information matrix. The second input is an a priori information matrix. This matrix is denoted by a Data Set number and an "A" in the measurable input. It differs from other information matrices in that it already contains an assumed σ. That is, while other information matrices have unity assumed variance and must be multiplied by some σ^2, the a priori matrix is simply added to the total information matrix as is.

Figure 3 shows an input data sheet for a priori inputs. Only the non-zero elements are called out: A_{11}, A_{12}, ..., etc. Any element which is not called out is assumed to be zero. For the type of program explained in the above example, each term is an element of a matrix of the form

$$A = \left[\rho_{ij}\sigma_i\sigma_j\right]^{-1}$$

where σ_i and σ_j are the a priori standard deviations of the ith and j^{th} parameters and ρ_{ij} is the a priori known correlation between terms. It should be remembered that the a priori information may contain terms involving biases and station location errors. Thus, if a station location (say Data Set 0055) is known to: $\sigma_{x''} = 10$ meters, $\sigma_{y''} = 65$ meters, and $\sigma_{z''} = 70$ meters, then the inverse of the squares of these quantities should be entered as a priori elements; in this case in elements $A_{9,9}$, $A_{10,10}$, and $A_{11,11}$.

The second case shown in Figure 2 shows how easily results may be obtained for various tracking times. In this case, only the TJs of the tracking stations have been changed from 10 minutes to 20 minutes.

Station: A priori

Data Set ☐☐☐☐

A_{11} =

A_{22} =

A_{33} =

A_{44} =

A_{55} =

A_{66} =

.

.

.

.

etc.

REMARKS:

Figure 3. Example A priori Input Data Sheet

The output of Program B which is stored on tape is $Q[\ ]^{-1}Q^T$ where the term inside of the brackets is the sum of all of the input information and a priori matrices. This stored data may, in turn, be used as an input to a following Program B. This allows tracking data on either side of a boost to be optimally combined. An example of this is shown in Case No. 3 of Figure 2. Here the result of tracking prior to a boost is recalled from a previous Program B by its Data Set Number, and old Case No. and a "P" in the measurable input to indicate that the input is the result of a previous Program B. This input is in the form of a covariance matrix and thus the a priori matrix which is added to it (Data Set 0085) must be of the same form and should correspond to the errors caused by boost uncertainty. The result of Program B is always of the form $Q[\ ]^{-1}Q^T$ and thus the output of the above example will be an information matrix. In this example instead of entering a Q matrix, the last input, Data Set 0038, is Q^T indicating the transpose of the Q matrix from Data Set 0038 (at 15.83 minutes). This has the effect of returning the reference from the time of the output of Data Set 0082 to zero time. The output is $Q^T[\ ]^{-1}Q$ instead of $Q[\ ]^{-1}Q^T$. Thus the resulting output of Case 3 is an information matrix just subsequent to a boost, but referenced to time zero. This result may be used as a priori information in a latter Program B which includes tracking after the boost. It is not possible to reference a data set within the data set being computed; thus three different data sets are required to track, boost, and track again.

If no Q matrix is specified in Program B, the output is simply the inverse of the weighted sum of the input matrices.

Output of Program B

The first output of Program B is a print-out of the input. A sample of this for the example Data Input Sheet (Figure 2) is shown below.

						6 x 6	BI	XS	YS	ZS	VL	GR	CL
Data Set	101	Case 5											
Data Set	54	Range	Time =	10.	SIGMA = .6		7	0	0	0	0	0	0
Data Set	54	Q	Time =	58.05		*							
Data Set	55	Range	Time =	10.	SIGMA = .6		8	9	10	11	0	0	0
Data Set	56	Range	Time =	10.	SIGMA = .6		12	13	14	15	0	0	0
Data Set	93	A	Time =	-0.			0	0	0	0	0	0	0

The first line gives the Data Set and Case Number. Also on the first line are descriptions of what is put into each row and column of the total information matrix before inversion. The first 6 x 6 elements of each contributing Data Set are followed by the seventh column, BI, indicating the bias information in the measurable. Following this is information regarding the three components of station location error in columns 8, 9, and 10. Column 11 is reserved for the velocity of light information (VL), followed by gravitational constant (GR) in column 12 and clock rate error (CL) in column 13.

After skipping a line, all of the Data Set inputs follow. The first three read as follows: "Old Data Set 0054 is Range information with TJ = 10 minutes. The range sigma is .6 meters. The first 6 x 6 elements are to be added directly to their corresponding elements in the total information matrix. The range bias information in Data Set 0054 column 7 is to be added to the total information matrix in column 7. Old Data Set 0054 has a Q matrix to be used which is calculated for 58.05 minutes after zero time. The * signifies that only the first 6 x 6 elements of this matrix are to be retained. Old Data Set 0055 is a Range matrix also with TJ= 10 minutes. Its bias terms (column 7) are to be placed in the eighth column of the total matrix and its 8th, 9th, and 10th columns,

corresponding to x", y", and z" station location errors, are to be placed in the 9th, 10th and 11th columns of the total information matrix."

Following the printout of the input data a few lines are skipped and the total matrix prior to inversion is printed. Following this a check of the matrix inversion is printed. This is $C\,C^{-1}$ and should ideally have "ones" down the main diagonal and zeros elsewhere.

Next appears the inverse itself. This is the inverse of the weighted sum of the information matrices, including a priori data.

The next matrix printed out is $Q[\ ]^{-1}Q^T$ which is usually the covariance matrix at some particular time of interest (the time corresponding to the Q matrix). This last matrix may be an information matrix when the inputs to Program B are "P's."

Following the $Q\ Cov\ Q^T$ matrix is the Rendezvous Miss, which is nothing more than

$$\text{REND MISS} = \left|P_{11} + P_{22} + P_{33}\right|^{1/2}$$

where the P's are terms of the $Q\ Cov\ Q^T$ matrix. This is a good estimate of the total positional uncertainty at the time of the Q matrix.

Next the square roots of the diagonal terms of $Q\ Cov\ Q^T$ are tabulated.

Finally there is a correlation matrix in which all diagonal terms are, of course, unity and the off-diagonal terms are ρ_{ij}, the correlation between the errors in the ith and j^{th} parameters.

Sample Problem and Results

Three Stations Tracking During an Earth Parking Orbit

This problem consists of three stations - Bermuda, Carnarvon, and Guaymas - each tracking a vehicle in a 200 kilometer circular Earth orbit. The vehicle is launched Eastward from Cape Kennedy, and burnout (and the initiation of tracking, T = 0) occurs at 27°

North Latitude and 60.5° West Longitude. The orbit is inclined to the Equator by 28.5 degrees. For the particular case presented herein, each radar has identical characteristics and tracks in range and x-y angles with the following precision for one second uncorrelated samples.

$$\sigma R = 15 \text{ meters}$$

$$\sigma \phi_1 = .8 \times 10^{-3} \text{ radians}$$

$$\sigma \phi_2 = .8 \times 10^{-3} \text{ radians}$$

No biases are assumed in any of these measurables, but each station is assumed to be located to the following a priori precision:

$$\left.\begin{array}{l} x'' = 10 \text{ meters} \\ y'' = z'' = 65.4 \text{ meters} \end{array}\right\} \text{A priori uncertainty for each station}$$

where x'' is perpendicular to the Earth's axis of rotation and z'' is North. The start of tracking for the Bermuda Station is at T= 0 (i.e. booster burnout) and the 5° horizon terminates tracking at 0.85 min. Carnarvon starts tracking at T= 42.60 minutes and stops at T= 48.02 minutes. Guaymas times are correspondingly 78.88 and 84.38 minutes.

Since three stations exist, three Program A's must be used; however, for simplicity only the Guaymas Station will be shown. The Data Set Numbers used for the three Program A's are:

Bermuda	=	Data Set 0152
Carnarvon	=	Data Set 0154
Guaymas	=	Data Set 0160

The input data sheet for the Guaymas Station is shown below:

Station: Guaymas

Data Set: 0 1 6 0

01	A 1	$.65781 \times 10^7$	m	Initial orbit radius
02	A 4	0.	m/sec	" " radial velocity
03	A 5	$.7795 \times 10^4$	m/sec	" " tangent velocity
04	GAM ID	1.		+ 1 if $\lvert\gamma\rvert < \frac{\pi}{2}$, - 1 otherwise
05	BETA	$.285 \times 10^2$	deg	Inclination of Earth-Moon orbit to Equator
06	XI	0.	deg	
07	ETA	0.	deg	Euler angles relating x' y' z'
08	ZETA	0.	deg	coordinates to $\tilde{x}\ \tilde{y}\ \tilde{z}$
09	LAMBDA	$.2806 \times 10^2$	deg	coordinates
10	ALPHA	$-.1109 \times 10^3$	deg	Station latitude
11	OMEGA E	$.729116 \times 10^{-4}$	rad/sec	" longitude
12	OMEGA M	0.	rad/sec	Earth angular rate
13	RHO E	$.63781 \times 10^7$	m	Earth-Moon angular rate
14	RHO M	0.	m	Radius of Earth
15	MU	$.399689 \times 10^{15}$	m^3/sec^2	Earth-Moon distance
18	BIG L	$.27 \times 10^2$	deg	Gravitational constant
19	SMA L	$.605 \times 10^2$	deg	Latitude of sublunar point at T = 0
20	SD IND	1.		Longitude of sublunar point at T = 0
21	S IND	1.		If 1, calculate information for $\dot{R}$
22	PS 1 IND	1.		" " " R
25	PS 2 IND	1.		" " " ϕ_1
26	PVW IND	1.		" " " ϕ_2
16	TJ	$.8438 \times 10^2$	min	, x-y mount. Maximum times of observation
23	Q IND	1.		If 1, calculate Q for all TJ
24	QF	0.	min	Values of time for which other Q wanted
27	TI (initial)	$.7938 \times 10^2$	min	Time of first observation
28	TI (interval)	.50	min	Interval between observations
29	DIMENSIONS	13		Desired size of inf. matrix

REMARKS: Earth Orbit

NOTE: Adjust σ for $\frac{1}{2}$ minute interval $(1/\sqrt{30})$

Since Program A computes as though a vehicle were in orbit about the Moon which in turn is in a circular orbit about the body containing the tracking stations (Earth), it is necessary to make the lunar coordinates coincide with the Earth-centered inertial coordinates for this problem. This is done simply by setting the Earth-Moon distance and angular rate equal to zero. By equating the inclination of the Earth-Moon orbit to that of the vehicle's orbit ($\beta = 28.5^{\circ}$) and by placing the sub-lunar point at the location of the vehicle at T = 0 (BIG L, SMA L) and by setting XI, ETA, and ZETA equal to zero, it is possible to convert the program to one which handles Earth orbital vehicles. The dynamic parameters A1, A4, and A5 are not affected. GAM ID is "1" since γ is less than 90°. All information matrices are to be used during some cases (although only R, ϕ_1 and ϕ_2 are used in this case) and thus SD, S, PS 1 and PS 2 indicators are set equal to "1." An x-y mount is assumed and thus PVW IND. = 1. The time interval is to be lengthened from 1.0 second to 0.5 minutes to save computer time. Thus TI (interval) is set equal to 0.5 min. and in Program B the measurable, σR, must be changed from 15 meters to 2.74 meters ($= 15\sqrt{1/30}$) and $\sigma\phi$ must be changed from 8×10^{-4} radians to 1.46×10^{-4} radians

$$(= 8 \times 10^{-4}\sqrt{1/30}).$$

The initial measurement is taken one interval after the initiation of tracking (78.88 min.) and thus TI (initial) is set equal to 79.38 minutes. Since station location errors and errors in other parameters may be of interest, a 13 x 13 information matrix is commanded.

The next input is an a priori information matrix used for the initial station location variances before tracking. In Program B the x", y", z" station location errors are parameters which will be estimated in the 7th, 8th, and 9th column and row for the first station (Bermuda) and in the 10th, 11th, and 12th positions for the second station, and in the 13th, 14th and 15th positions for the third station. Assuming a priori knowledge of the diagonal elements will give:

$$A_{7,7} = A_{10,10} = A_{13,13} = 1/10^2$$

$$A_{8,8} = A_{11,11} = A_{14,14} = 1/65.4^2$$

$$A_{9,9} = A_{12,12} = A_{15,15} = 1/65.4^2$$

However the format for an a priori input can only handle a 13 x 13 matrix. Thus only $A_{7,7}$ through $A_{13,13}$ are used as inputs, and a transfer is made in Program B by using two references to Data Set 0161.

Program B, Data Set 0165, Case 4 combines the three stations (Data Sets 0152, 0154, 0160) with the a priori information matrix (Data Set 0161). The measurables are range and two angles with the same radar accuracies from each station. The actual inputs for the various σ's have been weighted due to varying compression of the data samples taken, as explained previously. In the "Bias Indicator" column the first three lines state that the data regarding the x", y", z" components of the first station location errors are to be added and placed in the seventh, eighth, and ninth row and column of the total information matrix. The data regarding the other two stations is handled in a similar manner. The Q matrix from Data Set 0160 is to be reduced to a 6 x 6 (indicated by the *), and the first a priori matrix, Data Set 0161, is to be left alone, as is indicated by no entry in the Bias Indicator column. Actually, the desired a priori matrix is a 15 x 15, while the largest input allowable by the format is 13 x 13. This is handled by a simple transfer. Thus since the 8th and 9th diagonal terms of the derived a priori matrix are the same as the 14th and 15th, the Data Set is called in again and through the use of the bias indicator column, the appropriate terms are transferred.

This completes the required inputs for this first sample problem. The following is a discussion of facsimiles of the actual computer output (see Appendix A).

Station: Program B

Data Set | 0 | 1 | 6 | 5 |

Case Number	Data Set (4 digits)	Old Case No. or TJ	Measurable	σ	Bias Indicator
4	0152	0.85	R	4.34	x''/7 y''/8 z''/9
			ϕ_1	2.31×10^{-4}	same
			ϕ_2	2.31×10^{-4}	same
	0154	48.02	R	2.76	x''/10 y''/11 z''/12
			ϕ_1	1.47×10^{-4}	same
			ϕ_2	1.47×10^{-4}	same
	0160	84.38	R	2.74	x''/13 y''/14 z''/15
			ϕ_1	1.46×10^{-4}	same
			ϕ_2	1.46×10^{-4}	same
			Q		*
	0161		A		
			A		x''/14 y''/15

The first output of Program A is the input data (see page A-1) followed by the Q matrix at 84.38 minutes (see page A-2). Immediately after the Q matrix printout are sin 0 and cos 0 where 0 is the central angle traveled by the vehicle between T = 0 and T = 84.38 minutes. The information matrices for R, $\dot{R}$, ϕ_1 and ϕ_2 measurables follow on pages A-3 through A-6.

The a priori information matrix is printed out in its order (Data Set No. 0161 follows 0160 which was Program A for Guaymas). Only the non-zero terms are printed out. (page A-7).

The printout for Program B is shown starting on page A-8 with the input data. Following this is the total matrix prior to inversion, page A-9. Unfortunately, this facsimile did not quite fit the page. The

checkout of the inversion is shown on page A-10, and it can be seen that the inversion was quite accurate since the largest off-diagonal element is on the order of about 10^{-7}. The inverse or covariance matrix at T= 0 is on page A-11, and the covariance matrix at T= 84.38 minutes (at the termination of tracking over Guaymas) is on page A-12. The RMS velocity at T = 84.38 minutes is the square root of the fourth plus fifth plus sixth diagonal terms (0.0365 meters/sec) while the RMS uncertainty in position is the square root of the first plus second plus third diagonal terms and is printed out on page A-13 as REND MISS. Also on A-13 is the correlation matrix.

Instructions for Preparation of Data and Program Execution

General:

The program is set up to run on an IBM 7090 or 7094, under a FORTRAN II monitor system. Core storage must be at least 32,768 words. Magnetic tape A-2 is the standard input tape, the standard output tape is A-3 (for subsequent listing off-line), and tape A-5 is used to store the results of Programs A and B for subsequent use, if desired. For a machine run, it is necessary only to prepare an input deck of cards according to the specifications below and place this deck following the program deck and a FORTRAN "DATA" card.

Programs A and B may be employed in any sequence, subject only to the restriction that all data accessed using Program B must have previously been stored on tape A-5 (in either Program A or B).

Program A Input Data Cards:

Program A is composed of two parts. One calculates the partials of the measurables and stores these information matrices on magnetic tape. The other is used to store a priori data on magnetic tape. Figure 4 shows examples of input for each part, and references in the following paragraphs will be to that figure.

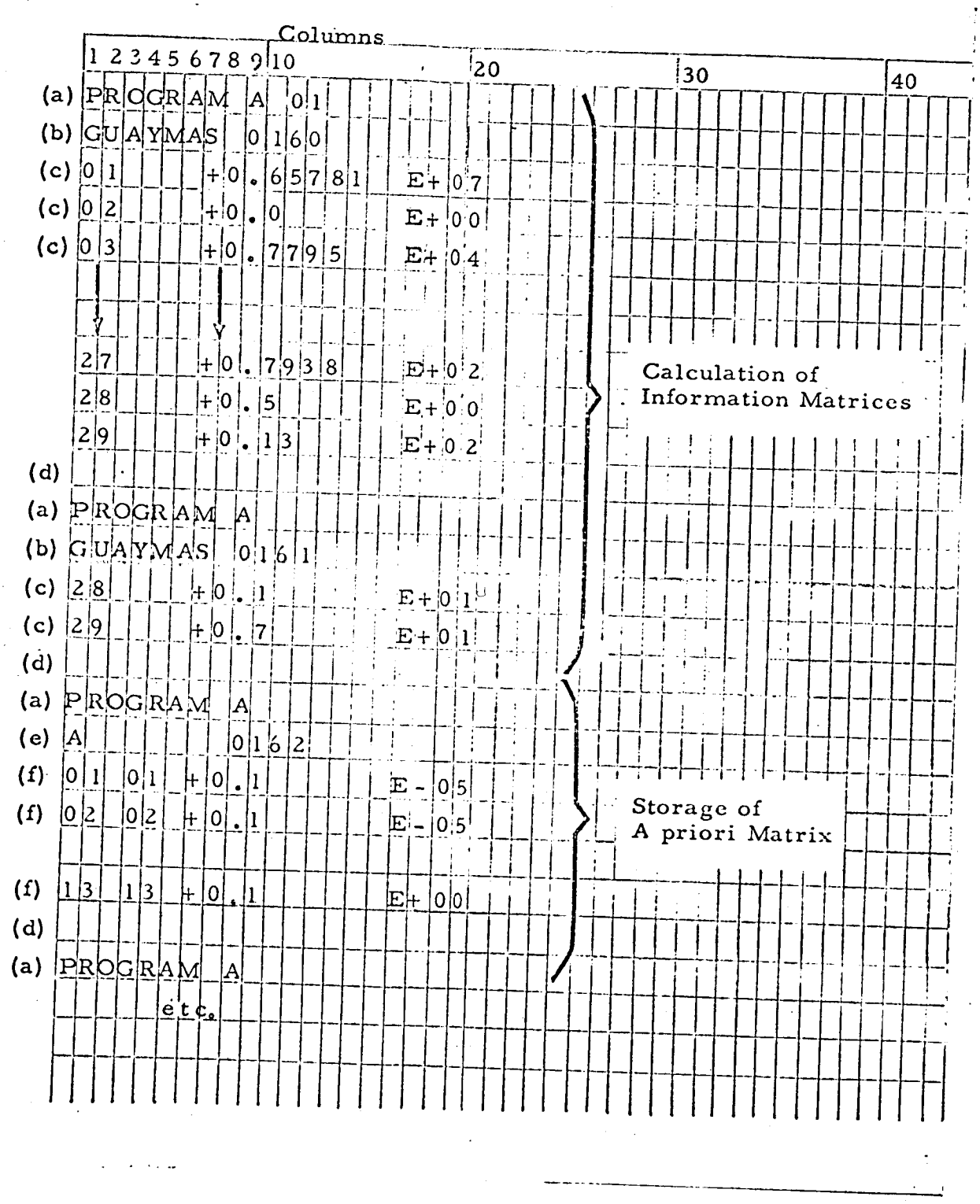


Figure 4

Sample Card Input Deck for Program A

Program A - Partials:

The first card must contain PROGRAM A in columns 1 - 9. Columns 11 - 12 may be blank or contain 01 (see cards (a)). If blank, the results of these calculations will be stored on tape following the results of any previous data sets already on the tape. If 01, the tape will be rewound and the results stored as the first block(s). Thus, an 01 must be used when (1) data is to be stored on the tape for the first time, or (2) it is desired to discard all previous results and start a new tape. The second card contains the station name in columns 1 - 8 and the data set in columns 9 - 12 (see cards (b)). Following this card are the input quantities such as orbit parameters and physical constants (see cards (c)), one input per card. Columns 1 - 2 contain the integer identifying the input quantity, and columns 7 - 20 contain the quantity itself, in floating point exponential format (± x.xxxxxxxE±xx). Read in of the input parameters is terminated by a blank card (see cards (d)). If Program A is used more than once in any computer run, for the second and subsequent data sets, it is not necessary to input all 29 quantities (cards (c)). Only those quantities which are changed over the preceding data set need be entered.

Special Note: Since multiple entries are allowed for observation time(s) (TJ: input number 16), whenever any any TJ change is entered, the entire preceding table of TJ's is destroyed.

Program A - A priori Data:

The first card must be a PROGRAM A card (see above). The second card must contain an "A" in column 1, and the data set in columns 9 - 12 (see card (e)). Following are cards containing the elements of the a priori matrix, one element per card (see cards (f)). Columns 1 - 2 contain the row number, columns 4 - 5 contain the column number, and columns 7 - 20 contain the element itself, in floating point exponential format (±x.xxxxxxxE±xx). Only non-zero elements need be entered. Read in of the elements is terminated by a blank card (see card (d)).

If the same procedure is followed except that a "Q" is placed in column 1 of the second card, instead of an "A," the program will store the desired matrix on tape labeled as a Q matrix. This capability is sometimes useful when employing the program in non-standard ways.

Program B Input Data Cards:

References in the following paragraph are to Figure 5. The first card of each group must contain PROGRAM B in columns 1 - 9 (see cards (a)). The second card must contain the data set in columns 1 - 4, and the case number (within data set) in columns 11 - 12 (see cards (b)). Following this are the card(s) which specify the data to be accessed on magnetic tape (see cards (c)). The card format is as follows:

Columns	1 - 4	Data set	
"	6 - 7	Case number (if any). Data on tape resulting from Program A calculations will have no case number.	
"	11 - 15	Measurable. These columns must contain one of the following:	
	RANGE	=	range data
	R RATE	=	range rate data
	ANG 1	=	angle 1 data
	ANG 2	=	angle 2 data
	Q	=	Q matrix
	A	=	A priori matrix
	P	=	Previously stored result from Program B.
"	18 - 31	Observation time, in floating point exponential format (±x.xxxxxxxE±xx). No entry is required in this field when the measurable field is either A or P.	

Columns	34 - 47	Sigma (σ) in floating point exponential format. No entry is required in this field when the measurable field is either A, Q, or P. In these cases, sigma is assumed to be 1.0.

The remainder of the card is used to specify the positioning in the final object matrix, to be inverted, of the rows and columns of each individual matrix. The fields used are:

Column(s)	50	6 x 6 indicator
"	53 - 54	Bias
"	56 - 57	Station location x error
"	59 - 60	" " y "
"	62 - 63	" " z "
"	65 - 66	Speed of light
"	68 - 69	Gravitational constant
"	71 - 72	Clock rate error

Positioning of the rows and columns resolves down to three distinct cases:

(1) The entire matrix (regardless of size) is to be used without adjustment. This is accomplished by leaving <u>all</u> fields blank.

(2) Only the first 6 rows and columns of the matrix are to be used. This is accomplished by a 6 in column 50.

(3) Some positioning of rows and columns > 6 is required. For this case column 50 is left blank, and the row-column position in the object matrix must be specified in the appropriate field of the card for <u>each</u> row-column. A blank field will result in that row-column being disregarded. For example, see Figure 5, Data Set 0152 Range matrix:

Information Matrix								Object Matrix
Row-column		8	(stat.	loc.	x	error)		Goes to row-column 7
"	"	9	("	"	y	"	)	Goes to row-column 8
"	"	10	("	"	z	"	)	Goes to row-column 9

All other rows and columns > 6 make no contribution to the object matrix. The process of accessing matrices from tape and adding to the final matrix is continued until a blank card is encountered (see cards (d)).

Columns

	1 2 3 4 5 6 7 8 9	10	20	30	40	50	60	70
						BI XS YS ZS VL GR CL		
(a)	PROGRAM B							
(b)	0165	04						
(c)	0152	RANGE	+0.85	E+00	+0.434	E+01	07 08 09	
(c)	0152	ANG 1	+0.85	E+00	+0.231	E-03	07 08 09	
(c)	0154	RRATE	+0.4802	E+02	+0.656	E+01	6	
(c)	0154	ANG 2	+0.4802	E+02	+0.147	E-03	10 11 12	
(c)	0160	RANGE	+0.4802	E+02	+0.276	E+01	10 11 12	
(c)	0160	Q	+0.8438	E+02			6	
(c)	0161	A						
(d)								
(a)	PROGRAM B							
(b)	0166	01						
(c)	0165 04	P						
(c)	0160	Q	+0.8438	E+02			6	
(d)								
(a)	PROGRAM B							
(b)	0166	02						
	etc.							

Figure 5

Sample Card Input Deck for Program B

APPENDIX A

FACSIMILES OF ACTUAL COMPUTER OUTPUT

(Corresponding to the
Sample Problem)

INPUT DATA

01	0.6578100E 07	A1
02	-0.	A4
03	0.7795000E 04	A5
04	0.1000000E 01	GAM ID
05	0.2850000E 02	BETA
06	-0.	XI
07	-0.	ETA
08	-0.	ZETA
09	0.2806000E 02	LAMBDA
10	-0.1109000E 03	ALPHA
11	0.7291160E-04	OMEGAE
12	-0.	OMEGAM
13	0.6378100E 07	RHOE
14	-0.	RHOM
15	0.3996890E 15	MU
16	0.8438000E 02	D TIME
18	0.2700000E 02	BIG L
19	-0.6050000E 02	SMA L
20	0.1000000E 01	SD IND
21	0.1000000E 01	S IND
22	0.1000000E 01	PS1IND
25	0.1000000E 01	PS2IND
23	0.1000000E 01	Q IND
24	-0.	QF IND
26	0.1000000E 01	PVWIND
27	0.7938000E 02	T INIT
28	0.5000000E 00	T INCR
29	0.1300000E 02	DIMENS

Q MATRIX 160 Q T = 84.38

-0.420366E 01	-0.112365E-01	0.	-0.246026E 03	-0.445772E 04	0.	0.
-0.181068E 02	0.961520E 00	0.	0.135615E 01	-0.155075E 05	0.	0.
0.	0.	0.959913E 00	0.	0.	-0.236544E 03	0.
0.207782E-01	0.138700E-03	-0.	0.107696E 01	0.180730E 02	-0.	0.
-0.611689E-02	0.305517E-03	-0.	-0.224730E-01	-0.431922E 01	-0.	0.
0.	0.	0.332148E-03	0.	0.	0.959913E 00	0.
0.	0.	0.	0.	0.	0.	0.100000E 01
0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.

0.	0.	0.	0.	0.573066E-07
0.	0.	0.	0.	0.198609E-06
0.	0.	0.	0.	0.
0.	0.	0.	0.	-0.229658E-09
0.	0.	0.	0.	0.678764E-10
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.100000E 01	0.	0.	0.	0.
0.	0.100000E 01	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.100000E 01	0.	0.
0.	0.	0.	0.100000E 01	0.
0.	0.	0.	0.	0.100000E 01

SIN(THETA) = -0.2803021E-00 COS(THETA) = 0.9599118E 00

GUAYMAS 160 RRATE T = 84.38

0.517860E 00	-0.232930E-01	-0.120006E-03	0.411017E 01	0.458167E 03	0.234147E-02	-0.120778E 01
-0.232930E-01	0.114601E-02	-0.403932E-05	-0.910844E-01	-0.204139E 02	0.397757E-02	0.527621E-01
-0.120006E-03	-0.403932E-05	0.959125E-06	-0.997269E-02	-0.124881E-00	-0.401830E-03	0.322321E-03
0.411017E 01	-0.910844E-01	-0.997269E-02	0.122111E 03	0.382161E 04	0.392003E 01	-0.109839E 02
0.458167E 03	-0.204139E 02	-0.124881E-00	0.382161E 04	0.405739E 06	0.101619E 02	-0.106911E 04
0.234147E-02	0.397757E-02	-0.401830E-03	0.392003E 01	0.101619E 02	0.173024E-00	-0.257869E-01
-0.120778E 01	0.527621E-01	0.322321E-03	-0.109839E 02	-0.106911E 04	-0.257869E-01	0.110000E 02
-0.151191E-02	0.246604E-03	-0.171145E-04	0.158605E-00	-0.983882E 00	0.752897E-02	0.263642E-02
0.289460E-01	-0.128974E-02	-0.790106E-05	0.241423E-00	0.256336E 02	0.645920E-03	-0.676000E-01
0.195535E-02	0.219188E-04	-0.111951E-04	0.120469E-00	0.194758E 01	0.464394E-02	-0.510656E-02
-0.363141E 03	0.182976E 03	-0.140004E 02	0.154864E 06	0.506978E 04	0.622257E 04	-0.102506E 03
-0.581758E-08	0.259434E-09	0.156687E-11	-0.483091E-07	-0.515143E-05	-0.120678E-09	0.135764E-07
-0.	0.	-0.	0.	-0.	0.	0.

-0.151191E-02	0.289460E-01	0.195535E-02	-0.363141E 03	-0.581758E-08	-0.
0.246604E-03	-0.128974E-02	0.219188E-04	0.182976E 03	0.259434E-09	0.
-0.171145E-04	-0.790106E-05	-0.111951E-04	-0.140004E 02	0.156687E-11	-0.
0.158605E-00	0.241423E-00	0.120469E-00	0.154864E 06	-0.483091E-07	0.
-0.983882E 00	0.256336E 02	0.194758E 01	0.506978E 04	-0.515143E-05	-0.
0.752897E-02	0.645920E-03	0.464394E-02	0.622257E 04	-0.120678E-09	0.
0.263642E-02	-0.676000E-01	-0.510656E-02	-0.102506E 03	0.135764E-07	0.
0.332970E-03	-0.620440E-04	0.196386E-03	0.280564E 03	0.128701E-10	0.
-0.620440E-04	0.161948E-02	0.123113E-03	-0.957698E 00	-0.325457E-09	-0.
0.196386E-03	0.123113E-03	0.131691E-03	0.170730E 03	-0.244970E-10	0.
0.280564E 03	-0.957698E 00	0.170730E 03	0.441033E 09	0.582899E-06	0.
0.128701E-10	-0.325457E-09	-0.244970E-10	0.582899E-06	0.654056E-16	0.
0.	-0.	0.	0.	0.	0.

GUAYMAS 160 RANGE T = 84.38

0.273992E 04	-0.122205E 03	-0.594819E 00	0.226232E 05	0.242388E 07	-0.102239E-00	0.466371E 01
-0.122205E 03	0.577214E 01	-0.151090E-01	-0.695126E 03	-0.107456E 06	0.172936E 02	-0.136921E 01
-0.594819E 00	-0.151090E-01	0.633969E-02	-0.460834E 02	-0.612156E 03	-0.253780E 01	0.223054E-00
0.226232E 05	-0.695126E 03	-0.460834E 02	0.493484E 06	0.206518E 08	0.170677E 05	-0.114291E 04
0.242388E 07	-0.107456E 06	-0.612156E 03	0.206518E 08	0.214563E 10	0.355274E 05	0.164439E 04
-0.102239E-00	0.172936E 02	-0.253780E 01	0.170677E 05	0.355274E 05	0.103880E 04	-0.893508E 02
0.466371E 01	-0.136921E 01	0.223054E-00	-0.114291E 04	0.164439E 04	-0.893508E 02	0.110000E 02
-0.837136E 01	0.108999E 01	-0.101044E-00	0.636837E 03	-0.593329E 04	0.422073E 02	-0.350773E 01
0.153154E 03	-0.678292E 01	-0.400628E-01	0.131177E 04	0.135587E 06	0.279345E 01	0.331555E-01
0.101196E 02	-0.114006E-01	-0.652900E-01	0.516920E 03	0.985618E 04	0.258910E 02	-0.212249E 01
0.217930E 07	-0.420009E 06	0.107289E 06	-0.340142E 09	0.116246E 10	-0.414452E 08	0.711599E 07
-0.307729E-04	0.136354E-05	0.796988E-08	-0.262932E-03	-0.272418E-01	-0.527866E-06	-0.856462E-08
0.749774E 04	-0.604624E 04	0.109995E 04	-0.574898E 07	-0.556838E 07	-0.439285E 06	0.540408E 05

-0.837136E 01	0.153154E 03	0.101196E 02	0.217930E 07	-0.307729E-04	0.749774E 04	
0.108999E 01	-0.678292E 01	-0.114006E-01	-0.420009E 06	0.136354E-05	-0.604624E 04	
-0.101044E-00	-0.400628E-01	-0.652900E-01	0.107289E 06	0.796988E-08	0.109995E 04	
0.636837E 03	0.131177E 04	0.516920E 03	-0.340142E 09	-0.262932E-03	-0.574898E 07	
-0.593329E 04	0.135587E 06	0.985618E 04	0.116246E 10	-0.272418E-01	-0.556838E 07	
0.422073E 02	0.279345E 01	0.258910E 02	-0.414452E 08	-0.527866E-06	-0.439285E 06	
-0.350773E 01	0.331555E-01	-0.212249E 01	0.711599E 07	-0.856462E-08	0.540408E 05	
0.174641E 01	-0.353629E-00	0.102485E 01	-0.153641E 07	0.724538E-07	-0.171989E 05	
-0.353629E-00	0.856847E 01	0.635788E 00	0.250441E 05	-0.172155E-05	-0.699641E 03	
0.102485E 01	0.635788E 00	0.685119E 00	-0.926620E 06	-0.126888E-06	-0.104924E 05	
-0.153641E 07	0.250441E 05	-0.926620E 06	0.565103E 13	-0.526943E-02	0.349581E 11	
0.724538E-07	-0.172155E-05	-0.126888E-06	-0.526943E-02	0.345889E-12	0.131206E-03	
-0.171989E 05	-0.699641E 03	-0.104924E 05	0.349581E 11	0.131206E-03	0.265591E 09	

GUAYMAS 160 ANGLE 1 T = 84.38

0.777635E-07	-0.212987E-08	-0.191595E-07	0.169431E-05	0.657094E-04	0.770531E-05	0.403974E-02
-0.212987E-08	0.684807E-09	0.296998E-09	0.556561E-06	-0.646571E-06	0.175001E-06	-0.482646E-04
-0.191595E-07	0.296998E-09	0.519161E-08	-0.639823E-06	-0.166840E-04	-0.222506E-05	-0.109843E-02
0.169431E-05	0.556561E-06	-0.639823E-06	0.617285E-03	0.254217E-02	0.540963E-03	0.148712E-00
0.657094E-04	-0.646571E-06	-0.166840E-04	0.254217E-02	0.576610E-01	0.725800E-02	0.354393E 01
0.770531E-05	0.175001E-06	-0.222506E-05	0.540963E-03	0.725800E-02	0.109376E-02	0.477348E-00
0.403974E-02	-0.482646E-04	-0.109843E-02	0.148712E-00	0.354393E 01	0.477348E-00	0.232840E 03
-0.124865E-07	0.138037E-08	0.284417E-08	0.726177E-06	-0.866775E-05	-0.667203E-06	-0.575355E-03
0.200473E-23	0.588371E-24	-0.852773E-24	0.663841E-21	0.290246E-20	0.656413E-21	0.194165E-18
0.177681E-07	0.541002E-09	-0.517934E-08	0.137971E-05	0.169886E-04	0.260092E-05	0.111403E-02
0.	-0.	-0.	-0.	0.	0.	0.
-0.839628E-15	0.805473E-17	0.213179E-15	-0.326823E-13	-0.737152E-12	-0.928310E-13	-0.452875E-10
0.	-0.	-0.	-0.	0.	0.	0.

-0.124865E-07	0.200473E-23	0.177681E-07	0.	-0.839628E-15	0.
0.138037E-08	0.588371E-24	0.541002E-09	-0.	0.805473E-17	-0.
0.284417E-08	-0.852773E-24	-0.517934E-08	-0.	0.213179E-15	-0.
0.726177E-06	0.663841E-21	0.137971E-05	-0.	-0.326823E-13	-0.
-0.866775E-05	0.290246E-20	0.169886E-04	0.	-0.737152E-12	0.
-0.667203E-06	0.656413E-21	0.260092E-05	0.	-0.928310E-13	0.
-0.575355E-03	0.194165E-18	0.111403E-02	0.	-0.452875E-10	0.
0.378040E-08	0.697514E-24	-0.130989E-08	-0.	0.110382E-15	-0.
0.697514E-24	0.	0.165186E-23	-0.	0.	-0.
-0.130989E-08	0.165186E-23	0.621839E-08	0.	-0.217327E-15	0.
-0.	-0.	0.	0.	-0.	0.
0.110382E-15	0.	-0.217327E-15	-0.	0.942401E-23	-0.
-0.	-0.	0.	0.	-0.	0.

GUAYMAS 160 ANGLE 2 T = 84.38

0.127411E-07	-0.543580E-09	0.948757E-10	0.126188E-06	0.112492E-04	-0.384116E-07	-0.243531E-03
-0.543580E-09	0.277934E-10	-0.414230E-11	-0.942436E-09	-0.471037E-06	0.144121E-08	0.929302E-05
0.948757E-10	-0.414230E-11	0.216563E-11	0.810332E-09	0.826692E-07	-0.850411E-09	-0.475176E-05
0.126188E-06	-0.942436E-09	0.810332E-09	0.553648E-05	0.120021E-03	-0.556210E-06	-0.339144E-02
0.112492E-04	-0.471037E-06	0.826692E-07	0.120021E-03	0.994934E-02	-0.339354E-04	-0.215236E-00
-0.384116E-07	0.144121E-08	-0.850411E-09	-0.556210E-06	-0.339354E-04	0.368960E-06	0.194702E-02
-0.243531E-03	0.929302E-05	-0.475176E-05	-0.339144E-02	-0.215236E-00	0.194702E-02	0.110000E 02
0.423472E-11	0.932447E-11	-0.188185E-12	0.921578E-08	0.221468E-07	-0.411683E-09	-0.226188E-05
0.683840E-09	-0.286539E-10	0.476585E-11	0.728437E-08	0.604986E-06	-0.196083E-08	-0.125342E-04
-0.554916E-10	0.748638E-11	-0.240846E-11	0.443993E-08	-0.379004E-07	0.698592E-09	0.370615E-05
-0.	0.	-0.	0.	-0.	0.	0.
-0.142909E-15	0.598175E-17	-0.105100E-17	-0.152688E-14	-0.126405E-12	0.431522E-15	0.273659E-11
-0.	0.	-0.	0.	-0.	0.	0.

0.423472E-11	0.683840E-09	-0.554916E-10	-0.	-0.142909E-15	-0.
0.932447E-11	-0.286539E-10	0.748638E-11	0.	0.598175E-17	0.
-0.188185E-12	0.476585E-11	-0.240846E-11	-0.	-0.105100E-17	-0.
0.921578E-08	0.728437E-08	0.443993E-08	0.	-0.152688E-14	0.
0.221468E-07	0.604986E-06	-0.379004E-07	-0.	-0.126405E-12	-0.
-0.411683E-09	-0.196083E-08	0.698592E-09	0.	0.431522E-15	0.
-0.226188E-05	-0.125342E-04	0.370615E-05	0.	0.273659E-11	0.
0.196523E-10	0.131979E-11	0.105955E-10	0.	-0.285964E-18	0.
0.131979E-11	0.368364E-10	-0.198350E-11	-0.	-0.768573E-17	-0.
0.105955E-10	-0.198350E-11	0.840048E-11	0.	0.480060E-18	0.
0.	-0.	0.	0.	0.	0.
-0.285964E-18	-0.768573E-17	0.480060E-18	0.	0.160589E-23	0.
0.	-0.	0.	0.	0.	0.

A PRIORI 161

```
A( 7, 7)  =   1.0000000E-02
A( 8, 8)  =   0.2340000E-03
A( 9, 9)  =   0.2340000E-03
A(10,10)  =   1.0000000E-02
A(11,11)  =   0.2340000E-03
A(12,12)  =   0.2340000E-03
A(13,13)  =   1.0000000E-02
```

SET	165	CASE 4					6X6	BI	XS	YS	ZS	VL	GR	CL
SET	152	RANGE	TIME =	0.85	SIGMA =	0.4340000E 01		0	7	8	9	0	0	0
SET	152	ANG 1	TIME =	0.85	SIGMA =	0.2310000E-03		0	7	8	9	0	0	0
SET	152	ANG 2	TIME =	0.85	SIGMA =	0.2310000E-03		0	7	8	9	0	0	0
SET	154	RANGE	TIME =	48.02	SIGMA =	0.2760000E 01		0	10	11	12	0	0	0
SET	154	ANG 1	TIME =	48.02	SIGMA =	0.1470000E-03		0	10	11	12	0	0	0
SET	154	ANG 2	TIME =	48.02	SIGMA =	0.1470000E-03		0	10	11	12	0	0	0
SET	160	RANGE	TIME =	84.38	SIGMA =	0.2740000E 01		0	13	14	15	0	0	0
SET	160	ANG 1	TIME =	84.38	SIGMA =	0.1460000E-03		0	13	14	15	0	0	0
SET	160	ANG 2	TIME =	84.38	SIGMA =	0.1460000E-03		0	13	14	15	0	0	0
SET	160	Q	TIME =	84.38	SIGMA =	-0.	*							
SET	161	A	TIME =	-0.	SIGMA =	-0.								
SET	161	A	TIME =	-0.	SIGMA =	-0.		0	14	15	0	0	0	0

MATRIX PRIOR TO INVERSION

473528E 03	0.141748E 02	0.121178E-00	0.374542E 05	0.417116E 06	0.346394E 03	-0.368340E-01	-0.473704E-01
141748E 02	0.102100E 02	0.381049E-02	0.103405E 05	0.119259E 05	-0.273998E 02	-0.101294E-00	-0.137217E-0
121178E-00	0.381049E-02	0.573620E 00	0.955943E 02	0.285371E 03	-0.707732E 02	0.818678E-01	0.106594E-0
374542E 05	0.103405E 05	0.955943E 02	0.117929E 08	0.324692E 08	-0.100703E 05	-0.851109E 00	-0.125704E 01
417116E 06	0.119259E 05	0.285371E 03	0.324692E 08	0.367920E 09	0.354892E 06	-0.268842E 01	-0.411390E 01
346394E 03	-0.273998E 02	-0.707732E 02	-0.100703E 05	0.354892E 06	0.572240E 05	0.191607E 01	0.286991E 01
368340E-01	-0.101294E-00	0.818678E-01	-0.851109E 00	-0.268842E 01	0.191607E 01	0.807802E-01	0.916900E-01
473704E-01	-0.137217E-00	0.106594E-00	-0.125704E 01	-0.411390E 01	0.286991E 01	0.916900E-01	0.124846E-01
361409E-01	0.102098E-00	-0.818695E-01	0.862108E 00	0.279816E 01	-0.198641E 01	-0.701750E-01	-0.924406E-01
365172E-00	0.302547E-00	-0.105712E-00	0.334056E 03	0.170889E 03	-0.128334E 02	0.	0.
996050E 01	0.306994E 01	-0.145782E-01	0.344881E 04	0.854252E 04	-0.193305E 02	0.	0.
347555E 01	0.700870E 00	0.318104E-00	0.800949E 03	0.325052E 04	0.386255E 02	0.	0.
170063E 01	0.210380E-00	0.119961E-00	0.119325E 03	-0.119590E 04	-0.256980E 02	0.	0.
204319E 02	-0.904819E 00	-0.511272E-02	0.175068E 03	0.180883E 05	0.280094E-00	0.	0.
217886E 01	0.242127E-01	-0.251789E-00	0.133788E 03	0.210804E 04	0.125499E 03	0.	0.

361409E-01	0.365172E-00	0.996050E 01	0.347555E 01	-0.170063E 01	0.204319E 02	0.217886E 01
102098E-00	0.302547E-00	0.306994E 01	0.700870E 00	0.210380E-00	-0.904819E 00	0.242127E-01
818695E-01	-0.105712E-00	-0.145782E-01	0.318104E-00	0.119961E-00	-0.511272E-02	-0.251789E-00
862108E 00	0.334056E 03	0.344881E 04	0.800949E 03	0.119325E 03	0.175068E 03	0.133788E 03
279816E 01	0.170889E 03	0.854252E 04	0.325052E 04	-0.119590E 04	0.180883E 05	0.210804E 04
198641E 01	-0.128334E 02	-0.193305E 02	0.386255E 02	-0.256980E 02	0.280094E-00	0.125499E 03
701750E-01	0.	0.	0.	0.	0.	0.
924406E-01	0.	0.	0.	0.	0.	0.
708468E-01	0.	0.	0.	0.	0.	0.
	0.148665E-00	0.121068E-00	-0.160476E-00	0.	0.	0.
	0.121068E-00	0.103031E 01	0.166073E-00	0.	0.	0.
	-0.160476E-00	0.166073E-00	0.484401E-00	0.	0.	0.
	0.	0.	0.	0.420891E-00	-0.470409E-01	0.755548E-01
	0.	0.	0.	-0.470409E-01	0.114327E 01	0.845929E-01
	0.	0.	0.	0.755548E-01	0.845929E-01	0.383609E-00

CHECKOUT OF MATRIX INVERSION - FOLLOWING IS (C)*(C)-1

.100000E 01	0.130285E-09	0.379714E-10	-0.177192E-12	-0.136557E-13	-0.810463E-14	-0.213163E-12	0.154159E-0
.174660E-11	0.100000E 01	0.374456E-11	-0.150695E-13	-0.172085E-14	-0.770217E-15	-0.	0.218918E-1
.177636E-13	0.355271E-12	1.000000E 00	-0.187350E-15	-0.190820E-16	0.693889E-16	0.144329E-13	0.348166E-1
.449290E-08	0.310829E-07	0.607542E-08	1.000000E 00	-0.419664E-11	-0.998313E-12	0.125056E-09	0.377331E-0
.277942E-07	0.133878E-06	0.360887E-07	-0.227487E-09	1.000000E 00	-0.926548E-11	0.411092E-09	0.192085E-
.236469E-10	0.101863E-09	0.436557E-10	-0.746070E-13	-0.799361E-14	1.000000E 00	-0.272848E-11	0.582077E-1
.470735E-13	0.525802E-12	0.177636E-13	-0.444089E-15	-0.329597E-16	-0.867362E-18	0.100000E 01	0.270006E-1
.683897E-13	0.525802E-12	0.106581E-13	-0.582867E-15	-0.503070E-16	-0.173472E-17	0.266454E-14	0.100000E 0
.355271E-13	-0.554223E-12	-0.177636E-13	0.333067E-15	0.425007E-16	-0.173472E-17	-0.310862E-14	-0.383693E-1
.452971E-13	0.127898E-12	0.710543E-13	-0.166533E-15	-0.312250E-16	-0.147451E-16	-0.	0.426326E-1
.118305E-11	0.106439E-10	0.147793E-11	-0.871525E-14	-0.112237E-14	-0.379904E-15	0.687228E-13	0.106013E-1
.202505E-12	0.224532E-11	0.440536E-12	-0.116573E-14	-0.284495E-15	-0.117961E-15	0.355271E-14	0.153477E-1
.710543E-13	-0.245137E-12	-0.603961E-13	0.392048E-15	0.104083E-16	0.242861E-16	0.421885E-14	-0.369482E-1
.194511E-12	0.424194E-11	0.121503E-11	-0.347292E-14	-0.170870E-15	-0.343475E-15	-0.672795E-13	0.301981E-1
.138556E-12	0.639488E-12	0.191847E-12	-0.860423E-15	-0.100614E-15	-0.111022E-15	-0.932587E-14	0.653699E-1

.528644E-11	-0.503064E-11	0.363798E-11	-0.391083E-10	-0.156035E-10	0.245564E-10	0.416094E-10
.194067E-11	0.160538E-12	0.812683E-13	-0.348521E-11	-0.405009E-12	0.846434E-12	0.350386E-11
.781597E-13	0.266454E-13	-0.781597E-13	0.817124E-13	0.373035E-13	0.163425E-12	-0.149214E-12
.134605E-08	0.600267E-10	0.126056E-08	-0.655928E-08	-0.115688E-08	0.162254E-08	0.704313E-08
.625732E-08	-0.486034E-08	0.873115E-10	-0.401342E-07	-0.960426E-08	0.237487E-07	0.350410E-07
.236469E-10	-0.500222E-11	-0.727596E-11	-0.472937E-10	-0.145519E-10	-0.363798E-11	0.509317E-10
.710543E-13	0.399680E-14	-0.461853E-13	-0.426326E-13	0.488498E-14	0.639488E-13	0.195399E-13
.639488E-13	0.399680E-14	-0.497380E-13	-0.284217E-13	0.710543E-14	0.781597E-13	0.710543E-14
.100000E 01	-0.444089E-14	0.497380E-13	0.230926E-13	-0.532907E-14	-0.781597E-13	-0.159872E-13
.710543E-14	0.100000E 01	0.461853E-13	-0.426326E-13	-0.106581E-13	0.390799E-13	0.888178E-13
.827782E-12	-0.248690E-13	1.000000E 00	-0.197531E-11	-0.254019E-12	0.813571E-12	0.160227E-11
.923706E-13	-0.213163E-13	0.	1.000000E 00	-0.959233E-13	0.170530E-12	0.405009E-12
.728306E-13	0.226485E-13	0.328626E-13	0.852651E-13	0.100000E 01	-0.408562E-13	-0.959233E-13
.632383E-12	-0.914824E-13	-0.219380E-12	-0.781597E-12	-0.646594E-12	0.100000E 01	0.102318E-11
.213163E-13	-0.213163E-13	0.	-0.255795E-12	-0.817124E-13	0.426326E-13	0.100000E 01

INVERSE

.100292E 03	0.525995E 03	0.745443E 02	-0.523329E 00	-0.826468E-01	-0.178585E-01	0.703515E 01	0.482035E 0
.525995E 03	0.417691E 04	0.352307E 03	-0.412337E 01	-0.380657E-00	-0.714955E-01	0.457576E 02	0.382804E 0
.745443E 02	0.352307E 03	0.347811E 03	-0.432408E-00	-0.479045E-01	-0.101128E-00	-0.157604E 02	0.344064E 0
.523329E 00	-0.412337E 01	-0.432408E-00	0.417839E-02	0.369933E-03	0.555125E-04	-0.397697E-01	-0.378020E 0
.826468E-01	-0.380657E-00	-0.479045E-01	0.369933E-03	0.730897E-04	0.750876E-05	-0.656781E-02	-0.347447E-0
.178585E-01	-0.714955E-01	-0.101128E-00	0.555125E-04	0.750876E-05	0.320231E-03	0.669558E-02	-0.926650E-0
.703515E 01	0.457576E 02	-0.157604E 02	-0.397697E-01	-0.656781E-02	0.669558E-02	0.907914E 02	0.376682E 0
.482035E 03	0.382804E 04	0.344064E 03	-0.378020E 01	-0.347447E-00	-0.926650E-01	0.376682E 02	0.374787E 0
.779812E 02	-0.777261E 03	0.293825E 03	0.673851E 00	0.683409E-01	-0.110989E-00	0.522678E 02	-0.380301E 0
.207225E 02	0.186543E 02	-0.571128E 02	-0.436816E-01	0.258835E-01	0.101627E-02	0.195093E 01	0.148462E 0
.826982E 02	-0.490748E 03	0.131066E 03	0.207337E-00	0.764892E-01	0.128083E-00	-0.187177E 02	-0.454146E 0
.867790E 02	-0.496415E 03	-0.292358E 03	0.523563E 00	0.548234E-01	0.867034E-01	0.100280E 02	-0.473225E 0
.193013E 02	0.488017E 01	-0.759610E 02	-0.166661E-01	-0.198280E-01	0.651628E-01	0.757735E 01	-0.571431E 0
.661601E 01	0.530466E 03	-0.253038E 03	-0.370992E-00	-0.338933E-01	0.150133E-00	0.215916E 02	0.462436E 0
.833428E 02	0.415339E 03	0.300551E 03	-0.474316E-00	-0.597341E-01	-0.171759E-00	-0.116740E 02	0.403472E 0

.779812E 02	-0.207225E 02	-0.826982E 02	-0.867790E 02	0.193013E 02	0.661601E 01	0.833428E 02
.777261E 03	0.186543E 02	-0.490748E 03	-0.496415E 03	0.488017E 01	0.530466E 03	0.415339E 03
.293825E 03	-0.571128E 02	0.131066E 03	-0.292358E 03	-0.759610E 02	-0.253038E 03	0.300551E 03
.673851E 00	-0.436816E-01	0.207337E-00	0.523563E 00	-0.166661E-01	-0.370992E-00	-0.474316E-00
.683409E-01	0.258835E-01	0.764892E-01	0.548234E-01	-0.198280E-01	-0.338933E-01	-0.597341E-01
.110989E-00	0.101627E-02	0.128083E-00	0.867034E-01	0.651628E-01	0.150133E-00	-0.171759E-00
.522678E 02	0.195093E 01	-0.187177E 02	0.100280E 02	0.757735E 01	0.215916E 02	-0.116740E 02
.380301E 03	0.148462E 02	-0.454146E 03	-0.473225E 03	-0.571431E 01	0.462436E 03	0.403472E 03
.105510E 04	-0.614691E 02	0.287808E 03	-0.191821E 03	-0.101796E 03	-0.425409E 03	0.224448E 03
.614691E 02	0.623193E 02	0.588844E 02	0.581134E 02	0.163870E 02	-0.137154E 02	-0.485007E 02
.287808E 03	0.588844E 02	0.923108E 03	0.540515E 02	0.716043E 01	-0.157863E 03	0.855791E 02
.191821E 03	0.581134E 02	0.540515E 02	0.295160E 03	0.627246E 02	0.229653E 03	-0.242896E 03
.101796E 03	0.163870E 02	0.716043E 01	0.627246E 02	0.635215E 02	-0.170050E 02	-0.751029E 02
.425409E 03	-0.137154E 02	-0.157863E 03	0.229653E 03	-0.170050E 02	0.906013E 03	-0.167065E 03
.224448E 03	-0.485007E 02	0.855791E 02	-0.242896E 03	-0.751029E 02	-0.167065E 03	0.301787E 03

(Q)(COV)(QT)

983915E 02	0.183886E 03	-0.429378E 01	-0.130736E-00	0.335795E-02	0.284598E-01	0.897422E 01	0.409522E
183886E 03	0.942893E 03	-0.290643E 03	-0.836104E 00	0.313775E-00	0.434540E-01	0.184093E 02	0.335562E
429378E 01	-0.290643E 03	0.384326E 03	0.299498E-00	-0.139141E-00	-0.470555E-01	-0.167124E 02	0.352191E
130736E-00	-0.836104E 00	0.299498E-00	0.898828E-03	-0.326746E-03	-0.896076E-04	-0.900611E-02	0.196218E-
335795E-02	0.313775E-00	-0.139141E-00	-0.326746E-03	0.164217E-03	0.781349E-05	0.208100E-03	-0.193362E-
284598E-01	0.434540E-01	-0.470555E-01	-0.896076E-04	0.781349E-05	0.268957E-03	0.119237E-02	0.253299E-
897422E 01	0.184093E 02	-0.167124E 02	-0.900611E-02	0.208100E-03	0.119237E-02	0.907914E 02	0.376682E
409522E 03	0.335562E 03	0.352191E 03	0.196218E-00	-0.193362E-00	0.253299E-01	0.376682E 02	0.374787E
133889E 03	-0.394248E 03	0.308300E 03	0.232722E-00	-0.707883E-01	-0.894620E-02	0.522678E 02	-0.380301E
177337E 02	-0.829357E 01	-0.550637E 02	-0.723954E-02	0.216414E-01	-0.179944E-01	0.195093E 01	0.148462E
388277E 02	-0.160342E 03	0.955150E 02	-0.180702E-00	0.208898E-01	0.166481E-00	-0.187177E 02	-0.454146E
282940E 01	0.244510E 03	-0.301147E 03	-0.317283E-00	0.130593E-00	-0.138783E-01	0.100280E 02	-0.473225E
112966E 02	-0.373327E 02	-0.883298E 02	0.254239E-01	-0.305571E-01	0.373203E-01	0.757735E 01	-0.571431E
208588E 03	0.915356E 03	-0.278407E 03	-0.801053E 00	0.276327E-00	0.600687E-01	0.215916E 02	0.462436E
279594E 02	-0.184029E 03	0.329131E 03	0.198926E-00	-0.114241E-00	-0.650464E-01	-0.116740E 02	0.403472E

133889E 03	-0.177337E 02	-0.388277E 02	-0.282940E 01	0.112966E 02	0.208588E 03	0.279594E 02
394248E 03	-0.829357E 01	-0.160342E 03	0.244510E 03	-0.373327E 02	0.915356E 03	-0.184029E 03
308300E 03	-0.550637E 02	0.955150E 02	-0.301147E 03	-0.883298E 02	-0.278407E 03	0.329131E 03
232722E-00	-0.723954E-02	-0.180702E-00	-0.317283E-00	0.254239E-01	-0.801053E 00	0.198926E-00
707883E-01	0.216414E-01	0.208898E-01	0.130593E-00	-0.305571E-01	0.276327E-00	-0.114241E-00
894620E-02	-0.179944E-01	0.166481E-00	-0.138783E-01	0.373203E-01	0.600687E-01	-0.650464E-01
522678E 02	0.195093E 01	-0.187177E 02	0.100280E 02	0.757735E 01	0.215916E 02	-0.116740E 02
380301E 03	0.148462E 02	-0.454146E 03	-0.473225E 03	-0.571431E 01	0.462436E 03	0.403472E 03
105510E 04	-0.614691E 02	0.287808E 03	-0.191821E 03	-0.101796E 03	-0.425409E 03	0.224448E 03
614691E 02	0.623193E 02	0.588844E 02	0.581134E 02	0.163870E 02	-0.137154E 02	-0.485007E 02
287808E 03	0.588844E 02	0.923108E 03	0.540515E 02	0.716043E 01	-0.157863E 03	0.855791E 02
191821E 03	0.581134E 02	0.540515E 02	0.295160E 03	0.627246E 02	0.229653E 03	-0.242896E 03
101796E 03	0.163870E 02	0.716043E 01	0.627246E 02	0.635215E 02	-0.170050E 02	-0.751029E 02
425409E 03	-0.137154E 02	-0.157863E 03	0.229653E 03	-0.170050E 02	0.906013E 03	-0.167065E 03
224448E 03	-0.485007E 02	0.855791E 02	-0.242896E 03	-0.751029E 02	-0.167065E 03	0.301787E 03

MISS TIME = 0.1033891E 01 MINIMUM MISS = 0.2197084E 02 REND MISS = 0.3775726E 02

SQRT((2,2)+(3,3)) = 0.3643102E 02
SQRT(1, 1) = 0.9919250E 01
SQRT(2, 2) = 0.3070656E 02
SQRT(3, 3) = 0.1960425E 02
SQRT(4, 4) = 0.2998046E-01
SQRT(5, 5) = 0.1281471E-01
SQRT(6, 6) = 0.1639990E-01
SQRT(7, 7) = 0.9528450E 01
SQRT(8, 8) = 0.6121984E 02
SQRT(9, 9) = 0.3248236E 02
SQRT(10,10) = 0.7894256E 01
SQRT(11,11) = 0.3038269E 02
SQRT(12,12) = 0.1718023E 02
SQRT(13,13) = 0.7970035E 01
SQRT(14,14) = 0.3010005E 02
SQRT(15,15) = 0.1737203E 02

1.000	0.604	-0.022	-0.440	0.026	0.175	0.095	0.674	-0.416	-0.226	-0.129	-0.017	0.143	0.699	0.162
0.604	1.000	-0.483	-0.908	0.797	0.086	0.063	0.179	-0.395	-0.034	-0.172	0.463	-0.153	0.990	-0.345
-0.022	-0.483	1.000	0.510	-0.554	-0.146	-0.089	0.293	0.484	-0.356	0.160	-0.894	-0.565	-0.472	0.966
-0.440	-0.908	0.510	1.000	-0.850	-0.182	-0.032	0.107	0.239	-0.031	-0.198	-0.616	0.106	-0.888	0.382
0.026	0.797	-0.554	-0.850	1.000	0.037	0.002	-0.246	-0.170	0.214	0.054	0.593	-0.299	0.716	-0.513
0.175	0.086	-0.146	-0.182	0.037	1.000	0.008	0.025	-0.017	-0.139	0.334	-0.049	0.286	0.122	-0.228
0.095	0.063	-0.089	-0.032	0.002	0.008	1.000	0.065	0.169	0.026	-0.065	0.061	0.100	0.075	-0.071
0.674	0.179	0.293	0.107	-0.246	0.025	0.065	1.000	-0.191	0.031	-0.244	-0.450	-0.012	0.251	0.379
-0.416	-0.395	0.484	0.239	-0.170	-0.017	0.169	-0.191	1.000	-0.240	0.292	-0.344	-0.393	-0.435	0.398
-0.226	-0.034	-0.356	-0.031	0.214	-0.139	0.026	0.031	-0.240	1.000	0.246	0.428	0.260	-0.058	-0.354
-0.129	-0.172	0.160	-0.198	0.054	0.334	-0.065	-0.244	0.292	0.246	1.000	0.104	0.030	-0.173	0.162
-0.017	0.463	-0.894	-0.616	0.593	-0.049	0.061	-0.450	-0.344	0.428	0.104	1.000	0.458	0.444	-0.814
0.143	-0.153	-0.565	0.106	-0.299	0.286	0.100	-0.012	-0.393	0.260	0.030	0.458	1.000	-0.071	-0.542
0.699	0.990	-0.472	-0.888	0.716	0.122	0.075	0.251	-0.435	-0.058	-0.173	0.444	-0.071	1.000	-0.319
0.162	-0.345	0.966	0.382	-0.513	-0.228	-0.071	0.379	0.398	-0.354	0.162	-0.814	-0.542	-0.319	1.000

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APOLLO NOTE NO. 187

H. Engel
7 April 1964

EFFECT OF POOR STATION GEOMETRY

Most of the MSFN stations named in Bissett-Berman Apollo Note No. 181 are located near a great circle through Cape Kennedy and with its maximum North latitude at Cape Kennedy. This is a reasonable distribution of stations for Earth orbit injection and Earth orbit tracking. However, it results in a poor station geometry for tracking vehicles in the vicinity of the Moon.

In order to determine how serious a problem this results in, we have chosen a worst case and compared the rendezvous prediction results obtained with those obtained previously with a far better configuration of stations.

The most ill-conditioned geometry appears to occur with the Moon at a declination of 28°N and a sublunar point at 28.0°N 112.5°W. In this situation the stations for which the Moon is more than 5° above the horizon are Guaymas, Corpus Christie, Houston, Cape Kennedy, Bermuda, Antigua, Hawaii and Goldstone. We see no reason to penalize ourselves unduly by choosing other than the best three stations from among these. Therefore, we have used Antigua, Goldstone and Hawaii.

The results are shown in Figure 2. This worst station geometry increases the error in prediction rendezvous position by a factor of 4 over those predicted with a far more favorable geometry of stations.

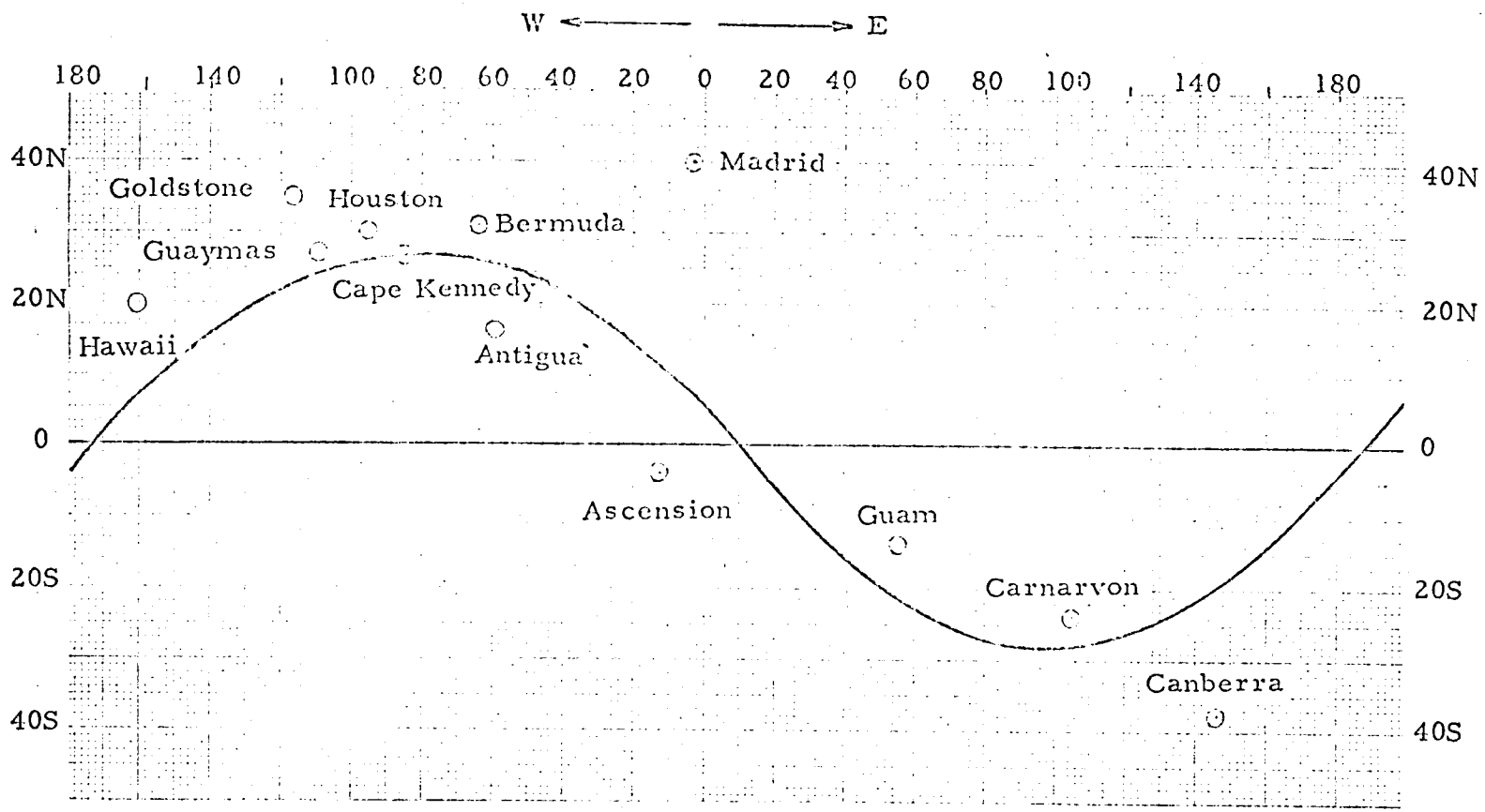
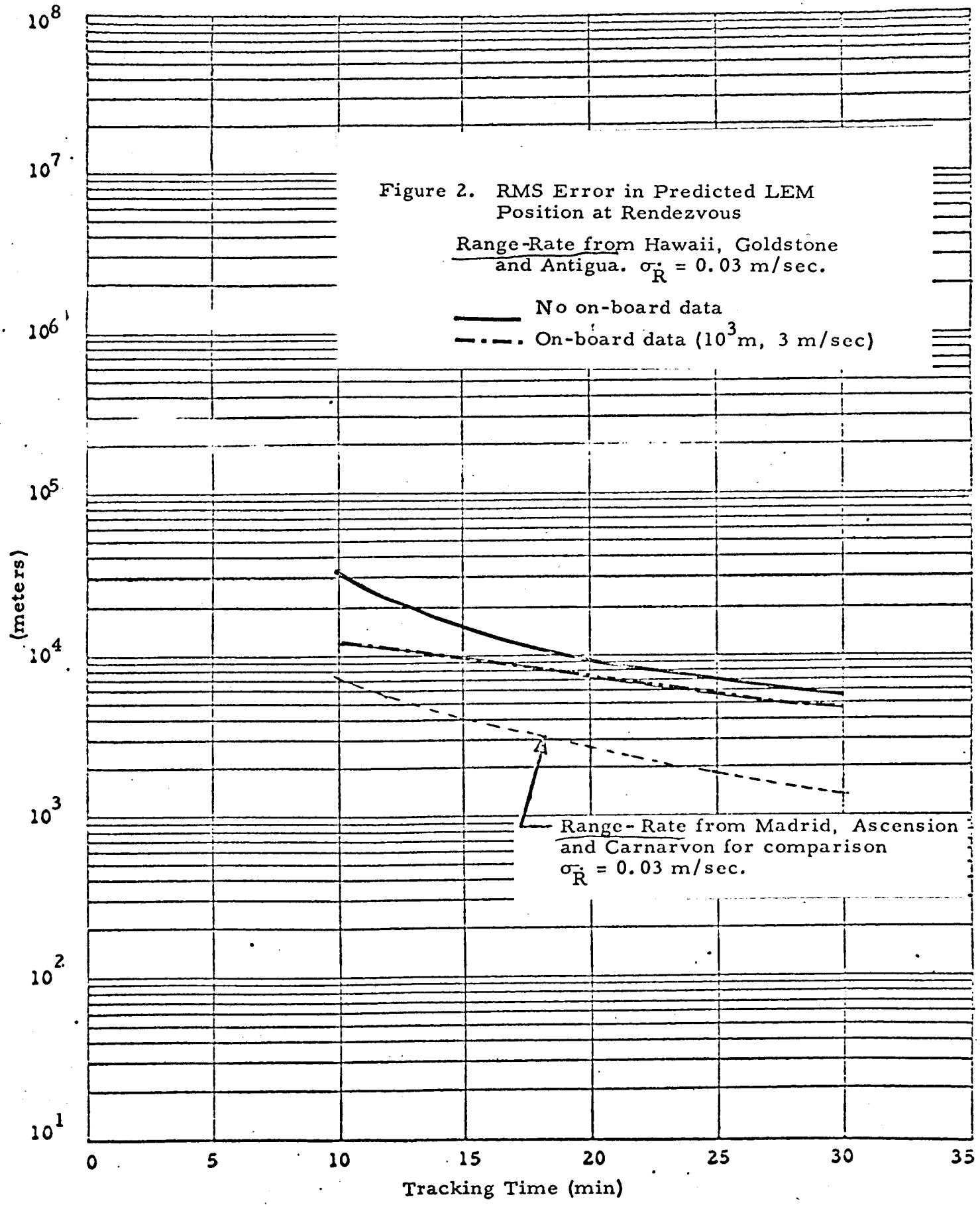


Figure 1.

Station Geometry Relation to Orbit
(Launch from Cape Kennedy)



Data Sets 0019, 0189

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APOLLO NOTE NO. 188

J. R. Holdsworth
8 April 1964

SOME ARITHMETIC RESULTS RELATING TO THE DEGRADATION OF THE PROBABILITY OF MISSION SUCCESS BECAUSE OF AN UNFORTUNATE CHOICE OF A MEASURE OF EFFECTIVENESS FOR THE SIMPLIFIED CORRECTIVE BOOST PROBLEM

The purpose of this note is to present the results of some simple arithmetic computations which will lend quantitative insight into the comments made in Note 183. The notation used will parallel that used in the aforementioned note.

Specifically, we shall assume that our real desire is to maximize the probability that the terminal value of some scalar state variable y lies in the interval $\left[-\tau, \tau\right]$ where τ is some known positive number and subject to the conditions formulated in the second section of Note 183. However, we shall assume that the actual correction made was that which minimized the mean square deviation from the desired zero configuration. As mentioned in the earlier note, the optimal strategies for the two different criteria are not the same. Thus, it is of interest to investigate numerically the difference between Prob $\left\{|y| \leq \tau \mid C_{opt}\right\}$ and Prob $\left\{|y| \leq \tau \mid C^*_{opt}\right\}$ as a function of initial displacement and autopilot error where C_{opt} and C^*_{opt} are the various optimal commanded changes for the two aforementioned measures of effectiveness and are defined by equations (15) and (24) of Note 183 respectively.

In what follows, we shall consider the mission a success if the value of the terminal state variable y is such that:

$$|y| \leq \tau \tag{1}$$

Furthermore, we shall write $P_s(C_{opt})$ and $P_s(C^*_{opt})$ for the success probabilities using the two different procedures.

From equations (9) and (15) of Note 183 we may write:

$$P_s(C_{opt}) = \frac{1}{\sqrt{2\pi}\,\sigma_e\,|C_{opt}|} \int_{-\tau}^{\tau} \exp - \frac{1}{2\sigma_e^2 C^2} (y - x - C_{opt})^2 \, dy \tag{2}$$

where:

$$C_{opt} = \frac{\tau - \sqrt{\tau^2 + 2k\sigma_e^2 \tau x}}{k\sigma_e^2} \tag{3}$$

and

$$k = \log \frac{x + \tau}{x - \tau}. \tag{4}$$

In equation (4) x is the initial value of the state variable which is assumed to be a perfectly known positive number greater than τ. As mentioned before, if $x < \tau$ the problem is trivial since by choosing $C_{opt} = 0$ we may obtain:

$$P_s(C_{opt} = 0,\ x < \tau) = 1 \tag{5}$$

For computational purposes it is convenient to transform the integral given by equation (2) so as to express it in terms of certain dimensionless variables. If we begin by making the change of variable

$$u = \frac{y - x - C_{opt}}{\sigma_e\,|C|} \tag{6}$$

then (2) may be written:

$$P_s(C_{opt}) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\tau + x + C_{opt}}{\sigma_e |C|}}^{\frac{\tau - x - C_{opt}}{\sigma_e |C|}} e^{-1/2\, u^2}\, du \tag{7}$$

If we now define the dimensionless quantities

$$z = \tau / x \tag{8}$$

$$\alpha(z) = \frac{|C_{opt}|}{x} = \frac{z}{k(z)\,\sigma_e^{\,2}} \left(\sqrt{1 + 2\,\frac{k(z)\sigma_e^{\,2}}{z}} - 1 \right) \tag{9}$$

$$k(z) = \log \frac{1 + z}{1 - z} \tag{10}$$

$$\ell_1(z, \sigma_e) = \frac{1}{\sigma_e} - \left(\frac{1 + z}{\alpha(z)\sigma_e} \right) \tag{11}$$

$$\ell_2(z, \sigma_e) = \frac{1}{\sigma_e} - \left(\frac{1 - z}{\alpha(z)\sigma_e} \right) \tag{12}$$

then $P_s(C_{opt})$ may be written in a natural way in terms of these dimensionless quantities as:

$$P_s(C_{opt}, z, \sigma_e) = \frac{1}{\sqrt{2\pi}} \int_{\ell_1(z, \sigma_e)}^{\ell_2(z, \sigma_e)} e^{-1/2\, u^2}\, du \tag{13}$$

Equation (13) is interesting not only because it is in a convenient form for computations but because it exhibits the fact that $P_s(C_{opt})$ depends only upon certain dimensionless variables and the autopilot error which was also defined to be dimensionless in Note 183.

We may proceed in an entirely comparable way for the reduction of the integral defining $P_s(C^*_{opt})$ by merely replacing C_{opt} by C^*_{opt} in equation (12) and recalling from equation (24) of Note 183 that

$$C^*_{opt} = - \frac{x}{1 + \sigma_e^2} \tag{14}$$

Then it is an easy exercise to show that by defining the analogous set of dimensionless quantities given below:

$$\ell_1^*(z) = - \frac{1}{\sigma_e} \left[\sigma_e^2 + z\,(1 + \sigma_e^2) \right] \tag{15}$$

$$\ell_2^*(z) = - \frac{1}{\sigma_e} \left[z(1 + \sigma_e^2) - \sigma_e^2 \right] \tag{16}$$

that the transformed integral defining $P_s(C^*_{opt})$ may be written as:

$$P_s(C^*_{opt}, z, \sigma_e) = \frac{1}{\sqrt{2\pi}} \int_{\ell_1^*(z,\ \sigma_e)}^{\ell_2^*(z,\ \sigma_e)} e^{-1/2\, u^2}\, du \tag{17}$$

Now as we have mentioned before, since C^*_{opt} is the optimal correction, if one wishes to minimize the expected square departure of the terminal state variable y from the zero position, it is not optimal for the effectiveness measure $P_s\left\{|y| \leq \tau\right\}$. Thus for fixed z and σ_e, we shall find that:

$$P_s(C_{opt}, z, \sigma_e) \geqslant P_s(C^*_{opt}, z, \sigma_e) \tag{18}$$

It is the behavior of the magnitude of this inequality with respect to z and σ_e that this note seeks to explore numerically.

This will be done in the following way. A fixed value of z determines two functions of the dimensionless autopilot error; namely, $P_x(C_{opt}, z, \sigma_e)$ and $P_s(C^*_{opt}, z, \sigma_e)$. That is, the family of curves will be parametrized by z which from prior considerations must range between zero and one; each member function of the family being plotted as a function of the dimensionless autopilot error σ_e.

Now the dimensionless $z = \tau/x$ is essentially reciprocal initial distance from zero expressed in tolerance units, i.e., the ratio of the semi-length of the tolerance window to the initial displacement from the desired configuration. It is perhaps appropriate to say a word about a possible interpretation of the standard deviation σ_e of the dimensionless autopilot error. Since the dimensionless quantity e has a zero mean gaussian distribution, then σ_e is dimensionless so that as σ_e varies, we may think of a varying family of gaussian distributions referred to the standard normal distribution with unit mean and unit standard deviation.

Referred to this standard normal distribution, σ_e may be regarded as a measure of sharpness or diffuseness relative to the standard distribution with $\sigma_e = 1$. Thus distributions with $\sigma_e < 1$ are sharper and in a sense "less random" than the standard distribution with $\sigma_e = 1$. That is, as $\sigma_e \to 0$ the total probability mass of the e disbribution tends to cluster in a very narrow band about the line $e = 0$ until in the limit where $\sigma_e = 0$, the distribution degenerates; and since all of its unit probability mass is compressed into line $e = 0$, we may say in a sense that the distribution of e loses its randomness.

On the other hand, for values of $\sigma_e > 1$, the distribution of e is more diffuse or more random than that of the standardized gaussian variate. As σ_e becomes very large, the distribution of e loses its sharpness and tends to distribute its probability mass more uniformly along the entire real line thus producing relatively large uncertainties in e.

Thus when we plot our functions z or 1 - z may be interpreted as a fractional initial distance from the initial success state corresponding to $-\tau \leq x \leq \tau$, while the parameter σ_e may be profitably interpreted as a measure of the uncertainty inherent in our dimensionless autopilot error.

There is one other doctrine that is of comparative interest: namely, the doctrine that the commanded correction should be such as to remove the observed discrepancy. Expressed quantitatively, this doctrine says that the commanded change is given by

$$C = -x \tag{19}$$

If we denote the probability of getting into the tolerance band using this maneuver by $P_s(C = -x, z, \sigma_e)$, then direct substitution into equation (9) of Note 183 shows that we may write:

$$P_s(C = -x) = \frac{1}{\sqrt{2\pi}\,|x|\,\sigma_e} \int_{-\tau}^{\tau} \exp - \frac{1}{2x^2\sigma_e^2}\, y^2\, dy. \tag{20}$$

Or for computational convenience we may make the change of variable

$$u = \frac{y}{|x|\,\sigma_e} \tag{21}$$

and write equation (21) as:

$$P_s(C = -x) = \frac{1}{\sqrt{2\pi}} \int_{-\tau/|x|\sigma_e}^{\tau/|x|\sigma_e} e^{-1/2\, u^2} du \tag{22}$$

or:

$$P_s(C = -x,\ z,\ \sigma_e) = \frac{1}{\sqrt{2\pi}} \int_{-z/\sigma_e}^{z/\sigma_e} e^{-1/2\, u^2} du$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{z/\sigma_e} e^{-1/2\, u^2} du \tag{23}$$

Again since the corrective maneuver given by equation (19) is not optimal for maximizing P_s, it follows that the following inequality will exist for all pairs of z and σ_e.

$$P_s(C_{opt},\ z,\ \sigma_e) \geq P_s(C = -x,\ z,\ \sigma_e) \tag{24}$$

In general, however, we may not write an inequality relation between $P_s(C^*_{opt},\ z,\ \sigma_e)$ and $P_s(C = -x,\ z,\ \sigma_e)$ which will be uniformly valid for all values of z and σ_e, since the direction of any such inequality may be shown to depend upon the particular pair z, σ_e for which the probabilities are evaluated. However, as was shown in Note 183 when $\sigma_e^2 << 1$, then the doctrines C_{opt} and C^*_{opt} both approach $C = -x$, so that in the limit we have:

$$\lim_{\sigma_e^2 \to 0} \left[P_s(C_{opt}) = P_s(C^*_{opt}) = P_s(C = -x) \right] \quad (25)$$

Equation (25) may be paraphrased by saying that in the case where σ_e is very small, sensible procedures under one criterion are sensible under another.

The numerical comparisons we have mentioned are largely self-explanatory and are shown in Figures 1 and 2. The cases we have considered are admittedly extreme to accentuate the consequences.

The rather interesting thing is that for both of the situations we have considered, there seems to be a greater difference between the C^*_{opt} and $C = -x$ doctrines as between C_{opt} and $C = -x$, when the measure of effectiveness is a probability such as we have considered. Thus it may well be worthwhile considering more physically realistic situations and examining maneuvering doctrines other than those which seek to remove the observed discrepancies.

Finally, we note that we could have inverted this problem and assumed that what we really wished to do was to minimize E_y^2. We then could plot $Ey^2(C_{opt})$, $Ey^2(C^*_{opt})$ and Ey^2 $(C = -x)$ versus τ and σ_e and would find in this case, of course, that the C^*_{opt} doctrine was the best. This, however, was not done as the computation of $Ey^2(C_{opt})$ requires a tedious, numerical integration.

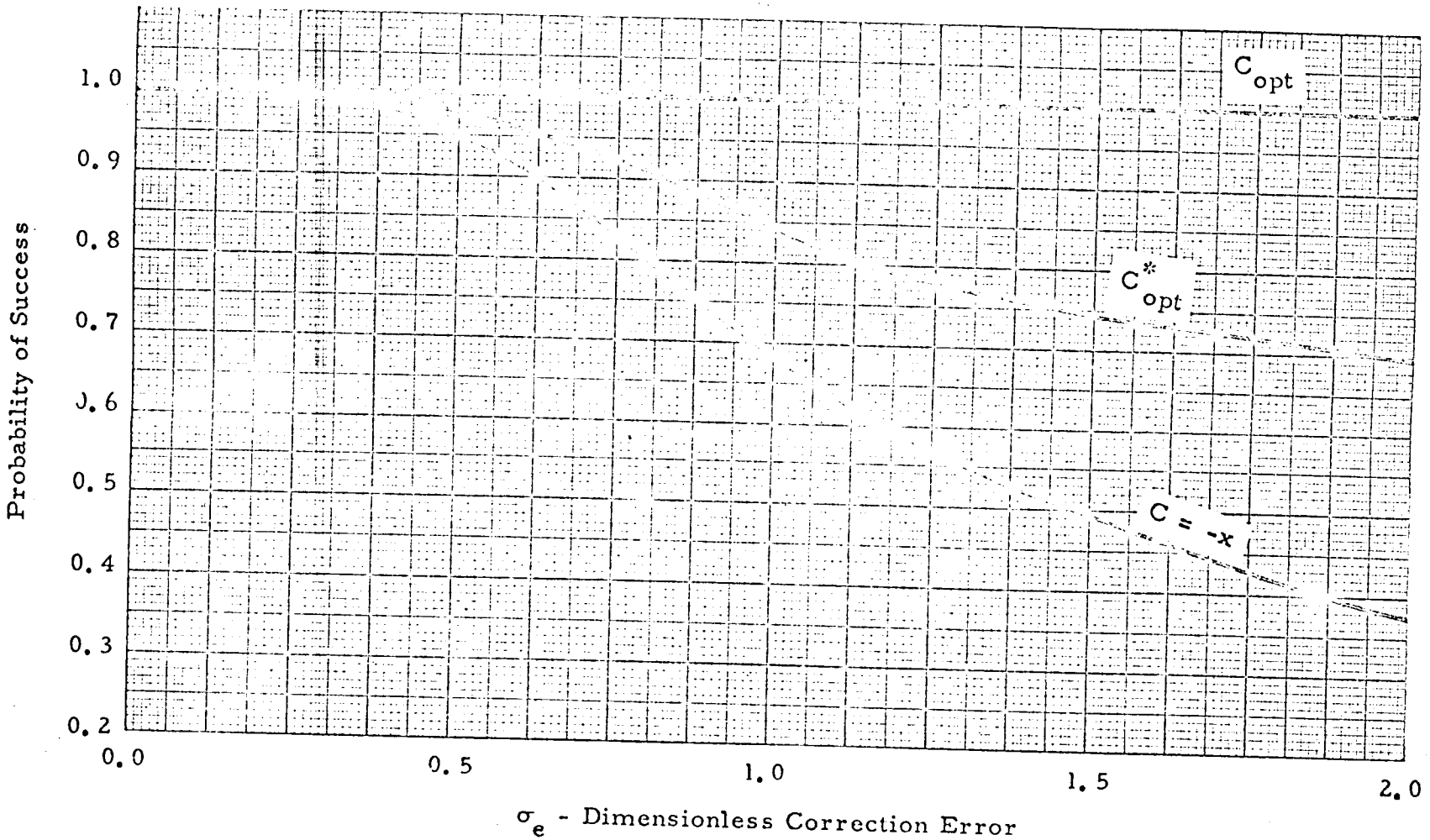

Figure 1

Probability of Success Versus Autopilot Error
For the Three Maneuver Doctrines For z = 1.0

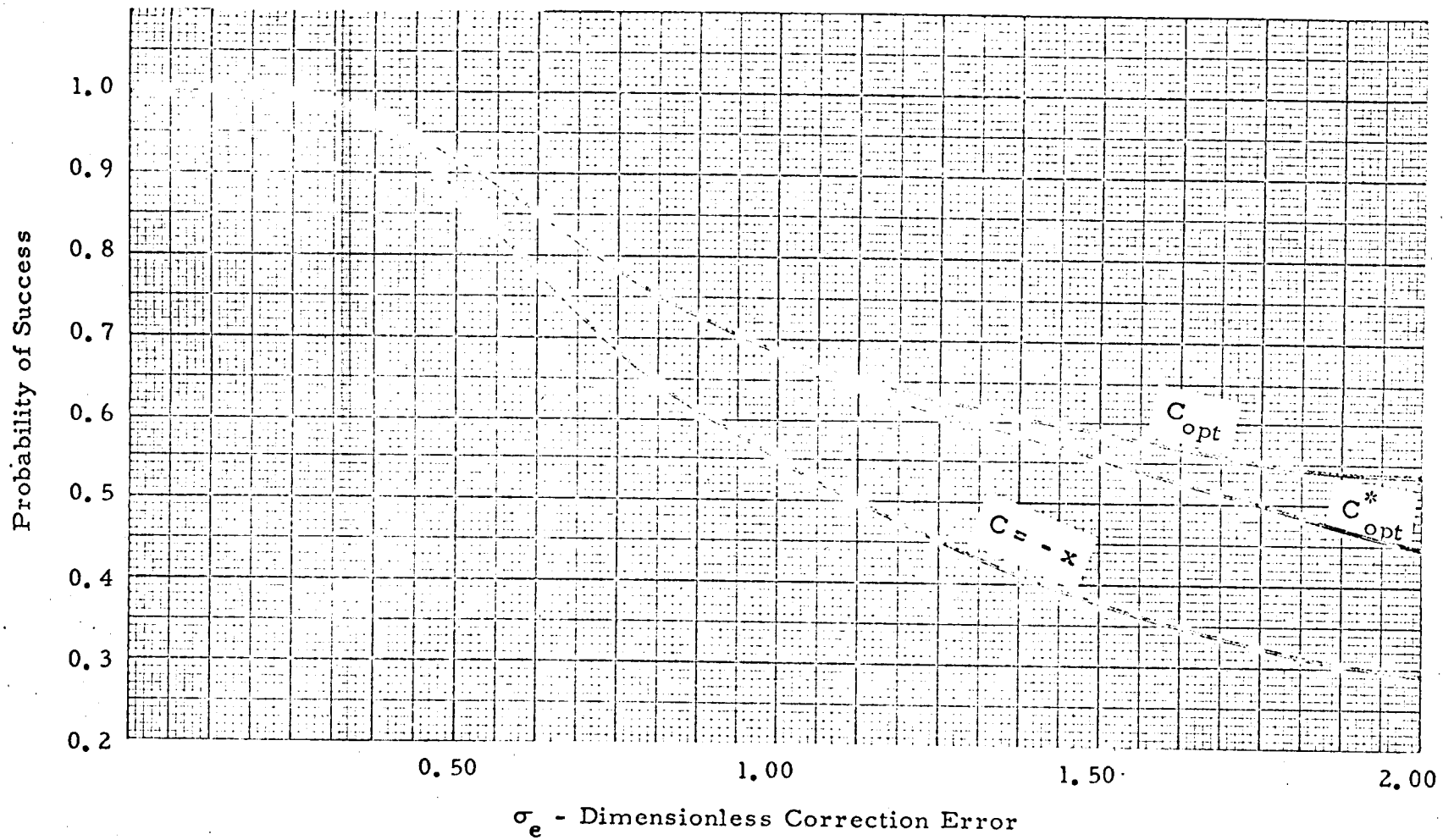

Figure 2

Probability of Success Versus Autopilot Error
For the Three Maneuver Doctrines For z = .75

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APOLLO NOTE NO. 189

L. S. Lustick
14 April 1964

ROTATION OF COVARIANCE MATRICES

Purpose

The purpose of this note is to describe a computer program for obtaining covariance matrices in arbitrary rotated coordinate systems.

Introduction

For cases when boosts occur and when transit from one sphere of influence to another are to be considered, it is convenient to be able to obtain the covariance matrices in rotated coordinate systems. The purpose of this note is to describe an addition to the orbit determination program which will allow this flexibility.

Method

Let x_1, x_2, x_3 be the coordinates in the new rotated coordinate system.

y_1, y_2, y_3 be the coordinates in the old coordinate system.

$$t_{ij} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$ = transformation that rotates a vector in the y coordinate system to a vector in the x coordinate system.

Let the covariance matrix before rotation, $[y]$ be partitioned as shown below.

$[y]$ =

	1 2 3	4 5 6	7 8 9 10 11 12 13
1 2 3	A'	B'	D'
4 5 6	B'^T	C'	E'
7 8 9 10 11 12 13	D'^T	E'^T	F'

The corresponding covariance matrix in the rotated x coordinate system is presented below.

$[x]$ =

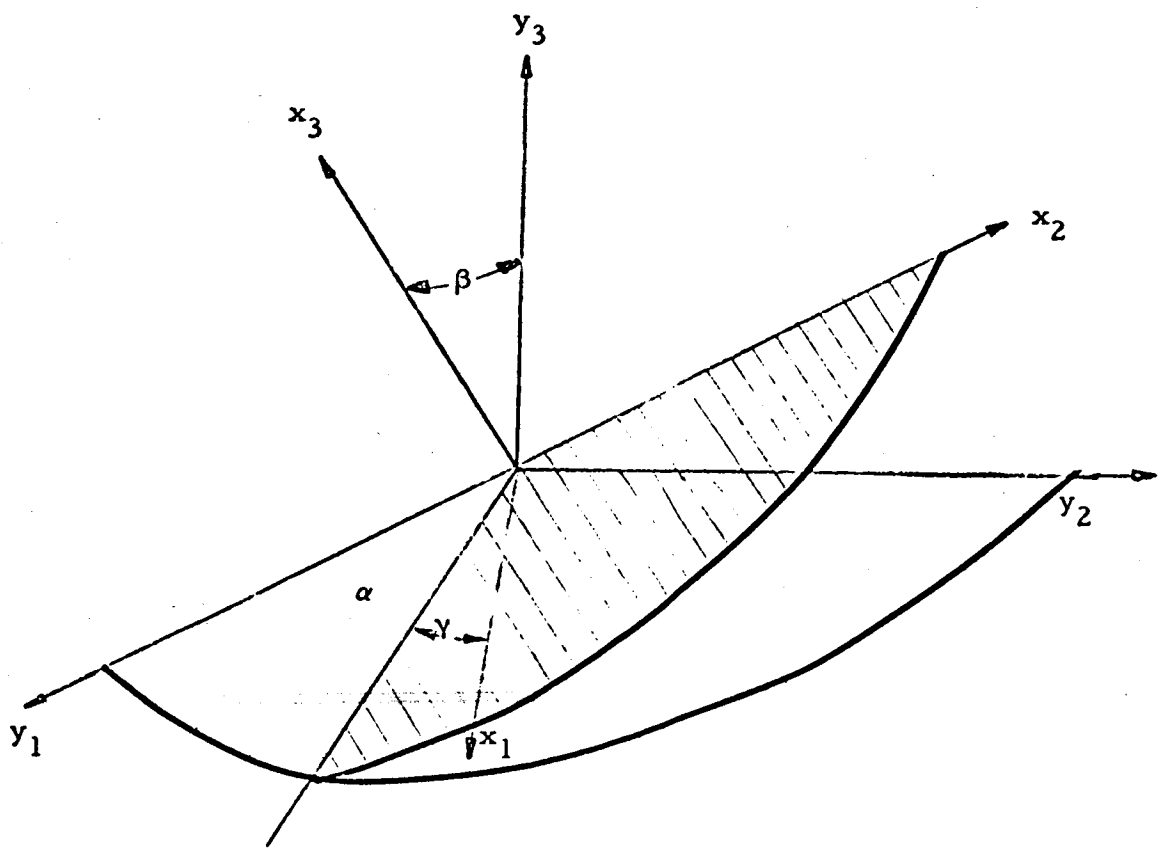
	1 2 3	4 5 6	7 8 9 10 11 12 13
1 2 3	A= $t(A')t^T$	B= $t(B')t^T$	D= t(D')
4 5 6	B^T	C= $t(C')t^T$	E= t(E')
7 8 9 10 11 12 13	D^T	E^T	F'

Let us consider two optional forms of inputting the rotation matrix.

1. The nine elements of the rotation matrix, t_{ij}, can be inputted directly.

2. Alternately it will be assumed that the input is the three Euler angles defining the rotation.

If we designate the Euler angles in this write up as α, β, γ with the significance pictured below.



then the t_{ij} matrix can be calculated as shown below.

$$t_{ij} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1)$$

Computer Program

Consider as a starting point that the matrix $Q \text{ cov } (T=0)\, Q^T$ has been calculated as is now done in program 2. The following is a description of modifications to program 2.

1. Is the covariance matrix to be obtained in a rotated coordinate system?

Yes	No
Go to step (2).	Continue on with program 2 in the present manner.

2. Are Euler angles used?

Yes	No
Calc t_{ij} from matrix equation (1)	Obtain t_{ij} elements from input data
Go to step (3).	

3. Obtain elements of rotated covariance matrix as shown on page 2.

4. Continue on with program 2 in the manner now done subsequent to obtaining $Q \text{ cov } (T=0) Q^T$.

The output of the program $Q \text{ cov } (T=0) Q^T$ in the rotated coordinate system will be stored on the output tape and can be operated on in subsequent runs as already provided for in the use of previous results.

Additional Input

A. Control Constants

1. Is matrix to be rotated?
2. Are Euler angles used?

B. Variable Input

1. Euler angles (α, β, γ) or the nine elements of the transformation matrix t_{ij}.

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APOLLO NOTE NO. 190

H. Dale
20 April 1964

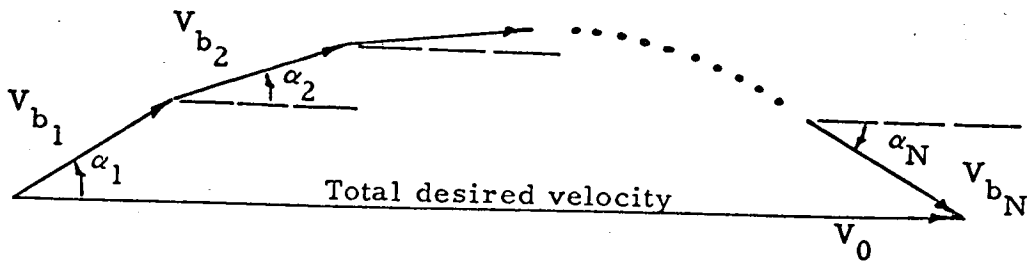
BOOST VELOCITY PENALTY FOR GYRO DRIFT AND MSFN ERRORS DURING THE LEM ASCENT BURN

This note presents a preliminary analysis of the way in which the required boost velocity during lunar ascent is affected by guidance and navigational errors. It is necessary to show that the guidance system employed does not cause an excessive velocity penalty. For the system analyzed, consisting of three MSFN stations with an on-board attitude reference which drifts at 0.2 degrees per second, it would appear that a conservative estimate of the total boost penalty to the "rendezvous box" is less than 30 ft/sec.

For a conservative estimate of the ability of the MSFN during powered flight reference is made to Apollo Notes 179 and 187. It appears that even with extremely poor station geometry, the largest component of velocity error (the out-of-plane component) will be less than 50 ft/sec. In the analysis of this note, the above figure is taken to be the unsmoothed, mean deviation error in velocity normal to the desired velocity to-be-gained for any single set of MSFN measurements.

The other source of error is the platform angular drift between boost attitude corrections. For this note strapped down gyros will be assumed whose drift rate is zero mean random, with a mean deviation of 0.2 degrees per second.

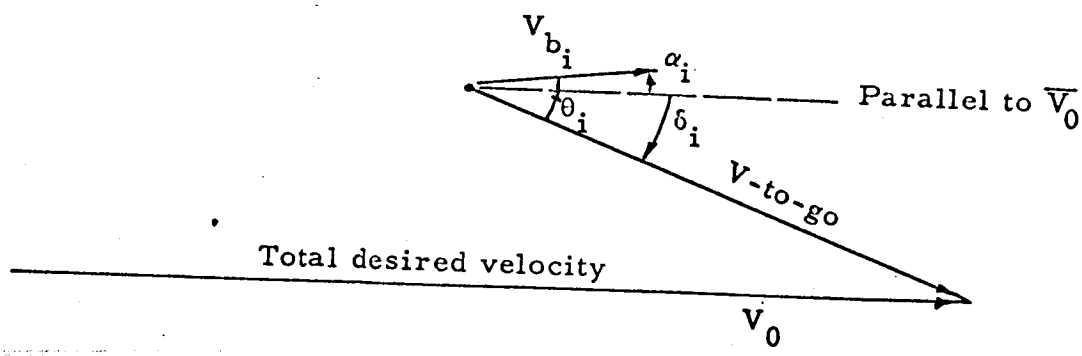
The analysis of the total boost cost during the powered portion of the ascent follows. Assume N equal boost attitude corrections



$$\text{Boost cost} \equiv \Delta V_b = V_{b_1} + V_{b_2} \ldots V_{b_N} - V_0$$

$$= V_{b_1}(1 - \cos \alpha_1) + V_{b_2}(1 - \cos \alpha_2) \ldots + V_{b_N}(1 - \cos \alpha_N)$$

Note that α is measured with respect to the horizontal, while a better angle to deal with is θ measured with respect to the velocity-to-go.



Now we may redefine ΔV_b

$$\Delta V_b = \sum_{i=1}^{N} V_{b_i}(1 - \cos \alpha_i) = \sum_{i=1}^{N} V_{b_i}(1 - \cos [\theta_i - \delta_i]) \cong \sum_{i=1}^{N} V_{b_1} \frac{(\theta_i - \delta_i)^2}{2}$$

$$\cong \frac{V_b}{2N} \sum_{i=1}^{N} (\theta_i - \delta_i)^2$$

where V_b is the actual boost and N is the total number of equal increments. Now an equation for δ_i may be generated.

$$\delta_i = \frac{\sum_{j=1}^{j=i-1} V_{b_j} \sin \alpha_j}{(N-i+1)V_{b_j}} \cong \frac{\sum_{j=1}^{j=i-1} \theta_j - \delta_j}{N-i+1} \quad \text{where } \delta_1 = 0$$

Putting this equation for δ_i in the equation for ΔV_b gives

$$\Delta V_b = \frac{V_b}{2N} \sum_{i=1}^{N} \left[\theta_i - \left(\frac{\sum_{j=1}^{j=i-1} \theta_j - \delta_i}{N-i+1} \right) \right]^2 ; \quad \delta_1 = 0$$

The average value of ΔV_b can be found by taking expected values of the above relationship. Under this assumption cross products (e.g. $\theta_i \theta_j$) vanish. A recursion relationship for $\overline{\Delta V_b}$ may be generaged by writing out the expression for N = 2, 3, 4, and 5.

N = 2:

$$\overline{\Delta V_b} = \frac{V_b}{2N} \left[\theta_2^2 + 2\theta_1^2 \right]$$

N = 3:

$$\overline{\Delta V_b} = \frac{V_b}{2N} \left[\theta_3^2 + 2\theta_2^2 + \frac{3}{2} \theta_1^2 \right]$$

N = 4:

$$\overline{\Delta V_b} = \frac{V_b}{2N} \left[\theta_4^2 + 2\theta_3^2 + \frac{3}{2} \theta_2^2 + \frac{4}{3} \theta_1^2 \right]$$

$N = 5$:

$$\overline{\Delta V_b} = \frac{V_b}{2N}\left[\theta_5^{\,2} + 2\theta_4^{\,2} + \frac{3}{2}\,\theta_3^{\,2} + \frac{4}{3}\,\theta_2^{\,2} + \frac{5}{4}\,\theta_1^{\,2}\right]$$

or in general, for any N

$$\boxed{\overline{\Delta V_b} = \frac{V_b}{2N}\left[\theta_N^{\,2} + \sum_{K=2}^{N} \frac{K}{K-1}\,\theta_{N+1-K}^{2}\right]}$$

At this point it is of interest to consider the type of errors that cause θ_i.

First there are errors caused by gyro drifts. With a random drift rate of ω which is assumed to be uncorrelated with time, the average error in θ caused by gyro drift is

$$\overline{\theta_g} = \frac{\sigma_\omega t}{2\sqrt{M}} \quad \text{where} \quad \begin{cases} \sigma_\omega = \text{one sigma drift rate} \\ t = \text{time} \\ M = \text{number of uncorrelated intervals} \end{cases}$$

Assuming that the correlation time is equal to the incremental boost time gives

$$\overline{\theta_g^{\,2}} = \frac{\sigma_\omega^{\,2}\left(\frac{T}{N}\right)^2}{4} \quad ; \quad \text{since } M = 1 \text{ by assumption.}$$

Thus the component of $\overline{\Delta V_b}$ due to gyro drift may be expressed as

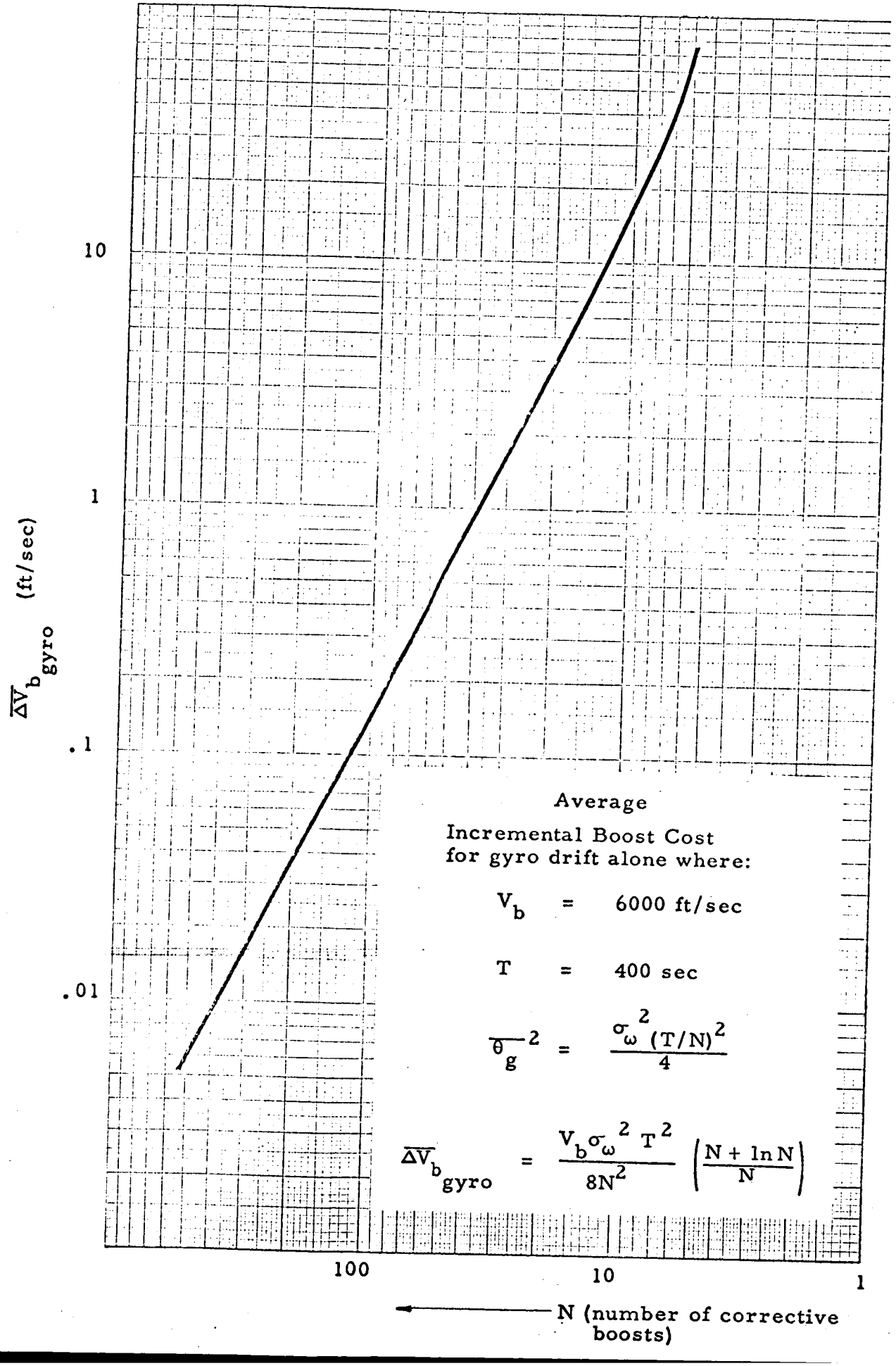
$$\overline{\Delta V_{b_{gyro}}} = \frac{V_b}{2N}\left[\frac{\sigma_\omega^{\,2}}{4}\left(\frac{T}{N}\right)^2\right]\left[1 + \sum_{K=2}^{N}\frac{K}{K-1}\right]$$

The summation may be approximated by an integral giving

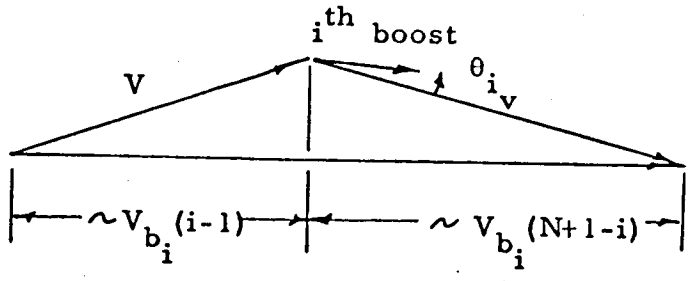
$$\overline{\Delta V_{b_{gyro}}} = \frac{V_b}{2N}\,\frac{\sigma_\omega^{\,2}}{4}\left(\frac{T}{N}\right)^2\left[1 + \int_{x=2}^{x=N+1}\frac{x\,dx}{x-1}\right]$$

$$= \frac{V_b}{8N}\,\sigma_\omega^{\,2}\left(\frac{T}{N}\right)^2 (N + \ln N) = \frac{V_b\sigma_\omega^{\,2}T^2}{8N^2}\left(\frac{N + \ln N}{N}\right)$$

For V_b = 6000 ft/sec., σ_ω = .2°/sec., and T = 400 sec., the following ΔV_b results as a function of the number of corrections, N:


10
1
.1
.01
$\overline{\Delta V}_{b_{gyro}}$ (ft/sec)
Average
Incremental Boost Cost
for gyro drift alone where:
V_b = 6000 ft/sec
T = 400 sec
$\overline{\theta_g}^2 = \frac{\sigma_\omega^2 (T/N)^2}{4}$
$\overline{\Delta V}_{b_{gyro}} = \frac{V_b \sigma_\omega^2 T^2}{8N^2} \left(\frac{N + \ln N}{N}\right)$
100
10
1
N (number of corrective boosts)

Next are the errors caused by an inability in measuring the out-of-plane component of velocity. Using three MSFN stations, the out-of-plane component is the most difficult one to come by.



Given that the measurement error is v with mean deviation σ_v and zero mean, then the mean deviation in the pointing of the axis due to this effect is

$$\sqrt{\overline{\theta_{i_v}^{2}}} = \frac{\sigma_v}{V_{b_i}(N+1-i)}$$

where as before

$$\overline{\Delta V}_{b_v} = \frac{V_b}{2N}\left[\overline{\theta_N^2} + \sum_{K=2}^{N} \frac{K}{K-1}\,\overline{\theta_{N+1-K}^2}\right]$$

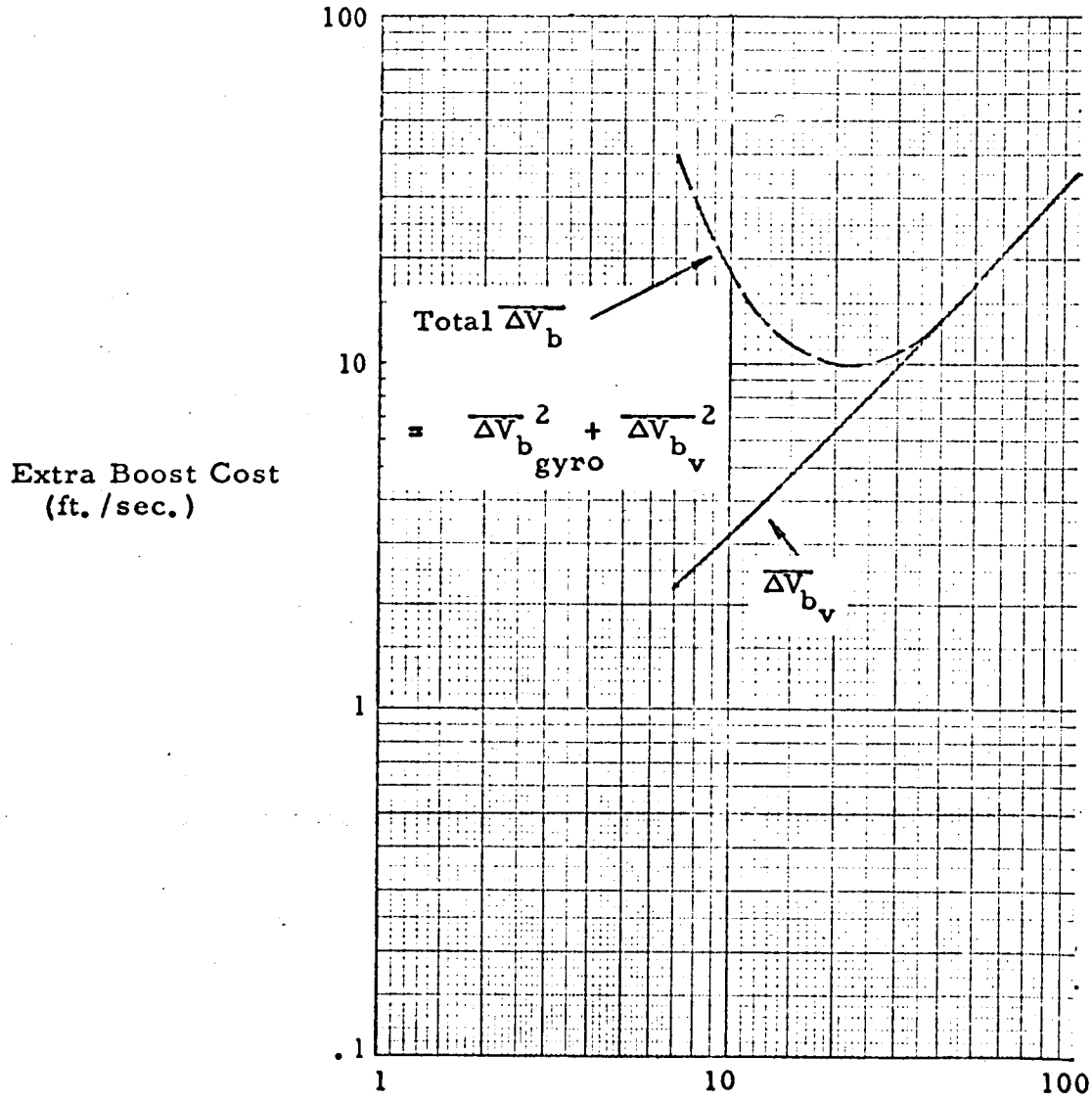
Thus combining these two expressions gives

$$\overline{\Delta V}_{b_v} = \frac{V_b}{2N}\left[\frac{\sigma_v}{V_{b_i}}^{2} + \sum_{K=2}^{N} \frac{K}{K-1}\left(\frac{\sigma_v}{KV_{b_i}}\right)^2\right]$$

and by substituting an integral for the summation this gives

$$\overline{\Delta V}_{b_v} = \frac{\sigma_v^2 N}{2V_b}\left[1 + \ln\left(\frac{2N}{N+1}\right)\right]$$

This may be plotted as a function of N for reasonable assumptions:
σ_v = 50 ft/sec., V_b = 6000 ft/sec.



Average Incremental Boost Cost for Measurement Error Alone (V_b = 6000 ft/sec., σ_v = 50 ft/sec.)

It should be noted that since the two causes for θ_i are independent, that is measurement error and gyro drift, then their effects may be added in quadrature:

$$\overline{\Delta V_{b_{total}}} = \left[\overline{\Delta V_{b_{gyro}}}^{\,2} + \overline{\Delta V_{b_v}}^{\,2} \right]^{1/2}$$

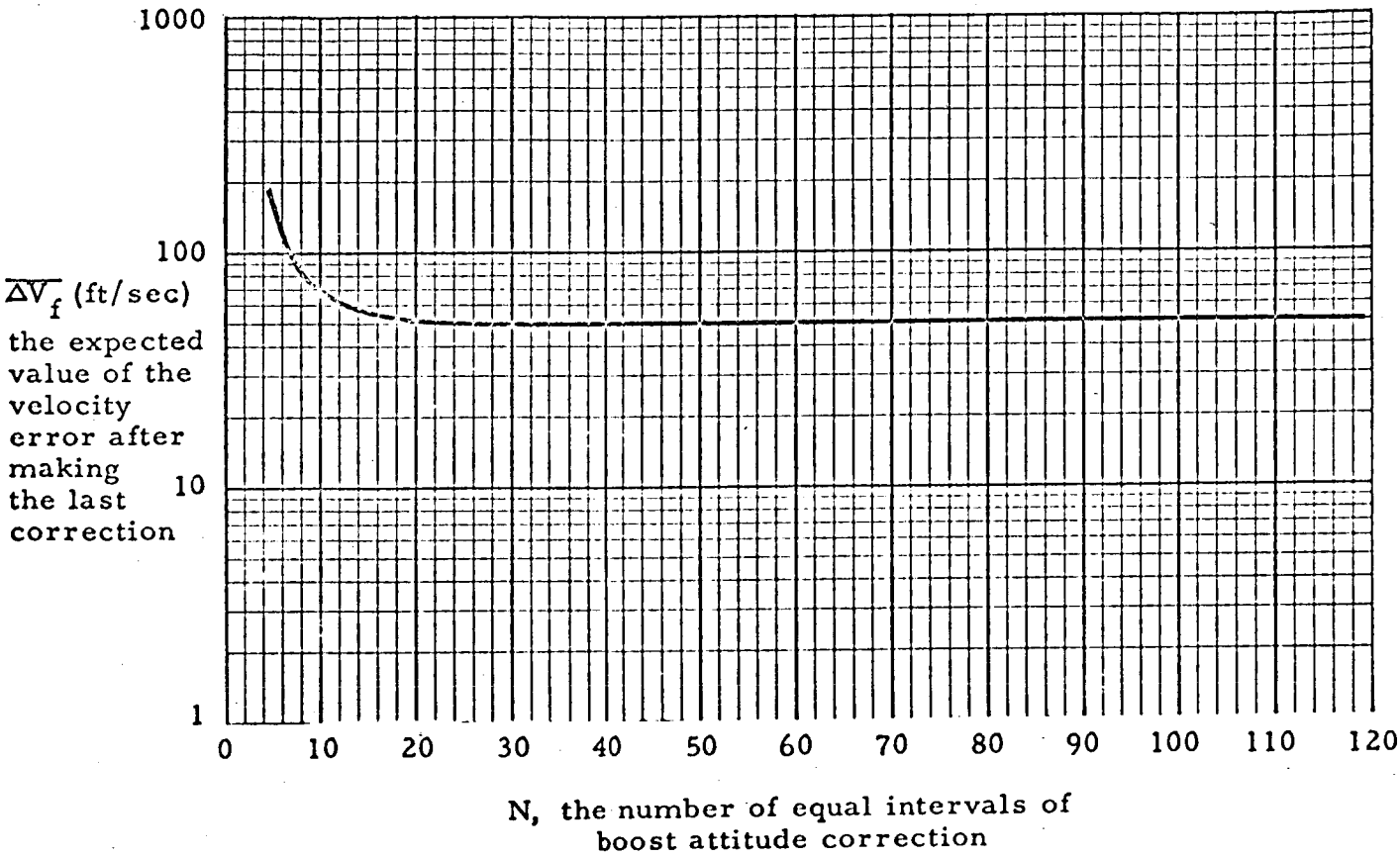
This is also shown in the last figure. It can be seen that for the numbers used, a minimum boost cost of 10 ft/sec. occurs for around twenty equal boost attitude corrections. It is, of course, not necessary that all boost be equal and the figures would imply that it might be a good idea to make fewer attitude corrections near the end of the flight since it is here that measurement errors are expensive.

A problem exists in the fact that the above analysis does not include as part of the boost cost the error made in the very last boost. This error should be expected to be the measurement error vectorially summed with the gyro drift error during the last boost.

$$\overline{\Delta V_{final}}^{\,2} = \sigma_v^2 + \overline{\theta_g}^{\,2} \Delta V_{b_i}^{\,2}$$

$$= \sigma_v^2 + \frac{\sigma_\omega^2 \left(\frac{T}{N}\right)^2}{4} \cdot \left(\frac{V_b}{N}\right)^2$$

Thus the expected value of ΔV_f can be plotted as a function of N



The conclusion which one must draw is that for any number of correction intervals greater than about ten, the expected termination velocity error will be equal to the expected value of the measurement error. In the case of the above assumptions one would expect a terminal error of about 50 ft/sec.

An interesting way of avoiding the penalty of correcting this error consists of shutting off the engine in a preplanned manner about 100 ft/sec. short of the value required for rendezvous. This will place the LEM in a near circular parking orbit. Two minutes of tracking with the non-accelerating LEM should yield extremely accurate orbit parameters determined by the MSFN. That is, after tracking the coasting LEM

for two minutes, the uncertainty of the out-of-plane velocity error should be at least an order of magnitude better than the previously assumed 50 ft/sec. Now, assuming that the LEM has been inertially aligned, a correction can be made which will combine, in quadrature, the 50 ft/sec. error with the 100 ft/sec. yet to be gained along the track. The total boost would thus be

$$V = \sqrt{100^2 + 50^2} = 110 \text{ ft/sec.}$$

and the cost would be only 10 ft/sec. Since the largest errors will occur normal to the track, and since the nominal trajectory to rendezvous is a Hohmann transfer, it is quite possible to split the error in two and take half of it out at perilune and the other half at apolune (i. e., rendezvous, where the injection velocity is also 100 ft/sec.). This will reduce the cost of an initial 50 ft/sec. error to 6 ft/sec. Of course, this can only be done for 180 degree transfers.

Thus the total "expected" boost velocity penalty should be the sum of the extra boost during the thrusting ascent and the correction due to the final cut-off errors discussed above. This average cost of 10 + 10 = 20 ft/sec. says nothing about the variance in the boost penalty. However, it is probably safe to assume that 30 ft/sec. is a reasonable pad. Further analysis should define this number more exactly, yet it is safe to say that a 0.5% required pad is quite small in comparison with the usual fuel pads which are provided to rocket engines.

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 191

L. Horowitz
20 April 1964

AZIMUTH-ELEVATION CONVERSION

Introduction:

This note describes modifications for converting the existing radar angle error analysis program to azimuth-elevation analysis. The criterion for selecting "best" mechanization is that of requiring minimum overall program modification.

Formulation:

The so-called "x-y" radar angles ϕ_1, ϕ_2, together with the azimuth-elevation angles A, E, are given in Figure 1:

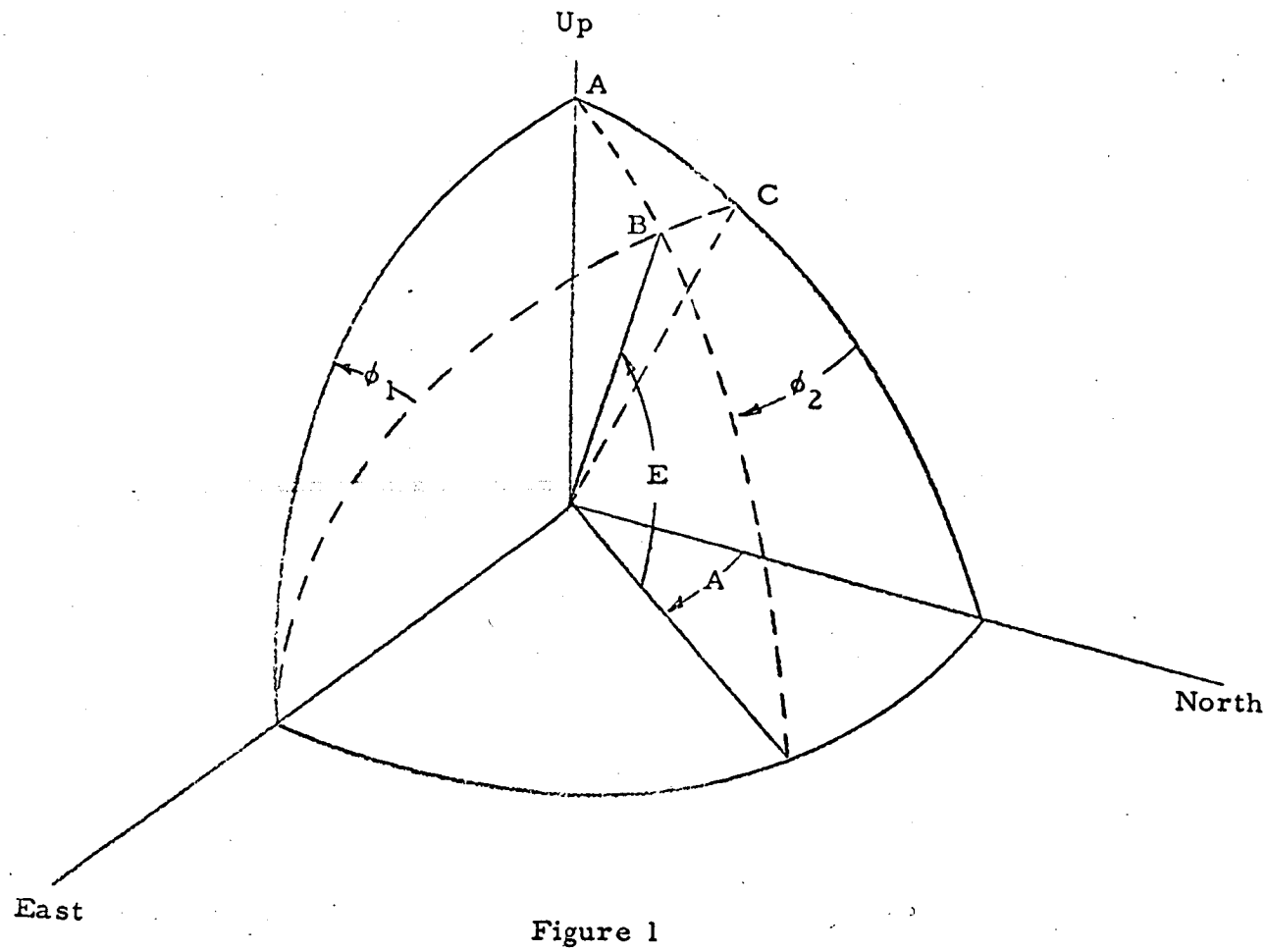


Figure 1

Spherical triangle ABC is shown more clearly in Figure 2:

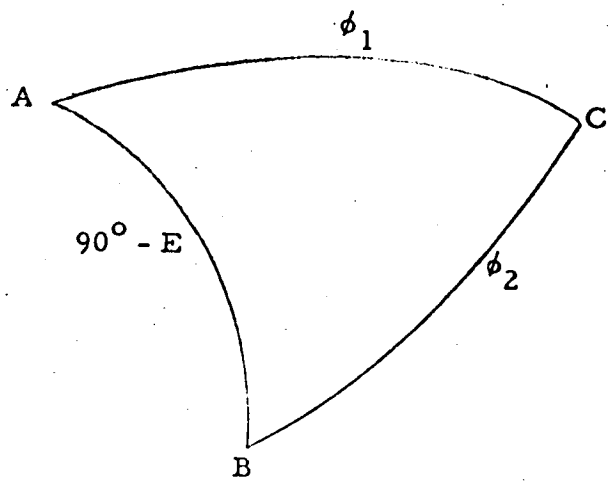


Figure 2

Using Napier's Analogies and the Spherical Law of Sines, we have

$$\tan A = \csc \phi_1 \tan \phi_2 \tag{1}$$

$$\sin E = \cos \phi_2 \cos \phi_1 \tag{2}$$

together with the inverse relations

$$\tan \phi_1 = \cos A \cot E \tag{3}$$

$$\sin \phi_2 = \sin A \cos E \tag{4}$$

Now the existing radar-angle program computes information matrices C^{ℓ} from elements of the form

$$C_{ij}^{\ell} = \sum_k \frac{\partial \phi_\ell(k)}{\partial a_i} \frac{\partial \phi_\ell(k)}{\partial a_j} \tag{5}$$

where ϕ_ℓ, $\ell = 1, 2$ are the two radar angles and a_i are the orbit parameters whose covariance matrices are to be determined. Assuming ϕ_1, ϕ_2 data to be independent, the composite information matrix C is just $C^1 + C^2$.

In order to modify this program to accommodate azimuth-elevation data instead of radar angle data, we observe

$$\frac{\partial \xi_\ell}{\partial a_i} = \frac{\partial \xi_\ell}{\partial \phi_1} \frac{\partial \phi_1}{\partial a_i} + \frac{\partial \xi_\ell}{\partial \phi_2} \frac{\partial \phi_2}{\partial a_i}, \quad \ell = 1, 2 \tag{6}$$

where we have written $\xi_1 = A$, $\xi_2 = E$ for convenience. Now the terms $\frac{\partial \phi_i}{\partial a_j}$ are already provided by the existing program; the only remaining terms are the four quantities $\frac{\partial \xi_i}{\partial \phi_j}$; i, j = 1, 2. These are given immediately by (1) and (2):

$$\begin{aligned}
\frac{\partial \xi_1}{\partial \phi_1} &= - \frac{\csc \phi_1 \cot \phi_1 \tan \phi_2}{\sec^2 A} \\
\frac{\partial \xi_1}{\partial \phi_2} &= \frac{\csc \phi_1 \sec^2 \phi_2}{\sec^2 A} \\
\frac{\partial \xi_2}{\partial \phi_1} &= - \frac{\cos \phi_2 \sin \phi_1}{\cos E} \\
\frac{\partial \xi_2}{\partial \phi_2} &= - \frac{\sin \phi_2 \cos \phi_1}{\cos E}
\end{aligned} \tag{7}$$

where

$$\sec A = \sqrt{1 + \csc^2 \phi_1 \tan^2 \phi_2}$$

$$\cos E = \sqrt{1 - \cos^2 \phi_1 \cos^2 \phi_2} \tag{8}$$

The information matrices are then added and the covariance matrix of the orbit parameters obtained in the usual fashion without further modification to the program.

Caution must be exercised in evaluation of the terms (6), since $\frac{\partial \xi_2}{\partial \phi_1}$, $\frac{\partial \xi_2}{\partial \phi_2}$ have vanishingly small denominators (cos E) as $E \rightarrow \frac{\pi}{2}$. Appendix I provides a solution to this problem by expressing the terms (6) in less volatile form.

APPENDIX I

To circumvent the limiting problem described above, we write the last two terms of (7) as:

$$\frac{\partial \xi_2}{\partial \phi_1} = - \frac{\cos \phi_2 \sin \phi_1}{\sqrt{1 - \cos^2 \phi_1 \cos^2 \phi_2}}$$

$$\frac{\partial \xi_2}{\partial \phi_2} = - \frac{\sin \phi_2 \cos \phi_1}{\sqrt{1 - \cos^2 \phi_1 \cos^2 \phi_2}}$$

and observe from Figure 1 that

$$\phi_1 \rightarrow \frac{\pi}{2} - E \text{ continuously as } E \rightarrow \frac{\pi}{2}$$

$$\phi_2 \rightarrow A \text{ continuously as } A \rightarrow 0$$

so that by physical argument, we can expect the terms (9) to remain bounded for these critical values of the arguments. Dividing numerator and denominator of each expression by the offending term, we obtain

$$\frac{\partial \xi_2}{\partial \phi_1} = \frac{-1}{\sqrt{1 + \csc^2 \phi_1 \tan^2 \phi_2}}$$

$$\frac{\partial \xi_2}{\partial \phi_2} = \frac{-1}{\sqrt{1 + \tan^2 \phi_1 \csc^2 \phi_2}} \quad .$$

In this form, we see that $\frac{\partial \xi_2}{\partial \phi_1}$, $\frac{\partial \xi_2}{\partial \phi_2}$ do not exceed 1 in magnitude for any values of the arguments.

The first two terms of (7) do possess poles at critical values of ϕ_1, ϕ_2, however, and are being investigated further.

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 192

H. Engel
21 April 1964

ACCURACY OF POSITION AND VELOCITY DETERMINATION DURING BOOST, III

The results of Bissett-Berman Apollo Note No. 185 are used to obtain expressions for matrices M, P, Q, and ϕ.

There are some notation changes. Instead of

$$P_{00,\,n+1} = \mathcal{P}_{00,\,n+1} - (\mathcal{M}_{0,\,n+1}\mathcal{P}_{00,\,n+1})^T(\mathcal{M}_{0,\,n+1}\mathcal{P}_{00,\,n+1}\mathcal{M}^T_{0,\,n+1} + \mathcal{Q}_{n+1})^{-1} \cdot (\mathcal{M}_{0,\,n+1}\mathcal{P}_{00,\,n+1})$$

we write

$$U_n = P_n - (M_nP_n)^T(M_nP_nM_n^T + Q_n)^{-1}(M_nP_n)$$

and instead of

$$\mathcal{P}_{00,\,n+1} = \phi_{00,\,n}P_{00,\,n}\phi^T_{00,\,n} + \phi_{12,\,n}Q_n\phi^T_{12,\,n}$$

we write

$$P_{n+1} = \phi_n U_n \phi_n^T + \psi_n R_n \psi_n^T$$

Here Q_n is the error covariance matrix of the random noise on measurement of position or velocity, and R_n is the error covariance matrix of the random noise on acceleration measurements. P_n is the error covariance matrix of the state vector at time t_n, prior to measurements at that time, and U_n is the error covariance matrix of the state

vector at time t_n, after measurements at that time.

As a first example we consider the following situation. A vehicle moves in the x, y plane, with accelerations a_x and a_y along its axes. The attitude reference of the vehicle is assumed to be in error by a fixed, unknown angle $\delta\theta$. The vehicle accelerometers are assumed to be perfect and noise-free. The measurables are vehicle acceleration in vehicle coordinates, and $\dot{x}$ and $\dot{y}$ in the reference coordinates. The state vector is

$$X = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \theta \end{bmatrix}$$

The initial velocity is assumed known perfectly, and the platform tilt known with a standard deviation σ_θ.

In these circumstances

$$\ddot{x}_n = a_{xn} - a_{yn}\theta$$

$$\ddot{y}_n = a_{xn}\theta + a_{yn}$$

$$\dot{x}_{n+1} = \left[a_{xn} - a_{yn}\theta\right]\tau + \dot{x}_n$$

$$\dot{y}_{n+1} = \left[a_{xn}\theta + a_{yn}\right]\tau + \dot{y}_n$$

The predicted values of $\dot{x}_{n+1}$ and $\dot{y}_{n+1}$ are

$$\dot{x}_{n+1,\,p} = \left[a_{xn} - a_{yn}(\theta + \delta\theta_n)\right]\tau + \dot{x}_n + \delta\dot{x}_n$$

$$\dot{y}_{n+1,\,p} = \left[a_{xn}(\theta + \delta\theta_n) + a_{yn}\right]\tau + \dot{y}_n + \delta\dot{y}_n$$

in which $\delta\dot{x}_n$ and $\delta\dot{y}_n$ are the errors in the estimates of $\dot{x}_n$ and $\dot{y}_n$. Then

$$\dot{x}_{n+1,\,p} = \left[a_{xn} - a_{yn}\theta\right]\tau + \dot{x}_n + \delta\dot{x}_n - a_{yn}\tau\,\delta\theta_n$$

$$\dot{y}_{n+1,\,p} = \left[a_{xn}\theta + a_{yn}\right]\tau + \dot{y}_n + \delta\dot{y}_n + a_{xn}\tau\,\delta\theta_n$$

or

$$\delta\dot{x}_{n+1,\,p} = \delta\dot{x}_n - a_{yn}\,\tau\,\delta\theta_n$$

$$\delta\dot{y}_{n+1,\,p} = \delta\dot{y}_n + a_{xn}\,\tau\,\delta\theta_n$$

$$\delta\theta_{n+1,\,p} = \delta\theta_n$$

We may then write

$$\delta X_{n+1,\,p} = \phi_n\,\delta X_n$$

in which

$$\phi_n = \begin{bmatrix} 1 & 0 & -a_{yn}\tau \\ 0 & 1 & a_{xn}\tau \\ 0 & 0 & 1 \end{bmatrix}$$

The expected values of δX_n and $\delta X_{n+1,\,p}$ are zero and their covariance matrices are U_n and P_{n+1}.

The measurables $\dot{x}_{m,\,n}$ and $\dot{y}_{m,\,n}$ depend only on $\dot{x}_n$ and $\dot{y}_n$, and we have

$$\begin{bmatrix} \dot{x}_{m,n} \\ \dot{y}_{m,n} \end{bmatrix} = M_n X_n + [\text{Noise}]$$

Thus

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

A particular case of this example has been run on a digital computer, using

$$P_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

$$Q_n = \begin{bmatrix} .01 & 0 \\ 0 & .01 \end{bmatrix}$$

and $a_{xn} = 5\ m/sec^2$, $a_{yn} = 0\ m/sec^2$, and $\tau = 1$ sec., corresponding to a constant acceleration in the x direction. Then

$$\phi_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

The results are shown in Figure 1. The uncertainty in platform orientation decreases rapidly. The uncertainty in the y component of velocity jumps from the a priori value of 0 to a maximum of 0.098 m/sec. in one second, and then decreases slowly. There is no uncertainty

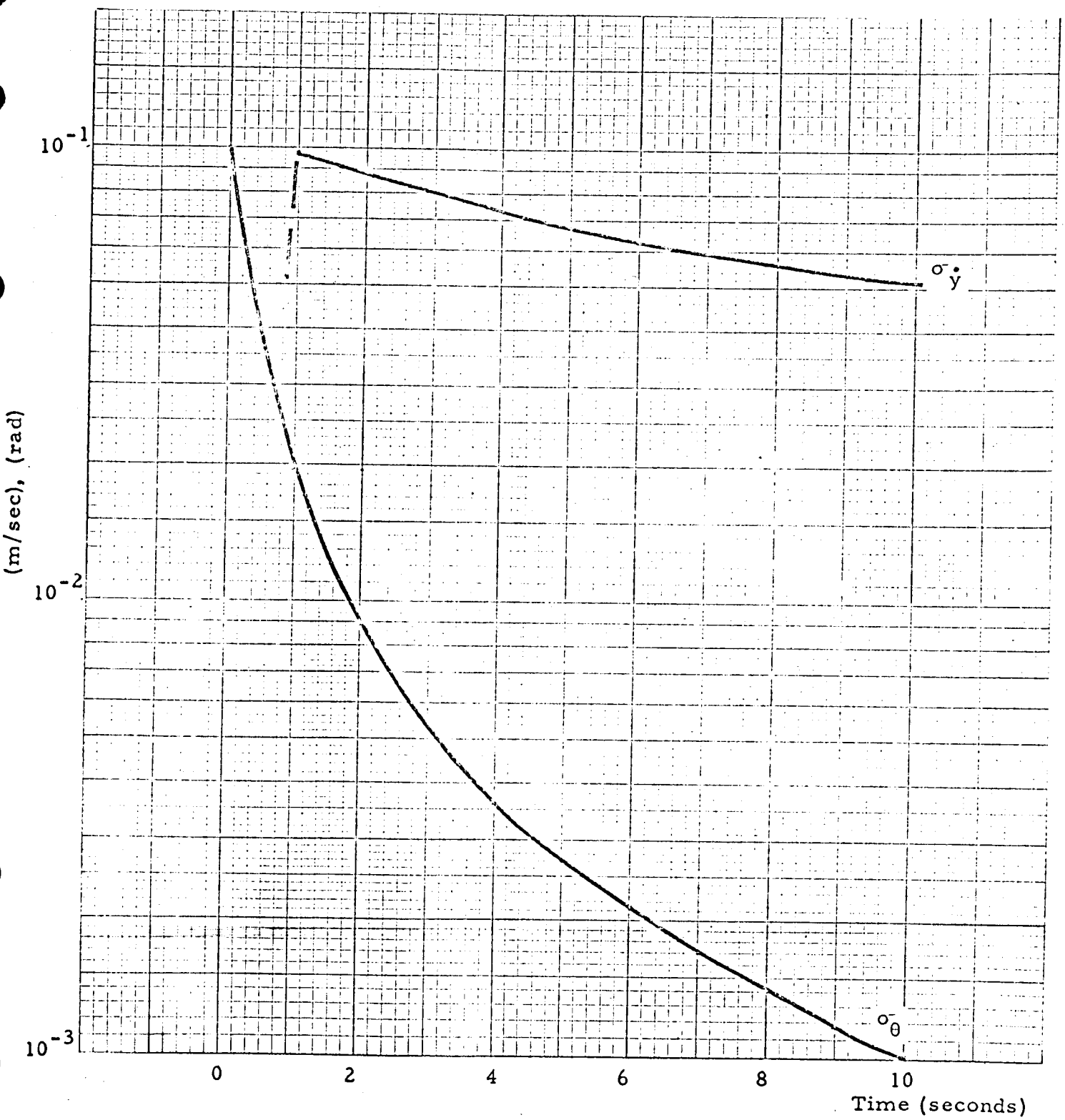


Figure 1
Test Case 5

in the x component of velocity because we have assumed perfect accelerometers and because we have used a small angle approximation, replacing cos $\delta\theta$ with unity.

The first iteration in this case has also been hand computed and the results compared with the computer output. We have

$$MP_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .01 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$MP_0M^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$MP_0M^T + Q = \begin{bmatrix} .01 & 0 \\ 0 & .01 \end{bmatrix}$$

$$\left[MP_0M^T + Q\right]^{-1} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

$$U_0 = P_0 - (MP_0)^T \left[MP_0M^T + Q\right]^{-1} (MP_0)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .01 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .01 \end{bmatrix}$$

$$\phi U_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .01 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & .05 \\ 0 & 0 & .01 \end{bmatrix}$$

$$P_1 = \phi U_0 \phi^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & .05 \\ 0 & 0 & .01 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .25 & .05 \\ 0 & .05 & .01 \end{bmatrix}$$

$$MP_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & .25 & .05 \\ 0 & .05 & .01 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .25 & .05 \end{bmatrix}$$

$$MP_1M^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .25 & .05 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & .25 \end{bmatrix}$$

$$MP_1M^T + Q = \begin{bmatrix} .01 & 0 \\ 0 & .26 \end{bmatrix}$$

$$\left[MP_1M^T + Q\right]^{-1} = \frac{1}{.26} \begin{bmatrix} 26 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(MP_1)^T \left[MP_1M^T + Q\right] = \begin{bmatrix} 0 & 0 \\ 0 & .25 \\ 0 & .05 \end{bmatrix} \begin{bmatrix} 26 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{.26} = \frac{1}{26} \begin{bmatrix} 0 & 0 \\ 0 & 25 \\ 0 & 5 \end{bmatrix}$$

$$(MP_1)^T \left[MP_1 M^T + Q \right]^{-1} (MP_1) = \frac{1}{26} \begin{bmatrix} 0 & 0 \\ 0 & 25 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & .25 & .05 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6.25 & 1.25 \\ 0 & 1.25 & .25 \end{bmatrix}$$

$$U_1 = P_1 - (MP_1)^T \left[MP, M^T + Q \right] (MP_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .25 & .05 \\ 0 & .05 & .01 \end{bmatrix} - \frac{1}{26} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6.25 & 1.25 \\ 0 & 1.25 & .25 \end{bmatrix}$$

$$= \frac{1}{26} \begin{bmatrix} 0 & 0 & 0 \\ 0 & .25 & .0005 \\ 0 & .0005 & .0001 \end{bmatrix}$$

Then

$$\sigma_{\dot{y}} = (.25/26)^{1/2} = 0.098058068$$

and

$$\sigma_{\dot{\theta}} = (.01/26)^{1/2} = 0.019611614$$

The corresponding computer outputs are

$$\sigma_{\dot{y}} = 0.09805806$$

and

$$\sigma_{\dot{x}} = 0.01961162$$

As a second example, we consider the following situation. A vehicle moves in three dimensions. Its on-board accelerometers

measure its acceleration perfectly with respect to its stable platform, but the platform has initial alignment errors θ_{x_0}, θ_{y_0} and θ_{z_0} and fixed drift rates ω_x, ω_y and ω_z. The range rate of the vehicle is measured from three stations. The state vector X is

$$X = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \theta_{x_0} \\ \theta_{y_0} \\ \theta_{z_0} \\ \omega_x \\ \omega_y \\ w_z \end{bmatrix}$$

Then,

$$\ddot{x}_n = a_{xn} - \theta_{zn} + \theta_{yn} a_{zn}$$

$$= a_{xn} - (\theta_{z0} + \omega_{zn} t_n)\, a_{yn} + (\theta_{y0} + \omega_y t_n)\, a_{zn}$$

$$\dot{x}_{n+1} = \dot{x}_n + \ddot{x}_n \tau$$

$$x_{n+1} = x_n + \dot{x}_n \tau + \ddot{x}_n \frac{\tau^2}{2}$$

and the predicted values of $\dot{x}_{n+1}$ and x_{n+1} are

$$\dot{x}_{n+1,p} = \dot{x}_{n+1} + \delta\dot{x}_n - \tau a_{yn}(\delta\theta_{z0} + t_n\delta\omega_z) + \tau a_{zn}(\delta\theta_{y0} + t_n\delta\omega_y)$$

$$x_{n+1,p} = x_n + \delta x_n + \tau\delta\dot{x}_n - \frac{\tau^2}{2} a_{yn}(\delta\theta_{z0} + t_n\delta\omega_z) + \frac{\tau^2}{2} a_{yn}(\delta\theta_{y0} + t_n\delta\omega_y)$$

Similar expressions may be obtained for the predicted values of $\dot{y}_{n+1}$, y_{n+1}, $\dot{z}_{n+1}$ and z_{n+1}. Then we can write

$$\delta X_{n+1,p} = \phi_n \, \delta X_n$$

where

$$\phi_n = \begin{bmatrix}
1 & 0 & 0 & \tau & 0 & 0 & 0 & \frac{\tau^2 a_{zn}}{2} & -\frac{\tau^2 a_{yn}}{2} & 0 & \frac{\tau^2 a_{zn} t_n}{2} & -\frac{\tau^2 a_{yn} t_n}{2} \\
0 & 1 & 0 & 0 & \tau & 0 & -\frac{\tau^2 a_{zn}}{2} & 0 & \frac{\tau^2 a_{xn}}{2} & -\frac{\tau^2 a_{zn} t_n}{2} & 0 & \frac{\tau^2 a_{zn} t_n}{2} \\
0 & 0 & 1 & 0 & 0 & \tau & \frac{\tau^2 a_{yn}}{2} & -\frac{\tau^2 a_{xn}}{2} & 0 & \frac{\tau^2 a_{yn} t_n}{2} & -\frac{\tau^2 a_n t_n}{2} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \tau a_{zn} & -\tau a_{yn} & 0 & \tau a_{zn} t_n & -\tau a_{yn} t_n \\
0 & 0 & 0 & 0 & 1 & 0 & -\tau a_{zn} & 0 & \tau a_{xn} & -\tau a_{zn} t_n & 0 & \tau a_{xn} t_n \\
0 & 0 & 0 & 0 & 0 & 1 & \tau a_{yn} & -\tau a_{xn} & 0 & \tau a_{yn} t_n & -\tau a_{xn} t_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

In order to find the M_n matrix, let us temporarily change coordinate systems, substituting x_1, x_2, x_3 for x, y, z. We let the vehicle be at x_1, x_2, x_3 and the jth observing station at x_{1j}, x_{2j}, x_{3j}. Then, letting all summations be from 1 to 3,

$$r^2 = \sum_i (x_i - x_{ij})^2$$

$$r\dot{r} = \sum_i (x_i - x_{ij})\dot{x}_i$$

assuming that the station locations are fixed. Then

$$r\frac{\partial r}{\partial x_k} = x_k - x_{k_j}$$

or

$$\frac{\partial r}{\partial x_k} = \frac{x_k - x_{k_j}}{r}$$

Also

$$\frac{\partial \dot{r}}{\partial \dot{x}_k} = \frac{x_k - x_{k_j}}{r}$$

Further,

$$r\frac{\partial \dot{r}}{\partial x_k} + \dot{r}\frac{\partial r}{\partial x_k} = \dot{x}_k$$

or

$$\frac{\partial \dot{r}}{\partial x_k} = \frac{\dot{x}_k}{r} - \frac{\dot{r}}{r}\frac{\partial r}{\partial x_k}$$

and $\partial \dot{r}/\partial x_k$ and $\partial r/\partial s_k$ are the elements of the M_n matrix. Returning to our normal notation, and letting

x, y, z be coordinates of vehicle

x_j, y_j, z_j be coordinates of jth station,

we have

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

$$r_2^2 = (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2$$

$$r_3^2 = (x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2$$

$$\frac{\dot{r}_1}{r_1} = \frac{(x - x_1)\dot{x} + (y - y_1)\dot{y} + (z - z_1)\dot{z}}{r_1^2}$$

$$\frac{\dot{r}_2}{r_2} = \frac{(x - x_2)\dot{x} + (y - y_2)\dot{y} + (z - z_2)\dot{z}}{r_2^2}$$

$$\frac{\dot{r}_3}{r_3} = \frac{(x - x_3)\dot{x} + (y - y_3)\dot{y} + (z - z_3)\dot{z}}{r_3^2}$$

and

$$M = \begin{bmatrix} \frac{\dot{x}}{r_1} - \frac{\dot{r}_1}{r_1}\frac{x-x_1}{r_1}, & \frac{\dot{y}}{y_1} - \frac{\dot{r}_1}{r_1}\frac{y-y_1}{r_1}, & \frac{\dot{z}}{r_1} - \frac{\dot{r}_1}{r_1}\frac{z-z_1}{r_1}, & \frac{x-x_1}{r_1}, & \frac{y-y_1}{r_1}, & \frac{z-z_1}{r_1}, & 0, & 0, & 0, & 0, & 0, & 0, \\ \frac{\dot{x}}{r_1} - \frac{\dot{r}_2}{r_2}\frac{x-x_2}{r_2}, & \frac{\dot{y}}{r_2} - \frac{\dot{r}_2}{r_2}\frac{y-y_2}{r_2}, & \frac{\dot{z}}{r_2} - \frac{\dot{r}_2}{r_2}\frac{z-z_2}{r_2}, & \frac{x-x_2}{r_2}, & \frac{y-y_2}{r_2}, & \frac{z-z_2}{r_2}, & 0, & 0, & 0, & 0, & 0, & 0, \\ \frac{\dot{x}}{r_3} - \frac{\dot{r}_3}{r_3}\frac{x-x_3}{r_3}, & \frac{\dot{y}}{r_3} - \frac{\dot{r}_3}{r_3}\frac{y-y_3}{r_3}, & \frac{\dot{z}}{r_3} - \frac{\dot{r}_3}{r_3}\frac{z-z_3}{r_3}, & \frac{x-x_3}{r_3}, & \frac{y-y_3}{r_3}, & \frac{z-z_3}{r_3}, & 0, & 0, & 0, & 0, & 0, & 0, \end{bmatrix}$$

A particular case of this example has been run. In this case the vehicle is initially at

$$(x, y, z) = (0.38 \times 10^9, 0, 0) \text{ meters and the three}$$

stations are at

$$(x_1, y_1, z_1) = (0, 2.4 \times 10^6, 0)$$

$$(x_2, y_2, z_2) = (0, -2.1 \times 10^6, 1.2 \times 10^6) \quad \text{meters}$$

$$(x_3, y_3, z_3) = (0, -2.1 \times 10^6, -1.2 \times 10^6) ;$$

i.e., the stations are in the plane $x = 0$, symmetrically distributed about the origin and at a distance of 2.4×10^6 meters from the origin.

The vehicle has an initial velocity of 800 meters/sec. in the y direction, and accelerates in that direction at 5 meters/sec^2. The observation interval is 1 second.

The initial range to the vehicle is very, very nearly $x(0)$ ($= 0.38 \times 10^9$ meters). In 10 seconds the vehicle travels 8,250 meters. At the end of this time the distance from any station to the vehicle is very, very nearly

$$r_j = \left[(0.38 \times 10^9)^2 + (8250)^2\right]^{1/2}$$

$$= 0.38 \times 10^9 \text{ meters}$$

At the end of 10 seconds

$$x, y, z = (0.38 \times 10^9, 8.25 \times 10^3, 0) \text{ meters}$$

$$\dot{x}, \dot{y}, \dot{z} = (0, 850, 0) \text{ meters/second}$$

At 10 seconds, then, letting x_0, y_0, z_0 be the initial values of x, y and z,

$$\frac{x-x_1}{r_1} = 1, \quad \frac{y-y_1}{r_1} = \frac{8.25 \times 10^3 - 2.4 \times 10^6}{0.38 \times 10^9} \cong \frac{-2.4 \times 10^6}{0.38 \times 10^9}, \quad \frac{z-z_1}{r_1} = 0$$

$$\frac{x-x_2}{r_2} = 1, \quad \frac{y-y_2}{r_2} = \frac{8.25 \times 10^3 + 2.1 \times 10^6}{.38 \times 10^9} \cong \frac{2.1 \times 10^6}{0.38 \times 10^9}, \quad \frac{z-z_2}{r_2} = \frac{-1.2 \times 10^6}{.38 \times 10^9}$$

$$\frac{x-x_3}{r_3} = 1, \quad \frac{y-y_3}{r_3} = \frac{8.25 \times 10^3 + 2.1 \times 10^6}{.38 \times 10^9} \cong \frac{2.1 \times 10^6}{.38 \times 10^9}, \quad \frac{z-z_3}{r_3} = \frac{1.2 \times 10^6}{.38 \times 10^9}$$

$$\frac{\dot{r}_1}{r_1} = \frac{(y-y_1)\dot{y}}{r_1^2} \cong \frac{-2.4 \times 10^6}{.38 \times 10^9} \cdot \frac{850.}{.38 \times 10^9} = -1.41 \times 10^{-9}$$

$$\frac{\dot{r}_2}{r_2} = \frac{(y-y_2)\dot{y}}{r_2^2} \cong \frac{2.1 \times 10^6}{.38 \times 10^9} \cdot \frac{850.}{.38 \times 10^9} = 1.23 \times 10^{-8}$$

$$\frac{\dot{r}_3}{r_3} \cong 1.23 \times 10^{-8}$$

$$M_{11} = 0 + 1.23 \times 10^{-9} \times 1 = 1.23 \times 10^{-8}$$

$$M_{12} = \frac{850}{.38 \times 10^9} - 1.41 \times 10^{-9} \frac{2.4 \times 10^6}{0.38 \times 10^9} = \frac{850}{.38 \times 10^9} = 2.24 \times 10^{-6}$$

and so on, the values of M_{ij} at 10 seconds being very little different from those at time 0,

$$M_{11} = 1.33 \times 10^{-8}$$

$$M_{12} = 2.10 \times 10^{-6}, \text{ etc.}$$

Thus we may use fixed values for M_{ij}, with very little error. We take the values at time 0

$$M = \begin{bmatrix} 1.33\times10^{-8} & 2.10\times10^{-6} & 0 & 1 & -6.32\times10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.16\times10^{-8} & 2.10\times10^{-6} & 3.68\times10^{-11} & 1 & 5.53\times10^{-3} & -3.16\times10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.16\times10^{-8} & 2.10\times10^{-6} & -3.68\times10^{-11} & 1 & 5.53\times10^{-3} & 3.16\times10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The a priori error covariance matrix was chosen with all elements zero except

$$P_{0,11} = P_{0,22} = P_{0,33} = 10^2 \text{, meters}^2$$

$$P_{0,77} = P_{0,88} = P_{0,99} = 6.4 \times 10^{-5} \text{ rad}^2$$

$$P_{0,10,10} = P_{o,11,11} = P_{o,12,12} = 10^{-2} \text{ rad}^2/\text{sec}^2$$

and the measurement noise covariance matrix was

$$Q = \begin{bmatrix} 10^{-4} & 0 & 0 \\ 0 & 10^{-4} & 0 \\ 0 & 0 & 10^{-4} \end{bmatrix}$$

The computations were done on a computer using single precision, but blew up completely after 8 iterations because of accumulated round-off errors. These results are shown in Figure 2.

Next the computations were done on a computer using double precision and 10 iterations were computed with negligible round-off errors. These results appear in Figure 3. The standard deviations

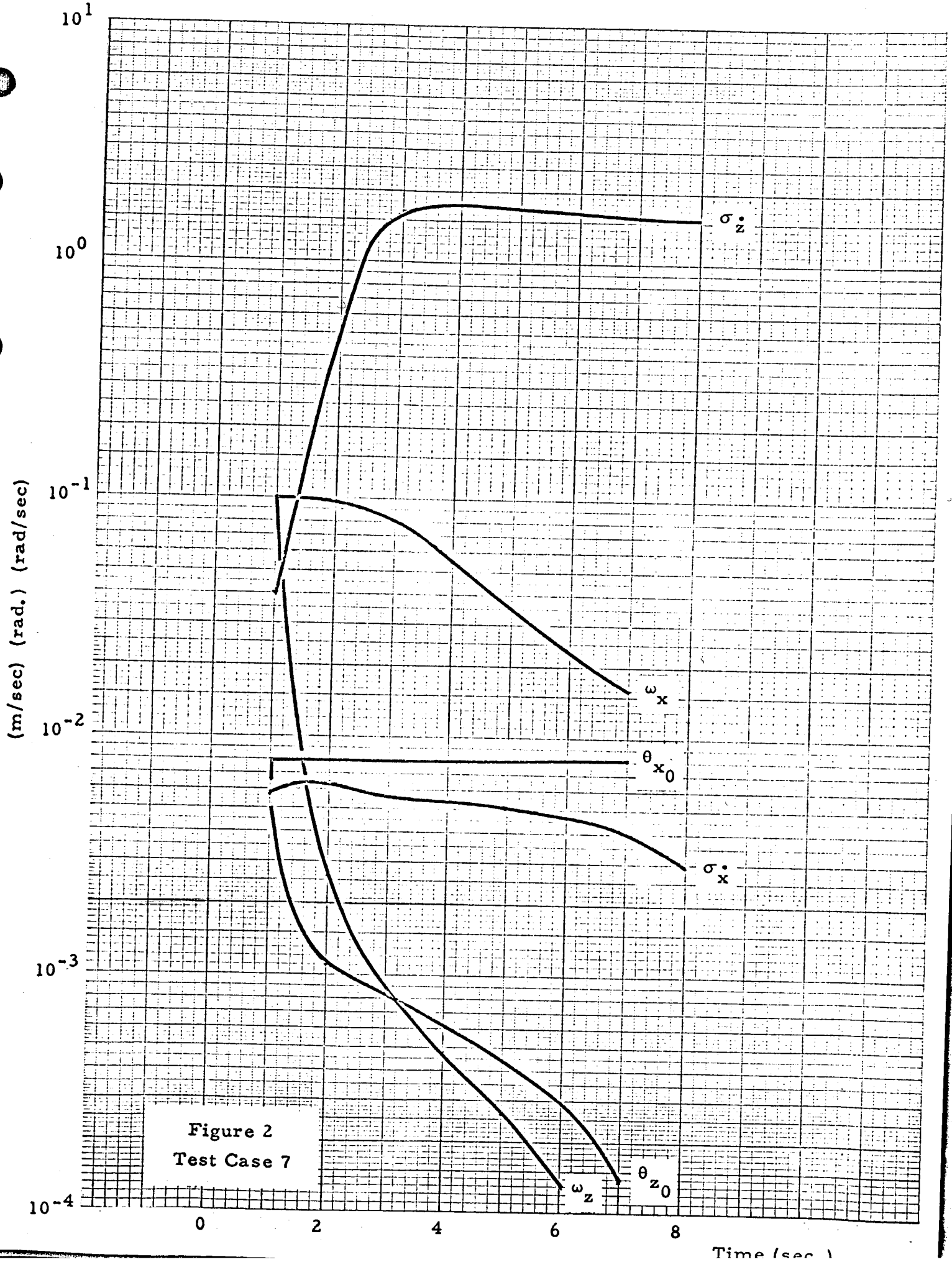


Figure 2
Test Case 7

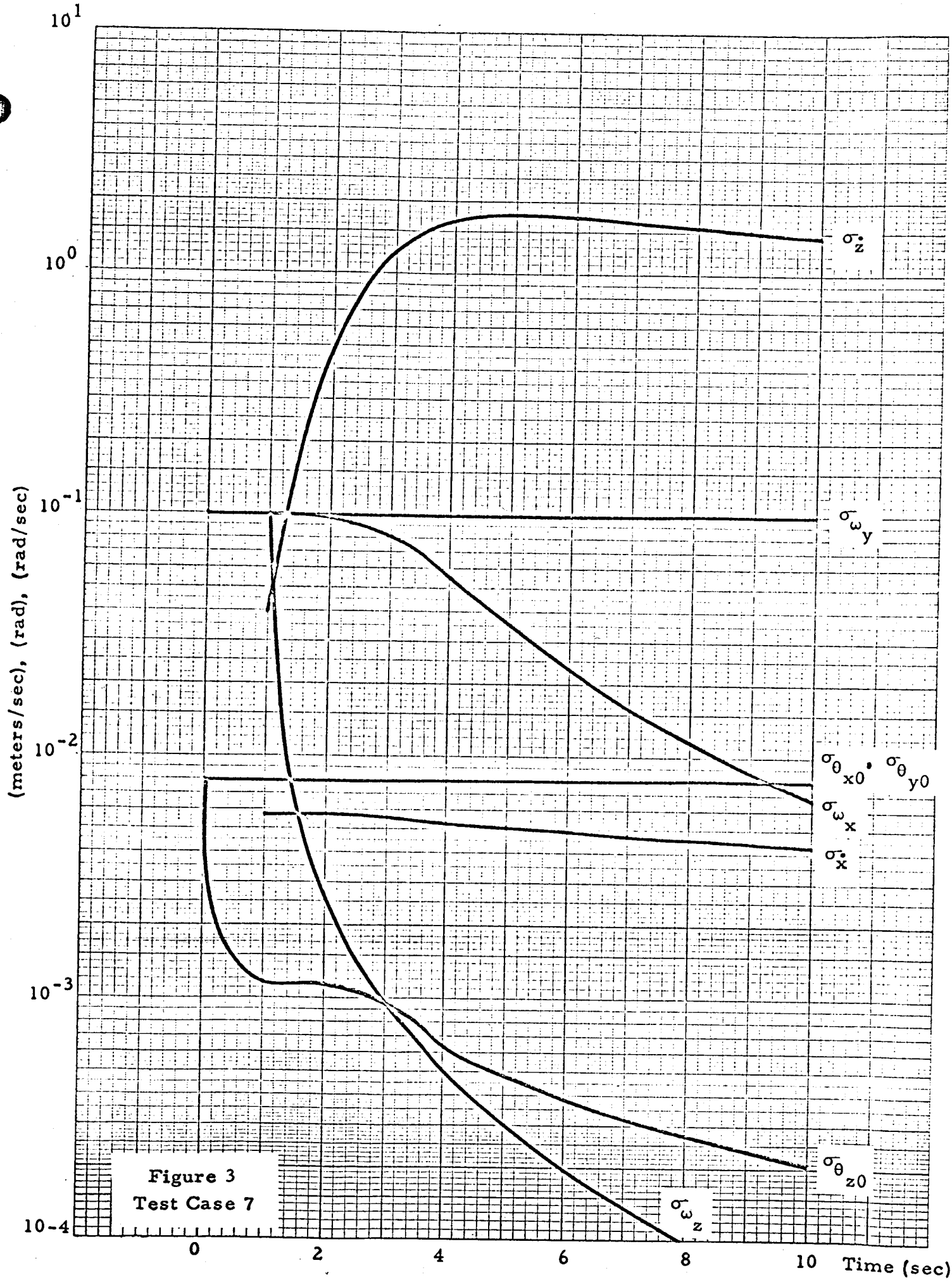


Figure 3
Test Case 7

of the position errors in the x, y and z directions are not shown, but remain approximately 10 meters. The standard deviation in $\dot{y}$ is not shown; since the accelerometers are assumed perfect, the cosines of the tilt angles equal to unity and $\sigma_{\dot{y}_0}$ equal to zero, $\sigma_{\dot{y}}$ remains zero.

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APOLLO NOTE NO. 193

H. Engel
22 April 1964

ACCURACY OF POSITION AND VELOCITY DETERMINATION DURING BOOST, IV

It has been pointed out that present platform misalignment rather than initial platform misalignment should be computed. Also the effects of a gravity field should be included. Further, since we intend to ignore the rotation of the Earth and the motion of the Moon about the Earth in this analysis, we must show that this action is justified.

We have

$$\ddot{x} = a_x - \theta_z a_y + \theta_y a_z + g_x$$

$$\ddot{y} = \theta_z a_x + ay - \theta_x a_z + g_y$$

$$\ddot{z} = -\theta_y a_x + \theta_x a_y + a_z + g_z$$

Now

$$\bar{g} = -\mu\bar{r}/r^3$$

and

$$\delta\bar{g} = \frac{\mu}{r^3}\left(-\delta\bar{r} + 3\frac{\bar{r}}{r}\delta r\right)$$

Further, if $\delta\bar{r}$ is kept to less than, say, 500 meters,

$$|\delta\bar{g}| < \frac{3.9969 \times 10^{14}}{(6.3781 \times 10^{6})^3}(500 + 3[500])$$

$$< 2.22 \times 10^{-3} \text{ meters/sec}^2$$

which in 400 seconds would alter the velocity by only 0.888 meters/second,

and the position by only 178 meters. This is a very loose upper bound, since the error has been assumed to persist over the entire powered flight, and because the magnitudes of the error components have been added directly. Hence if the computations of position and velocity uncertainty, ignoring $\delta\bar{g}$, lead to position errors of 500 meters or less, we have a certain upper bound to the error from this source.

The predicted value of $\delta\omega_{j,\ n+1}$

$$\delta\omega_{j,\ n+1,\ p} = \delta\omega_{j,\ n}$$

and the predicted value of $\delta\theta_{j,\ n+1}$ is

$$\delta\theta_{j,\ n+1,\ p} = \delta\theta_{j,\ n} + \tau\,\delta\omega_{j,\ n}$$

Then the predicted values of the remaining quantities are

$$\delta\dot{x}_{n+1,\ p} = \delta\dot{x}_n + \tau(-a_y\delta\theta_z + a_z\delta\theta_y) + \frac{\tau^2}{2}(\delta a_y\delta\omega_z + a_z\delta\omega_y)$$

$$\delta\dot{y}_{n+1,\ p} = \delta\dot{y}_n + \tau(ax\delta\theta_z - a_z\delta\theta_x) + \frac{\tau^2}{2}(a_x\delta\omega_z - a_z\delta\omega_x)$$

$$\delta\dot{z}_{n+1,\ p} = \delta\dot{z}_n + \tau(-a_x\delta\theta_y + a_y\delta\theta_x) + \frac{\tau^2}{2}(-a_x\delta\omega_y + a_y\delta\omega_x)$$

$$\delta x_{n+1,\ p} = \delta x_n + \tau\delta\dot{x}_n + \frac{\tau^2}{2}(-a_y\delta\theta_z + a_z\delta\theta_y) + \frac{\tau^3}{6}(-a_y\delta\omega_z + a_z\delta\omega_y)$$

$$\delta y_{n+1,\ p} = \delta y_n + \tau\,\delta\dot{y}_n + \frac{\tau^2}{2}(a_x\delta\theta_z - a_z\delta\theta_x) + \frac{\tau^3}{6}(a_x\delta\omega_z - a_z\delta\omega_x)$$

$$\delta z_{n+1,\ p} = \delta z_n + \tau\,\delta\dot{z}_n + \frac{\tau^2}{2}(-a_x\delta\theta_y + a_y\delta\theta_x) + \frac{\tau^3}{6}(-a_x\delta\omega_y + a_y\delta\omega_x)$$

and the resultant ϕ_n matrix is

$$
\phi_n = \begin{bmatrix}
1 & 0 & 0 & \tau & 0 & 0 & 0 & \frac{a_z\tau^2}{2} & \frac{-a_y\tau^2}{2} & 0 & \frac{a_z\tau^3}{6} & \frac{-a_y\tau^3}{6} \\
0 & 1 & 0 & 0 & \tau & 0 & \frac{-a_z\tau^2}{2} & 0 & \frac{a_x\tau^2}{2} & \frac{-a_z\tau^3}{6} & 0 & \frac{a_x\tau^3}{6} \\
0 & 0 & 1 & 0 & 0 & \tau & \frac{a_y\tau^2}{2} & \frac{-a_x\tau^2}{2} & 0 & \frac{a_y\tau^3}{6} & \frac{-a_x\tau^3}{6} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & a_z\tau & -a_y\tau & 0 & \frac{a_z\tau^2}{2} & \frac{-a_y\tau^2}{2} \\
0 & 0 & 0 & 0 & 1 & 0 & -a_x\tau & 0 & a_x\tau & \frac{-a_z\tau^2}{2} & 0 & \frac{a_x\tau^2}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & a_y\tau & -a_x\tau & 0 & \frac{a_y\tau^2}{2} & \frac{-a_x\tau^2}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tau & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tau \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

The other matrices of Apollo Note No. 192 remain unchanged.

In order to demonstrate that for the short time of ascent boost (400 seconds) the rotation of the Earth and of the Moon about the Earth may be ignored, rendezvous computations have been performed using actual ω_M and ω_E, and also using $\omega_M = \omega_E = 0$ for comparison. The rendezvous miss is only slightly changed, as shown in Figure 1.

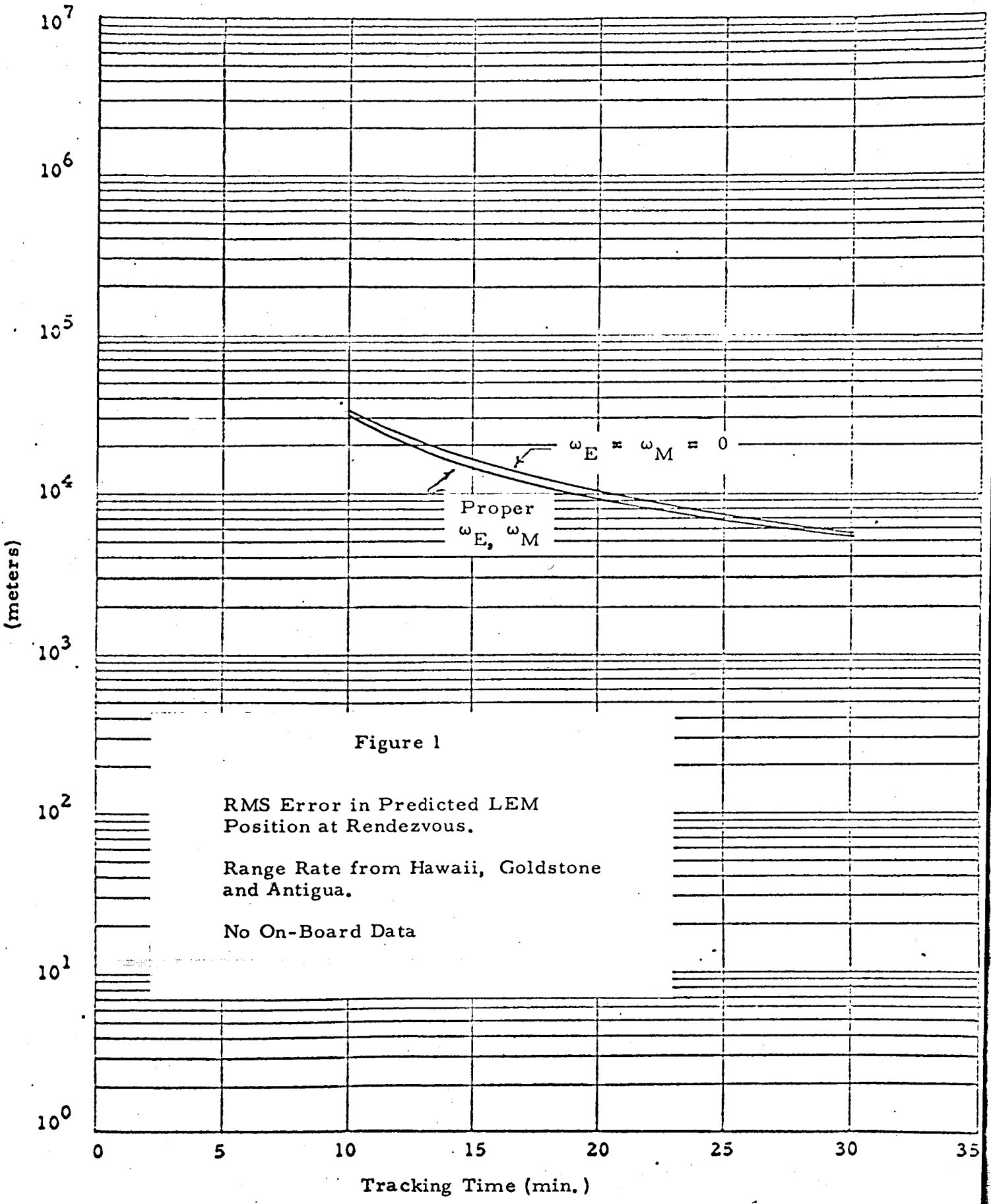

Figure 1

RMS Error in Predicted LEM Position at Rendezvous.

Range Rate from Hawaii, Goldstone and Antigua.

No On-Board Data

Data Sets 0189, 0193

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APOLLO NOTE NO. 194

L. Horowitz
21 April 1964

ANGLE COMPOSITION

Introduction

This note combines the azimuth-elevation conversion results of Apollo Note No. 191 with the kinematic equations (Section 3.3 of Apollo Final Report, NASw-688, 2 March 1964) for radar angle data to obtain the composition of azimuth-elevation from x-y angle data.

Formulation

The radar angles ϕ_1, ϕ_2 together with the azimuthal angle A and elevation angle E are illustrated in Figure 1 of Apollo Note No. 191. Letting $\xi_1 = A$ and $\xi_2 = E$ as before, we have the results of Note 191:

$$\frac{\partial \xi_1}{\partial \phi_1} = - \frac{\csc \phi_1 \cot \phi_1 \tan \phi_2}{1 + \csc^2 \phi_1 \tan^2 \phi_2} \tag{1}$$

$$\frac{\partial \xi_1}{\partial \phi_2} = \frac{\csc \phi_1 \sec^2 \phi_2}{1 + \csc^2 \phi_1 \tan^2 \phi_2} \tag{2}$$

$$\frac{\partial \xi_2}{\partial \phi_1} = - \frac{1}{\sqrt{1 + \csc^2 \phi_1 \tan^2 \phi_2}} \tag{3}$$

$$\frac{\partial \xi_2}{\partial \phi_2} = - \frac{1}{\sqrt{1 + \tan^2 \phi_1 \csc^2 \phi_2}} \, . \tag{4}$$

Equations (3) and (4) present no problem since $\left| \frac{\partial \xi_2}{\partial \phi_i} \right| \leq 1, \; i = 1, 2.$

Equations (1) and (2) do have poles for critical values of ϕ_1, ϕ_2 however and must therefore be combined with the expressions for $\frac{\partial \phi_i}{\partial a_j}$ to keep the products finite. Here a_j are the orbit parameters whose covariance matrices are to be determined:

$$\frac{\partial \xi_\ell}{\partial a_i} = \frac{\partial \xi_\ell}{\partial \phi_1} \frac{\partial \phi_1}{\partial a_i} + \frac{\partial \xi_\ell}{\partial \phi_2} \frac{\partial \phi_2}{\partial a_i} \; ; \; \ell = 1, 2. \tag{5}$$

From the kinematics developed in the Final Report,

$$\frac{\partial \phi_1}{\partial a_j} = \sin \phi_1 \cos \phi_1 \left(\frac{1}{v} \frac{\partial v}{\partial a_j} - \frac{1}{w} \frac{\partial w}{\partial a_j} \right) \tag{6}$$

$$\frac{\partial \phi_2}{\partial a_j} = \sin \phi_2 \cos \phi_2 \left(\frac{1}{v} \frac{\partial v}{\partial a_j} - \frac{1}{u} \frac{\partial u}{\partial a_j} - \cot \phi_1 \frac{\partial \phi_1}{\partial a_j} \right) \tag{7}$$

where u, v, w are axes moving with the observing station; u East, v North and w up.

In terms of u, v, w coordinates,

$$\begin{aligned} \frac{\partial \phi_1}{\partial a_j} &= \frac{1}{v^2 + w^2} \left[v \frac{\partial w}{\partial a_j} - \frac{\partial v}{\partial a_j} \right] \\ \frac{\partial \phi_2}{\partial a_j} &= -\sqrt{v^2 + w^2} \left[\frac{\partial u}{\partial a_j} - \frac{u}{v} \frac{\partial v}{\partial a_j} - \frac{wu}{w^2 + v^2} \left(\frac{\partial w}{\partial a_j} - \frac{w}{v} \frac{\partial v}{\partial a_j} \right) \right] \end{aligned} \tag{8}$$

as explained in Apollo Note No. 94. Equations (3) and (4) become

$$\frac{\partial \xi_2}{\partial \phi_1} = - \frac{v}{\sqrt{1 + v^2}}$$

$$\frac{\partial \xi_2}{\partial \phi_2} = \frac{-w}{\sqrt{w^2 + v^2 (u^2 + v^2 + w^2)}} \tag{9}$$

so that $\frac{\partial \xi_2}{\partial \phi_1}$, $\frac{\partial \xi_2}{\partial \phi_2}$ remain bounded for all real values of the arguments u, v, w. The final form for computation results when (8) and (9) are used in (5):

$$\frac{\partial \xi_2}{\partial \phi_1} \frac{\partial \phi_1}{\partial a_i} = \frac{-v}{(v^2 + w^2)\sqrt{1 + v^2}} \left[v \frac{\partial w}{\partial a_i} - w \frac{\partial v}{\partial a_i} \right]$$

$$\frac{\partial \xi_2}{\partial \phi_2} \frac{\partial \phi_2}{\partial a_i} = \frac{w \sqrt{v^2 + w^2}}{\sqrt{w^2 + v^2 (u^2 + v^2 + w^2)}} \left[\frac{\partial u}{\partial a_i} - \frac{u}{v} \frac{\partial v}{\partial a_i} - \frac{wu}{v^2 + w^2} \left(\frac{\partial w}{\partial a_i} - \frac{w}{v} \frac{\partial v}{\partial a_i} \right) \right],$$

a form in which the poles of numerator and denominator in (3) and (4) cancel to produce a well-behaved and manageable result.

The relations between u, v, w and ϕ_1, ϕ_2 are just

$$\left.\begin{aligned}
\cos \phi_1 &= \frac{w}{\sqrt{v^2 + w^2}} \\
\sin \phi_1 &= \frac{-v}{\sqrt{v^2 + w^2}} \\
\cos \phi_2 &= \frac{v}{\sqrt{v^2 + u^2 \sin^2 \phi_1}} \\
\sin \phi_2 &= \frac{-u \sin \phi_1}{\sqrt{v^2 + u^2 \sin^2 \phi_1}}
\end{aligned}\right\} \tag{10}$$

as shown in Note 94.

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APOLLO NOTE NO. 195

H. Engel
23 April 1964

ACCURACY OF POSITION AND VELOCITY DETERMINATION DURING BOOST, V

Noise in the accelerometer measurements is introduced to find expressions for ψ_n and R_n.

We have, typically,

$$\ddot{x}_n = a_{xn} - \theta_{zn} a_{yn} + \theta_{yn} a_{zn}$$

$$\dot{x}_{n+1} = \dot{x}_n + \ddot{x}_n \tau$$

and

$$\dot{x}_{n+1,p} = \dot{x}_n + \delta\dot{x}_n - \tau(a_{yn} + n_{yn})(\delta\theta_{zn} + \frac{\tau}{2}\delta\omega_z) + \tau(a_{zn} + n_{zn})(\delta\theta_{yn} + \frac{\tau}{2}\delta\omega_y)$$

$$+ \tau n_{xn} - \tau\theta_{zn} n_{yn} + \tau\theta_{yn} n_{zn}$$

in which n_{xn}, n_{yn} and n_{zn} represent the noise in the accelerometer measurements. Also,

$$x_{n+1,p} = x_n + \delta x_n + \tau\delta\dot{x}_n + \frac{\tau^2}{2} n_{xn} - \frac{\tau^2}{2}(a_{yn} + n_{yn})(\delta\theta_{zn} + \frac{\tau}{3}\omega_z)$$

$$+ \frac{\tau^2}{2}(a_{zn} + n_{zn})(\delta\theta_{yn} + \frac{\tau}{3}\delta\omega_y) + \frac{\tau^2}{2} n_{xn} - \frac{\tau^2}{2}\theta_{zn} n_{yn} + \frac{\tau^2}{2}\theta_{yn} n_{zn}$$

Ignoring terms that are the product of noise and errors in platform angle, we may write

$$\delta X_{n+1,p} = \phi_n \delta X_n + \psi_n \delta Y_n$$

in which ϕ_n and δX_n are defined as before,

$$\delta Y_n = \begin{bmatrix} n_{xn} \\ n_{yn} \\ n_{zn} \end{bmatrix}$$

and, letting the reference value of 0 be zero,

$$\psi = \psi_n = \begin{bmatrix} \frac{\tau^2}{2} & 0 & 0 \\ 0 & \frac{\tau^2}{2} & 0 \\ 0 & 0 & \frac{\tau^2}{2} \\ \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, R_n is the error covariance matrix of the accelerometer noise,

$$R = R_n = \begin{bmatrix} \sigma_{ax}^2 & 0 & 0 \\ 0 & \sigma_{ay}^2 & 0 \\ 0 & 0 & \sigma_{az}^2 \end{bmatrix}$$

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APOLLO NOTE NO. 196

H. Epstein
24 April 1964

AN APPROXIMATE TECHNIQUE TO EVALUATE THE EFFECT OF ERRORS IN MEASURED VALUES OF THE NORMALIZED AUTO-CORRELATION FUNCTION FOR A PROBABILITY DISTRIBUTION OF FIRST ORDER DEPENDENCE

The values of λ for a given order of dependence are determined from the normalized auto-correlation function using the following matrix equations:

$$[R]\,[\lambda] = [\rho]$$

$$[\lambda] = [R]^{-1}[\rho]$$

where

$$[\lambda] = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{q-1} \\ \lambda_q \end{bmatrix}$$

$$[\rho] = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{q-1} \\ \rho_q \end{bmatrix}$$

$$\left[R \right] = \left[C_{ij} \right]$$

$$C_{ij} = C_{ji} = 1 \qquad \text{for } i = j$$

$$= \rho_m \qquad |j-i| = m$$

Letting $[R + \Delta R]\ [\lambda + \Delta\lambda] = [\rho + \Delta\rho]$

where R, λ, and ρ are true values.

ΔR, $\Delta\lambda$, and $\Delta\rho$ are error quantities.

Then to the first order approximation

$$[R]\ [\Delta\lambda] = [\Delta\rho] - [\Delta R]\ [\lambda]$$

or

$$[\Delta\lambda] = [R]^{-1}\{[\Delta\rho] - [\Delta R]\ [\lambda]\}$$

is the desired result.

For first order dependence, $\lambda_1 = \rho_1 = \rho$ and all other λ's are zero. For a matrix order of 4 or greater

$$[\Delta\rho] - [\Delta R]\ [\lambda] = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 - \rho\Delta\rho_1 \\ \vdots \\ \Delta\rho_{q-1} - \rho\,\Delta\rho_{q-2} \\ \Delta\rho_q - \rho\,\Delta\rho_{q-1} \end{bmatrix}$$

Letting $[R] = [C_{ij}]$

where

$$C_{ji} = C_{ij} = \rho^m \qquad \text{for } |j - i| = m$$

then

$$[R]^{-1} = [D_{ij}]$$

where

$$D_{ij} = D_{ji} = 0 \qquad |j - i| \geq 2$$

$$= \frac{-\rho}{1 - \rho^2} \qquad |j - i| = 1$$

$$D_{11} = D_{qq} = \frac{1}{(1 - \rho^2)}$$

$$D_{ii} = \frac{1 + \rho^2}{1 - \rho^2} \qquad i \neq 1, q$$

Then for $q \geq 4$

$$(1 - \rho^2)\,\Delta\lambda_1 = (1 + \rho^2)\,\Delta\rho_1 - \rho\,\Delta\rho_2$$

$$(1 - \rho^2)\,\Delta\lambda_2 = -\rho\,(2 + \rho^2)\,\Delta\rho_1 + (1 + 2\rho^2)\,\Delta\rho_2 - \rho\,\Delta\rho_3$$

$$(1 - \rho^2)\,\Delta\lambda_m = \rho^2\,\Delta\rho_{m-2} - \rho\,(2 + \rho^2)\,\Delta\rho_{m-1}$$

$$+ (1 + 2\rho^2)\,\Delta\rho_m - \rho\,\Delta\rho_{m+1}$$

$$\text{for } 3 \leq m \leq q-1$$

$$(1 - \rho^2)\,\Delta\lambda_q = \rho^2\,\Delta\rho_{q-2} - 2\rho\,\Delta\rho_{q-1} + \Delta\rho_q$$

It is now possible to write the covariance matrix for the $\Delta\lambda$ errors $(\overline{\Delta\lambda_i \ \Delta\lambda_j})$ in terms of the covariance matrix of the $\Delta\rho$ error $(\overline{\Delta\rho_i \ \Delta\rho_j})$. The completion of this effort involves the calculation of $\overline{\Delta\rho_i \Delta\rho_j}$ in terms of the errors in the original data to determine $\overline{\Delta\lambda_i \Delta\lambda_j}$.

This analysis implies that $\phi(0)$ is exactly known. The analysis needs to be extended to allow for errors in ϕ (0). The approximate analysis checks well with some computer results.

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 197

H. Epstein
29 April 1964

THE COVARIANCE MATRIX FOR ERRORS IN THE AUTO-CORRELATION FUNCTION AND LINEAR REGRESSION COEFFICIENTS

I. Approximate Covariance Matrix for Errors in Linear Regression Coefficients.

Writing the linear regression expression as

$$x_n = \sum_{i=1}^{q} \lambda_i x_{n-i} + e_n$$

or

$$e_n = x_n - \sum_{i=1}^{q} \lambda_i x_{n-i}$$

For the situation that e_n is gaussian and

$$\overline{e_n} = 0$$

$$\overline{e_n e_{n-i}} = \phi_e \qquad i = 0$$

$$= 0 \qquad i \neq 0$$

and defining a likelihood function ($\mathcal{L}$) as the negative of the logarithm of the probability distribution.

Then $$\mathcal{L} = \frac{1}{2\phi_e} \sum_{n=1}^{N} \left[x_n - \sum_{i=1}^{q-1} \lambda_i x_{n-i} \right]^2 + \frac{N}{2} \ln \phi_e + \frac{N}{2} \ln (2\pi)$$

The estimated values of ϕ_e and the λ_i's are determined by differentiation as shown in the following equations.

$$\frac{1}{N} \sum_{n=1}^{N} \left[x_{n-k} x_n - x_{n-k} \sum_{i=1}^{q} \hat{\lambda}_i x_{n-i} \right] = 0$$

$$\hat{\phi}_e = \frac{1}{N} \sum_{n=1}^{N} \left[x_n - \sum_{i=1}^{q} \hat{\lambda}_i x_{n-i} \right]^2$$

$$= \frac{1}{N} \sum_{n=1}^{N} \left[x_n^2 - x_n \sum_{i=1}^{q} \hat{\lambda}_i x_{n-i} \right]$$

The expected values of these quantities are given by

$$\sum_{i=1}^{q} \lambda_i \phi(i-k) = \phi(k)$$

$$E(\hat{\phi}_e) = \phi(0) - \sum_{i=1}^{q} \lambda_i \phi(i)$$

where

$$E(\hat{\lambda}_i) = \lambda_i$$

$$E(x_n x_{n-i}) = \phi(i)$$

The technique outlined in Apollo Note No. 196 only needs slight modification to arrive at the desired linear error coefficients as follows.

$$\left[P\right]\left[\lambda\right] = \left[\phi\right]$$

$$\left[\lambda\right] = \left[P\right]^{-1}\left[\phi\right]$$

where

$$\left[\lambda\right] = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{q-1} \\ \lambda_q \end{bmatrix}$$

$$\left[\phi\right] = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{q-1} \\ \phi_q \end{bmatrix}$$

$$\left[P \right] = \left[C_{ij} \right] \qquad \begin{array}{l} 1 \leq i \leq q \\ 1 \leq j \leq q \end{array}$$

where

$$C_{ij} = C_{ji} = \phi(m) = \phi_o \, \rho(m) \qquad |j-i| = m$$

Letting $\left[P + \Delta P \right] \left[\lambda + \Delta \lambda \right] = \left[\phi + \Delta \phi \right]$

where P, λ, and ϕ are true values

ΔP, $\Delta \lambda$, and $\Delta \phi$ are error quantities.

Then to a first order approximation

$$\left[\Delta \lambda \right] = \left[P \right]^{-1} \left\{ \left[\Delta \phi \right] - \left[\Delta P \right] \left[\lambda \right] \right\}$$

is a suitable expression for the errors in λ.

Actually the most desirable expression is in terms of the covariance matrix for the errors in λ.

$$\left[\Delta \lambda^T \right] = \left\{ \left[\Delta \phi^T \right] - \left[\lambda^T \right] \left[\Delta P \right] \right\} \left[P \right]^{-1}$$

since $\left[\Delta P^T \right] = \left[\Delta P \right]$ and $\left[P^T \right] = \left[P \right]$

then $\left[\Delta \lambda \right] \left[\Delta \lambda^T \right] = \left[P \right]^{-1} \left[Q \right] \left[P \right]^{-1}$

where

$$\left[Q \right] = \left[F_{mn} \right]$$

$$= \left\{ \left[\Delta \phi \right] - \left[\Delta P \right] \left[\lambda \right] \right\} \left\{ \left[\Delta \phi^T \right] - \left[\lambda^T \right] \left[\Delta P \right] \right\}$$

The expected value of the errors in λ are given by

$$E\left(\left[\Delta\lambda\right]\left[\Delta\lambda^T\right]\right) = \overline{\left[\Delta\lambda\right]\left[\Delta\lambda^T\right]} = \left[P\right]^{-1}\overline{\left[Q\right]}\left[P\right]^{-1}$$

The expected value of the Q matrix can be written from terms of the covariance matrix for the errors in the auto-correlation function. These covariance matrices can be compared to determine which representation has smaller errors. Following the derivation of the covariance matrix for the errors in the auto-correlation function, the solution for three simple examples will be given.

II. Covariance Matrix for Errors in Auto-correlation Function

$$\hat{\phi}(m) = \frac{1}{N}\sum_{i=1}^{N} x_i\, x_{i+m}$$

Then $$\overline{\Delta\phi_m\,\Delta\phi_n} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\phi(j-i)\;\phi(j-i+n-m) + \phi(j-i+n)\,\phi(j-i-m)\right]$$

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_m\,\Delta\phi_n} = \rho(r) + \rho(m)\,\rho(n) + \sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)$$

$$\left[\rho(k)\,\rho(k+r) + \rho(k)\,\rho(k-r) + \rho(k+n)\,\rho(k-m) + \rho(k-n)\,\rho(k+m)\right]$$

where

$$n = m+r$$

Since $\overline{\Delta\phi_m\,\Delta\phi_n} = \overline{\Delta\phi_n\,\Delta\phi_m}$, at no loss in generality one can consider m as the smaller value and make $r \geq 0$.

Case 1a. $\rho = 1$ Over The Entire Data Set.

Then $\overline{\Delta\phi_m \, \Delta\phi_n} = 2\,\phi_o^2$

This result is equivalent to calculation of the difference between the fourth moment and the square of the second moment of a quantity generated from an unbiased gaussian distribution. As a slight extension, consider $\rho \leq \mu \leq 0$. Then, $\overline{\Delta\phi_m \, \Delta\phi_n} \geq 2\,\phi_o^2\,\mu^2$ and $\sqrt{\Delta\phi_m \, \Delta\phi_n} \geq \phi_o \mu \sqrt{2}$. This indicates that the data set must be well decorrelated to arrive at suitable accuracy for estimates for the correlation function.

Case 1b. Uncorrelated Noise $\rho(k) = 1$ for $k=0$, $= 0$ $k \neq 0$

$$\text{Then } \overline{\Delta\phi_m \, \Delta\phi_{m+r}} = \frac{2\phi_o^2}{N} \qquad r = 0 \text{ and } m=0$$

$$= \frac{\phi_o^2}{N} \qquad r = 0 \text{ and } m \neq 0$$

$$= 0 \qquad r \neq 0$$

The errors in the computed values of the auto-correlation are uncorrelated for this case as one should expect. It is of interest to note that the variance of the estimate of ϕ_o is twice the variance in any other value as one would further expect.

Case 2. $\phi(n) = \phi_o \, \rho^{|n|}$ n any integer or zero.

After extensive manipulation,

$$\overline{\Delta\phi_m \, \Delta\phi_{m+r}} = \frac{\phi_o^2 \, \rho^r}{N} \, G(\rho, N, m, r)$$

where $G(\rho, m, r, N) = 1 + \rho^{2m} + r\left(1 - \frac{r+1}{2N}\right) + m\rho^{2m}\left(1 - \frac{m+1}{2N}\right) + n\rho^{2m}\left(1 - \frac{n+1}{2N}\right)$

$$+ \frac{\rho^2}{(1-\rho^2)^2}\left[1 - \rho^2 - \frac{1}{N}(1-\rho^{2N})\right]$$

$$+ \frac{\rho^2}{(1-\rho^2)^2}\left[\left(1 - \frac{r}{N}\right)\left(1 - \rho^2\right) - \frac{1}{N}\left(1 - \rho^{2(N-r)}\right)\right]$$

$$+ \frac{\rho^{2m+2}}{(1-\rho^2)^2}\left[\left(1 - \frac{m}{N}\right)\left(1 - \rho^2\right) - \frac{1}{N}\left(1 - \rho^{2(N-m)}\right)\right]$$

$$+ \frac{\rho^{2m+2}}{(1-\rho^2)^2}\left[\left(1 - \frac{n}{N}\right)\left(1 - \rho^2\right) - \frac{1}{N}\left(1 - \rho^{2(N-m)}\right)\right]$$

for $-N \ln \rho \gg 1$, $m \ll N$, and $n \ll N$

$$G(\rho, m, r, N) \simeq G(\rho, m, r) = \left(r + \frac{1+\rho^2}{1-\rho^2}\right)\left(1 + \rho^{2m}\right) + 2m\rho^{2m}$$

Letting the standard deviation of the normalized error in the auto-correlation function (σ_m) be defined by the following equation:

$$\sigma_m = \frac{1}{\phi_o}\sqrt{\overline{\Delta\phi_m^2}}$$

then

$$\sigma_m \simeq \sqrt{\frac{1}{N}\left[\frac{(1+\rho^2)(1+\rho^{2m})}{1-\rho^2} + 2m\rho^{2m}\right]}$$

Denoted σ_∞ by the value of σ_m for m sufficiently large, then

$$\sigma_\infty \simeq \sqrt{\frac{1+\rho^2}{N(1-\rho^2)}}$$

$$\sigma_o \simeq \sqrt{\frac{2\,(1+\rho^2)}{N\,(1-\rho^2)}}$$

and $$\sigma_1 \simeq \sqrt{\frac{1}{N}\left[\frac{(1+\rho^2)^2}{(1-\rho^2)} + 2\rho^2\right]}$$

Numerical values for σ_o, σ_1, and σ_∞ for $0 \leq \rho \leq .9$ and $N = 1000$ are indicated in Figure 1. The results indicate that $\sigma_o \geq \sigma_1 \geq \sigma_\infty$. The form of the equation is such that the maximum standard deviation of the error occurs for σ_o and the minimum standard deviation of the error for σ_∞. The ratio of these two quantities $\left(\frac{\sigma_o}{\sigma_\infty}\right) = \sqrt{2}$. Since the bounds are rather close, for many applications it should not be necessary to calculate to a greater accuracy and for such cases the simple expressions can be used.

The nature of the correlation coefficient between errors can be simply approximated for m sufficiently large.

Defining $\rho_\phi(r) = \dfrac{\overline{\Delta\phi_m \Delta\phi_n}}{\sigma_m \sigma_n}$

then for sufficiently large m

$$\rho_\phi(r) \simeq \left[1 + \frac{r\,(1-\rho^2)}{1+\rho^2}\right] \rho^r$$

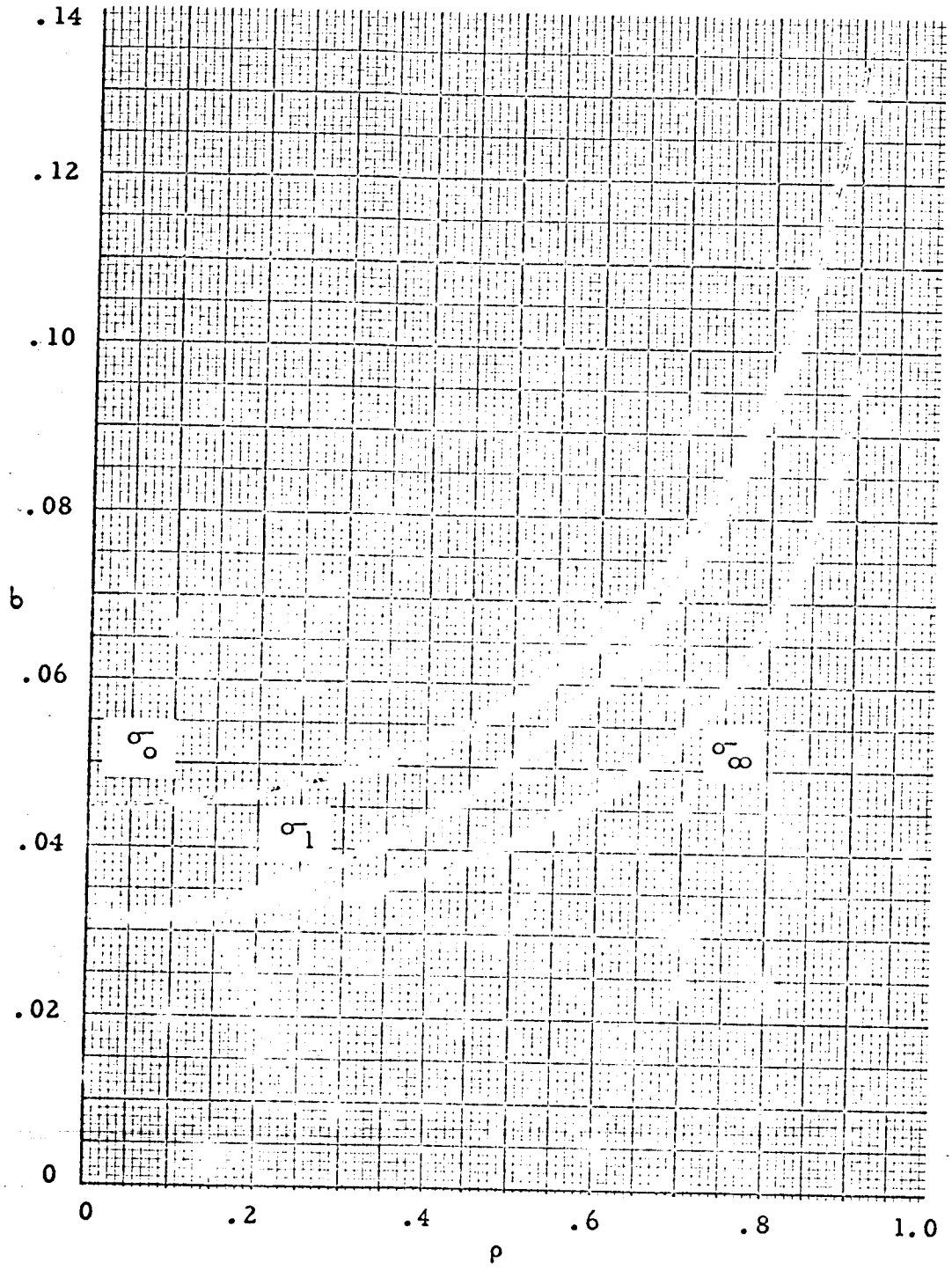

Figure 1.

Variation of the Normalized Standard Deviation of the Errors in the Auto-correlation Function with ρ (for N = 1000)

and a simple example being $\rho_\phi(1) \simeq \frac{2\rho}{1+\rho^2}$.

It will be shown in the next portion that the correlation between errors in λ is less than the correlation between errors in the auto-correlation function aside from the situation of zero order dependence in which case both type errors are uncorrelated. To this point, the statement is only proven for first order dependence with $\rho < 1$ and with the further restraint that the linear approximation is satisfactory (small error condition).

III. Sample Cases for Covariance Matrix of Errors in λ

Case 1. No Dependence (all λ's = 0)

By inspection, since $m \geq 1$

$$\overline{\Delta\lambda_m \, \Delta\lambda_{m+r}} = \frac{1}{\phi_o^2} \, \overline{\Delta\phi_m \, \Delta\phi_{m+r}} = \frac{1}{N} \qquad \text{for } r=0$$

$$= 0 \qquad \text{for } r \neq 0$$

This covariance matrix is diagonal and the diagonal elements are all equal. As might be expected, for this situation regression techniques offer no improvement over the use of the correlation function.

Case 2. First Order Dependence ($\lambda_1 = \rho$ and all other λ's zero)

$$\left[P \right] = \left[C_{ij} \right]$$

where $C_{ij} = \phi_o \, \rho^m \qquad |j-i| = m$

$$\left[P \right]^{-1} = \left[D_{ij} \right]$$

$$D_{ij} = D_{ji} = 0 \qquad |j-i| \geq 2$$

$$= \frac{-\rho}{\phi_o(1-\rho^2)} \qquad |j-i| = 1$$

$$D_{11} = D_{qq} = \frac{1}{\phi_o(1-\rho^2)}$$

$$D_{ii} = \frac{1+\rho^2}{\phi_o\,(1-\rho^2)} \qquad i \neq 1 \text{ or } q$$

where i is the row of the matrix involved.

From the first portion,

$$\left[Q \right] = \left[F_{mn} \right]$$

Let $r = n-m$ with $r \geq 0$

It can be shown that

$$\overline{F_{mn}} = \overline{\Delta\phi_m \, \Delta\phi_{m+r}} - \rho \left[\overline{\Delta\phi_{m-1} \, \Delta\phi_{m+r}} + \overline{\Delta\phi_m \, \Delta\phi_{m+r-1}} \right] + \rho^2 \, \overline{\Delta\phi_{m-1} \, \Delta\phi_{m+r-1}}$$

$$= \frac{\phi_o^2 \, \rho^r}{N} (1-\rho^2)\,(1-\frac{r}{N}) \qquad \text{for } r = |n-m|$$

where m is the row of the matrix involved.

To a first order approximation, since $r << N$

$$\overline{F_{mn}} \simeq \frac{\phi_o^2 \rho^r}{N} (1-\rho^2)$$

$$\therefore \overline{[\Delta\lambda][\Delta\lambda^T]} \simeq \frac{\phi_o(1-\rho^2)}{N} \left[P\right]^{-1}$$

$$\overline{\Delta\lambda_1^2} = \overline{\Delta\lambda_q^2} \simeq \frac{1}{N}$$

$$\overline{\Delta\lambda_m^2} \simeq \frac{1+\rho^2}{N} \qquad m \neq i, q$$

$$\overline{\Delta\lambda_m \Delta\lambda_{m+1}} \simeq \frac{-\rho}{N} \qquad 1 \leq m \leq q-1$$

$$\overline{\Delta\lambda_m \Delta\lambda_{m+r}} \simeq 0 \qquad r \geq 2$$

The form of the λ error covariance matrix is such that error quantities separated by greater than one integer are uncorrelated, errors separated by one integer have negative correlation for a positive ρ and positive correlation for a negative ρ. This indicates that for first order dependence, the computed errors in λ will tend to oscillate each lag in sense for a positive ρ and remain with the same sense for a negative ρ. The form of these equations is such as to clearly indicate the advantage of tests for order of dependence over use of the correlation function (at least for first order dependence). These results are clearly compatible with the viewpoint that the "best" performance is obtained when the fewest descriptors with the required accuracy are employed.

Letting $\sigma_1 = \sqrt{\overline{\Delta\lambda_1^2}} = \sqrt{\overline{\Delta\lambda_q^2}}$

and

$$\sigma_k = \sqrt{\overline{\Delta\lambda_k^2}}$$

then

$$\sigma_1 \simeq N^{-\frac{1}{2}}$$

$$\sigma_k \simeq \sqrt{\frac{1+\rho^2}{N}} \simeq \sigma_1 \sqrt{1+\rho^2}$$

Numerical values for these standard deviations are indicated in Figure 2 as a function of ρ for N = 1000. These standard deviations are never greater than the normalized standard deviations for the errors in the auto-correlation function. The ratio of the standard deviations for the two types of errors vary between a factor of one and $\sqrt{\frac{2(1+\rho^2)}{(1-\rho^2)}}$.

Defining $\rho_\lambda(1) = \dfrac{\overline{\Delta\lambda_1 \Delta\lambda_2}}{\sigma_1 \sigma_k} = \dfrac{\overline{\Delta\lambda_{q-1} \Delta\lambda_q}}{\sigma_1 \sigma_k}$

and

$$\rho_\lambda(k) = \frac{\overline{\Delta\lambda_k \Delta\lambda_{k+1}}}{\sigma_k^2} \qquad 2 \leq k \leq q-2$$

then

$$\rho_\lambda(1) \simeq \frac{-\rho}{\sqrt{1+\rho^2}}$$

and

$$\rho_\lambda(k) \simeq \frac{-\rho}{1+\rho^2}$$

The magnitude of these quantities as a function of the magnitude of ρ, with N = 1000, are indicated in Figure 3. The maximum value

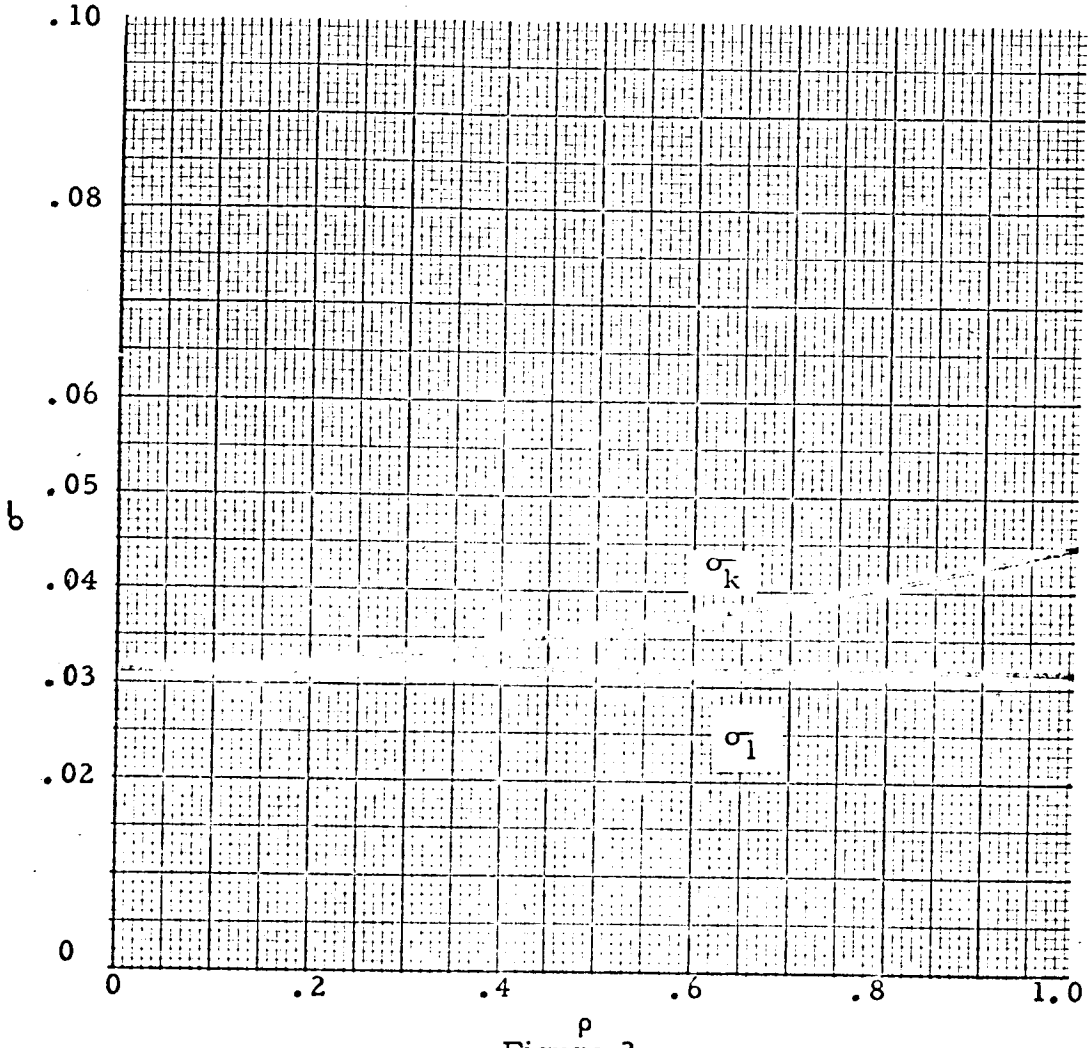

Figure 2.

Standard Deviation of the Errors in λ as a Function of ρ (for N = 1000)

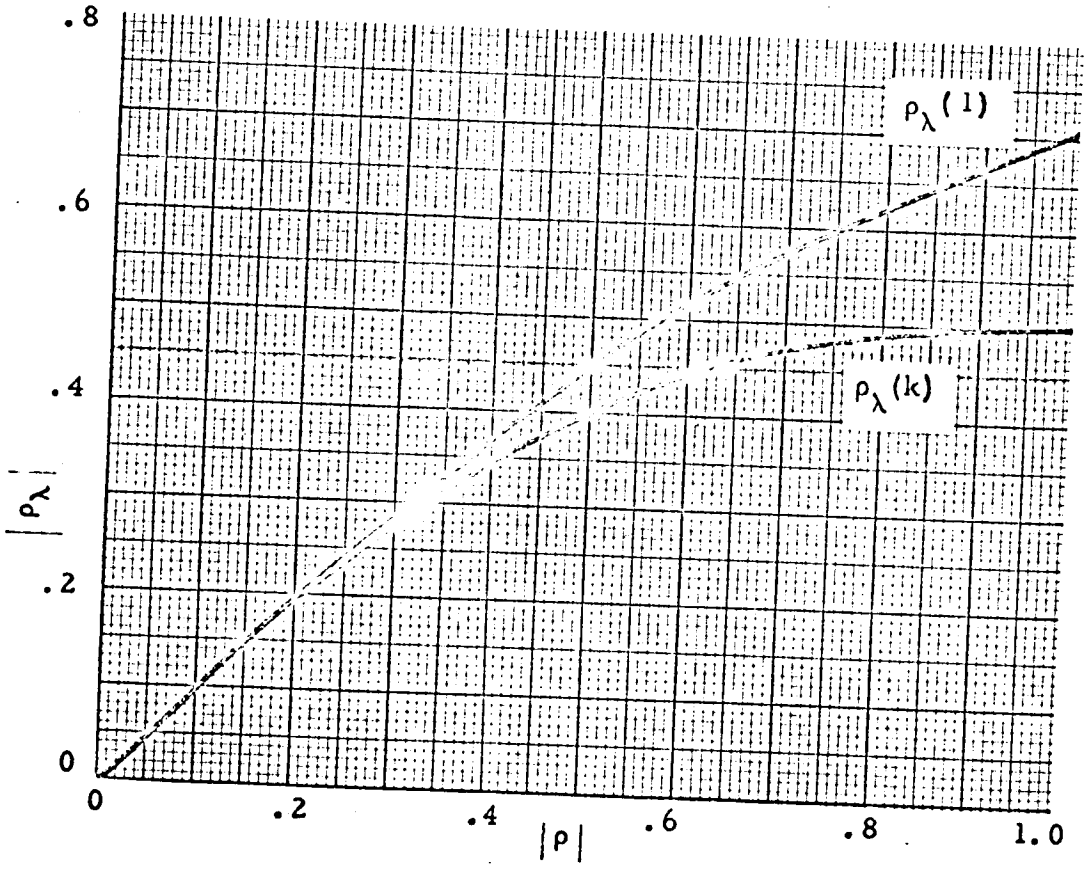

Figure 3.
Magnitude of One Lag Correlation
Between Errors in λ as a
Function of ρ (N = 1000)

of $\left|\rho_\lambda(1)\right| \simeq 2^{-\frac{1}{2}}$ and the maximum value of $\left|\rho_\lambda(k)\right| \simeq .5$.

These results indicate that for first order dependence not only are fewer error quantities correlated for the λ representation (only the adjacent error terms) than the auto-correlation function representation (all terms) but, in addition, less correlation is found for the correlated quantities. The results obtained are encouraging for the use of order of dependence when low orders of dependence are present.

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 198

H. Engel
30 April 1964

DETERMINATION OF POSITION AND VELOCITY IN POWERED FLIGHT BY DOPPLER MEASUREMENTS ALONE

Let us first consider a one-dimensional case to show that the manner in which acceleration varies with time is of no importance insofar as the accuracy with which position can be calculated. We have that the position, y, is

$$y = y_o + \int_0^t \dot{y}\, dt.$$

A Doppler radar, used to determine $\dot{y}$, really finds the average value of $\dot{y}$ in the interval t_i, t_{i+1} as

$$\dot{y}_{i,\,i+1} = \frac{1}{t_{i+1} - t_i} (y_{i+1} - y_i) + n_i$$

in which $\dot{y}_{i,\,i+1}$ is this average value and n_i is the noise in this measurement. We assume $E(n_i) = 0$ and $E(n_i\, n_j) = \delta_{ij}\, \sigma^2$.

Then the computed value of y_m is y_{mc},

$$y_{mc} = y_o + \delta y_o + \sum_{i=0}^{n-1} \dot{y}_{i,\,i+1} (t_{i+1} - t_i)$$

$$= y_o + \delta y_o + \sum_{i=0}^{m-1} (y_{i+1} - y_i) + \sum_{i=0}^{m-1} n_i (t_{i+1} - t_i)$$

$$= y_m + \delta y_o + \sum_{i=0}^{m-1} n_i (t_{i+1} - t_i)$$

in which δy_o is the error in y_o. Let us now take observations at one second intervals, so that

$$\delta y_m = y_{mc} - y_m = \delta y_o + \sum_{i=0}^{m-1} n_i$$

Assuming that $E(\delta y_o) = 0$, and δy_o uncorrelated with n_i, we find

$$E(\delta y_m) = 0$$

$$E(\delta y_m^2) = \delta y_o^2 + m\,\sigma^2$$

and this is <u>not dependent</u> on the manner in which the acceleration varies.

Next we show that in the case of a LEM lifting off the Moon, the errors in the three components of velocity computed on the basis of Doppler measurements from three Earth stations and estimates of vehicle position are not sensitive to errors in the estimates of vehicle position.

In the notation of Apollo Note No. 192, we have

$$r_j \dot{r}_j = \sum_{i=1}^{3} (x_i - x_{ij})\,\dot{x}_i \,.$$

Then, considering only errors in position, and the effect they have on computed velocity, we have

$$\dot{r}_j \, dr_j = \sum_{i=1}^{3} (x_i - x_{ij})\, d\dot{x}_i + \sum_{i=1}^{3} \dot{x}_i \, dx_i$$

Now, since

$$r_j^2 = \sum_{i=1}^{3} (x_i - x_{ij})^2$$

it follows that

$$r_j \; dr_j = \sum_{i=1}^{3} (x_i - x_{ij})\, dx_i$$

so by substitution in the equation for $\dot{r}_j \; dr_j$ we find

$$\sum_{i=1}^{3} \left[(x_i - x_{ij}) \frac{\dot{r}_j}{r_j} - \dot{x}_i \right] dx_i = \sum_{i=1}^{3} (x_i - x_{ij})\, d\dot{x}_i$$

or

$$- \sum_{i=1}^{3} \frac{\partial \dot{r}_j}{\partial x_i} \, dx_i = \sum_{i=1}^{3} \frac{\partial \dot{r}_j}{\partial \dot{x}_i} \, d\dot{x}_i$$

We can write this last equation in matrix form, j running from 1 to 3, as

$$- \left[\frac{\partial \dot{r}}{\partial x} \right] \left[dx \right] = \left[\frac{\partial \dot{r}}{\partial \dot{x}} \right] \left[d\dot{x} \right]$$

or

$$\left[d\dot{x} \right] = - \left[\frac{\partial \dot{r}}{\partial \dot{x}} \right]^{-1} \left[\frac{\partial \dot{r}}{\partial x} \right] \left[dx \right]$$

The expected values of $\left[dx \right]$ and $\left[d\dot{x} \right]$ are zero. Then,

$$E \left[d\dot{x}^T \; d\dot{x} \right] = E \left\{ \left[dx \right]^T \left[\left[\frac{\partial \dot{r}}{\partial \dot{x}} \right]^{-1} \left[\frac{\partial \dot{r}}{\partial x} \right] \right]^T \left[\frac{\partial \dot{r}}{\partial \dot{x}} \right]^{-1} \left[\frac{\partial \dot{r}}{\partial x} \right] \left[dx \right] \right\}$$

From Apollo Note No. 192 we may take

$$\left[\frac{\partial \dot{r}}{\partial \dot{x}}\right] = \begin{bmatrix} 1 & -6.32 \times 10^{-3} & 0 \\ 1 & 5.53 \times 10^{-3} & -3.16 \times 10^{-3} \\ 1 & 5.53 \times 10^{-3} & 3.16 \times 10^{-3} \end{bmatrix}$$

and

$$\left[\frac{\partial \dot{r}}{\partial x}\right] = \begin{bmatrix} 1.33 \times 10^{-8} & 2.10 \times 10^{-6} & 0 \\ -1.16 \times 10^{-8} & 2.10 \times 10^{-6} & 3.68 \times 10^{-11} \\ -1.16 \times 10^{-8} & 2.10 \times 10^{-6} & -3.68 \times 10^{-11} \end{bmatrix}$$

Then,

$$\left[\frac{\partial \dot{r}}{\partial \dot{x}}\right]^{-1} = \begin{bmatrix} .466 & .267 & .267 \\ -84.4 & 42.2 & 42.2 \\ 0 & -158. & 158. \end{bmatrix}$$

$$\left[\frac{\partial \dot{r}}{\partial \dot{x}}\right]^{-1} \left[\frac{\partial \dot{r}}{\partial x}\right] = \begin{bmatrix} 0 & 2.10 \times 10^{-6} & 0 \\ 2.10 \times 10^{-6} & 0 & 0 \\ 0 & 0 & 1.16 \times 10^{-8} \end{bmatrix}$$

$$\left[\frac{\partial \dot{r}}{\partial x}\right]^T \left[\frac{\partial \dot{r}}{\partial \dot{x}}\right]^{-T} \left[\frac{\partial \dot{r}}{\partial \dot{x}}\right] \left[\frac{\partial \dot{r}}{\partial x}\right]^{-1} = \begin{bmatrix} 4.41 \times 10^{-12} & 0 & 0 \\ 0 & 4.41 \times 10^{-12} & 0 \\ 0 & 0 & 1.33 \times 10^{-16} \end{bmatrix}$$

so

$$E\left[d\dot{x}^T \, d\dot{x}\right] = 4.41 \times 10^{-12} \; E\,(dx_1^2) + 4.41 \times 10^{-12} \; E\,(dx_2^2) + 1.33 \times 10^{-16} \; E\,(dx_3^2)$$

Thus even if the error in dx_1, dx_2 or dx_3 is as large as 10^4 meters, the resultant error in the computed velocity is 2.1 cm/sec. or less.

Continuing to use the example from Apollo Note No. 92, we now show that the position errors will be small compared to 10^4 meters. There are two parts to this demonstration. In the first part we find the errors in position due to noise on the radar measurements. In the second part we show that the errors due to using the wrong position in computing the gravitational acceleration are non-existent.

We have already found that the rms error in position due to radar noise and initial position error is

$$E\,(\delta y_m)^2 = \delta y_o^2 + m\,\sigma^2$$

If we assume that the initial position is known to 100 meters, and consider the worst case, at burnout after 400 seconds, and assume the range-range measurements accurate to 3 cm/sec., we find that due to the geometry σ is 4.8 m/sec. (=160 x 3 cm/sec), and

$$E(\delta y_{400}^2) = 10^4 + 400\,(4.8)^2$$

$$= 19,200$$

Thus the rms error in position at 400 seconds is 139 meters ($= [19,200]^{1/2}$) as compared to 100 meters at the start of boost.

Actually, we do not have to consider the Moon's gravitational field at all in determining the LEM velocity in this manner since the range-rate measurements include the effect of that field.

Thus we have shown that by using Doppler data accurate to 3 cm/sec. for one second measurements from three stations on the Earth separated from one another by at least 2400 km, and have a priori knowledge of the LEM position on the Moon's surface to 100 meters, then at any time up to burnout at 400 seconds we know the LEM velocity to better than 4.8 m/sec. and its position to better than 140 meters.

As a consequence it appears that a very simple data processing scheme, utilizing only three radars and no data from the on-board system will suffice to put the LEM in an excellent rendezvous transfer orbit.

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 199

J. R. Holdsworth
4 May 1964

A DERIVATION OF A POLARITY DIFFERENCE ESTIMATE OF THE NORMALIZED CORRELATION PARAMETER

The purpose of this note is to present a simple derivation of an estimate of the normalized correlation function of a stationary zero mean gaussian process which does not depend upon amplitude information but only upon the agreement or disagreement in sign of the lagged sample pairs.

We shall assume that we have polarity information, i.e., whether the observation is positive or negative, on N pairs of variates x_i, y_i for $i = 1, \ldots N$, where for any fixed i, y_i is to be regarded as an observation made at some fixed frame lag τ after the observation x_i has been made. By preserving only polarity information we mean that instead of recording the actual measurements x_i and y_i we merely record $\mathrm{sgn}(x_i)$ and $\mathrm{sgn}(y_i)$ where:

$$\mathrm{sgn}(x_i) = +1 \text{ if } x_i > 0 \text{ and } \mathrm{sgn}(x_i) = -1 \text{ if } x_i < 0.$$

Now $\mathrm{sgn}(x_i)\,\mathrm{sgn}(y_i) = \pm 1$ depending upon whether or not x_i and y_i have the same sign or not. Let N_+ denote the number of polarity coincidences in the lagged sample pairs; i.e., the number of times that $\mathrm{sgn}(x_i)\mathrm{sgn}(y_i) = 1$ and let N_- denote the number of anti-coincidences or the number of sample pairs such that $\mathrm{sgn}(x_i)\mathrm{sgn}(y_i) = -1$. Then we clearly have:

$$N = N_+ + N_- \tag{1}$$

For simplicity we shall assume that the successive sample pairs are independent or that the time between x_i and x_{i+1} is larger than either the correlation time or the constant lag τ between x_i and y_i. This is not strictly necessary but simplifies the derivation by allowing us to work with only bivariate distributions.

Since we have assumed the process to be stationary and gaussian with a zero mean, and if we denote the variance by o^2 and the coefficient of correlation between any x, y pair by ρ, then the joint probability density function of x and y may be written as:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}\,o^2} \exp - \frac{1}{2(1-\rho^2)o^2} (x^2 - 2\rho xy + y^2) \quad (2)$$

Now assume that we have a large number N of sample pairs x_i, y_i and that for illustrative purposes we plot them in a scatter diagram as shown below:

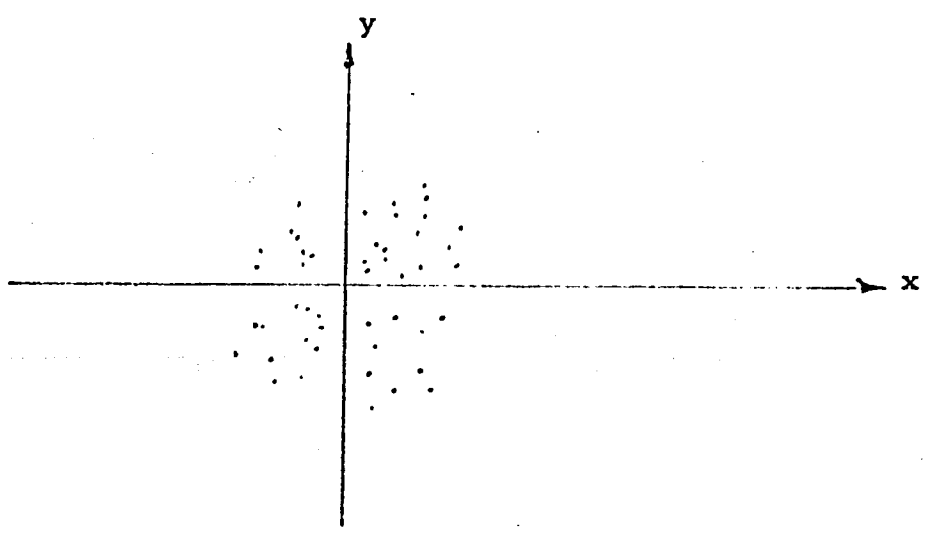

Figure 1

From Figure (1) we see that the x_i, y_i sample pairs which lie in the first or third quadrant are those for which $\mathrm{sgn}(x_i)\mathrm{sgn}(y_i) = 1$ so that N_+ is the number of sample pairs which lie in the first or third quadrants in the scatter diagram. Similarly the sample pairs x_i, y_i for which $\mathrm{sgn}(x_i)\mathrm{sgn}(y_i) = -1$ are those lying in either the second or fourth quadrants. Thus N_- is the number of sample pairs contained in the second or fourth quadrants.

Now intuitively if there is a strong positive correlation between x_i and y_i, we would expect the sample points to generally cluster in the first and third quadrants while if there is a strong negative correlation, we would expect the second and fourth quadrants to contain most of the sample pairs. Also, if the correlation were very weak, we would not expect any particular preferential clustering of the sample points among the quadrant pairs.

We shall now show how to obtain an estimate of ρ which is consistent and depends only upon the numbers N_+ and N, i. e., only upon gross sign information of the sample pairs. It will also be of some interest to note that this estimator does not depend upon the variance σ^2 which saves us the trouble of estimating σ^2 were it unknown.

From the law of large number we know that when N is sufficiently large that the sample ratio $\frac{N+}{N}$ will be approximately equal to the probability content of the first and third quadrant as determined by the probability density function in equation (2). Similarly we know that $\frac{N-}{N}$ will for large N, approximate the probability content of quadrants two and four. Hence for sufficiently large N we may write the following approximate relation:

$$\int_0^\infty \int_0^\infty f(x, y)\, dxdy - \int_{-\infty}^0 \int_0^\infty f(x, y)\, dxdy \approx \frac{1}{2}\left(\frac{N_+ - N_-}{N}\right) \tag{3}$$

where f(x, y) is given by equation (2).

Let the first and second integrals appearing on the left-hand side of equation (3) be denoted by I_1 and I_2 respectively. To evaluate I_1 and I_2, it is convenient to transform to dimensionless polar coordinates by making the following transformation:

$$x = \sqrt{1 - \rho^2}\ \sigma r \cos\theta$$

$$y = \sqrt{1 - \rho^2}\ \sigma r \sin\theta \quad \text{for } -\pi \leq \theta \leq \pi \tag{4}$$

Now let $g(r, \theta)$ be the probability density function of the transformed coordinates r and θ.

Then by the transformation law for probability distributions under changes of coordinates we know that:

$$g(r, \theta) = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| f(x, y) \tag{5}$$

where $\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right|$ is the absolute value of the Jacobian determinant corresponding to the transformation given by (4). An easy calculation shows that:

$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \begin{vmatrix} \sqrt{1 - \rho^2}\ \sigma \cos\theta & -\sqrt{1 - \rho^2}\ \sigma r \sin\theta \\ \sqrt{1 - \rho^2}\ \sigma \sin\theta & \sqrt{1 - \rho^2}\ \sigma r \cos\theta \end{vmatrix} \tag{6}$$

or:

$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \sigma^2 (1 - \rho^2)\, r \tag{7}$$

Now by substituting the expressions given in (4) for x and y into equation (2) and by substituting the result together with equation (7) in equation (5) we obtain:

$$g(r, \theta) = \frac{\sqrt{1 - \rho^2}\, r}{2\pi} \exp - \frac{1}{2}(1 - \rho \sin 2\theta)\, r^2 \tag{8}$$

From equation (8) we see that $g(r, \theta)$ is independent of the variance σ^2. Thus the integrals I_1 and I_2 will also be independent of σ^2. The estimate $\hat{\rho}$ will be obtained by explicitly evaluating I_1 and I_2 which will be only functions of ρ and then substituting these quantities into the approximate equation (3). $\hat{\rho}$ may then be obtained by solving (3) for $\hat{\rho}$ as a function of $\frac{N+ - N-}{N}$.

Now from equations (3) and (8) we have:

$$\begin{aligned} I_1 &= \int_0^{\pi/2}\int_0^{\infty} g(r, \theta)\, dr\, d\theta \\ &= \frac{\sqrt{1 - \rho^2}}{2\pi}\int_0^{\pi/2}\int_0^{\infty} r \exp - \frac{1}{2}(1 - \rho \sin 2\theta) r^2\, dr\, d\theta \\ &= \frac{\sqrt{1 - \rho^2}}{2\pi}\int_0^{\pi/2} \frac{d\theta}{1 - \rho \sin 2\theta} \\ &= \frac{1}{2\pi} \operatorname{arcos}(-\rho) \end{aligned} \tag{9}$$

Similarly for I_2 we have:

$$I_2 = \int_{-\pi/2}^{0}\int_0^{\infty} g(r, \theta)\, dr\, d\theta$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi} \int_{-\pi/2}^{0} \int_{0}^{\infty} r \exp - \frac{1}{2}(1 - \rho \sin 2\theta)\, r^2 dr\, d\theta$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi} \int_{-\pi/2}^{0} \frac{d\theta}{1 - \rho \sin 2\theta} \tag{10}$$

$$= \frac{1}{2\pi} \operatorname{arcos}(\rho)$$

Substituting equations (9) and (10) into (3) we have:

$$\frac{1}{2\pi}\left[\operatorname{arcos}(-\rho) - \operatorname{arcos}(\rho)\right] \approx \frac{1}{2}\left(\frac{N_+ - N_-}{N}\right) \tag{11}$$

However, since we have chosen our angular arguments to lie between $\pm\pi$, we have:

$$\operatorname{arcos}(-\rho) = \pi - \operatorname{arcos}(\rho) \tag{12}$$

which implies:

$$\operatorname{arcos}(\hat{\rho}) = \frac{\pi}{2} - \frac{\pi}{2}\left(\frac{N_+ - N_-}{N}\right) \tag{13}$$

from which

$$\hat{\rho} = \cos\left[\frac{\pi}{2} - \frac{\pi}{2}\left(\frac{N_+ - N_-}{N}\right)\right] \tag{14}$$

or finally:

$$\hat{\rho} = \sin \frac{\pi}{2} \left(\frac{N_+ - N_-}{N} \right) \tag{15}$$

Note that $\hat{\rho}$ depends only upon the single sample quantity $\frac{N_+ - N_-}{N}$. It is somewhat remarkable that any consistent estimator of the correlation coefficient can be obtained from such a tremendous compression of the observations. $\hat{\rho}$ is of course a statistic and will exhibit random fluctuations from sampe to sample. The consistency of the estimate $\hat{\rho}$ follows from the fact that the law of large numbers guarantees that for sufficiently large N, the approximation given by (3) may be made arbitrarily close. The derivation of the sampling distribution of $\hat{\rho}$ is very simple since it is essentially binominially distributed under our assumption of pair-wise independence. In the more general case where the pairs are correlated, the sampling distribution is much harder to compute. It is of course a less efficient estimate of ρ than is the Pearson product moment estimator r which requires, however, much more information to be recorded from the samples. Its advantage is that it is much quicker to compute or perhaps more importantly, it allows the estimation of the correlation function when only zero crossing information is available and in other situations where the basic data are consored to the extent that more conventional estimates can't be computed.

Finally , we repeat that the estimator $\hat{\rho}$ as given by (15) is consistent under much weaker assumptions than the pair-wise independence of lagged observations, an assumption which we never explicitly used anyway. The same derivation is essentially valid in this case. The effect of pair-wise sample correlation is to partially degrade the validity of the approximation given in (3) based upon the law of large numbers. In other words, in order that (3) hold to a given order of accuracy, it will be necessary to have a much larger value of N if the sample pairs (x_i, y_i) and (x_{i+1}, y_{i+1}) are correlated than if they are independent. From the point of view of the estimator, the effect of the pair-wise

correlation is to make the sampling distribution of the estimate more erratic and to reduce its rate of approach to asymptotic stability with an increasing sample size.

The estimator $\hat{\rho}$ is known in the literature and occasionally quoted; however, I have never seen the simple heuristic development which we have given here in print, although it is undoubtedly well known. It is given here mainly as a convenient reference since it does not appear to be accessible in any of the standard texts.

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APOLLO NOTE NO. 200

H. Epstein
1 May 1964

BOUNDS FOR λ's INVOLVED IN LINEAR REGRESSION THEORY FOR A STATIONARY PROCESS

Writing $x_n = \sum_{i=1}^{q} \lambda_i x_{n-i} + e_n$

The characteristic equation is given by

$$\rho^q - \sum_{i=1}^{q} \lambda_i \rho^{q-i} = 0$$

Let the roots of the characteristic equation be given by ρ_r, where

$r = 1, 2, 3 \ldots\ldots, q-1, q.$

For a stationary process, $|\rho_r| \leq 1.$

Making use of the properties of the roots of a polynomial equation,

For example,

$$\lambda_q = \prod_{r=1}^{q} \rho_r$$

$$|\lambda_q| = \prod_{r=1}^{k} |\rho_r| \leq 1$$

In general,

$$|\lambda_i| \leq \frac{q!}{i!\,(q-i)!} \qquad q \geq 1$$

For q an even number, the maximum possible $\lambda = \lambda_{\frac{q}{2}}$ which for large q, $(\lambda)_{\frac{q}{2}} \sim \frac{2^q}{\sqrt{2\pi q}}$. These results indicate that for large q's, the values for λ_i may be very large.

An example of a regressive model of this characteristic is

$$\lambda_i = \frac{-q!\ b^i}{i!\ (q-i)!}$$

$$x_n + \sum_{i=1}^{q} \frac{q!\ b^i\ x_{n-i}}{i!\ (q-i)!} = e_n$$

Defining $X(s) = \mathcal{L}\left\{x_n(t)\right\}$

$E(s) = \mathcal{L}\left\{e_n(t)\right\}$

$$E(s) = X(s)\left[1 + \sum_{i=1}^{q} \frac{q!\ b^i\ e^{-isT}}{i!\ (q-i)!}\right] = X(s)\left(1 + b\,e^{-sT}\right)^q$$

where T = interval between samples.

Replacing s by $j\omega$ and letting $W_x = X(j\omega)\ X(-j\omega)$

$W_e = E(j\omega)\ E(-j\omega)$

$$W_x = W_e\left[1 + b^2 + 2b\ \cos\omega T\right]^{-q}$$

Restricting the analysis to positive frequencies and taking aliasing into account,

$$f_{max} = \frac{\omega_{max}}{2\pi} = \frac{1}{2T}$$

$$f_k = \frac{k}{N} f_{max} \qquad k = i,\ 1, 2, \ldots\ldots, N$$

where N = total number of samples available.

Then for W_e a constant (white noise spectrum),

$$W_x(k) = W_e \left[1 + b^2 + 2b \cos \frac{\pi k}{N} \right]^{-2q}$$

for $b \geq 0$ $\quad W_x(N) = (W_x)_{max} = W_e (1-b)^{-2q}$

$$W_x(0) = (W_x)_{min} = W_e (1+b)^{-2q}$$

for $b \leq 0$ $\quad W_x(0) = (W_x)_{max} = W_e (1+b)^{-2q}$.

$$W_x(N) = (W_x)_{max} = W_e (1-b)^{-2q}$$

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APOLLO NOTE NO. 201

H. Epstein
6 May 1964

RECEIVER NOISE LIMITATION TO DIFFERENTIAL RANGE ACCURACY

This note indicates a conservative limit to the accuracy which is imposed on the differential range measurements by receiver noise for the Apollo mission. The most severe condition to be encountered will occur for the spacecraft transmitting to Earth under the following conditions: one, spacecraft employing an omni-direction antenna; two, a 30 feet receiving antenna instead of the 85 feet antenna; three, maximum spacecraft separation from Earth ($\sim 4 \times 10^8$ meters); four, shortest sampling time (one second); and, five, parametric amplifiers rather than masers for the low noise receiver (noise figure about 0 db). In addition, a conservative estimate of the effective noise figure (which will be defined later in this note) is taken as 17 db. The standard deviation of the error in differential range is shown to be about .08 cm under this situation for a 5 watt transmitter in the spacecraft. Clearly receiver noise is not an important error source for differential range accuracy in the doppler mode of operation. The mathematical model will now be derived. Equation (9), Apollo Note 112, indicates the following result for an unmodulated CW transmitter waveform,

$$\Delta R = \frac{\lambda \sqrt{3}}{2\pi \sqrt{\frac{2E_o}{N_o}}} \tag{1}$$

In addition, for this situation

$$\frac{E_o}{N_o} = \frac{KT_s}{F_{eff}} \tag{2}$$

and

$$K = \frac{P_T G_T}{(k T)} \left(\frac{D}{4R} \right)^2 \tag{3}$$

where

ΔR = standard deviation of the differential range error

λ = transmitter wavelength ($\sim$13 cm)

E_o = energy received during the sampling time (watt-seconds)

N_o = noise power density (watt-seconds)

T_s = sampling time (1 second)

F_{eff} = effective noise figure = 17 db

P_T = transmitter power level = 5 watts

D = receiving antenna dish diameter = 10 meters

R = separation between spacecraft and station = 4×10^8 meters

k = Boltzman constant

T = temperature reference for noise figure

kT = 4×10^{-21} watt seconds

G_T = transmitting antenna power gain = 0 db.

then ΔR = .081 cm for the above stated numerical values.

To satisfactorily conclude this note the selected value for F_{eff} needs to be justified as conservative. F_{eff} is taken to include the following factors: one, receiver noise figure component associated with internal receiver noise and external background noise (0 db); two, transmitting losses (2 db); receiver losses (1 db); imperfect process loss (3 db); receiving antenna efficiency considered as a loss (3 db); transmission media losses, transmitting antenna degradation, and other system degradation factors (8db). The total for F_{eff} is then 17 db.

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 202

L. S. Lustick
6 May 1964

SELF-CORRELATED MEASUREMENTS

Purpose

The purpose of this report is to indicate how to determine the errors in the orbit parameters when the sequence of measurements are correlated.

Introduction

Previous Apollo notes and the existing program for orbit determination assumes that successive measurements of a measurable are uncorrelated. The purpose of this Apollo Note is to indicate how this analysis can be extended in the event that successive measurements are correlated.

Method

Consider a random sequence of numbers generated by the auto-regressive model shown in Equation (1).

$$e_i = \sum_{j=1}^{q} \lambda_j \, e_{i-j} + \xi_i \tag{1}$$

where:

ξ_i is a zero mean gaussian variable.

$$\overline{\xi_i \, \xi_{i+m}} = 0 \qquad m \neq 0$$

$$= \sigma_\xi^2 \qquad m = 0$$

The order of dependence of a process generated as shown in Equation ( ) is q, where the term order of dependence is defined in Equation (2).

$$P\left[e_n/e_{n-1}, e_{n-2} \ldots, e_1 \right] = P\left[e_n/e_{n-1} \ldots, e_{n-q} \right] \qquad (2)$$

The joint distribution of a set of measurables e_i can be defined recursively as shown in Equation (3),

$$P\left[e_1, e_2 \ldots, e_q \right] \times P\left[e_{q+1}/e_1, e_2 \ldots, e_q \right] = P\left[e_1, e_2 \ldots, e_{q+1} \right] \qquad (3)$$

The above equation can be used to define the likelihood of a set of measurables and the orbit parameters can be determined which maximize the likelihood of the data set. This procedure is illustrated for an order of dependence equal to one in the following analysis.

Consider:

$$e_i = \lambda_1 \; e_{i-1} + \xi_i \qquad (4)$$

where: e_i = error in measurable at time $t = t_i$

since $e_i^2 = \lambda_1^2 \; e_{i-1}^2 + \xi_i^2 + 2 \, \xi_i \, \lambda_1 \; e_{i-1}$ and $\overline{\xi_i \; e_{i-1}} = 0$

then

$$\sigma_e^2 = \frac{\sigma_\xi^2}{1-\lambda_1^2} \qquad (5)$$

let

m_i = measurement at time $t = t_i$

m_{c_i} = computed value at time $t = t_i$

The correlation function for this process is exponential, that is;

$$\rho_n = \lambda_1 \, \rho_{n-1}$$

More sophisticated correlation functions can be obtained with higher orders of dependence.

The likelihood expression, $\mathcal{L}$, is presented in Equation (6).

$$\mathcal{L} = \frac{1}{2\sigma_{\xi}^{2}} \left\{ (1-\lambda_1^2) \left[m_1 - m_{c_1} \right]^2 + \sum_{i=2}^{N} \left[(m_i - m_{c_i}) - \lambda_1 (m_{i-1} - m_{c_{i-1}}) \right]^2 \right\} \quad (6)$$

Linearizing we have:

$$m_1 - m_{c_1} = m_t + e_1 - \left(m_{c_1} + \sum_k \frac{\partial m_{c_1}}{\partial a_k} \Delta a_k \right) = e_1 - \sum_k \frac{\partial m_{c_1}}{\partial a_k} \Delta a_k$$

$$m_i - m_{c_i} = e_i - \sum_k \frac{\partial m_{c_{i-1}}}{\partial a_k} \Delta a_k$$

$$m_i - m_{c_{i-1}} = e_{i-1} - \sum_k \frac{\partial m_{c_{i-1}}}{\partial a_k} \Delta a_k$$

$$m_i - m_{c_i} - \lambda_1 \left(m_{i-1} - m_{c_{i-1}} \right) = \xi_i - \sum_k \left[\frac{\partial m_{c_i}}{\partial a_k} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_k} \right] \Delta a_k$$

Rewriting Equation (6) we have;

$$\mathcal{L} = \frac{1}{2\sigma_{\xi}^{2}} \left\{ (1-\lambda_1^2) \left[e_1 - \sum_k \frac{\partial m_{c_1}}{\partial a_k} \Delta a_k \right]^2 \right.$$

$$\left. + \sum_{i=2}^{N} \left[\xi_i - \sum_k \left(\frac{\partial m_{c_i}}{\partial a_k} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_k} \right) \Delta a_k \right]^2 \right\}$$

$$\frac{\partial \mathcal{L}}{\partial a_j} = 0 = (1-\lambda_1)^2 \frac{\partial m_{c_1}}{\partial a_j}\left[e_1 - \sum_k \frac{\partial m_{c_1}}{\partial a_k} \Delta a_k \right]$$

$$+ \sum_{i=2}^{N} \left(\frac{\partial m_{c_i}}{\partial a_j} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_j} \right) \left[\xi_i - \sum_k \left(\frac{\partial m_{c_i}}{\partial a_k} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_k} \right) \Delta a_k \right]$$

Let

$$C_{jk} = \sum_{i=2}^{N} \left(\frac{\partial m_{c_i}}{\partial a_j} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_j} \right) \left(\frac{\partial m_{c_i}}{\partial a_k} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_k} \right)$$

$$F_{jk} = (1-\lambda_1^2) \frac{\partial m_{c_1}}{\partial a_j} \frac{\partial m_{c_1}}{\partial a_k}$$

Now writing the matrix equation defining Δa_k

$$(F + C)(\Delta a_k) = (1-\lambda_1^2) \; e_1 \begin{pmatrix} \frac{\partial m_{c_1}}{\partial a_1} \\ \frac{\partial m_{c_1}}{\partial a_2} \\ \vdots \\ \frac{\partial m_{c_1}}{\partial a_6} \end{pmatrix} + \begin{pmatrix} \sum_{i=2}^{N} \left(\frac{\partial m_{c_i}}{\partial a_1} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_1} \right) \xi_i \\ \vdots \\ \sum_{i=2}^{N} \left(\frac{\partial m_{c_i}}{\partial a_6} - \lambda_1 \frac{\partial m_{c_{i-1}}}{\partial a_6} \right) \xi_i \end{pmatrix}$$

$$(F + C)(\Delta a_k) = (1-\lambda_1^2)\ e_1\ (A) + (B)$$

$$\Delta a_k = (F + C)^{-1} \left[(1-\lambda_1^2)\ e_1\ A + B \right]$$

$$\overline{\Delta a_k\, \Delta a_k^T} = (F + C)^{-1} \overline{\left[(1-\lambda_1^2)\, e_1\ A + B \right] \left[A^T\, e_1\ (1-\lambda_1^2) + B^T \right]}\ (F + C)^{-1}$$

$$= (F + C)^{-1} \left[(1-\lambda_1^2)^2\ \sigma_e^2\ \frac{F}{(1-\lambda_1^2)} + \sigma_\xi^2\ C \right] (F + C)^{-1}$$

$$= \sigma_\xi^2\ (F + C)^{-1} \left[F + C \right] (F + C)^{-1}$$

$$\overline{\Delta a_k\, \Delta a_k^T} = \sigma_\xi^2\ (F + C)^{-1} \tag{7}$$

or

$$\overline{\Delta a_k\, \Delta a_k^T} = (1 - \lambda_1^2)\ \sigma_e^2\ (F + C)^{-1} \tag{7a}$$

If $\lambda_1 = 0$, the order of dependence is zero, and the expression for $\overline{\Delta a_k\, \Delta a_k^T}$ is as shown in Equation (8).

$$\overline{\Delta a_k\, \Delta a_k^T}\quad \sigma_e^2\ (G)^{-1}$$

where

$$G_{jk} = \sum_{i=1}^{N} \frac{\partial m_{c_i}}{\partial a_j}\ \frac{\partial m_{c_i}}{\partial a_k} \tag{8}$$

This is the result that one would expect for independent data.

Comparison of Correlated with Uncorrelated Data

It will be instructive to investigate the effect of correlation upon the errors in the determination of the orbit parameters.

Neglecting end effects the information matrix with correlation (order of dependence of one) is given in Equation (9).

$$C_{jk} = \sum \left[\frac{\partial m_i}{\partial a_j} - \lambda_1 \frac{\partial m_{i-1}}{\partial a_j} \right] \left[\frac{\partial m_i}{\partial a_k} - \lambda_1 \frac{\partial m_{i-1}}{\partial a_k} \right] \tag{9}$$

$$C_{jk} = (1 + \lambda_1^2)\, G - 2\lambda_1 \sum \frac{\partial m_{i-1}}{\partial a_j} \frac{\partial m_i}{\partial a_k}$$

$$\text{where } G = \sum \frac{\partial m_i}{\partial a_j} \frac{\partial m_i}{\partial a_k}$$

Let us now define $\hat{C}$ as shown below

$$\hat{C}_{jk} = \sum \left[\frac{\partial m_i}{\partial a_j} - \frac{\partial m_{i-1}}{\partial a_j} \right] \left[\frac{\partial m_i}{\partial a_k} - \frac{\partial m_{i-1}}{\partial a_k} \right]$$

$$\text{then } \hat{C} = 2G - 2 \sum \frac{\partial m_{i-1}}{\partial a_j} \frac{\partial m_i}{\partial a_k}$$

or

$$\sum \frac{\partial m_{i-1}}{\partial a_j} \frac{\partial m_i}{\partial a_k} = G - \frac{\hat{C}}{2}$$

Substituting then in the expression for C_{jk}, we have;

$$C_{jk} = (1 - \lambda_1)^2 \; G + \lambda_1 \hat{C}$$

$\hat{C}$ is approximately the information matrix obtained if the measurable is the derivative of the measurable used to obtain G. We know therefore that if we drop out the term involving $\hat{C}$, we will obtain a pessimistic covariance matrix for the errors in the orbit parameters. This result is presented in Equation (10).

$$\begin{aligned} \overline{\Delta a_k \, \Delta a_k^T} &\leq \left(\frac{1+\lambda}{1-\lambda}\right) \sigma_e^2 \; G^{-1} \\ &\leq \left(\frac{1+\lambda}{1-\lambda}\right) \left[\overline{\Delta a_k \; \Delta a_k^T} \right]_{\text{uncorrelated data}} \end{aligned} \tag{10}$$

ADDENDUM TO APOLLO NOTE NO. 203

H. Epstein

The present technique employed by JPL for calculation of differential range is an excellent technique. The major deficiency for Apollo application stems from the magnitude of the quantization error. This problem area is most significant for short counting time intervals. This error can be considerably reduced for Apollo. With no change, the standard deviation of this type error will be reduced by a factor of almost 2-1/2 due to the increased carrier frequency employed for Apollo. Two other rather straightforward techniques can be employed to further reduce this error. A technique which counts both positive and negative zero crossing, properly implemented, would reduce this error by an additional factor of two. Another technique would involve a multiplier for the doppler frequency prior to counting. The limitation imposed here would be associated with the multiplier noise generated. A multiplication ratio of about 10 seems reasonable. This indicates that quantization error can be reduced by about two order of magnitude for Apollo application.

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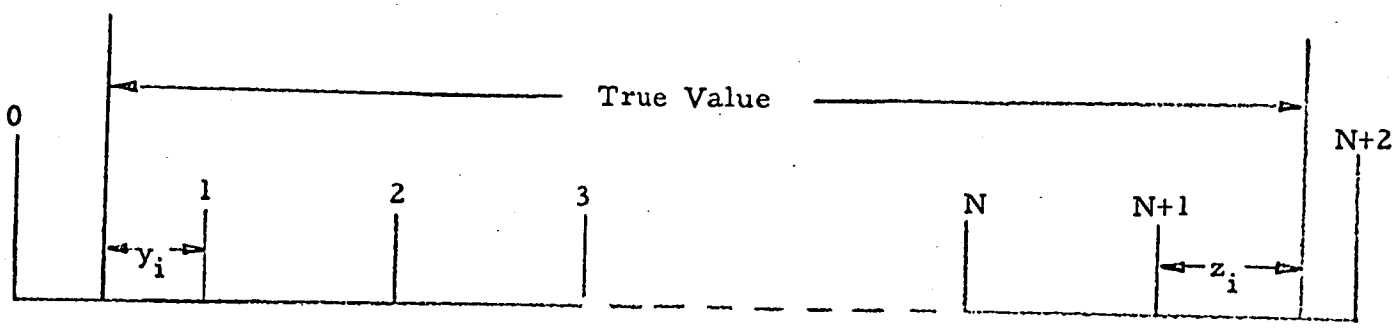
APOLLO NOTE NO. 203

H. Epstein
8 May 1964

SOME CHARACTERISTICS OF QUANTIZATION ERRORS

This note is concerned with some properties of the quantization error associated with a simple fixed time period system for measurement of differential range by zero crossing techniques. All other error sources are considered as being negligible for the purpose of this analysis. The quantization error is most simply represented by considering the error in the count involved. The figure below indicates the model for the process.



Then $\Delta x_i = x_{True} - x_{meas} = y_i + z_i - 1$ (1)

Considering y_i and z_i each as random variables with uniform probability of an error occurring between zero and one count.

Then $\bar{y} = \bar{z} = \int_0^1 y\, dy = \frac{1}{2}$ (2)

Δx_i can now be written as

$$\Delta x_i = y_i - \frac{1}{2} + z_i - \frac{1}{2} = (y_i - \bar{y}) + (z_i - \bar{z}) \quad (3)$$

then $\overline{\Delta x^2} = \overline{(y_i - \bar{y})^2} + \overline{(z_i - \bar{z})^2} + 2\,\overline{(y_i - \bar{y})\,(z_i - \bar{z})}$ (4)

For $\overline{(y_i - \bar{y})}\;\overline{(z_i - \bar{z})} = 0$

$$\sigma_x^2 = \overline{\Delta x^2} = 2\,\overline{(y_i - \bar{y})^2} \tag{5}$$

$$\overline{(y_i - \bar{y})^2} = \overline{(z_i - \bar{z})^2} = \int_0^1 (y - \tfrac{1}{2})^2\,dy = \frac{1}{12} \tag{6}$$

and $\sigma_x = \dfrac{1}{\sqrt{6}} = .408$ is the desired result (7)

This indicates that the standard deviation of the quantization error for this situation is about .4 counts. The associated differential range error ($\sigma_{\Delta R}$) is given for a two-way doppler mode by the following equation.

$$\sigma_{\Delta R} = \frac{\bar{\lambda}}{2\sqrt{6}} \tag{8}$$

where

$\bar{\lambda}$ = average wavelength of the ground-based and spacecraft transmitter frequencies.

The corresponding formulas for the average range-rate error ($\sigma_{\dot{k}}$) and the frequency error (σ_f) are given by

$$\sigma_{\dot{R}} = \frac{\bar{\lambda}}{2T\sqrt{6}} \tag{9}$$

where

T = counting interval

$$\sigma_f = \frac{1}{T\sqrt{6}} \tag{10}$$

The transmitting and receiving frequencies for the present L-band DSIF are 890 and 960 mc respectively. The associated differential range error is 6.63 cm ($\bar{\lambda}$ = 32.5 cm). The frequency and range-rate errors for T = 1, 10, 50, and 60 seconds are indicated in the table below.

TABLE 1.

Frequency and Range-Rate Quantization Errors

T (seconds)	σ_f (cps)	$\sigma_{\dot{R}}$ (cm/sec.)
1	.408	6.64
10	.0408	.664
50	.00817	.133
60	.00680	.110

The correlation function associated with this model for quantization error model assumed can easily be calculated. Both continuous counting and interrupted counting techniques will be treated. The value of the correlation function for both cases with lag values greater than one is zero. As a consequence, it is only necessary to calculate the variance and the one lag values.

Letting

$$x_n = e_n - e_{n-1} \tag{11}$$

then

$$\sigma_x^2 = \phi_o = 2\,\sigma_e^2 \tag{12}$$

$$\text{and } \phi(1) = \overline{(e_n - e_{n-1})(e_{n-1-\delta} - e_{n-2-\delta})} \tag{13}$$

where

$\delta = 0$ for continuous counting

$\delta > 0$ interrupted counting techniques

$$\text{then } \phi(1) = -\overline{(e_{n-1})(e_{n-1-\Delta})} \tag{14a}$$

$$= -\sigma_e^2 \qquad \text{for } \delta = 0 \tag{14b}$$

$$= 0 \qquad \text{for } \delta > 0 \tag{14c}$$

or

$$\rho(1) = -\frac{1}{2} \qquad \text{for } \delta = 0 \tag{15a}$$

$$= 0 \qquad \text{for } \delta > 0 \tag{15b}$$

One simple test for such an hypotheses with actual measurement data is to establish confidence limits in terms of the variance of the errors in the auto-correlation function with the assumed model. The covariance matrix for the errors in the auto-correlation function was shown in Apollo Note No. 197 to be of the following form.

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_m\,\Delta\phi_{m+r}} = \rho(r) + \rho(m)\,\rho(m+r) + \sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\Big[\rho(k)\,\rho(k+r) + \rho(k)\,\rho(k-r) + \rho(k+m+r)\,\rho(k-m) + \rho(k-m-r)\,\rho(k+m)\Big] \tag{16}$$

The following simplification can be made for the quantization error model.

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_m\,\Delta\phi_{m+r}} = \rho(r) + \rho(m)\,\rho(m+r) + \left(1 - \frac{1}{N}\right)\Big[\rho(1)\,\rho(1+r) + \rho(1)\,\rho(r-1) + \rho(1+m+r)\,\rho(m-1) + \rho(m+r-1)\,\rho(m+1)\Big] \tag{17}$$

where m is the row of the matrix involved and $r \geq 0$. Other terms can be found from symmetry $(\overline{\Delta\phi_i\,\Delta\phi_j} = \overline{\Delta\phi_j\,\Delta\phi_i})$.

Letting $\rho(1) = \rho$, the following expressions are found

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_o^2} = 2 + 4\rho^2\left(1 - \frac{1}{N}\right) \simeq 2 + 4\rho^2 \tag{18}$$

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_1^2} = 1 + \rho^2\left(3 - \frac{2}{N}\right) \simeq 1 + 3\rho^2 \tag{19}$$

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_m^2} = 1 + 2\rho^2\left(1 - \frac{1}{N}\right) \simeq 1 + 2\rho^2 \qquad \text{for } m \geq 2 \tag{20}$$

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_o\,\Delta\phi_1} = 2\rho\left(2 - \frac{1}{N}\right) \simeq 4\rho \tag{21}$$

$$\frac{N}{\phi_o^2}\,\overline{\Delta\phi_m\,\Delta\phi_{m+1}} = \rho\left(2 - \frac{1}{N}\right) \simeq 2\rho \qquad \text{for } m \geq 1 \tag{22}$$

$$\frac{N}{\phi_o^2} \overline{\Delta\phi_o \Delta\phi_2} = 2\rho^2 (1 - \frac{1}{N}) \simeq 2\rho^2 \tag{23}$$

$$\frac{N}{\phi_o^2} \overline{\Delta\phi_m \Delta\phi_{m+2}} = \rho^2 (1 - \frac{1}{N}) \simeq \rho^2 \qquad \text{for } m \geq 1 \tag{24}$$

$$\overline{\Delta\phi_m \Delta\phi_{m+r}} = 0 \qquad \text{for } r \geq 3 \tag{25}$$

Which for the interrupted counting technique

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_o^2} \simeq \frac{2}{N} \tag{26}$$

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_m^2} \simeq \frac{1}{N} \qquad m \geq 1 \tag{27}$$

$$\overline{\Delta\phi_m \Delta\phi_{m+r}} = 0 \qquad r \geq 1 \tag{28}$$

and for the continuous counting technique ($\rho = -\frac{1}{2}$)

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_o^2} \simeq \frac{3}{N} \tag{29}$$

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_1^2} \simeq \frac{1.75}{N} \tag{30}$$

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_m^2} \simeq \frac{1.5}{N} \qquad m \geq 2 \tag{31}$$

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_o \Delta\phi_1} \simeq \frac{-2}{N} \tag{32}$$

$$\frac{1}{\phi_o} \overline{\Delta\phi_m \Delta\phi_{m+1}} \simeq \frac{-1}{N} \qquad m \geq 1 \tag{33}$$

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_o \Delta\phi_2} \simeq \frac{1}{2N} \tag{34}$$

$$\frac{1}{\phi_o^2} \overline{\Delta\phi_m \Delta\phi_{m+2}} \simeq \frac{1}{4N} \qquad m \geq 1 \tag{35}$$

$$\overline{\Delta\phi_m \Delta\phi_{m+r}} = 0 \qquad r \geq 3 \tag{36}$$

A normalized standard deviation of the errors in the autocorrelation function (σ_k) is defined by the following expression.

$$\sigma_k = \frac{1}{\phi_o} \sqrt{\overline{\Delta\phi_k^2}}$$

Figures 1 and 2 indicate numerical values for σ_k as a function of N for the interrupted and continuous counting techniques.

The normalized correlation coefficient between errors for the continuous techniques is defined below

$$\rho(k, r) = \frac{\overline{\Delta\phi_k \, \Delta\phi_{k+r}}}{\phi_o^2 \; \sigma_k \; \sigma_{k+r}}$$

$$= 1 \qquad r = 0$$

$$= 0 \qquad r \geq 3$$

Other values are indicated in the table below.

Table 2.
Normalized Correlation Coefficient Between
Errors for Continuous Counting

k	ρ(k, 1)	ρ(k, 2)
0	-.872	.236
1	-.617	.154
≥ 2	-.667	.167

This table clearly indicates the strong correlation between adjacent errors.

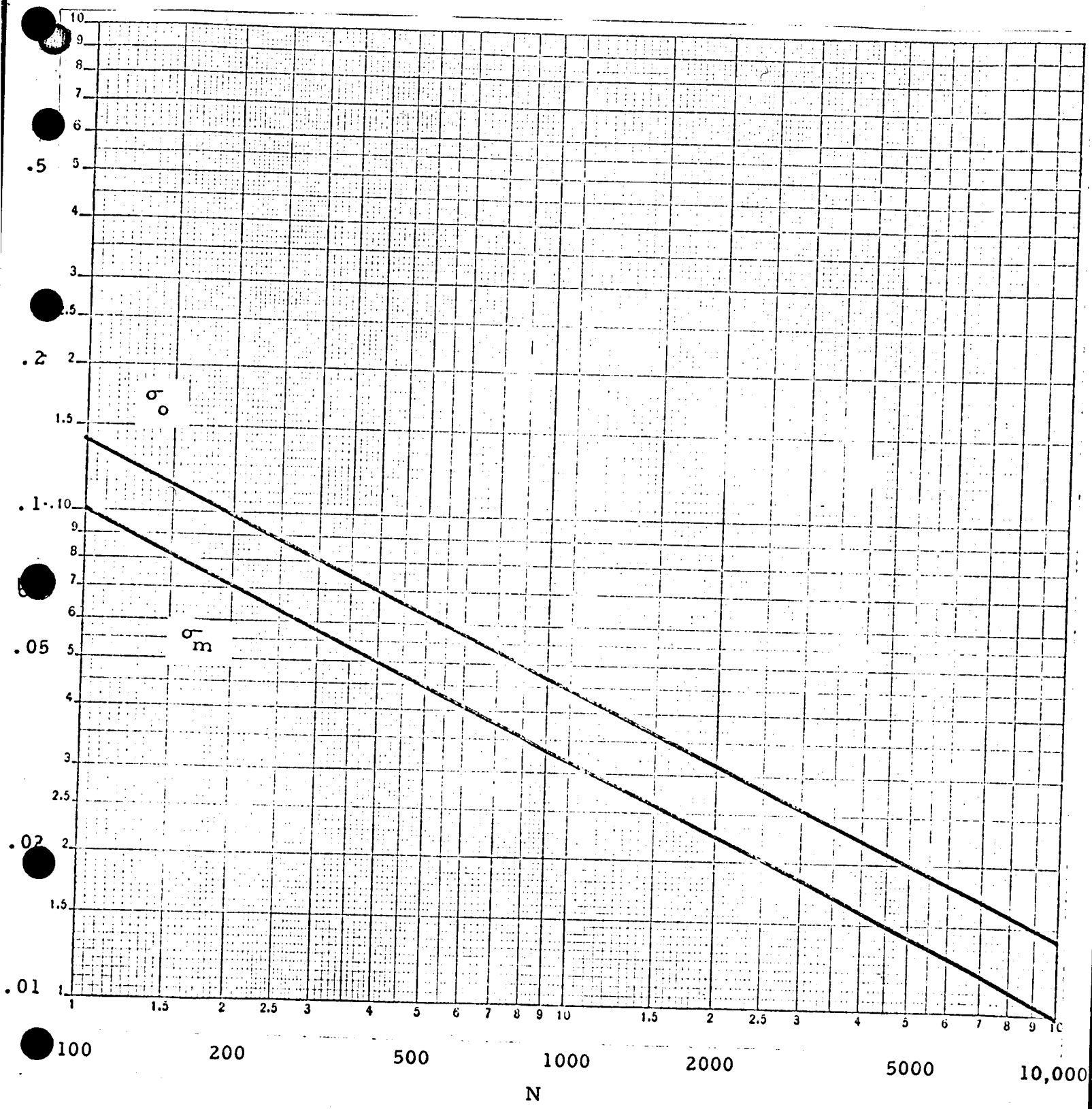


Figure 1.

Normalized Standard Deviation of Errors in Auto-correlation Function with Interrupted Counting Technique.

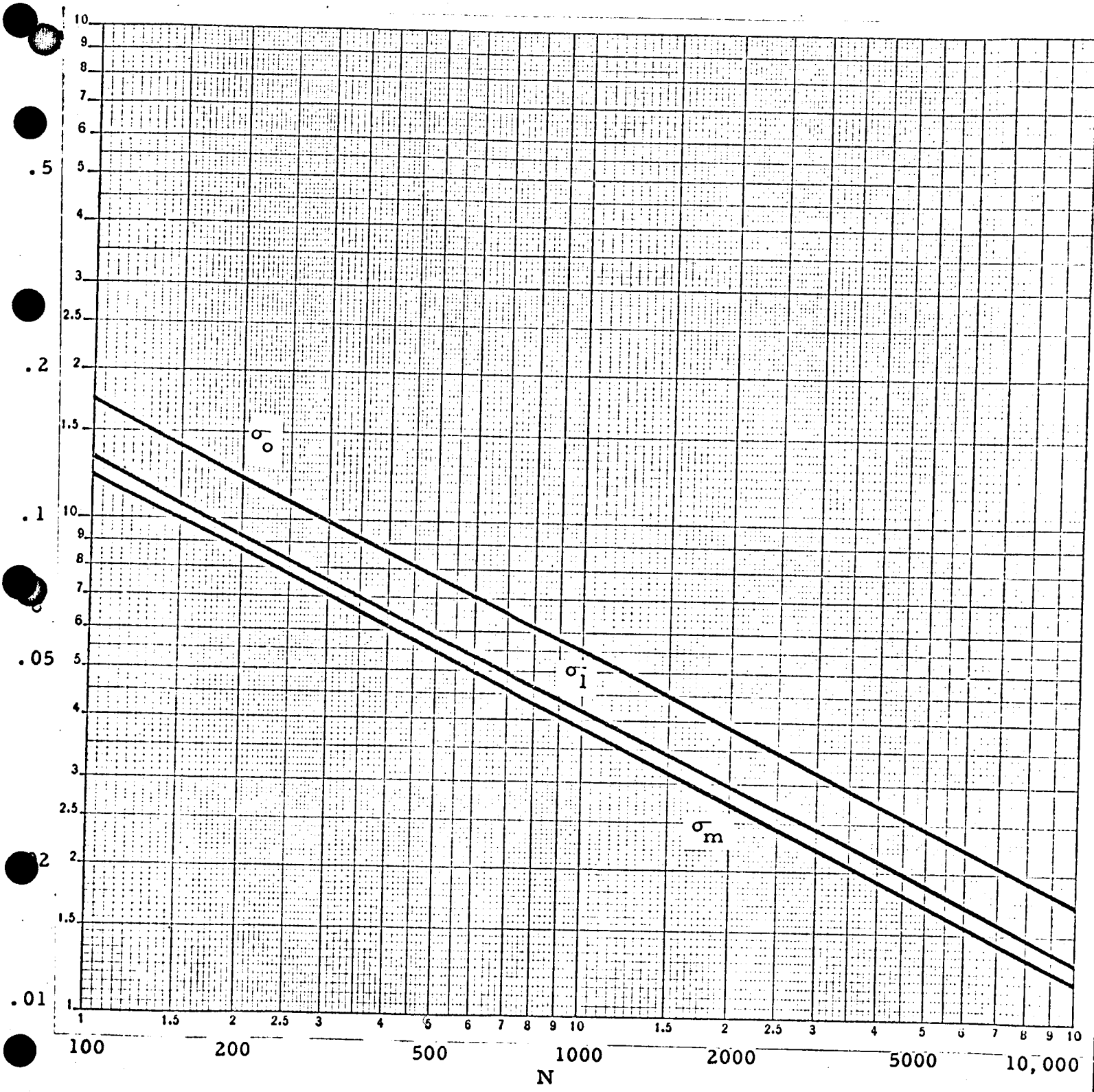


Figure 2.

Normalized Standard Deviation of Errors in Auto-correlation Function with Continuous Counting Techniques

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APOLLO NOTE NO. 204

J. R. Holdsworth
12 May 1964

THE ESTIMATION OF THE PARAMETERS OF AN AUTOREGRESSIVE STATIONARY STOCHASTIC PROCESS OF FINITE ORDER

Some previous Apollo Notes have dealt with the problem of estimating functionals of stationary stochastic processes such as the power spectral density function and/or autocorrelation function. These problems generally possess the property that only very minimal a priori assumptions are made about the underlying structure of the process such as their stationarity and gaussian nature, etc.

The purpose of the present note is to consider a class of problems where the a priori assumptions about the nature of the time series are much stronger. In fact, we shall assume that the probabilistic structure of the time series is known with the exception of a finite number of parameters whose values are to be estimated from observations made upon the time series. The name "finite parameter scheme" has been used to denote this general problem area which actually comprises two major sub-problems. The first which will not concern us in this note is the problem of estimating the parameters in a deterministic component of known functional form from measurements of this component which have been contaminated by additive noise.

The problem which will concern us in this note is the estimation of the parameters of stochastic processes which are solutions of linear time invariant stochastic difference equations or differential equations depending upon whether the process is defined for discrete

time instants or continuous time, respectively. For the class of problems which we shall consider our problem is equivalent to estimating a finite set of unknown parameters appearing in the correlation function or power spectral density function of known functional form from observations made upon the process.

Before discussing the general case it will be helpful to consider one or two specific examples. We shall first assume that we are observing a time series x_n at equally spaced instants in time and that it is known that x_n satisfies the following first order stochastic difference equation

$$x_k = cx_{k-1} + e_k \tag{1}$$

where c is a positive constant less than 1 and the e_k are independent gaussian variates with zero mean and variance σ_e^2. On the basis of observing $x_1, \ldots x_N$ we wish to estimate the quantities c and σ_e^2.

Successive iteration of equation (1) shows that we may write:

$$x_k = c^{k-m} x_m + \sum_{i=0}^{k-m-1} c^i e_{k-i} \tag{2}$$

If $|c| < 1$ then we have a stable system in the sense that values of x_m as $m \to -\infty$ have a vanishingly small influence on x_k so that setting $m = -\infty$ in equation (2) yields

$$x_k = \sum_{i=0}^{\infty} c^i e_{k-i} \tag{3}$$

From (3) we see that since x_k is a linear combination of gaussian variates that x_k must be a gaussian process. Moreover, since $E(e_k) = 0$ for all k then $E(x_k) = 0$ also, so that the structure of the time series x_k will be completely determined by its autocovariance function.

We now derive an equation for $\phi_x(\tau)$ where ϕ_x is the autocovariance function of x for some integer lag τ. By definition:

$$\phi_x(\tau) = E\left[x_k \, x_{k+\tau}\right] \tag{4}$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c^{i+j} \, E\left(e_{k+\tau-i} \, e_{k-i}\right) \tag{5}$$

But since the e_k are zero mean and independent

$$E\left[e_{k+\tau-j} \, e_{k-i}\right] = \delta_{i,\, j-\tau} \, \sigma_e^2 \tag{6}$$

so that

$$\phi_x(\tau) = \sum_{i=0}^{\infty} c^{2i+\tau} \, \sigma_e^2$$

$$= \left(\frac{1}{1-c^2}\right) c^{\tau} \, \sigma_e^2 \tag{7}$$

Since $\sigma_x^2 = \phi_x(0)$, we see from (7) that

$$\sigma_x^2 = \sigma_e^2 / 1 - c^2 \tag{8}$$

so that knowledge of c and σ_e^2 gives us complete distributional knowledge about the process x_n.

To estimate c and σ_e^2 given the observed values $x_1, \ldots x_N$, we proceed as follows: Since the quantities e_k are identically distributed gaussian variates, the joint density of $e_2, \ldots e_N$ may be written

$$L(e_2, \ldots e_N) = \frac{1}{(2\pi)^{\frac{N-1}{2}} \sigma_n^{N-1}} \exp - \frac{1}{2\sigma_n^2} \sum_{k=2}^{N} e_k^2 \tag{9}$$

or from equation (1):

$$L = \frac{1}{(2\pi)^{\frac{N-1}{2}} \sigma_e^{N-1}} \exp - \frac{1}{2\sigma_e^2} \sum_{k=2}^{N} (x_k - cx_{k-1})^2 \qquad (10)$$

The maximum likelihood estimates $\hat{c}$ and $\hat{\sigma}_e^2$ of c and σ_e^2 are obtained from the solution of the equations:

$$\frac{\partial L}{\partial c} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \sigma_e^2} = 0 \qquad (11)$$

and are easily found to be:

$$\hat{c} = \frac{\sum_{k=2}^{N} x_k x_{k-1}}{\sum_{k=2}^{N} \left(x_{k-1}\right)^2}, \quad \hat{\sigma}_e^2 = \frac{1}{N-1} \sum_{k=2}^{N} \hat{e}_k^2 \qquad (12)$$

where

$$\hat{e}_k = x_k - \hat{c}\, x_{k-1} \qquad (13)$$

The distribution of the estimators $\hat{c}$ and $\hat{\sigma}_e^2$ is difficult to calculate explicitly because of the correlation between successive observations on the x_n process although the distribution of $\hat{\sigma}_e^2/\sigma_e^2$ is approximately chi square with N - 2 degrees of freedom. It is, however, known that in the present case, e.g., ref. (1), the estimators $\hat{c}$ and $\hat{\sigma}_e^2$ are asymptotically with N, jointly normally distributed with a covariance matrix given by:

$$\operatorname{cov}(\hat{c}, \hat{\sigma}_e^2) = \begin{pmatrix} E\left(-\frac{\partial^2 \log L}{\partial c^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial c \partial \sigma_e^2}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial c \partial \sigma_e^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial (\sigma_e^2)^2}\right) \end{pmatrix}^{-1} \tag{14}$$

Differentiating equation (10) logarithmically and taking expected values, it is a straightforward task to show that:

$$\begin{aligned} E\left(-\frac{\partial^2 \log L}{\partial c^2}\right) &= \frac{N-1}{1-c^2} \\ E\left(-\frac{\partial^2 \log L}{\partial (\sigma_e^2)^2}\right) &= \frac{N-1}{2\sigma_e^4} \end{aligned} \tag{15}$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial c \partial \sigma_e^2}\right) = 0 \tag{16}$$

Thus equations (14), (15), and (16) reveal the interesting fact that the estimators $\hat{c}$ and $\hat{\sigma}_e^2$ are asymptotically independently distributed as gaussian random variables with

$$\begin{aligned} \operatorname{Var} \hat{c} &\approx \frac{1-c^2}{N} \\ \text{and} \quad \operatorname{Var} \hat{\sigma}_e^2 &\approx \frac{\sigma_e^4}{N/2} \end{aligned} \tag{17}$$

respectively.

Knowing that the form at the autocorrelation function is given by (7), it is reasonable to estimate this function by

$$\hat{\phi}(\tau) = \frac{\hat{c}^{\tau}\hat{\sigma}_e^2}{1 - \hat{c}^2} \tag{18}$$

Now by replacing $\hat{c}$ and $\hat{\sigma}_e^2$ in (18) by their values as given in equations (12) and (13), we may obtain an expression for $\hat{\phi}(\tau)$ as a function of the observed data. It is interesting to note that no matter what τ is that the estimator $\hat{\phi}(\tau)$ depends only upon the zeroth and first order average lagged products of the observation.

As a further illustration of the additional simplicity afforded by the assumption that the observations satisfy an autoregressive equation, it is of interest to note that the variance of the correlation function estimator (18) is approximately

$$\text{Var } \phi(\tau) \approx \sigma_e^4 \frac{c^{2\tau} - 2}{N} \left(\tau + \frac{2c^2}{1 - c^2} \right)$$

Comparison of this expression with the variance of the lag product estimator given in note (3) shows that the a priori knowledge that the process satisfies a given stochastic difference equation can be a very powerful piece of information.

It is important to realize that the particular manner in which the data is processed to obtain the estimator $\hat{c}$ as given by equation (12) depends upon our assumption about the gaussian nature of the residual process e_k. To illustrate this we shall consider the problem of deriving the maximum likelihood estimator for the parameter c for the case where the e_k quantities are independently and identically distributed according to a Rayleigh distribution law. That is instead of hypothesizing a gaussian distribution, we assume that the probability density function of each e_k may be written:

$$\begin{aligned} f(e_k) &= \frac{e_k}{\theta^2} \exp - \frac{e_k^2}{2\theta^2} \qquad e_k > 0 \\ &= 0 \qquad e_k \leq 0 \end{aligned} \tag{18}$$

where we assume for convenience that the parameter θ of the residual process is known and need not be estimated from the observations.

Again we assume that our sequence x_n satisfies equation (1) and that we wish to estimate c on the basis of observing $x_1, \ldots x_N$ for some positive integer N. Using (1) and (18), and recalling that the e_k's are independently distributed, we may write the likelihood function of the observed data as:

$$L(x_1, \ldots x_N) = \prod_{k=2}^{N} \frac{(x_k - cx_{k-1})}{\theta^2} \exp - \frac{1}{2\theta^2} \sum_{k=2}^{N} (x_k - cx_{k-1})^2 \tag{19}$$

or more conveniently:

$$\log L(x_1, \ldots x_N) = \log \frac{1}{\theta^{N-1}} + \sum_{k=2}^{N} \log \frac{x_k - cx_{k-1}}{\theta} - \frac{1}{2\theta^2} \sum_{k=2}^{N} (x_k - cx_{k-1})^2 \tag{20}$$

To obtain the maximum likelihood estimate for c, we wish to choose the estimate $\hat{c}$ as that function of the data $x_1, \ldots x_N$ which maximizes (20). Thus setting

$$\frac{\partial \log L}{\partial c} = 0 \tag{21}$$

and performing some simple algebra, we obtain:

$$- \sum_{k=2}^{N} \frac{x_{k-1}}{x_k - \hat{c}x_{k-1}} + \frac{1}{\theta^2} \sum_{k=2}^{N} x_{k-1}(x_k - \hat{c}x_{k-1}) = 0. \tag{22}$$

That is, the maximum likelihood estimator $\hat{c}$ is obtained by solving (22) for $\hat{c}$ as a function of the observations $x_1, \ldots x_N$ and the assumed known constant θ. However, from equation (22) we see that in order to obtain the estimator $\hat{c}$, it is necessary in general to solve an Nth degree polynomial equation which cannot be done in closed analytical form when N is large.

Even though a closed form expression for $\hat{c}$ may not be obtained from equation (22), it is not difficult to show that the estimator given by (12) in the gaussian case is not a root so that we conclude that the form of the maximum likelihood estimator for c depends upon how the e_k are distributed.

Somewhat more generally, we now assume that we are observing a discrete time series that satisfies a linear autoregressive scheme of order k. Specifically we assume that the nth observation satisfies the equation

$$x_n = \sum_{j=1}^{k} a_j x_{n-j} + e_n \tag{23}$$

where the a_j's are constant and the e_n's are independent zero mean gaussian variables with variance σ_e^2. There is no real loss of generality in assuming the e_n quantities to have zero means, but it does afford a certain notational advantage. To obtain an expression for the autocorrelation function of the x_n process, we multiply both sides of equation (23) by x_ℓ where ℓ is some integer less than n to obtain:

$$x_\ell x_n = \sum_{j=1}^{k} a_j x_\ell x_{n-j} + x_\ell e_n \tag{24}$$

Taking the expected value of equation (24) and noting that since $\ell < n$ that x_ℓ and e_n are independent, we obtain the following equation for the autocorrelation function:

$$\phi(n-\ell) = \sum_{j=1}^{k} a_j \phi(n-\ell-j)$$

or

$$\phi(n-\ell) - \sum_{j=1}^{k} a_j \phi(n-\ell-j) = 0 \tag{25}$$

Replacing $n - \ell$ by ℓ we see that from (25) the autocorrelation function of the observations must satisfy the kth order constant coefficient difference equation given by:

$$\phi(\ell) - \sum_{j=1}^{k} a_j \phi(\ell - j) = 0 \tag{26}$$

It is perhaps worthwhile to note that if the random function is observed in continuous time and is known to satisfy that k^{th} order constant coefficient stochastic differential equation given by

$$\sum_{j=1}^{k} a_j \frac{d^j}{dt^j} x(t) + x(t) = e(t) \tag{27}$$

where the $e(t)$ are independent random pulses, then the correlation function for the continuous time process satisfies the homogenous differential equation given by

$$\sum_{j=1}^{k} a_j \frac{d^j}{dt^j} \phi(t) + \phi(t) = 0 \tag{28}$$

Thus for example, the equation for the angular displacement of a pendulum in a turbulent fluid is usually assumed, for small displacements, to satisfy the stochastic differential equation

$$\frac{d^2\theta}{dt^2} + 2\alpha \frac{d\theta}{dt} + (\omega_0{}^2 + d^2)\, \theta\,(t) = I(t) \tag{29}$$

where $I(t)$ is the impulsive random exciting force per unit mass. The correlation function $\phi(\tau)$ will then satisfy:

$$\phi''(\tau) + 2\alpha\phi'(\tau) + (\omega_0{}^2 + j^2)\, \phi(\tau) = 0 \tag{30}$$

and hence will generally consist of exponentially damped sinusoidal functions.

Returning to equation (26) for the discrete case, we know from the theory of difference equations , ref. (1), that the autocorrelation function $\phi(\ell)$ may be written

$$\phi(\ell) = \sum_{j=1}^{k} c_j r_j^{\ell} \tag{31}$$

where $r_1, \ldots r_k$ are the distinct roots of the polynomial equation

$$r^k - \sum_{j=1}^{k} a_j r^{k-j} = 0 \tag{32}$$

and the c_j' terms are to be determined from appropriate initial conditions. If equation (32)has repeated roots then equation (31) must be modified somewhat. For example, if the root r_1 were a double root, all other roots of (32) being simple, then (31) would be written:

$$\phi(\ell) = (c_1 + c_2 \ell) r_1^{\ell} + \sum_{j=2}^{k} c_j r_j^{\ell} \tag{33}$$

It is of some interest also to investigate the nature of the power spectral density function of a discrete time process x_n satisfying (23). If Δ denotes the difference operator such that $\Delta x_j = \Delta x_{j-1}$, then equation (23) may be written as:

$$Lx_n = e_n \tag{34}$$

where:

$$L = 1 - \sum_{j=1}^{k} a_j \Delta^j$$

Since L is a discrete time, constant coefficient linear filter, its frequency response function is given by

$$y(\omega) = \frac{1}{1 - \sum_{j=1}^{k} a_j e^{i\omega j}} \tag{35}$$

so that since the e_n are independent with variance σ_e^2 the power spectral density function of the x_n process is:

$$P(\omega) = \frac{1}{2\pi} \frac{\sigma^2}{|y(\omega)|^2} \quad - \pi \leq \omega \leq \pi \tag{36}$$

The angular frequency may be limited between $-\pi$ and π in (36) because we have tacitly assumed dimensionless time. If, however, the time between observations is Δ seconds, then the power spectral density function should be written

$$P(f) = \frac{1}{2\pi} \frac{\sigma^2/\Delta}{|y(2\pi f)|^2} \quad \text{for } -\frac{1}{2\Delta} \leq f \leq \frac{1}{2\Delta} \tag{37}$$

where f is frequency in cycles per second.

The significant thing about equations (37) and 31) is that both the correlation function and the power spectral density function of the x process are known for any time lag or any frequency once the constants $a_1, \ldots a_k$ are known, thus when we know that our process satisfies an

autoregressive scheme of a known order, the problem of estimating a functional of a random process such as its correlation function reduces to the much easier problem of estimating a finite number of parameters.

We shall conclude by deriving the equations for the estimators of $a_1, \ldots a_k$ together with an asympotic expression for the covariance matrix of the estimator errors. We assume that x_n satisfies (23) and that $x_1, \ldots x_N$ have been observed where $N \gg k$. Then since the e_n are gaussian and independent and if L denotes the logarithm of the joint density function of the data and if we neglect inessential constants, we may write:

$$L = \sum_{n=k+1}^{N} \left(x_n - \sum_{j=1}^{k} a_j x_{n-j}\right)^2 \tag{38}$$

The maximum likelihood estimates $\hat{a}_j$, $j = 1, \ldots k$ are those functions of the observed data which minimuze (38).

Thus differentiating (38) with respect to $a_1, a_2, \ldots a_k$ and equating the results to zero, we obtain the following system of equations for the estimators:

$$\sum_{i=1}^{k} \left(\sum_{n=k+1}^{N} x_{n-i} x_{n-j} \right) \hat{a}_i = \sum_{n=k+1}^{N} x_n x_{n-j} \tag{39}$$

$$\text{for} \quad j = 1, \ldots k.$$

The system of equations given by (39) may be conveniently written in vector and matrix notation as:

$$c\,\hat{a} = b$$

where

$$\hat{a} = \begin{pmatrix} \hat{a}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{a}_k \end{pmatrix}, \quad b = \begin{pmatrix} \sum_{n=k+1}^{N} x_n x_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sum_{n=k+1}^{N} x_n x_{n-k} \end{pmatrix} \tag{40}$$

and where c is the kxk square matrix whose i, j^{th} element is given by

$$c_{ij} = \sum_{n=k+1}^{N} x_{n-j} x_{n-i} \tag{41}.$$

With this notation, the expression for the estimator vector $\hat{a}$ may be written:

$$\hat{a} = c^{-1} b \tag{42}$$

Comparison of the equations (39) through (42) shows that the estimators of the parameters $a_1, \ldots a_k$, and hence the estimators of the correlation and power spectral density functions will be functions of the sample mean lagged products of the observations with lags hanging from 1 to k.

Now the estimator vector is a non-linear function of the observed data so that it is a rather difficult problem to calculate expressions for the variance and correlation between the estimator errors exactly. However, since the $\hat{a}_i$ are maximum likelihood estimators, it is relatively easy to compute an expression for the asymptotic covariance

matrix of the estimator errors because of a well-known theorem which states that under fairly general conditions , ref. 2, the covariance matrix of the minimum likelihood estimator errors is approximately given by the inverse of the matrix whose i, j^{th} term is

$- E\left(\frac{\partial^2 L}{\partial a_i \partial a_j}\right)$ where E is the expected value operator and L is the logarithm of the likelihood function. The approximation becomes exact in the limit as the number of observations tends toward infinity.

Since the logarithm of the likelihood function of our sample, including the constants, may be written:

$$L = \frac{N-1}{2} \log \frac{1}{2\pi\sigma_e^2} - \frac{1}{2\sigma_e^2} \sum_{n=k+1}^{N} \left(x_n - \sum_{j=1}^{k} a_j x_{n-j} \right)^2 \tag{43}$$

Computing $\frac{\partial^2 L}{\partial a_i \partial a_j}$ from (43) we obtain

$$\frac{\partial^2 L}{\partial a_i \partial a_j} = - \frac{1}{\sigma_e^2} \sum_{n=k+1}^{N} x_{n-i} x_{n-j} \tag{44}$$

Thus taking the expected value of (44) yields:

$$E\left(- \frac{\partial^2 L}{\partial a_i \partial a_j}\right) = \frac{1}{\sigma_e^2} \sum_{n=k+1}^{N} \phi(i-j) = \frac{N-k}{\sigma_e^2} \phi(i-j) \tag{45}$$

where ϕ is the autocorrelation function of the observed process. Since the asymptotic expression for the covariance matrix of the estimator errors is the inverse of the matrix whose i, j^{th} element is given in

equation (45) we finally have:

$$\mathrm{Cov}\,(\hat{a} - a) = \left[E\ (\hat{a} - a)\,(\hat{a} - a)^T\right] \approx \frac{\sigma_e^2}{N-k}\left[\phi\,(i-j)\right]^{-1} \tag{46}$$

Summary:

In this note we have attempted to show how certain kinds of a priori knowledge of stochastic processes can greatly reduce the problems associated with the estimation of certain functionals of the process such as the correlation function or spectral density function by reducing the problem to one of finite parameter estimation. In addition to the practical advantages afforded by this information, we also obtain a certain theoretical peace of mind since stationary random processes which satisfy linear difference or differential equations are ergodic, ref. (3), so that time averages are legitimate estimates of distributional averages of properties of the functions.

Furthermore, many of the remarks we have made are also true when the x_n process satisfies an equation such as (23) where the e_n are no longer assumed to be independent but merely samples from a stationary process with known statistics. In this case, the difference or differential equation satisfied by the correlation function will no longer be homogeneous, but aside from this analytical inconvenience, the same basic approach should still be fruitful.

REFERENCES

1. Goldberg, Introduction To Difference Equations, Chapter 3, Science Editions, Inc., New York, 1961.

2. Bartlett, An Introduction to Stochastic Processes, Chapter 8, Cambridge University Press.

3. Grenander and Rosenblatt, Statistical Analysis of Stationary Time Series, Chapter 1, Section 7, John Wiley and Sons, 1957.

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APOLLO NOTE NO. 205

H. Epstein
12 May 1964

THE IMPLICATION OF JPL PUBLISHED ACCURACY DATA ON ATOMIC STANDARDS

JPL has made available some experimental results[1/] concerning the errors present in some atomic standards evaluated for use with the DSIF. Results are available both for Atomichron and Rubidium standards. The Rubidium standard data is the data of interest for the Apollo mission. This data is of further importance in assessing the error contribution of the frequency and timing reference on presently available measurement made by the DSIF. Tables 1 and 2 below indicate the numerical values obtained by JPL.

Table 1.
Manufacturers' Specifications for Atomic Frequency Standards

Type	Long-term Stability	Short-term Stability	Spectral Purity
Atomichron	5×10^{-10}	5×10^{-10} (5 sec)	—
Rubidium Standard	5×10^{-11}	5×10^{-11} (1 sec)	2 cps at 24 kMc

1/ Space Program Summary No. 37-17, Volume III, dated 1 October, 1962.

Table 2.
Frequency Stability Measurements of
Atomic Frequency Standards

Measurement	Atomichron	V4700 Rb Standard	Prototype Rb Standard
~ 50 msec average from one period histogram	$\pm 4.7 \times 10^{-10}$	$\pm 5 \times 10^{-11}$	$\pm 3 \times 10^{-11}$
~ 500 msec average from ten period histogram	$\pm 4.3 \times 10^{-10}$	$\pm 2 \times 10^{-11}$	$\pm 1.4 \times 10^{-11}$
~ 5 sec average from hundred period histogram	$\pm 1.1 \times 10^{-10}$	$\pm 0.5 \times 10^{-11}$	$\pm 0.4 \times 10^{-11}$
~ 8 min average from Analog Spectrum Analyzer	$\pm 1.1 \times 10^{-10}$	$\pm 0.4 \times 10^{-11}$	$\pm 0.5 \times 10^{-11}$
~ 40 min average from Digital Spectrum Analyzer	$\pm 3.7 \times 10^{-10}$	$\pm 0.4 \times 10^{-11}$	$\pm 0.5 \times 10^{-11}$

Our experience here at The Bissett-Berman Corporation (after processing data from manufacturers of Rubidium standards) has indicated that for the running times (T) used that the standard deviation of the error in running time (σ_T) can be adequately represented by the following simple expression.

$$\sigma_T^2 = a + bT + cT^2$$

Frequency errors (Δf) can be related to time errors (ΔT) by the following expression.

$$\frac{\Delta f}{f} = \frac{\Delta T}{T}$$

The values indicated by JPL are values for $\frac{\Delta f}{f}$. These values indicate the total errors for both clocks. If equal errors can be associated with each clock, the variance for a single clock should be reduced by a factor of 1/2. At the present time, for frequency standards of this quality, it is only possible to refer to relative frequency or timing accuracy. The total σ_T divided by T is indicated in Figure 1. The quantity 2T is used in place of T for graphical convenience. A simple empirical curve which fits the data reasonably well is given below. (This function is the solid curve indicated in Figure 1).

$$\frac{\sigma_T}{T} = 10^{-11} \sqrt{\frac{1}{T} + .2}$$

This approximate expression will be used to calculate associated timing errors (σ_T), differential range errors ($\sigma_{\Delta R}$), and range-rate errors ($\sigma_{\dot{R}}$). Running time of interest at this time are 1, 10, 50, and 60 seconds. The average wavelength for the L-band DSIF is about 32.5 cm. $\sigma_{\Delta R}$ and $\sigma_{\dot{R}}$ are given for the two-way doppler mode. The values obtained can be reduced by a factor of $\frac{1}{\sqrt{2}}$ if the measurement errors are attributed equal to both clocks. Numerical values are indicated in the table below.

Table 3.
Effect of Clock Errors on L-band DSIF

T (second)	σ_T (seconds)	$\sigma_{\Delta R}$ (cm)	$\sigma_{\Delta \dot{R}}$ (cm/sec)
1	1.1×10^{-11}	.16	.16
10	5.5×10^{-11}	.82	.082
50	2.3×10^{-10}	3.5	.071
60	2.8×10^{-10}	4.2	.070

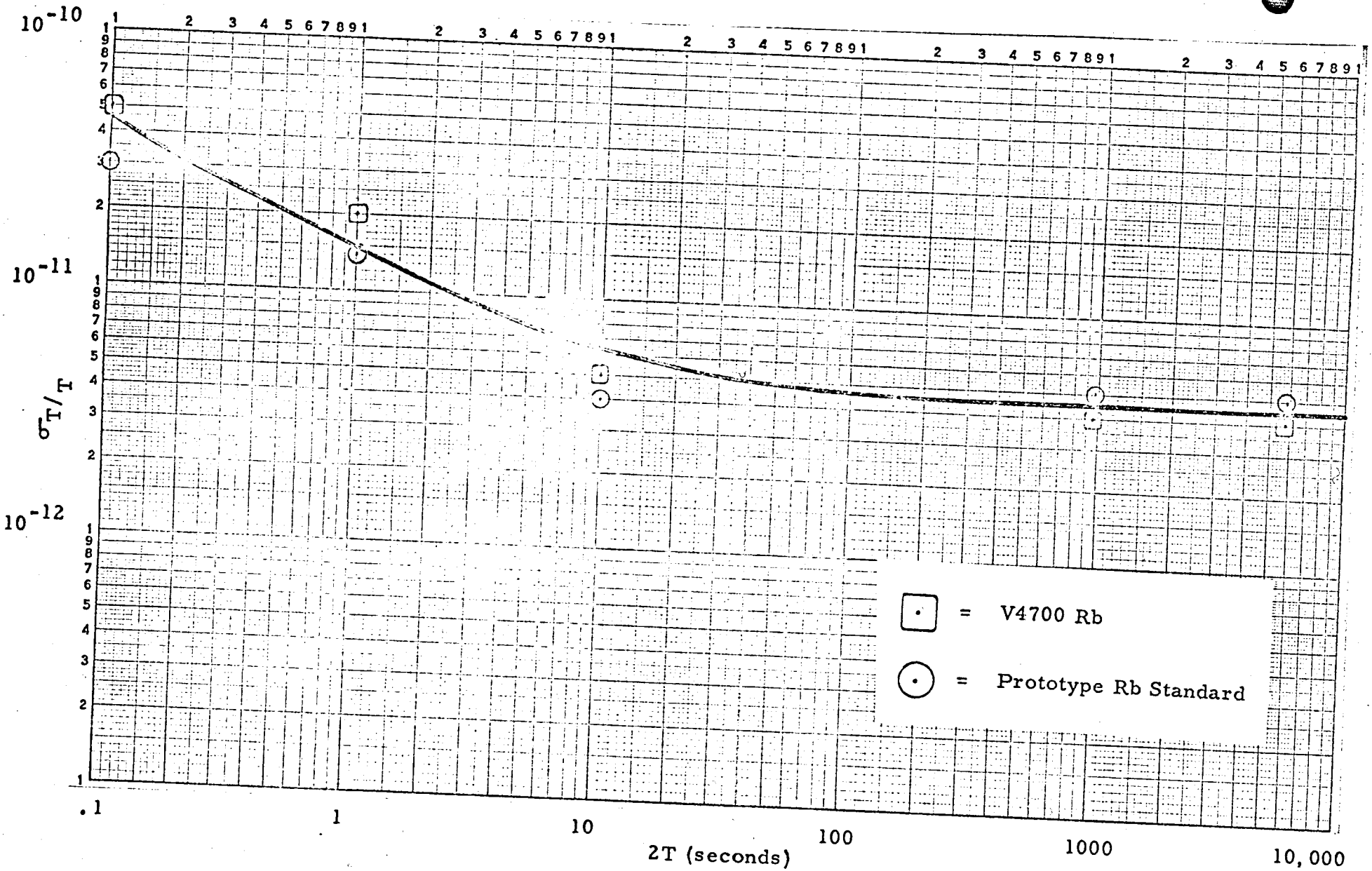

Figure 1. Frequency Accuracy Variation with Running Time.

Apollo Note No. 203 indicated that the differential range error due to random quantization noise was 6.64 cm for the L-band DSIF. Recalling that the tabular might well be reduced by a factor of about .7 when one clock only is used, it can not be noted that this type quantization error well dominates the clock error for running time less than 1 minute. Improved techniques to reduce quantization errors should be accompanied by improved techniques to reduce the effect of drift error in the present atomic standards. These results indicated that for long running times a low frequency drift rate is the dominant clock error source.

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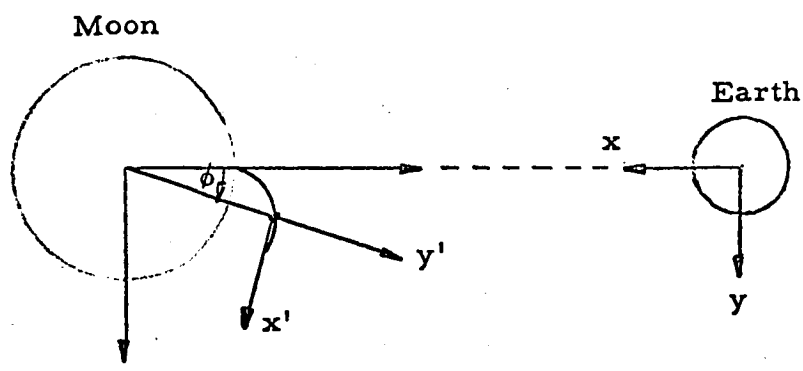
APOLLO NOTE NO. 206

H. Engel/
H. Dale
19 May 1964

SOME PRELIMINARY ESTIMATES OF THE UNCERTAINTY OF THE STATE VECTOR DURING LUNAR ASCENT USING THE SCHMIDT-KALMAN TECHNIQUE

The Schmidt-Kalman program is discussed in Apollo Notes 192 and 193. The basic computer program has been written and this note presents the results of some preliminary trial cases. These cases tend to be simplified such that the effects of various errors may be compared. Thus in one case, gyro drift errors were assumed to exist while accelerometers were assumed to be perfect; in other cases, the accelerometers were assumed to possess varying amounts of noise while the rate gyros were assumed to be perfect. Due to the rather long computing time required for each data interval (about 4 seconds of computing time for each one second interval of flight) only short portions of the 397 second actual trajectory were used. Later work may consist of combined errors and longer flight times.

The ϕ and M matrices of Apollo Notes 192 and 193 depend upon the trajectory. The trajectory used is based upon curves fitted to a NASA trajectory.



The NASA trajectory is (in ft/sec.^{2}).

$$\ddot{x}' = \begin{cases} 0 \ , & 0 \leq t \leq 8 \text{ sec.} \\ 8.55 + 0.027t \ , & 8 \leq t \leq 397 \text{ sec.} \end{cases}$$

$$\ddot{y}' = \begin{cases} 6 \ , & 0 \leq t \leq 8 \text{ sec.} \\ 1.64 - .0092t \ , & 8 \leq t \leq 325 \text{ sec.} \\ -1.26 \ , & 325 \leq t \leq 397 \text{ sec.} \end{cases}$$

$$\phi = \begin{cases} 0 \ , & 0 \leq t < 8 \text{ sec.} \\ 1.08 \times 10^{-6} (t-8)^2 \text{ rad/sec.}, & 8 \leq t \leq 397 \text{ sec.} \end{cases}$$

From this the thrust components of acceleration in the Earth-centered inertial frame may be written (expressed in meters/sec.^{2})

$$a_x = \ddot{x}' \phi - \ddot{y}' - g \quad , \text{ where } g = \text{lunar gravitation}$$

$$= \left[-11.32 \right] .3048 \quad \text{for } 0 \leq t < 8 \text{ sec.}$$

$$= \left[-6.96 + .00905\, t + 8.8 \times 10^{-6}\, t^2 + 2.9 \times 10^{-8}\, t^3 \right] .3048$$

$$\text{for } 8 \leq t \leq 325$$

$$= \left[-4.06 - 1.5 \times 10^{-4} t + 8.8 \times 10^{-6}\, t^2 + 2.9 \times 10^{-8} t^3 \right] .3048$$

$$\text{for } 325 \leq t \leq 397$$

$$a_y = \ddot{y}' \phi + \ddot{x}'$$

$$= 0 \qquad \text{for } 0 \leq t < 8$$

$$= \left[8.55 + .027t + 7.7 \times 10^{-6} t^2 - 10^{-8} t^3\right] .3048$$

$$\text{for } 8 \leq t < 325$$

$$= \left[8.55 + .027t + 4.39 \times 10^{-6} t^2\right] .3048$$

$$\text{for } 325 \leq t \leq 397$$

$$a_z = 0$$

From these thrust components, the ϕ matrix may be generated. This matrix is the same as shown in Apollo Note No. 193 with τ, the interval between samples, taken as one second. This matrix is shown on the next page.

The ϕ Matrix

1	0	0	1	0	0	0	0	$\frac{-a_y}{2}$	0	0	$\frac{-a_y}{6}$
0	1	0	0	1	0	0	0	$\frac{a_x}{2}$	0	0	$\frac{a_x}{6}$
0	0	1	0	0	1	$\frac{a_y}{2}$	$\frac{-a_x}{2}$	0	$\frac{a_y}{6}$	$\frac{-a_x}{6}$	0
0	0	0	1	0	0	0	0	$-a_y$	0	0	$\frac{-a_y}{2}$
0	0	0	0	1	0	0	0	a_x	0	0	$\frac{a_x}{2}$
0	0	0	0	0	1	a_y	$-a_x$	0	$\frac{a_y}{2}$	$\frac{-a_x}{2}$	0
0	0	0	0	0	0	1	0	0	1	0	0
0	0	0	0	0	0	0	1	0	0	1	0
0	0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1

The velocity in the inertial frame may be generated by integrating the total acceleration.

$$\dot{x} = \int_0^t (\ddot{x}'\phi - \ddot{y}')\,dt$$

$$= -6t \qquad \text{for } 0 \leq t < 8 \text{ sec.}$$

$$= \left[-35.1 - 1.64t + .00452t^2 + 2.9 \times 10^{-6}t^3 - .72 \times 10^{-8}t^4\right] .3048$$

$$\text{for } 8 \leq t < 325$$

$$= \left[-494.3 + 1.26t - .73 \times 10^{-4}t^2 + 2.96 \times 10^{-6}t^3 + .73 \times 10^{-8}t^4\right] .3048$$

$$\text{for } 325 \leq t \leq 397$$

$$\dot{y} = \int_0^t (\ddot{y}'\phi + \ddot{x}')\,dt = 0 \qquad \text{for } 0 \leq t < 8 \text{ sec.}$$

$$= \left[-67.5 + 8.55t + .0135t^2 + .62 \times 10^{-6}t^3 - .25 \times 10^{-8}t^4\right] .3048$$

$$\text{for } 8 \leq t < 325$$

$$= \left[-58.5 + 8.55t + .0135t^2 - .454 \times 10^{-6}t^3\right] .3048$$

$$\text{for } 325 \leq t \leq 397$$

$$\dot{z} = 0 \qquad \text{for all } t$$

Integrating the velocity components gives position.

$$x = \int_0^t \dot{x}\,dt + x_o$$

$$= .38 \times 10^9 - 3t^2 \qquad \text{for } 0 \leq t < 8 \text{ sec.}$$

$$= .38 \times 10^9 + .3048 \left[140.4 - 35.1t - .82t^2 + .0015t^3 + .72 \times 10^{-6}t^4 + .144 \times 10^{-8}t^5\right]$$

$$\text{for } 8 \leq t < 325$$

$$x = .38 \times 10^9 + .3048 \Big[48,344.1 - 494.3t + 63t^2 - 24.3 \times 10^{-6} t^3 + 73.2 \times 10^{-8} t^4 + 14.6 \times 10^{-10} t^5 \Big]$$

for $325 \leq t \leq 397$

$$y = \int_0^t \dot{y}\, dt = 0 \qquad \text{for } 0 \leq t < 8 \text{ sec.}$$

$$= \Big[264.1 - 67.5t + 4.275t^2 + 45. \times 10^{-4} t^3 + .155 \times 10^{-6} t^4 - .0313 \times 10^{-8} t^5 \Big] .3048 \qquad \text{for } 8 \leq t < 325$$

$$= \Big[-4271.3 - 58.5t + 4.275t^2 + 4.5 \times 10^{-3} t^3 - 11.35 \times 10^{-8} t^4 \Big] .3048 \qquad \text{for } 325 \leq t \leq 397$$

$$z = 0 \qquad \text{for all } t$$

The position of the three radars are assumed to be on the vertices of an equilateral triangle:

$$x_1, y_1, z_1 = 0,\ 2.4 \times 10^6 \text{ meters},\ 0$$

$$x_2, y_2, z_2 = 0,\ -2.1 \times 10^6,\ 1.2 \times 10^6$$

$$x_3, y_3, z_3 = 0,\ -2.1 \times 10^6,\ -1.2 \times 10^6$$

Now the range from each station to the vehicle is

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 + z_i^2, \qquad i = 1, 2, 3$$

which allows the measurable, range-rate, to be expressed.

$$\dot{r}_i = \frac{(x - x_i)\,\dot{x} + (y - y_i)\,\dot{y}}{r_i^2} \qquad , \; i = 1, 2, 3$$

Now the components of the M matrix are: (showing only non-zero terms of a 3 x 12 matrix)

$$M_{11} = \frac{\dot{x}}{r_1} - \frac{\dot{r}_1\,(x - x_1)}{r_1^2} \qquad M_{24} = \frac{x - x_2}{r_2}$$

$$M_{12} = \frac{\dot{y}}{r_1} - \frac{\dot{r}_1\,(y - y_1)}{r_1^2} \qquad M_{25} = \frac{y - y_2}{r_2}$$

$$M_{13} = \frac{\dot{r}_1\,z_1}{r_1^2} \qquad M_{26} = \frac{-z_2}{r_2}$$

$$M_{14} = \frac{x - x_1}{r_1} \qquad M_{31} = \frac{\dot{x}}{r_3} - \frac{\dot{r}_3\,(x - x_3)}{r_3^2}$$

$$M_{15} = \frac{y - y_1}{r_1} \qquad M_{32} = \frac{\dot{y}}{r_3} - \frac{\dot{r}_3\,(y - y_3)}{r_3^2}$$

$$M_{16} = \frac{-z_1}{r_1} \qquad M_{33} = \frac{\dot{r}_3\,z_3}{r_3^2}$$

$$M_{21} = \frac{\dot{x}}{r_2} - \frac{\dot{r}_2\,(x - x_2)}{r_2^2} \qquad M_{34} = \frac{x - x_3}{r_3}$$

$$M_{22} = \frac{\dot{y}}{r_2} - \frac{\dot{r}_2\,(y - y_2)}{r_2^2} \qquad M_{35} = \frac{y - y_3}{r_3}$$

$$M_{23} = \frac{\dot{r}_2\,z_2}{r_2^2} \qquad M_{36} = \frac{-z_3}{r_3}$$

The Q matrix used in this program assumes a one sigma error in the measurable, range-rate, of 3 cm/sec. Thus the only non-zero elements of the 3 x 3 matrix are:

$$Q_{11} = Q_{22} = Q_{33} = (.03)^2 = 9 \times 10^{-4} \text{ m}^2/\text{sec}^2$$

The P matrix describes the a priori uncertainty in the estimators involved in the problem. The initial position is assumed to be known with an uncertainty of 10 meters:

$$P_{11} = P_{22} = P_{33} = 10^2 \text{ meters}^2$$

The initial velocity of the vehicle is in fact zero and this is known. Thus, all other terms of the P matrix up to and including P_{66} are zero. The one sigma error in the position of the platform is 8 mils in each axis:

$$P_{77} = P_{88} = P_{99} = 64 \times 10^{-6} \text{ rad.}^2$$

and the one sigma error in the rate gyros is known to 1 mil/sec. a priori, corresponding to use of the autopilot as an attitude reference:

$$P_{10,10} = P_{11,11} = P_{12,12} = 10^{-6} \text{ rad}^2/\text{sec}^2.$$

All other terms are zero.

In the first case to be studied, Data Set 1, no accelerometer errors were assumed to exist, thus the ψ and R matrices were not used.

In the next four cases, accelerometer errors were studied under the assumption of no gyro drift rate. Thus for these cases the estimator matrix is reduced to a 6 x 6 since no ω or θ errors would

exist. As a further consequence of this, the ϕ matrix is reduced to a 6 x 6, the M matrix is reduced to a 3 x 6 and the P matrix is reduced to a 6 x 6. However for these cases of accelerometer random error the covariance matrix of these errors, R, must be used. Only diagonal terms of R are non-zero and express the variances in the three orthogonal accelerometers.

Case	Accelerometer Error	$R_{11} = R_{22} = R_{33} =$
P 2	2 meters/sec^2	4
P 3	1	1
P 4	.5	.25
P 5	.2	.04

Case P1 was run for 150 seconds while P2 through P5 were run for only 15 seconds.

Results

For the case of gyro drift without accelerometer error, P1, the Figure 1 shows that the x and y error remain constant at ten meters while the out-of-plane component grows to 70 meters by 150 seconds. It would appear from the shape of the curve that the z component might grow to about 140 meters by burnout. In any case, no real problem appears to exist in position uncertainty.

Figures 2 and 3 show the uncertainty in velocity components for case P1. Figure 3 is probably the most meaningful curve of the bunch, showing that the velocity error, which is almost all in the out-of-plane component, tends to stabilize after about a minute to a value near a meter per second. This number should scale with the assumed error in measuring range-rate, 3 cm/sec for this case.

Figures 4 through 7 show the uncertainty in angle and angular rate of the platform for the P1 case. It would appear that although the drift rate uncertainty may be held to within one mil per second, the

uncertainty of vehicle attitude tends to grow with time and were it not for the crew or other attitude acquiring device, future boosts could not be made on the basis of MSFN obtained attitude data in a straightforward manner.

Figures 8 through 13 show the cases P2, P4, and P5 for which accelerometer errors are assumed to exist without gyro drift or initial platform angular error. A mistake was made in the data inputs for P3 and thus this case is missing. What is interesting is the fact that the major velocity error will again be in the out-of-plane direction. The $\dot{x}$ error will never get larger than $\sqrt{3}$ times the range-rate error or .017 meters per second. For the case of 2 meters/sec.^{2} random accelerometer error, P2, the maximum velocity error $\left(\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right)$ of about 3.8 meter per second is reached by 10 seconds. For the case of 0.5 meter/sec.^{2} random errors in the accelerometers, P4, the maximum velocity error appears to be near 2.1 meter/sec. occurring after about 18-20 seconds. Although not enough time was studied, it would appear that by reducing the accelerometer errors to 0.2 meters/sec^2 the affect of accelerometer errors would approximate the affect of the gyro drift errors studied, i.e., the total velocity uncertainty might be held to about a meter per second.

Future runs might include combined gyro and accelerometer errors for flight times longer than 15 seconds.

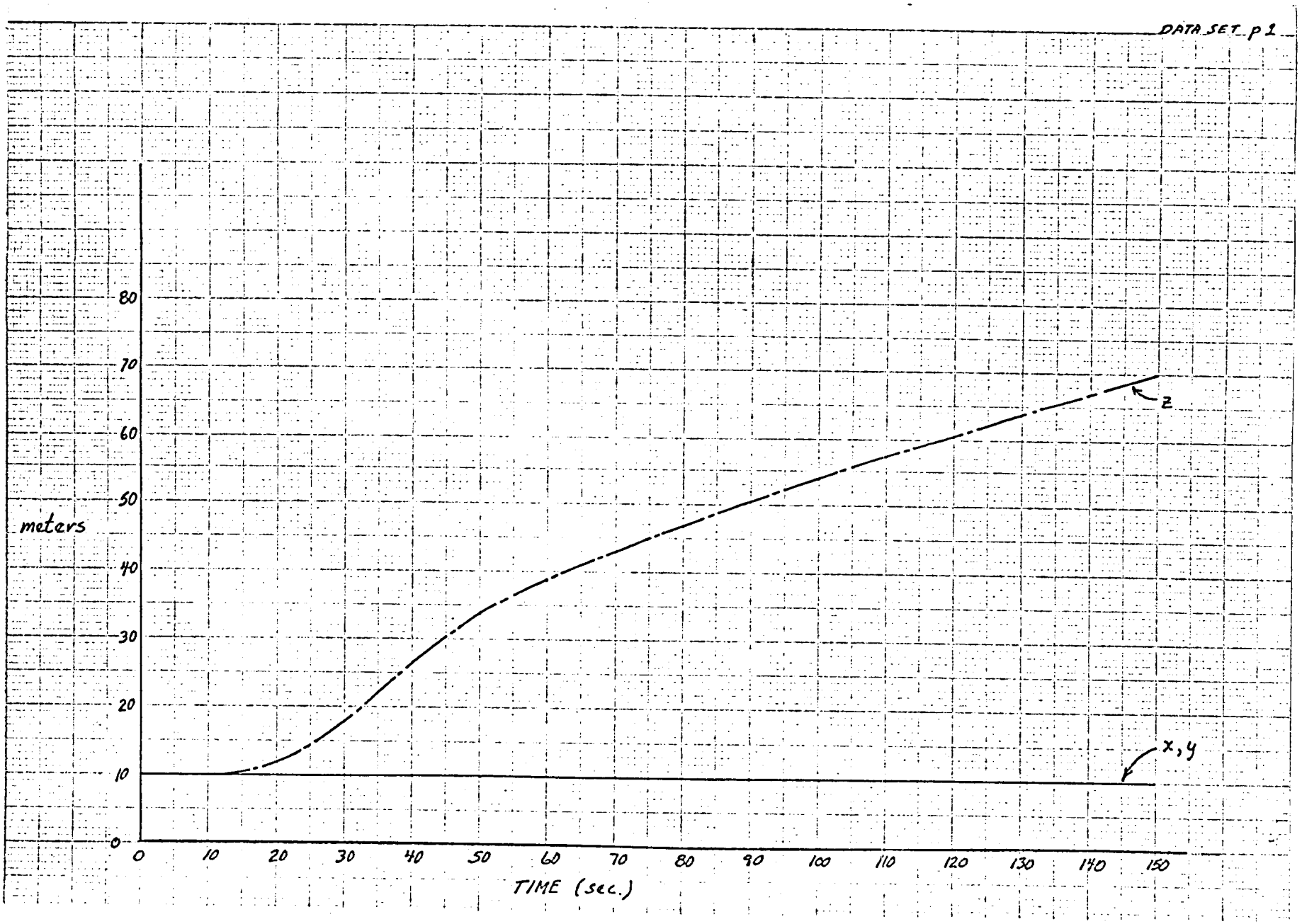


Figure 1.

Expected Uncertainty in Position for the Case of Gyro Drift without Accelerometer Error

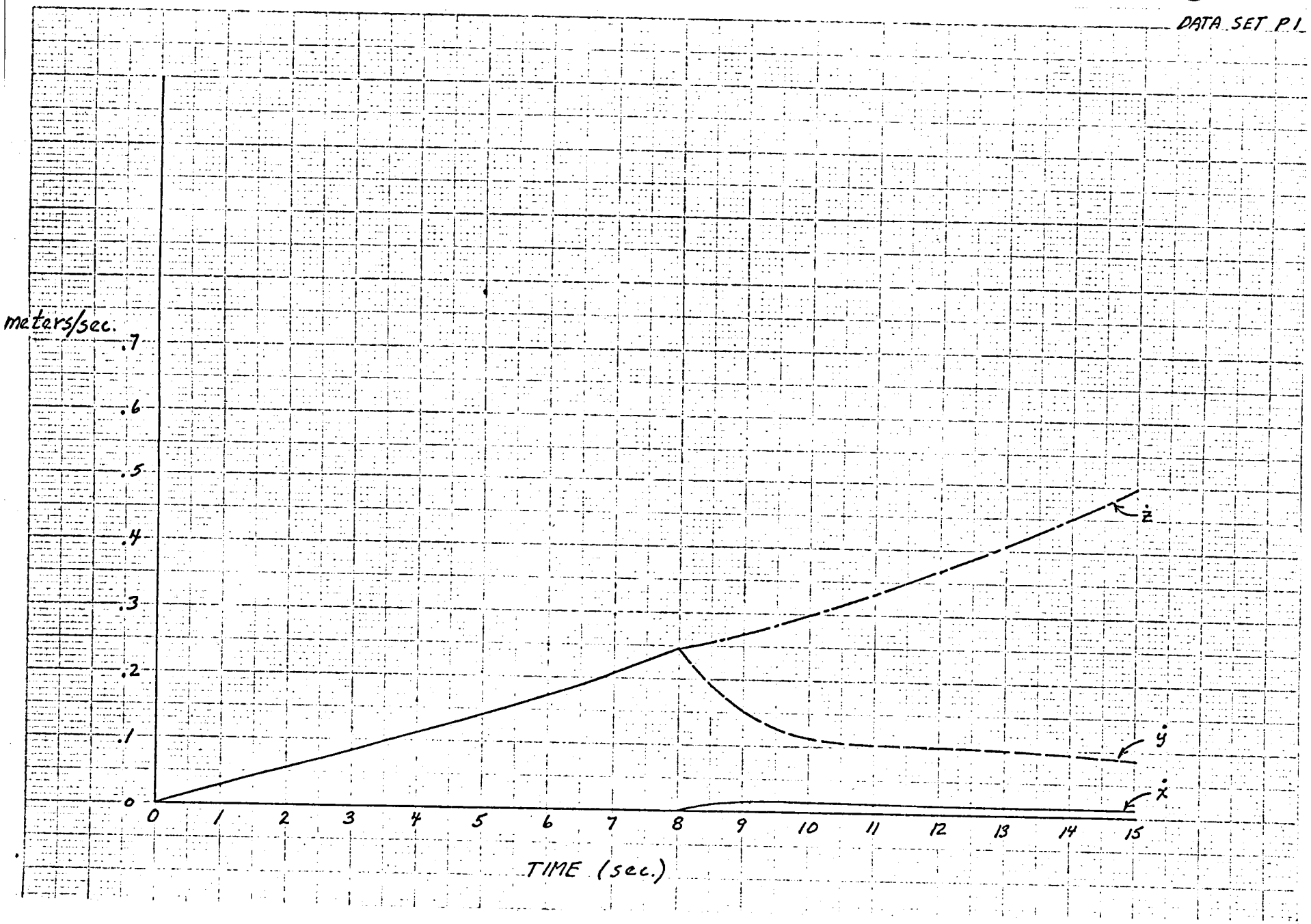


Figure 2.

Expected Uncertainty in Velocity
Gyro Drift without Accelerometer Error

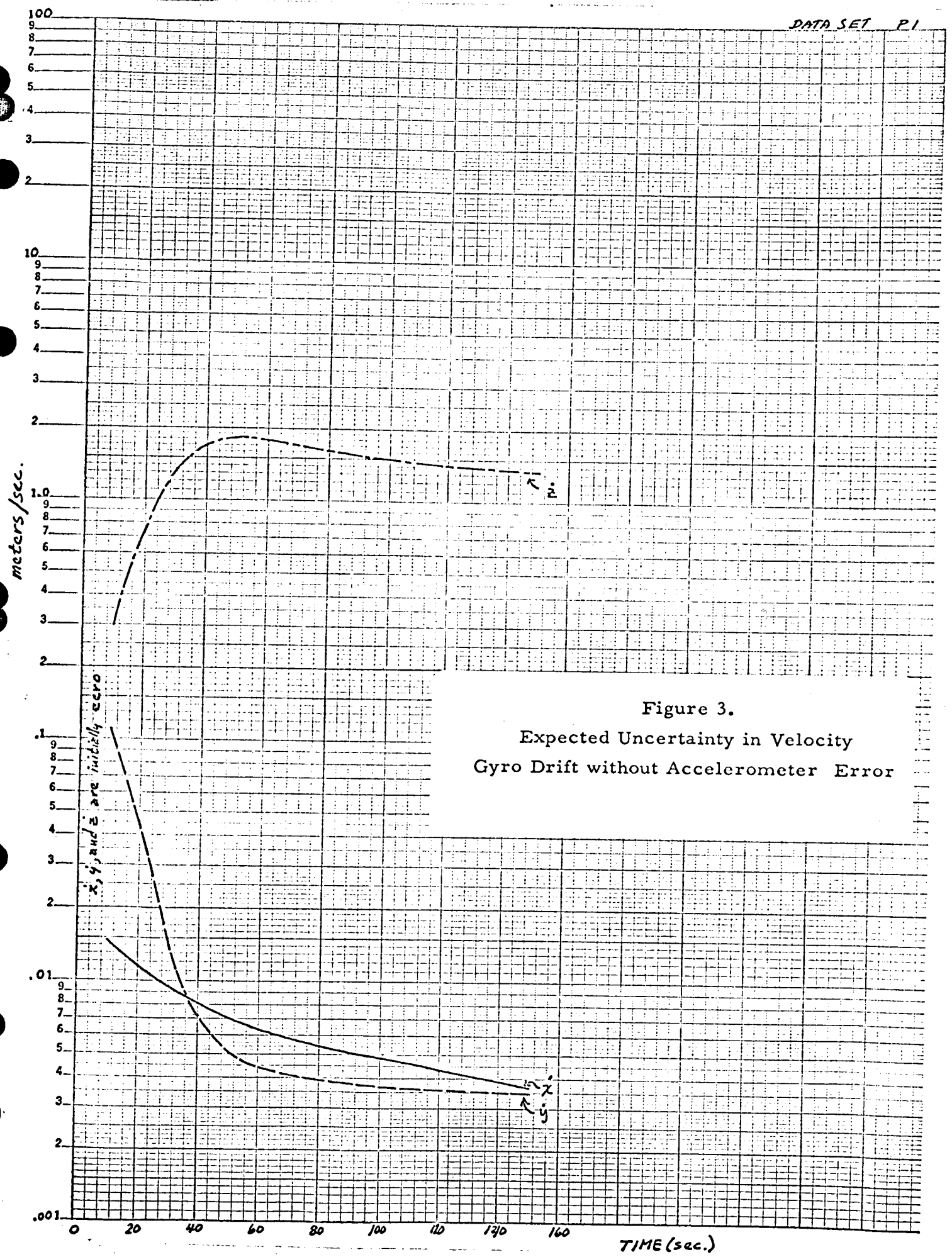


Figure 3.
Expected Uncertainty in Velocity
Gyro Drift without Accelerometer Error

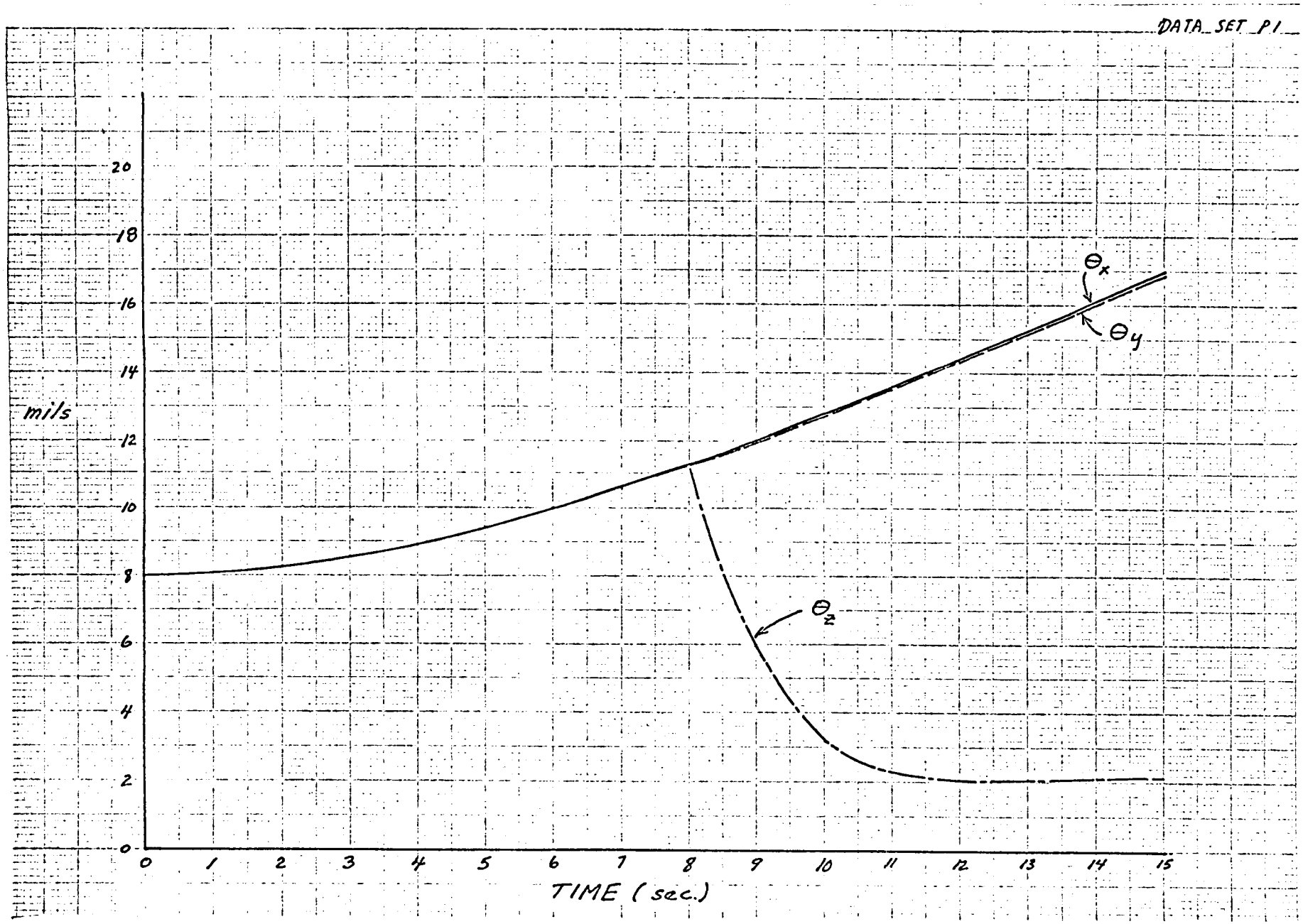


Figure 4.

Expected Uncertainty in the Platform Angle
Gyro Drift without Accelerometer Error

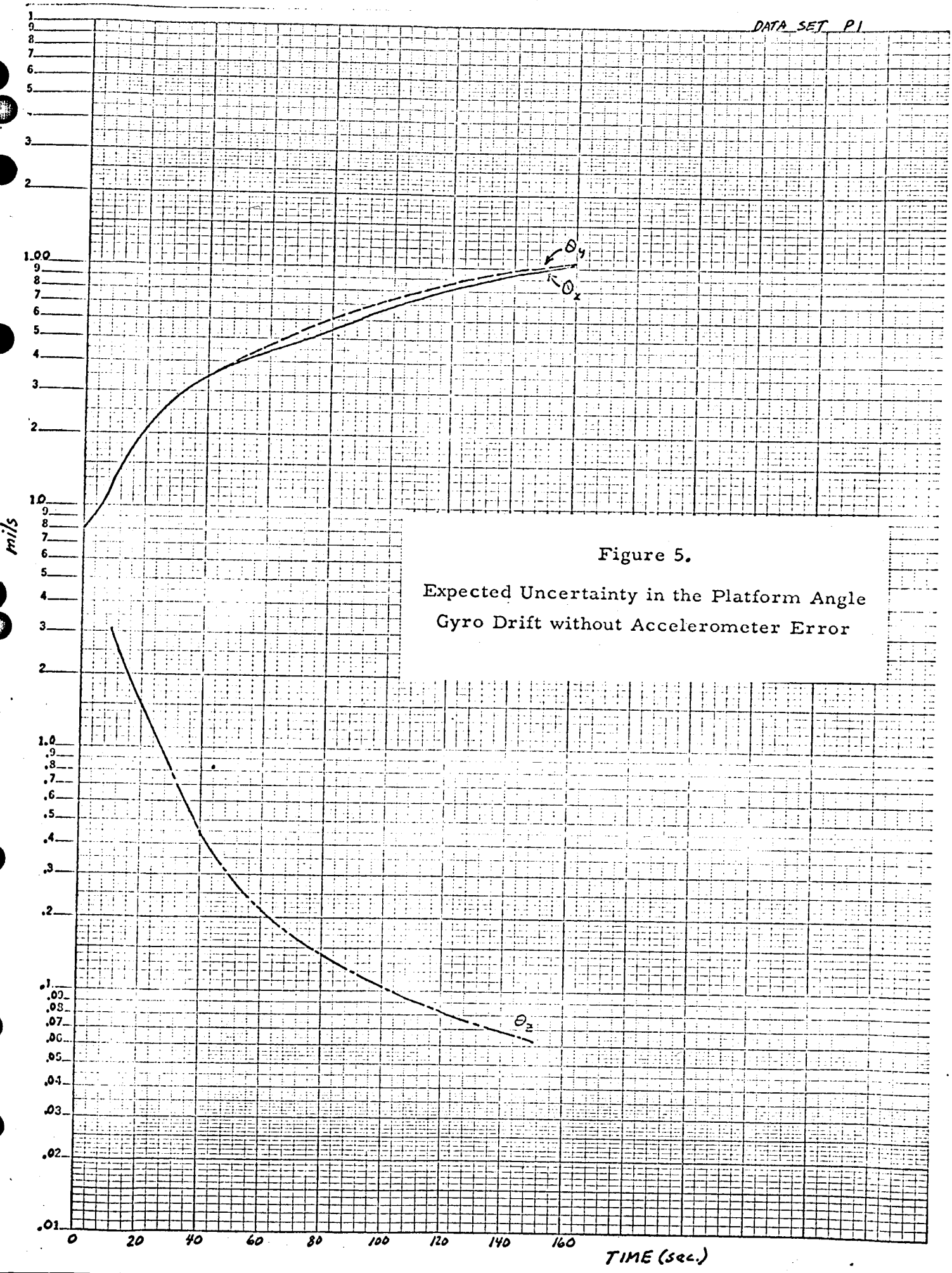


Figure 5.

Expected Uncertainty in the Platform Angle Gyro Drift without Accelerometer Error

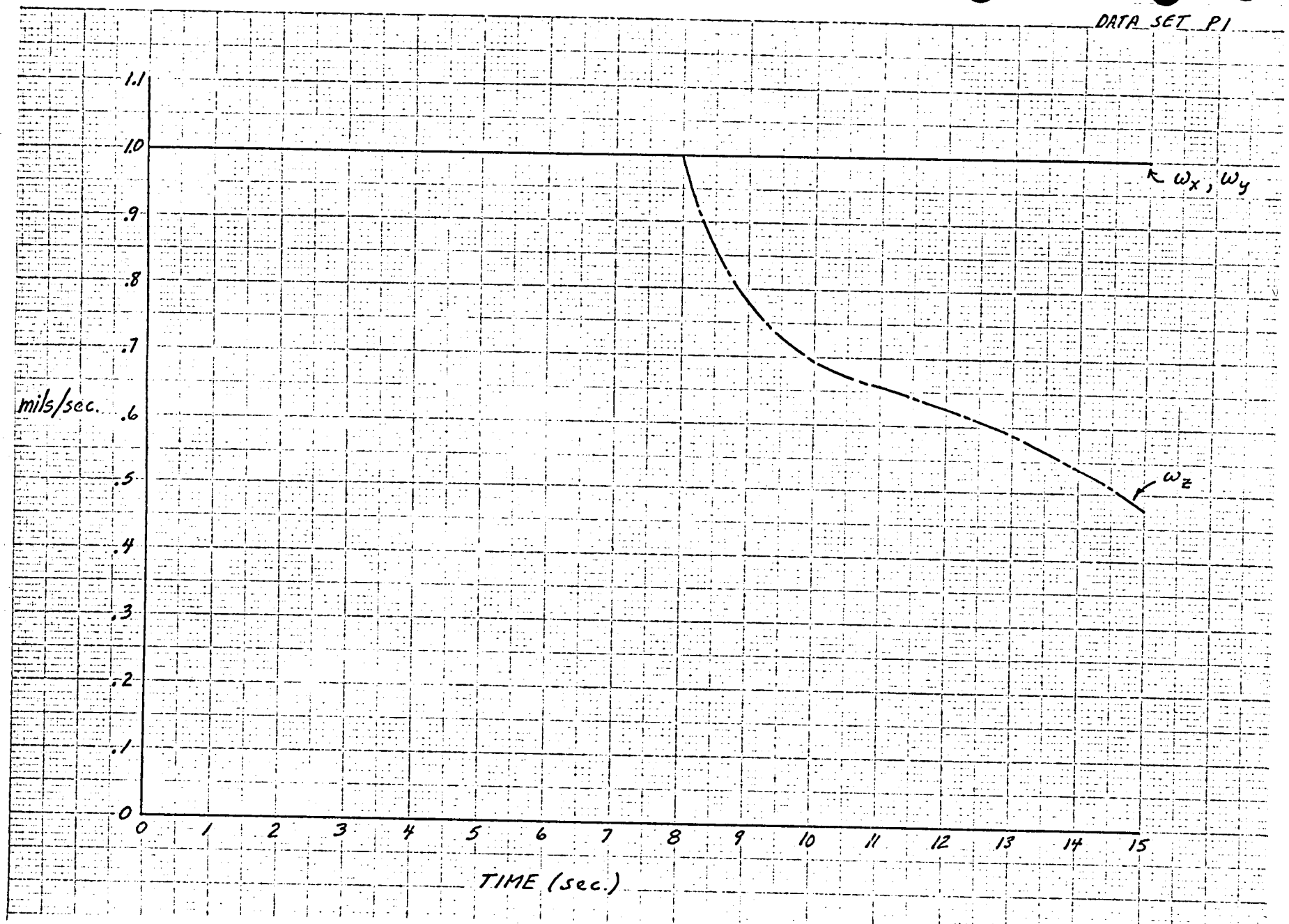


Figure 6.

Expected Uncertainty in the Gyro Drift Rate

Gyro Drift without Accelerometer Error

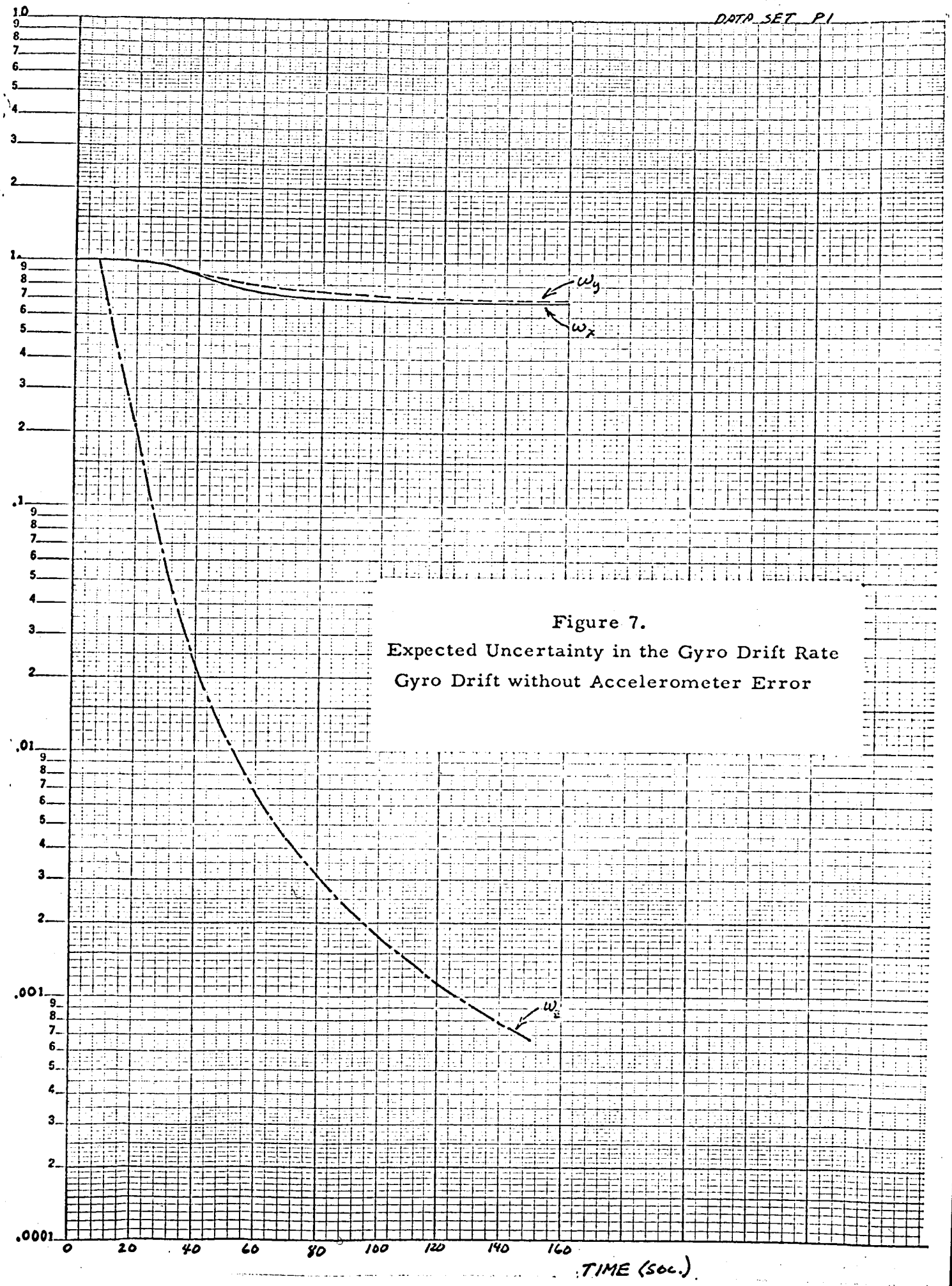


Figure 7.
Expected Uncertainty in the Gyro Drift Rate
Gyro Drift without Accelerometer Error

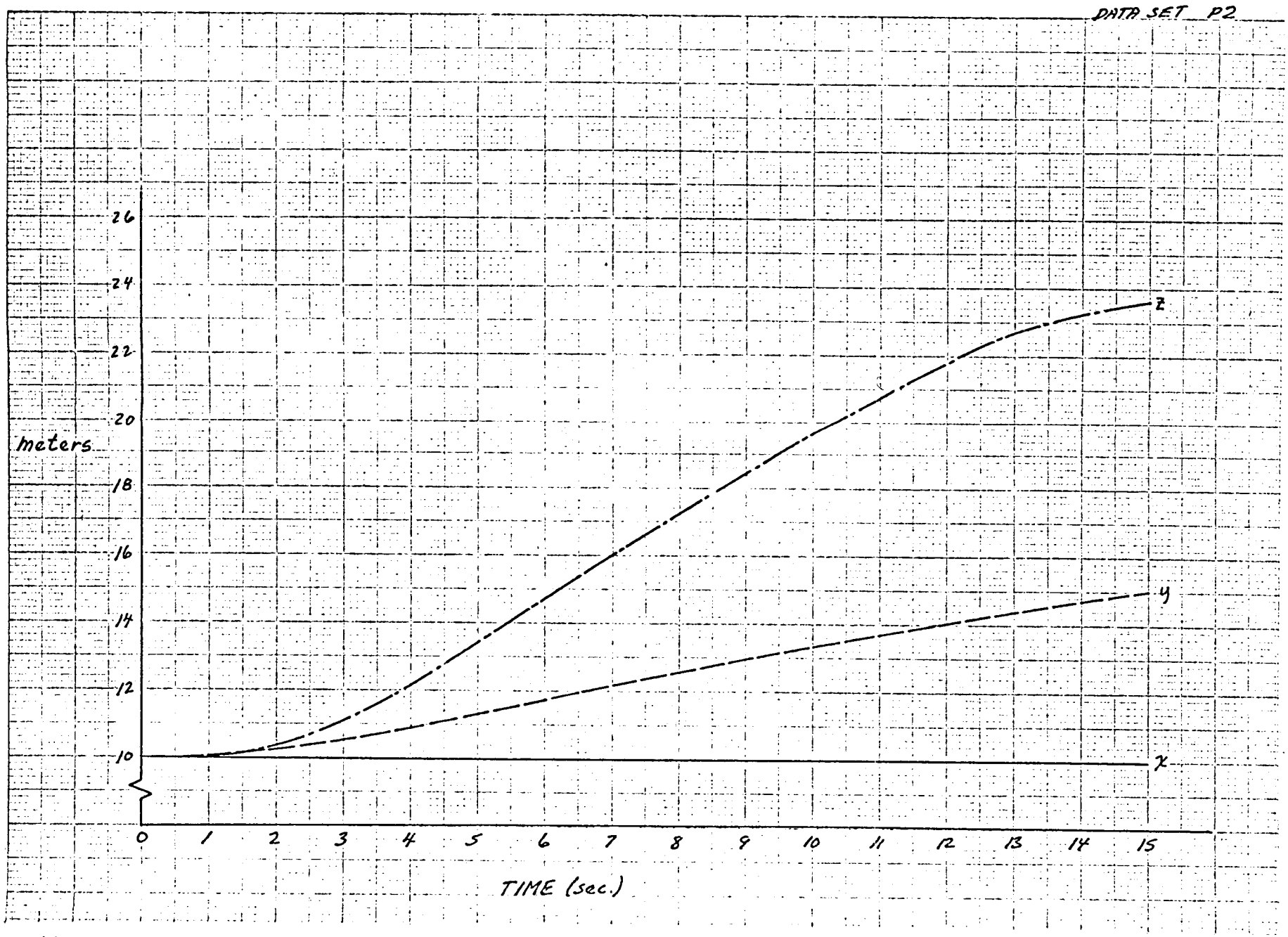


Figure 8.
Expected Uncertainty in Position -
Accelerometer Errors of 2 m/sec.^{2} with Perfect Gyros

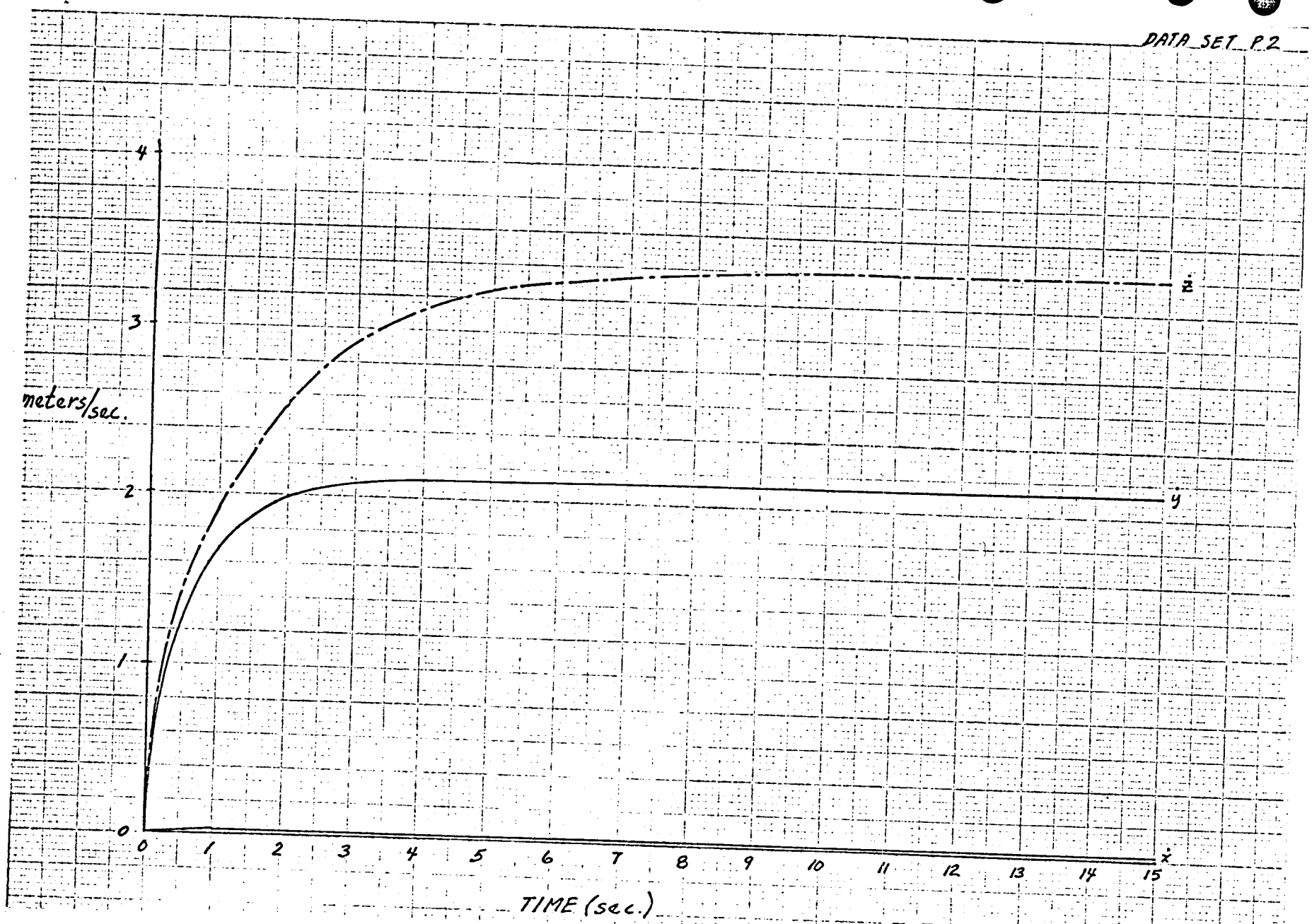


Figure 9.
Expected Uncertainty in Velocity -
Accelerometer Errors of 2 m/sec.^{2} with Perfect Gyros

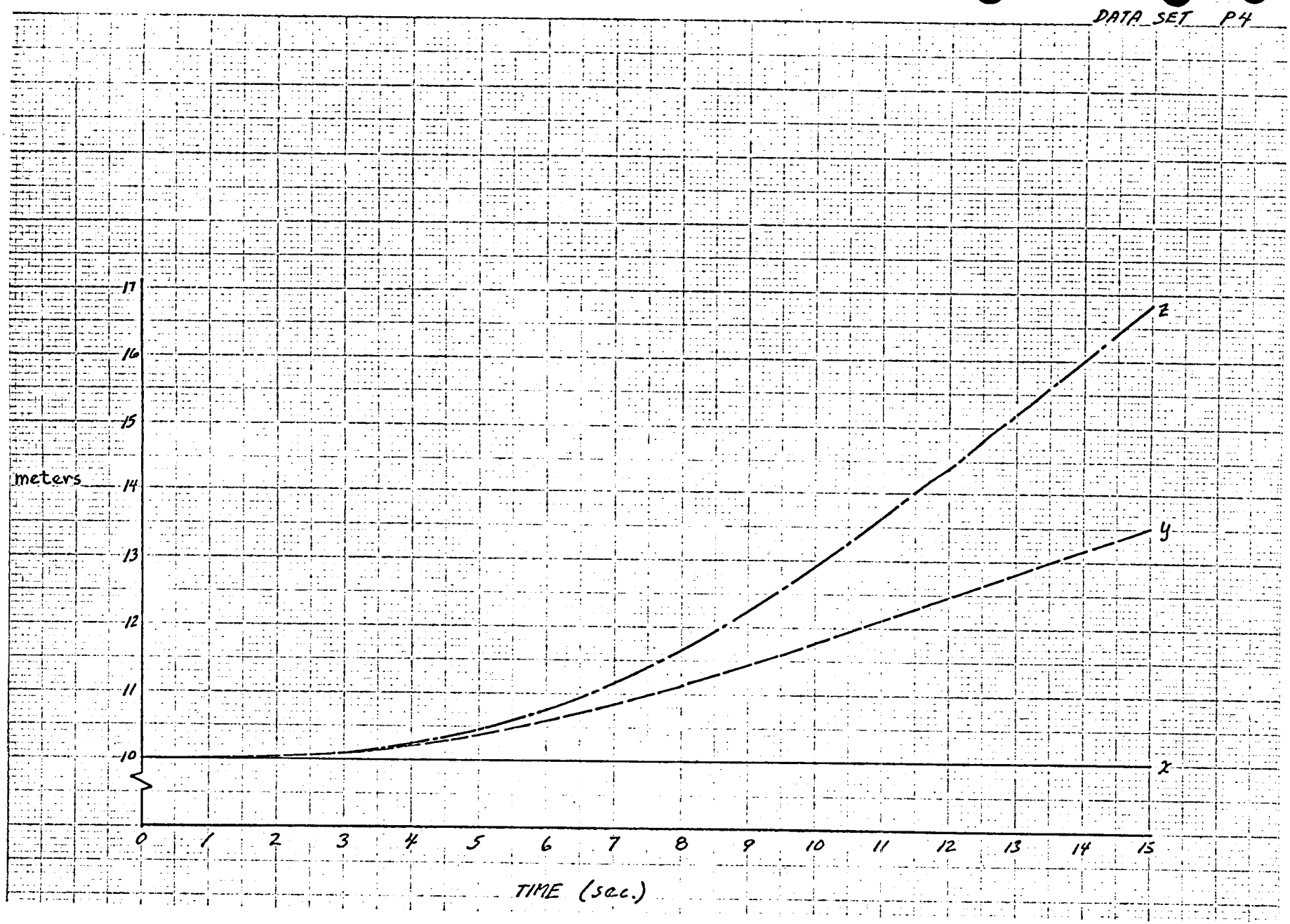


Figure 10.
Expected Uncertainty in Position -
Accelerometer Errors of 0.5 m/sec.^{2} with Perfect Gyros

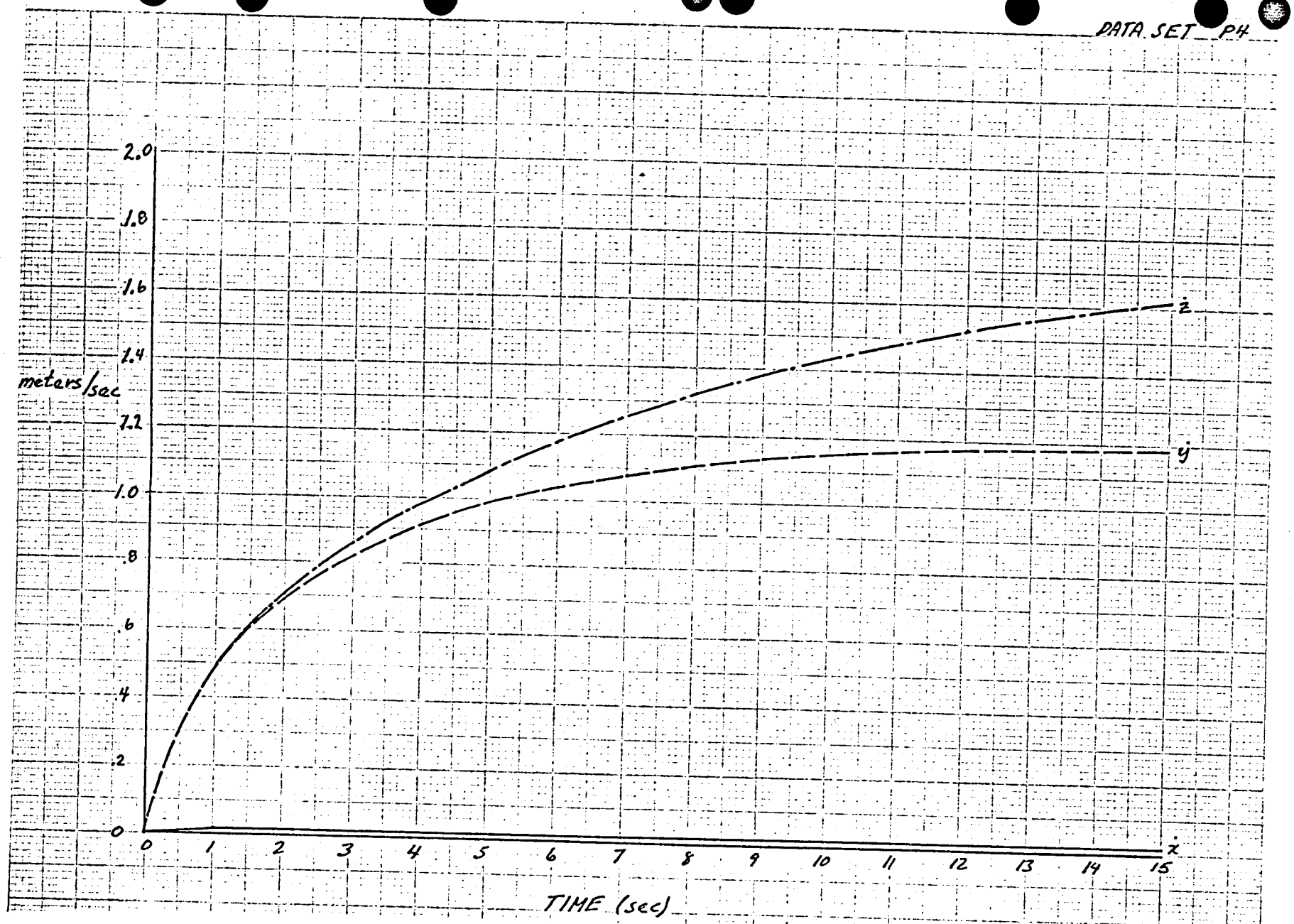


Figure 11.
Expected Uncertainty in Velocity -
Accelerometer Errors of 0.5 m/sec.^{2} with Perfect Gyros

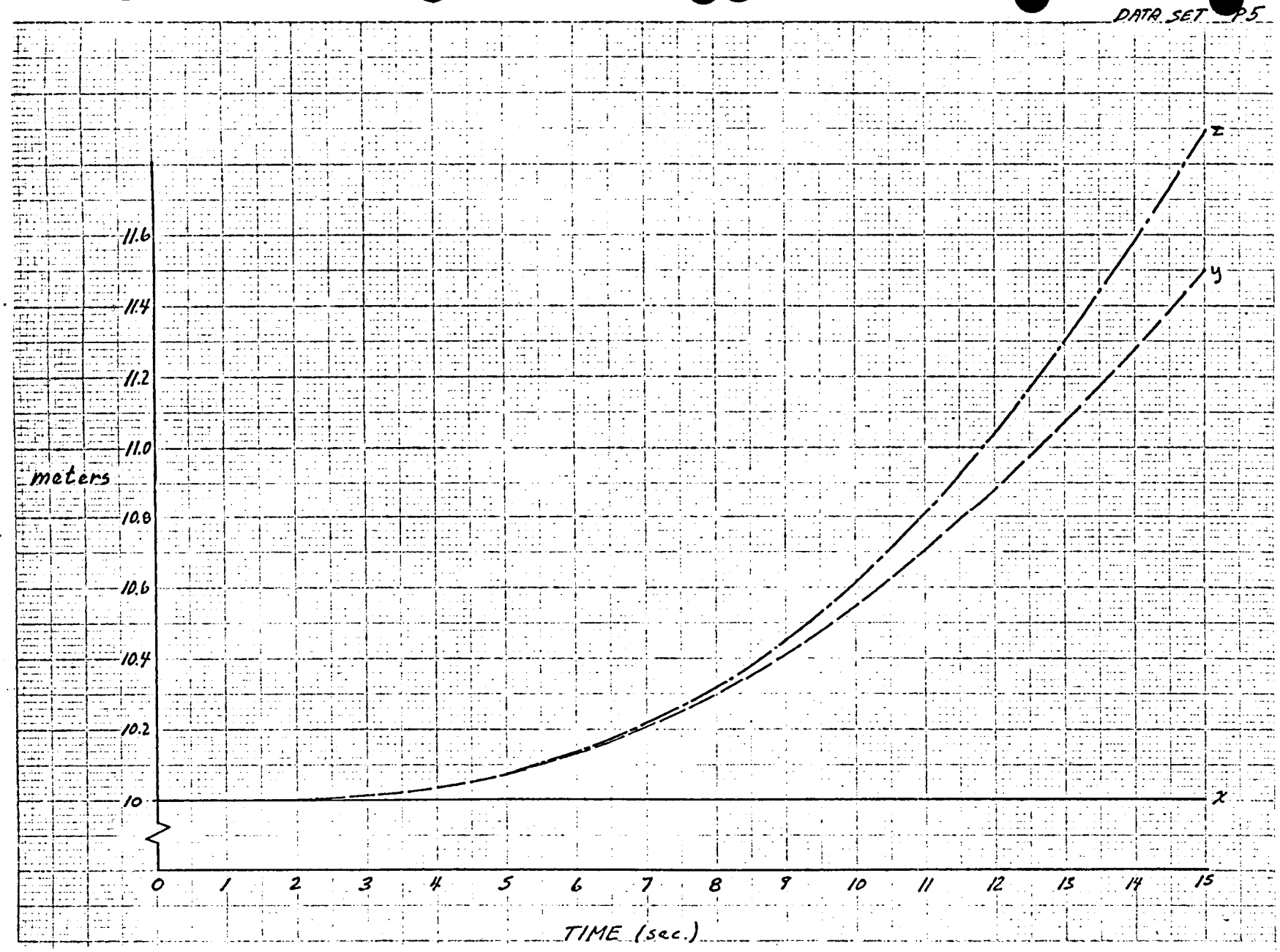


Figure 12.
Expected Uncertainty in Position -
Accelerometer Errors of 0.2 m/sec.^{2} with Perfect Gyros

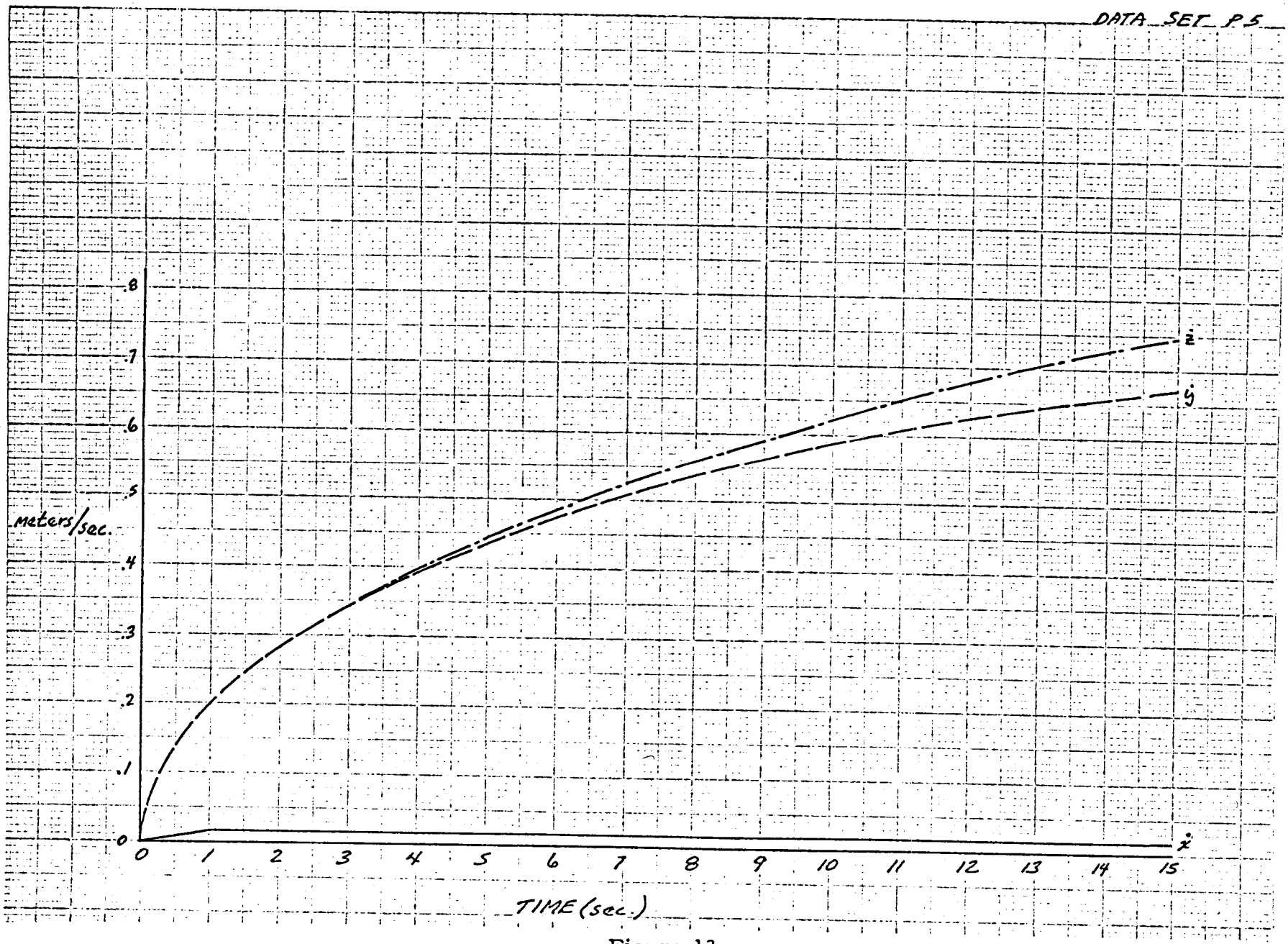


Figure 13.
Expected Uncertainty in Velocity -
Accelerometer Errors of 0.2 m/sec.^{2} with Perfect Gyros

The Bissett-Berman Corporation 2941 Nebraska Avenue, Santa Monica, California EXbrook 4-3270

APOLLO NOTE NO. 207

L. S. Lustick
18 May 1964

EXTENSION OF SCHMIDT-KALMAN METHOD FOR AUTO-CORRELATED DATA

Purpose

The purpose of this note is to indicate how the Schmidt-Kalman method can be extended to include cases when the measurable is auto-correlated.

Method

Consider a random sequence of numbers generated by the auto-regressive model shown in Equation (1).

$$e_i = \sum_{j=1}^{q} \lambda_j e_{i-j} + \mathcal{E}_i \tag{1}$$

where:

$\mathcal{E}_i$ is a zero mean gaussian variable.

$$\overline{\mathcal{E}_i \mathcal{E}_{i+m}} = 0 \qquad m \neq 0$$

$$= \sigma_{\mathcal{E}}^2 \qquad m = 0$$

The noise for this process is said to have order of dependence q (see Apollo Note No. 202), and with proper choice of λ_j can represent infinite memory random processes. The Schmidt-Kalman method will be illustrated using an order of dependence equal to one.

The correlation function for an order of dependence one (q=1) is exponential as shown in Equation (2).

$$\rho_n = \lambda \rho_{n-1} \tag{2}$$

Let:

$\hat{a}_o$ = a priori estimate of state vector

$a_1 = \phi_{1,o} \hat{a}_o$ = estimate of state vector at time, t_1 based on a priori state vector

$a_2 = \phi_{2,o} \hat{a}_o$ = estimate of state vector at time, t_2, based on a priori state vector

$\delta a_1 = \phi_{1,o} \delta \hat{a}_o$ = errors in state vector at time, t_1, based on a priori state vector

$\delta a_2 = \phi_{2,o} \delta \hat{a}_o$ = errors in state vector at time, t_2, based on a priori state vector

y_1 = predicted measurable at time, t_2 - λ predicted measurable at time, t_1

$-$ [measured value at time, t_2 $-$ λ measured value at time, t_1]

$$y_1 = \left[M_2 \phi_{2,o} \hat{a}_o - \lambda M_1 \phi_{1,o} \hat{a}_o \right] - \left[m_2 - \lambda m_1 \right]$$

$$\delta y_1 = \left[M_2 \phi_{2,o} - \lambda M_1 \phi_{1,o} \right] \delta \hat{a}_o - \xi_{1,2}$$

where:

M = matrix that transforms state variables into measurable

m = values of measurable

$$A_{11} = \overline{\delta a_1 \, \delta a_1^T} = \phi_{1,o} \, \mathrm{cov}\,(\hat{a}_o) \, \phi_{1,o}^T$$

$$A_{12} = \overline{\delta a_1 \, \delta y_1^T} = \phi_{1,o} \, \delta \hat{a}_o \left[(M_2 \, \phi_{2,o} - \lambda M_1 \, \phi_{1,o}) \, \delta a_o \right]^T$$

$$A_{12} = \phi_{1,o} \, \mathrm{cov}\,(\hat{a}_o) \left[M_2 \, \phi_{2,o} - \lambda M_1 \, \phi_{1,o} \right]^T$$

$$A_{22} = \overline{\delta y_1 \, \delta y_1^T} = \left[M_2 \, \phi_{2,o} - \lambda M_1 \, \phi_{1,o} \right]$$

$$\mathrm{cov}\,(\hat{a}_o) \left[M_2 \, \phi_{2,o} - \lambda M_1 \, \phi_{1,o} \right]^T + \overline{\varepsilon_{1,2} \, \varepsilon_{1,2}^T}$$

$\hat{a}_1$ = estimate of state vector at time t_1, subsequent to the measurements at time t_2 and t_1

$$\left. \hat{a}_1 = \phi_{1,o} \, \hat{a}_o - A_{12} \, A_{22}^{-1} \, y_1 \right\}$$ Fundamental Theorem of Schmidt-Kalman Method

$$\overline{\delta \hat{a}_1 \, \delta \hat{a}_1^T} = A_{11} - A_{12} \, A_{22}^{-1} \, A_{21}$$

<u>Recursively</u>

$a_2 = \phi_{2,1} \, \hat{a}_1$ predicted state vector at time, t_2, based on updated orbit parameters at time, t_1

$a_3 = \phi_{3,1} \, \hat{a}_1$ predicted state vector at time, t_3, based on updated orbit parameters at time, t_1

$$\delta a_2 = \phi_{2,1} \, \delta \hat{a}_1$$

$$\delta a_3 = \phi_{3,1}\, \delta \hat{a}_1$$

$$y_2 = \left[M_3\, \phi_{3,1}\, \hat{a}_1 - \lambda M_2\, \phi_{2,1}\, \hat{a}_1 \right] - \left[m_3 - \lambda m_1 \right]$$

$$\delta y_2 = \left[M_3\, \phi_{3,1} - \lambda M_2\, \phi_{2,1} \right] \delta \hat{a}_1 - \mathcal{E}_{2,3}$$

$$A_{11} = \overline{\delta a_2\, \delta a_2^T} = \phi_{2,1}\, \mathrm{cov}\,(\hat{a}_1)\, \phi_{2,1}^T$$

$$A_{12} = \phi_{2,1}\, \mathrm{cov}\,(\hat{a}_1) \left[M_3\, \phi_{3,1} - \lambda\, M_2\, \phi_{2,1} \right]^T$$

$$A_{22} = \overline{\delta y_2\, \delta y_2^T} = \left[M_3\, \phi_{3,1} - \lambda M_2\, \phi_{2,1} \right] \mathrm{cov}\,(\hat{a}_1) \left[m_3\, \phi_{3,1} - \lambda m_2\, \phi_{2,1} \right]^T + \overline{\mathcal{E}_{2,3}\, \mathcal{E}_{2,3}^T}$$

$$\hat{a}_2 = \phi_{2,1}\, \hat{a}_1 - A_{12}\, A_{22}^{-1}\, y_2$$

$$\overline{\delta \hat{a}_2\, \delta \hat{a}_2^T} = A_{11} - A_{12}\, A_{22}^{-1}\, A_{21}$$

In the manner described above we can recursively define successive estimates of the orbit parameters. This same procedure can be extended in a similar manner to higher orders of dependence, q.

The case for finite memory is not illustrated but can be handled in a straightforward manner.

Check on Method

Consider the problem of estimating a constant given three measurements m_1, m_2 and m_3 assuming that the noise is related as shown in Equation (1) (auto-regressive model-first order dependence).

Following the method described in Apollo Note 202, the likelihood expression is as shown in Equation (3).

$$\mathcal{L} = \frac{\left\{ \left[(m_2 - a) - \lambda (m_1 - a) \right]^2 + \left[(m_3 - a) - \lambda (m_2 - a) \right]^2 \right\}}{\sigma_{\xi}^2} + \frac{(a - \hat{a}_o)^2}{\sigma_{\hat{a}_o}^2} \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial a} = 0 = (1-\lambda) \left[(m_2 - \lambda m_1) + (m_3 - \lambda m_2) \right] - 2 (1-\lambda)^2 a - \frac{\sigma_{\xi}^2}{\sigma_{\hat{a}_o}^2} a + \frac{\sigma_{\xi}^2}{\sigma_{\hat{a}_o}^2} \hat{a}_o$$

Solving this equation for a, we obtain

$$a = \frac{\sigma_{\hat{a}_o}^2 (1-\lambda) \left[(m_2 - \lambda m_1) + (m_3 - \lambda m_2) \right] + \sigma_{\xi}^2 \hat{a}_o}{2 (1-\lambda)^2 \sigma_{\hat{a}_o}^2 + \sigma_{\xi}^2} \tag{4}$$

Alternately solving the same problem with the Schmidt-Kalman procedure described previously we obtain:

$$\hat{a}_1 = \frac{\sigma_{\xi}^2 \hat{a}_o + (1-\lambda) \sigma_{\hat{a}_o}^2 (m_2 - \lambda m_1)}{(1-\lambda)^2 \sigma_{\hat{a}_o}^2 + \sigma_{\xi}^2} \tag{5}$$

$$\sigma_{\hat{a}_1}^2 = \frac{\sigma_{\xi}^2 \sigma_{\hat{a}_o}^2}{(1-\lambda)^2 \sigma_{\hat{a}_o}^2 + \sigma_{\xi}^2} \tag{6}$$

Recursively

$$\hat{a}_2 = \frac{\sigma_{\xi}^2 \hat{a}_1 + (1-\lambda)\ \sigma_{\hat{a}_1}^2 (m_3 - \lambda m_2)}{(1-\lambda)^2\ \sigma_{\hat{a}_1}^2 + \sigma_{\xi}^2} \tag{7}$$

Substitution of Equation (5) and (6) into (7) gives

$$\hat{a}_2 = \frac{\sigma_{\hat{a}_o}^2 (1-\lambda) \left[(m_2 - \lambda m_1) + (m_3 - \lambda m_2) \right] + \sigma_{\xi}^2\ \hat{a}_o}{2\ (1-\lambda)^2\ \sigma_{\hat{a}_o}^2 + \sigma_{\xi}^2} \tag{8}$$

The expression for the estimate of the constant with the same data is identical (comparison of Equation (4) with Equation (8).

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APOLLO NOTE NO. 208

C. Siska
21 May 1964

A CALCULATION PROCEDURE FOR OBTAINING COVARIANCE ELEMENTS FOR MANY TRACKING STATIONS

1. Introduction

A customary method for obtaining the covariance matrix is to process the tracking data for many stations in a "simultaneous" fashion. That is to say, if there are J orbital parameters (orbital elements, gravitational constant, etc.) common to Q tracking stations, each with M unique station parameters (location, bias in the measurables, clock rate, etc.), obtaining covariance elements for the orbital parameters leads to the formidable operation of manipulating a $J + Q \times M$ square matrix.

An alternative procedure for obtaining the covariance elements for the orbital parameter is to process the tracking data for each station "sequentially." In this method, we process the data for the first station, which means manipulating a $J + M$ square matrix, and then use the $J \times J$ matrix for the orbital parameters as a priori information to the second station and so forth. In this way one manipulates at most a $J + M$ square matrix, and after a sequence of Q stations, one obtains the same $J \times J$ covariance elements for the orbital parameters as the "simultaneous" method. The "sequential" technique, however, does not yield the same estimates for the station parameters.

In the following, the equivalence of the two methods in obtaining the $J \times J$ covariance elements is shown, and two procedures are indicated for correcting the estimates of the station parameters obtained by the "sequential" method.

2. "Simultaneous" Procedure

To recapitulate briefly, the maximum likelihood technique minimizes the weighted sum of the squared errors as follows (for simplicity we take only one type of measurable and N measurements per station),

$$\min_{\hat{a}_j,\ \hat{x}_{qm}} \sum_{q=1}^{Q} \sum_{i=1}^{N} \left[m_q(i) - \hat{f}_q(\hat{a}_j, \hat{x}_{qm}, i) \right]^2$$

where

$m_q(i)$ = (i)th measurement of the (q)th station normalized by the noise sigma σ_q.

$\hat{f}_q(\hat{a}_j, \hat{x}_{qm}, i)$ = estimated value of $m_q(i)$ normalized by σ_q.

$\hat{a}_j$ = estimated orbital parameter $(j=1....,J)$

$\hat{x}_{qm}$ = estimated station parameter unique to (q) station $(m=1....M)$

Differentiating the above function with a parameter u yields the set of J + Q x M extremal constraints,

$$\sum_{q=1}^{Q} \sum_{i=1}^{N} \left\{ \left[m_q(i) - f_q(i) \right] - \left[\hat{f}_q(i) - f_q(i) \right] \right\} \left\{ \frac{\partial f_q(i)}{\partial u} + \left[\frac{\partial \hat{f}_q(i)}{\partial u} - \frac{\partial f_q(i)}{\partial u} \right] \right\} = 0 \qquad (1)$$

where

$u = a_1....a_J;\ x_{q1}....x_{qM};\ q=1....Q$

$f_q(i) = f_q(a_j, x_{qm}, i)$ = actual value of the measurable

Expanding Equation (1) to first order about the values a_j, x_{qm}, considering the normalized noise $n_q(i) = m_q(i) - f_q(i)$ to be first order, yields the two sets of equations,

$$\sum_{q=1}^{Q} \sum_{i=1}^{N} \left\{ n_q(i) + \sum_{j=1}^{J} \frac{\partial f_q(i)}{\partial a_j} \hat{\Delta a}_j + \sum_{m=1}^{M} \frac{\partial f_q(i)}{\partial x_{qm}} \hat{\Delta x}_{qm} \right\} \frac{\partial f_q(i)}{\partial a_k} = 0; \; k=1....J \quad (2)$$

$$\sum_{q=1}^{Q} \sum_{i=1}^{N} \left\{ n_q(i) + \sum_{j=1}^{J} \frac{\partial f_q(i)}{\partial a_j} \hat{\Delta a}_j + \sum_{m=1}^{M} \frac{\partial f_q(i)}{\partial x_{qm}} \hat{\Delta x}_{qm} \right\} \frac{\partial f_q(i)}{\partial x_{pr}} = 0; \; \begin{matrix} p=1...Q \\ r=1...M \end{matrix} \quad (3)$$

Note in Equation (3) that $\partial f_q(i)/\partial x_{pr} = 0$ for $p \neq q$, and thus, there are Q sets of equations, each set containing the M station parameters of one station only.

Rewriting Equation (2) in matrix form,

$$\sum_{q=1}^{Q} A_q + \sum_{q=1}^{Q} B_{qj} \hat{\Delta a}_j + \sum_{q=1}^{Q} C_q \Delta x_q = 0 \quad (4)$$

where the elements of the matrices A_q, B_q, C_q, $\hat{\Delta a}$, $\hat{\Delta x}_q$ are

$$A_{qk} = \left| \sum_{i=1}^{N} n_q(i) \frac{\partial f_q(i)}{\partial a_k} \right| ; \quad \begin{matrix} J \times 1 \text{ matrix} \\ k = 1....J \end{matrix}$$

$$B_{qjk} = \left| \sum_{i=1}^{N} \frac{\partial f_q(i)}{\partial a_j} \frac{\partial f_q(i)}{\partial a_k} \right| ; \quad \begin{matrix} J \times J \text{ matrix} \\ j, k=1....J \end{matrix}$$

$$C_{qkm} = \left| \sum_{i=1}^{N} \frac{\partial f_q(i)}{\partial x_{qm}} \frac{\partial f_q(i)}{\partial a_k} \right| \quad ; \quad \begin{array}{l} J \times M \text{ matrix} \\ k = 1 \ldots J \\ m = 1 \ldots M \end{array}$$

$$\hat{\Delta a} = \left| \hat{\Delta a}_j \right| \quad ; \; J \times 1 \text{ matrix, } j = 1 \ldots J$$

$$\hat{\Delta x}_q = \left| \hat{\Delta x}_{qm} \right| \quad ; \; M \times 1 \text{ matrix, } m = 1 \ldots M$$

In a similar fashion, Equation (3) yields a set of Q matrix equations,

$$\overline{A}_q + \overline{B}_q \hat{\Delta a} + C_q \hat{\Delta x}_q = 0 \; ; \; q = 1 \ldots Q \tag{5}$$

where

$$\overline{A}_{qr} = \left| \sum_{i=1}^{N} n_q(i) \frac{\partial f_q(i)}{\partial x_{qr}} \right| \quad ; \quad \begin{array}{l} M \times 1 \text{ matrix} \\ r = 1 \ldots M \end{array}$$

$$\overline{B}_{qrj} = \left| \sum_{i=1}^{N} \frac{\partial f_q(i)}{\partial a_j} \frac{\partial f_q(i)}{\partial x_{qr}} \right| \quad ; \quad \begin{array}{l} M \times J \text{ matrix} \\ j = 1 \ldots J \\ r = 1 \ldots M \end{array}$$

$$\overline{C}_{qrm} = \left| \sum_{i=1}^{N} \frac{\partial f_q(i)}{\partial x_{qm}} \frac{\partial f_q(i)}{\partial x_{qr}} \right| \quad ; \quad \begin{array}{l} M \times M \text{ matrix} \\ m, \; r = 1 \ldots M \end{array}$$

$$\overline{B}_q = C_q^T$$

Solving Equation (5) for the station parameters yields

$$\hat{\Delta x}_q = - \overline{C}_q^{-1} \overline{A}_q - \overline{C}_q^{-1} \overline{B}_q \hat{\Delta a} \quad ; \quad q=1 \ldots Q \tag{6}$$

and substituting into Equation (4),

$$\sum_{q=1}^{Q} (A_q - C_q \overline{C}_q^{-1} \overline{A}_q) + \sum_{q=1}^{Q} (B_q - C_q \overline{C}_q^{-1} \overline{B}_q) \hat{\Delta a} = 0 \tag{7}$$

Manipulating Equation (7) will yield the desired covariance elements (J x J matrix) for the orbital parameters.

Writing Equation (7) as

$$\sum_{q=1}^{Q} F_q + \sum_{q=1}^{Q} G_q \hat{\Delta a} = 0 \tag{8}$$

the desired covariance matrix is (since G_q is a symmetric matrix)

$$\overline{\hat{\Delta a} \; \hat{\Delta a}^T} = \left(\sum_{q=1}^{Q} G_q \right)^{-1} \sum_{q=1}^{Q} \sum_{s=1}^{Q} \overline{F_q F_s^T} \left(\sum_{q=1}^{Q} G_q \right)^{-1} \tag{9}$$

Since $\overline{n_q(i) \; n_s(j)} = 1$, if $q = s$ and $i = j$, and zero otherwise, the middle term becomes

$\sum_{q=1}^{Q} \overline{F_q F_q^T}$, and in view of the definitions in Equations (4) and (5),

$$\begin{aligned} \overline{A_q \; A_q^T} &= B_q \\ \overline{A_q \; \overline{A}_q^T} &= C_q \end{aligned} \tag{10}$$

$$\overline{\overline{A}_q \; A_q^T} = \overline{B}_q$$

$$\overline{\overline{A}_q \; A_q^T} = \overline{C}_q$$

so that the following applies

$$\sum_{q=1}^{Q} \sum_{s=1}^{Q} \overline{F_q \, F_s^T} = \left(\sum_{q=1}^{Q} G_q \right) \tag{11}$$

Thus, the desired result is

$$\overline{\Delta\hat{a} \; \Delta\hat{a}^T} = \left(\sum_{q=1}^{Q} G_q \right)^{-1} = \left[\sum_{q=1}^{Q} (B_q - C_q \; \overline{C}_q^{-1} \; \overline{B}_q) \right]^{-1} \tag{12}$$

This equation is to be compared to that developed by the "sequential" technique.

3. "Sequential" Procedure

In this procedure, the past tracking history determines an a priori orbit and each station in turn updates the a priori with its own tracking data. In this case, the maximum likelihood techniques determines a function to be minimized which is also the weighted sum of the squared errors but which includes the errors resulting from the a priori estimate.

Thus, for the (q)th station the function is

$$\min_{\hat{a}_{qj}, \hat{x}_{qm}} \left\{ \sum_{i=1}^{N} \left[m_q(i) - \hat{f}_q (\hat{a}_{qj}, \hat{x}_{qm}, i) \right]^2 + \sum_{j=1}^{J} \sum_{u=1}^{J} \left[H_{q-1}^{uj} (\hat{a}_{q-1}) \right] (\hat{a}_{qj} - \hat{a}_{q-1,j})(\hat{a}_{qu} - \hat{a}_{q-1,u}) \right\}$$

where

$\hat{a}_{q-1,j}$ = a priori estimate for the J orbital parameters for tracking data from (q-1) stations.

$\hat{a}_{qj}$ = estimate for the orbital parameters including the tracking data for the (q)th station.

$H^{uj}_{q-1}(\hat{a}_{q-1})$ = u, j <u>element</u> in the <u>inverse</u> of the covariance matrix (J x J) for the orbital parameter estimates $\hat{a}_{q-1}$ for q-1 stations (i.e., the weighting factor).

The resulting extremal conditions are similar to Equations (2) and (3),

$$\sum_{i=1}^{N}\left\{n_q(i) + \sum_{j=1}^{J}\frac{\partial f_q(i)}{\partial a_{qj}}\hat{\Delta a}_{qj} + \sum_{m=1}^{M}\frac{\partial f_q(i)}{\partial x_{qm}}\hat{\Delta x}_{qm}\right\}\frac{\partial f_q(i)}{\partial a_{qk}} + \sum_{j=1}^{J} H^{uj}_{q-1}(\hat{a}_{qj} - \hat{a}_{q-1,j}) = 0 \qquad (13)$$

$$k = 1 \ldots J$$

$$\sum_{i=1}^{N}\left\{n_q(i) + \sum_{j=1}^{J}\frac{\partial f_q(i)}{\partial a_{qj}}\hat{\Delta a}_{qj} + \sum_{m=1}^{M}\frac{\partial f_q(i)}{\partial x_{qm}}\hat{\Delta x}_{qm}\right\}\frac{\partial f_q(i)}{\partial x_{qr}} = 0 \qquad (14)$$

$$r = 1 \ldots M$$

Equation (14) is identical to Equation (5) for the (q)th station with the solution for $\hat{\Delta x}_q$ given by Equation (6).

If we write $\hat{a}_{qj} - \hat{a}_{q-1,j} = (\hat{a}_{qj} - a_{qj}) - (a_{qj} - \hat{a}_{q-1,j}) = \hat{\Delta a}_{qj} - \hat{\Delta a}_{q-1,j}$ and collect terms in Equation (13), then the result is similar to Equation (4) without the summation signs but with the first two matrix coefficients slightly altered by the a priori term. Specifically Equation (13) becomes

$$(A_q - H_{q-1}\,\hat{\Delta a}_{q-1}) + (B_q + H_{q-1})\,\hat{\Delta a}_q + C_q\,\hat{\Delta x}_q = 0 \tag{15}$$

where H_{q-1} is now the inverse of the covariance matrix (J x J) for the orbital parameters for q-1 stations, with H_o defined as zero.

Combining Equations (6) and (15) leads to an equation similar to Equation (8) or Equation (7),

$$(F_q - H_{q-1}\,\hat{\Delta a}_{q-1}) + (G_q + H_{q-1})\,\hat{\Delta a}_q = 0 \tag{16}$$

In a progression similar to the one which led to Equation (12), we obtain the recurrence relation,

$$\overline{\hat{\Delta a}_q\ \hat{\Delta a}_q^T} = (G_q + H_{q-1})^{-1}\,(G_q + H_{q-1}\ \overline{\hat{\Delta a}_{q-1}\ \hat{\Delta a}_{q-1}^T}\ (G_q + H_{q-1})^{-1\,T} \tag{17}$$

Starting with the first station q=1, we have, since $H_o = 0$,

$$\overline{\hat{\Delta a}_1\ \hat{\Delta a}_1^T} = G_1^{-1}$$

and therefore

$$H_1 = (\overline{\hat{\Delta a}_1\ \hat{\Delta a}_1^T})^{-1} = G_1 \tag{18}$$

For the second station

$$\hat{\Delta a}_2\ \hat{\Delta a}_2^T = (G_2 + H_1)^{-1}\,(G_2 + H_1\ \overline{\hat{\Delta a}_1\ \hat{\Delta a}_1^T}\ H_1^T)\,(G_2 + H_1)^{-1\,T}$$

and since G_q is a symmetric matrix, substitution of H_1 yields

$$\overline{\hat{\Delta a}_2 \; \hat{\Delta a}_2^T} = (G_2 + G_1)^{-1} \tag{19}$$

After Q stations the result is

$$\overline{\hat{\Delta a}_Q \; \hat{\Delta a}_Q^T} = (G_1 + G_2 + \ldots G_Q)^{-1} = \left(\sum_{q=1}^{Q} G_q \right)^{-1} \tag{20}$$

Equations (12) and (20) are identical, which shows the equivalence of the two procedures in calculating the covariance matrix for the orbital parameters. The same result applies if more than one type of measurable with various number of measurements is assumed per station.

4. Updating the Station Parameters

The variance for the station parameters for the q(th) station can be obtained using Equations (6), (7), and (10) to yield

$$\overline{\hat{\Delta x}_q \; \hat{\Delta x}_q^T} = \overline{C}_q^{-1} \left[1 + \overline{B}_q \; \overline{\Delta a_Q \, \Delta a_Q^T} \; \overline{B}_q^T \, \overline{C}_q^{-1} \right] \tag{21}$$

In the "sequential" method, the variance for the orbital parameters in the second term is calculated using the tracking data for the first q stations only, instead of all the Q stations.

To update the station parameters, consider the following matrix equation for the (q) th station,

$$\left|\begin{array}{c:c} \underset{(J \times J)}{X} & \underset{(J \times M)}{C_q} \\ \hdashline \underset{(M \times J)}{\overline{B}_q} & \underset{(M \times M)}{\overline{C}_q} \end{array}\right| \cdot \left|\begin{array}{c:c} \underset{(J \times J)}{a} & \underset{(J \times M)}{b} \\ \hdashline \underset{(M \times J)}{c} & \underset{(M \times M)}{d} \end{array}\right| \equiv 1$$

The items of interest are matrices a and d which are related to the first matrix elements as follows:

$$a = \left(X - C_q \, \overline{C}_q^{-1} \, \overline{B}_q\right)^{-1}$$

$$d = \overline{C}_q^{-1} \left(1 - \overline{B}_q \, X^{-1} \, C_q \, \overline{C}_q^{-1}\right)^{-1} \tag{23}$$

Note that the matrix a is the covariance matrix for the orbit parameters using only the (q)th station data according to Equation (12), i.e., defining X equal to B_q.

Letting matrix d equal the matrix $\overline{\Delta\hat{x}_q \, \Delta\hat{x}_q^T}$ in Equation (21) requires that

$$1 + \overline{B}_q \, \overline{(\Delta\hat{a}_Q \, \Delta\hat{a}_Q^T)} \, \overline{B}_q^T \, \overline{C}_q^{-1} = \left(1 - \overline{B}_q \, X^{-1} \, C_q \, \overline{C}_q^{-1}\right)^{-1} \tag{24}$$

Solving Equation (24) for the matrix X yields

$$X = \overline{(\Delta\hat{a}_Q \, \Delta\hat{a}_Q^T)}^{-1} - \overline{B}_q^T \, \overline{C}_q^{-1} \, \overline{B}_q \tag{25}$$

Since $\overline{B}_q^T = C_q$ from Equation (5), combining Equation (25) with the first equation of Equation (23) with X defined equal to B_q yields

$$X = \left[\overline{(\hat{\Delta a}_Q \; \hat{\Delta a}_Q^T)}^{-1} - a^{-1} \right] + B_q \tag{26}$$

Thus to obtain the updated station parameters defined in Equation (21) we need to invert the following matrix and retain the resulting lower right M x M matrix.

$$\left| \begin{array}{c:c} \overline{(\hat{\Delta a}_Q \; \hat{\Delta a}_Q^T)}^{-1} - a^{-1} + B_q \; (J \times J) & C_q \; (J \times M) \\ \hdashline \overline{B}_q \; (M \times J) & \overline{C}_q \; (M \times M) \end{array} \right|$$

where a^{-1} is obtained by inverting the matrix

$$\left| \begin{array}{c:c} B_q \; (J \times J) & C_q \; (J \times M) \\ \hdashline \overline{B}_q \; (M \times J) & \overline{C}_q \; (M \times M) \end{array} \right|$$

and then inverting the upper left J x J matrix alone.

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APOLLO NOTE NO. 209

H. Engel
25 May 1964

ABILITY OF MSFN TO MONITOR LEM IN POWERED FLIGHT, I

It appears that the only way in which the MSFN can detect anything but a catastrophic failure in the IMU-AGC is by comparison of the IMU-AGC estimate of velocity telemetered from the LEM with the velocity of the LEM determined by the MSFN radar alone. This means that velocity errors of the order of 2 cm/sec. in the direction of the Earth-Moon line, and of the order of 5 m/sec. in the two orthogonal directions can be detected, using three MSFN radars in the three-way Doppler mode (Reference Bissett-Berman Apollo Note No. 198). The particular values of detectable velocity errors are, of course, dependent on the spacing of the MSFN stations.

Errors of these magnitudes in velocity at burnout would result in misses at the nominal time of rendezvous using a Hohmann transfer of the order of

9 meters for a 2 cm/sec. error in altitude rate

1140 meters for a 5 m/sec. error in velocity along track

~0 meters for a 5 m/sec. error in velocity cross track.

The IMU gyros have a nominal drift rate of 10 meru. A rough calculation, based on a velocity to be gained at 1500 m/sec. in a boost time of 400 seconds, indicates that this drift rate would lead to a velocity error of the order of 0.2 m/sec. at burnout.

$$10 \text{ meru } 7.3 \times 10^{-8} \frac{\text{rad}}{\text{sec meru}} \frac{400}{2} \text{ sec. } 1500 \frac{\text{m}}{\text{sec}} = 0.2 \text{ m/sec.}$$

An error of this order of magnitude can be detected in one second by the MSFN radar if in the Earth-Moon direction, but not in the two orthogonal directions.

In order to obtain a better estimate of the gyro drift errors detectable by the MSFN we must perform a somewhat more sophisticated analysis, such as that described below.

Let us assume that the on-board system has perfect gyros and perfect accelerometers up to some time, and then one gyro starts to drift. Let us also assume, for the moment, that the vehicle is not boosting and that its velocity is in the direction of the fixed inertial direction y, common to the MSFN and IMU coordinate systems. If at this time the IMU undergoes a sudden rotation as a result of a gyro failure so that that platform axis that was in the y direction is now in the y' direction, the on-board system still indicates that the velocity is in the y direction, since it is not realized that the platform has been rotated. Hence, in non-boosting flight after this rotation the MSFN and the on-board system will indicate the same velocity and position. It is only when the vehicle boosts that a disparity occurs between the position and velocity determined by the on-board system and by the MSFN.

Now let us see what happens in the case of boost. Let us assume that the vehicle boost acceleration has a fixed magnitude a, and a fixed inertial direction, and that at some time, which we shall arbitrarily call time zero, a gyro with its sensitive axis perpendicular to the direction of boost starts to drift at a fixed rate ω. Then the IMU sees an apparent acceleration $a\omega t$ normal to the actual acceleration and computes a false velocity, v,

$$v = \int_0^t a\omega t \, dt = \frac{a\omega t^2}{2}$$

in this direction. This velocity may be compared with the radar estimate, V, of the corresponding component of velocity, which will

have a mean value 0, since there is actually no boost in this direction. The radar components of velocity are computed from measurements of range change in a period τ divided by τ and modified by the station-vehicle geometry. It is convenient, then, to express time in terms of τ. We have

$$t_i = i\tau.$$

Then

$$v_i = \frac{a\omega\tau^2}{2} i^2$$

and

$$V_i = 0 + n_i$$

in which n_i is the random error in V_i due to radar noise.

Gyro drift is determined by comparison of V_i and v_i. The difference should be of the order of the RMS error in velocity, $\sigma = \left(\sqrt{\overline{n_i^2}}\right)$, determined from the radar measurements, so that if $v_i - V_i$ is substantially larger than σ, it is an indication that there is a systematic error in the on-board system or the radar. We assume that the error is in the on-board system.

Simple comparison of $v_i - V_i$ with σ, however, is not the best we can do. We can detect smaller on-board system errors if we look at many successive values of $v_i - V_i$. Let us define, ξ,

$$\xi_i = v_i - V_i.$$

Then

$$\xi_i = \frac{a\omega\tau^2}{2} i^2 - n_i$$

The mean square value of ξ_i over the interval i equal 1 to N is

$$\text{Letting } \overline{\xi^2} = \frac{1}{N} \sum_{i=1}^{N} \left[\frac{a\omega\tau^2}{2} i^2 - n_i \right]^2$$

$$\text{then } E\,\overline{\xi^2} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{a\omega\tau^2}{2} \right)^2 i^4 + \sigma^2$$

$$= \frac{a^2 \omega^2 \tau^4}{120} (N + 1)(2N + 1)(3N^2 + 3N - 1) + \sigma^2$$

which for N large, say larger than 8, is

$$E\left[\overline{\xi^2}\right] = \frac{a^2 \omega^2 \tau^4}{20} N^4 + \sigma^2$$

If we now consider a case corresponding to part of the LEM ascent boost, with $a = 5$ m/sec.^{2}, $\tau = 1$ sec., and $\sigma = 5$ m/sec. for errors in the cross-range direction, we find

$$E\left[\overline{\xi^2}\right] = 1.25\ \omega^2 N^4 + 25.$$

If we say we shall consider a gyro drift error detectable when $\overline{\xi^2}$ is as great as $2\sigma^2$, then we find that on the average

$$1.25\ \omega^2 N^4 = 25$$

or

$$\omega = \frac{\sqrt{20}}{N^2} \ \frac{\text{rad.}}{\text{sec.}}$$

This is shown in Figure 1.

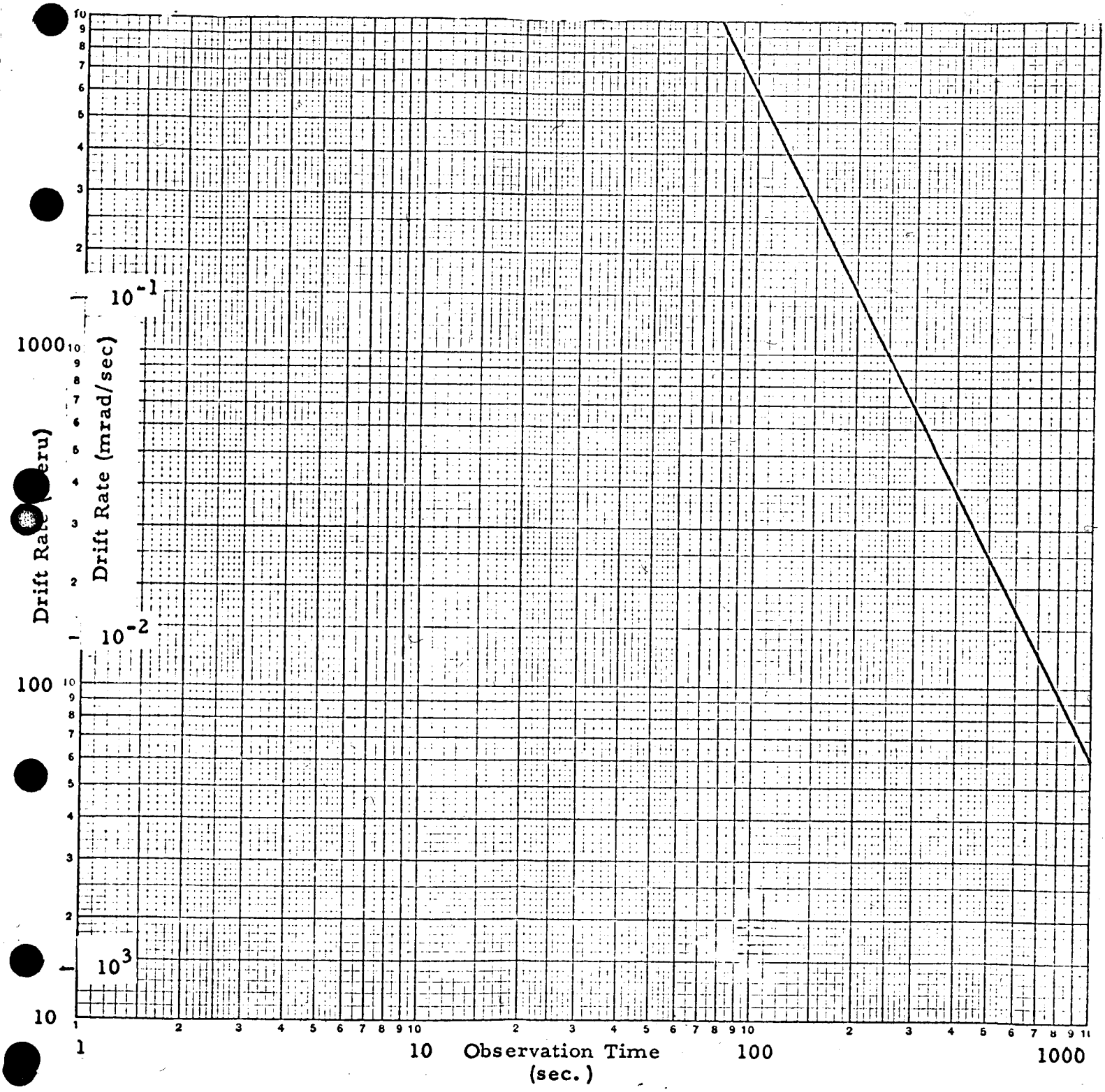
Drift Rate (mrad/sec)
Observation Time (sec.)
1000
100
10
1
10
100
1000
10^{-1}
10^{-2}
10^{3}

Figure 1.

Thus for drift about an axis that is unfavorable for the detection of drift, a drift rate of 200 meru can be detected in 10 seconds, or a drift rate of 20 meru can be detected in 100 seconds.

A better way to look at the errors that can be detected is to realize that substitution leads to

$$E\left[\overline{\xi^2}\right] = \frac{v^2}{5} + \sigma^2$$

so that no matter what the drift rate, if we say that the drift is detectable when $\overline{\xi^2}$ is twice σ^2, then the velocity error when the drift is detected is 2.14σ. In the case in which σ is 5 m/sec., the error in computed velocity when the drift is detected is 11.2 m/sec.

Now, do we have to wait until $\overline{\xi^2}$ is twice σ^2 before we can decide that there is a gyro drift? In order to answer this question we must examine the uncertainty in the observed value of $\overline{\xi^2}$. Returning to the original expression for $\overline{\xi^2}$, we have

$$\overline{\xi^2} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{a\omega\tau^2}{2}\right)^2 i^4 + \frac{1}{N}\sum_{i=1}^{N} n_i^2$$

Now, if n_i is uncorrelated, random Gaussian with variance σ^2, then

$$E\left(\frac{1}{N}\sum_{i=1}^{N}\right) n_i^2 = \sigma^2$$

The variance in the observed value of $\overline{\xi^2}$ is $2\sigma^4/N$. Thus, if $\left(\overline{\xi^2}\right)^2$ exceeds σ^4 by 2 or 3 times $2\sigma^4/N$, this may be taken as evidence of a gyro drift rate significantly different from 0. We have, for $N > 8$,

$$E\left(\overline{\xi^2}\right) \cong \frac{v^2}{5} + \sigma^2$$

so

$$E\left[\left(\overline{\xi^2} - \sigma^2\right)^2\right] = \left(\frac{v^2}{5}\right)^2 = \frac{2\,\sigma^4}{N} .$$

The error in velocity that can be detected is, of course, proportional to σ. One cannot guarantee with absolute certainty that the observations do or do not indicate an error in velocity. Instead, it is necessary to state results in terms of the probability that an error in velocity will be detected and the probability that we will think there is an error in velocity when there is not (false alarm probability). This has been done and is shown in Figure 2 for detection probabilities of 0.70, 0.80, and 0.90, and false alarm probabilities of 0.01 and 0.05. In terms of the expected radar capabilities the figure indicates that the following velocity errors can be detected with probability 0.90 and false alarm probability 0.01.

Error Direction	Observation Time (sec.)		
	10	30	100
Earth-Moon	0.086 m/sec.	0.060	0.041
Orthogonal	14.3	10.0	6.75

It should be remembered that these figures have been derived assuming a constant thrust acceleration. If the thrust acceleration is time-varying, as in the actual LEM ascent, some small changes in these figures will result.

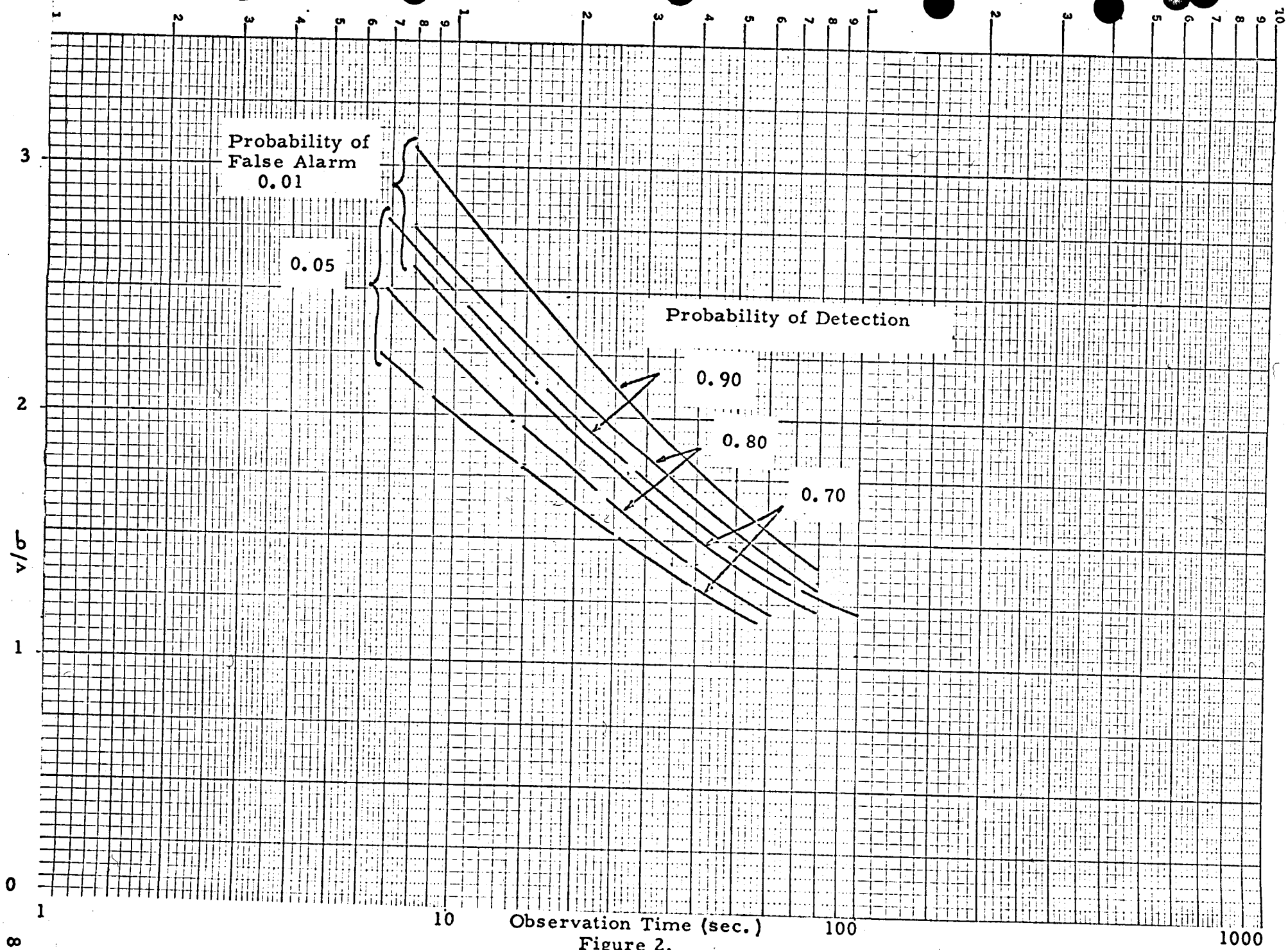
Probability of
False Alarm
0.01
0.05
Probability of Detection
0.90
0.80
0.70
v/σ
3
2
1
0
1
10
100
1000
Observation Time (sec.)

Figure 2.

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APOLLO NOTE NO. 210

H. Epstein
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SIGNAL-TO-NOISE RATIO AND BIAS ERROR LIMITATIONS TO MSFN RADARS

In Apollo Note No. 201, it was shown that receiver noise does not present a severe accuracy limitation provided that a good signal-to-noise ratio is present in the spacecraft and ground-based radar receiving phase-locked loops. The primary requirement imposed on signal-to-noise ratio is to insure that successful tracking is maintained for the required tracking period. These typically involve an effective signal-to-noise ratio of about 16 db, (more specifically, 12 to 20 db). The parameters required for signal-to-noise ratio for a reference system are taken from the parameters given in Apollo Note No. 201. Other results may be obtained from scaling. The reference system has a 30 db signal-to-noise ratio for 1 second smoothing time.

Some component values which comprise F_{eff} are modified as a consequence of a discussion with John Painter (NASA, Houston, Information System Engineering Office). System degradation or the tolerance problem is presently under investigation at NASA, Houston. The revised numerical results and rationale are indicated. An unfavorable situation occurs with the Earth radars using uncooled parametric amplifiers for the low noise receiver (as is contemplated presently for the radars with 30 feet dish antennas) and the portion of the Moon within the main lobe of the antenna illuminated by the Sun. The main lobe of the antenna is always considered as not encompassing any portion of the Sun. The internal receiver noise and external background noise is then about 3 db for this situation. Lunar background noise account for the additional 3 db. Polarization loss treated separately accounts for a 3 db loss when the omni-directional antenna system is employed by the spacecraft and a 1 db loss when the spacecraft directional antenna is employed. The 3 db loss is a consequence

of linear polarization for the omni-directional antenna system and circular polarization for the directional antennas (both spacecraft and ground-based). The 1 db factor is to account for the different effective ellipticity for the directional antenna systems. Considering the spacecraft using an omni antenna and the 30 feet ground-based radars in the unfavorable noise background condition, a total F_{eff} of 17 db can then only allow for media loss, additional antenna degradation, and other system degradation factor of about 2 db total. Media losses should be negligible at these frequencies. The actual F_{eff} at the input to the phase-locked loop will be 3 db greater than the value indicated since the processing loss occurs following the phase-locked loop. With this factor in mind, an F_{eff} of 17 db actually allows for 5 db instead of 2 db additional degradation. Considering that conservative and quite detailed losses have been assumed, a 5 db safety factor is extremely conservative.

This note specifically deals with the doppler mode of radar tracking. The omni-directional antenna system is assumed employed by the spacecraft and the ground-based radars are assumed to use 30 foot antennas. The signal-to-noise performance can be increased by 22 db and 28 db for two feet and four feet spacecraft antennas respectively. The two situations treated below are for a ground-based transmitter and a spaceborne transmitter. The carrier tracking loop is considered as being a second order loop with a damping coefficient of about 0.7.

The ground-based transmitting system is treated first. The parameters which require scaling with the net change in db are indicated in the following table with the net signal-to-noise ratio $\left(\frac{S}{N}\right)$ indicated.

Table 1.

Net Signal-to-Noise Ratio for Ground-Based Transmitter

Reference Signal-to-Noise Ratio	+ 30 db
Transmitter Power (10 kw)	+ 33 db
Receiver Noise Figure (11 db)	- 8 db
Effective Bandwidth (1 kc)	- 30 db
Net Signal-to-Noise Ratio	+ 25 db

The associated steady-state range (bias) error for a constant acceleration of 10 meters per sec.^{2} is 10^{-3} cm. These numbers are quite evidently satisfactory for this application. The VCO output is multiplied by a factor 120 in frequency for transmission at the desired S-band frequency. This increases the bias error to .12 cm while $\frac{S}{N}$ is decreased to + 4 db. The multiplier, to a first order approximately, simply increases the noise bandwidth to 120 kc with the same spectral density as before for band-limited white noise. As a consequence, the change in $\frac{S}{N}$ is unimportant provided that the ground-based receiver tracking loop is narrow as compared to the spacecraft tracking loop. There will be a slight decrease in effective spacecraft transmitter power for this situation.

Considering now the situation with the spacecraft acting as the transmitter, the performance will be found to deteriorate somewhat under extreme conditions. John Painter indicated that the nominal spacecraft transmitter power as 20 watts. He further stated that the

carrier would be suppressed by no more than 7 to 8 db. 8 db suppression of 20 watts would change the net $\frac{S}{N}$ by -2 db from the reference system. A 40 cps ground-based receiver bandwidth would further reduce the signal-to-noise ratio by 16 db yielding a net $\frac{S}{N}$ of + 12 db which should be satisfactory. The associated bias error is .625 cm.

These results have indicated that 30 feet ground-based antenna systems can be quite satisfactory for Apollo tracking requirements. Naturally, one would take advantage of improved performance (e.g., directional antennas) where it is available rather than operate in a degraded mode.

One other important factor needs to be considered— solar noise. A severe deterioration in $\frac{S}{N}$ will occur if the situation arises where the main lobe of the ground-based receiving system must, from geometrical considerations, encompass a substantial portion of the Sun. This is particularly true for disturbed sun conditions. Directional spacecraft antennas should be required for this condition.