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COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Two-Stage Vehicle Optimization

TM- 67-1013-3

DATE- July 18, 1967

FILING CASE NO(S)- 233

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ABSTRACT

This memorandum considers the problem of minimizing the gross weight of a two-stage vehicle subject to a given value of total characteristic velocity, ΔV, and fixed payload. The general optimization problem is first solved in implicit form, and a complete explicit solution is obtained for the case where the two stages have the same specific impulse. Based on this special solution, the more general case of different specific impulses is solved in closed form, leading to a first order and a third order formula.

Two cases are considered: (1) selected stage propellant fraction, and (2) linear dependence of dry stage weight on both gross weight (or thrust) and propellant weight. These are shown to be mathematically equivalent in the optimization procedure.

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TECHNICAL MEMORANDUM

INTRODUCTION

This memorandum considers the two-stage weight optimization problem, with the obvious associated task of minimizing total stage weight for a given ΔV and payload (assuming that intrinsic mass fraction characteristics and specific impulses are known). Presented herein are the general implicit solutions for the stage mass ratio, from which ΔV_1 and ΔV_2 may be immediately deduced, and closed form solutions of a series of special cases of fundamental importance, including the perturbed general solution for small differences in specific impulse.

GROWTH FACTOR CONSIDERATIONS

The stage Growth Factor, G, is defined as*

$$G = \frac{W_G}{P} \quad .$$

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G can be defined separately for each stage or for the combination of stages. The combined factor may readily be expressed in terms of the individual stage factors G_1 and G_2, as follows:

$$G_1 = \frac{W_{G_1}}{P_1} \tag{1a}$$

*All notation is defined in List of Symbols.

and

$$G_2 = \frac{W_{G_2}}{P_2} \tag{1b}$$

In a multi-stage vehicle $P_1 = W_{G_2}$, so that, upon multiplying (1a) and (1b),

$$G_1 G_2 = \frac{W_{G_1}}{P_2} \quad .$$

But W_{G_1} is the combined stage gross weight, and P_2 the combined stage payload, so that, by definition,

$$G = G_1 G_2 \quad . \tag{2}$$

Hence, total stage weight normalized to the payload may be considered as the product of the individual, decoupled stage growth factors.

STAGE PARAMETER REPRESENTATION FOR PRELIMINARY STAGE DESIGN

Two approaches to preliminary stage sizing are particularly convenient. In the first technique it is assumed that the propellant fraction parameters λ_1 and λ_2 may be closely approximated as constants independent of stage weight. Often this is done based on observed past performances of similar systems, and a priori knowledge of parameter variation as a function of stage size and thrust.

The second technique is somewhat more detailed in that it incorporates the sensitivity of major subsystems to both gross stage weight and propellant weight in the optimization procedure. Gross weight (or thrust) is fundamental for sizing engine weight, stage thrust structure and control systems allowances. Correspondingly, propellant weight governs propellant container size, and pressurization and residual weights.

The analytical representations of the two techniques are summarized below:

<u>Case 1</u> $W_G = \lambda(W_G - P) + (1 - \lambda)(W_G - P) + P$

<u>Case 2</u> $W_G = W_P + K_G W_G + K_P W_P + P$

It can be readily shown that in both cases G_i may be reduced to the form

$$G_i = \frac{\beta_i r_i}{1 - \alpha_i r_i} \tag{3}$$

where r_i is the stage mass ratio and α_i and β_i are coefficients, defined as follows:

<u>TABLE 1</u>

	α_i	β_i
Case 1	$1 - \lambda_i$	λ_i
Case 2	$\frac{K_G + K_P}{1 + K_P}$	$\frac{1}{1 + K_P}$

The procedures developed herein are therefore equally suitable to both methods of stage optimization.

GROWTH FACTOR OPTIMIZATION

If the total characteristic velocity for a particular mission is ΔV, then the first stage must provide, say, ΔV_1 and the second satge ΔV_2 so that

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{4}$$

The gross weight of the vehicle will depend on how ΔV is divided between the two stages. From the basic rocket equation

$$\Delta V_i = g_o I_i \ln r_i \quad (i = 1,2) \tag{5}$$

Now, for a given ΔV, (4) and (5) give

$$r_2 = R\, r_1^{-\rho} \tag{6}$$

where

$$R = e^{\Delta V/g_o I_2} \tag{7}$$

and

$$\rho = I_1/I_2 \tag{8}$$

Hence for given values of ΔV, I_1 and I_2, a relationship between r_1 and r_2 is established. Thus the gross weight (or, equivalently, the growth factor) may now be minimized with respect to r_1 or r_2. For convenience r_1 is chosen as the independent variable. From (2) and (3),

$$G = \beta_1 \beta_2 \frac{r_1 r_2}{(1 - \alpha_1 r_1)(1 - \alpha_2 r_2)} \tag{9}$$

Differentiating this with respect to r_1 and equating the resulting expression to zero yields

$$\rho(1 - \alpha_1 r_1) = 1 - \alpha_2 r_2 \tag{10}$$

as a relationship r_1 and r_2 have to satisfy for a weight-optimized two-stage vehicle. Combining equations (10) and (6) now gives the "optimal equation"

$$\alpha_1 \rho \, r_1^{(1+\rho)} + (1-\rho) r_1^{\rho} - \alpha_2 K = 0 \tag{11}$$

This equation implicitly gives the solution to the optimization problem for, once r_1 is found from (11), r_2 may be found from (6) and ΔV_1 and ΔV_2 from (5). A special solution of the problem will be discussed next.

COMPLETE SOLUTION FOR SPECIAL CASE

In this section we consider the case where the two stages have the same specific impulse, I. Thus

$$\rho = 1$$

and the optimal equation becomes

$$\alpha_1 r_1^2 - \alpha_2 R = 0$$

The solution is

$$r_1 = \sqrt{\frac{\alpha_2}{\alpha_1} R} \tag{12}$$

In view of (7) and (6) this gives

$$\left.\begin{aligned} r_1 &= \sqrt{\frac{\alpha_2}{\alpha_1}}\, e^{\Delta V/2g_o I} \\ r_2 &= \sqrt{\frac{\alpha_1}{\alpha_2}}\, e^{\Delta V/2g_o I} \end{aligned}\right\} \tag{13}$$

It immediately follows from (5) that

$$\left.\begin{aligned} \Delta V_1 &= \frac{1}{2}\left(\Delta V + g_o I \ln \frac{\alpha_2}{\alpha_1}\right) \\ \Delta V_2 &= \frac{1}{2}\left(\Delta V - g_o I \ln \frac{\alpha_2}{\alpha_1}\right) \end{aligned}\right\} \tag{14}$$

If, in addition, the two stages have the same value of ϵ, the result is

$$\Delta V_1 = \Delta V_2 = \frac{1}{2}\,\Delta V \tag{15}$$

as would be expected for this case (see Reference 1).

In the next section the more general case is treated where the difference between the specific impulses for the two stages is small relative to their respective values. This case is of particular interest since slight variation in specific impulse often results due to atmospheric effects and variation in delivered values between engines of different thrust.

PERTURBATION SOLUTION OF THE OPTIMAL EQUATION

As noted before let

$$r_c = \sqrt{\frac{\alpha_2}{\alpha_1}\,R} \tag{16}$$

be the solution of the optimal equation when $\rho = 1$. Write

$$r_1 = r_o + \delta r$$

and

$$\rho = 1 + \varepsilon$$

where

$$\varepsilon = \frac{I_1 - I_2}{I_2} \tag{17}$$

is a small number and δr is the variation of r_1 about r_o. Now equation (11) can be written in the form

$$\alpha_1(1+\varepsilon)r_o^{2+\varepsilon}\left(1 + \frac{\delta r}{r_o}\right)^2\left(1 + \frac{\delta r}{r_o}\right)^\varepsilon -$$

$$-\varepsilon r_o^{1+\varepsilon}\left(1 + \frac{\delta r}{r_o}\right)\left(1 + \frac{\delta r}{r_o}\right)^\varepsilon - \alpha_1 r_o^2 = 0 \tag{18}$$

which is then to be solved for the variation δr. By a truncated binomial series approximation,

$$\left(1 + \frac{\delta r}{r_o}\right)^\varepsilon = 1 + \varepsilon\frac{\delta r}{r_o} - \frac{\varepsilon}{2}\left(\frac{\delta r}{r_o}\right)^2. \tag{19}$$

This is valid up to fourth order terms in combinations of ε and $\frac{\delta r}{r_o}$. When this is substituted into equation (18), the terms expanded and rearranged by powers of $\left(\frac{\delta r}{r_o}\right)$, there results a quadratic equation of the form

$$A\left(\frac{\delta r}{r_o}\right)^2 + B\left(\frac{\delta r}{r_o}\right) + C = 0 \tag{20}$$

where

$$\left.\begin{aligned} A &= \frac{\alpha_1}{2}(1+\varepsilon)(2+2\varepsilon)r_o^{2+\varepsilon} - \frac{\varepsilon^2}{2} r_o^{1+\varepsilon} \\ B &= \alpha_1(1+\varepsilon)(2+\varepsilon)r_o^{2+\varepsilon} - \varepsilon(1+\varepsilon)r_o^{1+\varepsilon} \\ C &= \alpha_1(1+\varepsilon)r_o^{2+\varepsilon} - \varepsilon r_o^{1+\varepsilon} - \alpha_1 r_o^2 \end{aligned}\right\} \tag{21}$$

so that

$$r_1 = r_o\left(1 + \frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) \tag{22}$$

This is a third order solution of the optimal equation in closed form.

For hand computation a first order formula can be obtained by returning to equation (18) and, in the subsequent expansions, neglecting all terms of order two in ε, $(\frac{\delta r}{r_o})$ and combinations. The result is a linear equation, instead of the quadratic form (20), whose solution is

$$r_1 = \frac{\varepsilon}{2\alpha_1} + \frac{1}{2} r_o^{1-\varepsilon} + \frac{1-\varepsilon}{2} r_o \tag{23}$$

Using a first order approximation

$$r_o^{-\varepsilon} = 1 - \varepsilon \ln r_o + o[(\varepsilon \ln r_o)^2]$$

the above can be reduced to

$$\boxed{r_1 = r_o + \frac{\varepsilon}{2}\left[\frac{1}{\alpha_1} - r_o\,(1 + \ln r_o)\right]} \qquad (24)$$

EXAMPLE

Using equation (24) a numerical example was constructed with

$$\lambda_1 = 0.92^*,\ I_1 = 360 \text{ sec}$$

$$\lambda_2 = 0.88^*,\ I_2 = 330 \text{ sec}$$

This combination was considered with total velocity changes of 10,000, 20,000, and 30,000 fps. The results of computations were as shown in Figures 1 to 3.

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*Using Case I as in Table 1.

In addition to the first order solution, the results obtained by two simple averaging approximations are included for comparison. These approximations are as follows:

$$1)\quad \tilde{I}_1 = \tilde{I}_2 = \frac{I_1 + I_2}{2} \quad \text{and}$$

$$2)\quad \tilde{I}_1 = \tilde{I}_2 = \frac{I_1 + I_2}{2}, \text{ and } \tilde{\lambda}_1 = \tilde{\lambda}_2 = \frac{\lambda_1 + \lambda_2}{2}$$

The first type allows equation (14) to be employed and the second type assumes $\Delta V_1 = \Delta V_2$ as shown in equation (15). Upon estimation of the ΔV's, the correct stage λ's and I_{sp}'s can be used for the individual stage calculations. Averaging the specific impulse yields a 1-1/2% increase in growth factor compared to the first order solution. Averaging both the specific impulse and mass fraction yields a 4-1/2 to 12% increase in the growth factor.

To display the gross stage sensitivity to the allotted stage velocity increments, stage growth factor is plotted as a function of $\Delta V_1/\Delta V$. Note in Figure 1 that the results of the averaging approximations could lead to the conclusion that two stages are desired. The first order solution correctly indicates a single stage to be optimum.

As noted in Figures 2 and 3, the region in the vicinity of the optimum point is fairly "flat." As a consequence of low growth factor sensitivity to stage velocity allocation in the region, it may be advantageous to choose an off optimum point to improve the second-stage propellant fraction. For example, sizing the second stage of Figure 2 for 5815 fps may result in an unrealistically small vehicle, and higher propellant fractions than anticipated. In this case, equation (24), used with modified values of λ_1 and λ_2, may be helpful as a design tool to enable a rapid convergence to a practical solution of the optimization problem. (This iteration procedure is not required in the Case 2 approach (see Table 1).)

An error estimate was made for the case shown in Figure 2. By using the numerical value of r_1, a residual can be computed

from equation (11). Comparing this residual to a derivative of equation (11) with respect to r_1 shows that $dr_1 = +.01$; i.e., using formula (24) results in a value of r_1 which, for this example, is approximately 1/4% too large. However, when the effect of this correction to r_1 is traced through the computation of G, the result indicates a correction in G of less than 0.0003; i.e., a relative error

$$\frac{dG}{G} < 0.004\%$$

Finally, the resultant relative error in the first stage velocity increment is

$$\frac{d(\Delta V_1)}{\Delta V_1} = 0.2\%$$

SUMMARY

The optimal equation (11) expresses a condition for the first stage mass fraction, r_1, under which the gross weight of a two-stage vehicle will be minimized. For a given value of total characteristic velocity, ΔV, this is an implicit solution for the optimization problem.

Based on the exact solution for the special case of equal specific impulses, explicit solutions to the optimal equation are obtained in closed form. Numerical examples and comparisons with common approximations show that the first order formula gives good results.

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Attachments
List of Symbols
Reference
Figures 1-3

LIST OF SYMBOLS

Symbol

g_o = gravitational acceleration at sea level

G = growth factor

I = specific impulse

K_G = gross weight proportionality factor

K_P = propellant weight proportionality factor

P = payload weight

r = mass ratio (initial/burnout)

R = $\exp(\Delta V/g_o I_2)$

W_G = gross weight

W_P = propellant weight

α, β = coefficients defined in Table 1

δr = variation in mass fraction

ΔV = characteristic velocity

ε = perturbation in ρ

λ = propellant fraction (propellant weight/stage weight)

ρ = I_1/I_2

Subscripts

0 = condition corresponding to $I_1 = I_2$

1,2 = referring to first or second stage

REFERENCE

1. Malina, F. J., and M. Summerfield: The Problem of Escape from the Earth by Rocket. Journal of the Aeronautical Sciences, Vol. 14, August, 1947, p. 471.

[illegible]	λ_1	λ_2	I_1 ([illegible])	I_2 ([illegible])	ΔV_1 ([illegible])	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

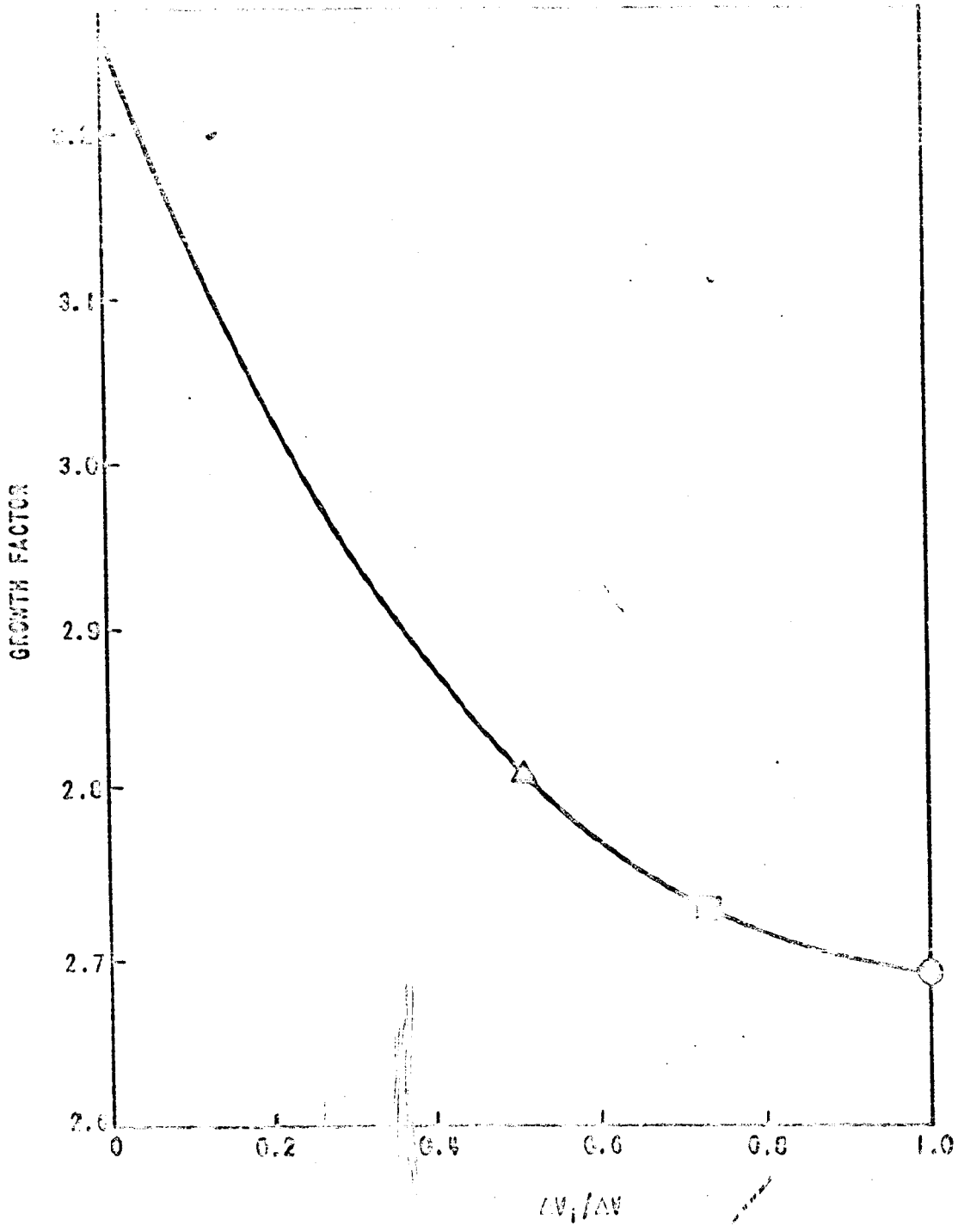
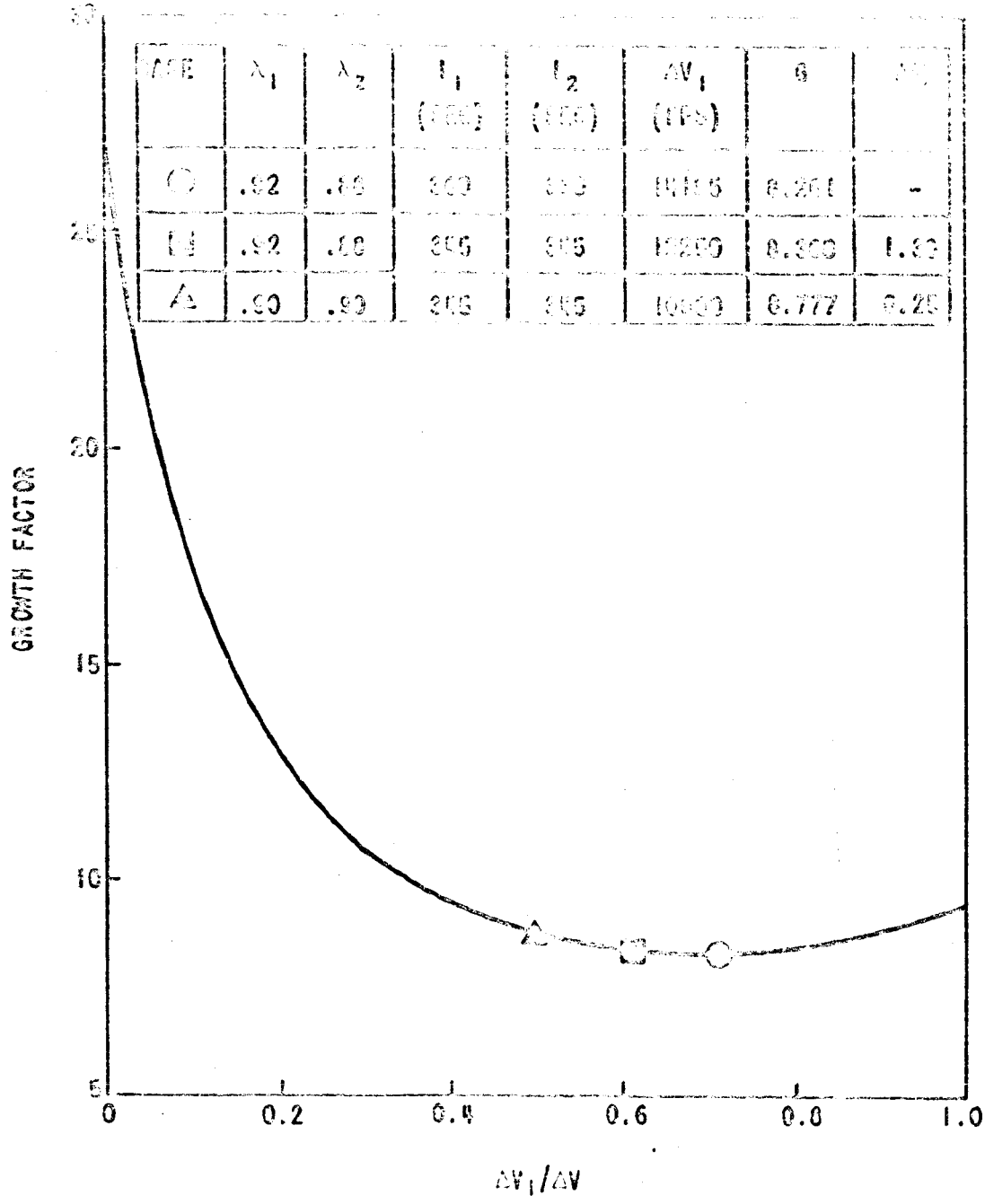


FIGURE 1 - GROWTH FACTOR VERSUS STAGE 1 VELOCITY CHANGE FOR A TOTAL VELOCITY CHANGE OF 10,000 FPS



CASE	λ_1	λ_2	I_1 (SEC)	I_2 (SEC)	ΔV_1 (FPS)	G	[illegible]
○	.92	.88	360	[illegible]	[illegible]	8.261	-
□	.92	.88	355	355	[illegible]	8.300	1.3[illegible]
△	.90	.90	355	355	10000	8.777	0.25

FIGURE 2 - GROWTH FACTOR VERSUS STAGE I VELOCITY CHANGE FOR A TOTAL VELOCITY CHANGE OF 20,000 FPS

[illegible]	λ_1	λ_2	t_1 (sec)	t_2 (sec)	ΔV_1 (fps)	θ	[illegible]
[illegible]	.82	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	–
[illegible]	.82	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.50	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

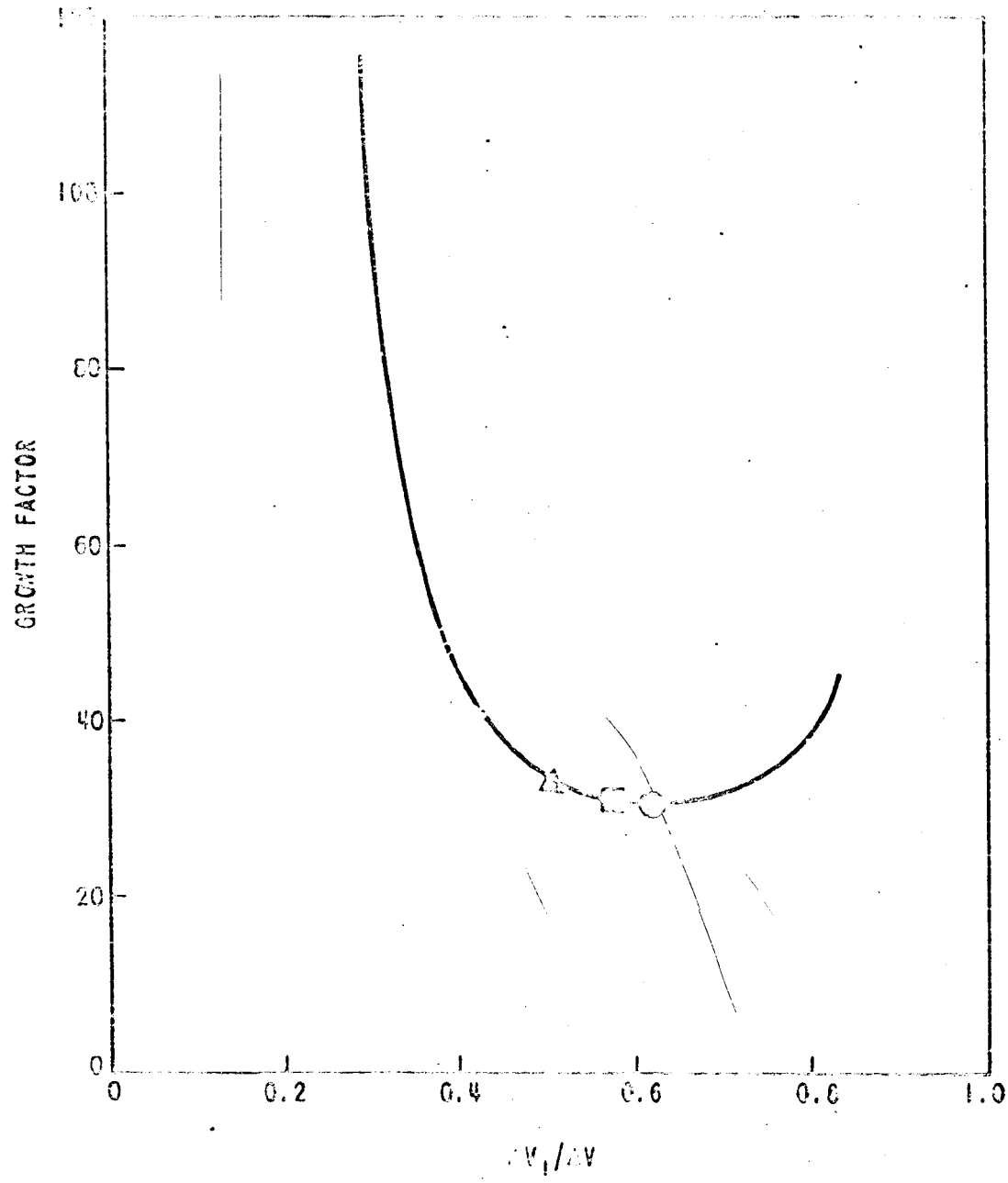


FIGURE 3 - GROWTH FACTOR VERSUS STAGE I VELOCITY CHANGE FOR A TOTAL VELOCITY CHANGE OF 30,000 FPS