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APOLLO MISSION SIMULATOR

-MATHEMATICAL OUTLINES-

(U)

NAS 9-150

10 September 1962

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Prepared by

TRAINING SYSTEMS
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FOREWARD

In the development of training equipment designed to support the training requirements of the Apollo program, a study of the various spacecraft flight problems has been made. These studies involve the complete lunar mission and include lunar environment as well as midcourse and earth environment studies.

Through the use of ordinary dynamics and mathematical techniques, an effort has been made to describe the flight of the Apollo vehicle from launch to lunar orbit and return. This description is intended to provide a criteria for analog and digital computer design.

The descriptions included utilize standard simulation expressions for flying vehicles. Much of the data has been derived from formerly proven techniques in mathematical description, which have been used for computer design criteria. Other data included has been obtained from NAA preliminary Apollo vehicle analog and digital computer studies.

It is assumed that Ephemeris data is available to establish stellar locations and Earth-Moon and Earth-Sun vectors for both dynamic and visual considerations. Consequently, no tables of data are presented.

Part One includes the math models describing the complete mission

and Part Two will present, in a subsequent issue, the Computer Mechanization Equations for these models.

Subsequently, parts of this report will be revised and used to establish Mathematical Models for various phases of the Apollo Mission required by the Part Task Trainers.

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flight. No aerodynamic terms are included and the moon is not considered a triaxial ellipsoid, but spherical for simulation purposes.

Similarly, the Midcourse Model sums Model II of Earth and Lunar Environment with aerodynamic and oblateness terms eliminated. Ephemeris data can be included for complete description.

The Launch Escape Propulsion System Model formulates the propulsion and torques that are added to the equations of motion to produce the abort trajectories of the vehicle. This is true for any time during launch until the system is jettisoned.

The Rendezvous Model formulates the required trajectories for rendezvous in both orbital transfer and homing phases. Complete capability is provided for in the Earth Environment while only emergency homing capability is provided for in the Lunar Environment. Homing is described as steering the Ferry Vehicle to a docking position using either telemetry or visual reference techinques.

The Functional Block Diagram integrates all models into one coherent system which includes all flight phases and controls (system variables).

The equations of motion governing the Apollo spacecraft trajectory have been described mathematically, in all modes of the mission, along with the visual cues required at every point and are integrated as one complete system.

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SUMMARY

Six Mathematical Models have been formulated which describes the Apollo Mission Simulator requirements. They can be considered as Earth and Lunar Environments in general, with Launch Escape, Midcourse, Rendezvous, and Visual Models included.

Two models for the Earth Environment has been established which makes use of spherical coordinates referred to the earth's center. For each model, the coordinates used in the translation equations do not depend upon the vehicle's angular orientation. Hence, the components of the vehicle's velocity in these coordinate systems cannot change value rapidly unless the velocity vector of the vehicle changes magnitude or direction rapidly. This along with placing the Euler Transformation "gimbal lock" in the roll axes instead of the pitch axes improves computer dynamic characteristics. The latter allows tumbling of the Apollo vehicle in the pitch plane which will provide a better overall simulation of vehicle flight. Both models are suitable for digital and hybrid analog - digital computation. Model I is preferred, mainly because the mathematical expressions are less complex. However, Model I possesses an indeterminate point which arrises when simulation requires a "flight" over a pole. Therefore, Model II should be used if a polar or near polar flight is anticipated.

The model for the Lunar Environment is presented which is similar to Model II of the Earth Environment and contains polar flight capabilities which is considered essential for emergency rendezvous

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PART TWO

COMPUTER MECHANIZATION EQUATIONS

(To Be Developed)

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SCOPE:

This document defines and establishes the mathematical models required for the design of the Apollo Mission Simulator. The complete lunar mission is described in terms of the various encountered environments.

An overall functional model is established in terms of operational equipment. Computer equipment interface requirements are identified providing a criteria for analog and digital systems design.

APPLICABLE DOCUMENTS:

Air Force Documents

ASD Technical Report 61-171 (I)	Flight Simulation of Orbital and Reentry Vehicles, Part I - Development of Equations of Motion in Six Degrees of Motion, October 1961

NASA Documents

Letter	To John W. Paup (Apollo Program Manager) from NASA Resident Apollo Project Office, dated May 23, 1962 (page 2)

Minneapolis Honeywell, Inc. Documents

Launch Escape System Design Data	Volume 1 - March 9, 1962
	Volume 2 - April 21, 1962

North American Aviation, Inc. Documents

MD 59-272	Analytical Study of Satellite Rendezvous-Final Report, October 20, 1960, Aerospace Laboratories
SID 62-1091	Apollo Navigation and Trajectory Control Trainer, Report No. 1, August 29, 1962

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LIST OF SYMBOLS

a	Semi-major axis (feet or miles) (rendezvous)
$\bar{a}$	Acceleration vector of vehicle relative to Earth Inertial axes.
$\bar{a}_L$	Acceleration vector of vehicle relative to lunar Inertial axes.
A	Binormal velocity component (FT/SEC) ferry vehicle
B	Binormal velocity component (FT/SEC) target vehicle
C	Circumferential velocity component (FT/SEC)
C_1	Any point on X_B
C.G.	Center of gravity
C.G.'	Center of gravity after fuel burnout
C.M.	Command Module
C_1M	Radius of curvature of C.M. cone in existing plane
C.S.	Celestial sphere
d_1	Distance from astronaut's eye to center of window
$\bar{d}_1$	With appropriate subscripts, center line of sight vectors
e	Eccentricity
E_1	Astronaut's right eye
E_2	Astronaut's left eye
f	Parameter describing Earth oblateness
$f(FB)$	Function of total fuel burned
f_r	True anomaly $\theta - \omega$ (radians) (rendezvous)
$\bar{F}$	Aerodynamic force vector
F	With appropriate subscripts, components of $\bar{F}$ referred to selected axis
F_1	Field of view (windows)
$F_{x_1}, F_{y_1}, F_{z_1}$	Thrust or perturbation accelerations (homing) (FT/SEC^2)

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$\bar{G}$ Gravitational force vector of vehicle

g_e Gravational acceleration at Earths surface

g_L Gravational acceleration at moon's surface

h Altitude above Earth's surface

h_L Altitude above moon's surface

h_r Angular momentum per unit mass (FT^2/SEC) (Rendezvous)

$\bar{H}$ Control force vector

H With appropriate subscripts, components of $\bar{H}$ referred to selected axis

H velocity thrust coefficient (homing)

i Inclination (rendezvous)

i,j,k, Unit vectors along the X,Y,Z, axes respectively; with appropriate subscripts, unit vectors along axes in other sets

i_r, j_r, k_r, Unit vectors in a circumferential, binormal radial system (rendezvous)

I Total impulse function (FT/SEC) (rendezvous)

I_1 Impulse at departure point (FT/SEC) (rendezvous)

I_2 Impulse at arrival point (FT/SEC) (rendezvous)

I_{XX}, I_{YY}, I_{ZZ}, Moments of Inertia of vehicle about the X_B, Y_B, Z_B axis respectively

I_{XZ} Product of inertia of vehicle about X_B and Z_B axis

J With appropriate subscripts, components of control moment

K Earth gravity constant

K_L Lunar gravity constant

K_r Proportional thrust coefficient (rendezvous)

[redacted]

L	Lead thrust coefficient (rendezvous)
L,M,N	Components of the aerodynamic moment, referred to the X_B, Y_B, Z_B axis respectively
LES	Launch escape system
m	Vehicle mass
M	Median to window
n	Rate of change of mean anomaly (average angular velocity-radians/SEC)
N	Normal to window
p	Semi-latus rectum (feet or miles) (ferry vehicle transfer ellipse)
p_1	Semi-latus rectum (feet or miles) (ferry vehicle elliptical trajectory)
p_2	Semi-latus rectum (feet or miles) (target vehicle elliptical trajectory)
$\bar{P}$	Propulsive force vector
$\bar{P}_{LES}$	Launch escape system propulsion vector
P_{LES}	With appropriate subscripts, components of $\bar{P}_{LES}$ referred to selected axis
P_{SFM}	Solid fuel motor propulsion
P,Q,R	Components of angular velocity of vehicle relative to inertial axes, referred to the body axis
$\bar{r}$	Radius vector from earth center to vehicle centroid
r	Length of $\bar{r}$
r	With appropriate subscripts, reaction control radius
$\bar{r}_L$	Radius vector from lunar center to vehicle centroid
r_L	Length of $\bar{r}_L$
r_1	Radius from Earth center to ferry vehicle centroid
r_2	Radius from Earth center to target vehicle centroid

r_w	Radius of window
δ_r	Difference between appropriate r and R_o
R	Radial velocity component (FT/SEC) (rendezvous)
R_o	Earth's radius in the equatorial plane (feet)
R_{oL}	Moon's radius in the equatorial plane (feet)
R_E	Earth's radius at a local point on the Earth's surface (feet)
R_{EL}	Moon's radius at a local point on the moon's surface (feet)
R_1	Arithmetic mean of the Earth's radius at pole and equator (feet)
R_{1L}	Arithmetic mean of the moon's radius at pole and equator (feet)
R_{XSFM}	Radius along the X-axis from solid fuel motor to center of gravity line
S	Laplace transform frequency variable (rendezvous)
S/C	Spacecraft
SM	Service module
SL	Space laboratory
t	Time
T	With appropriate subscripts, components of propulsive moment
T_{LES}	With appropriate subscripts, torque about coordinate axes (LES)
$\bar{V}$	Velocity vector of vehicle relative to inertial axes
$\bar{V}_a$	Velocity vector of vehicle relative to the air
V_a	With appropriate subscripts, components of $\bar{V}_a$ referred to selected axis
V_a	Magnitude of $\bar{V}_a$
$\bar{V}_E$	Velocity vector of vehicle relative to the X_E, Y_E, Z_E frame
$\bar{V}_L$	Velocity vector of vehicle relative to the X_L, Y_L, Z_L, frame
$\bar{V}_W$	Wind velocity vector relative the X_E, Y_E, Z_E, frame

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V_k With appropriate subscripts, components of $\bar{V}_k$ referred to selected axis

w Command module window

X,Y,Z Inertial axis. With appropriate subscripts, other axis systems

x_1, y_1, z_1 Rendezvous axes system

X(S),Y(S), Z(S) Laplace transform coordinate axis

α Angle of attack

α_1 Angle between transfer orbit plane and initial orbit plane (radians)

α_2 Angle between transfer orbit plane and terminal orbit plane (radians)

α_{1S} Angle of rotation of sextant about the Z_B axis

α_{1T} Angle of rotation of scanning telescope about the Z_B axis

α_{1W}, α_{2W} Angles of rotation of window about the Z_B axis

β Angle of sideslip

β_{1S} Angle of rotation of sextant about the Y_B axis

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β_{1T} Angle of rotation of scanning telescope about the Y_B axis

β_{1W}, β_{2W} Angles of totation of window about the Y_B axis

φ, θ, ψ Euler angles establishing vehicle roll, pitch, and yaw orientation, respectively, in earth environment; with appropriate subscripts, visual instrument orientation.

$\varphi_L, \theta_L, \psi_L$ Euler angles establishing vehicle roll, pitch and yaw orientation respectively, in lunar environment.

φ_1 Angle from reference axis to postition in initial orbit (radius)

φ_2 Angle from reference axis to position in terminal orbit (radius)

θ_1 Ferry vehicle polar angle

θ_2 Target vehicle polar angle

θ_3 Angle from ascending node to position in transfer orbit (radius)

$\Delta\theta$ $\theta_2 - \theta_1$ (radius)

λ Angle establishing initial direction of nominal trajectory (Earth)

λ_F Angle establishing local direction of nominal trajectory (Earth)

λ_L Angle establishing initial direction of nominal trajectory (Lunar)

λ_{FL} Angle establishing local direction of nominal trajectory (Lunar)

μ Universal gravity constant

Φ Geocentric latitude of vehicle

Φ_i Geocentric latitude of launch point

Φ_F Spherical angular coordinate of vehicle measured normal to nominal earth trajectory plane

Φ_{FL} Spherical angular coordinate of vehicle measured normal to nominal moon trajectory plane

Φ_{iL} Selenocentric latitude of launch point

[redacted]

Φ_L Selenocentric latitude of vehicle

Φ Geocentric longitude of vehicle measured from launch point

Φ_F Spherical angular coordinate of vehicle measured in nominal earth trajectory plane

Φ_{FL} Spherical angular coordinate of vehicle measured in nominal moon trajectory plane

Φ_L Selenocentric longitude of vehicle measured from launch point

ω Argument of perigee, angle from reference axis to perigee point (Radians)

$\bar{\omega}_B$ Angular velocity vector of vehicle relative to earth inertial frame

$\bar{\omega}_{BL}$ Angular velocity vector of vehicle realtive to lunar inertial frame

$\bar{\omega}_E$ Angular velocity vector of X_E, Y_E, Z_E frame relative to earth inertial frame

$\bar{\omega}_F$ Angular velocity vector of X_F, Y_F, Z_F frame relative to earth inertial frame

$\bar{\omega}_{FE}$ Angular velocity vector of X_F, Y_F, Z_F frame relative to X_E, Y_E, Z_E frame

$\bar{\omega}_G$ Angular velocity vector of X_G, Y_G, Z_G frame relative to earth inertial frame

$\bar{\omega}_L$ Angular velocity vector of X_L, Y_L, Z_L frame relative to lunar inertial frame

$\bar{\omega}_S$ Angular velocity vector of X_S, Y_S, Z_S frame relative to lunar inertial frame

$\bar{\omega}_V$ Angular velocity vector of X_V, Y_V, Z_V frame relative to earth inertial frame

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$\bar{\omega}_{VL}$ Angular velocity vector of X_{VL}, Y_{VL}, Z_{VL} frame relative to lunar inertial frame

Ω_E Earth's rotational velocity

Ω_L Lunar rotational velocity

δ_1 Angle between the normal and median to the C.M. window

γ Angle between the element of the cone of C.M. body and X_B, or one half the vertex angle of C.M. cone

ε Angle between E_1 M and C_1 M

SUBSCRIPTS

β Body axes in earth environment

β_L Body axes in lunar environment

C Command Module

CS Command Module and service module

E Earth rotating axes

F Earth - Vehicle geocentric axes referred to nominal trajectory plane

FL Lunar - Vehicle orbit plane selenocentric axes

G Earth - Vehicle geocentric axes referred to equatorial plane

L Lunar rotating axes

LEM Lunar excursion module

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LES	Launch escape system
s	Sextant
SS	Service module and space laboratory
T	Scanning telescope
V	Vehicle geocentric axes
VL	Vehicle selenocentric axes
x,y,z	Components along body axes X_B, Y_B, Z_B respectively
r,Φ,Ψ	Components along X_G, Y_G, Z_G (lunar environment: X_S, Y_S, Z_S) axes respectively
r, Φ_F, Ψ_F	Components along X_F, Y_F, Z_F axes respectively
W	Vehicle wind axes
w	Vehicle windows
1	Transfer orbit at departure point
2	Transfer orbit at arrival point
11	Initial orbit at departure point
22	Initial orbit at arrival point

NOTE: When applied to orbital elements (p, e, ω, i, Ω), subscript 1 refers to the initial orbit and subscript 2 refers to the terminal orbit. Orbital elements of transfer orbit are denoted without subscripts.

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Section I: EARTH ENVIRONMENT

Axes and Coordinates (See FIGURE 1)

INERTIAL AXES (X,Y,Z)

1. Origin at earth center.
2. Unit vectors (i,j,k).
3. Z - axis coincident with earth polar axis, positive North.
4. X - Z plane contains initial position of vehicle.

EARTH AXES (X_E,Y_E,Z_E)

1. Origin at earth center.
2. Unit vectors (i_E, j_E, k_E).
3. Z_E - axis coincident with earth polar axis, positive North.
4. Initial position coincident with inertial axes.

EARTH - VEHICLE ORBIT PLANE GEOCENTRIC AXES (X_F,Y_F,Z_F).

1. Origin at earth center.
2. X_F - axis passes through vehicle centroid.
3. X_F - Y_F plane is nominal trajectory plane.
4. Y_F points essentially in direction of flight.
5. Z_F points left when looking in direction of flight.
6. Unit vectors (i_F, j_F, k_F).

EARTH - VEHICLE GEOCENTRIC AXES (X_G,Y_G,Z_G)

1. Origin at earth center.
2. X_G - axis passes through vehicle centroid.

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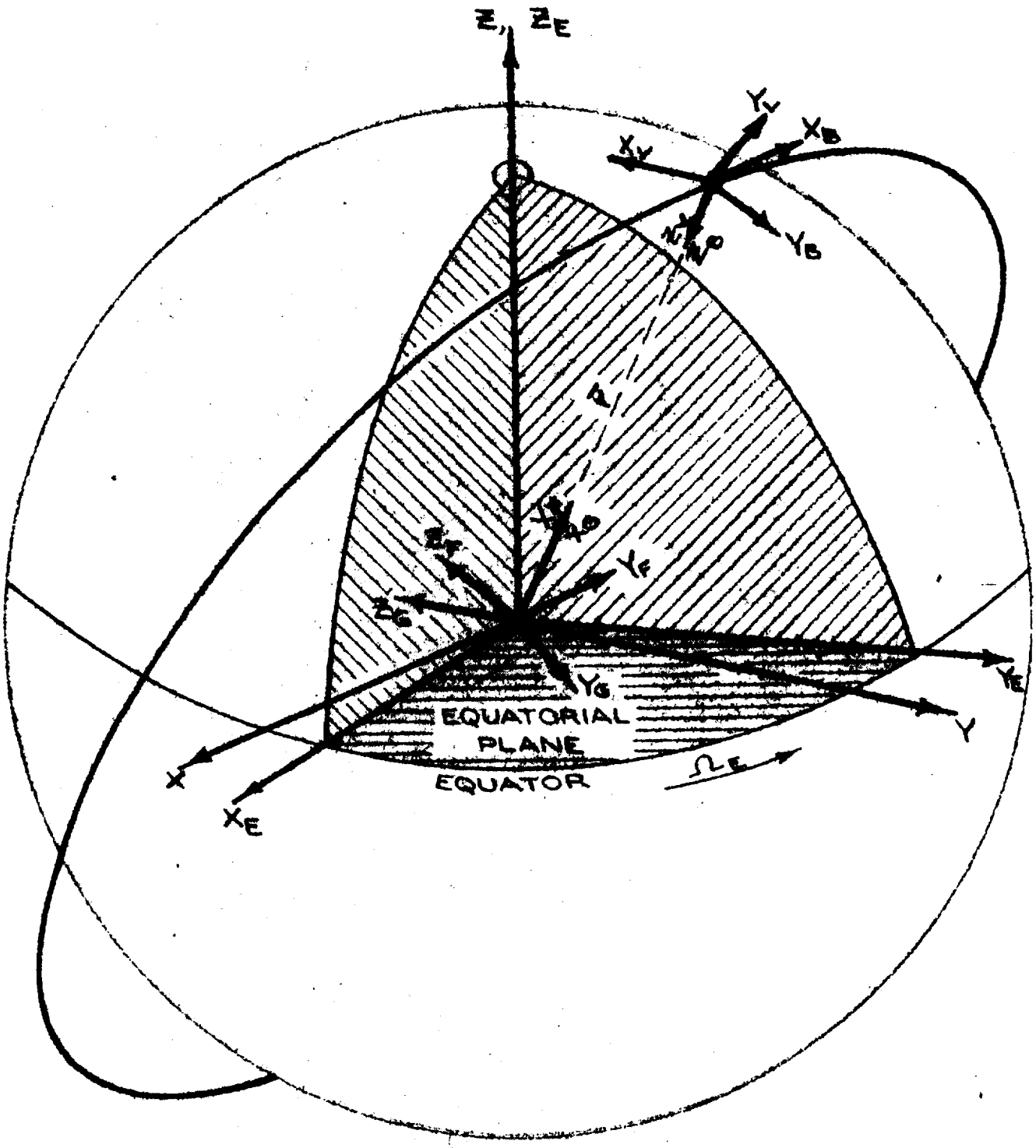

NOTE: WIND AXES, (X_W, Y_W, Z_W), NOT SHOWN.

EARTH ENVIRONMENT
FIGURE 1

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3. X_G-Z_G plane contains earth polar axis.

4. Y_G-Axis lies in the equatorial plane.

5. Z_G-Axis is positive North of equatorial plane.

6. Unit vectors (i_G, j_G, k_G).

VEHICLE BODY AXES (X_B, Y_B, Z_B).

1. Origin at vehicle centroid.

2. X_B -Z_B plane coincident with plane of symmertry of vehicle.

3. Z_B axis positive downward, normal to X_B axis.

4. X_B-axis positive forward.

5. Y_B -axis positive right looking forward and normal to X_B.

6. Unit vectors (i_B, j_B, k_B).

VEHICLE WIND AXES (X_W, Y_W, Z_W).

1. Origin at vehicle centroid.

2. X_W - axis points in direction of vehicle velocity relative to air.

3. Z_W - axis lies in plane of symmetry of vehicle.

4. Y_W - axis positive right looking toward positive X_W and normal to X_W.

5. These axes are reached from X_B, Y_B, Z_B axes by rotation.

   a) About Y_B -axis through angle ($-\alpha$).

   b) About Z_B -axis through angle (β).

6. Unit vectors (i_W, j_W, k_W).

VEHICLE GEOCENTRIC AXES (X_V, Y_V, Z_V)

1. Origin at vehicle centroid.

[redacted]

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2. X_V -axis positive North.

3. Y_V -axis positive East.

4. Z_V-axis passes through earth center.

5. X_V-Z_V plane contains earth polar axis.

6. Unit vectors (i_V, j_V, k_V).

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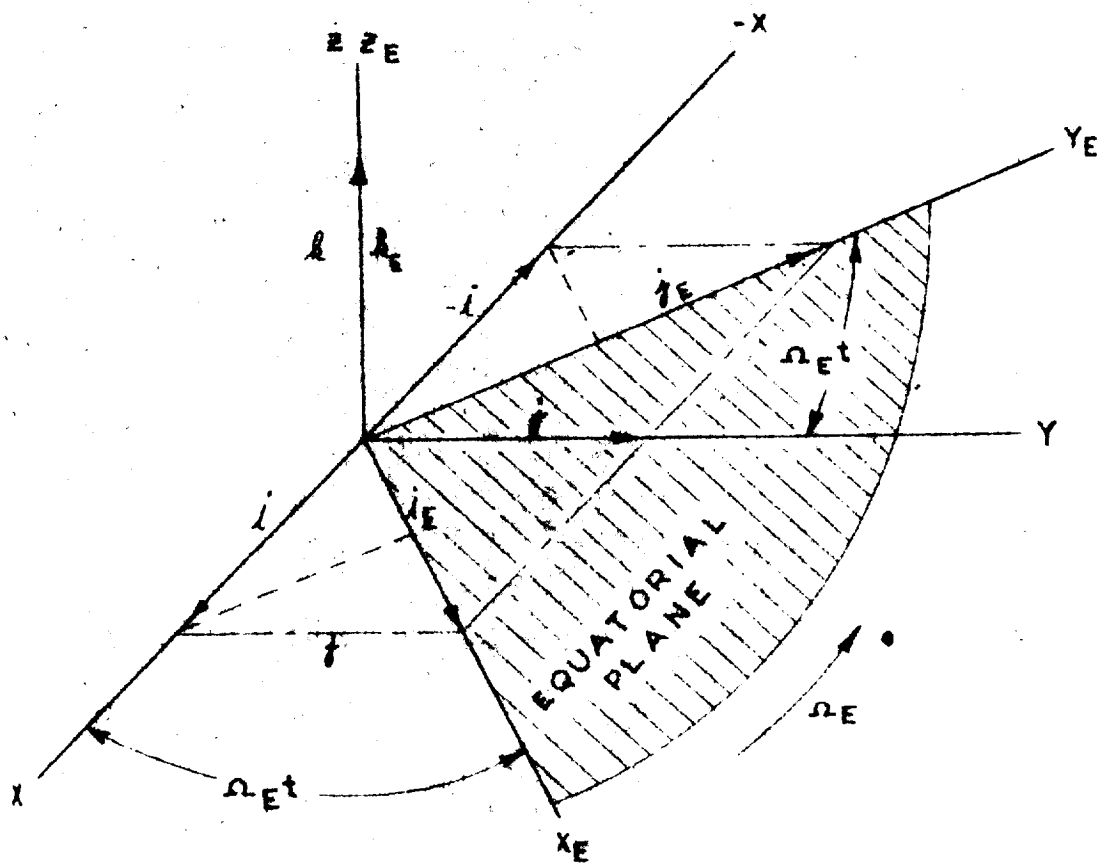
Transformation of Axes

Transformation from Inertial to Earth axes

Inertial axes: X,Y,Z

Earth axes: X_E, Y_E, Z_E

Unit Vectors: (i, j, k) And i_E, j_E, k_E)



$$i_E = i \cos\Omega_E t + j \sin\Omega_E t + k\,(o)$$

$$j_E = -i \sin\Omega_E t + j \cos\Omega_E t + k\,(o)$$

$$k_E = i\,(o) + j\,(o) + k\,(1)$$

$$\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\Omega_E t & \sin\Omega_E t & 0 \\ -\sin\Omega_E t & \cos\Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[1]} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

FIGURE 2

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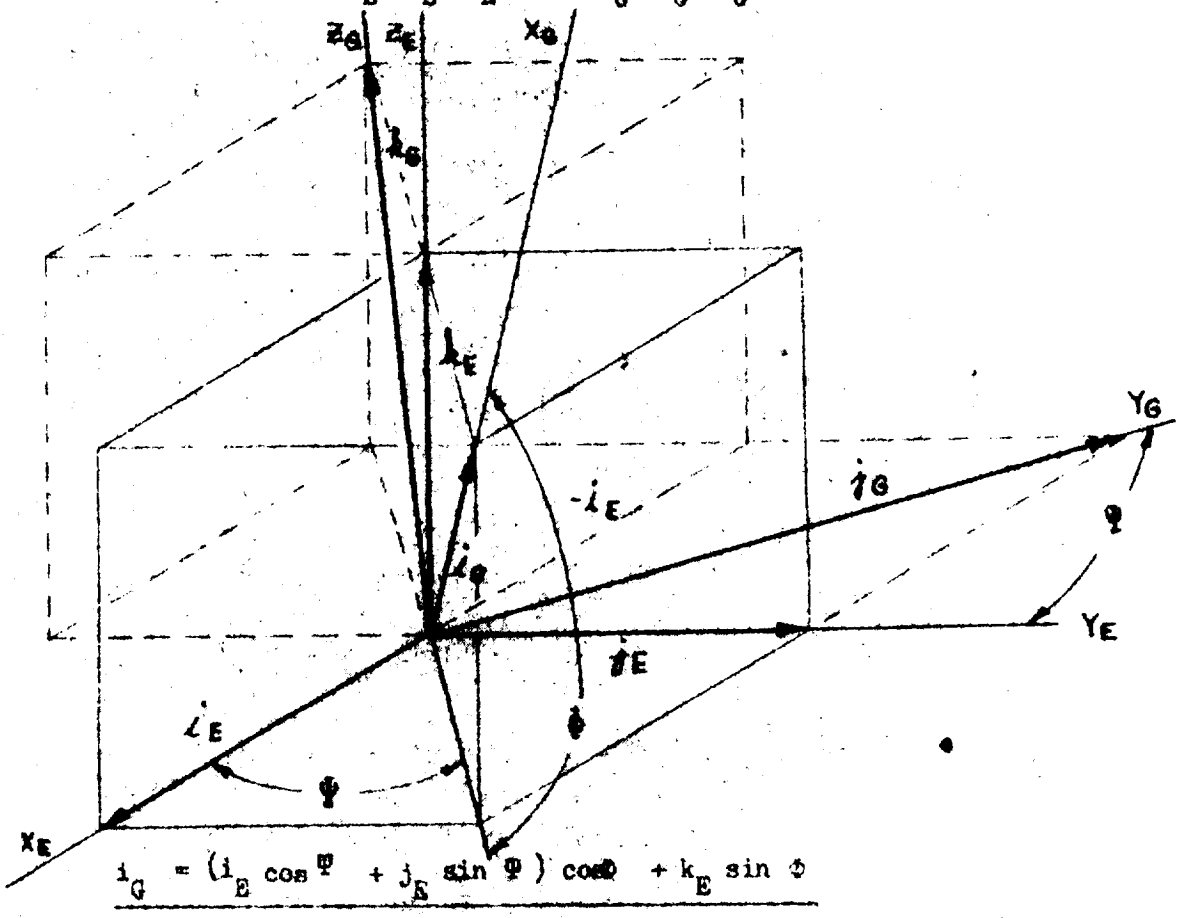
Rotation from Earth axes to Earth-vehicle geocentric axes

Earth Rotating Axes (X_E, Y_E, Z_E).

Earth-vehicle geocentric axes (X_G, Y_G, Z_G).

Unit vectors: (i_E, j_E, k_E) And (i_G, j_G, k_G).



$$i_G = (i_E \cos\Psi + j_E \sin\Psi)\cos\Phi + k_E \sin\Phi$$

$$j_G = -i_E \sin\Psi + j_E \cos\Psi + k_E(0)$$

$$k_G = (-i_E \cos\Psi - j_E \sin\Psi)\sin\Phi + k_E \cos\Phi$$

$$\begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix}}_{[3]} \underbrace{\begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[2]} \begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix}$$

FIGURE 3

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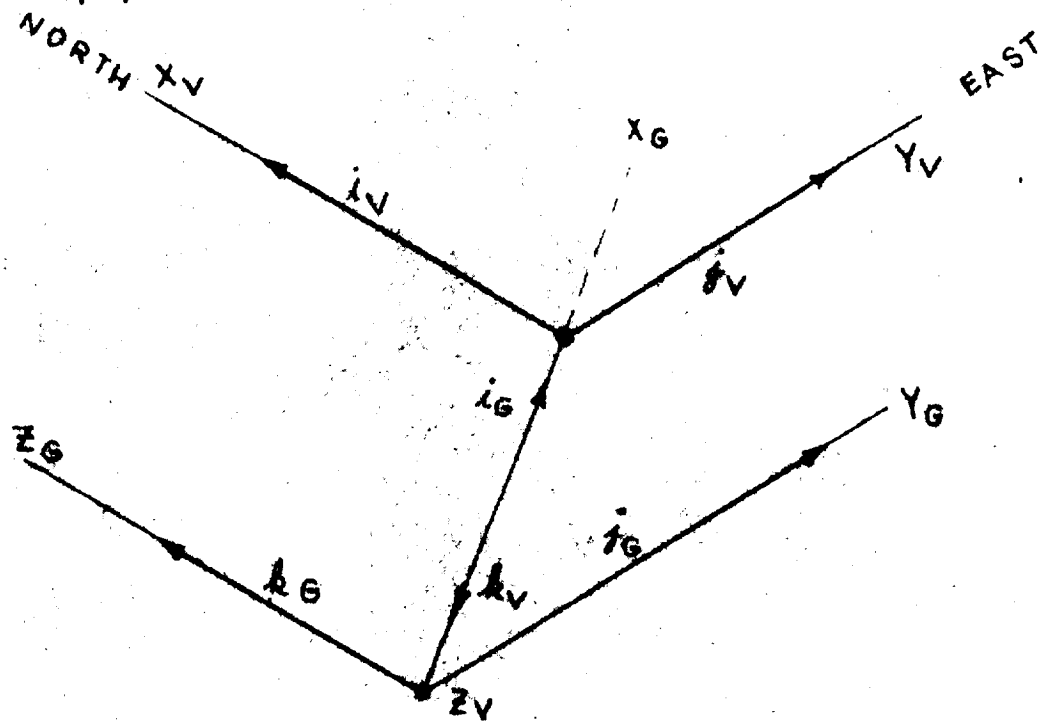
Rotation from Earth - Vehicle geocentric axes to vehicle geocentric axes

Earth - Vehicle geocentric axes (X_G, Y_G, Z_G)

Unit vectors: (i_G, j_G, k_G) and (i_V, j_V, k_V)

X_G-Z_G Plane contains Earth polar axis

X_V-Z_V Plane contains Earth polar axis



$$i_V = i_G(o) + j_G(o) + k_G(1)$$

$$j_V = i_G(o) + j_G(1) + k_G(o)$$

$$k_V = i_G(-1) + j_G(o) + k_G(o)$$

$$\begin{Bmatrix} i_V \\ j_V \\ k_V \end{Bmatrix} = \underbrace{\begin{bmatrix} o & o & 1 \\ o & 1 & o \\ -1 & o & o \end{bmatrix}}_{[4]} \cdot \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix}$$

FIGURE 4

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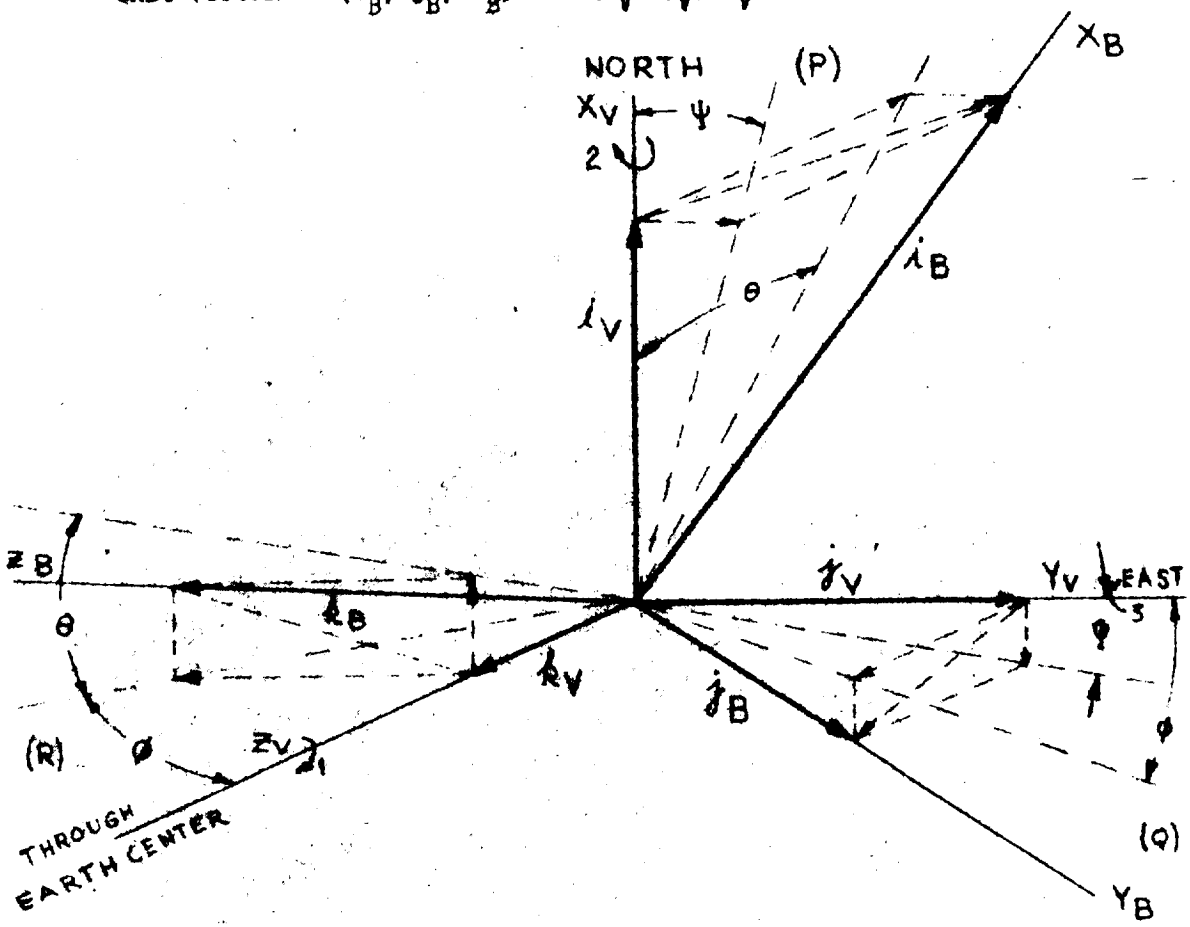
Orientation of Vehicle Body Axes Relative to the Vehicle Geocentric Axes

Vehicle body axes (X_B, Y_B, Z_B)

Vehicle geocentric axes (X_V, Y_V, Z_V)

Unit vectors: (i_B, j_B, k_B) and (i_V, j_V, k_V)



$$\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{[7]} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}}_{[6]} \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[5]} \begin{Bmatrix} i_V \\ j_V \\ k_V \end{Bmatrix}$$

FIGURE 5

[redacted]

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The orientation of vehicle body axes relative to the vehicle geocentric axes, as shown in figure 5, is accomplished in three steps as follows:

<u>Step No.1</u>: Rotation about the $Z_{B''}$-axis through angle ψ as shown in figure 5 A, where (B") denotes intermediate position of X_B, Y_B, Z_B.

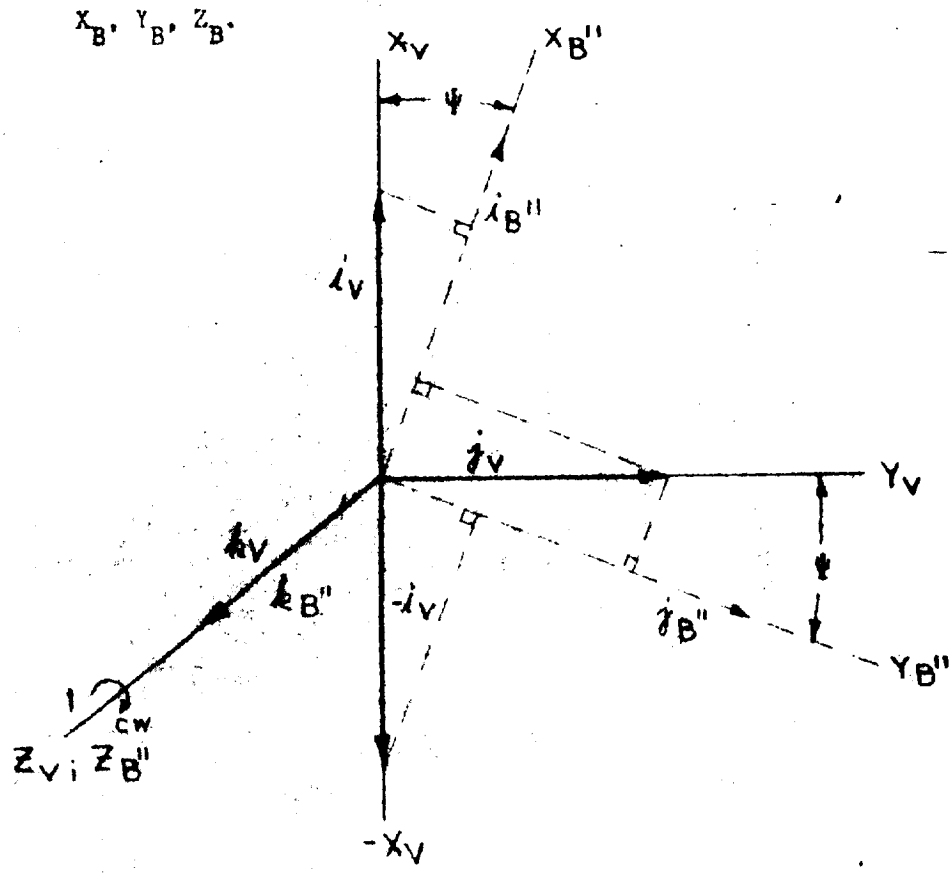

FIGURE 5 A

$$i_{B''} = i_V \cos\psi + j_V \sin\psi + k_V(o)$$

$$j_{B''} = -i_V \sin\psi + j_V \cos\psi + k_V(o)$$

$$k_{B''} = i_V(o) + j_V(o) + k_V(1)$$

$$\begin{Bmatrix} i_{B''} \\ j_{B''} \\ k_{B''} \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & o \\ -\sin\psi & \cos\psi & o \\ o & o & 1 \end{bmatrix}}_{[5]} \begin{Bmatrix} i_V \\ j_V \\ k_V \end{Bmatrix}$$

[redacted]

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Step No. 2: Rotation about the $X_{B'}$-axis through angle φ as shown in figure 5 B, where (B') denotes intermediate position of X_B, Y_B, Z_B.

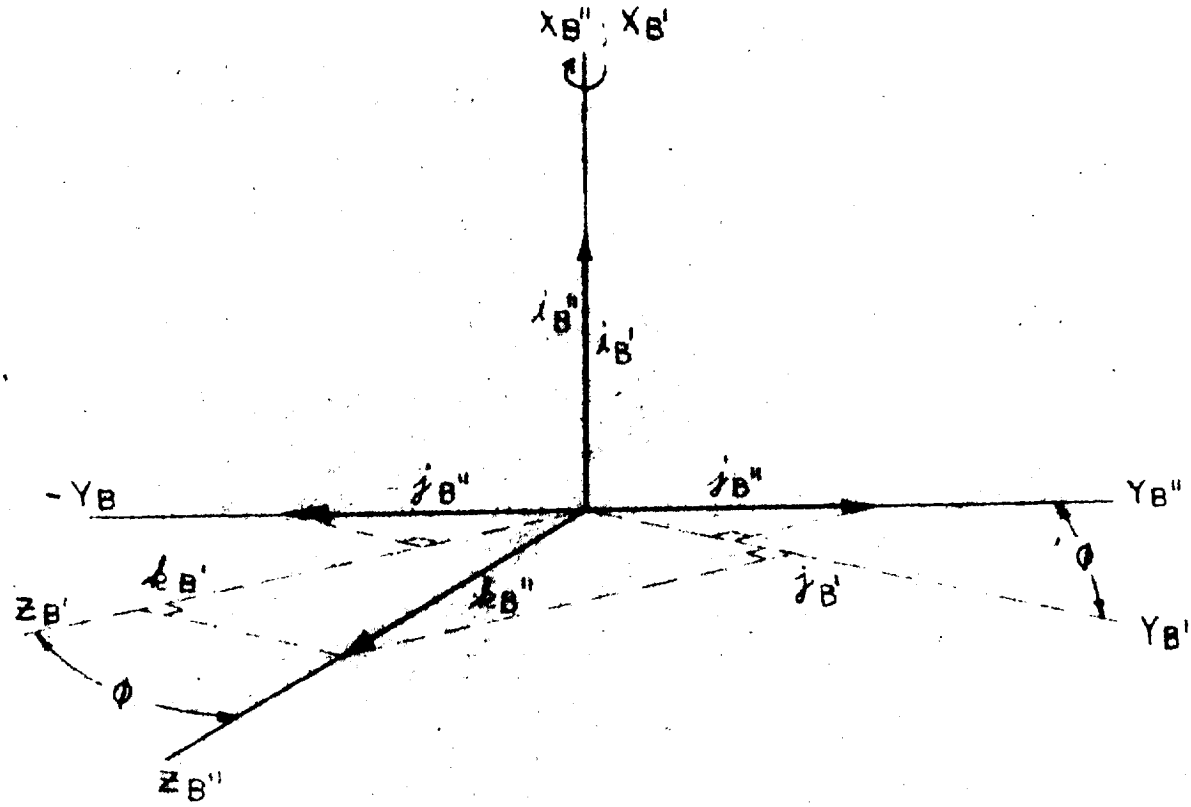

FIGURE 5 B

$$i_{B'} = i_{B''}(1) + j_{B''}(o) + k_{B''}(o)$$
$$j_{B'} = i_{B''}(o) + j_{B''}\cos\varphi + k_{B''}\sin\varphi$$
$$k_{B'} = i_{B''}(o) - j_{B''}\sin\varphi + k_{B''}\cos\varphi$$

$$\begin{Bmatrix} i_{B'} \\ j_{B'} \\ k_{B'} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & o & o \\ o & \cos\varphi & \sin\varphi \\ o & -\sin\varphi & \cos\varphi \end{bmatrix}}_{[6]} \begin{Bmatrix} i_{B''} \\ j_{B''} \\ k_{B''} \end{Bmatrix}$$

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Step No. 3: Rotation about the Y_B- axis through angle θ as shown in figure 5 C.

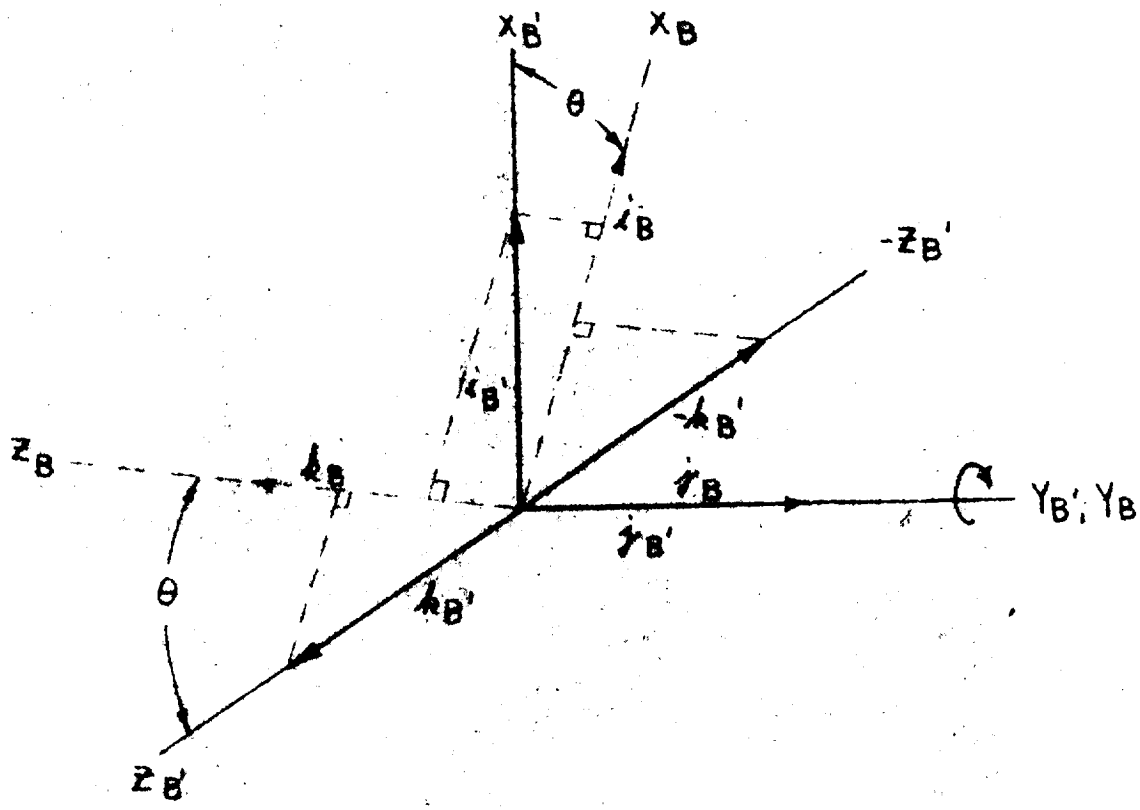

FIGURE 5 C

$$i_B = i_{B'} \cos\theta + j_{B'}(o) - k_{B'} \sin\theta$$

$$j_B = i_{B'}(o) + j_{B'}(1) + k_{B'}(o)$$

$$k_B = i_{B'} \sin\theta + j_{B'}(o) + k_{B'} \cos\theta$$

$$\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & o & -\sin\theta \\ o & 1 & o \\ \sin\theta & o & \cos\theta \end{bmatrix}}_{[7]} \begin{Bmatrix} i_{B'} \\ j_{B'} \\ k_{B'} \end{Bmatrix}$$

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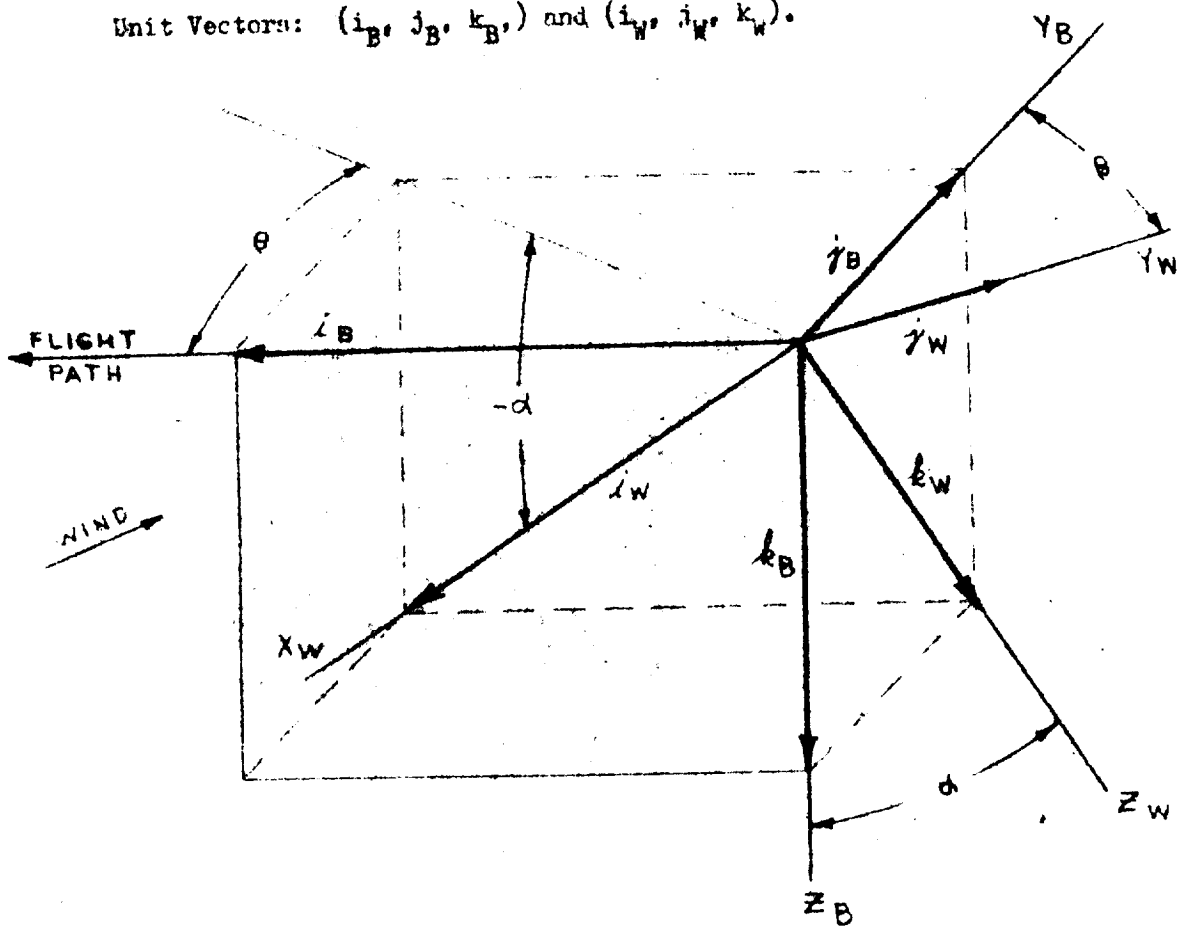
Orientation of Vehicle Body Relative to Wind Axes

Vehicle body axes (X_B, Y_B, Z_B).

Vehicle wind axes (X_W, Y_W, Z_W).

Unit Vectors: $(i_B, j_B, k_B,)$ and (i_W, j_W, k_W).



$$i_W = (i_B \cos\alpha + k_B \sin\alpha) \cos\beta + j_B \sin\beta$$

$$j_W = j_B \cos\beta - (i_B \cos\alpha + k_B \sin\alpha) \sin\beta$$

$$k_W = -i_B \sin\alpha + k_B \cos\alpha$$

$$\begin{Bmatrix} i_W \\ j_W \\ k_W \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[9]} \underbrace{\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}}_{[8]} \begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix}$$

FIGURE 6

[redacted]

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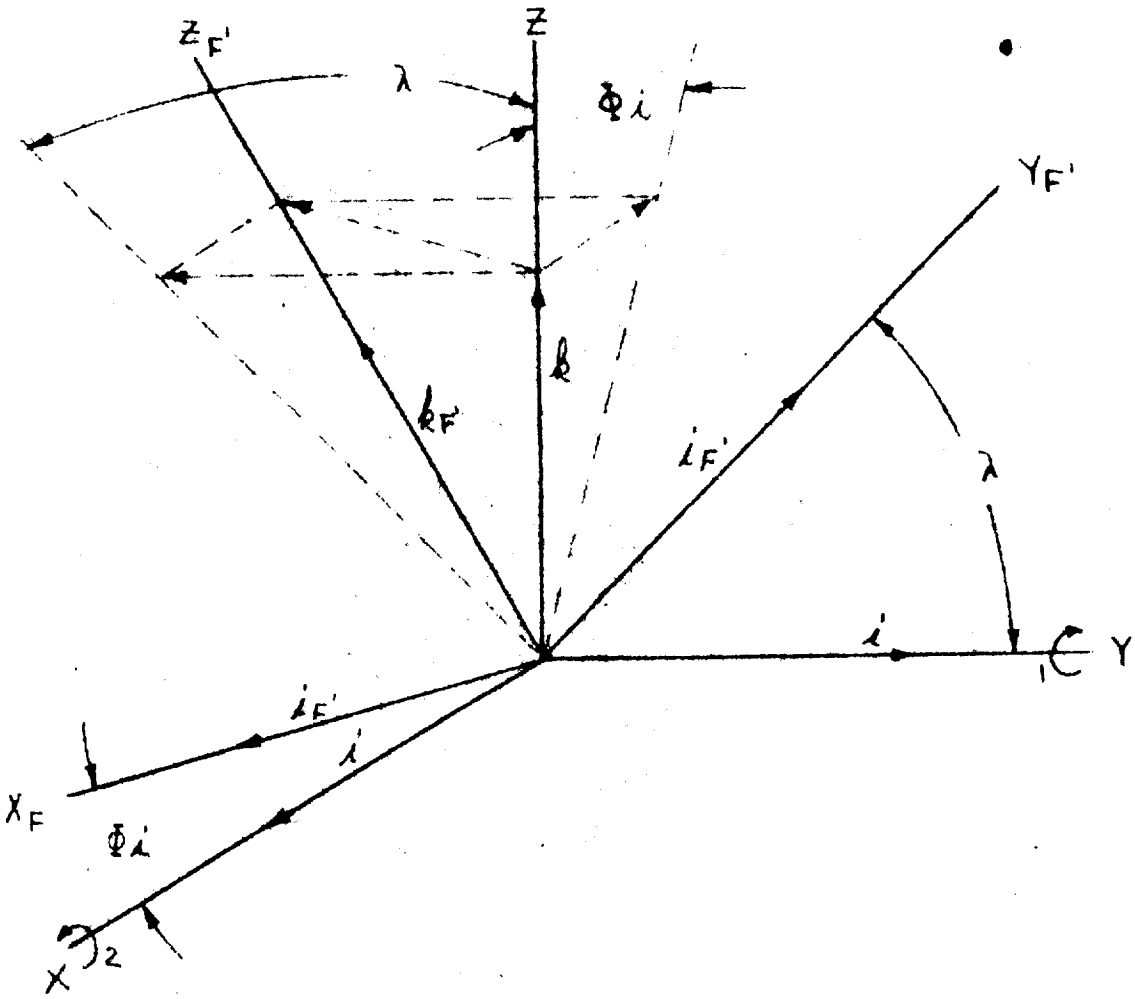
Orientation of Orbit Plane Relative to Inertial Axes

Inertial axes (X,Y,Z)

$X_{F'}$, $Y_{F'}$, $Z_{F'}$ - intermediate position of X_F, Y_F, Z_F.

Unit Vectors: (i, j, k) and $i_{F'}$, $j_{F'}$, $k_{F'}$)



Φ_i - Geocentric latitude of launch point.

λ - Angle establishing initial direction of nominal trajectory.

FIGURE 7 A

[redacted]

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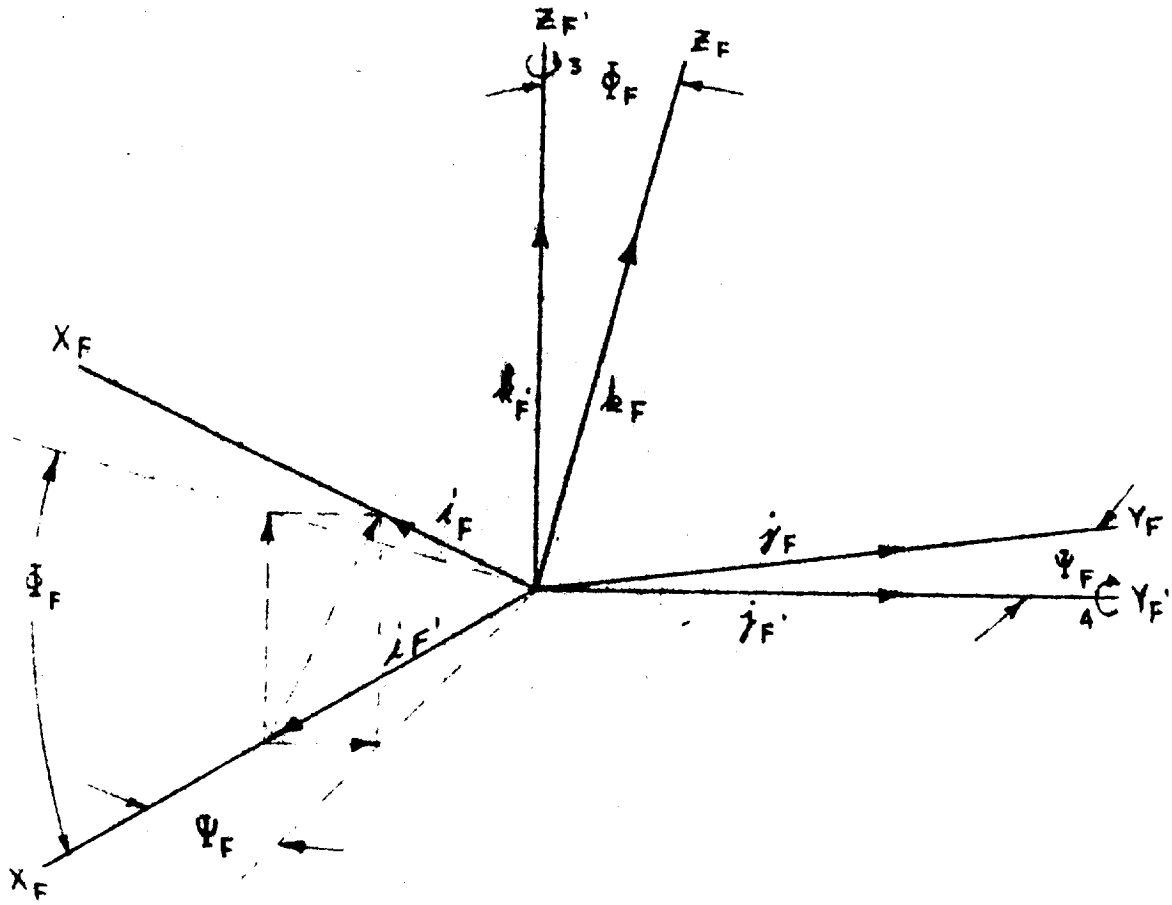
Earth-Vehicle orbit plane geocentric axes (X_F, Y_F, Z_F)

$X_{F'}$, $Y_{F'}$, $Z_{F'}$ - Intermediate position of X_F, Y_F, Z_F

Unit vectors: (i_F, j_F, k_F) and (i_F', j_F', k_F')



Ψ_F - Spherical angular coordinate of vehicle measured in nominal trajectory plane, about Z - axes.

Φ_F - Spherical angular coordinate of vehicle measured normal to nominal trajectory, about Y - axes.

FIGURE 7B

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The orientation of orbit plane relative to inertial axes is accomplished in four steps as shown in figure 7A and figure 7B and is as follows:

Step No. 1: Rotation about the Y - axis through angle Φ_i as shown in figure 8A.

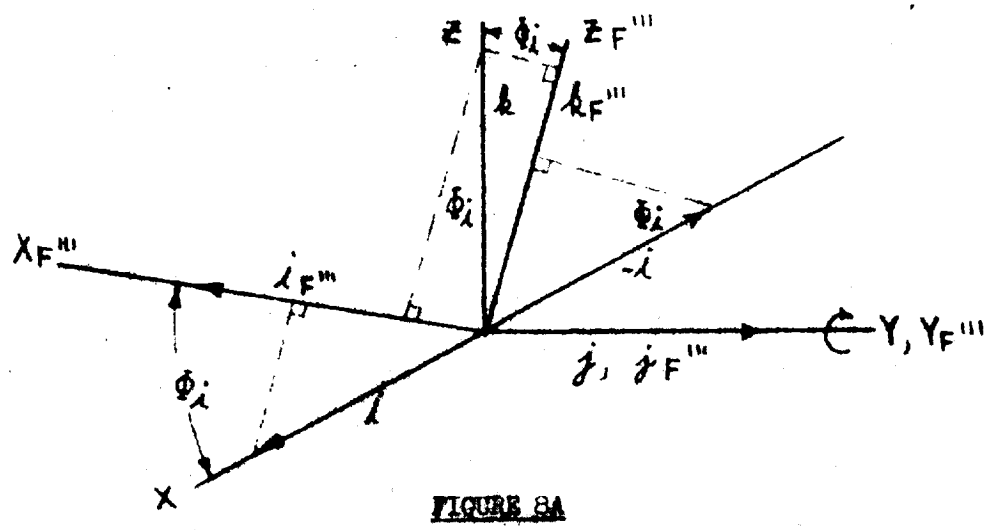

FIGURE 8A

(REFERENCE FIGURE 7A)

Computing for $i_{F'''}$, $j_{F'''}$, $k_{F'''}$ we obtain the following:

$$\begin{aligned} i_{F'''} &= i\cos\Phi_i + k\sin\Phi_i \\ j_{F'''} &= j \\ k_{F'''} &= i(-\sin\Phi_i) + k\cos\Phi_i \end{aligned} = \begin{Bmatrix} i_{F'''} \\ j_{F'''} \\ k_{F'''} \end{Bmatrix} = \begin{bmatrix} \cos\Phi_i & 0 & \sin\Phi_i \\ 0 & 1 & 0 \\ -\sin\Phi_i & 0 & \cos\Phi_i \end{bmatrix} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

Step No. 2: Rotation about the $X_{F'''}$-axis through angle λ as shown in figure 8B.

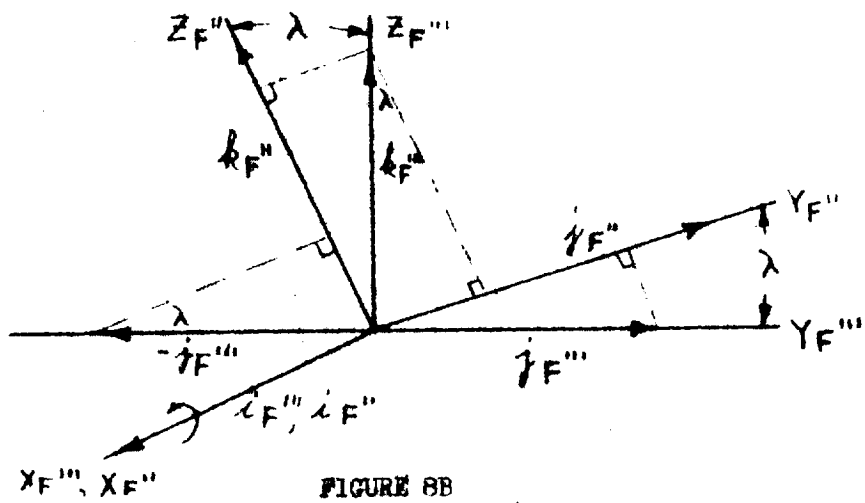

FIGURE 8B

(REFERENCE FIGURE 7A)

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Computing for $i_{F''}$, $j_{F''}$, $k_{F''}$ we obtain the following:

$$\begin{aligned} i_{F''} &= i_{F''} \\ j_{F''} &= j_{F''} \cos\lambda + k_{F''} \sin\lambda \\ k_{F''} &= -j_{F''} \sin\lambda + k_{F''} \cos\lambda \end{aligned} = \begin{Bmatrix} i_{F''} \\ j_{F''} \\ k_{F''} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & \sin\lambda \\ 0 & -\sin\lambda & \cos\lambda \end{bmatrix}}_{[11]} \begin{matrix} i_{F''} \\ j_{F''} \\ k_{F''} \end{matrix}$$

STEP NO. 3: Rotation about the $Z_{F''}$ - axis through angle Ψ_F as shown in figure 8C.

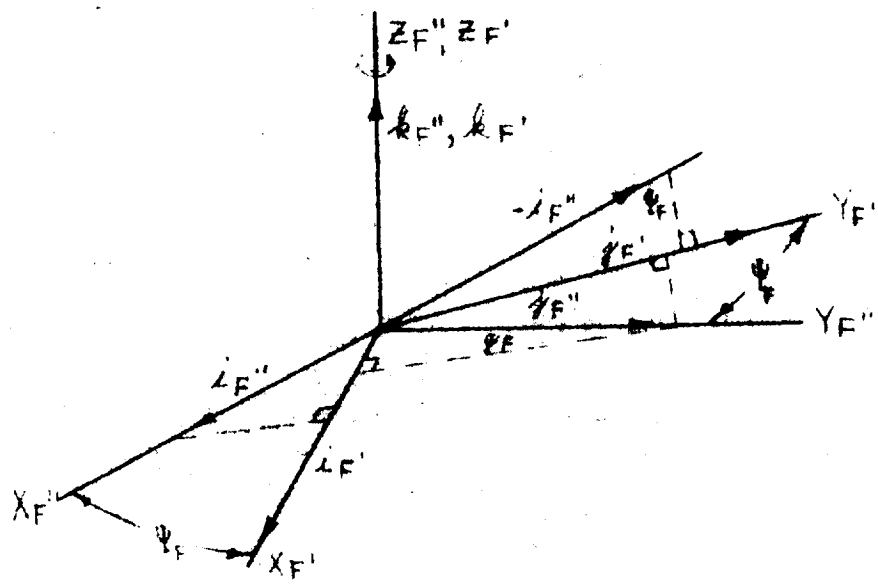

FIGURE 8C

(REFERENCE FIGURE 7B)

Computing for $i_{F'}$, $j_{F'}$, $k_{F'}$ we obtain the following:

$$\begin{aligned} i_{F'} &= i_{F''} \cos\Phi_F + j_{F''} \sin\Phi_F \\ j_{F'} &= j_{F''} \cos\Phi_F - i_{F''} \sin\Phi_F \\ k_{F'} &= k_{F''} \end{aligned} = \begin{Bmatrix} i_{F'} \\ j_{F'} \\ k_{F'} \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\Phi_F & \sin\Phi_F & 0 \\ -\sin\Phi_F & \cos\Phi_F & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[12]} \begin{Bmatrix} i_{F''} \\ j_{F''} \\ k_{F''} \end{Bmatrix}$$

STEP NO. 4: Rotation about the $Y_{F'}$ - axis thtough angle Φ_F as shown in figure 8D.

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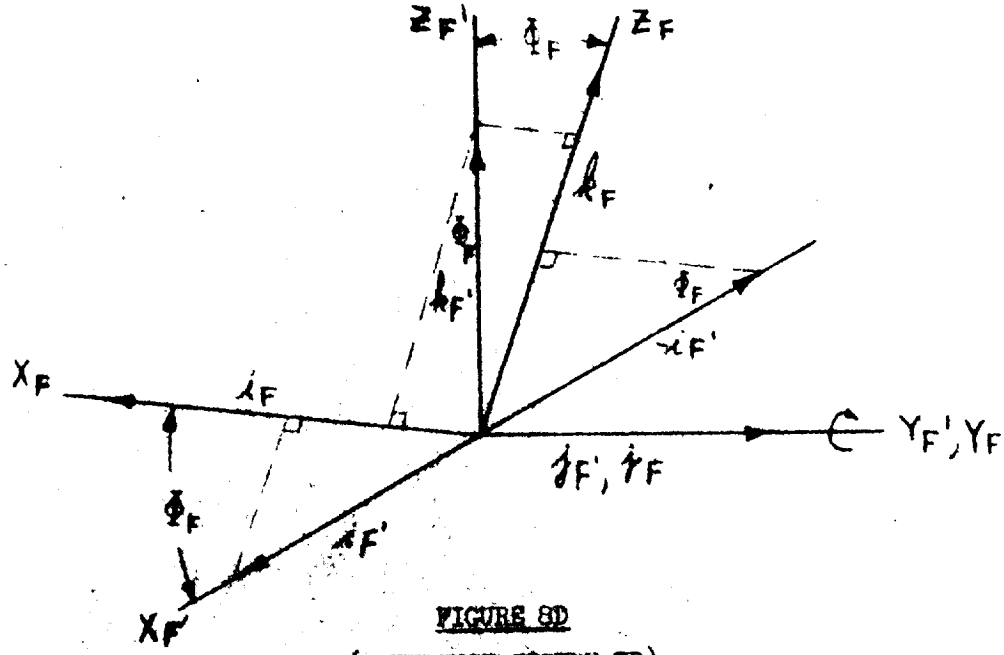

FIGURE 8D

(REFERENCE FIGURE 7B)

Computing for i_F, j_F, k_F we obtain the following:

$$\begin{aligned} i_F &= i_{F'} \cos\Phi_F + k_{F'} \sin\Phi_F \\ j_F &= j_{F'} \\ k_F &= k_{F'} \cos\Phi_F - i_{F'} \sin\Phi_F \end{aligned} \qquad \begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = \begin{bmatrix} \cos\Phi_F & 0 & \sin\Phi_F \\ 0 & 1 & 0 \\ -\sin\Phi_F & 0 & \cos\Phi_F \end{bmatrix} \begin{Bmatrix} i_{F'} \\ j_{F'} \\ k_{F'} \end{Bmatrix} \qquad [13]$$

Finally by substitution in step no. 1 through step no. 4 we obtain the following results:

STEP NO. 1

$$i_{F'''} = i \cos\Phi_i + k \sin\Phi_i$$

$$j_{F'''} = j$$

$$k_{F'''} = i(-\sin\Phi_i) + k \cos\Phi_i$$

STEP NO. 2

$$i_{F''} = i \cos\Phi_i + k \sin\Phi_i$$

$$j_{F''} = j \cos\lambda + [i(-\sin\Phi_i) + k \cos\Phi_i] \sin\lambda$$

$$k_{F''} = -j \sin\lambda + [i(-\sin\Phi_i) + k \cos\Phi_i] \cos\lambda$$

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STEP NO. 3

$$i_{F'} = (i \cos\Phi_i + k \sin\Phi_i)\cos\Psi_F + (j \cos\lambda + [i(-\sin\Phi_i) + k \cos\Phi_i] \sin\lambda)\sin\Psi_F$$

$$j_{F'} = (j \cos\lambda + [i(-\sin\Phi_i) + k \cos\Phi_i] \sin\lambda) \cos\Psi_F - (i \cos\Phi_i + k \sin\Phi_i) \sin\Psi_F$$

$$k_{F'} = -j \sin\lambda + [i(-\sin\Phi_i) + k \cos\Phi_i] \cos\lambda$$

STEP NO. 4

$$i_F = [[(i \cos\Phi_i + k \sin\Phi_i) \cos\Psi_F + (j \cos\lambda + [i(-\sin\Phi_i) + k\cos\Phi_i] \sin\lambda) \sin\Psi_F]] \cos\Phi_F + [[-j\sin\lambda + [i(-\sin\Phi_i) + k \cos\Phi_i] \cos\lambda]] \sin\Phi_F$$

$$j_F = (j\cos\lambda + [i(-\sin\Phi_i) + k\cos\Phi_i] \sin\lambda) \cos\Psi_F - (i \cos\Phi_i + k \sin\Phi_i) \sin\Psi_F$$

$$k_F = (-j\sin\lambda + [i(-\sin\Phi_i) + k\cos\Phi_i] \cos\lambda) \cos\Phi_F - [[(i \cos\Phi_i + k \sin\Phi_i) \cos\Psi_F + (j\cos\lambda + [i(-\sin\Phi_i) + k\cos\Phi_i] \sin\lambda) \sin\Psi_F]] \sin\Phi_F$$

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STEP NO. 5

CONVERSION OF STEP NO. 4 INTO MATRIX FORM

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = \begin{bmatrix} \cos\Phi_i \cos\Psi_F \cos\Phi_F & \cos\lambda \sin\Psi_F \cos\Phi_F & \sin\Phi_i \cos\Psi_F \cos\Phi_F \\ -\sin\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F & -\sin\lambda \sin\Phi_F & +\cos\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F \\ -\sin\Phi_i \cos\lambda \sin\Phi_F & & +\cos\Phi_i \cos\lambda \sin\Phi_F \\ -\sin\Phi_i \sin\lambda \cos\Psi_F & \cos\lambda \cos\Psi_F & \cos\Phi_i \sin\lambda \cos\Psi_F \\ -\cos\Phi_i \sin\Psi_F & & -\sin\Phi_i \sin\Psi_F \\ -\sin\Phi_i \cos\lambda \cos\Phi_F & -\sin\lambda \cos\Phi_F & \cos\Phi_i \cos\lambda \cos\Phi_F \\ -\cos\Phi_i \cos\Psi_F \sin\Phi_F & -\cos\lambda \sin\Psi_F \sin\Phi_F & -\sin\Phi_i \cos\Psi_F \sin\Psi_F \\ +\sin\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F & & -\cos\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F \end{bmatrix} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\Phi_F & 0 & \sin\Phi_F \\ 0 & 1 & 0 \\ -\sin\Phi_F & 0 & \cos\Phi_F \end{bmatrix}}_{[13]} \underbrace{\begin{bmatrix} \cos\Psi_F & \sin\Psi_F & 0 \\ -\sin\Psi_F & \cos\Psi_F & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[12]} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & \sin\lambda \\ 0 & -\sin\lambda & \cos\lambda \end{bmatrix}}_{[11]} \underbrace{\begin{bmatrix} \cos\Phi_i & 0 & \sin\Phi_i \\ 0 & 1 & 0 \\ -\sin\Phi_i & 0 & \cos\Phi_i \end{bmatrix}}_{[10]} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

THE ABOVE X_F, Y_F, Z_F ARE THE NEW EARTH-VEHICLE GEOCENTRIC AXES

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The Orientation of the X_F, Y_F, Z_F axes Relative to the Earth Rotating axes X_E, Y_E, Z_E is obtained as follows:

From figure 2 we have the matrix form

$$\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} = \begin{bmatrix} \cos \Omega_E t & \sin \Omega_E t & 0 \\ -\sin \Omega_E t & \cos \Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

Transpostition of this matrix, as shown in figure 7, results in the matrix form

$$\begin{Bmatrix} i \\ j \\ k \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos \Omega_E t & -\sin \Omega_E t & 0 \\ \sin \Omega_E t & \cos \Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[1]} \begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix}$$

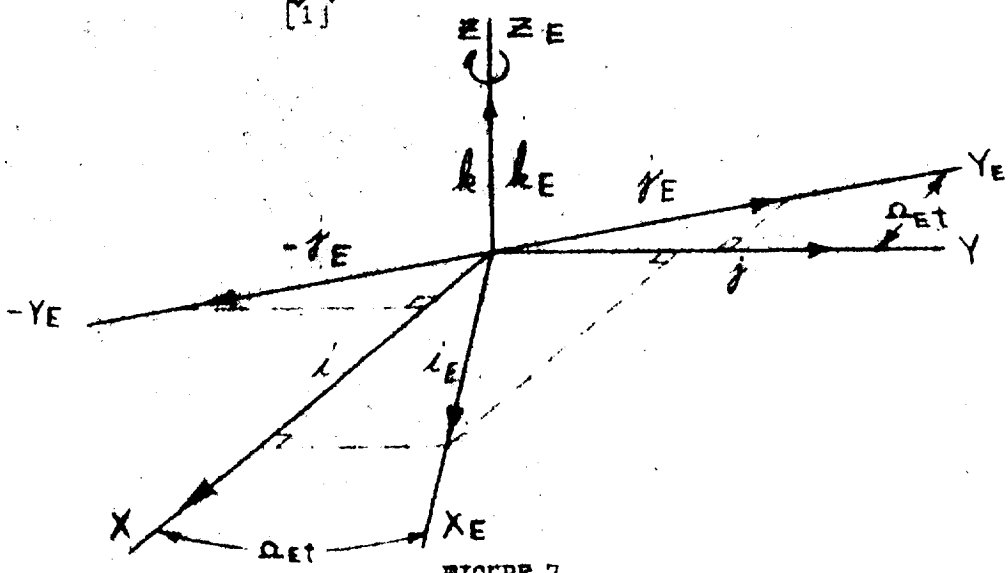

FIGURE 7

$$i = i_E \cos \Omega_E t - j_E \sin \Omega_E t + k_E(0)$$
$$j = i_E \sin \Omega_E t + j_E \cos \Omega_E t + k_E(0)$$
$$k = i_E(0) + j_E(0) + k_E(1)$$

Now performing matrix multiplication of step no. 5 matrix with the transformation matrix of figure 3 obtained above we obtain the final resultant matrix:

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$$
\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix}
=
\begin{bmatrix}
\cos\Phi_i \cos\Psi_F \cos\Phi_F \cos\Omega_E t & -\sin\Omega_E t \cos\Phi_i \cos\Psi_F \cos\Phi_F & \sin\Phi_i \cos\Psi_F \cos\Phi_F \\
-\sin\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F \cos\Omega_E t & +\sin\Omega_E t \sin\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F & +\cos\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F \\
-\sin\Phi_i \cos\lambda \sin\Phi_F \cos\Omega_E t & +\sin\Omega_E t \sin\Phi_i \cos\lambda \sin\Phi_F & +\cos\Phi_i \cos\lambda \sin\Phi_F \\
+\cos\lambda \sin\Psi_F \cos\Phi_F \sin\Omega_E t & +\cos\Omega_E t \cos\lambda \sin\Psi_F \cos\Phi_F & \\
-\sin\lambda \sin\Phi_F \sin\Omega_E t & -\cos\Omega_E t \sin\lambda \sin\Phi_F & \\
-\cos\Omega_E t \sin\Phi_i \sin\lambda \cos\Psi_F & \sin\Omega_E t \sin\Phi_i \sin\lambda \cos\Psi_F & \cos\Phi_i \sin\lambda \cos\Psi_F \\
-\cos\Omega_E t \cos\Phi_i \sin\Psi_F & +\sin\Omega_E t \cos\Phi_i \sin\Psi_F & -\sin\Phi_i \sin\Psi_F \\
+\sin\Omega_E t \cos\lambda \cos\Psi_F & +\cos\Omega_E t \cos\lambda \cos\Psi_F & \\
-\cos\Omega_E t \sin\Phi_i \cos\lambda \cos\Phi_F & \sin\Omega_E t \sin\Phi_i \cos\lambda \cos\Phi_F & \cos\Phi_i \cos\lambda \cos\Phi_F \\
-\cos\Omega_E t \cos\Phi_i \cos\Psi_F \sin\Phi_F & +\sin\Omega_E t \cos\Phi_i \cos\Psi_F \sin\Phi_F & -\sin\Phi_i \cos\Psi_F \sin\Phi_F \\
+\cos\Omega_E t \sin\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F & -\sin\Omega_E t \sin\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F & -\cos\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F \\
-\sin\Omega_E t \sin\lambda \cos\Phi_F & -\cos\Omega_E t \sin\lambda \cos\Phi_F & \\
-\sin\Omega_E t \cos\lambda \sin\Psi_F \sin\Phi_F & -\cos\Omega_E t \cos\lambda \sin\Psi_F \sin\Phi_F &
\end{bmatrix}
\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix}
$$

$$
\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix}
=
\underbrace{\begin{bmatrix} \cos\Phi_F & 0 & \sin\Phi_F \\ 0 & 1 & 0 \\ -\sin\Phi_F & 0 & \cos\Phi_F \end{bmatrix}}_{[13]}
\underbrace{\begin{bmatrix} \cos\Psi_F & \sin\Psi_F & 0 \\ -\sin\Psi_F & \cos\Psi_F & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[12]}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & \sin\lambda \\ 0 & -\sin\lambda & \cos\lambda \end{bmatrix}}_{[11]}
\underbrace{\begin{bmatrix} \cos\Phi_i & 0 & \sin\Phi_i \\ 0 & 1 & 0 \\ -\sin\Phi_i & 0 & \cos\Phi_i \end{bmatrix}}_{[10]}
\underbrace{\begin{bmatrix} \cos\Omega_E t & -\sin\Omega_E t & 0 \\ \sin\Omega_E t & \cos\Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[1]}
\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix}
$$

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Vehicle Body Axes Relative To New Geocentric Axes

The orientation of the vehicle body axes, X_B, Y_B, Z_B, relative to the new geocentric axes is determined by means of Euler Angles ψ, φ, θ similar to those used previously, except that they are now referred to the new geocentric axes. Thus, since X_F, Y_F, Z_F axes to X_V, Y_V, Z_V axes has the same result as X_G, Y_G, Z_G axes to X_V, Y_V, Z_V axes from Figure 4 and Figure 5 we obtain:

$$\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} = \underset{[7]}{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}} \underset{[6]}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}} \underset{[5]}{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{[4]}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}} \begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix}$$

Determination Of Geocentric Latitude And Longitude

In the course of the analysis it will be desirable to determine the geocentric latitude Φ and longitude Ψ. A relationship between Φ and Ψ on the one hand and Φ_F and Ψ_F on the other is thus required. We proceed to establish such a relationship by writing the transformation,

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = [13]\ [12]\ [11]\ [10]\ [1]'\ [2]'\ [3]' \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix}$$

Since $i_F = i_G$, we may write

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = [13]\ [12]\ [11]\ [10]\ [1]'\ [2]'\ [3]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

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Or

$$[1]' \; [2]' \; [3]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = [0]' \; [1]' \; [2]' \; [3]' \cdot \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

Which yields the following relations:

(1) $\underline{\cos(\Psi + \Omega_E t)\cos\Phi = \cos\Phi_i \cos\Psi_F \cos\Phi_F}$

$\underline{-\sin\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F - \sin\Phi_i \cos\lambda \sin\Phi_F}$

(2) $\underline{\sin(\Psi + \Omega_E t)\cos\Phi = \cos\lambda \sin\Psi_F \cos\Phi_F - \sin\lambda \sin\Phi_F}$

(3) $\underline{\sin\Phi = \sin\Phi_i \cos\Psi_F \cos\Phi_F + \cos\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F}$

$\underline{+ \cos\Phi_i \cos\lambda \sin\Phi_F}$

From which the coordinates Ψ and Φ may be determined.

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Model One

Model One is that model which shows the relationship of rotation between the following axes:

Inertial (X,Y,Z), Earth (X_E, Y_E, Z_E), Earth-vehicle geocentric (X_G, Y_G, Z_G), vehicle body (X_B, Y_B, Z_B) Vehicle geocentric (X_V, Y_V, Z_V), and vehicle-wind (X_W, Y_W, Z_W).

Note: Model One does not include a polar orbit.

Differential Equations For Euler Angles Of Model One

A set of differnetial equations governing the euler angles, θ, φ, ψ, is now formulated.

The angular velocity, ω_B, of the vehicle relative to the inertial axes may be written as follows:

$$\bar{\omega}_B = P\,i_B + Q\,j_B + R\,k_B \tag{1}$$

Where P, Q and R are the components about the X_B, Y_B, Z_B axes respectively.

This equation may be written in the matrix form:

$$\left\{\omega_B\right\}_B = \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} \tag{2}$$

In which the elements are components of the vector and the subscript outside the bracket identifies the axes system with respect to which these components are taken. We now proceed to relate these components to the euler angles, θ, φ, ψ, the geocentric coordinates Φ and Ψ and their derivatives. In doing this we set up an alternative representative for $\bar{\omega}_B$.

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We first write, in matrix form, the angular velocity, $\bar{\omega}_E$, of the Earth rotating axes, X_E, Y_E, Z_E, relative to the inertial axes, but resolved abouth the Earth rotating axes. Thus:

$$\bar{\omega}_E = [1] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \Omega_E \end{Bmatrix} \tag{3}$$

The angular velocity of the Earth-vehicle geocentric axes, X_G, Y_G, Z_G, relative to the inertial axes, but resolved about Earth-vehicle geocentric axes is:

$$\begin{Bmatrix} \bar{\omega}_G \end{Bmatrix}_G = [3]\ [2] \begin{Bmatrix} \bar{\omega}_E \end{Bmatrix}_E + [3]\ [2] \begin{Bmatrix} 0 \\ 0 \\ \dot{\Phi} \end{Bmatrix} + [3] \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix}$$

Or:

$$\begin{Bmatrix} \bar{\omega}_G \end{Bmatrix}_G = [3] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E + \dot{\Phi} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix} \tag{4}$$

The angular velocity of the vehicle geocentric axes, X_V, Y_V, Z_V, relative to the inertial axes, but resolved about the vehicle geocentric axes is:

$$\begin{Bmatrix} \bar{\omega}_V \end{Bmatrix}_V = [4] \begin{Bmatrix} \bar{\omega}_G \end{Bmatrix}_G = [4]\ [3] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E + \dot{\Phi} \end{Bmatrix} + [4] \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix} \tag{5}$$

Finally, the angular velocity of the vehicle body axes, X_B, Y_B, Z_B, relative to the inertial axes, but resolved abouth the vehicle body axes is:

$$\begin{Bmatrix} \bar{\omega}_B \end{Bmatrix}_B = [7]\ [6]\ [5] \begin{Bmatrix} \bar{\omega}_V \end{Bmatrix}_V + [7]\ [6]\ [5] \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix} + [7]\ [6] \begin{Bmatrix} \dot{\Phi} \\ 0 \\ 0 \end{Bmatrix}$$

$$+ [7] \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix}$$

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Or:

$$\begin{Bmatrix} \bar{\omega}_B \end{Bmatrix}_B = [7]\ [6]\ [5]\ [4]\ [3] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E + \dot{\Psi} \end{Bmatrix} + [7]\ [6]\ [5]\ [4] \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix}$$

$$+ [7]\ [6] \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix} + [7] \begin{Bmatrix} \dot{\varphi} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} \qquad (6)$$

By equating and reducing equations (2) and (6) we have:

$$\begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} = [7]\ [6]\ [5]\ [4]\ [3] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E + \dot{\Psi} \end{Bmatrix} + [7]\ [6]\ [5]\ [4] \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta & \cos\varphi \\ 0 & 1 & \sin\varphi & \\ \sin\theta & 0 & \cos\theta & \cos\psi \end{bmatrix} \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} \qquad (7)$$

Or:

$$\begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & \cos\varphi \\ 0 & 1 & \sin\varphi & \\ \sin\theta & 0 & \cos\theta & \cos\varphi \end{bmatrix}^{-1} \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} - [7]\ [6]\ [5]\ [4]\ [3] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E + \dot{\Psi} \end{Bmatrix}$$

$$- [7]\ [6]\ [5]\ [4] \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix} \qquad (8)$$

Performing the indicated matrix inversion and matrix multiplications, and reducing, we finally obtain in scalar form the equations:

$$\dot{\varphi} = \frac{P \sin\theta \tan\varphi + Q - R\cos\theta \tan\varphi + \dot{\Phi} \cos\psi \sec\psi + (\Omega_E + \dot{\Psi}) \cos\Phi}{\sin\psi \sec\varphi} \qquad (9)$$

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$$\dot{\varphi} = P\cos\theta + R\sin\theta + \dot{\Phi}\sin\psi - (\Omega_E + \dot{\Psi})\cos\Phi\cos\psi \tag{10}$$

$$\dot{\psi} = -P\sin\theta\sec\varphi + R\cos\theta\sec\varphi - \dot{\Phi}\cos\psi\tan\varphi - (\Omega_E + \dot{\Psi})\cos\Phi\sin\psi\tan\varphi + (\Omega_E + \dot{\Psi})\sin\Phi \tag{11}$$

Translational Equations of Motion

Let the radius vector from the Earth's center to the vehicle centroid be $\bar{r}$. Then the velocity vector relative to the inertial axes is given by:

$$\bar{v} = \frac{d\bar{r}}{dt} \quad \left(\frac{\partial \bar{r}}{\partial t}\right)_G + \bar{\Omega}_G \times \bar{r} \tag{12}$$

Where $\left(\frac{\partial}{\partial t}\right)_G$ denotes a partial differentiation in which i_G, j_G, k_G, are held fixed, with $\bar{r} = r\, i_G$ Where $r = |\bar{r}|$ (13)

From Equation (4):

$$\bar{\Omega}_G = (\Omega_E + \dot{\Psi})\sin\Phi\, i_G - \dot{\Phi} j_G + (\Omega_E + \dot{\Psi})\cos\Phi\, k_G \tag{14}$$

Equation (12) becomes:

$$\bar{v} = \dot{r}\, i_G + r(\Omega_E + \dot{\Psi})\cos\Phi\, j_G + r\dot{\Phi} k_G \tag{15}$$

The acceleration vector relative to the inertial axes may now be written as follows:

$$\bar{a} = \frac{d\bar{v}}{dt} \quad \left(\frac{\partial \bar{v}}{\partial t}\right)_G + \bar{\omega}_G \times \bar{v}$$

$$\bar{a} = [\ddot{r} - r\dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2\cos^2\Phi]\, i_G$$

$$+ [\{2\dot{r}(\Omega_E + \dot{\Psi}) + r\ddot{\Psi}\}\cos\Phi - 2r\dot{\Phi}(\Omega_E + \dot{\Psi})\sin\Phi]\, j_G$$

$$+ [r\ddot{\Phi} + 2\dot{r}\dot{\Phi} + r(\Omega_E + \dot{\Psi})^2\sin\Phi\cos\Phi]\, k_G \tag{16}$$

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The gravity acceleration vector, including the effect of Earth oblateness is written as follows:

$$\frac{\bar{G}}{M} = \left[\frac{-K}{r^2} + \frac{6\,\mu K R_o^2}{r^4}\left(2-3\cos^2\Phi\right)\right] i_G + \left[\frac{-12\mu\, K R_o^2}{r^4}\sin\Phi\cos\Phi\right] k_G \quad (17)$$

Where $\bar{G}$ is the gravity force vector, m is the vehicle mass, R_o is the radius of the Earth at the equator (R_o = 20,926,428 feet), and K and μ are gravity constants with the following values:

$K = 0.14077500 \times 10^{17}\ ft^3 \cdot sec^{-2}$

$6\mu = 1.638 \times 10^{-3}$

The aerodynamic, propulsive and control forces are first computed with reference to vehicle body axes, and a transformation to Earth - vehicle geocentric axes is then effected.

The aerodynamic force vector is written as follows:

$$\bar{F} = F_x\, i_B + F_y\, j_B + F_z\, k_B \quad (18)$$

$$= F_r\, i_G + F_\Psi\, j_G + F_\Phi\, k_G$$

The propulsive force vector is:

$$\bar{P} = P_x\, i_B + P_y\, j_B + P_z\, k_B \quad (19)$$

$$= P_r\, i_G + P_\Psi\, j_G + P_\Phi\, k_G$$

And the control force vector is:

$$\bar{H} = H_x\, i_B + H_y\, j_B + H_z\, k_B \quad (20)$$

$$= H_r\, i_G + H_\Psi\, j_G + H_\Phi\, k_B$$

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Summing corresponding components from equations (18), (19), and (20), The following matrix equation represents the transformation from body axes to Earth geocentric axes

$$\begin{Bmatrix} F_{r\varphi} + P_r + H_r \\ F_\varphi + P_\varphi + H_\varphi \\ F_\Phi + P_\Phi + H_\Phi \end{Bmatrix} = [4]'[5]'[6]'[7]' \begin{Bmatrix} F_x + P_x + H_x \\ F_y + P_y + H_y \\ F_z + P_z + H_z \end{Bmatrix} \quad (21)$$

where the prime denotes matrix transposition. Upon multiplication, the transformation matrix becomes:

$$[4]'[5]'[6]'[7]' = \quad (22)$$

$$\begin{bmatrix} \sin\theta \cos\varphi & -\sin\varphi & -\cos\theta \cos\varphi \\ \sin\psi \cos\theta + \cos\psi \sin\theta \sin\varphi & \cos\psi \cos\varphi & \sin\psi \sin\theta - \cos\psi \cos\theta \sin\varphi \\ \cos\psi \cos\theta - \sin\psi \sin\theta \sin\varphi & -\sin\psi \cos\varphi & \cos\psi \sin\theta + \sin\psi \cos\theta \sin\varphi \end{bmatrix}$$

The corresponding transformation matrix in the case of conventional euler angles is obtained by interchanging matrices $[6]'$ and $[7]'$ in equation (22).

Summing applied, gravity and inertia force components in the direction of vehicle-geocentric axes, we have the following three translational equations of motion in terms of the coordinates r, Ψ, and Φ:

$$\ddot{r} - r\left\{\dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2 \cos^2\Phi\right\} = \frac{-K}{r^2} + \frac{6\mu K R_0^2}{r^4}(2 - 3\cos^2\Phi) + \frac{1}{M}(F_r + P_r + H_r) \quad (23)$$

$$\left\{r\ddot{\Psi} + 2\dot{r}(\Omega_E + \dot{\Psi})\cos\Phi - 2r\dot{\Phi}(\Omega_E + \dot{\Psi})\sin\Phi = \frac{1}{M}(F_\varphi + P_\varphi + H_\varphi)\right. \quad (24)$$

$$r\ddot{\Phi} + 2\dot{r}\dot{\Phi} + r(\Omega_E + \dot{\Psi})^2 \sin\Phi \cos\Phi = -\frac{12\mu K R_0^2}{r^4} \sin\Phi \cos\Phi + \frac{1}{M}(F_\Phi + P_\Phi + H_\Phi) \quad (25)$$

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Given the shape of the oblate Earth, approximately as follows:

$$R_E = R_o(1 - f \sin^2\Phi) \tag{26}$$

Where R_E is the distance from the Earth's center to a local point on the Earth's surface, and $f = 0.0033670034$

The altitude h can be determined from the relation,

$$h = r - R_o(1 - f\sin^2\Phi) \tag{27}$$

It can be seen from equation (24) that Ψ and its derivatives are indeterminate at $\Phi = 90^\circ$, thus precluding the use of the present equations for simulation of flight over a pole.

Rotational Equations Of Motion

The rotational equations of motion developed on the basis of moment equilibrium about the body axes are the same as those familiar in aircraft analysis. For a vehicle with the $X_b - Z_b$ plane a plane of symmetry, they are:

$$-\left[\dot{P} I_{xx} - (I_{yy} - I_{zz}) QR - I_{xz}(\dot{R} + PQ)\right] + L + T_x + J_x = 0 \tag{28}$$

$$-\left[\dot{Q} I_{yy} - (I_{zz} - I_{xx}) RP - I_{xz}(R^2 - P^2)\right] + M + T_y + J_y = 0 \tag{29}$$

$$-\left[\dot{R} I_{zz} - (I_{xx} - I_{yy}) PQ - I_{xz}(\dot{P} - QR)\right] + N + T_z + J_z = 0 \tag{30}$$

Where I_{xx}, I_{yy}, I_{zz}, I_{xz} are moments and products of inertia referred to the body axes; L, M, And N are components of the aerodynamic moment: T_x, T_y, and T_z are components of the propulsive moment; and J_x, J_y, and J_z are components of the control moment, all referred to the X_b, Y_b, and Z_b axes respectively.

Angle of Attack and Angle of Sideslip

With appropriate modification of equation (15), the velocity of the

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vehicle relative to the Earth rotating axes, X_E, Y_E, Z_E, becomes:

$$\bar{V}_E = \dot{r}\, i_G + r\,\dot{\Psi}\cos\Phi\, j_G + r\,\dot{\Phi}\, k_G \tag{31}$$

If we now allow a wind velocity given by

$$\bar{V}_w = V_{w_r}\, i_G + V_{w\Psi}\, j_G + V_{w\Phi}\, k_G \tag{32}$$

The velocity of the vehicle relative to the air becomes

$$\bar{V}a = Va_r\, i_G + Va_\Psi\, j_G + Va_\Phi\, k_G \tag{33}$$

Where $Va_r = \dot{r} - Vw_r$

$Va_\Psi = r\,\dot{\Psi}\cos\Phi - Vw_\Psi$

$Va_\Phi = r\,\dot{\Phi} - Vw_\Phi$

And the magnitude of this velocity is:

$$Va = \sqrt{Va_r^2 + Va_\Psi^2 + Va_\Phi^2}\,. \tag{34}$$

A transformation to body axes may be effected as follows:

$$\begin{Bmatrix} Va_x \\ Va_y \\ Va_z \end{Bmatrix} = [7]\,[6]\,[5]\,[4] \begin{Bmatrix} Va_r \\ Va_\Psi \\ Va_\Phi \end{Bmatrix} \tag{35}$$

Noting that

$\bar{V}a = Va\, i_w$

A transformation from wind axes to body axis is given by:

$$\begin{Bmatrix} Va_x \\ Va_y \\ Va_z \end{Bmatrix} = [8]'\,[9]' \begin{Bmatrix} Va \\ 0 \\ 0 \end{Bmatrix} = Va \begin{Bmatrix} \cos\alpha\ \cos\beta \\ \sin\beta \\ \sin\alpha\ \cos\beta \end{Bmatrix} \tag{37}$$

Equating the right hand sides of equations (35) and (37), we have:

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$$\begin{Bmatrix} \cos\alpha & \cos\beta \\ \sin\beta & \\ \sin\alpha & \cos\beta \end{Bmatrix} = [7]\,[6]\,[5]\,[4] \begin{Bmatrix} V_{a_r}/V_a \\ V_{a_\Psi}/V_a \\ V_{a_\Phi}/V_a \end{Bmatrix} \tag{38}$$

from which α and β may be determined. The transformation matrix in this equation is seen to be the transpose of the matrix given in equation (22). The corresponding transformation matrix in the case of conventional euler angles is obtained by interchanging matrices [6] and [7] in equation (38).

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Model Two

This model shows the relationship of rotation between the following axes: Inertial (X, Y, Z), Earth (X_E, Y_E, Z_E), Original Earth-vehicle geocentric (X_G, Y_G, Z_G), new Earth-vehicle geocentric (X_F, Y_F, Z_F), vehicle body (X_B, Y_B, Z_B), vehicle geocentric (X_V, Y_V, Z_V) and wind vehicle (X , Y , Z ,).

Note: Model Two includes a polar orbit

Differential Equations for the Euler Angles:

Following the notations and equations (1) thru (1') on page , and , the angular velocity of the X_F, Y_F, Z_F, axes relative to the inertial axes, but resolved about the X_F, Y_F, Z_F axes, is given by:

$$\left\{\bar{\omega}_F\right\}_F = [13]\begin{Bmatrix}0\\0\\\dot{\Phi}_F\end{Bmatrix} + \begin{Bmatrix}0\\-\Phi_F\\0\end{Bmatrix} \qquad (39)$$

Similarly, the angular velocity of the body axes relative to the inertial axes, but resolved about the body axes is given by:

$$\left\{\bar{\omega}_B\right\}_B = [1][6][5][4][13]\begin{Bmatrix}0\\0\\\dot{\Phi}_F\end{Bmatrix} + [7][6][5][4]\begin{Bmatrix}0\\-\dot{\Phi}_F\\0\end{Bmatrix}$$

$$+ [7][6]\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix} + [7]\begin{Bmatrix}\dot{\varphi}\\0\\0\end{Bmatrix} + \begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix} \qquad (40)$$

But,

$$\left\{\bar{\omega}_B\right\}_B = \begin{Bmatrix}P\\Q\\R\end{Bmatrix} \qquad (41)$$

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Therefore, equating the right-hand sides of equations (40) and (41) and proceeding as on page , and , we obtain the relations:

$$\dot{\theta} = P \sin\theta \tan\varphi + Q - R\cos\theta \tan\varphi + \dot{\Phi}_F \cos\Phi_F \sec\varphi + \dot{\Psi}_F \sin\Phi \sec\varphi \quad (42)$$

$$\dot{\varphi} = P\cos\theta + R\sin\theta + \dot{\Phi}_F \sin\psi - \dot{\Psi}_F \cos\Phi_F \cos\psi \quad (43)$$

$$\dot{\psi} = -P\sin\theta \sec\varphi + R\cos\theta \sec\varphi - \dot{\Phi}_F \cos\psi \tan\varphi - \dot{\Psi}_F \cos\Phi_F \sin\psi \tan\varphi + \dot{\Psi}_F \sin\Phi_F \quad (44)$$

Translational Equations of Motion:

Following the analysis on pages thru , the radius vector, velocity and acceleration may be written as follows:

$$\bar{r} = r\, i_F \quad (45)$$

$$\bar{V} = \dot{r}\, i_F + r\dot{\Psi}_F \cos\Phi_F\, j_F + r\dot{\Phi}_F\, k_F \quad (46)$$

$$\bar{a} = \left[\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2\Phi_F)\right] i_F + \left[(2\dot{r} + \dot{\Psi}_F + r\ddot{\Psi}_F)\cos\Phi_F - 2r\dot{\Phi}_F \dot{\Psi}_F \sin\Phi_F\right] j_F + \left[r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin\Phi_F \cos\Phi_F\right] k_F \quad (47)$$

The gravity acceleration vector must now be resolved into components along the X_F, Y_F, Z_F axes. This necessitates a transformation from the original to the new Earth-vehicle geocentric axes. Since the X_F axis is coincident with the X_G- axis, this transformation involves simply a rotation about the X_G axis, through an angle which we will denote by λ_F.

Thus:

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & \sin\lambda_F \\ 0 & -\sin\lambda_F & \cos\lambda_F \end{bmatrix} \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix} \quad (48)$$

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And

$$\frac{\bar{G}}{M} = \left[\frac{-K}{r^2} + \frac{6\mu K R_o^2}{r^4}\left(2-3\cos^2\Phi\right)\right] i_F$$

$$+ \left[-\frac{12K\mu R_o^2}{r^4}\sin\lambda_F \sin\Phi\cos\Phi\right] j_F$$

$$+ \left[-\frac{12K\mu R_o^2}{r^4}\cos\lambda_F \sin\Phi\cos\Phi\right] k_F \qquad (49)$$

From equation on page and equation (48) we can write

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & \sin\lambda_F \\ 0 & -\sin\lambda_F & \cos\lambda_F \end{bmatrix} = [13]\,[12]\,[11]\,[10]\,[1]'\,[2]'\,[3]' \qquad (50)$$

Or, inverting,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & -\sin\lambda_F \\ 0 & \sin\lambda_F & \cos\lambda_F \end{bmatrix} = [3]\,[2]\,[1]\,[10]'\,[11]'\,[12]'\,[13]' \qquad (51)$$

This may be rearranged in the form,

$$[2]'\,[3]' \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & -\sin\lambda_F \\ 0 & \sin\lambda_F & \cos\lambda_F \end{bmatrix} = [1]\,[10]'\,[11]'\,[12]'\,[13]' \qquad (52)$$

Equating the elements in the last row and second and third columns of the product matrices on both sides of equation (52), we have finally,

$$\cos\Phi \sin\lambda_F = -\sin\Phi_i \sin\Psi_F + \cos\Phi_i \sin\lambda_F \cos\Psi_F \qquad (53)$$

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$$\cos\Phi \cos\lambda_F = -\sin\Phi_i \sin\Phi_F \cos\Psi_F - \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F + \cos\Phi_i \cos\lambda \cos\Phi_F \tag{54}$$

We can now substitute equations (53) and (54) into equation (49) to obtain the gravity acceleration in the form,

$$\frac{\bar{G}}{M} = \left[-\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4}(2-3\cos^2\Phi)\right] i_F + \left[\frac{12K\mu R_o^2}{r^4}\sin\Phi(\sin\Phi_i \sin\Psi_F - \cos\Phi_i \sin\lambda \cos\Psi_F)\right] j_F + \left[\frac{12K\mu R_o^2}{r^4}\sin\Phi(\sin\Phi_i \sin\Phi_F \cos\Psi_F + \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F - \cos\Phi_i \cos\lambda \cos\Phi_F)\right] k_F \tag{55}$$

The aerodynamic, propulsive and control forces are again determined with reference to body axes and then transformed to vehicle geocentric axes, X_F, Y_F, Z_F, using the transformation matrix of equation (21), but recognizing that the euler angles here are referred to the X_F, Y_F, Z_F, axes.

The translational equations of motion may now be written as follows:

$$\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2\Phi_F) = -\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4}(2-3\cos^2\Phi) + \frac{1}{M}(F_r + P_r + H_r) \tag{56}$$

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$$r\ddot{\Psi}_F \cos\Phi_F + 2\dot{r}\dot{\Psi}_F \cos\Phi_F - 2r\dot{\Phi}_F\dot{\Psi}_F \sin\Phi_F$$

$$= \frac{12K\mu R_o^2}{r^4} \sin\Phi\,(\sin\Phi_i \sin\Psi_F - \cos\Phi_i \sin\lambda \cos\Psi_F)$$

$$+ \frac{1}{M}(F_{\Psi_F} + P_{\Psi_F} + H_{\Psi_F}) \qquad (27)$$

$$r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin\Phi_F \cos\Phi_F = \frac{12K\mu R_o^2}{r^4} \sin\Phi\,(\sin\Phi_i \sin\Phi_F$$

$$\cos\Psi_F + \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F - \cos\Phi_i \cos\lambda \cos\Phi_F)$$

$$+ \frac{1}{M}(F_{\Phi_F} + P_{\Phi_F} + H_{\Phi_F}) \qquad (28)$$

Upon solution of these equations, the altitude may again be determined from equation (27). Because of the increased complexity of the equations, even with r as a basic variable, a reformulation to introduce h as a basic variable is not carried out. However, the substitution indicated in the equation, $r = R_o + \sigma r$, may be made.

Rotational Equations of Motion:

The change in coordinate system does not affect the rotational equations of motion, and equations (28), (29) and (30) remain applicable.

Angle of Attack and Angle of Sideslip:

The velocity of the vehicle relative to the Earth rotating axes, X_E, Y_E, Z_E, may be written:

$$\bar{V}_E = \left(\frac{\sigma r}{\sigma t}\right)_F + \bar{\omega}_{FE} \times \bar{r} \qquad (49)$$

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In which $\bar{r}$ is given by equation (45), $\left(\frac{d}{dt}\right)_F$ denotes a partial differentiation in which i_F, j_F, k_F, are held fixed, and $\bar{\omega}_{FE}$ is the rotational velocity of the X_F, Y_F, Z_F frame relative to the X_E, Y_E, Z_E frame. In terms of components about the X_F, Y_F, Z_F axis $\bar{\omega}_{FE}$ is given by,

$$\left\{\bar{\omega}_{FE}\right\}_F = [13]\begin{Bmatrix}0\\0\\\dot{\Psi}_F\end{Bmatrix} + \begin{Bmatrix}0\\-\dot{\Phi}_F\\0\end{Bmatrix} - [13]\,[12]\,[11]\,[10]\begin{Bmatrix}0\\0\\\Omega_E\end{Bmatrix} \tag{60}$$

Or, after performing the indicated matrix multiplications,

$$\begin{aligned}\bar{\omega}_{FE} = {} & \Big\{\dot{\Psi}_F \sin\Phi_F - \Omega_E \sin\Phi_i \cos\Psi_F \cos\Phi_F \\ & -\Omega_E \cos\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F - \Omega_E \cos\Phi_i \cos\lambda \sin\Phi_F\Big\}\, i_F \\ & +\Big\{-\dot{\Phi}_F + \Omega_E \sin\Phi_i \sin\Psi_F - \Omega_E \cos\Phi_i \sin\lambda \cos\Psi_F\Big\}\, j_F \\ & +\Big\{\dot{\Psi}_F \cos\Phi_F + \Omega_E \sin\Phi_i \cos\Psi_F \sin\Phi_F \\ & +\Omega_E \cos\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F - \Omega_E \cos\Phi_i \cos\lambda \cos\Phi_F\Big\}\, k_F\end{aligned} \tag{61}$$

If we allow a wind velocity given by,

$$\bar{V}_w = V_{w_r}\, i_F + V_{w\Psi_F}\, j_F + V_{w\Phi_F}\, k_F \tag{62}$$

The velocity of the vehicle relative to the air becomes,

$$\bar{V}_a = \left(\frac{d\bar{r}}{dt}\right)_F + \bar{\omega}_{FE} \times \bar{r} - \bar{V}_w \tag{63}$$

Introducing equations (45), (61) and (62) into equations (63), we have finally,

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$$\bar{V}_a = V_{a_r}\, i_F + V_{a_{\Psi_F}}\, j_F + V_{a_{\Phi_F}}\, k_F \tag{64}$$

Where

$$V_{a_r} = \dot{r} - V_{w_r}$$

$$V_{a_{\omega_F}} = r(\dot{\Psi}_F \cos\Phi_F + \Omega_E \sin\Phi_i \cos\Psi_F \sin\Phi_F$$

$$+ \Omega_E \cos\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F - \Omega_E \cos\Phi_i \cos\lambda \cos\Phi_F) - V_{\omega_{\Psi_F}}$$

$$V_{a_{\Phi_F}} = r(\dot{\Phi}_F - \Omega_E \sin\Phi_i \sin\Psi_F + \Omega_E \cos\Phi_i \sin\lambda \cos\Psi_F) - V_{\omega_{\Phi_F}}$$

And

$$V_a = \sqrt{V^2_{a_r} + V^2_{a_{\Psi_F}} + V^2_{a_{\Phi_F}}} \tag{65}$$

α and β may now be determined as previously, on page with equation (38) being replaced by the relation

$$\begin{Bmatrix} \cos\alpha & \cos\beta \\ & \sin\beta \\ \sin\alpha & \cos\beta \end{Bmatrix} = [7][6][5][4] \begin{Bmatrix} V_{a_r}/V_a \\ V_{a_{\Psi_F}}/V_a \\ V_{a_{\Phi_F}}/V_a \end{Bmatrix} \tag{66}$$

Where again the transformation matrix is the transpose of the matrix in equation (42).

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Section II: LAUNCH ESCAPE PROPULSION SYSTEM

Introduction:

In the event of abort conditions of the Apollo mission during the pre-orbit phase, the launch escape system is utilized to lift the command module from the booster to safety.

Trajectory:

In general, the trajectory resulting from the system utilization is circular with an altitude of approximately 4000 ft. and a range of approximately 4000 ft.

Propulsion:

As shown in Figure 1, the primary propulsion of the system is produced by a single engine through four nozzels placed at an angle of 33° to the forward thrust vector. Since the extension of this vector passes through the C.G. at all times, it makes an angle with the X_B axis of $3^{\circ}15'$. As the fuel burns and the C.G. shifts to its final position C.G. The vector extended continues essentially to pass through the C.G. at any time T.

The initial C.G. is 6.78' from impulse point or center of the coordinate axes. As the fuel burns, the C.G. shifts to a point 10.70' away, measured along the X_B - axis. A solid fuel motor is located in the upper region of the escape tower normal to the X_B - axis. For further simplification, the propulsion vector of this motor is considered directed parallel to the Z_B - axis.

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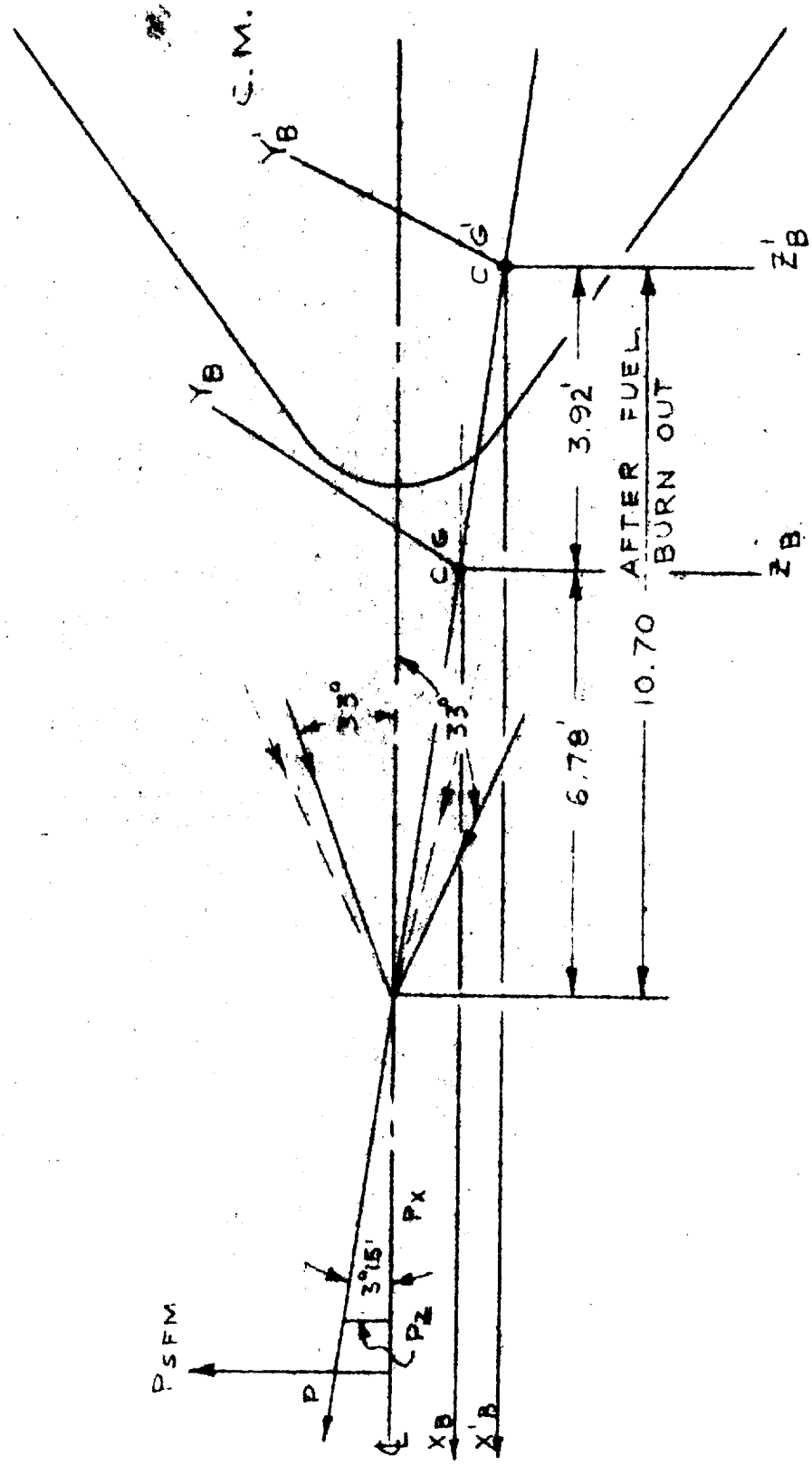

LAUNCH ESCAPE PROPULSION SYSTEM

FIGURE 1

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Consequently the propulsive forces along the three axis can be written:

$$P_{LES} = P \cos 3^{\circ}15'$$

$$P_{YLES} = 0$$

$$P_{ZLES} = P \sin 3^{\circ}15' + P_{SPM}$$

Torque:

The torque about the axes are:

$$T_{XLES} = 0$$

$$T_{YLES} = [6.78' + 3.92' f(FB)] P_{LES} \sin 3^{\circ}15' + [R_{XSPM} + 3.92' f(FB)] P_{SPM}$$

$$T_{ZLES} = 0$$

The propulsive forces above enter the equations of motion along with other forces already present, and alter the propulsive forces and moments commutation which in turn alters the translational equations, velocity in air computation, and geodetic commutation (See the Functional Block Diagram).

Similarly the torques above enter the Body axes Rotational Equations that in turn alter the Euler angles and consequently affect the equations of motion. Together the equations of motion reflect the new trajectory produced by the Launch Escape System at any time after lift - off until the tower is jettisoned.

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The tower jettison motor thrust vector is not considered because it does not affect the C.M. trajectory after the tower is released.

However it is intended to operate as part of the normal abort sequence as well as serve as a backup to the main launch escape motor for tower jettison during the second stage boost when abort is deemed unnecessary. In both cases it separates the tower from the command module and provides a safe clearance.

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Section III: LUNAR ENVIRONMENT

Axes and Coordinates (See Figure 1)

Inertial Axes (X, Y, Z)

1. Origin at lunar center
2. Z - axis coincident with lunar polar axis positive north
3. X - Z plane, parallel to Earth's X- Z plane
4. Unit vectors (i, j, k)

Lunar Axes (X_L, Y_L, Z_L)

1. Origin at lunar center
2. Z_L - axis coincident with lunar polar axis, positive north
3. Initial position coincident with inertial axes
4. Unit vectors (i_L, j_L, k_L)

Lunar-Vehicle selenocentric axes (X_s, Y_s, Z_s)

1. Origin at lunar center
2. X_s - axis passes through vehicle centroid
3. X_s - Z_s plane contains lunar polar axis
4. Y_s - axis lies in lunar equatorial plane
5. Z_s - axis is positive north of lunar equatorial plane
6. Unit vectors (i_s, j_s, k_s)

Vehicle Body Axes (X_{BL}, Y_{BL}, Z_{BL})

1. Origin at vehicle centroid
2. X_{BL} - Z_{BL} plane coincident with plane os symmetry of vehicle

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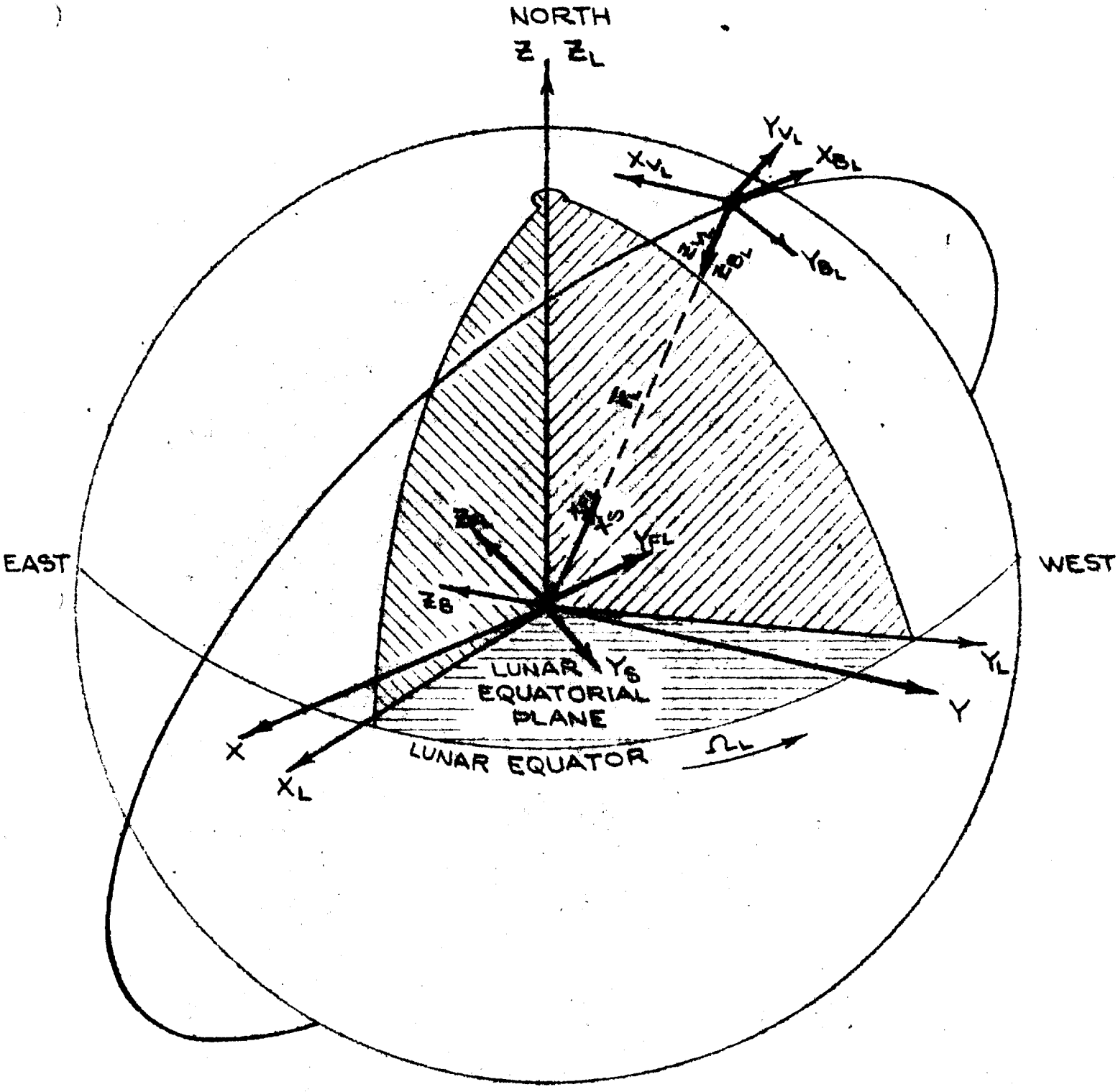

LUNAR ENVIRONMENT

FIGURE 1

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3. Z_{BL} - axis positive downward, normal to X_{BL} - axis
4. X_{BL} - axis positive forward
5. Y_{BL} - axis positive right looking forward and normal to X_{BL} - axis
6. Unit vectors (i_B, j_B, k_B)

Vehicle Selenocentric axes (X_{VL}, Y_{VL}, Z_{VL})

1. Origin at vehicle centroid
2. X_{VL} - axis positive north
3. Y_{VL} - axis positive west
4. Z_{VL} - axis passes through lunar center
5. X_{VL} - Z_{VL} - plane contains lunar polar axis
6. Unit vectors (i_{VL}, j_{VL}, k_{VL})

Lunar-Vehicle Orbit Plane Selenocentric Axes (X_{FL}, Y_{FL}, Z_{FL})

1. Origin at moon's center
2. X_{FL} - axes passes through vehicle centroid
3. X_{FL}-Y_{FL} plane is nominal trajectory plane
4. Y_{FL} points essentially in direction of flight
5. Z_{FL} points left when looking in direction of flight
6. Unit vectors, (i_{FL}, j_{FL}, k_{FL})

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Lunar Model

This model is formulated in a similar manner as Model Two of the Earth Environment Section. The aerodynamic and oblateness terms are omitted.

This model has a polar orbit capability. With a change in symbolism the equations presented here in are analogous to those, in Model Two.

Differential Equations For Euler Angles

$$\dot{\theta}_L = P \sin\theta_L \operatorname{Tan}\varphi_L + Q - R\cos\theta_L \operatorname{Tan}\varphi_L + \dot{\Phi}_{FL}\cos\psi_L \sec\varphi_L \tag{1}$$

$$+ \dot{\Psi}_{FL}\cos\Phi_{FL}\sin\psi_L \sec\varphi_L$$

$$\dot{\varphi}_L = P\cos\theta_L + R\sin\theta_L + \dot{\Phi}_{FL}\sin\psi_L - \dot{\Psi}_{FL}\cos\Phi_{FL}\cos\psi_L \tag{2}$$

$$\dot{\psi}_L = P\sin\theta_L \sec\varphi_L + R\cos\theta_L \sec\varphi_L - \dot{\Phi}_{FL}\cos\psi_L \operatorname{Tan}\varphi_L$$

$$- \dot{\Psi}_{FL}\cos\Phi_{FL}\sin\psi_L \operatorname{Tan}\varphi_L + \dot{\Psi}_{FL}\sin\Phi_{FL} \tag{3}$$

Translational Equations of Motion

$$\ddot{r}_L - r_L(\dot{\Phi}^2_{FL} + \dot{\Psi}^2_{FL}\cos^2\Phi_{FL}) = -\frac{K_L}{r_L^2} + \frac{1}{M}(P_{r_L} + H_{r_L}) \tag{4}$$

$$r_L\ddot{\Psi}_{FL}\cos\Phi_{FL} + 2\dot{r}_L\dot{\Psi}_{FL}\cos\Phi_{FL} - 2r_L\dot{\Phi}_{FL}\dot{\Psi}_{FL}\sin\Phi_{FL}$$

$$= \frac{1}{M}(P_{\Psi_{FL}} + H_{\Psi_{FL}}) \tag{5}$$

$$r_L\ddot{\Phi}_{FL} + 2\dot{r}_L\dot{\Phi}_{FL} + r_L\dot{\Psi}^2_{FL}\sin\Phi_{FL}\cos\Phi_{FL} = \frac{1}{M}(P_{\Phi_{FL}} + H_{\Phi_{FL}}) \tag{6}$$

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Rotational Equations of Motion

The rotational equations of motion are the same as those formulated for Earth Environment - Model One with the L, M, and N terms omitted.

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Section IV: MIDCOURSE ENVIRONMENT

Gravity Vectors

The vehicle to earth gravity vector is described in model two of the earth environment as:

$$\frac{-K\, i_F}{r} \quad \text{where } K = g_e \; R_0^2, \qquad K = 1.4077500 \times 10^{16} \text{ ft}^3 \text{ sec}^{-2} \tag{1}$$

Similarly the vehicle to moon vector can be described as:

$$\frac{-K_L}{r_L^2}\, i_{F_L} \quad \text{where } K_L = g_L \; R_0^2, \qquad K_L = 1.773 \times 10^{14} \text{ ft}^3 \text{ sec}^{-2} \tag{2}$$

Gravitational Equation

Relative to the "F" frame, these equations are summed up to form a three-body (earth, moon, vehicle) gravitational equation:

$$\frac{\bar{G}}{M} = \left(\frac{-K}{r^2}\right) i_F - \left(\frac{K_L}{r_L^2}\right) i_{FL} \tag{3}$$

Translational Equations

Following the analysis in Part I, Model II, the moon referenced translation equation of acceleration may be written as:

$$\bar{a}_L = \left[\ddot{r}_L - r_L\left(\dot{\Phi}^2_{FL} + \dot{\Psi}^2_{FL} \cos^2 \Phi_{FL}\right)\right] i_{FL} + \left[\left(2\dot{r}_L \dot{\Phi}_{FL} + r_L \ddot{\Phi}_{FL}\right) \cos \Phi_{FL} - 2r_L \dot{\Phi}_{FL} \dot{\Psi}_{FL} \sin \Phi_{FL}\right] j_{FL} + \left[r_L \dot{\Phi}_{FL} + 2\dot{r}_L \dot{\Phi}_{FL} + r_L \dot{\Psi}^2_{FL} \sin \Phi_{FL} \cos \Phi_{FL}\right] k_{FL} \tag{4}$$

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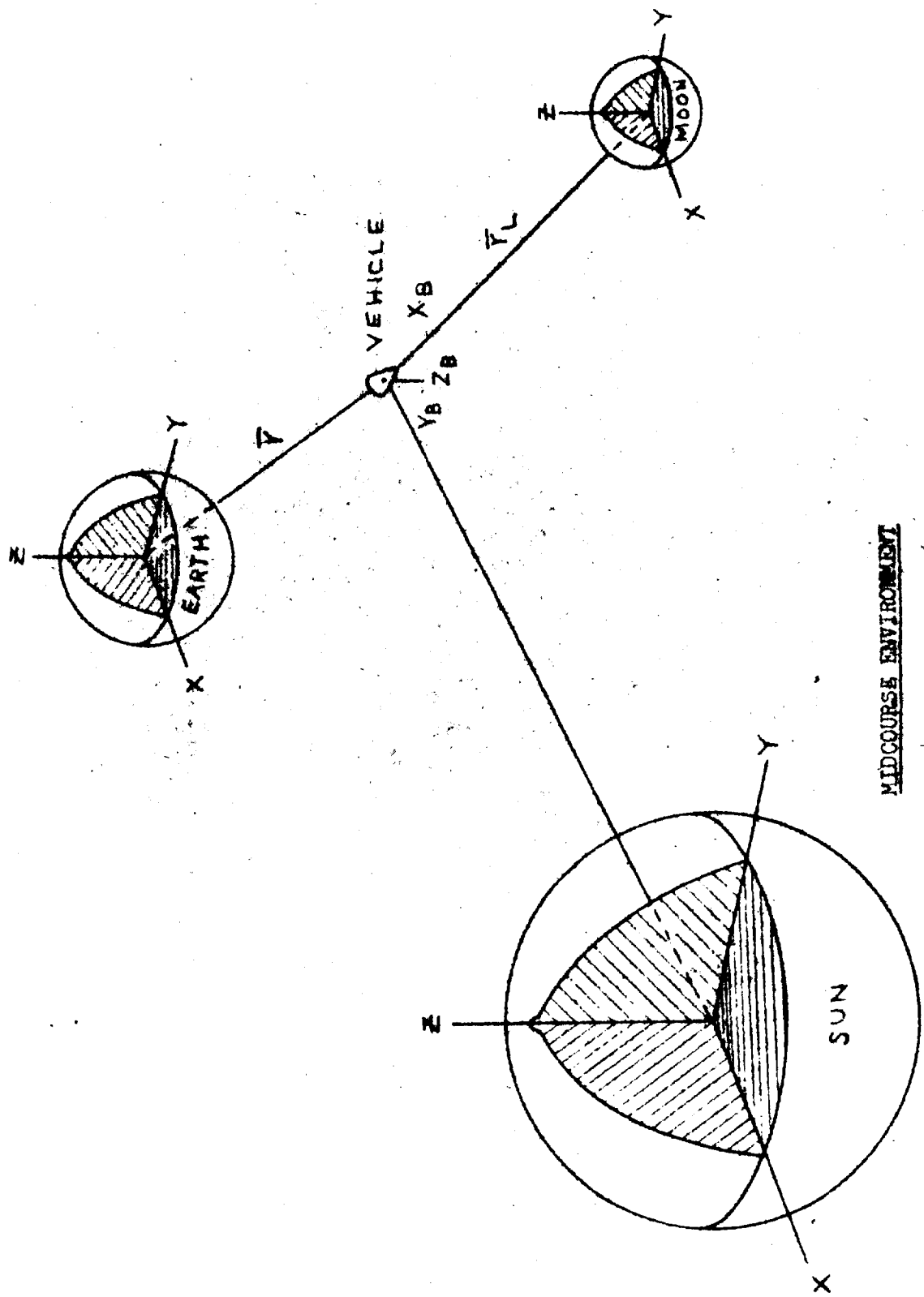

MIDCOURSE ENVIRONMENT

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Combing equations (3) and (4) above with the translating equations from model two of the earth environment, and disregarding aerodynamic and earth oblateness forces, the midcourse translational equations of motion may now be written as follows:

$$\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2 \Phi_F) + \ddot{r}_F - r_L(\dot{\Phi}_{FL}^2 + \dot{\Psi}_{FL}^2 \cos^2 \Phi_{FL}) = -\frac{K}{r^2} \tag{5}$$

$$+ \frac{K_L}{r_L^2} + \frac{1}{M}(P_r + H_r) + \frac{1}{M}(P_{r_L} + H_{r_L})$$

$$(2\dot{r}\dot{\Psi}_F + r\ddot{\Psi}_F)\cos\Phi_F - 2r\dot{\Phi}_F\dot{\Psi}_F \sin\Phi_F + (2\dot{r}_L\dot{\Psi}_{FL} + r_L\ddot{\Psi}_{FL})\cos\Phi_{FL} - 2r_L$$

$$\dot{\Phi}_{FL}\dot{\Psi}_{FL}\sin\Phi_{FL} = \frac{1}{M}(P\Psi_F + H\Psi_F) + \frac{1}{M}(P\Psi_{FL} + H\Psi_{FL}) \tag{6}$$

$$r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin\Phi_F \cos\Phi_F + r_L\ddot{\Phi}_{FL} + 2\dot{r}_L\dot{\Phi}_{FL} + r_L\dot{\Psi}_{FL}^2 \sin\Phi_{FL}\cos\Phi_{FL}$$

$$= \frac{1}{M}(P\Phi_F + H\Phi_F) + \frac{1}{M}(P\Phi_{FL} + H\Phi_{FL}) \tag{7}$$

To simplify the solution of these equations, they may be seperated into two sets as follows:

$$\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2 \Phi_F) = \frac{-K}{r^2} + \frac{1}{M}(P_r + H_r)$$

$$(2\dot{r}\dot{\Psi}_F + r\ddot{\Psi}_F)\cos\Phi_F - 2r\dot{\Phi}_F\dot{\Psi}_F \sin\Phi_F = \frac{1}{M}(P\Psi_F + H\Psi_F) \tag{8}$$

$$r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin\Phi_F \cos\Phi_F = \frac{1}{M}(P\Phi_F + H\Phi_F)$$

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$$\ddot{r}_L - r_L(\dot{\Phi}^2_{FL} + \dot{\Psi}^2_{FL}\cos^2\Phi_{FL}) = -\frac{K_L}{r_L^2} + \frac{1}{M}(P_{r_L} + H_{r_L})$$

$$(2\dot{r}_L\dot{\Psi}_{FL} + r_L\ddot{\Psi}_{FL})\cos\Phi_{FL} - 2r_L\dot{\Phi}_{FL}\dot{\Psi}_{FL}\sin\Phi_{FL} = \frac{1}{M}(P\Psi_{FL} + H\Psi_{FL}) \qquad (9)$$

$$r_L\ddot{\Phi}_{FL} + 2\dot{r}_L\dot{\Phi}_{FL} + r_L\dot{\Psi}^2_{FL}\sin\Phi_{FL}\cos\Phi_{FL} = \frac{1}{M}(P\Phi_{FL} + H\Phi_{FL})$$

Therefore, the midcourse equations are merely a summation of Model II Earth and Lunar Environments with aerodynamic and oblateness gravitational terms eliminated. Consequently, both Earth and Lunar referenced computations proceed simultaneously and the results along with ephemeris data establishing Stellar Locations, Earth-Moon and Earth-Sun Vectors produces a complete dynamic description of translunar and transearth flight.

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Section V: RENDEZVOUS

Introduction:

Rendezvous may be considered to take place in three parts. First the launch and boost phase, in which the ferry vehicle is placed in an orbit, as nearly identical as possible, to that of the target vehicle. Secondly, the orbital transfer phase, in which the ferry vehicle propels itself into the trajectory and general vicinity of the target vehicle. Thirdly, the homing phase, in which the ferry vehicle is guided visually or electronically into docking position with the appropriate on board propulsion systems. The equations presented in this report will deal with the transfer and homing phases of the satellite rendezvous problem, because of their application to the Apollo Project.

Orbital Transfer

To effect an orbital transfer requires a two step impulse to change the ferry vehicle from its initial orbit with elements p_1, e_1, ω_1, i_1, and Ω_1 to the target vehicle orbit with elements p_2, e_2, ω_2, i_2 and Ω_2. If the terminal orbit is used as a reference, $i = o$. Also, the symmetry of the central gravity field makes the orientation of the line of nodes of no value, therefor $\Omega_1 = o$. Consequently, since the value of Ω_2 is arbitrary, the problem is specified by seven parameters: an initial orbit with elements p_1, e_1, ω_1, i_1, and a terminal orbit with elements p_2, e_2, ω_2.

The three parameter family of transfers between two points can be determined using the rule: Given two points not collinear with the origin, and a quantity "p" greater than zero, their exists a unique conic passing

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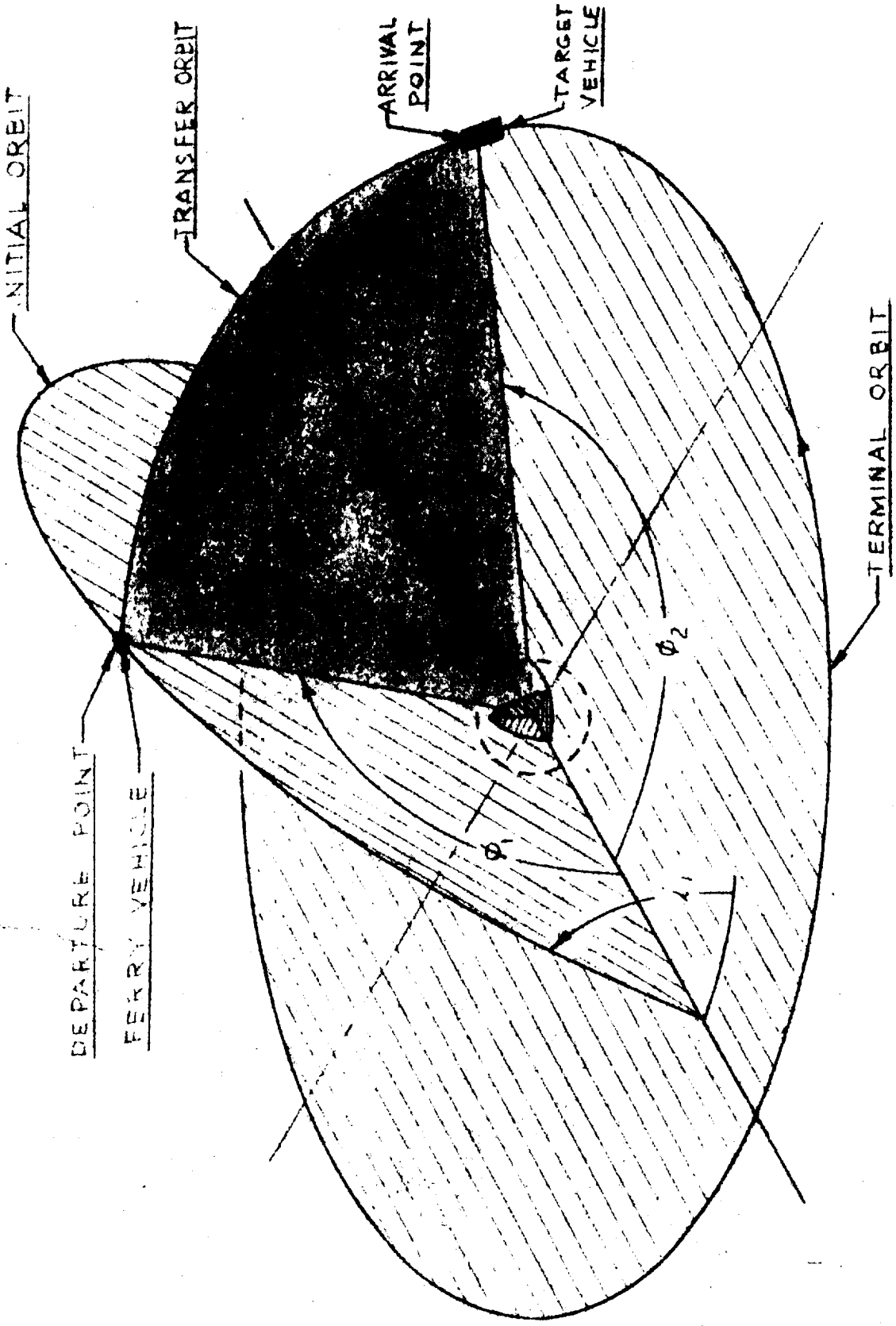

TRANSFER GEOMETRY
FIGURE 1

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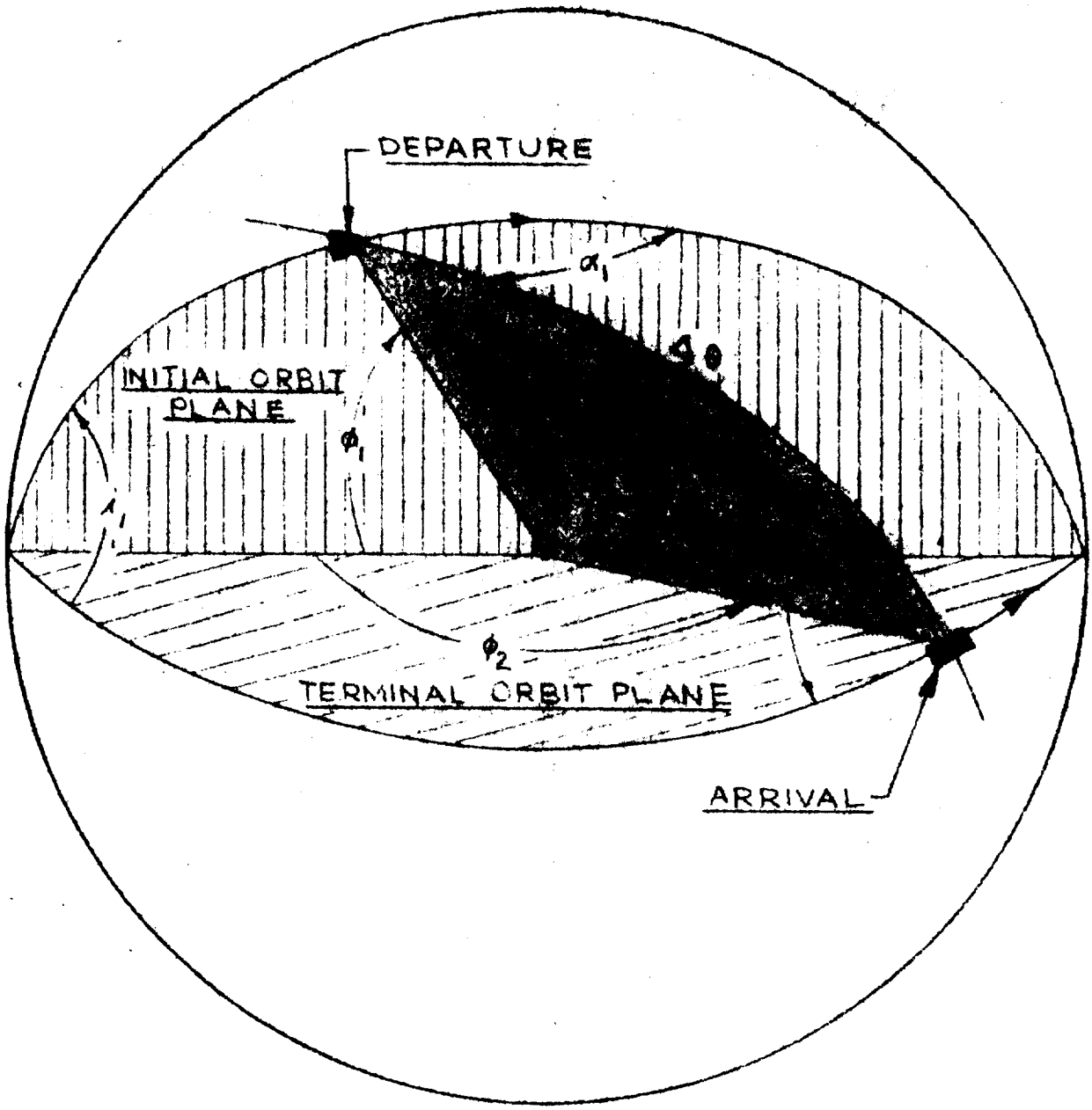

PROJECTION ON UNIT SPHERE

FIGURE 2

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through the two points, with the origin as focus, and "p" as its simi-latus rectum.

Let $r_1\,\theta_1$ and $r_2\,\theta_2$ be the two points. It is necessary to obtain $e \geq 0$ and ω such that:

$$r_1 = \frac{p}{1 + e\cos(\theta_1 - \omega)}; \qquad r_2 = \frac{p}{1 + e\cos(\theta_2 - \omega)} \tag{1}$$

And:

$$\frac{p}{r_1} - 1 = e\cos(\theta_1 - \omega); \qquad \frac{p}{r_2} - 1 = e\cos(\theta_2 - \omega) \tag{2}$$

Let $\frac{p}{r_1} - 1 = A$ And $\frac{p}{r_2} - 1 = B$,

Then:

$$A = e\cos(\theta_1 - \omega); \qquad B = e\cos(\theta_2 - \omega) \tag{3}$$

Or:

$$\frac{A}{e} = \cos(\theta_1 - \omega); \qquad \frac{B}{e} = \cos(\theta_2 - \omega) \tag{4}$$

Using the spherical trigonometric identity:

$$\sin(\theta_1 - \omega)\sin(\theta_2 - \theta_1) = \cos(\theta_1 - \omega)\cos(\theta_2 - \theta_1) - \cos(\theta_2 - \omega) \tag{5}$$

$$\sin(\theta_1 - \omega)\sin(\theta_2 - \theta_1) = \frac{A}{e}\cos(\theta_2 - \theta_1) - \frac{B}{e} \tag{6}$$

$$\sin(\theta_1 - \omega) = \frac{\frac{A}{e}\cos(\theta_2 - \theta_1) - B}{\sin(\theta_2 - \theta_1)} \tag{7}$$

Using equations (4) and (7):

$$\sin^2(\theta_1 - \omega) + \cos^2(\theta_1 - \omega) = 1$$

$$\frac{A^2\cos^2(\theta_2 - \theta_1) - 2AB\cos(\theta_2 - \theta_1) + B^2}{e^2\sin^2(\theta_2 - \theta_1)} + \frac{A^2}{e^2} = 1$$

Let $\theta_2 - \theta_1 = \Delta\theta$ Then:

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$$\frac{A^2 - A^2 \sin^2\Delta\theta - 2AB\cos\Delta\theta + B^2 + A^2 \sin^2\Delta\theta}{\sin^2\Delta\theta} = e^2$$

$$e = \sqrt{\frac{A^2 - 2AB\cos\Delta\theta + B^2}{\left|\sin\Delta\theta\,2\right|}} \tag{8}$$

Equations (4), (7), and (8) form the solution.

The conic demonstrated by these equations is unique, but as an orbit it may be traversed in either of two directions. However, we shall assume that the path which traverses an angle of less than 180° shall be selected in all cases in that it is the shortest route.

Transfer Geometry (See Figures 1 and 2)

Transfer geometry can now be defined. Given the initial and terminal orbit elements (p_1, e_1, ω_1, p_2, e_2, ω_2) a transfer orbit is prescribed by selecting two angles and a distance. These are φ_1, the position of the departure point on the initial orbit, φ_2, the position of the arrival point on the terminal orbit, and p the simi-latus rectum of the transfer conic. The conic is undefined for values of $\varphi_2 - \varphi_1 = 180° = 0$.

Impulse Function

A double-valued impulse function of the variables φ_1, φ_2, and p may now be defined. The radial and circumferential velocities of a conic are given by:

$$R = \sqrt{K/p}\left(e \quad \sin(\theta_3 - w)\right) \tag{9}$$

$$C = \sqrt{K/p}\left(1 + e \quad \cos(\theta_3 - w)\right) \tag{10}$$

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The impulse function is then defined by:

$$I = I_1 + I_2$$

Or:

$$I = \sqrt{(R_1-R_{11})^2 + (C_1-C_{11}\cos\alpha_1)^2} + \sqrt{(C_{11}\sin\alpha_1)^2 + (R_2-R_{22})^2 + (C_2-C_{22}\cos\alpha_2)^2 + (C_{22}\sin\alpha_2)^2} \qquad (11)$$

The various realtionships in the transfer geometry can be seen by refering to Figure 1 and 2.

The partial derivatives of the implicit impulse function with respect to φ_1, φ_2, and p may now be written as:

$$\partial I/\partial\varphi_1 = \frac{R_1-R_{11}}{I_1 \sin\Delta\theta}\left[R_2\cos\alpha_1 - R_{11}\cos\Delta\theta\sqrt{p/p_1} + \left(\sqrt{K/p_1} - C_{11}\right)\sin\Delta\theta\right]$$

$$+ \frac{C_1-C_{11}\cos\alpha_1}{I_1}\left[-R_{11}\sqrt{p/p_1}\right] + C_{11}-C_1\cos\alpha_1\,\frac{[-R_{11}]}{I_1}$$

$$+ \frac{C_1C_{11}}{I_1\sin^3\Delta\theta}\left[\sin^2 i_1 \sin^2\varphi_2\cos\Delta\theta\right]$$

$$+ \frac{R_2-R_{22}}{I_2\sin\Delta\theta}\left[R_1\cos\alpha_1 - R_{11}\sqrt{p/p_1}\right] + \frac{C_2C_{22}}{I\sin^3\Delta\theta}\left[-\sin^2 i_1\sin\varphi_1\right.$$

$$\left.\sin\varphi_2\right] \qquad (12)$$

$$\partial I/\partial\varphi_2 = \frac{R_1-R_{11}}{I_1\sin\Delta\theta}\left[-R_2\cos\alpha_2 + R_{22}\sqrt{p/p_2}\right] + \frac{C_1C_{11}}{I_1\sin^3\Delta\theta}\left[-\sin^2 i_1\sin\varphi_1\right.$$

$$\left.\sin\varphi_2\right]$$

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$$+ \frac{R_2 - R_{22}}{I_2 \sin\Delta\theta} \left[-R_1 \cos\alpha_2 + R_{22} \cos\Delta\theta \sqrt{p/p_2} + \left(\sqrt{K/p_2} - C_{12}\right) \sin\Delta\theta\right]$$

$$+ \frac{C_2 - C_{22} \cos\alpha_2}{I_2} \left[-R_{22} \sqrt{p/p_2}\right] + \frac{C_{22} - C_2 \cos\alpha_2}{I_2} \left[-R_{22}\right]$$

$$+ \frac{C_1 C_{22}}{I_2 \sin^3\Delta\theta} \left[-\sin^2\varphi_1 \sin^2 i_1 \cos\Delta\theta\right] \tag{13}$$

$$\partial\dot{\theta}/\partial p = \frac{R_1 - R_{11}}{2I_1 p \sin\Delta\theta} \left[R_1 \sin\Delta\theta - 2(1-\cos\Delta\theta) \sqrt{K/p}\right]$$

$$+ \frac{C_1 - C_{11} \cos\alpha_1}{2I_1 p} \left[C_1\right] + \frac{R_2 - R_{22}}{2I_2 p \sin\Delta\theta} \left[R_2 \sin\Delta\theta + 2(1-\cos\Delta\theta) \sqrt{K/p}\right]$$

$$+ \frac{C_2 - C_{22} \cos\alpha_2}{2I_2 p} \left[C_2\right] \tag{14}$$

Thus, a class of orbital transfer equations have been formulated and defined.

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Homing

Homing can be considered as the act of guiding a craft to a satellite target with initial position and velocity differences of approximately 20 miles and 100 ft/SEC respectively, to within .05 miles and 2.0 Ft/SEC. Thus, orbit transfer errors are corrected until the docking maneuver can be accomplished. A coordinate system with the origin at the target vehicle is oriented such that:

The x_1 - axis is in the target plane pointing forward

The y_1 - axis is directed binormally

The z_1 - axis is radial

Then, the first order equations of motion are:

$$F_{x_1} = \ddot{x}_1 + K/r^3(1-p/r)x_1 + (2h_r/r^2)\dot{z}_1 - \frac{2Ke \sin f}{r^3} z_1$$

$$F_{y_1} = \ddot{y}_1 + (K/r^3)y_1 \qquad (15)$$

$$F_{z_1} = \ddot{z}_1 - K/r^3 (2 + p/r) z_1 - 2h_r/r^2 \dot{x}_1 + \frac{2Ke \sin f}{r^3} x_1$$

Make the following substitutions:

$$h_r = \sqrt{KP} \; ; \qquad p = a(1-e^2); \qquad n = \sqrt{K/a^3} \; ; \qquad r = \frac{p}{1 + e \cos f}$$

And considering the path of motion to be circular by letting $e \rightarrow 0$, equation (15) becomes:

$$F_{x_1} = \ddot{x}_1 - en^2(x_1 \cos f + 2z_1 \sin f) + 2n(1+2e \cos f)\dot{z}_1$$

$$F_{y_1} = \ddot{y}_1 + n^2(1+3e \cos f) y_1 \qquad (16)$$

$$F_{z_1} = \ddot{z}_1 - 2en^2 x_1 \sin f - n^2(3 + 10e \cos f) z_1 - 2n(1+2e \cos f)\dot{x}_1$$

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Note that the "e" terms still remain but that the "e^2" and "e^3" terms have been canceled in as much as this simplifies the equations without any appreciable loss in accuracy. Equations (16) can be further simplified if the "e" terms are removed entirely. This has been shown to be permissible provided that rendezvous is completed within one orbit of the vehicle.

The equations can then be written:

$$F_{x_1} = \ddot{x}_1 + 2n\,\dot{z}_1$$

$$F_{y_1} = \ddot{y}_1 + n^2\,y_1 \qquad (17)$$

$$F_{z_1} = \ddot{z}_1 - 3n^2\,z_1 - 2n\,\dot{x}_1$$

Since these equations have constant coefficients they can be written in terms of Laplace transforms as follows:

$$X(s) = \frac{(x_o + 2n\,z_o)\left[s^2 + Hns + (K_r-3)n^2\right]}{D(s)} + \frac{x_o(S + Hn)\left[s^2+Hns+(K_r-3)n^2\right]}{D(s)}$$

$$-\frac{(\dot{z}_o - 2n\,x_o)\,(2ns + Ln^2)}{D\,(s)} - \frac{z_o(s + Hn)\,(2ns + Ln^2)}{D\,(s)} \qquad (18)$$

$$Y(s) = \frac{(S + Hn)\,y_o + \dot{y}_o}{s^2 + Hns + (K_r + 1)\,n^2} \qquad (19)$$

$$Z(s) = \frac{(\dot{z}_o - 2nx_o)\,(S^2 + Hns + K_r n^2)}{D\,(s)} + \frac{z_o(S + Hn)(S^2 + Hns + K_r n^2)}{D\,(s)}$$

$$+ \frac{(\dot{x}_o + 2n\,z_o)(2ns + Ln^2)}{D\,(s)} + \frac{x_o(S + Hn)(2ns + Ln^2)}{D\,(s)} \qquad (20)$$

Where:

$$D\,(s) = s^4 + 2Hns^3 + (H^2 + 2K_r + 1)n^2\,s^2 + \left[H(2K_r-3)+4L\right]n^3 s + (K_r^2 - 3K_r + L^2)\,n^4$$

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Integration of the complete equations of motion for both the target and ferry vehicle must be performed for purposes of rendezvous.

Steering

To accurately steer the vehicle to the point of rendezvous a "lead" term perpendicular to the line of sight must be included in the steering equations which are comprised of three parts:

(1) A line of sight thrust toward the target;

$$\underline{-K_r\, n^2(x_1\, i_r + y_1 j_r + z_1 k_r)}$$

(2) A "damping" term proportional to velocity in the moving system;

$$\underline{-Hn(\dot{x}_1 i_r + \dot{y}_1 j_r + \dot{z}_1 k_r)}$$

(3) A "lead" term perpendicular to the line of sight, and directed with a sense opposite to that of the vehicle's motion about the Earth or Moon;

$$\underline{Ln^2(x_1 k_r - z_1 i_r)}$$

Thus:

$$\underline{F_{x_1} = -K_r\, n^2\, x_1 - H_n \dot{x}_1 - Ln^2\, z_1}$$

$$\underline{F_{y_1} = -K_r n^2\, y_1 - Hn\, \dot{y}_1} \qquad (21)$$

$$\underline{F_{z_1} = -K_r\, n^2\, z_1 - Hn\, \dot{z}_1 + Ln^2\, x_1}$$

Are the steering equations for rendezvous.

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Conculsion

Rendezvous, in its transfer and homing phases, has been described mathematically. Of necessity, two vehicles must be in orbit prior to these phases. These vehicles are referred to as the target and ferry vehicles. The ferry vehicle's trajectory and location at any time shall be computed by the earth or lunar environment sections of the computer. However, for purposes of training, the target vehicle's trajectory and location can be preprogrammed or taped into the rendezvous portion of the computer. Then the ferry vehicle, using its onboard propulsion and reaction control capability performs the necessary rendezvous maneuvers.

By programming and computing simulated rendezvous problems on an IBM 7090, the optimum values for H, K_r, and L in the steering equations were found to be 6, 16, and 8 respectively. Optimum impulse quantities for particualr problems were also ascertained.

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Section VI: VISUAL SIMULATION

A complete mathematical description is at once both impractical and unnecessary. However, a generalized approach with as many mathematical details as possible incorporated, seems desirable for training systems requirements. Of necessity, the various means of visual observation must be considered separately.

Window Observation

At present, five windows are planned for the command module. In general, one in the center and two on the right and left. If the astronaut's right eye (E_1) is normal to the window's approximate center, his scope of observation is a field of view on the celestial sphere, see Figure I. Assuming the window to be flat, the field of view can be computed as a function of the window area and the distance from the eye to the center of the window. For example, if the window is circular with dimensions shown in Figure I, we have:

$$F_1 = 2 \arctan \frac{r_w}{d_1} \qquad (1)$$

Other window shapes would produce corresponding fields of view on the celestial sphere. All celestial bodies in this field of view will be seen by the astronaut. This includes the sun, moon, earth, fourth magnitude stars or brighter etc. Conversely all sources of light within the field enters the C.M. through the window, and can be seen by E_1.

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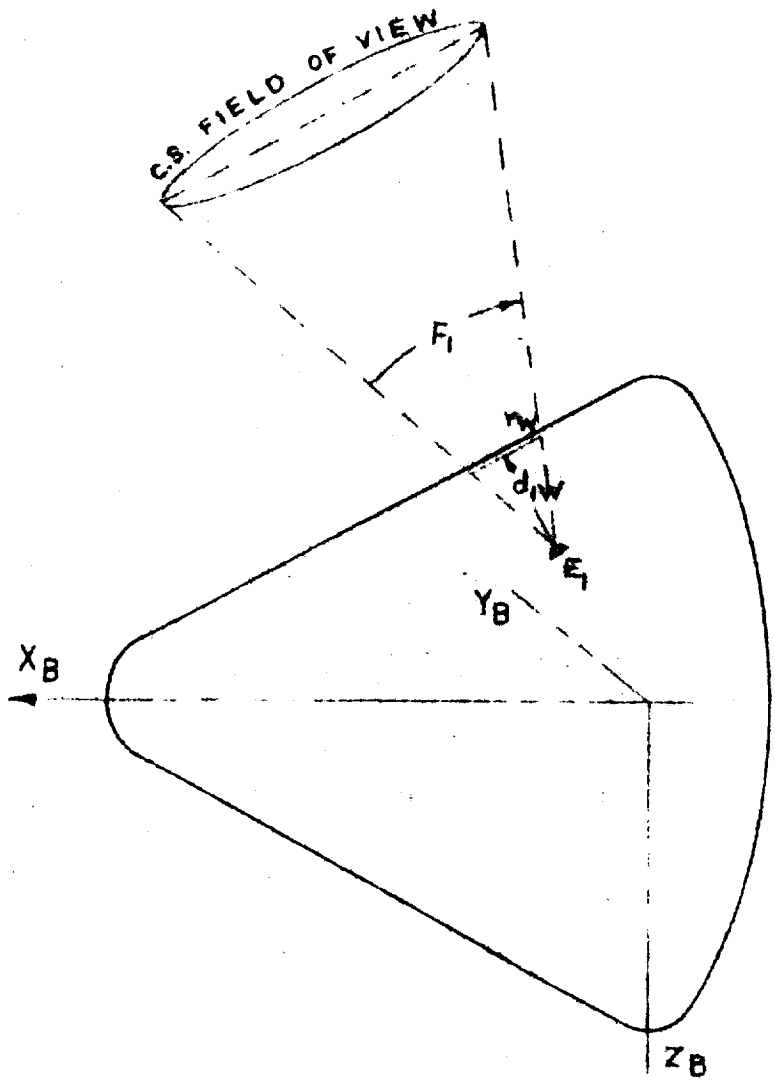

FIGURE I

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Vector d_{1w} can be described as the resultant of two rotations about the Z_B and Y_B axis respectively, as shown in Figure 2. The angles of rotation are α_1 and β_1 respectively. Note that X'_B, which is parallel to X_B, only through the astronaut's eye, can be considered the same line when projected on the celestial sphere or at infinity.

The possibility exists that $\alpha_1=0$ when viewing out the center window, but the general case requires both rotations. These can be written mathematically from the body axis as shown below. Note that w refers in general to any of the 5 windows, with the individual angles determined by command module design.

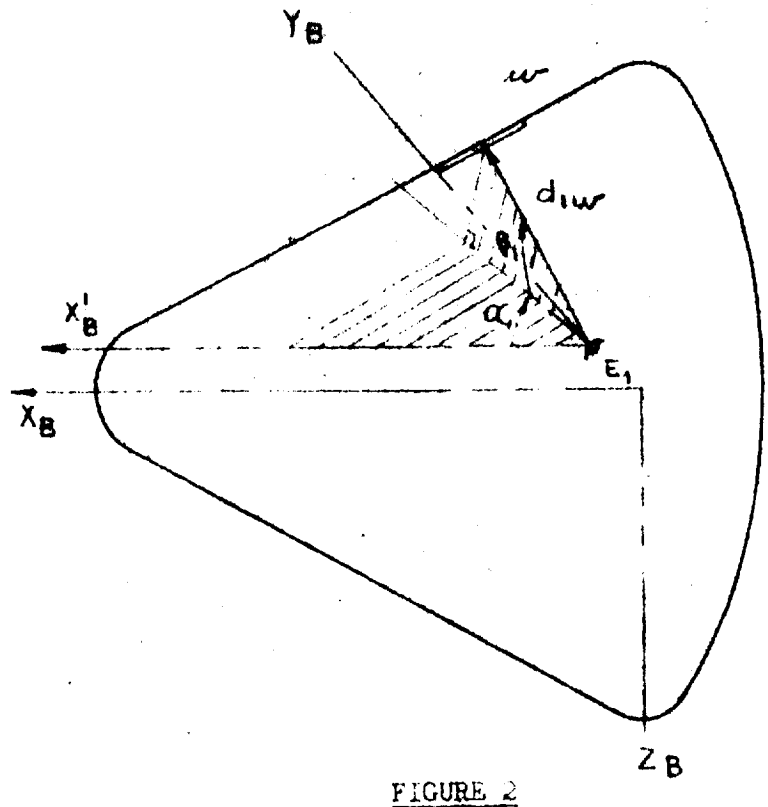

FIGURE 2

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Consequently, vector d_{1w} can be described as:

$$\bar{d}_{1w} = \begin{bmatrix} \cos\beta_{1w} & 0 & \sin\beta_{1w} \\ 0 & 1 & 0 \\ -\sin\beta_{1w} & 0 & \cos\beta_{1w} \end{bmatrix} \begin{bmatrix} \cos\alpha_{1w} & \sin\alpha_{1w} & 0 \\ -\sin\alpha_{1w} & \cos\alpha_{1w} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_B \\ j_B \\ k_B \end{bmatrix} \qquad (2)$$

In a similar manner the center line vector of the Scanning Telescope (d_{1T}) and the Sextant can be oriented to the command module body axis. The corresponding angles α_{1T}, β_{1T} and α_{1s} β_{1s}, vary however according to the orientation of the optical instrument. Essentially, relative to the Euler Angles, we have simply two more angular rotations described as:

$$\phi_w = \phi$$

$$\theta_w = \theta + \beta_{1w}$$

$$\psi_w = \psi + \alpha_{1w}$$

where α_{1w} and β_{1w} are the additional rotations about the Z_B and Y_B axes respectively. With appropriate subscripts, the orientation of the scanning telescope and sextant can similarly be described.

It now becomes necessary to generalize window observations. The first step is to move the eye from the normal to the window while keeping it in the vertical plane or X_B Y_B plane in the case of the central window. See Figure 3.

Assuming this window to be circular, observations from this point would cause the window to appear elliptical and distort the circular celestial sphere field of view into an ellipse.

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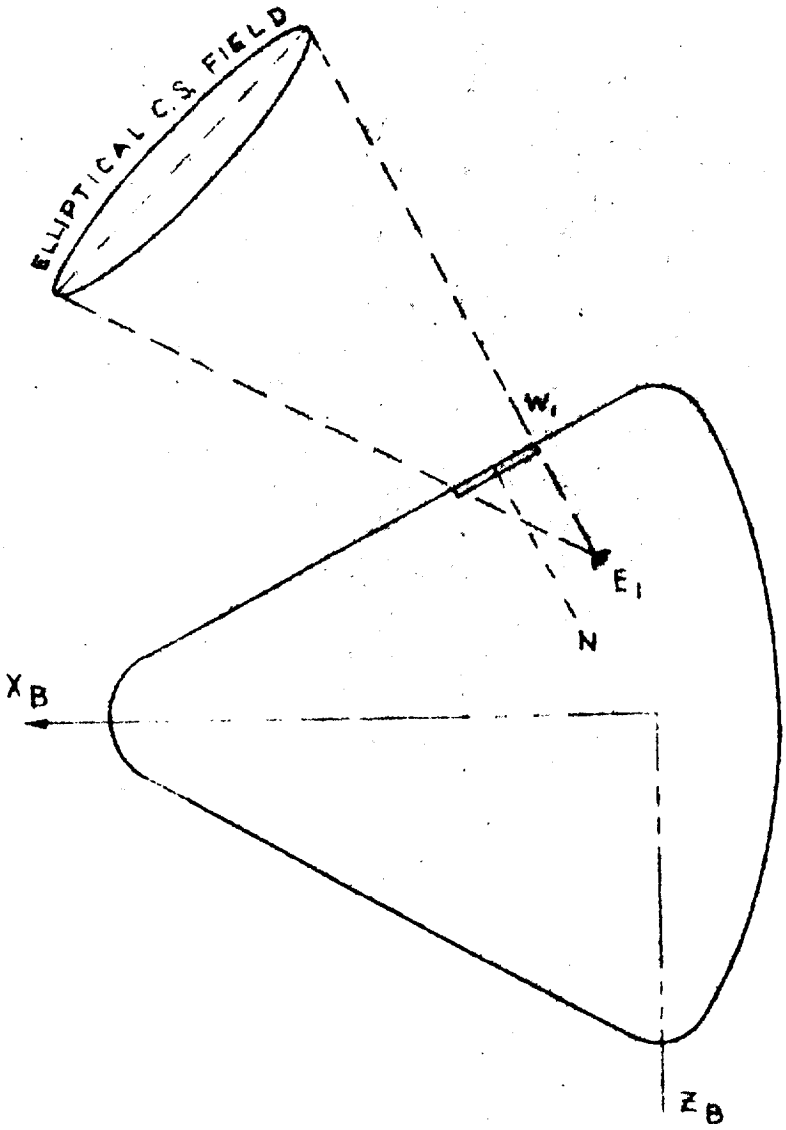

FIGURE 3

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The degree of distortion can be calculated from the angle $[\delta_1]$ between the normal and the median to the window (or plane of the window) from the observers eye, the ellipse is a $(90 - \delta_1)°$ with the major axis along the Y_B, as shown in Figure 4. The angle of rotation $\beta_1 = 90 - \gamma_1 + \delta_1$ where γ_1 is the angle between the element of the cone of the command module body and the X_B axis or one half the vertex angle of the command module cone. Since the right eye is held in the $X_B Z_B$ plane in this case, $\alpha_1 = 0$ and equation (3) may be written.

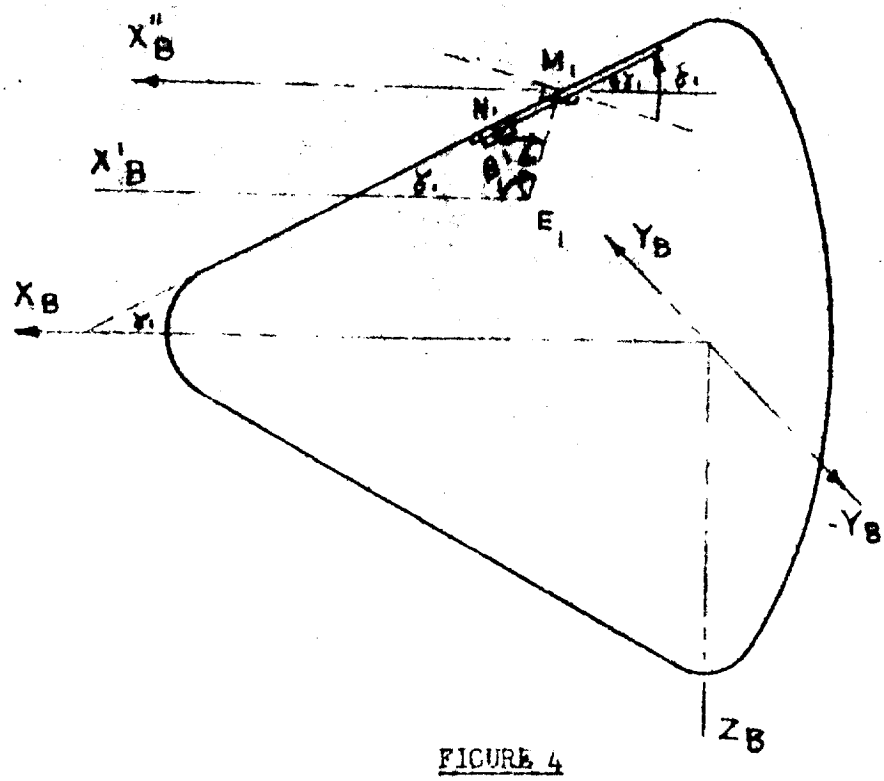

FIGURE 4

$$\begin{bmatrix} i_W \\ j_W \\ k_W \end{bmatrix} = \begin{bmatrix} \cos(90 - \gamma_1 + \delta_1) & 0 & \sin(90 - \gamma_1 + \delta_1) \\ 0 & 1 & 0 \\ -\sin(90 - \gamma_1 + \delta_1) & 0 & \cos(90 - \gamma_1 + \delta_1) \end{bmatrix} \begin{bmatrix} i_B \\ j_B \\ k_B \end{bmatrix} \quad (3)$$

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The next step is to move the eye (E_1) to any position in the Y_B Z_B plane or a parallel plane within the command module and sight the midpoint of any of the five windows as shown in Figure 5. C_1 is any point on the X_B axis and C_1M_1 is the radius of curvature of the C.M. cone in that particular plane. ϵ_1 is the angle between E_1M_1 and C_1M_1. The ellipse then is a $(90 - \epsilon_1)°$ ellipse. Combining these two results, the degree of the final ellipse after both rotations is $\frac{90 - \epsilon_1}{90}$ $[90 - \delta_1]$ with the major axis rotated ϵ_1 counter clockwise for $+\alpha_1$ and rotated ϵ_1 clockwise for $-\alpha_1$ rotations.

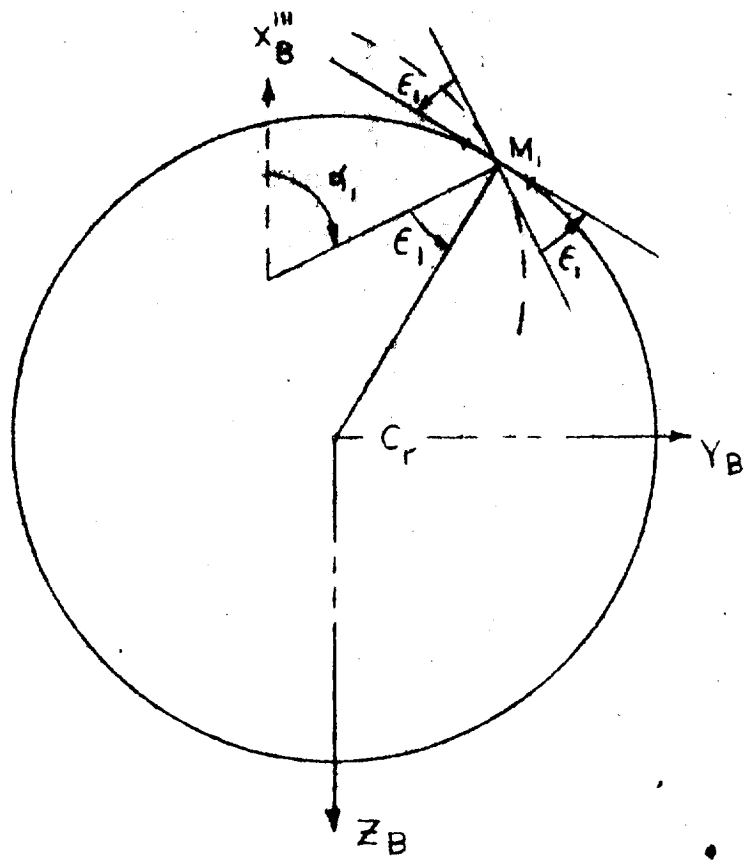

FIGURE 5

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The next consideration is a window observation with both eyes which shall be denoted as E_1 for right eye and E_2 for left eye. This normal situation produces the "binoncular view" as shown in Figure 6. Region $(E_1 + E_2)$ is observed by both eyes, while regions (E_1) and (E_2) are observed by the right and left eyes only, respectively. Thus the observation illustrated in Figure 1 would be as shown in Figure 6.

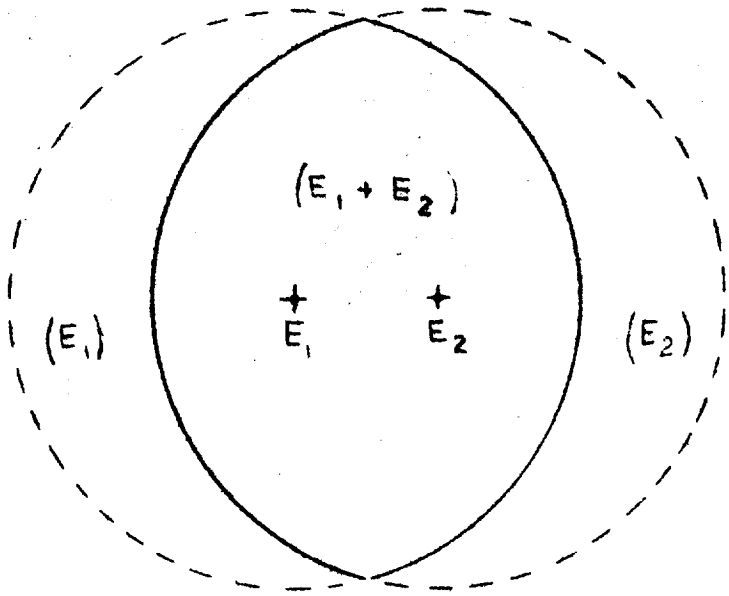

FIGURE 6

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After the α_1 and β_1 rotations, the observation would appear similar to the double ellipse shown in Figure 7. It is recognized that celestial sphere window observation is two dimensional in general, and thus with the exception of clarity condiserations, view is the same with one eye at a time added together, or both eyes at once. In general, it is only necessary to superimpose the trace of one eye on the other.

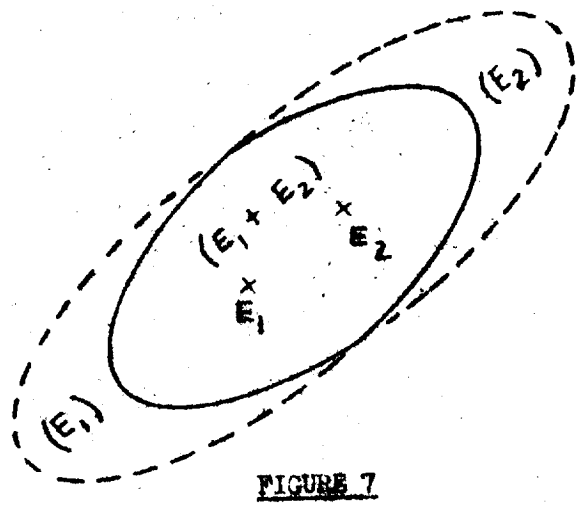

FIGURE 7

When E_1E_2 has any direction and $\alpha_1 \beta_1$ and $\alpha_2 \beta_2$ are any reasonable rotation angles, we have the general case shown in Figure 8.

If a window housing thickness is considered, another elliptical distortion occurs as shown in the shaded area in Figure 8.

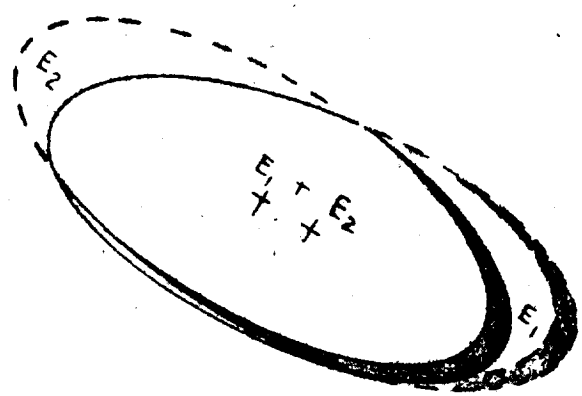

FIGURE 8

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Sunlight ingress or sun shafting will occur in general when the sun is in the same hemisphere as one of the windows, Reference Figure 9.

In the case of a circular window, the shaft will be an elliptical cylinder and will enter the craft as a vector, in the body frame, whose direction will depend upon the orientation of the craft relative to the suns position. Naturally, night conditions will prevail when the moon or earth pass between craft and sun. Hence, complete description can be formulated using appropriate vectors and ephemeris data. In all of these considerations, such items as window curvature and glass composition have not yet been established.

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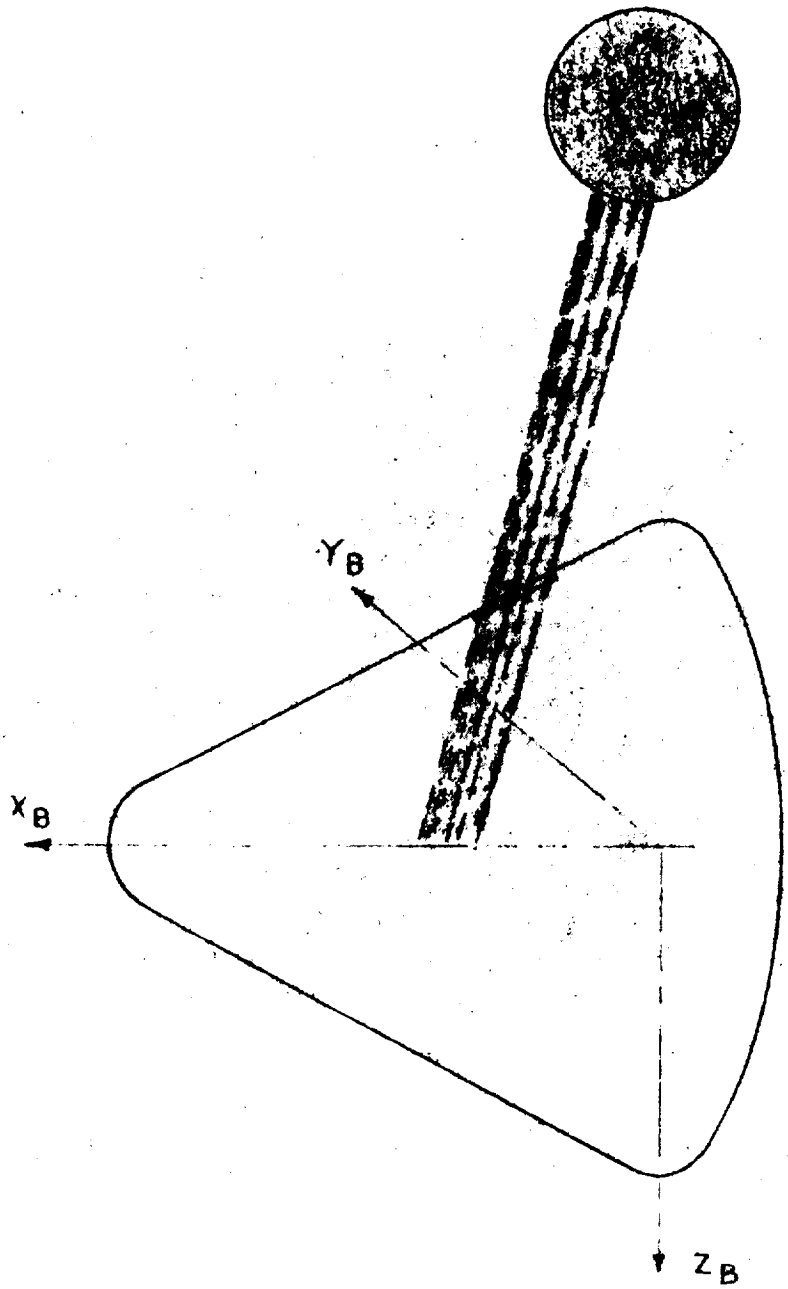

FIGURE 9

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Instrument Observation

Using the same rotations in equation 2, through 360° about the Z_B and Y_B axis, the following instruments will scan the celestial sphere with the given fields of view and magnifications:

INSTRUMENT	FIELD	MAGNIFICATION
Scanning Telescope	60° to 20°	1 to 3 (Zoom lens)
Sextant	2°	20

The field will be circular in these cases with a double superimposed image in the sextant for alignment.

The Scanning Telescope can be used for general surveillance when not being used for navigational purposes.

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Section VII: FUNCTIONAL BLOCK DIAGRAM (Integrated Model)

This model integrates all previous models into one coherent, complete system as shown on the enclosed print.

The Command Module represents the actual trainer with all of the simulated instruments and controls for the various flight phases. The computer interface integrates the C.M. with the Instructors Console, Flight Programer, and the equations of motion.

The various environments are listed except the midcourse environment which is assumed to be the combination of Earth and Lunar Environments. The results of the Launch Escape Model appear in the Propulsion System Computer Block.

Pertinent elements of these models are connected to the Visual Computer Interface where stellar, occultation and orientation equations are solved. These computations along with the euler angles relating each visual instrument to the flight trajectory are used to generate the visual cues in the Simulator.

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PART TWO

COMPUTER MECHANIZATION EQUATIONS

(To Be Developed)

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