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GRAVITY TORQUE ON AN ORBITING VEHICLE

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SUMMARY

The potential energy of a small body moving under the influence of only the earth's gravity field is derived. It is shown that the effects of earth and body oblateness are separate to terms of second order in the potential energy function. The invariance of the potential energy under orthogonal transformations is discussed and is used to simplify calculations of the terms in the potential function. The equations of motion of the vehicle are obtained in generalized and in body coordinates. The components of torque in body coordinates are obtained by a transformation determined by the invariance of rotational power. Two applications of the equations of motion in body coordinates are made. They are first used to show that an energy integral does not exist for these equations since they are derived with respect to the satellite mass center. Then they are applied in simplified form to an examination of the stability of a vehicle in a circular orbit.

INTRODUCTION

The utility of artificial satellites can be greatly enhanced by maintaining guidance and control over them. For this reason, a number of studies have been made of the problem of keeping the vehicle, or some of its contents, pointed in a particular direction. A review of the literature of the subject has been given by Roberson in reference 1.

The first problem in the study of satellite control is the quantitative determination of the torques disturbing the satellite, and the reaction of the satellite to the torques. Evaluations of the magnitudes and effects of many sources of torque, such as meteorites or electric and magnetic fields, must await more experimental information. But rough estimates given by Roberson in reference 2 indicate that, except for special designs, the latter two sources will be small compared to torques due to gravity and atmosphere. The effect of the atmosphere, which of course predominates at the lower altitudes, has been studied by DeBra (ref. 3) for certain satellite shapes.

The torque source most enduring for an orbiting vehicle, and most frequently mentioned, is the gravitational field of the earth. That a torque on an elongated satellite arises from this source was learned more than two centuries ago (ref. 4) in connection with the librations

of the moon. Roberson and Tatistcheff have recently made a more complete study of this source of torque in reference 5. However, some of the torque-generating terms retained in the potential function of this reference are small enough to be ignored in a stability study, particularly if the satellite has provisions for control. The present report shows that if the small terms, which arise from the oblateness of the earth, are neglected, an invariance property manifests itself that greatly reduces the labor of expressing the potential energy in terms of the moments and products of inertia about axes fixed in the satellite.

The purpose of the present report is to calculate the equations of motion of a satellite not for purposes of orbital studies but in a form convenient for stability studies. The equations will be derived twice; first in terms of the Euler angles chosen, then in terms of body coordinates. The equations of motion will be derived from the potential and kinetic energy functions in both cases.

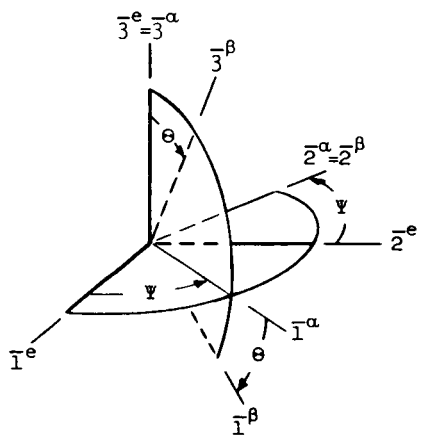
The order of development in this report is as follows. The potential and the kinetic energy functions are derived in a form independent of particular coordinate systems. A particular geometry is selected in the next section, and the functions are expressed in terms of the chosen parameters. Next the equations of motion are obtained in generalized coordinates by applying Euler's equation to the Lagrange function of the kinetic and potential energies. Then the equations of motion are obtained in terms of body coordinates. The inertial terms can be obtained directly from the kinetic energy, but more consideration must be given to expressing the torque, derived from the potential, in terms of body coordinates. The torque transformation law is obtained on the basis of the invariance of power and then is applied to give the torque components in body coordinates. The energy integral is briefly discussed in the next section and is shown not to exist. The final section of the report applies the derived equations to the stability of oscillation of a symmetrical body in a circular orbit.

SYMBOLS

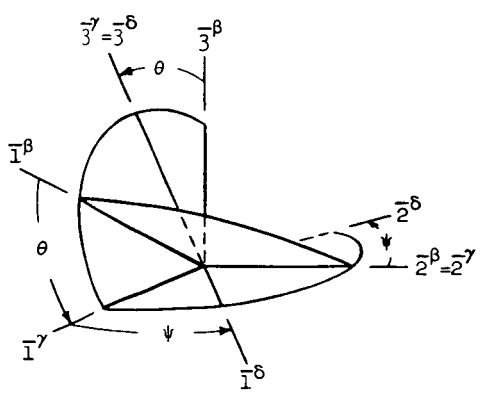
a	distance between earth and satellite mass centers
e	an inertial coordinate system centered at the earth's mass center
G	universal constant of gravitation
I_e	matrix representing earth inertia with respect to axes fixed in the earth
I_s	matrix representing satellite inertia with respect to axes fixed in the satellite

I_s'	matrix representing satellite inertia with respect to fixed earth coordinates
I_s''	matrix representing satellite inertia with respect to the β coordinate system
K.E.	kinetic energy
L	Lagrangian (K.E. - V)
L,M,N	components of gravity torque in body coordinates
M_e	mass of the earth
m	mass of the satellite
p,q,r	components of satellite angular velocity vector $\bar{\Omega}^{s-e}$ in satellite body coordinates
R	effective earth radius
s	as subscript or superscript, coordinate system fixed in the satellite
s	Laplace transform variable
$T_{\dot{\varphi}}, T_{\dot{\theta}}, T_{\dot{\psi}}$	components of torque, equal to $-\frac{\partial V}{\partial \varphi}$, $-\frac{\partial V}{\partial \theta}$, and $-\frac{\partial V}{\partial \psi}$, respectively
$\bar{u}$	satellite mass center velocity
u,v,w	components of $\bar{u}$ in the s coordinate system
V	potential energy function
V_{trans}	term of the potential energy function independent of φ, θ, and ψ
V_{rot}	term of the potential energy function containing φ, θ, and ψ
$\alpha,\beta,\gamma,\delta$	systems of coordinates intermediate to the inertial, e, and satellite, s, coordinates (see below)
∇	differential operator
Θ,Ψ	angles determining position of satellite mass center (see below)
θ,ψ,φ	angles determining the orientation of the satellite (see below)

μ	earth potential form factor
$\bar{\Omega}^{\beta-e}$	angular velocity line from earth to satellite mass center with respect to inertial space of β coordinate system
$\bar{\Omega}^{s-e}$	angular velocity of β coordinates with respect to inertial space
1,2,3	labels of any right-hand triad of unit vectors (as subscripts, the components of a vector associated with the pertinent unit vector)
$-$	a vector
( )	a rotation matrix, the rotation being specified by the angle enclosed



Sketch (a)



Sketch (b)

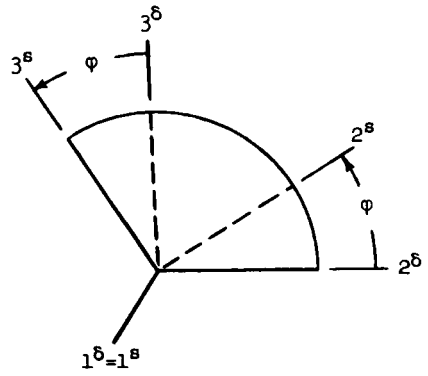
Definitions of Coordinates

Three coordinate systems are of chief interest in the report. They are: the e system, which is an inertial system fixed in a non-rotating earth; the β system, along the 3 axis on which the satellite mass center is located; and the s system, which is rigidly attached to the moving satellite.

The relationship between the e system and the β system is shown in sketch (a). The sketch shows that the β system is obtained from the e system by a rotation through the angle Ψ followed by a rotation through the angle Θ.

The relationship between the β system and the s system is shown in sketches (b) and (c). Sketch (b) shows that a γ system is obtained from the β system by the angle of rotation θ. A δ system is then obtained by the angle ψ as shown. Thus the axes 3^e, 3^α, 3^β, 3^γ, and 3^δ

lie in the same plane. Sketch (c) shows that the s system is obtained from the δ system by a rotation through the angle φ. These s coordinates are the satellite body coordinates.



Sketch (c)

DERIVATION OF THE LAGRANGIAN FUNCTION

The two methods commonly employed in classical mechanics for the formulation of the equations of motion of a dynamical system are the force method of Newton and the kinetic potential method of Lagrange. In the former method, each part of a body is isolated and its acceleration is expressed as being equal to the resultant of all impressed and constraining forces. When the various parts of the body are in a force field and have several degrees of freedom, or the interaction of the various parts is obscure, use of this method is a delicate operation with many hidden difficulties.

The Lagrangian method will be followed in this report because it is better suited to the present problem of calculating the equations of motion of an earth satellite. The first step to be taken in this method is the determination of the so-called kinetic potential, or the Lagrangian, a function that is simply the difference between the kinetic and the potential energies of the system.

The next step is the formulation of the equations of motion which depends on the type of coordinates in which the Lagrangian is expressed. If it is expressed in generalized, that is mutually independent, coordinates, the equation for each coordinate can be obtained from the Lagrangian by a fixed procedure. If nonintegrable coordinates are used, such as the angular rates about three orthogonal body axes in the aircraft moment equations, the procedure is less simple.

Potential Energy Derivation

This section contains the development of the potential energy function as far as it can proceed without the specification of a particular coordinate system. The derivation is patterned after the multipole expansion given in appendix VI of reference 6. This pattern is followed more closely in the appendix of this report than in this section. There the treatment is more explicit, being expressed in terms of the Cartesian components of the various particles making up the satellite. But this

greater explicitness obscures both the nature of the development and the invariance of form of terms which is exhibited by the more general treatment of this section.

The derivation consists of a Taylor's expansion of the inverse distance factor in the formula for the potential energy of two isolated masses, and an integration over the mass distributions of the earth and the satellite. The expressions simplify under the integration if the centers of mass of the two bodies are chosen as a reference. The result of the calculation is an equation with three terms. The first term gives the potential energy of the two bodies as though each were condensed to the position of their mass centers. The other two terms correct this simplification to account for the distributions of mass, one term for the earth, the other for the satellite.

The mass distribution of the satellite is a function of its orientation with respect to the earth, that is, of its rotational motion. To express this distribution in terms of quantities that can be measured in the body, the moments and products of inertia expressed in body coordinates, would be a lengthy task except for the simple form in which it appears in the potential. This simplicity will make it easy to show that the term preserves its form under orthogonal transformations. The invariance of form thus shown means that the factors in the term can be expressed in any consistent set of coordinates - body coordinates, for example - without any complications.

The potential energy of two separate mass distributions, the earth of mass M_e, and the satellite of mass m, is given by the expression

$$V = -G \iint_{mM} \frac{dm\ dM_e}{\rho} \tag{1}$$

where G is the universal gravity constant, ρ is the distance between elementary mass points, and dM_e and dm are the masses associated with each point of the two bodies. The function $f(\rho) = 1/\rho$ can be expanded in a Taylor's series about the center of mass of the satellite relative to the center of mass of the earth. The expansion can be expressed succinctly in the form

$$f(\rho) = f(a) + (\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)f(a) + \frac{1}{2}(\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)(\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)f(a) + \ldots \tag{2}$$

The quantities appearing in this expression are thought of as being expressed in Cartesian terms; thus

$$f(\rho) = [(x + \xi' - x')^2 + (y + \eta' - y')^2 + (z + \zeta' - z')^2]^{-1/2}$$

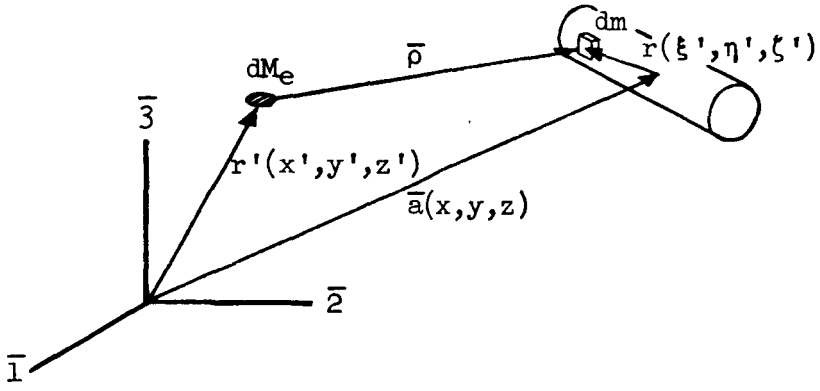
$$f(a) = (x^2 + y^2 + z^2)^{-1/2}$$

$$\bar{r} = \xi'\bar{1} + \eta'\bar{2} + \zeta'\bar{3}$$

$$\bar{r}' = x'\bar{1} + y'\bar{2} + z'\bar{3}$$

$$\nabla = \frac{\partial}{\partial x}\bar{1} + \frac{\partial}{\partial y}\bar{2} + \frac{\partial}{\partial z}\bar{3}$$

As shown in sketch (d), the coordinates (x',y',z') refer an earth mass point to its mass center. The coordinates (x,y,z) give the location of



Sketch (d)

the satellite mass center. The coordinates (ξ',η',ζ') refer a satellite mass point to its center of mass. The space in which the quantities and the operation of gradient appear is considered to be a fixed Cartesian space with unit vectors $\bar{1}$, $\bar{2}$, and $\bar{3}$. Because of the linear occurrence of x', x, and ξ', the operation $\partial/\partial x$ can be used for $-\partial/\partial x'$ and $\partial/\partial \xi'$, and $f(\rho)$ can be evaluated before the gradient operation.

To obtain the potential energy, one multiplies equation (2) by $-G\ dm\ dM_e$ and integrates over both sets of masses as indicated in equation (1). The following expressions result:

$$V_o = -G \iint \frac{dm\ dM_e}{a} = -\frac{GmM_e}{a} \tag{3a}$$

$$V_1 = -G \iint (\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)f(a)dm\, dM_e = 0 \tag{3b}$$

$$V_2 = -\frac{G}{2}\iint (\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)(\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)f(a)dm\, dM_e$$

$$= \frac{Gm}{2}\nabla\cdot I_e\nabla f(a) + \frac{GM_e}{2}\nabla\cdot I_s'\nabla f(a) \tag{3c}$$

The terms V_3 and higher that would appear in the expansion of V are neglected as small compared with the lower terms. Equation (3b) vanishes in the integration because both $\bar{r}$ and $\bar{r}'$ refer to their centers of mass. The quantities I_e and I_s' are representations of the inertia tensors of the earth and satellite, respectively, in the fixed coordinate system.

To complete the derivation of equation (3c), let it first be noted that

$$(\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)(\bar{r}\cdot\nabla - \bar{r}'\cdot\nabla)f(a) = \bar{r}\cdot\nabla[\bar{r}\cdot\nabla f(a)] + \bar{r}'\cdot\nabla[\bar{r}'\cdot\nabla f(a)] - \bar{r}\cdot\nabla[\bar{r}'\cdot\nabla f(a)] - \bar{r}'\cdot\nabla[\bar{r}\cdot\nabla f(a)]$$

Since the last two terms, when multiplied by $dm\, dM_e$ and integrated, vanish for the same reason as did equation (3b), attention can be centered on the first two terms on the right. Either of these terms can be transformed by the identity $\bar{b}\cdot\nabla\beta = \nabla\cdot\bar{b}\beta - \beta\nabla\cdot\bar{b}$ where β is a scalar and $\bar{b}$ any vector. Using this identity one sees that

$$\bar{r}\cdot\nabla[\bar{r}\cdot\nabla f(a)] = \nabla\cdot[\bar{r}\bar{r}\cdot\nabla f(a)] - \bar{r}\cdot\nabla[f(a)\nabla\cdot\bar{r}]$$

The divergence of $\bar{r}, \nabla\cdot\bar{r}$ is zero because the derivative is calculated with respect to the coordinates of $\bar{a}$, not of $\bar{r}$. Multiplying $\nabla\cdot[\bar{r}\bar{r}\cdot\nabla f(a)]$ by $-\frac{G}{2}\, dm\, dM_e$ and integrating, one has

$$-\frac{G}{2}\iint \nabla\cdot[\bar{r}\bar{r}\cdot\nabla f(a)]dm\, dM_e = -\frac{GM_e}{2}\nabla\cdot\int \bar{r}\bar{r}\cdot\nabla f(a)dm$$

$$= \frac{GM_e}{2}\nabla\cdot I_s'\nabla f(a)$$

which is the term appearing in equation (3c). The validity of the last step given is shown explicitly in the alternate derivation given in appendix A. Confirmation follows writing out the expression $\nabla\cdot\int \bar{r}\bar{r}\cdot\nabla f(a)dm$ and noting that $\nabla^2 f(a) = 0$. The change of sign results from the definitions of the moments and products of inertia, namely:

$$I_1' = \int (\eta'^2 + \zeta'^2)dm \qquad I_{12}' = -\int \xi'\eta' dm$$

$$I_2' = \int (\xi'^2 + \zeta'^2)dm \qquad I_{13}' = -\int \xi'\zeta' dm$$

$$I_3' = \int (\xi'^2 + \eta'^2)dm \qquad I_{23}' = -\int \eta'\zeta' dm$$

Now that the derivation has been completed, the potential energy function can be written. It is expressed by

$$V = -\frac{GM_e m}{a} + \frac{Gm}{2}\nabla\cdot I_e\nabla f(a) + \frac{GM_e}{2}\nabla\cdot I_s'\nabla f(a) \tag{4}$$

Before this expression can be used in the Lagrangian, its functional dependence on the coordinates must be made explicit. Since coordinates have not yet been chosen for the problem, this explication is postponed to a later section. In the meantime, two interesting points should be noted concerning the potential function given. One point is that, to the order of expansion of the potential energy as given here (determined by stopping the series expansion of $f(\rho)$ in eq. (2)), the effects of the mass distributions of earth and satellite are separate. This separation will later be seen to imply that the oblateness of the earth does not yield a component of torque about the satellite vehicle, but only affects its orbit. The neglected terms are of the order of a fraction of a percent of the terms retained.

The second point to be noted from equation (4) is its invariance. Each term of the potential function, being a scalar, should be invariant both in form and value under a simple rotation of coordinates. The first term on the right clearly is invariant, being merely a length. That the other terms are invariant also can be seen from what follows. Let ∇' be the gradient operator in another coordinate system related to the first in such a way by the orthogonal transformation, T, that $\nabla = T\nabla'$. Now, in the matrix formalism being used, the divergence operator is the transpose of the gradient $\nabla\cdot = \nabla^T$. It undergoes the transformation $\nabla^T = (T\nabla')^T$. Now, since the transpose and the inverse of the matrix T

are the same, the inertia matrix has the following transformation $T^T I T = I'$. The invariance of the expression follows from putting together these transformations in this way:

$$\nabla^T I \nabla f(a) = (T\nabla')^T I (T\nabla') f(a)$$

$$= \nabla'^T T^T I T \nabla' f(a)$$

$$= \nabla'^T I' \nabla' f(a)$$

This property will be useful later in simplifying the calculation of the potential term involving the satellite.

Kinetic Energy Derivation

This section contains the derivation of the expressions for the kinetic energy of the system. The expressions will be general, as before, in that the derivation will be carried only up to the point where a particular system of coordinates needs to be chosen.

The kinetic energy of the satellite is given by the expression $\text{K.E.} = \frac{1}{2}\int dm\bar{v}\cdot\bar{v}$. The vector $\bar{v}$ is the total velocity of the mass point dm, and can be expressed as $\bar{v} = \bar{u} + \bar{\Omega}^{s-e} \times \bar{r}$. Here $\bar{u}$ is the total velocity of the satellite mass center and $\bar{\Omega}^{s-e}$ is the angular velocity of the satellite. The superscript notation anticipates the notation to be used later in formulating the geometry of the problem. Then it will be seen that s refers to a coordinate system fixed in the satellite, and e refers to an inertial system centered at the center of mass of the earth. The kinetic energy expression becomes

$$\text{K.E.} = \frac{1}{2}\int dm(u^2 + 2\bar{u}\cdot\bar{\Omega}^{s-e} \times \bar{r} + \bar{\Omega}^{s-e} \times \bar{r}\cdot\bar{\Omega}^{s-e} \times \bar{r}) \tag{5}$$

Because $\bar{r}$ is measured from the mass center

$$\int dm\bar{u} \cdot \bar{\Omega}^{s-e} \times \bar{r} = \bar{u} \cdot \bar{\Omega}^{s-e} \times \int dm\bar{r} = 0$$

Also, since

$$\bar{\Omega}^{s-e} \times \bar{r} \cdot \bar{\Omega}^{s-e} \times \bar{r} = r^2(\bar{\Omega}^{s-e})^2 - (\bar{r} \cdot \bar{\Omega}^{s-e})^2$$

we have

$$\frac{1}{2}\int dm\bar{\Omega}^{s-e} \times \bar{r} \cdot \bar{\Omega}^{s-e} \times \bar{r} = \frac{1}{2}\,\bar{\Omega}^{s-e} \cdot I_s\bar{\Omega}^{s-e}$$

where I_s is a representation of the inertia tensor of the satellite. It differs from the quantity I_s' of the previous section in that it is given with respect to a set of coordinates fixed in the body, whereas the inertia representation of the previous section is given in a fixed space. The kinetic energy, in summary, is then

$$\text{K.E.} = \frac{m}{2}\,\bar{u}^2 + \frac{1}{2}\,\bar{\Omega}^{s-e} \cdot I_s\bar{\Omega}^{s-e} \tag{6}$$

This expression is the one that can be used in deriving the equations of motion in body coordinates.

For the generalized coordinate expression, the velocity $\bar{u}$ is further expanded to the expression

$$\bar{u} = \dot{\bar{a}} + \bar{\Omega}^{\beta-e} \times \bar{a}$$

Here $\bar{a}$ is the vector position of the satellite center of mass relative to the earth mass center. It is fixed in a coordinate system which is rotating about the earth mass center with an angular velocity $\bar{\Omega}^{\beta-e}$. The velocity $\dot{\bar{a}}$ refers to that part of the total velocity of the satellite mass center that could be observed by a person attached to the coordinates rotating with angular velocity $\bar{\Omega}^{\beta-e}$. Squaring $\bar{u}$ gives

$$\bar{u} \cdot \bar{u} = \dot{\bar{a}}^2 + 2\dot{\bar{a}} \cdot \bar{\Omega}^{\beta-e} \times \bar{a} + \bar{\Omega}^{\beta-e} \times \bar{a} \cdot \bar{\Omega}^{\beta-e} \times \bar{a} \tag{7}$$

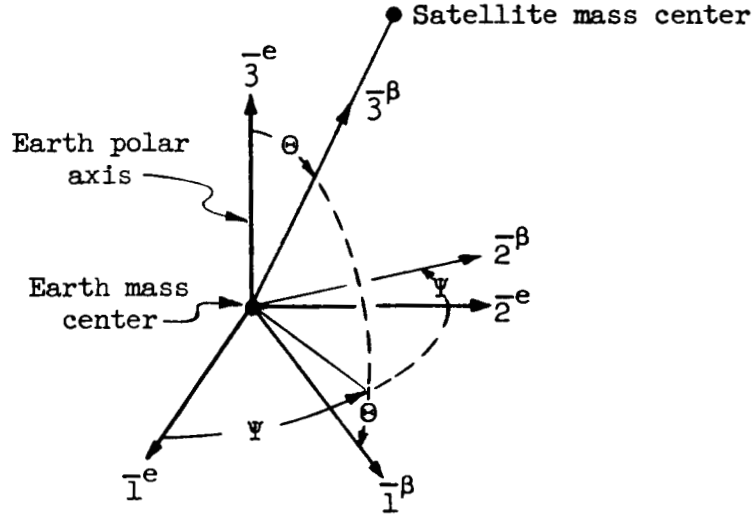
The middle term on the right is zero because $\dot{\bar{a}}$ and $\bar{a}$ are collinear. When the expression in (7) is put into equation (6), the kinetic energy expression appropriate for generalized coordinates is

$$\text{K.E.} = \frac{m}{2}\dot{\bar{a}}^2 + \frac{m}{2}\bar{\Omega}^{\beta-e} \times \bar{a} \cdot \bar{\Omega}^{\beta-e} \times \bar{a} + \frac{1}{2}\bar{\Omega}^{s-e} \cdot I_s\bar{\Omega}^{s-e} \tag{8}$$

Calculation of Kinematic and Inertial Quantities

The derivations in the previous two sections were developed as far as possible without specifying the coordinates of the particular problem. This section will be devoted to a choice of coordinates and to expressing the kinematic and inertial quantities appearing in the potential and kinetic energy expressions in terms of the parameters of the chosen coordinates.

The satellite has six degrees of freedom and requires six parameters to specify its motion. The first three parameters can be chosen to specify the location of the satellite mass center with respect to the nonrotating earth. A possible set of parameters is depicted in sketch (e).



Sketch (e)

The unit vector $\bar{3}^e$ lies along the earth's polar axis. The unit vectors $\bar{1}^e$ and $\bar{2}^e$ form a right-handed inertially fixed set with $\bar{3}^e$. An intermediate triad of unit vectors, not shown in the sketch, $\bar{1}^\alpha$, $\bar{2}^\alpha$, $\bar{3}^\alpha$, is obtained from the superscript e set $\bar{1}^e$, $\bar{2}^e$, $\bar{3}^e$ (which will be referred to as the e system henceforth) by a positive rotation about the $\bar{3}^e = \bar{3}^\alpha$ direction through the angle Ψ. From this set the β set is

obtained by a positive rotation about the $2^{\alpha} = 2^{\beta}$ direction through the angle Θ. The angles Ψ and Θ are to be generated in such a way that the vector $\bar{a}$ specifying the location of the satellite always lies along the 3^{β} direction. This choice of coordinate systems is not suited to orbital studies but is convenient for the study which is primarily interested in the rotational motion of the satellite.

The orientation of the satellite will be specified by the pitch, yaw, and roll angles θ, ψ, and φ shown in figure 1. Parts a and b of this figure show that a set of vectors $\bar{1}^{\gamma}$, $\bar{2}^{\gamma}$, $\bar{3}^{\gamma}$ is obtained from the β set by a positive pitch rotation θ about the $\bar{2}^{\beta} = \bar{2}^{\gamma}$ direction. Parts b and c show that a δ set is obtained in turn from the γ set by a positive yaw rotation ψ about the $\bar{3}^{\gamma} = \bar{3}^{\delta}$ direction. Finally, the figure shows that the s set is obtained from the δ set by a rolling through the angle φ about the $\bar{1}^{s} = \bar{1}^{\delta}$ direction. It should be noted that the definitions given here for the positive directions of the axes differ from those usually assumed for an airplane. If an airplane were heading in the direction of the 1^{s} axis, the 3^{s} axis would point up through the canopy, and the 2^{s} axis would point along the left wing of the airplane.

The following equations relate the different coordinates and transformations. The angles in parentheses represent rotation matrices (ref. 7).

$$\left.\begin{aligned}
\begin{bmatrix} \bar{1}^{\beta} \\ \bar{2}^{\beta} \\ \bar{3}^{\beta} \end{bmatrix} &= (\Theta) \begin{bmatrix} \bar{1}^{\alpha} \\ \bar{2}^{\alpha} \\ \bar{3}^{\alpha} \end{bmatrix} = (\Theta)(\Psi) \begin{bmatrix} \bar{1}^{e} \\ \bar{2}^{e} \\ \bar{3}^{e} \end{bmatrix} \\
\begin{bmatrix} \bar{1}^{s} \\ \bar{2}^{s} \\ \bar{3}^{s} \end{bmatrix} &= (\varphi) \begin{bmatrix} \bar{1}^{\delta} \\ \bar{2}^{\delta} \\ \bar{3}^{\delta} \end{bmatrix} = (\varphi)(\psi) \begin{bmatrix} \bar{1}^{\gamma} \\ \bar{2}^{\gamma} \\ \bar{3}^{\gamma} \end{bmatrix} = (\varphi)(\psi)(\theta) \begin{bmatrix} \bar{1}^{\beta} \\ \bar{2}^{\beta} \\ \bar{3}^{\beta} \end{bmatrix}
\end{aligned}\right\} \tag{9}$$

These equations summarize the geometry of the problem. The relationships shown in them will now be used to calculate the functional dependencies of the terms $\nabla \cdot I_s' \nabla f(a)$, $\bar{\Omega}^{\beta-e} \times \bar{a} \cdot \bar{\Omega}^{\beta-e} \times \bar{a}$, and $\bar{\Omega}^{s-e} \cdot I_s \bar{\Omega}^{s-e}$.

Calculation of $\nabla \cdot I_s' \nabla f(a)$.- Equation (A8) of appendix A shows that

$$\nabla \cdot I_s' \nabla f(a) = \frac{3}{a^5} [xyz] \begin{bmatrix} I_1' & I_{12}' & I_{13}' \\ I_{12}' & I_2' & I_{23}' \\ I_{13}' & I_{23}' & I_3' \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{1}{a^3} \text{trace } I_s' \tag{A8}$$

where xyz are the components of $\bar{a}$ in the e system, and the moments and products of inertia of the satellite are expressed in the same system. According to what has been said earlier, equation (A8) is unaltered if xyz are the components of $\bar{a}$ in the s system and the inertial quantities are those of the satellite in the same system. These inertial quantities are constants in the body, and some of them may be zero. For purposes of example, it will be assumed that the $x - z$ plane is a plane of symmetry and therefore $I_{12} = I_{23} = 0$.

All that remains to be done is to express the position vector $\bar{a}$ in satellite coordinates so that x, y, and z are expressed in terms of a, ψ, θ, and φ. Let $B = (\varphi)(\psi)(\theta)$ be the transformation matrix from the β coordinates to the s coordinates, so that $a_s = Ba_\beta$, where a_s is the matrix representation of the vector $\bar{a}$ in the s coordinates. Then $a_s^T = (Ba_\beta)^T = a_\beta^T B^T$. With this definition

$$[x_s y_s z_s] \begin{bmatrix} I_1 & 0 & I_{13} \\ 0 & I_2 & 0 \\ I_{13} & 0 & I_3 \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = [00a] \begin{bmatrix} I_1'' & I_{12}'' & I_{13}'' \\ I_{12}'' & I_2'' & I_{23}'' \\ I_{13}'' & I_{23}'' & I_3'' \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$

$$= a_s^T I_s a_s$$

$$= a_\beta^T I_s'' a_\beta$$

$$= a_\beta^T B^T I_s B a_\beta$$

Here $x_s y_s z_s$ are the components of $\bar{a}$ in the s system, and I'' is the inertia tensor expressed in the β system. Now, because of the form of the matrix a_β, only the element I_3'', the element of I_s'' in

the third row and third column, need be considered. For this reason, it will be easier to calculate $B^T I_s B$ instead of Ba_β. Of this transformation only the term $B_{3i}{}^T I_{ij} B_{j3}$ need be obtained where B_{j3} is the element in the jth row and third column of B. Since $B_{3i}{}^T = B_{i3}$, only the elements of the third column of B need be used. The matrix B is given by

$$B = (\varphi)(\psi)(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & s\varphi \\ 0 & -s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & \cdot & -s\theta c\psi \\ \cdot & \cdot & c\varphi s\theta s\psi + s\varphi c\theta \\ \cdot & \cdot & -s\varphi s\theta s\psi + c\varphi c\theta \end{bmatrix}$$

where c stands for cosine, and s stands for sine. Hence,

$$a_\beta{}^T B^T I_s B a_\beta = a^2 \left[I_1 (s\theta c\psi)^2 + I_2 (c\varphi s\theta s\psi + s\varphi c\theta)^2 + I_3 (-s\varphi s\theta s\psi + c\varphi c\theta)^2 + \right.$$

$$\left. 2I_{13} (-s\theta c\psi)(-s\varphi s\theta s\psi + c\varphi c\theta) \right] \tag{10}$$

Equation (10) gives the expression desired for the first term on the right in equation (A8). The second term on the right in equation (A8) is also invariant; that is,

$$\text{trace } I_s'' = \text{trace } I_s = I_1 + I_2 + I_3 \tag{11}$$

Proof of this assertion most easily follows the recognition that

$$I_1 + I_2 + I_3 = 2 \int dm r^2$$

and that the magnitude of the position vector $\bar{r}$ locating the mass point dm is invariant under a coordinate rotation. Using equations (10) and (11) in (A8), one gets

$$\nabla\cdot I_s'\nabla f(a) = \frac{3}{a^3}\left\{I_1(s\theta c\psi)^2 + I_2(c\varphi s\theta s\psi + s\varphi c\theta)^2 + I_3(-s\varphi s\theta s\psi + c\varphi c\theta)^2 + 2I_{13}\left[(-s\theta c\psi)(-s\varphi s\theta s\psi + c\varphi c\theta)\right] - \frac{1}{3}(I_1 + I_2 + I_3)\right\}$$

The same procedure could be repeated for the quantity $\nabla\cdot I_e\nabla f(a)$. But this term is given empirically by $M_e\mu(R/a)^2(3c^2\Theta - 1)$, where $R = 6.364\times10^6$ meters, and $\mu = 5.64\times10^{-4}$ (ref. 8). Substituting the above quantities into equation (4) for the potential energy, one has

$$V = -\frac{GM_e m}{a}\left[1 + \mu\left(\frac{R}{a}\right)^2(1 - 3c^2\Theta)\right] + \frac{3}{2}\frac{GM_e}{a^3}\left[I_1(s\theta c\psi)^2 + I_2(c\varphi s\theta s\psi + s\varphi c\theta)^2 + I_3(-s\varphi s\theta s\psi + c\varphi c\theta)^2 - 2I_{13}(s\theta c\psi)(-s\varphi s\theta s\psi + c\varphi c\theta) - \frac{1}{3}(I_1 + I_2 + I_3)\right] \tag{12}$$

Calculation of angular rates.- The kinetic energy expression contains the two angular velocities $\bar{\Omega}^{s-e}$ and $\bar{\Omega}^{\beta-e}$. The quantity $\bar{\Omega}^{s-e}$ can be written as the sum of the two angular velocities $\bar{\Omega}^{s-e} = \bar{\Omega}^{s-\beta} + \bar{\Omega}^{\beta-e}$, where $\bar{\Omega}^{s-\beta}$ is the angular velocity of the satellite coordinates with respect to the β coordinates. Hence expressions are to be found for the two angular velocities $\bar{\Omega}^{s-\beta}$ and $\bar{\Omega}^{\beta-e}$. The calculations in this section will be made in the following order. First $\bar{\Omega}^{\beta-e}$ will be expressed in the β system in terms of $\dot{\Psi}$ and $\dot{\Theta}$. Then it will be transformed into the s system. Then $\bar{\Omega}^{s-\beta}$ will be expressed in the s system in terms of $\dot{\varphi}$, $\dot{\psi}$, and $\dot{\theta}$, and the two angular velocities will be summed to give $\bar{\Omega}^{s-e}$.

The quantity $\bar{\Omega}^{\beta-e}$ is given in terms of its Euler parameters by the equation $\bar{\Omega}^{\beta-e} = \dot{\Theta}\bar{2}^{\beta} + \dot{\Psi}\bar{3}^{\alpha}$. The α and β systems are related as shown in the following table.

	$\bar{1}^{\alpha}$	$\bar{2}^{\alpha}$	$\bar{3}^{\alpha}$
$\bar{1}^{\beta}$	$c\Theta$	0	$-s\Theta$
$\bar{2}^{\beta}$	0	1	0
$\bar{3}^{\beta}$	$s\Theta$	0	$c\Theta$

Therefore

$$\bar{\Omega}^{\beta-e} = -\dot{\Psi}s\Theta\bar{1}^{\beta} + \dot{\Theta}\bar{2}^{\beta} + \dot{\Psi}c\Theta\bar{3}^{\beta} \tag{13}$$

The transformation from the β system to the s system is given by $(\varphi)(\psi)(\theta)$ which can be tabulated as follows:

	$\bar{1}^{\beta}$	$\bar{2}^{\beta}$	$\bar{3}^{\beta}$
$\bar{1}^{s}$	$c\theta c\psi$	$s\psi$	$-c\psi s\theta$
$\bar{2}^{s}$	$-c\varphi s\psi c\theta + s\varphi s\theta$	$c\varphi c\psi$	$c\varphi s\psi s\theta + s\varphi c\theta$
$\bar{3}^{s}$	$s\varphi s\psi c\theta + c\varphi s\theta$	$-s\varphi c\psi$	$-s\varphi s\psi s\theta + c\varphi c\theta$

(14)

Writing $\bar{\Omega}^{\beta-e}$ as $\bar{\Omega}^{\beta-e} = \Omega_{1s}^{\beta-e}\,\bar{1}^{s} + \Omega_{2s}^{\beta-e}\,\bar{2}^{s} + \Omega_{3s}^{\beta-e}\,\bar{3}^{s}$, one gets for its components

$$\left.\begin{aligned}
\Omega_{1s}^{\beta-e} &= -\dot{\Psi}c\psi s(\theta+\Theta) + \dot{\Theta}s\psi \\
\Omega_{2s}^{\beta-e} &= \dot{\Psi}c\varphi s\psi s(\theta+\Theta) + \dot{\Psi}s\varphi c(\theta+\Theta) + \dot{\Theta}c\varphi c\psi \\
\Omega_{3s}^{\beta-e} &= -\dot{\Psi}s\varphi s\psi s(\theta+\Theta) + \dot{\Psi}c\varphi c(\theta+\Theta) - \dot{\Theta}s\varphi c\psi
\end{aligned}\right\} \tag{15}$$

The determination of $\bar{\Omega}^{s-\beta}$ proceeds similarly. The quantity $\bar{\Omega}^{s-\beta}$ is given by $\bar{\Omega}^{s-\beta} = \dot{\varphi}\bar{1}^{s} + \dot{\psi}\bar{3}^{\delta} + \dot{\theta}\bar{2}^{\gamma}$. To express it in the s system, the transformations (φ), (ψ), and $(\varphi)(\psi)$ are needed. The coefficients of these transformations are tabulated below.

	$\bar{1}^{\delta}$	$\bar{2}^{\delta}$	$\bar{3}^{\delta}$
$\bar{1}^{s}$	1	0	0
$\bar{2}^{s}$	0	$c\varphi$	$s\varphi$
$\bar{3}^{s}$	0	$-s\varphi$	$c\varphi$

	$\bar{1}^{\gamma}$	$\bar{2}^{\gamma}$	$\bar{3}^{\gamma}$
$\bar{1}^{\delta}$	$c\psi$	$s\psi$	0
$\bar{2}^{\delta}$	$-s\psi$	$c\psi$	0
$\bar{3}^{\delta}$	0	0	1

	$\bar{1}^{\gamma}$	$\bar{2}^{\gamma}$	$\bar{3}^{\gamma}$
$\bar{1}^{s}$	$c\psi$	$s\psi$	0
$\bar{2}^{s}$	$-c\varphi s\psi$	$c\varphi c\psi$	$s\varphi$
$\bar{3}^{s}$	$s\varphi s\psi$	$-s\varphi c\psi$	$c\varphi$

(16)

Writing $\bar{\Omega}^{s-\beta}$ as $\bar{\Omega}^{s-\beta} = \Omega_{1s}^{s-\beta}\,\bar{1}^{s} + \Omega_{2s}^{s-\beta}\,\bar{2}^{s} + \Omega_{3s}^{s-\beta}\,\bar{3}^{s}$ and using (16) one gets for the components

$$\Omega_{1s}^{s-\beta} = \dot{\varphi} + \dot{\theta}s\psi \qquad \Omega_{2s}^{s-\beta} = \dot{\psi}s\varphi + \dot{\theta}c\psi c\varphi \qquad \Omega_{3s}^{s-\beta} = \dot{\psi}c\varphi - \dot{\theta}c\psi s\varphi \tag{17}$$

Equations (17) can also be written as matrix equations in the following alternate forms:

$$\begin{bmatrix} \Omega_{1s}^{s-\beta} \\ \Omega_{2s}^{s-\beta} \\ \Omega_{3s}^{s-\beta} \end{bmatrix} = S \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} ; \qquad \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = S^{-1} \begin{bmatrix} \Omega_{1s}^{s-\beta} \\ \Omega_{2s}^{s-\beta} \\ \Omega_{3s}^{s-\beta} \end{bmatrix} \tag{18}$$

where S and its inverse S^{-1} are given by

$$S = \begin{bmatrix} 1 & s\psi & 0 \\ 0 & c\varphi c\psi & s\varphi \\ 0 & -s\varphi c\psi & c\varphi \end{bmatrix} ; \qquad S^{-1} = \frac{1}{c\psi}\begin{bmatrix} c\psi & -s\psi c\varphi & s\psi s\varphi \\ 0 & c\varphi & -s\varphi \\ 0 & c\psi s\varphi & c\psi c\varphi \end{bmatrix} \tag{19}$$

The components of the total satellite angular velocity $\bar{\Omega}^{s-e} = \bar{\Omega}^{s-\beta} + \bar{\Omega}^{\beta-e}$ are obtained from equations (15) and (17). These components in the s system will figure so prominently in the equations of motion that they will be given special symbols. They will be called p, q, and r by

analogy with terminology common in aircraft dynamics. Thus

$$\bar{\Omega}^{s-e} = p\bar{1}^{s} + q\bar{2}^{s} + r\bar{3}^{s}$$

where

$$\left.\begin{aligned} p &= \dot{\varphi} + (\dot{\theta}+\dot{\Theta})s\psi - \dot{\Psi}c\psi s(\theta+\Theta) \\ q &= \dot{\psi}s\varphi + (\dot{\theta}+\dot{\Theta})c\psi c\varphi + \dot{\Psi}s\varphi c(\theta+\Theta) + \dot{\Psi}c\varphi s\psi s(\theta+\Theta) \\ r &= \dot{\psi}c\varphi - (\dot{\theta}+\dot{\Theta})c\psi s\varphi + \dot{\Psi}c\varphi c(\theta+\Theta) - \dot{\Psi}s\varphi s\psi s(\theta+\Theta) \end{aligned}\right\} \tag{20}$$

It should be borne in mind that q and r are positive along the positive directions of coordinates 2^s and 3^s. These are the negatives of what is usually considered positive with respect to an airplane.

Calculation of $\bar{\Omega}^{s-e} \cdot I_s\bar{\Omega}^{s-e}$ and $\bar{\Omega}^{\beta-e} \times \bar{a} \cdot \bar{\Omega}^{\beta-e} \times \bar{a}$.- The purpose of this subsection is to finish calculating the kinematic and inertial quantities appearing in the energy expressions (4) and (8). Up to this point we have chosen the particular geometry summarized in (9). Then the expression for $\nabla \cdot I_s{}'\nabla f(a)$ was calculated. It appears in equation (12). The first of the remaining quantities, $\bar{\Omega}^{s-e} \cdot I_s\bar{\Omega}^{s-e}$, has the familiar form

$$\bar{\Omega}^{s-e} \cdot I_s\bar{\Omega}^{s-e} = I_1p^2 + I_2q^2 + I_3r^2 + 2I_{13}pr \tag{21}$$

where p, q, and r are given in equations (20) and it is assumed that $I_{12} = I_{23} = 0$.

The final quantity to be evaluated is $\bar{\Omega}^{\beta-e} \times \bar{a} \cdot \bar{\Omega}^{\beta-e} \times \bar{a}$. In the β system $\bar{a} = a\bar{3}^{\beta}$ only. Hence using equation (13), one has $\bar{\Omega}^{\beta-e} \times \bar{a} = a(\dot{\Theta}\bar{1}^{\beta} + \dot{\Psi}s\Theta\bar{2}^{\beta})$. From this expression one gets, finally

$$\bar{\Omega}^{\beta-e} \times \bar{a} \cdot \bar{\Omega}^{\beta-e} \times \bar{a} = a^2(\dot{\Theta}^2 + \dot{\Psi}^2s^2\Theta) \tag{22}$$

THE EQUATIONS OF MOTION

The equations of motion will be obtained in this section. At first they will be obtained in terms of the generalized coordinates a, Ψ, Θ, θ, ψ, and φ. Then later they will be obtained in body coordinates in terms of components of linear and angular velocities in the body, and their derivatives. In order to obtain these equations, the kinetic and potential energy functions are needed. The potential energy is given in equation (12) only. But for the kinetic energy, equations (8), (20), (21), and (22) will be used for the generalized coordinates, and equations (6) and (21) for the body coordinates.

Equations in Generalized Coordinates

The equations of motion are derived in the following way. Let ν be any of the coordinates a, Ψ, Θ, ψ, θ, or φ. Then the equation for this coordinate is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\nu}}\right) = \frac{\partial L}{\partial \nu}$$

Where the Lagrangian function is $L = K.E. - V$. The potential function V is given in equation (12):

$$V = -\frac{GM_e m}{a}\left[1 + \mu\left(\frac{R}{\alpha}\right)^2 (1 - 3c^2\Theta)\right] + \frac{3}{2}\frac{GM_e}{a^3}\left[I_1(s\theta c\psi)^2 + I_2(c\varphi s\theta s\psi + s\varphi c\theta)^2 + \right.$$

$$\left. I_3(-s\varphi s\theta s\psi + c\varphi c\theta)^2 + 2I_{13}(-s\theta c\psi)(-s\varphi s\theta s\psi + c\varphi c\theta) - \frac{1}{3}(I_1 + I_2 + I_3)\right] \tag{12}$$

The kinetic energy, from equations (8), (20), (21), and (22), is given by

$$K.E. = \frac{m}{2}\dot{a}^2 + \frac{m}{2}a^2(\dot{\Theta}^2 + \dot{\Psi}^2 s^2\Theta) + \frac{1}{2}(I_1 p^2 + I_2 q^2 + I_3 r^2 + 2I_{13}pr) \tag{23}$$

where

$$p = \dot{\varphi} + (\dot{\theta}+\dot{\Theta})s\psi - \dot{\Psi}c\psi s(\theta+\Theta)$$

$$q = (\dot{\theta}+\dot{\Theta})c\varphi c\psi + \dot{\psi}s\varphi + \dot{\Psi}[s\psi c\varphi s(\theta+\Theta) + s\varphi c(\theta+\Theta)]$$

$$r = -(\dot{\theta}+\dot{\Theta})s\varphi c\psi + \dot{\psi}c\varphi + \dot{\Psi}[c\varphi c(\theta+\Theta) - s\varphi s\psi s(\theta+\Theta)]$$

The left- and right-hand sides of the equations of motion are given by:

for a,

$$\left.\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{a}} = m\ddot{a} \\
&\frac{\partial L}{\partial a} = ma(\dot{\Theta}^2 + \dot{\Psi}^2 s^2\Theta) - \frac{GM_e m}{a^2}\left[1 + 3\mu\left(\frac{R}{a}\right)^2(1 - 3c^2\Theta)\right] + \frac{9}{2}\frac{GM_e}{a^4}\Big\{I_1(s\theta c\psi)^2 + I_2(c\varphi s\theta s\psi + s\varphi c\theta)^2 + \\
&\qquad I_3(-s\varphi s\theta s\psi + c\varphi c\theta)^2 - 2I_{13}[s\theta c\psi(-s\varphi s\theta s\psi + c\varphi c\theta)] - \frac{1}{3}(I_1 + I_2 + I_3)\Big\}
\end{aligned}\right\} \quad (24)$$

for Ψ,

$$\left.\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\Psi}} = \frac{d}{dt}ma^2\dot{\Psi}s^2\dot{\Theta} + \frac{d}{dt}(I_1 p + I_{13}r)c\psi s(\theta+\Theta) + \frac{d}{dt}I_2 q[s\psi c\varphi s(\theta+\Theta) + s\varphi c(\theta+\Theta)] + \\
&\qquad \frac{d}{dt}(I_3 r + I_{13}p)[c\varphi c(\theta+\Theta) - s\varphi s\psi s(\theta+\Theta)] \\
&\frac{\partial L}{\partial \Psi} = 0
\end{aligned}\right\} \quad (25)$$

for Θ,

$$\left.\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\Theta}} = \frac{d}{dt}ma^2\dot{\Theta} + \frac{d}{dt}(I_1 p + I_{13}r)s\psi + \frac{d}{dt}I_2 q c\varphi c\psi - \frac{d}{dt}(I_3 r + I_{13}p)s\varphi c\psi \\
&\frac{\partial L}{\partial \Theta} = ma^2\dot{\Psi}^2 s\Theta c\Theta + (I_1 p + I_{13}r)\dot{\Psi}c\psi c(\theta+\Theta) + I_2 q\dot{\Psi}[s\psi c\varphi c(\theta+\Theta) - s\varphi s(\theta+\Theta)] - \\
&\qquad (I_3 r + I_{13}p)\dot{\Psi}[c\varphi s(\theta+\Theta) + s\varphi s\psi c(\theta+\Theta)] + 6GM_e m\mu\frac{R^2}{a^3}s\Theta c\Theta
\end{aligned}\right\} \quad (26)$$

for θ,

$$\left.\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= \frac{d}{dt}(I_1 p + I_{13} r) s\psi + \frac{d}{dt} I_2 q c\varphi c\psi - \frac{d}{dt}(I_3 r + I_{13} p) s\varphi c\psi \\
\frac{\partial L}{\partial \theta} &= (I_1 p + I_{13} r)\dot{\Psi} c\psi c(\theta+\Theta) + I_2 q \dot{\Psi}[s\psi c\varphi c(\theta+\Theta) - s\varphi s(\theta+\Theta)] - (I_3 r + I_{13} p)\dot{\Psi}[c\varphi s(\theta+\Theta) + s\varphi s\psi c(\theta+\Theta)] - \\
&\quad \frac{3GM_e}{a^3}\{s\theta c\psi[I_1 c\theta c\psi + I_{13}(s\varphi c\theta s\psi + c\varphi s\theta)] + I_2(c\varphi s\theta s\psi + s\varphi c\theta)(c\varphi c\theta s\psi - s\varphi s\theta) + \\
&\quad (s\varphi s\theta s\psi - c\varphi c\theta)[I_3(s\varphi c\theta s\psi + c\varphi s\theta) + I_{13} c\theta c\psi]\}
\end{aligned}\right\} \tag{27}$$

for ψ,

$$\left.\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} &= \frac{d}{dt}(I_3 r + I_{13} p) c\varphi + \frac{d}{dt} I_2 q s\varphi \\
\frac{\partial L}{\partial \psi} &= (I_1 p + I_{13} r)(q c\varphi - r s\varphi) - I_2 q(p - \dot{\varphi}) c\varphi + (I_3 r + I_{13} p)(p - \dot{\varphi}) s\varphi - \frac{3GM_e}{a^3}\{s\theta c\psi(I_{13} s\varphi s\theta c\psi - I_1 s\theta s\psi) + \\
&\quad I_2[c\varphi s\theta c\psi(c\varphi s\theta s\psi + s\varphi c\theta)] + [(c\varphi c\theta - s\varphi s\theta s\psi)(I_{13} s\theta s\psi - I_3 s\varphi s\theta c\psi)]\}
\end{aligned}\right\} \tag{28}$$

for φ,

$$\left.\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} &= I_1 \dot{p} + I_{13} \dot{r} \\
\frac{\partial L}{\partial \varphi} &= (I_2 - I_3) q r - I_{13} p q - \left\{\frac{3GM_e}{a^3}(c\varphi s\theta s\psi + s\varphi c\theta)[(I_2 - I_3)(c\varphi c\theta - s\varphi s\theta s\psi) + I_{13} s\theta c\psi]\right\}
\end{aligned}\right\} \tag{29}$$

Since the quantity μ characterizing the earth's oblateness does not appear in equations (27), (28), and (29), the oblateness of the earth does not generate a torque about a satellite to the order of approximation used. When the equations are written in terms of principal axes, so that $I_{13} = 0$, and when the satellite roll axis is an axis of symmetry so that $I_2 = I_3$, the equations reduce to the form

$$m\ddot{a} = ma(\dot{\Theta}^2 + \dot{\Psi}^2 s^2\Theta) - \frac{GM_e m}{a^2}\left[1 + 3\mu\left(\frac{R}{a}\right)^2(1 - 3c^2\Theta)\right] + \frac{9}{2}\frac{GM_e}{a^4}(I_1 - I_2)\left(s^2\theta c^2\psi - \frac{1}{3}\right) \tag{24a}$$

$$\frac{d}{dt} ma^2\dot{\Psi}s^2\Theta + \frac{d}{dt} I_1pc\psi s(\theta+\Theta) + \frac{d}{dt} I_2s\psi s(\theta+\Theta)(qc\varphi - rs\varphi) + \frac{d}{dt} I_2c(\theta+\Theta)(qs\varphi + rc\varphi) = 0 \tag{25a}$$

$$\frac{d}{dt} ma^2\dot{\Theta} + \frac{d}{dt} I_1ps\psi + \frac{d}{dt} I_2c\psi(qc\varphi - rs\varphi) = ma^2\dot{\Psi}^2 s\Theta c\Theta + I_1p\dot{\Psi}c\psi c(\theta+\Theta) - I_2\dot{\Psi}s(\theta+\Theta)(qs\varphi + rc\varphi) + I_2[\dot{\Psi}s\psi c(\theta+\Theta)(qc\varphi - rs\varphi)] + 6GM_e\mu \frac{R^2}{a^3} s\Theta c\Theta \tag{26a}$$

$$\frac{d}{dt} I_1ps\psi + \frac{d}{dt} I_2c\psi(qc\varphi - rs\varphi) = I_1p\dot{\Psi}c\psi c(\theta+\Theta) + I_2\dot{\Psi}s\psi c(\theta+\Theta)(qc\varphi - rs\varphi) - I_2\dot{\Psi}s(\theta+\Theta)(qs\varphi + rc\varphi) - \frac{3GM_e}{a^3}(I_1 - I_2)s\theta c\theta c^2\psi \tag{27a}$$

$$\frac{d}{dt} I_2(qs\varphi + rc\varphi) = I_1p(qc\varphi - rs\varphi) - I_2(p-\dot{\varphi})(qc\varphi - rs\varphi) + \frac{3GM_e}{a^3}(I_1 - I_2)s^2\theta c\psi s\psi \tag{28a}$$

$$I_1\dot{p} = 0 \tag{29a}$$

Equations in Body Coordinates

The Lagrangian formalism can be used again to derive the equations of motion in body coordinates. Since some of these coordinates are not mutually independent over the history of the motion, considerable complexity is introduced into the general procedure for finding the equations from the Lagrangian (see, e.g., sec. 91 of ref. 9). The procedure adopted in this section is equivalent to the general formulation, but may be somewhat easier to follow.

Force equations.- The kinetic energy expression is given by equations (6) and (21)

$$K.E. = \frac{m}{2}\bar{u}^2 + \frac{1}{2}(I_1p^2 + I_2q^2 + I_3r^2 + 2I_{13}pr)$$

(The linear velocity vector, $\bar{u}$, is given in components by $\bar{u} = u\bar{1}^s + v\bar{2}^s + w\bar{3}^s$. Except for the same caution concerning the positive senses of v and w

that was voiced earlier with respect to the q and r components of the angular velocity $\overline{\Omega}^{s-\beta}$, the labels u,v,w are those customary in aircraft dynamics.) The inertial terms can be obtained from the kinetic energy expression above after it has been noted that the quantities $\partial K.E./\partial\overline{u}$ and $\partial K.E./\partial\overline{\Omega}^{s-\beta}$ can denote the linear and angular momenta, respectively, of the satellite. The inertial terms of the force equation follow directly by differentiating the linear momentum totally with respect to time. This derivative operation is given in rotating coordinates by the right-hand side of the equations

$$\frac{d}{dt}\frac{\partial K.E.}{\partial\overline{u}} = \frac{\partial}{\partial t}\frac{\partial K.E.}{\partial\overline{u}} + \overline{\Omega}^{s-e} \times \frac{\partial K.E.}{\partial\overline{u}}$$

This operation gives rise to the terms usually appearing in the force equations and which will be found on the left in equation (32) below.

The right-hand side of the three force equations will be derived from the potential function given in equation (12). The total force is the negative gradient of this potential, and is given directly in the β set by the equation

$$\overline{F} = -\nabla V = -\frac{1}{a}\frac{\partial V}{\partial\Theta}\overline{1}^{\beta} - \frac{1}{as\Theta}\frac{\partial V}{\partial\Psi}\overline{2}^{\beta} - \frac{\partial V}{\partial a}\overline{3}^{\beta} \tag{30}$$

The values of the components of force are obtained by the appropriate operations on equation (12) and are given by

$$\left.\begin{aligned}
-\frac{1}{a}\frac{\partial V}{\partial\Theta} &= \frac{6GM_e m}{a^2}\mu\left(\frac{R}{a}\right)^2 c\Theta s\Theta \\
-\frac{1}{as\Theta}\frac{\partial V}{\partial\Psi} &= 0 \\
-\frac{\partial V}{\partial a} &= -\frac{GM_e m}{a^2}\left[1 + 3\mu\left(\frac{R}{a}\right)^2(1 - 3c^2\Theta)\right] + \frac{9}{2}\frac{GM_e}{a^4}\Big[I_1(s\theta c\psi)^2 + \\
&\quad I_2(c\varphi s\theta s\psi + s\varphi c\theta)^2 + I_3(c\varphi c\theta - s\varphi s\theta s\psi)^2 - \\
&\quad 2I_{13}s\theta c\psi(c\varphi c\theta - s\varphi s\theta s\psi) - \frac{1}{3}(I_1 + I_2 + I_3)\Big]
\end{aligned}\right\} \tag{31}$$

These components are resolved into the s coordinate system by the transformation $(\varphi)(\theta)(\psi)$. The direction cosines of the transformation can be taken from the table in (14) to give the right-hand sides of the following force equations:

$$\left.\begin{aligned}
m(\dot{u} + qw - rv) &= -\frac{1}{a}\frac{\partial V}{\partial \Theta} c\theta c\psi + \frac{\partial V}{\partial a} c\psi s\theta \\
m(\dot{v} + ru - pw) &= -\frac{1}{a}\frac{\partial V}{\partial \Theta}(s\varphi s\theta - c\varphi s\psi c\theta) - \frac{\partial V}{\partial a}(c\varphi s\psi s\theta + s\varphi c\theta) \\
m(\dot{w} + pv - qu) &= -\frac{1}{a}\frac{\partial V}{\partial \Theta}(s\varphi s\psi c\theta + c\varphi s\theta) - \frac{\partial V}{\partial a}(c\varphi c\theta - s\varphi s\psi s\theta)
\end{aligned}\right\} \qquad (32)$$

Moment equations.- As in the force equations, the inertial terms of the moment equations can be obtained from the kinetic energy expression after it has been noted that the quantity $\partial K.E./\partial\bar{\Omega}^{s-e}$ can denote the angular momentum of the satellite. The inertial terms of the moment equations follow directly by differentiating the angular momentum totally with respect to time. This derivative operation is given in rotating coordinates by the right-hand side of the equation

$$\frac{d}{dt}\frac{\partial K.E.}{\partial\bar{\Omega}^{s-e}} = \frac{\partial}{\partial t}\frac{\partial K.E.}{\partial\bar{\Omega}^{s-e}} + \bar{\Omega}^{s-e} \times \frac{\partial K.E.}{\partial\bar{\Omega}^{s-e}}$$

This operation gives rise to the terms usually appearing in the moment equations and which will be found on the left in the equations below.

$$\left.\begin{aligned}
I_1\dot{p} + qr(I_3 - I_2) + (\dot{r} + pq)I_{13} &= L \\
I_2\dot{q} + rp(I_1 - I_3) + (r^2 - p^2)I_{13} &= M \\
I_3\dot{r} + pq(I_2 - I_1) + (\dot{p} - qr)I_{13} &= N
\end{aligned}\right\} \qquad (33)$$

The quantities appearing on the right in these equations, L, M, and N, are the components of torque applied to the satellite. They, like the components of force in equations (32), can be obtained by differentiating the potential energy expression with respect to the proper parameters, in this case φ, ψ, and θ. These partial derivatives, or their negatives $-\partial V/\partial\varphi$, $-\partial V/\partial\psi$, and $-\partial V/\partial\theta$ are the components of torque in the coordinate system in which the differentiations are performed. This is

a nonorthogonal system based on the unit vectors $\bar{1}^s$, the direction of $\dot{\bar{\varphi}}$; $\bar{3}^\delta$, the direction of $\dot{\bar{\psi}}$; and $\bar{2}^\gamma$, the direction of $\dot{\bar{\theta}}$. It is necessary now to determine the law of transformation that will bring these components of torque into the body coordinate system.

The usual procedure in determining a transformation law is to look for an expression containing components of torque and whose transformation law is known. The rotational power of the system, given by the equation

$$-\frac{d}{dt} V_{rot} = -\frac{\partial V}{\partial \varphi}\dot{\varphi} - \frac{\partial V}{\partial \psi}\dot{\psi} - \frac{\partial V}{\partial \theta}\dot{\theta} \tag{34}$$

is a suitable expression. It is a scalar, entirely independent of coordinate systems. Since the transformation of the angular velocity $\bar{\Omega}^{s-\beta} = \dot{\varphi}\bar{1}^s + \dot{\psi}\bar{3}^\delta + \dot{\theta}\bar{2}^\gamma$ is known from equations (17) and (18), the transformation of the torque components can be deduced. If the components of torque $-\partial V/\partial\varphi$, $-\partial V/\partial\psi$, and $-\partial V/\partial\theta$ are defined as $T_{\dot{\varphi}}$, $T_{\dot{\psi}}$, and $T_{\dot{\theta}}$, respectively, equation (34) can be written in the form of the product of a row and a column matrix as follows:

$$-\frac{d}{dt} V_{rot} = [T_{\dot{\varphi}} T_{\dot{\theta}} T_{\dot{\psi}}] \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [LMN] \begin{bmatrix} \Omega_{1s}^{s-\beta} \\ \Omega_{2s}^{s-\beta} \\ \Omega_{3s}^{s-\beta} \end{bmatrix} \tag{35}$$

Now equations (18) and (19) give the equalities

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = S^{-1} \begin{bmatrix} \Omega_{1s}^{s-\beta} \\ \Omega_{2s}^{s-\beta} \\ \Omega_{3s}^{s-\beta} \end{bmatrix} ; \qquad S^{-1} = \frac{1}{c\psi} \begin{bmatrix} c\psi & -s\psi c\varphi & s\psi s\varphi \\ 0 & c\varphi & -s\varphi \\ 0 & c\psi s\varphi & c\psi c\varphi \end{bmatrix}$$

After this transformation has been substituted into equation (35), a slight rearrangement gives the equation

$$\left\{ [T_{\dot{\varphi}} T_{\dot{\theta}} T_{\dot{\psi}}] \begin{bmatrix} c\psi & -s\psi c\varphi & s\psi s\varphi \\ 0 & c\varphi & -s\varphi \\ 0 & c\psi s\varphi & c\psi c\varphi \end{bmatrix} - c\psi[LMN] \right\} \begin{bmatrix} \Omega_{1s}^{s-\beta} \\ \Omega_{2s}^{s-\beta} \\ \Omega_{3s}^{s-\beta} \end{bmatrix} = 0 \tag{36}$$

Since the angular velocity components can have any values, equation (36) gives the following values of L, M, and N:

$$\left.\begin{aligned} L &= T_{\dot{\varphi}} \\ Mc\psi &= -T_{\dot{\varphi}} s\psi c\varphi + T_{\dot{\theta}} c\varphi + T_{\dot{\psi}} s\varphi c\psi \\ Nc\psi &= T_{\dot{\varphi}} s\psi s\varphi - T_{\dot{\theta}} s\varphi + T_{\dot{\psi}} c\psi c\varphi \end{aligned}\right\} \tag{37}$$

where

$$\left.\begin{aligned} -T_{\dot{\varphi}} &= \frac{\partial V}{\partial \varphi} = \frac{3GM_e}{a^3} [(I_1 - I_3)(c\varphi s\theta s\psi + s\varphi c\theta)(c\varphi c\theta - s\varphi s\theta s\psi) + \\ &\qquad I_{13} s\theta c\psi (s\varphi c\theta + c\varphi s\theta s\psi)] \\ -T_{\dot{\theta}} &= \frac{\partial V}{\partial \theta} = \frac{3GM_e}{a^3} [I_1 c^2\psi s\theta c\theta + I_2(c\varphi s\theta s\psi + s\varphi c\theta)(c\varphi c\theta s\psi - s\varphi s\theta) - \\ &\qquad I_3(c\varphi c\theta - s\varphi s\theta s\psi)(c\varphi s\theta + s\varphi c\theta s\psi) - I_{13} c\theta c\psi (c\varphi c\theta - s\varphi s\theta s\psi) + \\ &\qquad I_{13} s\theta c\psi (c\varphi s\theta + s\varphi c\theta s\psi)] \\ -T_{\dot{\psi}} &= \frac{\partial V}{\partial \psi} = \frac{3GM_e}{a^3} [-I_1 s^2\theta c\psi s\psi + I_2 c\varphi s\theta c\psi (c\varphi s\theta s\psi + s\varphi c\theta) - \\ &\qquad I_3 s\varphi s\theta c\psi (c\varphi c\theta - s\varphi s\theta s\psi) + I_{13} s^2\theta s\psi c\psi (c\varphi c\theta - s\varphi s\theta s\psi) + \\ &\qquad I_{13} s^2\theta c^2\psi s\varphi] \end{aligned}\right\} \tag{38}$$

In the case mentioned before, where $I_2 = I_3$ and $I_{13} = 0$, equations (37) and (38) reduce to the more manageable expression

$$\left.\begin{aligned} L &= 0 \\ M &= -\frac{3GM_e}{a^3}(I_1 - I_2)s\theta c\psi(c\theta c\varphi - s\theta s\psi s\varphi) \\ N &= \frac{3GM_e}{a^3}(I_1 - I_2)s\theta c\psi(s\theta s\psi c\varphi + c\theta s\varphi) \end{aligned}\right\} \tag{39}$$

The latter equations can also be obtained directly from (37) and the potential function which for this case becomes

$$V_{rot} = \frac{3}{2}\frac{GM_e}{a^3}(I_1 - I_2)\left(s^2\theta c^2\psi - \frac{1}{3}\right)$$

The first of equations (39) deserves some attention. It says that the torque about the body roll axis, the axis of symmetry, vanishes and therefore this component of momentum is conserved. In fact, equations (33) show that the component of angular velocity p is constant in this case. By choosing the coordinates in the order ψ,θ,φ, instead of the order θ,ψ,φ as we have done, the rotational potential can be brought into the form

$$V_{rot} = \frac{3}{2}\frac{GM_e}{a^3}(I_1 - I_2)\left(s^2\theta - \frac{1}{3}\right)$$

If, in addition, the rotational energy were to be conserved, that is, if equations (33) could be integrated once, the whole rotational problem could be solved by quadratures. The problem would resemble that of a symmetrical top with one point fixed and acted on by gravity. That this integral does not exist, however, will be seen in the following section.

ENERGY INTEGRAL

When one has obtained equations of motion from a force field derived from a potential function which is not a function of time, he usually expects to be able to integrate the equations and show that the total energy is a constant. The force equations given in equations (32) fit readily into the proper form, as will be shown. It will also be shown, however, that the torque equations do not, so that the sum of the two partial integrals is not a constant. The reason is that the equations are referred to the center of mass of the body, not to the center of gravity. As the reference point of forces and torques is shifted, the

forces remain the same, but the torques are changed by a term consisting of the vector cross product of the displacement of the reference point and the forces that acted on the original point.

The equations (32) can be written in the form

$$\left.\begin{aligned} m(\dot{u} + qw - rv) &= F_{1s} \\ m(\dot{v} + ru - pw) &= F_{2s} \\ m(\dot{w} + pv - qu) &= F_{3s} \end{aligned}\right\} \tag{40}$$

Multiplying the first by u, the second by v, the third by w, and adding, one has

$$m(u\dot{u} + v\dot{v} + w\dot{w}) = F_{1s}u + F_{2s}v + F_{3s}w$$

This can be written in the form

$$\frac{d}{dt}\frac{m}{2}(u^2 + v^2 + w^2) = [F_{1s}F_{2s}F_{3s}]\begin{bmatrix} u \\ v \\ w \end{bmatrix} = [F_{1\beta}F_{2\beta}F_{3\beta}]\begin{bmatrix} a\dot{\Theta} \\ as\Theta\dot{\Psi} \\ \dot{a} \end{bmatrix} \tag{41}$$

The last expression on the right follows from the invariance of power previously discussed. It shows the forces and velocities written in the β set of coordinates. Comparison with equation (30) shows that it can be written $-\overline{v} \cdot \nabla V$, where V is the potential energy and $\overline{v}$ is the vector velocity. Since V is not an explicit function of time,

$$-\overline{v} \cdot \nabla V = -\frac{dV}{dt_{trans}} \tag{42}$$

where the subscript on the derivative on the right means that the translational parameters alone are being considered. Referring to equation (41), one sees that the expression on the left is the time derivative of the translational part of the kinetic energy. Hence, using equation (42), equation (41) can be written in the form

$$\frac{d}{dt} K.E._{trans} + \frac{dV_{trans}}{dt} = 0 \tag{43}$$

This equation shows the proper form to be integrated and to equal a constant. If the potential energy did not contain the rotational parameters φ and θ as it does, equation (43) would be immediately integrable.

We turn now to the moment equations

$$\left.\begin{aligned} I_1\dot{p} + qr(I_3 - I_2) + (\dot{r} + pq)I_{13} &= L \\ I_2\dot{q} + rp(I_1 - I_3) + (r^2 - p^2)I_{13} &= M \\ I_3\dot{r} + pq(I_2 - I_1) + (\dot{p} - qr)I_{13} &= N \end{aligned}\right\} \tag{33}$$

Multiplying the first of these equations by p, the second by q, the third by r, and adding them, one obtains the equation

$$I_1 p\dot{p} + I_2 q\dot{q} + I_3 r\dot{r} + I_{13}(\dot{p}r + p\dot{r}) = Lp + Mq + Nr$$

which can be written in the form

$$\frac{d}{dt}\frac{1}{2}\left(I_1 p^2 + I_2 q^2 + I_3 r^2 + I_{13} pr\right) = [LMN]\begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{44}$$

The right-hand side of (44) can be expanded to the form

$$[LMN]\begin{bmatrix} \Omega_{1s}^{s-\beta} \\ \Omega_{2s}^{s-\beta} \\ \Omega_{3s}^{s-\beta} \end{bmatrix} + [LMN]\begin{bmatrix} \Omega_{1s}^{\beta-e} \\ \Omega_{2s}^{\beta-e} \\ \Omega_{3s}^{\beta-e} \end{bmatrix}$$

Comparing the first term above with equations (34) and (35) shows that it is $-\frac{d}{dt} V_{rot}$. Equation (44), therefore, can be written in the form

$$\frac{d}{dt}\,\mathrm{K.E.}_{rot} + \frac{d}{dt}\,V_{rot} = [LMN]\begin{bmatrix}\Omega_{1s}^{\beta-e}\\ \Omega_{2s}^{\beta-e}\\ \Omega_{3s}^{\beta-e}\end{bmatrix} \tag{45}$$

Adding equations (43) and (45) gives

$$\frac{d}{dt}\,(\mathrm{K.E.} + V) = [LMN]\begin{bmatrix}\Omega_{1s}^{\beta-e}\\ \Omega_{2s}^{\beta-e}\\ \Omega_{3s}^{\beta-e}\end{bmatrix} \tag{46}$$

The power term on the right above spoils the constancy of the total energy given within the parentheses on the left. If the torques, which are small, were zero, then V_{rot} would also be zero and the satellite would have exactly the Poinsot motion described in reference 10. If, for some reason or another, the right-hand side of equation (46) disappeared but V_{rot} did not, and if the satellite had such rotational symmetry that $I_2 = I_1$, then we would have the situation mentioned at the end of the preceding section. The equations of motion would again be integrable and the solution could be expressed in a manner analogous, but not identical, to that concerning a heavy symmetrical top such as given in reference 11.

SMALL OSCILLATIONS OF A SYMMETRIC VEHICLE IN A CIRCULAR ORBIT

The stability of small oscillations about an equilibrium position of the satellite can be studied by using equations (20), (33), and (39). Let the satellite be symmetrical, and the equations be written in principal axes, so that $I_2 = I_3$ and $I_{13} = 0$. Furthermore, for simplicity, assume a polar orbit. Then since $\dot{\Psi} = \Psi = 0$ equations (20) become

$$\begin{bmatrix}p\\ q\\ r\end{bmatrix} = \begin{bmatrix}1 & s\psi & 0\\ 0 & c\psi c\varphi & s\varphi\\ 0 & -c\psi s\varphi & c\varphi\end{bmatrix}\begin{bmatrix}\dot{\varphi}\\ \dot{\theta}+\dot{\Theta}\\ \dot{\psi}\end{bmatrix} \tag{47}$$

which can be inverted to

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} + \dot{\Theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{c\psi} \begin{bmatrix} c\psi & -s\psi c\varphi & s\psi s\varphi \\ 0 & c\varphi & -s\varphi \\ 0 & c\psi s\varphi & c\psi c\varphi \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{48}$$

Equations (33) and (39) take the form

$$I_1\dot{p} = 0 \tag{49}$$

$$I_2\dot{q} + rp(I_1 - I_2) = -\frac{3GM_e}{a^3}(I_1 - I_2)s\theta c\psi(c\theta c\varphi - s\theta s\psi s\varphi) \tag{50}$$

$$I_2\dot{r} + pq(I_2 - I_1) = \frac{3GM_e}{a^3}(I_1 - I_2)s\theta c\psi(c\theta s\varphi + s\theta s\psi c\varphi) \tag{51}$$

The first equation (49) shows that p is constant. Multiplying (50) by $c\varphi$ and (51) by $-s\varphi$ and adding gives

$$I_2(\dot{q}c\varphi - \dot{r}s\varphi) - I_2p(qs\varphi + rc\varphi) + I_1p(qs\varphi + rc\varphi) = \frac{3GM_e}{a^3}(I_1 - I_2)c\psi s\theta c\theta$$

Because of equation (47), this equation can be written as

$$I_2(\dot{q}c\varphi - \dot{r}s\varphi) - I_2\dot{\varphi}(qs\varphi + rc\varphi) - I_2(\dot{\theta} + \dot{\Theta})s\psi(qs\varphi + rc\varphi) + I_1p(qs\varphi + rc\varphi)$$
$$= -\frac{3GM_e}{a^3}(I_1 - I_2)c\psi s\theta c\theta$$

and using equation (48) this becomes

$$\frac{d}{dt} I_2(\dot{\theta} + \dot{\Theta})c\psi - I_2(\dot{\theta} + \dot{\Theta})\dot{\psi}s\psi + I_1p\dot{\psi} = -\frac{3GM_e}{a^3}(I_1 - I_2)c\psi s\theta c\theta \tag{52}$$

Again, multiplying (50) by $s\varphi$, and (51) by $c\varphi$, adding, and using equation (47) gives

$$I_2(\dot{q}s\varphi + \dot{r}c\varphi) + I_2\dot{\varphi}(qc\varphi - rs\varphi) + I_2(\dot{\theta} + \dot{\Theta})s\psi(qc\varphi - rs\varphi) - I_1p(qc\varphi - rs\varphi)$$

$$= \frac{3GM_e}{a^3}(I_1 - I_2)s^2\theta c\psi s\psi$$

With the help of equation (48), the equation becomes

$$I_2\ddot{\psi} + I_2(\dot{\theta} + \dot{\Theta})^2 s\psi c\psi - I_1p(\dot{\theta} + \dot{\Theta})c\psi = \frac{3GM_e}{a^3}(I_1 - I_2)s^2\theta c\psi s\psi \tag{53}$$

Equations (52) and (53) can be used to examine two cases. In the first case the angles and angular rates are assumed small so that second and higher order terms can be neglected. Then, taking a circular orbit, so that $\dot{\Theta}$ and $|\bar{a}|$ are constants and neglecting the effects of φ, ψ, and θ on the orbit the equations become

$$\left.\begin{aligned} I_2\ddot{\theta} + I_1p\dot{\psi} &= -\frac{3GM_e}{a^3}(I_1 - I_2)\theta \\ I_2\ddot{\psi} + I_2\dot{\Theta}^2\psi - I_1p\dot{\theta} &= I_1p\dot{\Theta} \end{aligned}\right\} \tag{54}$$

These equations indicate that the motion will be stable if $I_1 > I_2$. Then the motion consists of undamped oscillations in both yaw and pitch. The yaw and pitch motions are coupled by the arbitrary spin, p, of the body. The frequencies of the oscillations are obtained from the system determinant in the following manner. Dividing equations (54) by I_2 and taking their Laplace transform gives the system of equations

$$\left.\begin{aligned} \left[s^2 + \frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right)\right]\theta + \frac{I_1}{I_2}ps\psi &= f_1(s) \\ -\frac{I_1}{I_2}ps\theta + (s^2 + \dot{\Theta}^2)\psi &= f_2(s) \end{aligned}\right\} \tag{55}$$

The s in these equations is the transform variable. The dependent variables are understood to be functions of s. The quantities f_1 and f_2 are functions of the constant $\dot{\Theta}$ and of the initial conditions. The system determinant is

$$s^4 + \left[\frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right) + \left(\frac{I_1}{I_2}p\right)^2 + \dot{\Theta}^2\right] s^2 + \frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right)\dot{\Theta}^2$$

This expression can be written in the factored form $(s^2 + \omega_1{}^2)(s^2 + \omega_2{}^2)$ where

$$\omega_1{}^2 = \frac{1}{2}\left[\frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right) + \left(\frac{I_1}{I_2}p\right)^2 + \dot{\Theta}^2\right] + \left\{\frac{1}{4}\left[\frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right) + \left(\frac{I_1}{I_2}p\right)^2 + \dot{\Theta}^2\right]^2 - \frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right)\dot{\Theta}^2\right\}^{1/2} \tag{56}$$

and

$$\omega_2{}^2 = \frac{1}{2}\left[\frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right) + \left(\frac{I_1}{I_2}p\right)^2 + \dot{\Theta}^2\right] - \left\{\frac{1}{4}\left[\frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right) + \left(\frac{I_1}{I_2}p\right)^2 + \dot{\Theta}^2\right]^2 - \frac{3GM_e}{a^3}\left(\frac{I_1 - I_2}{I_2}\right)\dot{\Theta}^2\right\}^{1/2} \tag{57}$$

Since $\omega_1{}^2$ and $\omega_2{}^2$ are positive and real, the four roots of the determinant are imaginary and occur in conjugate pairs. The motion consists of two oscillatory modes in yaw and pitch with the frequencies ω_1 and ω_2.

The variation of these frequencies with inertia moment ratio, I_1/I_2, for various values of spin is shown by the solid and dashed lines in figure 2. The dependence of these frequencies on the parameter I_1/I_2 and $I_1p/I_2\dot{\Theta}$ can be obtained from equations (56) and (57) after it has been realized that the relationship $GM_e/a^3 = \dot{\Theta}^2$ holds for circular orbits. Then these equations take the forms

$$\left(\frac{\omega_1}{\dot{\Theta}}\right)^2 = \frac{1}{2}\left[3\frac{I_1}{I_2} - 2 + \left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]\left\{1 + \sqrt{1 - \frac{12\left(\frac{I_1}{I_2} - 1\right)}{\left[3\frac{I_1}{I_2} - 2 + \left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]^2}}\right\} \tag{58}$$

and

$$\left(\frac{\omega_2}{\dot{\Theta}}\right)^2 = \frac{1}{2}\left[3\,\frac{I_1}{I_2} - 2 + \left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]\left\{1 - \sqrt{1 - \frac{12\left(\frac{I_1}{I_2} - 1\right)}{\left[3\,\frac{I_1}{I_2} - 2 + \left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]^2}}\right\} \tag{59}$$

Equation (58) is shown by the solid lines in figure 2 for inertia moment ratios greater than unity. The frequency ratio can be seen to tend toward a linearly increasing dependence on the inertia moment ratio for a given value of the spin parameter $I_1p/I_2\dot{\Theta}$. The frequency ratio also increases with the spin parameter.

Equation (59) is shown by the dashed lines in figure 2 for inertia moment ratios greater than unity. These curves rise from zero frequency at $I_1 = I_2$, the value for a sphere, to a horizontal asymptote that approaches unity as the value of spin vanishes, or for constant values of the parameter $I_1p/I_2\dot{\Theta}$. As the spin increases, however, these curves go toward zero.

The second case of interest has an equilibrium position in the neighborhood of $\theta = \pi/2$. Letting $\theta = \gamma - \pi/2$ and then assuming the angles and angular rates to be small, and bearing in mind that $GM_e/a^3 = \dot{\Theta}^2$, one gets for equations (52) and (53)

$$\left.\begin{aligned} \ddot{\gamma} + \frac{I_1 p}{I_2}\,\dot{\psi} &= 3\dot{\Theta}^2\left(\frac{I_1}{I_2} - 1\right)\gamma \\ \ddot{\psi} + \dot{\Theta}^2\psi - \frac{I_1 p}{I_2}\,\dot{\gamma} &= \frac{I_1 p}{I_2}\,\dot{\Theta} + 3\dot{\Theta}^2\left(\frac{I_1}{I_2} - 1\right)\psi \end{aligned}\right\} \tag{60}$$

This time the oscillations are stable if the inertia moment ratio is less than unity. Then the motion is again purely oscillatory with the yaw and pitch axes coupled by the arbitrary spin. The frequency ratios are given by the square roots of the following positive real numbers

$$\left(\frac{\omega_3}{\dot{\Theta}}\right)^2 = \frac{1}{2}\left[6\left(1-\frac{I_1}{I_2}\right)+1+\left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]\left\{1+\sqrt{1-\frac{12\left(1-\frac{I_1}{I_2}\right)\left[3\left(1-\frac{I_1}{I_2}\right)+1\right]}{\left[6\left(1-\frac{I_1}{I_2}\right)+1+\left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]^2}}\right\} \tag{61}$$

and

$$\left(\frac{\omega_4}{\dot{\Theta}}\right)^2 = \frac{1}{2}\left[6\left(1-\frac{I_1}{I_2}\right)+1+\left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]\left\{1-\sqrt{1-\frac{12\left(1-\frac{I_1}{I_2}\right)\left[3\left(1-\frac{I_1}{I_2}\right)+1\right]}{\left[6\left(1-\frac{I_1}{I_2}\right)+1+\left(\frac{I_1 p}{I_2\dot{\Theta}}\right)^2\right]^2}}\right\} \tag{62}$$

The variation of these frequencies (solid lines for eq. (61) and dashed lines for eq. (62)) with I_1/I_2 and $I_1 p/I_2\dot{\Theta}$ is shown in figure 3.

As the moments of inertia I_1 and I_2 approach each other, equations (58) and (61) also approach each other. At the same time equations (59) and (62) approach zero. This inertial condition is that of a sphere. The torques disappear and an oscillation is superimposed on a constant rate of rotation. Although this description following from the solution of equations (54) and (60) for $\dot{\theta}$ and $\dot{\psi}$ is qualitatively valid, the small angle assumptions underlying these are patently violated.

For a final remark on equations (54) and (60), it can be noted that the yaw equation term $I_1 p\dot{\Theta}$ displaces the equilibrium position of the yaw angle ψ from zero by an amount proportional to the spin, p, of the body and its orbital frequency $\dot{\Theta}$.

CONCLUDING REMARKS

The purpose of this report has been to calculate the equations of motion of an artificial satellite, not for orbital studies, but in a form convenient for stability studies. For this reason, expressions for the kinetic energy and potential function were derived, and the force and moment equations derived from them. Two sets of equations were derived.

The first was obtained directly from the Lagrangian function of the kinetic and potential energies. They are the equations in generalized coordinates. The second set of equations is in body coordinates.

The potential function was derived by expanding the gravitational potential of two mass points in a Taylor's series to include all the mass points of the earth and the satellite. The expansion, which was stopped so as to include only second-order effects, was found to give an expression for the potential that was easy to evaluate in terms of inertia quantities fixed in the body. To the order of expansion used for the potential energy function, the effects of the earth's oblateness and vehicle's mass distribution are separate. Because of this separation, the oblateness of the earth affects only the orbit of the satellite, and does not give rise to torques about the vehicle itself.

The existence of a gravity gradient, however, combines with the fact that the mass of the vehicle is distributed in space to give rise to torques acting about the vehicle. Some care is required in transforming these torques into body axes since in being obtained from the potential function they are expressed in a nonorthogonal coordinate system. The transformation law is derived in this report from the scalar invariance of the rotational power.

The center of mass of the vehicle is chosen as the reference point in the equations of motion. This choice is convenient since it is fixed in the body. But it is shown to have the disadvantage of eliminating the total energy as a first integral of the motion.

Both sets of the equations of motion show that all coordinates are mutually coupled. When they are written in principal body axes, they show that the torque about any axis of symmetry vanishes.

Under the assumptions of mass symmetry, circular orbit, and small angles, the body moment equations show that the pitch and yaw coordinates are coupled by any spin about the axis of symmetry. There is a position of stable equilibrium for a body in which the axis of least inertia moment points toward the center of the earth.

Ames Research Center
National Aeronautics and Space Administration
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APPENDIX A

ALTERNATE DERIVATION OF POTENTIAL ENERGY FUNCTION

The derivation of the potential energy function that appears in the text is concise, but the following derivation may be more illuminating. In the interest of simplicity, the earth will be considered spherical. Since the extension to an oblate earth would follow the same pattern, little is lost by this assumption.

The potential energy of earth-satellite system is given by

$$V = -GM_e \int \frac{dm}{\rho} \tag{A1}$$

where ρ, the distance from the center of mass of the earth to mass point dm, is given by

$$\rho = [(x+\xi')^2 + (y+\eta')^2 + (z+\zeta')^2]^{1/2} \tag{A2}$$

The coordinates of the center of mass of the satellite are x, y, and z in a stationary coordinate system. The coordinates of mass element dm are ξ, η, and ζ relative to the satellite mass center. With ρ^{-1} denoted by $f(\rho)$, this function can be expanded in terms of a Taylor's series about the satellite mass center. The expansion has the form

$$\begin{aligned} f(\rho) = f(a) &+ \xi' \frac{\partial f(a)}{\partial x} + \eta' \frac{\partial f(a)}{\partial y} + \zeta' \frac{\partial f(a)}{\partial z} + \\ &\frac{1}{2}\left\{\xi' \frac{\partial}{\partial x}\left[\xi' \frac{\partial f(a)}{\partial x} + \eta' \frac{\partial f(a)}{\partial y} + \zeta' \frac{\partial f(a)}{\partial z}\right] + \right. \\ &\eta' \frac{\partial}{\partial y}\left[\xi' \frac{\partial f(a)}{\partial x} + \eta' \frac{\partial f(a)}{\partial y} + \zeta' \frac{\partial f(a)}{\partial z}\right] + \\ &\left.\zeta' \frac{\partial}{\partial z}\left[\xi' \frac{\partial f(a)}{\partial x} + \eta' \frac{\partial f(a)}{\partial y} + \zeta' \frac{\partial f(a)}{\partial z}\right]\right\} \end{aligned} \tag{A3}$$

Normally a typical term of this expansion would appear as, say, $\xi' \, \partial f(a)/\partial \xi' |_{\rho=a}$ rather than $\xi' \, \partial f(a)/\partial x$. But an examination of the form of (A2) shows that the operations are equivalent.

Multiplying equation (A3) by dm, rearranging, and integrating gives

$$\int f(\rho)dm = \frac{m}{a} + \left\{\frac{\partial f}{\partial x}\int \xi' dm + \frac{\partial f}{\partial y}\int \eta' dm + \frac{\partial f}{\partial z}\int \zeta' dm\right\} +$$

$$\frac{1}{2}\left\{\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\int \xi'^2 dm + \frac{\partial f}{\partial y}\int \xi'\eta' dm + \frac{\partial f}{\partial z}\int \xi'\zeta' dm\right] +\right.$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\int \xi'\eta' dm + \frac{\partial f}{\partial y}\int \eta'^2 dm + \frac{\partial f}{\partial z}\int \eta'\zeta' dm\right] +$$

$$\left.\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial x}\int \xi'\zeta' dm + \frac{\partial f}{\partial y}\int \zeta'\eta' dm + \frac{\partial f}{\partial z}\int \zeta'^2 dm\right]\right\} \qquad \text{(A4)}$$

The operation $\partial f/\partial$ in this equation stands for $\partial f(a)/\partial$. The terms in the first braces in (A4) vanish because ξ', η', and ζ' refer dm to its center of mass. The quantities in brackets within the second pair of braces now receive attention. The term $\frac{\partial f}{\partial x}\int (\eta'^2 + \zeta'^2)dm$ is added to and subtracted from the quantity within the first brackets. Similarly $\frac{\partial f}{\partial y}\int (\xi'^2 + \zeta'^2)dm$ and $\frac{\partial f}{\partial z}\int (\xi'^2 + \eta'^2)dm$ are added and subtracted in the second and third brackets, respectively. Equation (A4) becomes

$$\int f(\rho)dm = \frac{m}{a} + \frac{1}{2}\left\{\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right]\int\left[\xi'^2 + \eta'^2 + \zeta'^2\right] dm\right\} -$$

$$\frac{1}{2}\left\{\frac{\partial}{\partial x}\left[I_1' \frac{\partial f}{\partial x} + I_{12}' \frac{\partial f}{\partial y} + I_{13}' \frac{\partial f}{\partial z}\right] +\right.$$

$$\frac{\partial}{\partial y}\left[I_{12}' \frac{\partial f}{\partial x} + I_2' \frac{\partial f}{\partial y} + I_{23}' \frac{\partial f}{\partial z}\right] +$$

$$\left.\frac{\partial}{\partial z}\left[I_{13}' \frac{\partial f}{\partial x} + I_{23}' \frac{\partial f}{\partial y} + I_3' \frac{\partial f}{\partial z}\right]\right\} \qquad \text{(A5)}$$

where

$$I_1' = \int (\eta'^2 + \zeta'^2)dm \qquad I_{12}' = -\int \xi'\eta' dm$$

$$I_2' = \int (\xi'^2 + \zeta'^2)dm \qquad I_{13}' = -\int \xi'\zeta' dm$$

$$I_3' = \int (\xi'^2 + \eta'^2)dm \qquad I_{23}' = -\int \eta'\zeta' dm$$

Now, since $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0$, the first term in braces in (A5) is zero. The resulting equation can be expressed in the form

$$\left.\begin{aligned} \int f(\rho)dm &= \frac{m}{a} - \frac{1}{2}\left[\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right]\begin{bmatrix} I_1' & I_{12}' & I_{13}' \\ I_{12}' & I_2' & I_{23}' \\ I_{13}' & I_{23}' & I_3' \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f(a) \\ &= \frac{m}{a} - \frac{1}{2}\nabla\cdot I_s'\nabla f(a) \end{aligned}\right\} \quad (A6)$$

the potential energy, therefore, is

$$V(\rho) = -\frac{GM_e m}{a} + \frac{GM}{2}\nabla\cdot I_s'\nabla f(a) \qquad (A7)$$

There remains to evaluate the derivatives in equation (A5). Since

$$f = [x^2 + y^2 + z^2]^{-1/2} = a^{-1}$$

we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{a^3}\left[3\left(\frac{x}{a}\right)^2 - 1\right]; \qquad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{3}{a^3}\left(\frac{x}{a}\right)\left(\frac{y}{a}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{a^3}\left[3\left(\frac{y}{a}\right)^2 - 1\right]; \qquad \frac{\partial^2 f}{\partial x\,\partial z} = \frac{3}{a^3}\left(\frac{x}{a}\right)\left(\frac{z}{a}\right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{a^3}\left[3\left(\frac{z}{a}\right)^2 - 1\right]; \qquad \frac{\partial^2 f}{\partial y\,\partial z} = \frac{3}{a^3}\left(\frac{y}{a}\right)\left(\frac{z}{a}\right)$$

Substituting these values into the expression for $\nabla \cdot I_s' \nabla f(a)$ gives

$$\nabla \cdot I_s' \nabla f(a) = \frac{3}{a^5}\,[xyz]\begin{bmatrix} I_1' & I_{12}' & I_{13}' \\ I_{12}' & I_2' & I_{23}' \\ I_{13}' & I_{23}' & I_3' \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{1}{a^3}\,\text{trace } I_s' \qquad \text{(A8)}$$

where the trace of the matrix is just the sum of the main diagonal terms:

$$\text{trace } I_s' = [I_1' + I_2' + I_3']$$

REFERENCES

1. Roberson, Robert E.: A Review of the Current Status of Satellite Attitude Control. Presented at the IAS-ARS Regional Meeting on Space Exploration, San Diego, Calif., Aug. 1958.

2. Roberson, Robert E.: Attitude Control of a Satellite Vehicle - An Outline of the Problems. Presented at the Eighth International Astronautical Congress, Barcelona, Spain, ARS Paper 485-57, Oct. 1957.

3. DeBra, D. B.: The Effect of Aerodynamic Forces on Satellite Attitude. Presented at the Western Regional Meeting of the American Astronautical Society, Preprint No. 58-32, Aug. 18-19, 1958.

4. Routh, Edward John: The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies. Ch. XII, 6th ed., 1905. Published in 1955 by Dover Pub.

5. Roberson, R. E., and Tatistcheff, D.: The Potential Energy of a Small Rigid Body in the Gravitational Field of an Oblate Spheroid. Jour. Franklin Inst., vol. 262, no. 3, Sept. 1956, pp. 209-214.

6. Slater, J. C., and Frank, Nathaniel H.: Electromagnetism. Appendices IV and VI. McGraw-Hill Book Co., Inc., 1947.

7. Doolin, Brian F.: The Application of Matrix Methods to Coordinate Transformations Occurring in Systems Studies Involving Large Motions of Aircraft. NACA TN 3968, 1957.

8. Wheelon, Albert D.: Free Flight of a Ballistic Missile. Ramo-Wooldridge Corp., Mar. 7, 1956. (Also pub. as ARDC WDD doc. 7-4766)

9. Corben, H. C., and Stehle, Philip.: Classical Mechanics. Ch. 16. John Wiley & Sons, Inc., 1950.

10. Bracewell, R. N., and Gariott, O. K.: Rotation of Artificial Earth Satellites. Nature, vol. 182, no. 4638, Sept. 20, 1958, pp. 760-762.

11. Webster, Arthur Gordon: The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies. 3rd ed., Hafner Pub. Co., N.Y., 1949, pp. 91-92.

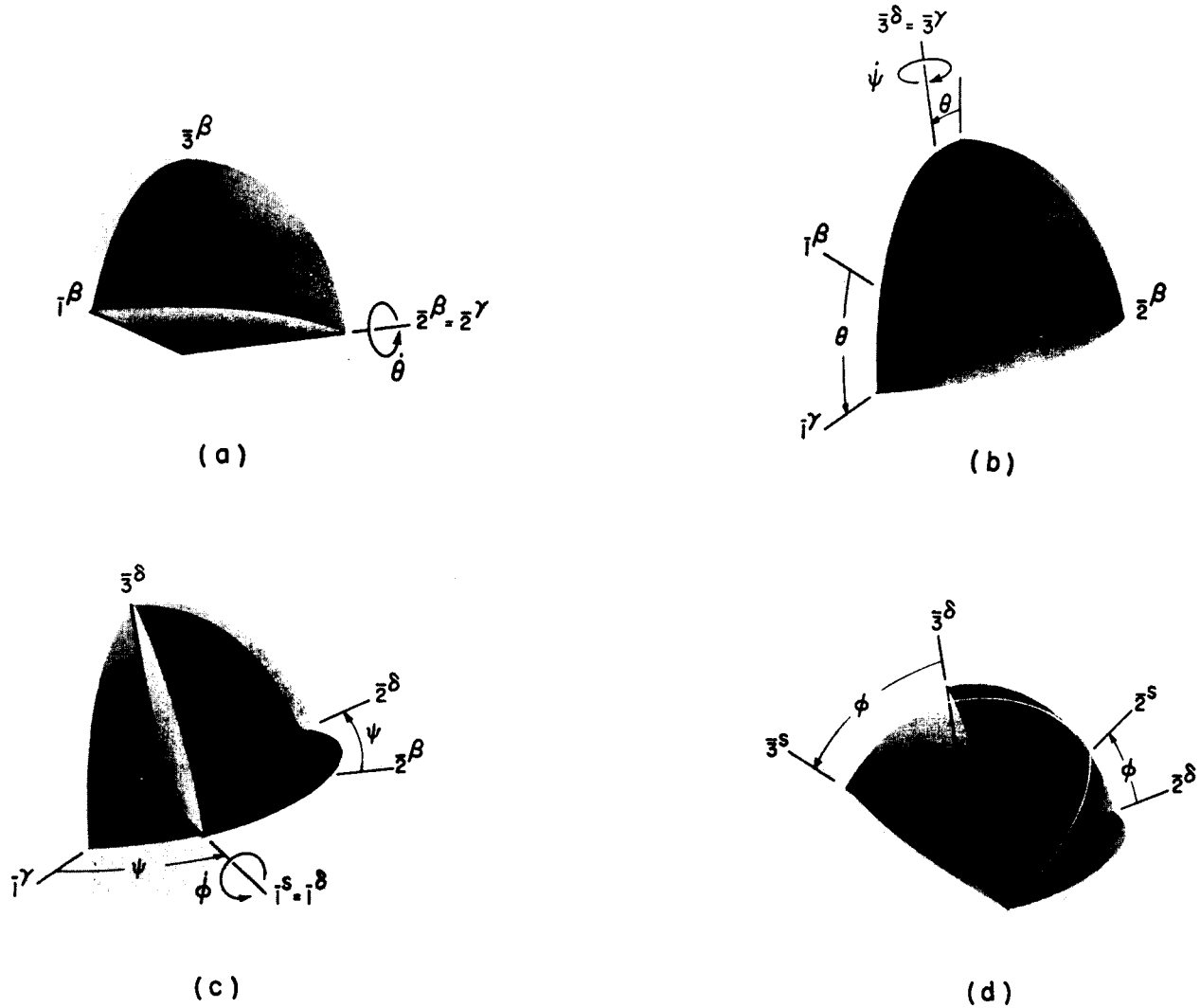

Figure 1.- Satellite orientation angles.

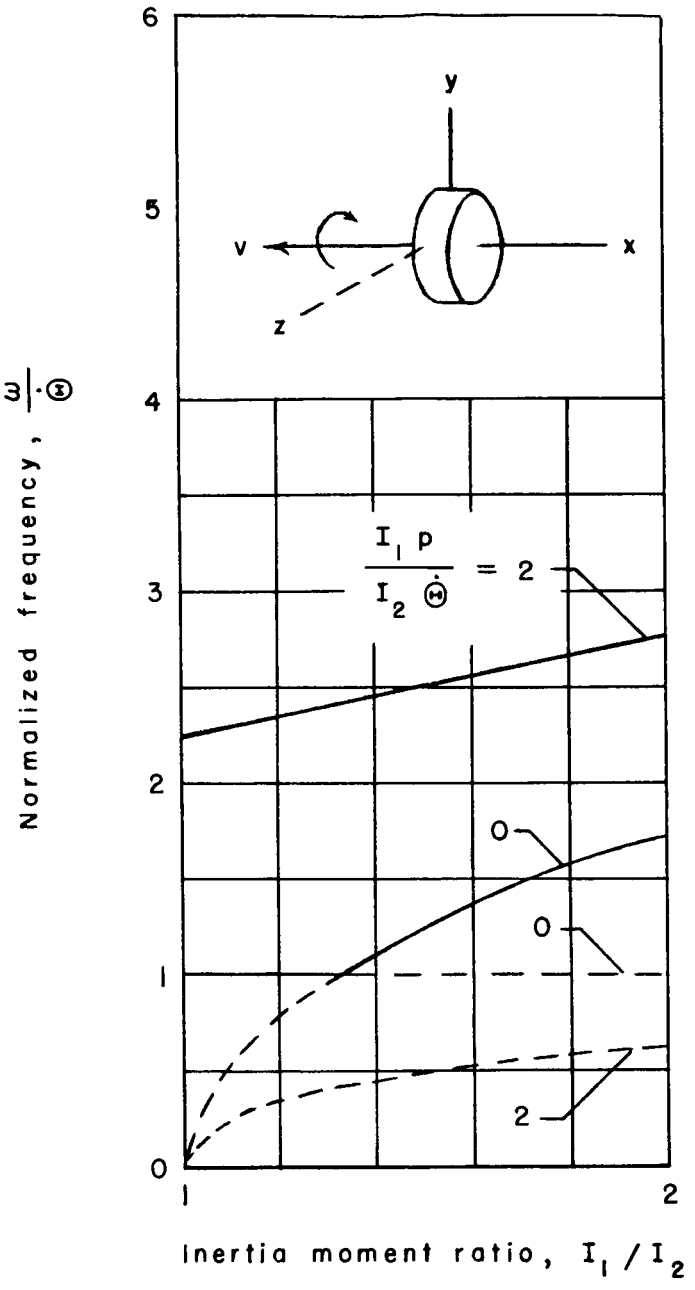

Figure 2.- Oscillation frequency variation for inertia ratios greater than unity; circular orbit.

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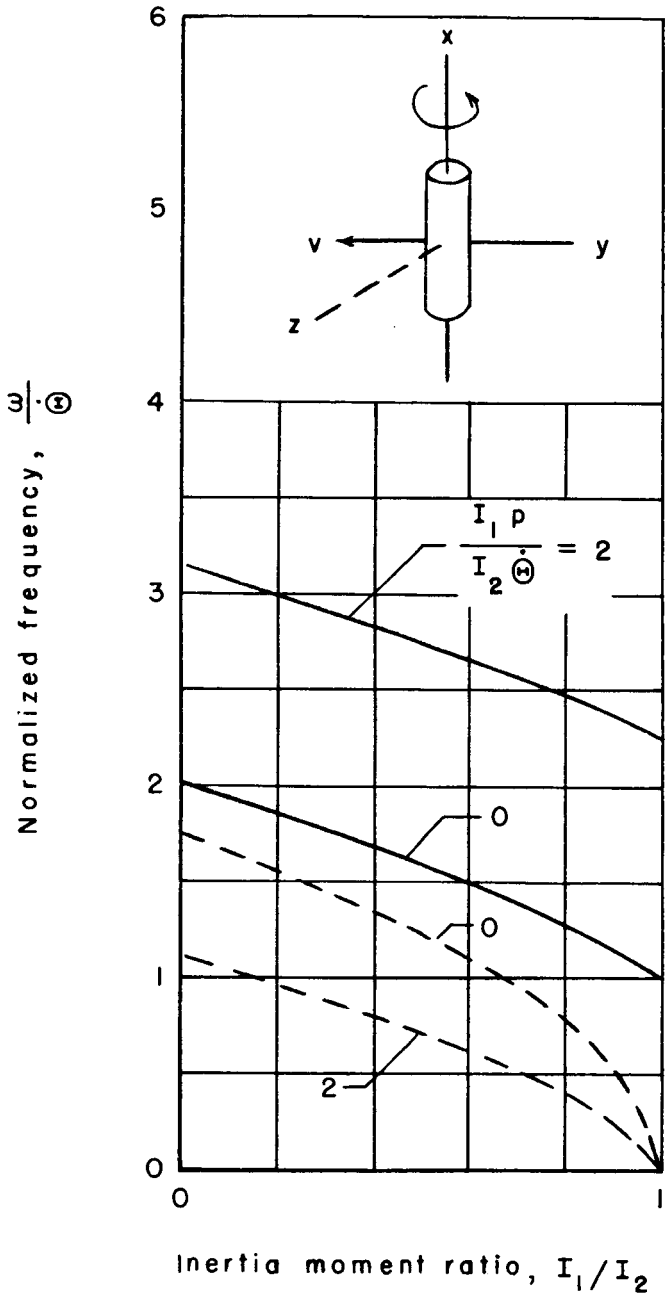

Figure 3.- Oscillation frequency variation for inertia ratios less than unity; circular orbit.