Recursive Multibody Dynamics and Discrete-Time Optimal Control

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Abstract

A recursive algorithm is developed for the solution of the simulation dynamics problem for a chain of rigid bodies. Arbitrary joint constraints are permitted, that is, joints may allow translational and/or rotational degrees of freedom. The recursive procedure is shown to be identical to that encountered in a discrete-time optimal control problem. For each relevant quantity in the multibody dynamics problem, there exists an analog in the context of optimal control. The performance index that is minimized in the control problem is identified as Gibbs' function for the chain of bodies.

1 Introduction

The need to predict the motion of robotic systems in terrestrial and space applications has focused attention on the area of multibody dynamics. In this paper, we treat the simulation dynamics of a chain of rigid bodies. Given the external force distribution and control influences acting on the chain, we show how its subsequent motion, namely, the joint accelerations, can be determined using a recursive procedure. The key to its solution is the elimination of the constraint forces which exist at each joint. Our method in this regard is a generalization of that used by Featherstone [1983] for single degree of freedom joints, although Featherstone [1987] has explained how the extension to general constraints can be effected.

Recently, Rodriguez [1987] has pointed out the similarity between the equations describing a chain of hinged bodies and those that arise in discrete-time optimal estimation and smoothing problems. His approach has utilized the correspondence with optimal filtering (the Kalman filter) and smoothing (the Bryson-Frazier smoother). Here, we show that the equations are identical in form to an optimal control problem. In fact, there is a one-to-one correspondence between the elements of the multibody dynamics problem and the control problem. The feedback solution for the control in terms of the state is precisely that which yields the joint accelerations in terms of the body accelerations. The analogy is further uncovered by identifying the performance index (written in terms of the chain dynamics) as Gibbs' [1879] function.

The major benefit of a recursive solution of the simulation problem is its computational efficiency. One avoids dealing with the system of equations describing the system in its entirety. This would involve Gaussian elimination of the global mass matrix at each time step. The computational consequences of this can be quite substantial since the number of calculations involved in a recursive solution grows linearly with the number of bodies whereas the Gaussian elimination obeys a cubic relationship.

2 Equations of Motion

Let us consider a chain of contiguous bodies \( B_0, B_1, \ldots, B_N \) as shown in Figure 1. Interbody joints may permit arbitrary relative (rotational and/or translational) motion. Each joint therefore possesses at least
one degree of freedom and at most six. For convenience, we shall assume interbody translations to be small; however, the extension to large translations can be incorporated into the present formulation. For additional details on the derivation of the equations of motion, the reader should consult SINCARSIN & HUGHES [1989].

Figure 1: A Chain of Rigid Bodies

The motion of $B_n$ is defined by the velocity $\mathbf{v}_n$ of $O_n$ and the angular velocity $\omega_n$ of $B_n$. (See Figure 2.) Both $\mathbf{v}_n$ and $\omega_n$ are measured with respect to inertial space but are expressed in $\mathcal{F}_n$, a reference frame attached to $B_n$. We shall define

$$\mathbf{v}_n \triangleq \begin{bmatrix} \mathbf{v}_n \\ \omega_n \end{bmatrix}$$  \hspace{1cm} (1) $\triangleq$

as the generalized velocity (cf. twist velocity) of $B_n$ at $O_n$. We furthermore introduce the accompanying definition for a generalized force (cf. wrench) acting at $O_n$:

$$f^{n-1}_n \triangleq \begin{bmatrix} f^{n-1}_n \\ g^{n-1}_n \end{bmatrix}$$  \hspace{1cm} (2) $\triangleq$

where $f^{n-1}_n$ and $g^{n-1}_n$ are the reaction forces and torques on $B_n$ due to $B_{n-1}$ as expressed in $\mathcal{F}_n$.

Figure 2: Reference Frame

The resulting equation of motion for $B_n$ can be written as

$$\mathcal{M}_n \ddot{\mathbf{v}}_n = f_{nT} + f_{nl}$$  \hspace{1cm} (3) $=+$
where

\[ \mathcal{M}_n = \begin{bmatrix} m_n 1 & -v_n^x \\ \omega_n \times v_n & J_n \end{bmatrix} \]

is the (constant) mass matrix corresponding to \( B_n \), that is, \( m_n \), \( \omega_n \) and \( J_n \) are the zeroeth (mass), first and second moments of inertia (about \( O_n \)) of \( B_n \). Also, \( f_{nT} \) is the total external (generalized) force acting on \( B_n \), including interbody forces, and \( f_{nI} \), which accounts for the nonlinear inertial terms, can be neatly written as

\[ f_{nI} = (v_n^x)^T \mathcal{M}_n v_n \]  

(4)

where

\[ v_n^x \triangleq \begin{bmatrix} \omega_n^x \\ \omega_n^x \end{bmatrix} \]

and \( (\cdot)^x \) operating on a Cartesian \((3\times1)\) column matrix, such as \( v_n \), \( \omega_n \) or \( c_n \), is the matrix equivalent of the vector cross product. In a rate-linear model, one would set \( f_{nI} \equiv 0 \).

**Interbody Constraints**

The set of equations (3) does not yet describe a chain of bodies since it does not take into consideration the interbody constraints imposed by the joints. To do so, we begin by observing that

\[ v_n = T_{n,n-1} v_{n-1} + v_{n,int} \]  

(5)

which introduces the relative interbody generalized velocity \( v_{n,int} \) of \( B_n \) with respect to \( B_{n-1} \). In addition,

\[ T_{n,n-1} = \begin{bmatrix} C_{n,n-1} & -C_{n,n-1} r_{n-1}^n \times \\ \cdot & C_{n,n-1} \end{bmatrix} \]

is the generalized transformation matrix between \( B_{n-1} \) and \( B_n \); \( C_{n,n-1} \) is the rotation matrix from \( \mathcal{F}_{n-1} \) to \( \mathcal{F}_n \) and \( r_{n-1}^n \) is the position of \( O_n \) with respect to \( O_{n-1} \). The geometric constraints imposed by the joints can thus be expressed formally as

\[ v_{n,int} = \mathcal{P}_n v_{n\gamma} \]  

(6)

where \( \mathcal{P}_n \) is a projection matrix and \( v_{n\gamma} \) is the column of free joint (rate) variables. The absolute velocities \( v_n \) can be obtained recursively from \( v_{n-1} \) and \( v_{n\gamma} \).

We also note that

\[ f_{nT} = T_{n+1,n}^T f_{n+1}^n - f_{n}^{n-1} + f_{n,ext} \]  

(7)

where \( f_{n,ext} \) is due to solely to external influences. Furthermore, the generalized interbody forces \( f_{n}^{n-1} \) can be expressed as a sum of control forces \( f_{n,c} \) and constraint forces \( f_{n,\square} \), i.e.,

\[ f_{n}^{n-1} = -\mathcal{P}_n f_{n,c} - \mathcal{Q}_n f_{n,\square} \]  

(8)

The projection matrix \( \mathcal{Q}_n \) is the complement of \( \mathcal{P}_n \).

**Projection Matrices**

A few words are perhaps in order regarding the projection matrices. First, as a simple yet very important example, consider a joint with a single rotational degree of freedom about, say, the third axis of an appropriately chosen reference frame. The corresponding projection matrix \( \mathcal{P}_n \) is

\[ \mathcal{P}_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \]

We may also add that \( v_{n\gamma} = \gamma_3 \), where \( \gamma_3 \) is the angle of rotation.

In general, \( \mathcal{P}_n \) is not constant, as above, but rather is dependent on configuration. Contemplation of a universal joint will quickly reveal this fact. The columns of \( \mathcal{P}_n \) are in general not orthonormal but

\[ \mathcal{P}_n^T \mathcal{P}_n = I_n \]  

(9)
where is \( T_n \) is nonsingular. The complementary projection matrix \( Q_n \) satisfies

\[
P_n^T Q_n = 0
\]

Without loss in generality, the columns of \( Q_n \) can be taken as orthonormal.

**Kinematical Equations**

The kinematical equations accompanying the dynamical equations (3) can be summarized in terms of \( T_{n,n-1} \):

\[
T_{n,n-1} = -v_{n,int} T_{n,n-1}
\]

If we express \( v_{n,int} \) as

\[
v_{n,int} = \begin{bmatrix} v_{n,int} \\
\omega_{n,int} \end{bmatrix}
\]

we can extract from (11),

\[
\dot{C}_{n,n-1} = -\omega_{n,int} C_{n,n-1}
\]

For physical reasons, Euler angles make for the most convenient and expedient representation of rotational joint degrees of freedom. Interbody translation is given by the integration of \( v_{n,int} \) and would be reflected in \( r_{n-1}^n \).

### 3 Rate-Linear Simulation Dynamics

The recursive method presented here is a generalization of Featherstone's method applicable to rigid multi-body chains with arbitrary interbody constraints. The development, in fact, runs parallel to a similar generalization of Armstrong's recursive method [D'ELEUTERIO 1989]. The essential difference is that the former is based on an affine relationship of the total interbody force to the absolute (generalized) body acceleration while the latter relates explicitly only the interbody constraint force. The generalized Featherstone approach is particularly appealing because of its direct analogy to the discrete-time optimal problem. As shall be demonstrated, however, a simple equivalence exists between the two schemes.

Let us begin, for explanatory purposes, by considering the rate-linear model, that is, we shall set \( f_{nI} \equiv 0 \) in (3) leaving

\[
M_n \dot{v}_n = f_{nT}
\]

The extension to the nonlinear case (and, in fact, to elastic multibody trees) will be straightforward from here, although not totally without some algebraic effort.

**Recursion for \( f_{n-1}^n \)**

We conjecture that the interbody forces \( f_{n-1}^n \) can be written as

\[
-f_{n-1}^n = \Psi_n \dot{v}_n + \psi_n
\]

which is a generalization of Featherstone's hypothesis. Note that \( \Psi_n \) is, in effect, a mass matrix and \( \psi_n \) is a generalized force quantity. The recursive algorithm is based on this result and the fact that \( \Psi_n \) and \( \psi_n \) can be determined recursively from \( B_N \) to \( B_0 \).

The proof of (15) is by induction:
Step I. For $B_N$, (14) becomes

$$M_N \ddot{q}_N = -f_{N+1}^{n-1} + f_{N,\text{ext}}$$

wherein it has been observed that

$$f_{N+1}^N \equiv 0$$

since $B_N$ is the (free) terminal body. It is immediately obvious that if we set

$$\Psi_N = M_N, \quad \psi_N = -f_{N,\text{ext}}$$

(15) is satisfied for $n = N$.

Step II. We assume that

$$-f_{n+1}^n = \Psi_n \dot{q}_{n+1} + \psi_{n+1}$$

Step III. Given (18), we shall show that (14) follows. Substituting (7) into (14) yields

$$M_n \dot{q}_n = \mathcal{T}_{n+1,n}^T f_{n+1}^n - f_{n-1}^n + f_{n,\text{ext}}$$

Now,

$$\dot{q}_{n+1} = \mathcal{T}_{n+1,n} \dot{q}_n + \mathcal{P}_{n+1} \dot{q}_{n+1,\gamma}$$

and

$$\dot{q}_{n+1} = \mathcal{T}_{n+1,n} \dot{q}_n + \mathcal{P}_{n+1} \dot{q}_{n+1,\gamma}$$

(Note that the terms involving the time derivatives of $\mathcal{T}_{n+1,n}$ and $\mathcal{P}_{n+1}$ are omitted since they are nonlinear rate terms.) Substituting (21) and (8) in (19) and premultiplying by $\mathcal{P}_{n+1}^T$ gives

$$\mathcal{I}_{n+1} \dot{f}_{n+1,c} = \Psi_{n+1,\gamma} \dot{q}_{n+1,\gamma} + \mathcal{P}_{n+1}^T \Psi_{n+1} \mathcal{T}_{n+1,n} \dot{q}_n + \psi_{n+1,\gamma}$$

where, in general,

$$\Psi_{n,\gamma} = \mathcal{P}_{n}^T \Psi_{n} \mathcal{P}_{n}, \quad \psi_{n,\gamma} = \mathcal{P}_{n}^T \psi_{n}\mathcal{P}_{n}$$

Solving for $\dot{q}_{n+1,\gamma}$ from (22), inserting back into (21) and using the result with (18) in (19) eventually leads to

$$-f_{n-1}^n = \{M_n + \mathcal{T}_{n+1,n}^T (\Psi_{n+1} - \Psi_{n+1} \mathcal{P}_{n+1} \mathcal{P}_{n+1}^T \Psi_{n+1}) \mathcal{T}_{n+1,n} \} \dot{q}_n + \{\mathcal{T}_{n+1,n} \Psi_{n+1} \mathcal{P}_{n+1}^T \mathcal{P}_{n+1}^T (\mathcal{I}_{n+1} \dot{f}_{n+1,c} - \psi_{n+1,\gamma}) + \psi_{n+1,\gamma} \} - f_{n,\text{ext}}$$

Hence, we can identify

$$\Psi_n = M_n + \mathcal{T}_{n+1,n}^T (\Psi_{n+1} - \Psi_{n+1} \mathcal{P}_{n+1} \mathcal{P}_{n+1}^T \Psi_{n+1}) \mathcal{T}_{n+1,n}$$

$$\psi_n = \mathcal{T}_{n+1,n} \Psi_{n+1} \mathcal{P}_{n+1}^T \mathcal{P}_{n+1}^T (\mathcal{I}_{n+1} \dot{f}_{n+1,c} - \psi_{n+1,\gamma}) + \psi_{n+1,\gamma} - f_{n,\text{ext}}$$

Step IV. By induction, then, (15) is proven. 

The matrix $\Psi_n$ has an attractive physical interpretation. It is the mass matrix (about $O_n$) of the part of the chain from $B_n$ to $B_N$ associated with the constrained degrees of freedom. FEATHERSTONE [1983] would refer to $\Psi_n$ as the articulated-body inertia. It should also be pointed out that $\Psi_n$, which is positive-definite, and $\psi_n$ are configuration-dependent.

Recursion for $\dot{q}_{n,\gamma}$

By the inductive nature of the proof for (15), it has been shown that the matrices $\Psi_n$ and $\psi_n$ can be evaluated recursively inward, i.e., from $B_N$ to $B_0$. Having done so, one can then perform outward recursion, from $B_0$ to $B_N$, to solve for $\dot{q}_{n,\gamma}$. This is evident from (22).

Rewriting (22) for $B_n$ instead of $B_{n+1}$ and solving explicitly for $\dot{q}_{n,\gamma}$ yields

$$\dot{q}_{n,\gamma} = \Psi_{n,\gamma}^{-1} (\mathcal{I}_{n} \dot{f}_{n,c} - \mathcal{P}_{n}^T \Psi_{n} \mathcal{T}_{n,n-1} \dot{q}_{n-1} - \psi_{n,P})$$

Examining this result, we see that at $B_n$ all the quantities on the right-hand side are known since $\dot{q}_{n-1}$ can be computed recursively from its inboard neighbor according to (21).
4 Nonlinear Simulation Dynamics

The extension to the nonlinear case can be had by simply persevering with the nonlinear rate terms in the preceding development. However, there is a much more palatable approach which is also not without significance in computational considerations.

Let [Golla 1988]

\[ \dot{v}_n = a_n + a_{n,\text{non}} \] (26)

such that

\[ a_n = T_{n,n-1}a_{n-1} + P_n \dot{v}_n \] (27)

Inserting (27) into (5) and differentiating reveals that we must have

\[ a_{n,\text{non}} = T_{n,n-1}a_{n-1,\text{non}} + \dot{T}_{n,n-1}v_{n-1} + P_n v_{n,\gamma} \] (28)

for (28) to hold. In essence, the acceleration quantities \( a_n \) account for the rate-linear effects and \( a_{n,\text{non}} \) for the nonlinear effects. Moreover, not only is \( a_n \) found recursively (outward) but \( a_{n,\text{non}} \) as well.

Upon substitution of (27) into the motion equation (3), we have

\[ M_n a_n = f_{n,T} + f_{n,I} + f_{n,\text{non}} \] (29)

where

\[ f_{n,\text{non}} \triangleq -M_n a_{n,\text{non}} \]

In fact, we could write (30) as

\[ M_n a_n = T^T_{n+1,n} f_{n+1}^n - f_n^{n-1} + f_{n,\text{net}} \] (30)

where

\[ f_{n,\text{net}} \triangleq f_{n,\text{ext}} + f_{n,I} + f_{n,\text{non}} \]

Comparing (31) to (19), we learn that the nonlinear dynamics model is of the identical form as the rate-linear model with \( \dot{v}_n \) replaced by \( a_n \) and \( f_{n,\text{ext}} \) replaced by \( f_{n,\text{net}} \). We can therefore apply the results obtained above directly to the nonlinear case.

Recursion for \( f_{n}^{n-1} \)

In general, then, for rigid multibody chains

\[ -f_n^{n-1} = \Psi_n a_n + \varphi_n \] (31)

Note that \( \Psi_n \) is the same as before; however,

\[ \varphi_n = T^T_{n+1,n} [\Psi_{n+1} P_{n+1} \Psi_{n+1,pp} (I_{n+1} f_{n+1,c} - \varphi_{n+1,p}) \varphi_{n+1}] - f_{n,\text{net}} \] (32)

with

\[ \varphi_{np} \triangleq P^T_n \varphi_n \]

and \( \varphi_N = -f_{N,\text{net}} \).

Recursion for \( \dot{v}_{n,\gamma} \)

The recursive relation for \( \dot{v}_{n,\gamma} \) can be expressed as

\[ \dot{v}_{n,\gamma} = \Psi_n^{-1} p (I_n f_{n,c} - P^T_n \Psi_n T_{n,n-1} a_{n-1} - \varphi_{np}) \] (33)

which reflects (26). It bears mentioning that the kinematical equations remain unchanged.
Relationship to Armstrong's Work

Before proceeding onward, it is worth pointing out that

\[ \Phi_n = \Psi_n Q_n - \Psi_n P_n \Psi_n P_n \Phi_n = Q_n \Phi_n Q_n^T \]

where

\[ \Phi_n = \Psi_n Q_n - \Psi_n P_n \Psi_n P_n \Phi_n P_n \]

and

\[ \Psi_n P_n \Psi_n P_n \Phi_n P_n \Phi_n = Q_n \Phi_n Q_n \]

Showing (34) requires invoking the identity

\[ \mathcal{P}_n T_n^{-1} \mathcal{P}_n + Q_n Q_n^T = 1 \]

By virtue of (34), we can rewrite the first of (24) as

\[ \Psi_n = \mathcal{M}_n + T_{n+1,n}^T Q_{n+1} \Phi_n + T_{n+1,n}^T T_{n+1,n} \]

which is a more streamlined expression.

The significance of \( \Phi_n \), however, lies in the fact that

\[ f_n,0 = \Phi_n Q_n^T a_n + \phi_n \]

where

\[ \phi_n = Q_n^T \nu_n + \Psi_n P_n \Psi_n P_n (I_n f_n,c - \nu_n,\nu) \]

This result is equivalent to Armstrong's method for rigid multibody chains with arbitrary joint constraints.

5 A Discrete-Time Optimal Control Problem

Diverting our attention from multibody dynamics momentarily, consider the following optimal control problem: minimize

\[ J = \sum_{k=0}^{N} \frac{1}{2} x_k^T M_k x_k + x_k^T h_k - u_{k-1}^T t_{k-1} \]

subject to the linear state equation

\[ x_{k+1} = A_k x_k + B_k u_k, \quad x_{-1} = 0 \]

Here, \( M_k \) is a sequence of positive-definite weighting matrices, and \( h_k \) and \( t_k \) are vector weighting sequences. Since \( u_N \) does not influence \( x_k, k \leq N \), we shall assume that \( t_N = 0 \). This problem is slightly different than the standard "linear-quadratic" version that one typically encounters. The cost functional in the present case is linear in the control variable.

Minimizing \( J \) subject to the state equation is a straightforward optimization problem. Introducing the lagrange multiplier or adjoint variable \( \lambda_k \), we define the augmented performance index as follows:

\[ J' \triangleq \sum_{k=0}^{N} \frac{1}{2} x_k^T M_k x_k + x_k^T h_k - u_{k-1}^T t_{k-1} + \lambda_k^T (x_k - A_k x_k + B_k u_k) \]

The necessary conditions for optimality,

\[ \frac{\partial J'}{\partial \lambda_{k+1}} = \frac{\partial J'}{\partial x_k} = \frac{\partial J'}{\partial u_k} = 0 \]
produce the two-point boundary value problem (TPBVP):

\[ x_{k+1} = A_k x_k + B_k u_k, \quad x_{-1} = 0 \]  \hspace{1cm} (40)

\[ \lambda_k = A_k^T \lambda_{k+1} - M_k x_k - h_k, \quad \lambda_{N+1} = 0 \]  \hspace{1cm} (41)

\[ t_k = -B_k^T \lambda_{k+1} \]  \hspace{1cm} (42)

We have taken \( \lambda_{N+1} = 0 \), without loss in generality, since \( t_N = 0 \). Hence, from (41), \( \lambda_N = -M_N x_N - h_N \) which supplies the basis for the inhomogeneous Riccati transformation, sometimes called the sweep method:

\[ \lambda_k = -S_k x_k - r_k \]  \hspace{1cm} (43)

with \( S_N = M_N \) and \( r_N = h_N \). Substituting (43) into the equation for \( t_k \) (42) and replacing \( x_{k+1} \) with the right side of (40), produces the feedback law

\[ u_k = -K_k x_k + R_k^{-1} [t_k - B_k^T r_{k+1}] \]  \hspace{1cm} (44)

where

\[ R_k \triangleq B_k^T S_{k+1} B_k, \quad K_k \triangleq R_k^{-1} B_k^T S_{k+1} A_k \]

The matrix \( R_k \) will be invertible if \( B_k \) is monic and \( S_{k+1} \) is positive-definite. Substituting the sweep solution (43) for \( \lambda_k \) and \( \lambda_{k+1} \) and using (40) for \( x_{k+1} \) and (44) for \( u_k \) gives

\[ [S_k - A_k^T (S_{k+1} - S_{k+1} B_k R_k^{-1} B_k^T S_{k+1}) A_k] x_k = -r_k + (A_k - B_k K_k)^T r_{k+1} + K_k^T t_k + h_k \]

Since this must hold for general \( x_k \), the coefficient of \( x_k \) must vanish as well as the right-hand side. Hence,

\[ S_k = A_k^T (S_{k+1} - S_{k+1} B_k R_k^{-1} B_k^T S_{k+1}) A_k + M_k \]  \hspace{1cm} (45)

which is the discrete-time matrix Riccati equation and

\[ r_k = (A_k - B_k K_k)^T r_{k+1} + K_k^T t_k + h_k \]  \hspace{1cm} (46)

We now return to the question of the invertibility of \( R_k \). The definitions of \( K_k \) and \( R_k \) reveal that \((A_k - B_k K_k)^T S_{k+1} B_k = 0\) which allows us to write the Riccati equation as

\[ S_k = (A_k - B_k K_k)^T S_{k+1} (A_k - B_k K_k) + M_k \]  \hspace{1cm} (47)

Since \( S_N = M_N \) is symmetric and positive-definite, \( S_k \) is symmetric and positive-definite (using backwards induction). Hence, \( R_k \) defined previously is positive-definite and is always invertible.

The optimal control policy can now be summarized as follows: one solves the Riccati equation (45) (or (47)) and the vector equation (46) backwards from \( k = N \) to \( k = 0 \) using the boundary conditions \( S_N = M_N \) and \( r_N = h_N \). The optimal control can then be calculated using (44) while propagating the state forward using the state equation (40).

6 Relationship Between Optimal Control and Recursive Dynamics

The TPBVP generated by the previous optimal control problem (40-42) is identical in form to that of the multibody dynamics problem (30), (33), and

\[ T_n f_{n,c} = -P_n^T f_n^{n-1} \]  \hspace{1cm} (48)

which follows from premultiplying (8) by \( P_n^T \) while recognizing (9) and (10). Therefore, we make the following identifications:

\[ x_k \mapsto a_n \quad \lambda_k \mapsto f_n^{n-1} \]

\[ u_k \mapsto \nu_{n+1,c} \quad h_k \mapsto -f_{n,n,c} \]

\[ A_k \mapsto T_{n+1,n} \quad M_k \mapsto \mathcal{M}_n \]

\[ B_k \mapsto P_{n+1} \quad t_k \mapsto T_{n+1} f_{n+1,c} \]

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Hence, the accelerations $a_n$ are analogous to the states, the interbody forces $f_{n-1}^n$ are analogous to the adjoint states, the joint accelerations $\dot{v}_n \gamma$ play the role of the control inputs, and the projection matrices $\mathcal{P}_n$ take the place of the input matrix $B_k$. It can be shown that the interbody transformation matrices $\mathcal{T}_{n+1,n}$ possess the properties of the state transition matrix thus completing the analogy. Comparing the transformation (43) with the generalization of Featherstone's solution (32) allows us to identify

$$S_k \mapsto \Psi_n, \quad r_k \mapsto \rho_n, \quad R_k \mapsto \Psi_{n+1,P} P.$$ 

We also emphasize that recursion in time ($k$) has been replaced by spatial recursion ($n$) at a given instant in time.

Using the above identifications, the performance index $J$ can be written as

$$J = \sum_{n=0}^{N} \frac{1}{2} \alpha_n^T \mathcal{M}_n a_n - f_{n,net}^T a_n - f_{n,c}^T \mathcal{I}_n \dot{v}_n \gamma$$

Hence, in the multibody dynamics problem one can minimize $J$ subject to the kinematical constraint equation (30) to arrive at the defining equations. Compare this with GIBBS' [1879] formulation of the dynamics of a system of $N$ particles with masses $m_n$, coordinates $z_n, y_n, z_n$, and subjected to forces $X_n, Y_n, Z_n$: minimize

$$\sum_{n=1}^{N} \frac{1}{2} m_n (\ddot{z}_n + \dot{y}_n^2 + \dot{z}_n^2) - X_n \ddot{x}_n - Y_n \dot{y}_n - Z_n \dot{z}_n$$

subject to the kinematical constraints.

In the work of RODRIGUEZ [1987], he points out the similarity between the equations describing a chain of hinged bodies and the TPBVP that arises in discrete-time, optimal estimation and smoothing problems. In his formulation, the bodies in the chain are numbered inwardly (i.e., the tip body is $B_0$ and the root body is $B_N$). Here, the numbering of the bodies is outward (the root body is $B_0$ and the tip body is $B_N$). With this convention, the equations are rendered dual to those of Rodriguez. As such, the corresponding discrete-time problem is not one of estimation and smoothing but one of control. It is interesting to note the dual relationships inherent in Rodriguez's work. The role of the state is played by the interbody forces and the adjoint states are the link accelerations, which are a juxtaposition of the results given above. The control torque at each joint plays the role of a measurement of the states whereas we have the joint accelerations acting as 'control inputs'.

### 7 Summary of the Recursive Algorithm

We now summarize the procedures for determining the motion of the chain of bodies. The control forces, $f_{n,c}(t)$, and external force distribution, $f_{n,ext}(t)$, are prescribed on the time interval of interest. Beginning with $t = 0$, we proceed as follows:

**Step 1.** At time $t$, the relative velocities $v_n(t)$ and the rotation matrices $C_{n,n-1}(t)$ are known.

**Step 2.** Outward recursion for the velocities $v_n$ and determination of $f_{n,net}$:

Do $n = 0$ to $N$;

- Generate $\mathcal{T}_{n,n-1}$ using $C_{n,n-1}$.
- $v_{n,int} = \mathcal{P}_n v_n \gamma$.
- $\mathcal{T}_{n,n-1} = -v_{n,int} \mathcal{T}_{n,n-1}$.
- $v_n = \mathcal{T}_{n,n-1} v_{n-1} + v_{n,int}$.
- $a_{n,non} = \mathcal{T}_{n,n-1} a_{n-1,non} + \mathcal{T}_{n,n-1} v_{n-1} + \mathcal{P}_n v_n \gamma$.
- $f_{n,fl} = (v_n^T \mathcal{M}_n v_n, f_{n,non} = -\mathcal{M}_n a_{n,non}$.
- $f_{n,net} = f_{n,ext} + f_{n,fl} + f_{n,non}$.

Next $n$. 293
Step 3. Inward Recursion for $\Psi_n$ and $\varphi_n$:

Set $\Psi_N = \mathcal{M}_N$ and $\varphi_N = -f_{N,\text{non}}$.

Do $n = N - 1$ to 0;

\[
\begin{align*}
\Psi_{n+1,P} &= \mathcal{P}_n^T\Psi_{n+1}P_{n+1}, \quad P_{n+1,P} = \mathcal{P}_n^T\varphi_{n+1} \\
\kappa_n &= \Psi_n^{-1}\mathcal{P}_n^T\mathcal{P}_n + \Gamma_n + \mathcal{M}_n \\
\Psi_n &= \mathcal{P}_n^T\Psi_{n+1}\mathcal{P}_n + \mathcal{M}_n \\
\varphi_n &= \mathcal{P}_n^T\Psi_n\mathcal{P}_n + \varphi_{n+1} + \kappa_n f_{n+1,c} - f_{n,\text{net}} \\
\text{Next } n.
\end{align*}
\]

If $\mathcal{P}_0 \neq 0$, $\Psi_0 P_{\varphi} = \mathcal{P}_0^T\Psi_0 P_{\varphi}, \quad \varphi_0 = \mathcal{P}_0^T\varphi_0$

Step 4. Outward Recursion for $\dot{v}_{n,\gamma}$:

If $\mathcal{P}_0 = 0$ ($B_0$ is constrained), then $\dot{v}_{0,\gamma} = a_0 = 0$.

Otherwise, $\dot{v}_{0,\gamma} = \Psi_0^{-1}[f_{0,c} - \varphi_{0,\varphi}], \quad a_0 = \mathcal{P}_0^T v_{0,\gamma}$.

Do $n = 1$ to $N$;

\[
\begin{align*}
\dot{v}_{n,\gamma} &= -\kappa_{n-1}a_{n-1} + \Psi_{n,P}^{-1}(f_{n,c} - \varphi_{n,P}) \\
\dot{C}_{n,n-1} &= -\omega_{n,n-1}^\kappa C_{n,n-1} \\
a_n &= T_{n,n-1}a_{n-1} + P_{n}\dot{v}_{n,\gamma} \\
\text{Next } n.
\end{align*}
\]

Step 5. Estimate $v_{n,\gamma}(t + \Delta t)$, $C_{n,n-1}(t + \Delta t)$ using some quadrature scheme.

Go back to Step 1 and replace $t$ with $t + \Delta t$.

This completes the summary of the recursive simulation procedure. Note that in a rate-linear simulation, one ignores the contributions of $f_{n,I}$ and $f_{n,\text{non}}$ to $f_{n,\text{net}}$ in Step 2. We have written the recursion for $\Psi_n$ and $\varphi_n$, in Step 3, in terms of the quantities $\kappa_n$ and $\Gamma_{n+1,n}$ since this leads to the most compact and efficient expressions. The fourth step produces the joint accelerations $\dot{v}_{n,\gamma}$ which can be integrated in conjunction with the kinematical relationships for the rotation matrices to produce the joint orientations/positions and velocities.

8 Concluding Remarks

Given the forces on a chain of rigid bodies, we have shown that the accelerations of the bodies can be determined using the recursive procedures of discrete-time optimal control. The underlying analogy that makes this possible yields great insight into the structure of the multibody dynamics problem.

There are many extensions of the present results, a few of which we shall briefly mention here. The analysis presented was limited to topological chains of rigid bodies. It is easily extended to topological tree configurations. The problem of flexible multibody dynamics has been considered by D'Eleuterio [1989] who shows that the structure of the equations is unaltered by flexibility. Indeed there is a one-to-one correspondence between the rigid and flexible problems. With this duality in hand, one can readily extend the present analysis to the problem of elastic multibody chains. Such an extension has been performed by Damaren & D'Eleuterio [1989].

References


