Verification of Floating-Point Software

D. N. Hoover
Odyssey Research Associates, Ithaca NY

Abstract

Floating point computation presents a number of problems for formal verification. Should one treat the actual details of floating point operations, or accept them as imprecisely defined? or should one ignore round-off error altogether, and behave as if floating point operations are perfectly accurate? There is the further problem that a numerical algorithm usually only approximately computes some mathematical function, and we often do not know just how good the approximation is, even in the absence of round-off error.

ORA has developed a theory of asymptotic correctness which allows one to verify floating point software with a minimum entanglement in these problems. We describe this theory and its implementation in the Ariel C verification system, also developed at ORA. We illustrate the theory using a simple program which finds a zero of a given function by bisection.
Verification of Floating-Point Software

Douglas Hoover

Odyssey Research Associates, Inc.
Difficulties

• Machine real arithmetic does not have nice mathematical properties

• Doesn’t match ideal arithmetic (overflow, round-off, underflow)

• Programs don’t satisfy the specification we’d like them to
Asymptotic Correctness

- Specify "ideal behavior" of the program (e.g. "program computes the square root of its input")

- Verify that if program is run on a sequence of machines converging to perfect accuracy, then program's behavior converges to ideal behavior
Advantages of the Asymptotic Approach

- Machine real arithmetic can be specified loosely
- Specifications can be written in terms of ideal behavior
- Verification does not require roundoff error analysis
- Verifies logical correctness — absence of "bugs" from inaccuracy of machine arithmetic that are not related to error magnitude.
Nonstandard analysis

\[ \mathbb{R} \subseteq \mathcal{R} \]

Standard part map

\[ st : \mathcal{R} \to \mathbb{R} \]

rounds off a finite nonstandard real to an infinitely close standard real.

**Continuity**

if is continuous at \( (a_1, \ldots, a_n) \)

\[ st(f(a_1, \ldots, a_n)) = f(st(a_1), \ldots, st(a_n)) \]

**Differentiation by algebraic manipulation**

Let \( st(\epsilon) = 0, \epsilon \neq 0 \). For all standard \( x \),

\[
\frac{d(x^2)}{dx} = st\left(\frac{(x + \epsilon)^2 - x^2}{\epsilon}\right) \\
= st\left(\frac{2\epsilon x + \epsilon^2}{\epsilon}\right) \\
= st(2x + \epsilon) \\
= 2x
\]
Nonstandard Analysis

- Asymptotic approach can be formalized naturally in nonstandard analysis using infinitesimals
- Primitive operations are assumed to return values which are infinitely close to the ideal values when the arguments and ideal answers are finite
- Programs are specified to have behaviors infinitely close to ideal behavior when inputs are finite
Finding Roots of a Continuous Function

- `find_zero` searches for a root of a user-supplied function $F$ by bisection.

- At each iteration, it tests to see if the values of $F$ at the left endpoint and the midpoint are of opposite sign, and changes one of the endpoints to the midpoint so as to keep a root between the two endpoints.

- The program terminates when it finds a root or when it reaches a user-supplied bound on the number of iterations.
float find_zero(left0, right0, maxit)
float left0, right0;
int maxit;
{
    float left, right, center;
    float cval, lval0, rval0;
    int numit;

    numit = 0;

    lval0 = F(left0);
    rval0 = F(right0);

    left = left0;
    right = right0;
    center = (left + right)/2.0;
    cval = F(center);

    while (cval != 0.0 && numit < maxit) {
        if (lval0 * cval < 0)
            right = center;
        else
            left = center;
        center = (left + right)/2.0;
        cval = F(center);
        lval0 = F(left);
        numit = numit + 1;
    }

    return(center);
}
Specification of find-zero

IF F is continuous and find-zero is started up with

- left0 and right0 not "large";
- maxit "large";
- $F(left0)$ and $F(right0)$ of opposite sign

THEN find-zero terminates normally (i.e. without an exception) and the value output is "close to" some zero of F.
Attempted Verification

• Proof of termination is easy.

• Proof that termination is normal is a bit harder. Must prove that no overflow happens. To prove this, must prove that the values of the endpoints stay in some range of numbers which are not "large".

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How would we prove that the program returns an approximation to a root?

- Prove when the program terminates, the endpoints are “close”. This follows from the fact that the program halves the interval a “large” number of times.

- Prove there’s always a root between the endpoints. This should follow from the way the program decides whether to move the left endpoint or the right. From this we’d get center “close to” a root.

Unfortunately, it’s not true that there’s always a root between the endpoints.
The Bug

• In the test statement, can have lval0 and cval of opposite sign, but have the product underflow to 0. This causes the program to move the wrong endpoint.

• Tests bear out this bug.
Possible Fixes

Several ways to fix this bug

- Change test to

\[(lval0 < 0 \&\& cval \geq 0) \lor (lval0 \geq 0 \&\& cval < 0)\]

- Change test so instead of always testing left endpoint against midpoint, it always tests the endpoint with the larger value of F against the midpoint. This doesn’t necessarily keep a root between the endpoints, but it delivers an approximation to a root anyway.
Ariel

- Verification system for subset of C including real arithmetic and some UNIX system calls.
- Implements nonstandard formalization of the asymptotic approach.
Semantic Verification

- Ariel verifies programs by generating a description of the program's denotation in a higher-order language (the *Clio metalanguage*).

- Specifications are statements about the denotation in the Clio metalanguage.

- Verification is a proof of the specification directly from the description of the denotation in Clio theorem prover.

- Specifications can be any statement about the program's denotation which can be expressed in the Clio, including termination.
C Semantics

• A “run” of the program is modeled as a sequence of events

• Events are:
  – the event of going into a certain state
  – terminating and returning a value
  – terminating and returning no value
  – raising an exception
  – an “unknown” event

• The semantics of the program is expressed as a collection of axioms saying which sequences of events can happen in the course of executing the program.
Sample Verifications

- ZBRENT — a program which finds zeros of a continuous function by bisection

- SWAP — a very simple program to swap the contents of 2 locations which contains a surprising bug

- HOSTILE BOOSTER — a suite of programs, developed by Applied Technology Associates for SDIO, that estimate hostile booster trajectories. This verification is currently in progress.

- SECURE DEVICE DRIVER — specification and verification of security for an Ethernet device driver. Currently in progress.