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ANALYSIS OF SPINNING IN A MONOPLANE WING BY THE INDUCTION METHOD AS COMPARED WITH THE STRIP METHOD

By L. Poggi

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ANALYSIS OF SPINNING IN A MONOPLANE WING BY THE INDUCTION METHOD AS COMPARED WITH THE STRIP METHOD*

By L. Poggi

INTRODUCTION

It is an established fact that the phenomenon of autorotation, which causes the greater percentage of airplane crashes can be verified when, the wing being wholly or in part beyond the stall, the rolling and yawing moments set up as a result of a rotation about the axis of roll are such as to foster this rotation which does not occur when the wing is below the stall.

This manifests the importance which attaches to the study of forces and moments produced by the rotation of the airplane about an axis located in its plane of symmetry.

The problem has been attacked by various authors, in particular by Fuchs and Schmidt who applied the so-called strip method which is based upon the assumption that the forces and moments per unit length acting in each section of the wing are equal to those on an infinite cylindrical wing of equal section in an air flow of intensity and direction resulting from the apparent relative motion of this section with respect to the surrounding air (reference 1). In other words they disregarded the induced velocities, which, however, are of such importance that their omission is bound to result in appreciable errors, as we attempt to prove in this report.

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*"Calcolo dell'autorotazione col metodo dell'induzione in ala monoplana e confronto col metodo della striscia." From L'Aerotecnica, October 1933, pp. 1255-1293. (This paper was awarded a prize of 3000 lire by the Italian Research Council.)

PART I. THEORETICAL ANALYSIS

Notation

The notation is prefaced by the following explanations:

a) Whenever we refer to a quantity in which the induced incidence had been omitted, the symbol for the quantity is over-scored.

b) The subscript $_o$ denotes the value of the pertinent quantity with respect to the wing center.

c) The signs x', y', κ', ζ', used in connection with vector quantities, represent the components along axis x parallel with the zero lift curve of the wing (constant airfoil section and constant angle of attack), of axis y at right angles to x and to the wing axis, of axis κ, parallel to the axis of motion and of axis ζ perpendicular to it and to the wing axis (fig. 1).

L = wing span,

$z = \frac{L}{2}\xi$, distance of a section of the wing from the center (fig. 1),

$l\,L$ = wing chord of any section: (l = nondimensional coefficient),

V = resultant velocity,

Ω = angular velocity of rotation of the wing,

$\omega = \left(\frac{\Omega}{V_{ox}}\right)\frac{L}{2}$, ratio of velocity at wing tip with rotation Ω to the component V_{ox} of the apparent velocity at the center of the wing.

V_i = induced velocity.

General Observations and Simplifications

fig/7 The study of an airfoil in translatory motion and at ~~low~~ angle of attack which is bound up with the so-called

~~second problem~~ of a wing of finite span (see Pistolesi, Aerodinamica, pp. 250, etc.) is more complicated because:

a) The apparent velocity is not constant at every point of the wing:

b) The angle-of-attack range in question is so large as to make it absolutely inadmissible, even in first approximation, to consider the lift curve as a straight line or to substitute for each angle its tangent or sine:

c) The vortices shed by the wing are helicoidal about the axis of motion rather than rectilinear.

Aside from these facts, there is the question of validity of the well-known relation which ties the strength of the free vortices shed by the wing to the lift distribution. And the reason for this is that this relationship is generally derived from the Kutta Joukowski and Stoke's theory on the conservation of the vortex theories the applicability of which does not appear to be admissible without the other.

On the other hand, when we consider the physical phenomenon of the formation of the vortices and bear in mind that it reflects the disequilibration caused by the pressure differences on the top and bottom surfaces of the wing in accord with the change of lift across the span, it is seen that Prandtl's method retains its validity even for the case of stalling, at least in approximation, as is conceded in the present report*.

A check of the above should be of interest and could be readily made in the wing tunnel by measuring the lift and drag on elliptical wings of varying fineness ratio at stalling. If this could be verified for such wings, the difference in fineness ratio should correspond to a difference (constant across the span) of the apparent angle of attack, and would be easily computed.

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*Various English experiments (references 2 and 3) demonstrate the validity of the Kutta Joukowski theory at stalling provided that, to the extent to which it is computed, the circulation cancels normally to the wake.

In view of the foregoing, together with the lack of more reliable criteria, the proposed problem is treated by the conventional method deduced from Prandtl's theory.

With this assumption it is noted that in consequence of consideration b) the lift and drag coefficients are no longer referred to the angle of attack α, a transcendental term of the velocity component, but rather, to its tangent $\frac{V_y}{V_x}$ denoted with A.

In addition, it is noted that in the expression for the circulation - instead of comparing the resultant velocity V, an irrational function of the component of motion - we compare its component on axis x, which, at least, when disregarding the induced velocity, can, as component of axis y, have $V_{o\xi}$ and ω_x ω_ξ as linear function of quantity V_{ox}, V_{oy} and $\omega_x : \omega_y$ or V_{ox}.

Then we write

$$\Gamma = \frac{1}{2} V_x \, l \, L \, P \tag{1}$$

P being a function of A,

which, compared with the current expression gives alone:

$$P = \frac{C_{p\,\infty}}{\cos \alpha}$$

The unit wing lift and the two components on axis x and y are:

$$\begin{aligned} &\frac{1}{2} V \, V_x \, l \, L \, P \\ &-\frac{1}{2} V_y \, V_x \, l \, L \, P \\ &\frac{1}{2} V_x \, V_x \, l \, L \, P \end{aligned} \tag{2}$$

Similarly the drag per unit of wing span and its two components on axis x and y are:

$$\frac{1}{2} V V_x l L R$$

$$\frac{1}{2} V_x V_x l L R \tag{3}$$

$$\frac{1}{2} V_y V_x l L R$$

Hence

$$R = \frac{C_{r\infty}}{\cos \alpha}$$

This obviates the extension in the case in which the components of this said reaction along any other two axes are considered. It is noted that, so long as the induced velocity is disregarded, it is:

$$\bar{V}_x = \bar{V}_{ox} (1 - \omega_y \xi); \; \bar{V}_y = \bar{V}_{ox} (\bar{A}_o + \omega_x \xi)$$
$$\bar{A} = \frac{A_o + \omega_x \xi}{1 - \omega_y \xi} \tag{4}$$

When the induced velocity is taken into account, we either change V or A. The possibility of simplifying the equations makes it possible to write the circulation as

$$\frac{\Gamma}{L} = \frac{1}{2} [\bar{V}_x \bar{P} + v_i P_i'] l \tag{5}$$

by comparing the values of V and A relative to the lift. P_i is a function of A and of the direction of the induced velocity v_i.

Lastly, to insure a real advantage, the coefficients P and R should be expressed in simple manner by means of the variable A.

Then we have

$$P = A_1 A + A_2 A^2 + A_3 A^3 + H_2 A^2 + H_3 A^3 + \ldots \tag{6}$$

$$R = B_o + B_1 A + B_2 A^2 + \ldots \tag{7}$$

Obviously the value of the above coefficients must be computed step by step on the basis of the P and R curves. (As to their order of magnitude, the reader is referred to the numerical example.)

Regarding item c), it is admitted that the free vortices coil around the axis of motion, as is rigorously observed when coincident with the axis of rotation. The results in appendix 1 then permit the substitution of rectilinear for helical vortices starting from the same point of the wing and tangent to the corresponding helical vortices.

Determination of the Induced Velocity of the System of Free Vortices

To begin with, it is admitted that, in spite of the high angles of attack the usual method of calculation can be applied to the induced angles of attack, and specifically that, at each point of the wing, they release a vortex of intensity equal to the derivative in that point of the adhering vorticity and that is of the circulation.

Then, with $v_{i\kappa}$ and $v_{i\zeta}$ as the induced velocity components along axis of motion κ and along axis ζ at right angles to it and of the axis of the wing, the approximation gives

$$d\, v_{i\kappa} = + \frac{1}{4\pi} \frac{d \frac{\Gamma}{L/2}}{\xi - \xi'} \sin(\tan^{-1})(\omega_\kappa\, \xi') \tag{8}$$

$$d\, v_{i\zeta} = - \frac{1}{4\pi} \frac{d \frac{\Gamma}{L/2}}{\xi - \xi'} \cos(\tan^{-1})(\omega_\kappa\, \xi') \tag{9}$$

ξ = reduced abscissa at the point in which the induced drag can be determined, and ξ' = that at the point where the vortex is released.*

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*Note:

$$\omega_\kappa = \frac{\Omega_\kappa}{V_{o\kappa}} \frac{L}{2} = \frac{\Omega}{V_o} \frac{L}{2}$$

Now we can write*

$$\cos \times (\tan^{-1}) \times x = 1 - 0.1\, x - 0.02 \times x^2$$
$$\sin \times (\tan^{-1}) \times x = 1.05 \times x - 0.35 \times x^2 \qquad (10)$$

which, expressed by a multiplicative coefficient, become

$$\frac{d\,\Gamma}{\xi - \xi'} (a\, \xi'^2 + b\, \xi' + c) =$$
$$= - d\,\Gamma \left[a\, \xi' + (a\, \xi + b) + \frac{a\, \xi^2 + b\, \xi + c}{\xi' - \xi} \right] \qquad (11)$$

By integrating (11) relative to ξ' we obtain - when considering that the circulation about the wing tips is certainly zero, otherwise we should have concentric vortices at these points -:

$$v_{i\kappa} = - \frac{1}{4\,\pi} \left[(1.05\, \xi\, \omega_\kappa - 0.35\, \xi^2\, \omega_\kappa{}^2) \int_{-1}^{1} \frac{1}{\xi - \xi'} \frac{\frac{d\,\Gamma}{\frac{1}{2} L}}{d\,\xi'}\, d\,\xi' - \right.$$
$$\left. - 0.35\, \omega_\kappa{}^2 \int_{-1}^{1} \frac{\Gamma}{\frac{1}{2} L}\, d\,\xi' \right]$$

$$v_{i\zeta} = \frac{1}{4\,\pi} \left[(1 - 0.1\, \xi\, \omega_\kappa - 0.20\, \xi^2\, \omega^2\, \kappa) \int_{-1}^{1} \frac{1}{\xi - \xi'} \frac{\frac{d\,\Gamma}{\frac{1}{2} L}}{d\,\xi'}\, d\,\xi' - \right.$$
$$\left. - 0.20\, \omega_\kappa{}^2 \int_{-1}^{1} \frac{\Gamma}{\frac{1}{2} L}\, d\,\xi' \right]$$

---

*Note the comparison between the values of (10) and the effective values of the particular functions in the appendix.

or for (10)

$$v_{i\kappa} = \frac{1}{4\pi}\left[\sin(\tan^{-1})(\omega_\kappa \xi)\int_{-1}^{1}\frac{1}{\xi - \xi}\frac{\frac{d\Gamma}{L/2}}{d\xi'}d\xi' - \right.$$

$$\left. - 0.35\,\omega_\kappa^2\int_{-1}^{1}\frac{\Gamma}{\frac{1}{2}L}d\xi'\right]$$

$$v_{i\zeta} = \frac{1}{4\pi}\left[\cos(\tan^{-1})(\omega_\kappa \xi)\int_{-1}^{1}\frac{1}{\xi - \xi'}\frac{\frac{d\Gamma}{\frac{1}{2}L}}{d\xi'}d\xi' - \right.$$

$$\left. - 0.20\,\omega_\kappa^2\int_{-1}^{1}\frac{\Gamma}{\frac{1}{2}L}d\xi'\right] \qquad (12)$$

and which is, with exception of the terms containing $\int_{-1}^{1}\Gamma\, d\xi'$ which usually are much smaller, precisely the resultant induced velocities obtained when the vortex filaments are parallel to that leaving at the particular point. Consequently they are perpendicular to the direction of the velocity in that point and their magnitude will be that of a system of coplanar vortices of identical strength.

Analysis of Circulation Distribution

We begin with the terms containing $\int_{-1}^{1}\Gamma\, d\xi'$. By substituting the vector sum $\overline{V}_0'$ for the apparent velocity of the center of pressure V_0 the circulation distribution can be computed when taking $\overline{V}_0'$ as the apparent velocity and considering for the induced velocity only the first terms occurring in the second term of (12), which corresponds to the induced velocity when the vortices are coplanar to that shed at the particular point.

On the basis of the defined distribution we can determine $\int_{-1}^{1} \Gamma \, d\,\xi'$ and thus arrive at the apparent velocity V_o.

The objectionable feature of this procedure lies in the difficulty of following the calculation based upon a sufficiently established apparent velocity, which has no other interest when we wish to pursue the calculations of the entire angle of attack range.

The term $\int_{-1}^{1} \Gamma \, d\,\xi'$ is mor tractable with the strip method, so that $\overline{V}_o'$ can be determined when the velocity $\overline{V}_o'$ is known. This method involves no appreciable error, as the terms are generally quite small in comparison with the remainder of (12).

Putting $\overline{V}_o'$ instead of $\overline{V}_o$ we can determine the induced velocities on the basis of the abstracted terms with $\int_{-1}^{1} \Gamma \, d\,\xi'$. The following figures always refer to a velocity at the center of the wing equal to $\overline{V}_o'$. However, for the sake of brevity we write $\overline{V}_o$. Then even the smallness of the terms enables us to differentiate between the two quantities, and we retain the notation v_i to indicate the part of the pertinent induced velocity.

As to the statement about the rectangularity of the induced velocity to the apparent velocity $\overline{V}$, the induced angle of attack results in $\frac{v_i}{V}$ and the effective velocity will practically be equal in magnitude to the apparent velocity, hence the increment due to them in the circulation will be $\frac{1}{2} v \frac{d\,C_{p\infty}}{d\,\alpha}$.

Compared with (5) it is

$$P_i' = \frac{d\,C_{p\infty}}{d\,\alpha} \tag{13}$$

which serves to calculate function P_i.

Now we write into (5) the value of the induced veloc-

ity derived from the preceding considerations, so that:

$$\frac{\Gamma}{L} = \frac{1}{2}\left[\overline{V}_{ox}\,(1 - \omega_y\,\xi)\,Pl + P_i'l\,v_i\right] =$$

$$= \frac{1}{2}\,V_{ox}\left[(1 - \omega_y\,\xi)\,Pl - \frac{1}{4\,\pi}\,\frac{P_i'l\int_{-1}^{1}\frac{1}{\xi - \xi'}\,\frac{d\,\frac{\Gamma}{\frac{1}{2}L}}{d\,\xi'}\,d\,\xi'}{V_{ox}}\right] \tag{14}$$

The result is an integral equation which can be treated with the method perfectly analogous to that employed in the resolution of the "second problem on wings of finite span" cited above.

For example, when Glauert's method (described in detail in Pistolesi's treatise) which is particularly appropriate, is used, we can put:

$$\Gamma = V_{ox}\,L\sum_{1}^{\infty} a_n \sin n\,\theta \tag{15}$$

whereby:

$$\cos\theta = \xi \tag{16}$$

Then we have (see above):

$$-\frac{1}{4\,\pi}\int_{-1}^{1}\frac{1}{\xi - \xi'}\,\frac{d\,\frac{\Gamma}{\frac{1}{2}L}}{d\,\xi'}\,d\,\xi' = \frac{-V_{ox}}{2\sin\theta}\sum_{1}^{\infty} n\,a_n \sin n\,\theta = v_i \tag{17}$$

and (14) becomes

$$\frac{\Gamma}{V_{ox}L} = \sum_{1}^{\infty} a_n \sin n\,\theta =$$

$$= \frac{1}{2}\left[(1 - \omega_y \cos\theta)\,P\,l - \frac{P_i'\,l}{2\sin\theta}\sum_{1}^{\infty} n\,a_n \sin n\,\theta\right]$$

or:

$$\sum_{1}^{\infty} a_n \sin n\theta \left[1 + \frac{P_1' l n}{4 \sin \theta}\right] = \frac{1}{2} P l (1 - \omega_y \cos \theta) \quad (18)$$

Note: Incidentally it will be seen that this integral equation is substantially the same and gives therefore the same values for the circulation as that relative to the "second problem" for a wing of finite span moving with uniform forward speed V_0 and for which the lift coefficient will be proportional to the total effective angle of attack (apparent γ_0 and induced γ_i) according to the ratio of proportionality A. The chord c and the apparent angle of attack γ_0 are defined from

$$\frac{c}{L} = \frac{1}{A} P_1' l$$

$$\gamma_0 = \frac{\bar{P}}{P_1'}(1 - \omega_y \cos \theta)$$

Thus it is apparent that the variation of the derivative of the lift curve corresponds to a congruent change in apparent angle of attack and wing chord.

The induced velocity then will be:

$$v_i = \frac{-V_{0x}}{2 \sin \theta} \sum_{1}^{\infty} n\, a_n \sin n\theta \quad (19)$$

Determination of Coefficients a_n

Equation (18) being satisfied for all points of the axis of the wing exactly defines the infinite coefficients a_n. Their determination requires in general the resolution of a system of infinite equations obtained opportunely by checking (18) for an infinite number of points on the wing axis.

In practice, since the coefficients a_n - being related to a development in Fourier series - approach zero when n = infinite, and usually are small enough to be negligible, when starting from $n = 6$ to 8, we can confine ourselves to the first terms of the development.

If they are m in number the relative m coefficients can be obtained in two ways:

The first, which has the advantage of not requiring an analytical term of the curve of coefficient P and P_1', simply consists in prescribing that (18) be complied with at m points of the wing axis (obviously the tips must also be identically satisfied) well chosen.

The second method, which we prefer, has the great advantage of giving Γ and $\overline{\Gamma}$ (circulation with or without considering the induction effect) in the same analytical form, and which, lending itself readily for comparison of both cases, is a natural extension of Munk's method for the elliptical wing at low angle of attack.

It requires the development of

$$\frac{1}{2}\,(1 - \omega_y \cos\theta)\,\overline{P}\,l$$

$$\frac{P_1'\,l}{4 \sin\theta}$$

in Fourier series, to wit:

$$\frac{1}{2}\,(1 - \omega_y \cos\theta)\,\overline{P}\,l = \sum_1^\infty \overline{a}_n \sin n\,\theta \tag{20}$$

$$\frac{P_1'\,l}{4 \sin\theta} = \sum_1^\infty t_n \cos n\,\theta \tag{21}$$

The first term of (18) can, as we know, be expressed with

$$\sum_1^\infty k_n \sin n\,\theta$$

wherein k_n is a linear function of a_n.

To comply with (18) it is:

$$k_n = \overline{a}_n$$

The result is a system of infinite equations of infinite unknown values a_n. Then, if the development contains only a limited number of terms (say, two or three), the problem is markedly simplified, with the result that a number of coefficients cancel

Note: In the above treatment it was tacitly assumed that the increment of the lift coefficient was in every point proportional to the induced angle of attack. Now, since the induced angles of attack can assume, as we know, appreciable values, this simplification consisting in final analysis, in substituting at each point of the lift curve the tangent of this curve, can involve considerable errors. Although we were not concerned with this as we did not wish to complicate our problem too much, it nevertheless should be interesting, even for the effects on other problems, to see how to obtain a closer approximation.

We have: V = velocity, α_o = apparent angle of attack

$$\Gamma = L\, l\, V\, \bar{C}_p = V\, L \sum_1^\infty \bar{a}_n \sin n\,\theta$$

the value of the circulation in a certain approximation (for example by strip method);

$$\bar{v}_i = \frac{-V}{2 \sin\theta} \sum_1^\infty n\, \bar{a}_n \sin n\,\theta$$

the corresponding value of the induced velocity by the same approximation;

$$\Gamma = L\, V\, \Sigma\, a_n \sin n\,\theta$$

the effective value of the circulation;

$$v_i = \frac{-V}{2 \sin\theta} \sum_1^\infty n\, a_n \sin n\,\theta$$

the effective value of the induced velocity.

We write

$$\frac{\Gamma}{L} = V \sum_1^\infty a_n \sin n\,\theta = \left\{ \bar{C}_p + \frac{d\,C_p}{d\,\alpha} \frac{(\bar{v}_i - v_i)}{V} \right\} l\, V$$

This equation is similar to (14), but the incremental terms and the quantity are much reduced, hence the approximation is better.

Induced Velocity*

Equation (19) yields:

$$v_{i\kappa} = v_i \sin(\tan^{-1})\frac{\omega_\kappa \cos\theta}{1 - \omega_\zeta \cos\theta}$$

$$v_{i\zeta} = v_i \cos(\tan^{-1})\frac{\omega_\kappa \cos\theta}{1 - \omega_\zeta \cos\theta}$$

which may be written in the form of:

$$v_{i\kappa} = \frac{V_{ox}}{2 \sin\theta} \sum_1^\infty l_n \sin n\theta$$

$$v_{i\zeta} = \frac{V_{ox}}{2 \sin\theta} \sum_1^\infty c_n \sin n\theta$$

The b_n and c_n values are easily computed, when bearing in mind (12) as functions of the coefficients a_n (see Part II).

Similar terms are obtained for the induced velocities projected on axis x and y.

Forces and Moments on the Wing

a) Forces and Moments Due to Lift.

First we abstract the induced velocities. Based upon (15) the circulation is then expressed as

$$\bar{\Gamma} = \bar{V}_{ox} L \Sigma \bar{a}_n \sin n\theta$$

*Contrary to the general rule the induced velocities are figured positive according to the positive direction of the corresponding axis.

The force components along axis κ and ζ due to the lift per unit length of the wing are:

$$
\left.
\begin{aligned}
P_\kappa &= -\rho \int_{-\frac{L}{2}}^{\frac{L}{2}} \overline{V}_\zeta \overline{\Gamma}\, dz = -\rho V_{ox} \frac{L}{2} \int_0^\pi \omega_\kappa \cos\theta\, \overline{\Gamma} \sin\theta\, d\theta \\
&= -\rho \frac{V_{ox}^2 L^2}{2} \omega_\kappa \int_0^\pi \sum_1^\infty \overline{a}_n \sin n\theta \cos\theta \sin\theta\, d\theta \\
P_\zeta &= \rho \int_{-\frac{L}{2}}^{\frac{L}{2}} \overline{V}_\kappa \overline{\Gamma}\, dz = \rho \overline{V}_{ox} \frac{L}{2} \int_0^\pi \left[\frac{1}{\cos\alpha_o} - \omega_\zeta \cos\theta\right] \overline{\Gamma} \sin\theta\, d\theta \\
&= \rho \frac{\overline{V}_{ox}^2 L^2}{2} \left[\frac{1}{\cos\alpha_o} \int_0^\pi \sum_1^\infty \overline{a}_n \sin n\theta \sin\theta\, d\theta - \right. \\
&\qquad \left. - \omega \int_0^\pi \sum_1^\infty \overline{a}_n \sin n\theta \cos\theta \sin\theta\, d\theta\right]
\end{aligned}
\right\} \quad (22)
$$

and the moment components are:

$$
\left.
\begin{aligned}
M_{p\zeta} &= -\rho \int_{-\frac{L}{2}}^{\frac{L}{2}} \overline{V}_\zeta \overline{\Gamma}\, dz = -\rho \frac{V_{ox}^2 L^3}{4} \left[\frac{\omega_\kappa}{2} \int_0^\pi \sum_1^\infty a_n \sin n\theta \sin 2\theta \cos\theta\, d\theta\right] \\
M_{p\kappa} &= \rho \int_{-\frac{L}{2}}^{\frac{L}{2}} \overline{V}_\kappa \overline{\Gamma} z\, dz = \frac{\rho V_{ox}^2 L^3}{4} \left[\frac{1}{\cos\alpha_o} \int_0^\pi \sum_1^\infty \overline{a}_n \sin n\theta \cos\theta \sin\theta\, d\theta - \right. \\
&\qquad \left. - \frac{\omega_\zeta}{2} \int_0^\pi \sum_1^\infty \overline{a}_n \sin n\theta \sin 2\theta \cos\theta\, d\theta\right]
\end{aligned}
\right\} \quad (23)
$$

and is

$$
\left.
\begin{aligned}
\int_0^\pi \sin n\theta \sin\theta\, d\theta &= \begin{cases} 0 & \text{for } n \neq 1 \\ \frac{\pi}{2} & \text{for } n = 1 \end{cases} \\
\int_0^\pi \sin m\theta \sin n\theta \cos\theta\, d\theta &\begin{cases} 0 & \text{for } |n - m| \neq 1 \\ \frac{\pi}{4} & \text{for } |n - m| = 1 \end{cases}
\end{aligned}
\right\} \quad (24)
$$

we finally arrive at

$$\left.\begin{aligned}
\bar{f}_{p\kappa} &= \frac{\bar{P}_\kappa}{\frac{1}{2}\rho S \bar{V}_o^2} = -\cos^2\alpha_o \frac{\pi\lambda}{2}\frac{\omega_\kappa}{2}\bar{a}_2 \\
\bar{f}_{p\zeta} &= \frac{\bar{P}_\zeta}{\frac{1}{2}\rho S \bar{V}_o^2} = \cos^2\alpha_o \frac{\pi\lambda}{2}\left\{\frac{a_1}{\cos\alpha_o} - \frac{\omega_\zeta}{2}\bar{a}_2\right\} \\
\bar{m}_{p\zeta} &= \frac{M_{p\zeta}}{\frac{1}{2}\rho \bar{V}_o^2 S^{3/2}} = -\cos^2\alpha_o \frac{\pi}{4}\lambda^{3/2} \times \frac{\omega_\kappa}{2}[\bar{a}_1 + \bar{a}_3] \\
\bar{m}_{p\kappa} &= \frac{M_{p\kappa}}{\frac{1}{2}\rho \bar{V}_o^2 S^{3/2}} = \cos^2\alpha_o \frac{\pi}{4}\lambda^{3/2}\left\{\frac{\bar{a}_2}{\cos\alpha_o} - \frac{\omega_\zeta}{2}[\bar{a}_1 + \bar{a}_3]\right\}
\end{aligned}\right\} \quad (25)$$

when taking into account that L can be replaced by $\sqrt{S\lambda}$ (S = area, λ = aspect ratio) and introducing the force and moment coefficients.

The induction due to (22) can be allowed for by substituting a_n for $\bar{a}_n$, likewise $V_{ox}\,\omega_\kappa \cos\theta$ and $V_{ox}\left(\frac{1}{\cos\alpha_o} - \omega_\zeta \cos\theta\right)$ can be replaced by the more complete equations

$$\begin{aligned}
&\bar{V}_{ox}\left(\omega_\kappa \cos\theta + \frac{\sum_1^\infty c_n \sin n\theta}{2\sin\theta}\right) \\
&\bar{V}_{ox}\left(\frac{1}{\cos\alpha_o} - \omega_\zeta \cos\theta + \frac{1}{2\sin\theta}\sum_1^\infty b_n \sin n\theta\right)
\end{aligned} \quad (26)$$

Then we have

$$\begin{aligned}
f_{p\kappa} &= -\cos^2\alpha_o \frac{\pi\lambda}{2}\left[\frac{1}{2}\omega_\kappa a_2 + \frac{1}{2}\sum_1^\infty a_n c_n\right] \\
f_{p\zeta} &= \cos^2\alpha_o \frac{\pi\lambda}{2}\left[\frac{a_1}{\cos\alpha_o} - \frac{1}{2}\omega_\zeta a_2 + \frac{1}{2}\sum_1^\infty a_n b_n\right]
\end{aligned} \quad (27)$$

and again

$$m_{p\zeta} = -\cos^2\alpha_0 \frac{\pi\lambda^{3/2}}{4}\left[\frac{1}{2}\omega_\kappa(a_1 + a_3) + \frac{1}{2}\sum_1^\infty(a_n c_{n+1} + c_n a_{n+1})\right]$$

$$m_{d\kappa} = \cos^2\alpha_0 \frac{\pi}{4}\lambda^{3/2}\left[\frac{a_2}{\cos\alpha_0} - \frac{1}{2}\omega_\zeta(a_1 + a_3) + \right. \tag{28}$$

$$\left. + \frac{1}{2}\sum_1^\infty(a_n c_{n+1} + c_n a_{n+1})\right]$$

Analogously, the comparison of (27) with (25) gives the induction effect:

$$f_{p\zeta} - \bar{f}_{p\zeta} = -\cos^2\alpha_0 \frac{\pi\lambda}{2}\left[\frac{1}{2}\omega_\kappa(a_2 - \bar{a}_2) + \frac{1}{2}\sum_1^\infty a_n c_n\right]$$

$$f_{p\zeta} - \bar{f}_{p\zeta} = \cos\alpha_0 \frac{\pi\lambda}{2}\left[\frac{a_1 - \bar{a}_1}{\cos\alpha_0} - \frac{1}{2}\omega_\zeta(a_2 - \bar{a}_2) + \frac{1}{2}\sum_1^\infty a_n b_n\right]$$

for the moments.

It will be observed that the terms containing the difference $a_1 - \bar{a}_1$ denote the effect of the change in circulation, whereas the terms containing b_n and c_n represent the effect - for equal circulation - of the directional change of the velocity.

b) Forces and Moments Due to Drag

On the basis of (3) we have:

$$d\,R_\kappa = \frac{1}{2}\rho\, V_\kappa\, V_{ox}\left(1 - \omega_y\xi + \frac{v_i}{\bar{V}_{ox}}\right)\frac{lL^2}{2}\,R\,d\xi$$

$$d\,R_\zeta = \frac{1}{2}\rho\, V_\kappa\, V_{ox}\left(1 - \omega_y\xi + \frac{v_i}{\bar{V}_{ox}}\right)\frac{lL^2}{2}\,R\,d\xi$$

Putting

$$\frac{1}{2}\left(1 - \omega_y\,\xi + \frac{v_i}{\bar{V}_{ox}}\right) = \sum_1^\infty d_n \sin n\,\theta$$

we similarly derive:

$$
\left.
\begin{aligned}
f_{r\kappa} &= \cos^2 \alpha_0 \frac{\pi \lambda}{2} \left[\frac{d_1}{\cos \alpha_0} - \frac{1}{2} \omega_\zeta d_3 + \frac{1}{2} \sum_1^\infty d_n \, b_n\right] \\
f_{r\zeta} &= \cos^2 \alpha_0 \frac{\pi \lambda}{2} \left[\frac{1}{2} \omega_\kappa \, d_3 + \frac{1}{2} \sum_1^\infty d_n \, c_n\right] \\
m_{r\zeta} &= \cos^2 \alpha_0 \frac{\pi}{4} \lambda^{3/2} \left[\frac{d_2}{\cos \alpha_0} - \frac{1}{2} \omega_\zeta (d_1 + d_3) + \right. \\
&\qquad \left. + \frac{1}{2} \sum_1^\infty (d_n \, b_{n+1} + d_{n+1} \, b_n)\right] \\
m_{r\kappa} &= \cos^2 \alpha_0 \frac{\pi}{4} \lambda^{3/3} \left[\frac{1}{2} \omega_\kappa (d_1 + d_3) + \frac{1}{2} \sum_1^\infty (d_n \, c_{n+1} \, c_n \, d_{n+1})\right]
\end{aligned}
\right\} \quad (29)
$$

We omit the terms of the corresponding coefficients for the case of negligible induced velocities since this is easily deduced from the above formulas.

Now let us note how by a similar method the force and moment coefficients with respect to axis x and y can be directly established, which in special cases, such as of rotation about axis x, may make the calculations more tractable.

We have with simple developments:

$$
\left.
\begin{aligned}
f_{px} &= - \cos^2 \alpha_0 \frac{\pi \lambda}{2} \left[\bar{A}_0 \, a_1 + \frac{1}{2} \omega_x a_2 + \frac{1}{2} \sum_1^\infty a_n \, c_n\right] \\
f_{py} &= \cos^2 \alpha_0 \frac{\pi \lambda}{2} \left[a_1 - \frac{1}{2} \omega_y \, a_2 + \frac{1}{2} \sum_1^\infty a_n \, b_n\right] \\
m_{py} &= -\cos^2 \alpha_0 \frac{\pi}{4} \lambda^{3/2} \left[A_0 a_2 + \frac{1}{2} \omega_x (a_1 + a_3) + \frac{1}{2} \sum_1^\infty (a_r c_{n+1} + c_n a_{n+1})\right] \\
m_{px} &= \cos^2 \alpha_0 \frac{\pi}{4} \lambda^{3/2} \left[a_2 - \frac{1}{2} \omega_y (a_1 + a_3) + \frac{1}{2} \sum_1^\infty (a_n b_{n+1} + b_n a_{n+1})\right]
\end{aligned}
\right\} \quad (30)
$$

Naturally b_n and c_n must be computed with respect to the new axes. The total force and moment coefficients are obtained by adding the moments due to lift and those due to drag.

PART II. NUMERICAL APPLICATION

The mathematical interpretation of the effect of the induced velocities on the moments acting on the wing consists in applying the results of the preceding analysis to the determination of the rolling and yawing moments of an elliptical wing of aspect ratio 6. This determination extends to various values of mean angle of attack and speed of rotation for both included and disregarded induced velocity.

The results of these calculations being given elsewhere, we give here only a brief summary of the analytical method employed in the determination of the various pertinent quantities.

Characteristic Wing Curves

The wing characteristics are defined on the basis of the curve of the lift P and drag R coefficients in terms of variable $A = \tan \alpha$ by means of the following equations (empirically stable in following the course of the corresponding experimental values relative to the wing taken into consideration in the work of Fuchs and Schmidt*):

$$P = 5.75\ A - 0.55\ A^2 - 17.50\ A^3 + 13.90\ A^4 \qquad (1')$$

$$R = 0.05 + 1.1\ A^4 \qquad (2')$$

These curves are illustrated in figure 2, while figure 3 gives the corresponding curves for the coefficients C_p and C_r (with conventional notation) in comparison with the above experimental figures.

As previously pointed out, the intersection of P and R with C_p and C_r is directly given from

$$\frac{P}{C_p} \quad \frac{R}{C_r} \quad \frac{1}{\cos \alpha}$$

α = absolute angle of attack

---

*As a matter of fact, this work treats the values of C_p and C_r for a finite aspect ratio. However, we are not concerned with this except as to the trend of the curve which at least at high angles of attack slightly varies with the aspect ratio.

Figure 2 gives only the curve of the P_1' values, readily obtainable from

$$P_1' = \frac{d\ C_p}{d\alpha} = \frac{1}{\cos\alpha}\frac{d\ P}{d\ A} - P \sin\alpha$$

Determination of $\bar{a}_n$

According to the explanation, the coefficient a_n are factors of the development of

$$\frac{1}{2}(1 - \omega_y \cos\theta)\ \bar{P}\ l$$

in Fourier series,

or, for the elliptical wing, of

$$\frac{2}{\pi\lambda}(1 - \omega_y \cos\theta)\ \bar{P} \sin\theta$$

This development is readily effected when $\omega_y = 0$. In the opposite case it is more expedient to express the variable

$$\bar{A} = \frac{A_o + \omega_x \cos\theta}{1 - \omega_y \cos\theta}$$

in the form of

$$-\frac{\omega_x}{\omega_y} + \frac{\bar{A}_o + \frac{\omega_x}{\omega_y}}{1 - \omega_y \cos\theta}$$

Then we substitute (1') for P which, written in (3), gives

$$\Sigma\ \bar{a}_n \sin n\theta = [A_o + \omega_x \cos\theta]\left[A_o + \frac{A_1}{1 - \omega_y \cos\theta} + \frac{A_2}{(1 - \omega_y \cos\theta)^2} + \frac{A_3}{(1 - \omega_y \cos\theta)^3}\right] \sin\theta$$

where the coefficients A, A_2, A_3, are simply functions of the mean angle of attack (or what is the same, of its tangent A) and of ratio $\frac{\omega_x}{\omega_y}$.

Thus the problem narrows down to the development of

$$\frac{\sin\theta}{1 - \omega_y \cos\theta}$$

$$\frac{\sin\theta}{(1 - \omega_y \cos\theta)^2}$$

$$\frac{\sin\theta}{(1 - \omega_y \cos\theta)^3}$$

in Fourier series, which is easily effected.

Approximate Evaluation of $\frac{P_i' l}{4 \sin\theta}$

In the particular case of the elliptical wing, this becomes

$$\frac{P_i'}{\pi\lambda}$$

The exact evaluation of this term involves considerable complications in the calculations, at least, when we wish to apply the method of calculation indicated in Part I, and which is followed in the present example.

As otherwise the term itself is relatively small with respect to unity, it is added in the first term of (18) so as to give it a sufficiently approximate evaluation. This may be done as in this particular case by constructing graphically by points the curve giving this term as function of the variable θ.

This accomplished, the curve is assimilated to a parabola after which the said term is readily put in the following form:

$$\frac{P_i'}{\pi\lambda} = t_0 + t_1 \cos\theta + t_2 \cos 2\theta$$

Calculation of a_n

With due regard to the preceding approximate term, we can write

$$\sum_1^\infty a_n \sin n\theta \left[1 + \frac{n\,l}{4 \sin\theta} P'\right] =$$

$$= \left[a_1 (1 + t_0) - \frac{1}{2} a_1 t_2 + a_2 t_1 + \frac{3}{2} a_3 t_2\right] \sin\theta +$$

$$+ \left[a_2 (1 + 2 t_0) + \frac{3}{2} t_1 a_3 + 2 a_4 t_2 + \frac{1}{2} a_1 t_1\right] \sin 2\theta + \ldots.$$

With (16) in mind, (18) gives:

$$\left(1 + t_0 - \frac{t_2}{2}\right) a_1 + t_1 a_2 + \frac{3}{2} t_2 a_3 = \bar{a}_1$$

$$\frac{1}{2} t_1 a_1 + (1 + 2 t_0) a_2 + \frac{3}{2} t_1 a_3 + 2 t_2 a_4 = \bar{a}_2 \qquad (3')$$

$$\frac{1}{2} t_2 a_1 + t_1 a_2 + (1 + 3 t_0) a_3 + 2 t_1 a_4 + \frac{5}{2} t_2 a_5 = \bar{a}_3$$

and for the following terms in general:

$$\frac{n-2}{2} t_2 a_{n-2} + \frac{n+1}{2} t_1 a_{n-1} + (1 + n t_0) a_n +$$

$$+ \frac{n+1}{2} t_1 a_{n+1} + \frac{n+2}{2} t_2 a_{n+2} = a_n \qquad (3'')$$

If the consideration is limited to the first m terms by assuming the others to be negligible, it results in a system of m equations of m unknown $a_1\ a_2 \ldots\ldots a_n$.

This can be resolved, for example, by a method of progressive approximations the first of which may be obtained by assuming all terms of the principal determinant to be zero, excepting those of its principal diagonal which usually have much higher coefficients than the other terms. Then the thus obtained values may be put in the terms first assumed as zero, considering them now as shown, which yields a second approximation. The procedure can be repeated to

obtain various progressive approximations. In the case, however, where t with index greater than 1 is zero it is more expedient to use the conventional method of resolution by substitution.

Calculation of Factors b_n and c_n

These coefficients are the results of the development of

$$\sin\left(\tan^{-1}\right)\left(\frac{\omega_K \cos\theta}{1-\omega_\zeta \cos\theta}\right)\sum_1^\infty n\, a_n \sin n\,\theta$$

$$\cos\left(\tan^{-1}\right)\left(\frac{\omega_K \cos\theta}{1-\omega_\zeta \cos\theta}\right)\sum_1^\infty n\, a_n \sin n\,\theta$$

By expressing $\frac{\omega_K \cos\theta}{1-\omega_\zeta \cos\theta}$ in the approximate form $\omega_K \cos\theta + \omega_K\,\omega_\zeta \cos^2\theta$ and introducing equation (10) while disregarding the terms with $\cos\theta$ at powers higher than two, we can write

$$\sin\left(\tan^{-1}\right)\frac{\omega_K \cos\theta}{1-\omega_\zeta \cos\theta} = \sim h_1 \cos\theta + h_2 \cos^2\theta$$

$$\cos\left(\tan^{-1}\right)\frac{\omega_K \cos\theta}{1-\omega_\zeta \cos\theta} = 1 + k_1 \cos\theta + k_2 \cos^2\theta$$

where h and k are constants which are easily computed.

Then we have

$$c_1 = \left(1+\frac{k_2}{4}\right)a_1 + k_1 a_2 + \frac{3}{4}k_2\, a_3$$

$$c_2 = \frac{1}{2}k_1 a_1 + 2\left(1+\frac{k_2}{2}\right)a_2 + \frac{3}{2}k_1 a_3 + k_2 a_4$$

and in general:

$$c_n = \frac{n-2}{4} k_2 a_{n-2} + \frac{n-1}{2} k_1 a_{n-1} + n\left(1 + \frac{k_1}{2}\right) a_n +$$

$$+ \frac{n+1}{2} k_1 a_{n+1} + \frac{n+2}{4} k_2 a_{n+2} ,$$

b_n can be obtained in the same manner by substituting k for h and deleting the terms 1 within [ ] (which can be called $k_0 = 1$, while $h_0 = 0$).

It should be noted that the above method can be extended to include the case where the directions in which the induced velocities are to be determined may be other than axis κ or ζ, and the particular case when in direction of axis x and y.

Coefficients $\bar{d}$ and d

The $\bar{d}$ coefficients can be determined in the same manner as the $\bar{a}$ coefficients. Now, however, the results are much more simplified becasue of the smaller number of terms in the R equation compared to that for P.

The d coefficients can be defined in the same way by any approximation considering only the first two terms of the series development of the induced angles of attack so that the effective velocities can still be assumed as being linear.

In this manner it is shown that practically any part included in formulas (25), (27), (28), and (29), can be calculated, which give the forces and moments along axis κ and ζ or along x and y. It may be found expedient to use the one or the other according to whether the calculations insure simpler results with one than with the other. To illustrate: if $\omega_y = 0$, it is preferable to use the second, if $\omega_\zeta = 0$, to use the first. From the results of the first, we can readily change to those of the second by a simple change of axes.

Numerical Calculation

<u>Case 1</u>.- $\bar{A}_0 = 0.6$, rotation about axis κ, which gives:

$$\frac{\omega_x}{\omega_y} = \frac{1}{\bar{A}_o}$$

$$\omega_x = 0.0833$$

$$\omega_x = 0.1666$$

$$\omega_x = 0.2600$$

Case 2.- $\bar{A}_o = 0.6$, rotation about axis $x(\omega_y = 0)$*

$$\omega_x = 0.10$$

$$\omega_x = 0.20$$

$$\omega_x = 0.30$$

Case 3.- $A_o = 0.3$, axis of rotation and angular velocity.

The above theory is, strictly speaking, applicable only to the first case, since, in other cases, it does not satisfy the system of vortices leaving the wing under the assumed conditions, i.e., helical vortices about the axis or rotation. However, it seemed advisable to effect the calculations for these cases also, so as to obtain some idea of the degree of influence of the induced angle of attack when the rotation is other than about the axis of motion.

The following tables show the results obtained in case 1, while figures 4, 5, and 6, give the curves of the moment coefficients for the three cases. It will be noted that, for case 1, these coefficients are plotted against axis κ and ζ, and for cases 2 and 3, against axis x and y. This may seem inconsistent, but it actually is not so, when it is observed that, in this manner, the moments about the axis of rotation and the rectangular axis to that of the wing are considered. It affords a better characterization of the wing reaction to the rotation. The curves and tables show for each case the influence of the induced angle of attack to be appreciable. It is interesting then to note how

*It is easily seen that the two cases of speed or rotation, in the first and second cases, approximately correspond to the same ratio between the resultant speed of rotation and velocity V_{ox}.

the results of this effect in contrast to that which a priori considerations would lead us to expect. Take, for example, the case $A_0 = 0.3$. Here we may be induced to consider the lift curve as being straight and to apply the methods which other authors employed from such hypotheses.

The result would be a decreased effect in induced angle of attack with a diminished rolling moment, contrary to that shown in figure 5. The fact is explained by observing that, while in the case of absolutely straight lift curve the first term (constant) of the development of the induced velocity in Fourier series results in a lower lift coefficient constant over the whole wing and consequently produces no rolling moment unless the curve is rigorously straight, but has, as in the case in question, a derivative less than the high angle of attack values; said decrease is less for the lowered wing, and produces a rolling moment of the same sign as that obtained by the strip method.

For correct interpretation of the results hereinafter it is specified that <u>the moments opposed to rotation are assumed to be positive</u>, i.e., in inverse direction of positive rotation.

The data in tables II and III may also appear complicated at first glance. This is due, especially with regard to the rolling moments, to the fact that the induced angle of attack has a marked influence, and which increases as the moment increases when assuming the value 0.260. However, we note that this is due to the fact that the rolling moment is the algebraic sum of two quantities, lift and drag, which, in this case, are of opposite sign and rather high value and such that, while one (lift quota) increases in absolute value as induction effect, the other decreases. In any case, the increment of the two terms is still noteworthy, even if taken separately. For example, the first increases by about 48 percent. An explanation of this fact is seen in the analysis of the equation system (18) which gives the coefficients a_n in terms of $\bar{a}_n$. From this, bearing in mind that coefficient t_2 assumes high values (about 0.133 in this case) as can be seen from the considerations of the curve of the P_i' coefficients, it appears, for example, that a_4, since it has no effect on coefficients $\bar{f}$ and $\bar{m}$, may have a profound effect on the coefficients a_1, a_2, a_3, and consequently on coefficients m and f.

The above increase of the coefficients seems perfectly natural when coordinated with other research data obtained by strip method. From this (see, for example, fig. 29, Luftfahrtforschung, February 1929) it appears as a small angle of attack change which may be due to the induced velocity raised by changes in moment coefficients of the order of those found.

As for the rate of rotation in the pertinent field, there are no appreciable values at which the rolling moment cancels, corresponding - that is, except as indicated - to the case of simple autorotation, at least, in cases of rotation about the axis of motion or about axis x, and for an angle of attack in the median plane corresponding to $A = 0.6$, we may deduce by extrapolation of the computed curves that this occurs at much higher rate of rotation than is obtained with the strip method.

The exact calculation of such rates of autorotation, which may be useful for experimental verification (model in wind tunnel mounted free to rotate about its axis of motion) was omitted here, simply because there are corresponding maximum values of the angle of attack greater than those at which the experimental curves of the wing coefficients are sensibly coincident with those given by the analytical expressions used for the calculation.

This, naturally, does not minimize the generalization of the procedure, because it is always possible to give analytical expressions with greater number of terms, which would coincide over a greater range of angle of attack with the experimental ones, or else because the analytical translation of the experimental curves is not an essential characteristic of the method.

However, it was not deemed necessary to effect ulterior calculations on the basis of the above modifications in order to determine the rate of zero moment, because if we separate the experimental verification from the above application they have not the great importance which they are supposed to have, they would not rigorously correspond to the true conditions of autorotation, for which, besides nullifying the rolling moments, it also requires the concurrent cancelling of the yawing moments. This, as may be readily proved, does not occur in the conditions of flight under consideration, wherein the path of the center of gravity is maintained in the plane of symmetry of the airplane. The

rate of autorotation requires in general the consideration of a rate of yaw, which complicates the problem to such an extent as to make it almost intractable.

TABLE I.

Principal Coefficients for the Determination of the Forces and Moments on a Wing

Case: Rotation about Axis of Motion $A = 0.6$

ω_K	0.083333	0.16666	0.260
$\bar{a}_1$	0.0665	0.0673	+ 0.0679
$\bar{a}_2$	- 0.0065	- 0.0126	- 0.0116
$\bar{a}_3$	- 0.0001	- 0.0005	0.000
$\bar{a}_4$	0.0010	0.0291	0.1382
$\bar{d}_1$	0.0477	0.0486	0.0504
$\bar{d}_2$	0.0113	0.0227	0.0352
$\bar{d}_3$	0.0005	0.0015	0.0251
a_1	0.0784	0.0715	0.0772
a_2	- 0.0084	- 0.0486	- 0.0178
a_3	- 0.0122	- 0.0227	- 0.0067
a_4	+ 0.0021	+ 0.0015	0.0205
d_1	0.0477	0.0441	0.0476
d_2	0.0113	0.0259	0.0391
d_3	0.0004	0.0018	0.0414

TABLE II

Moment Coefficients $\overline{m}_K$ and $\overline{m}_\zeta$

(Induced Velocity Disregarded)

ω_K	0.8333	0.1666	0.260
a) Moment coefficient $\overline{m}_K$ (rolling moment)			
Lift component	- 0.0320	- 0.0620	- 0.0580
Drag component	+ 0.0098	+ 0.0206	+ 0.0356
Total	- 0.0222	- 0.0414	- 0.0224
b) Moment coefficient $\overline{m}_\zeta$ (yawing moment)			
Lift component	- 0.0136	- 0.0278	- 0.0434
Drag component	+ 0.0559	+ 0.1130	+ 0.1740
Total	0.0423	0.0832	0.1386

TABLE III

Moment Coefficients m_κ and m_ζ

(Induced Velocity Included)

ω_κ	0.8333	0.1666	0.260
a) Moment coefficient m_κ			
Lift component	- 0.0410	- 0.0715	- 0.0859
Drag component	0.0086	0.0176	0.0305
Total	- 0.0324	- 0.0539	- 0.0554
b) Moment coefficient m_ζ			
Lift component	- 0.0183	- 0.0354	- 0.0505
Drag component	0.0656	0.1276	0.1932
Total	- 0.0473	- 0.0922	- 0.1427

APPENDIX

I. Calculation of Induced Velocities of a Vortex Filament Coiled Spirally Around an Axis

We refer to the notation given in figure 7 and confine ourselves to the determination of the velocities V_t' V_r' induced at points of axis X parallel to axis Y and Z.

Consider a vortex filament of strength Γ which leaves at a point on abscissa r of the same axis. Let V_0 be its rate of advance in direction of axis Z and Ω its rate of rotation about the same axis. Then the vortex filament has a pitch of $p = 2\pi \frac{V_0}{\Omega}$ and the inclination of its tangent on Z is $\frac{\Omega r}{V_0}$.

Then we have at a point of abscissa r' (Pistolesi, Aerodinamica, p. 472):

$$v_a' = \frac{\Gamma}{4\pi} r \int_0^\infty N^{-3/2}[r - r' \cos\theta] d\theta$$

$$v_t' = \frac{\Gamma}{4\pi} \frac{V_0}{\Omega} \int_0^\infty N^{-3/2} [r - r\cos\theta - r\theta\sin\theta] d\theta \tag{1}$$

with

$$N = r^2 + r'^2 - 2 r r' \cos\theta + \frac{V_0^2}{\Omega^2}\theta^2$$

By putting N in the form of

$$N = \left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right] + 2 r r' (1 - \cos\theta)$$

we can write

$$N^{-3/2} = \frac{1}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{3/2}} - \frac{3}{2} \frac{2 r r'(1 - \cos\theta)}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{5/2}}$$

in close approximation*.

This expression written in (1) affords:

$$v_a' = \frac{\Gamma}{4\pi} r \left[\int_0^\infty \frac{(r - r')\, d\theta}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{3/2}} + \right.$$

$$+ r' \int_0^\infty \frac{(1 - \cos\theta)}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{3/2}} d\theta -$$

$$\left. - 3 r r' \int_0^\infty \frac{(1 - \cos\theta)(r - r' \cos\theta)}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{5/2}} d\theta \right]$$

$$v_t' = \frac{\Gamma}{4\pi} \frac{V_0}{\Omega} \left[\int_0^\infty \frac{(r' - r)\, d\theta}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{3/2}} + \right.$$

$$+ r \int_0^\infty \frac{1 - \cos\theta - \theta \sin\theta}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{5/2}} -$$

$$\left. - 3 r r' \int_0^\infty \frac{(1 - \cos\theta\, [r' - r\, (\cos\theta + \theta \sin\theta)]}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{5/2}} d\theta \right]$$

On the other hand, we know that:

$$\int_0^\infty \frac{r' - r}{\left[(r - r')^2 + \left(\frac{V_0}{\Omega}\right)^2 \theta^2\right]^{3/2}} d\theta = \frac{\Omega}{V_0} \frac{1}{r' - r}$$

---

*The convergence of the above indicated development is insured from being consistently $1 - \cos < \frac{1}{2}\theta^2$ so that in the most unfavorable hypothesis the ratio of incremental to basic term results in $= r r' \left(\frac{\Omega}{V_0}\right)^2$, a ratio which, as pointed out elsewhere in the report, is considered < 0.25.

therefore we can write synthetically:

$$v_a' = \frac{\Gamma}{4\pi}\frac{\Omega}{V_0}r\left[\frac{-1}{r'-r} + \frac{V_0}{\Omega}\left(A_1 r' + B_1 r r'\right)\right]$$

$$v_t' = \frac{\Gamma}{4\pi}\left[\frac{1}{r'-r} + \frac{V_0}{\Omega}\left(A_2 r + B_2 r r'\right)\right]$$

These expressions are useful for comparing the helical vortex having the same initial velocity with those of a straight vortex leaving at the same point of the vortex.

These are clearly expressed in the formula

$$\left.\begin{aligned} \overline{v}_a' &= \frac{-1}{r'-r}\frac{\Gamma}{4\pi}\sin\left(\tan^{-1}\right)\frac{\Omega r}{V_0} = \frac{-1}{r'-r}\frac{\Gamma}{4\pi}\cos\left(\tan^{-1}\frac{V_0}{\Omega r}\right) \\ \overline{v}_t &= \frac{1}{r'-r}\frac{\Gamma}{4\pi}\cos\left(\tan^{-1}\right)\frac{\Omega r}{V_0} \end{aligned}\right\}(2)$$

Now we shall show how the calculation of coefficients A and B may be carried out. To compute these integrals ranging between 0 and ∞ it is advisable to divide them into two parts, in one of which ranging near 0 and a finite number suitably selected, we can develop the numerator in series of variable θ, while in the other, ranging between said number and infinity, the term $\left[(r-r')^2 + \left(\frac{V_0}{\Omega}\right)^2\right]^{-3/2}$ can be developed in similar fashion. In this manner we can return to the calculation of the indicated integrals. Obviously, in the above series developments it suffices to consider any appropriate finite number of terms.

For example, the terms A_2 may be approximated at

$$A_2 = \sim\int_0^2 \frac{-\frac{\theta^2}{2} + \frac{\theta^4}{8} - \frac{\theta^6}{128} + \frac{\theta^8}{5760}}{(a + b\,\theta)^{3/2}}\,d\theta + \int_2^\infty \frac{1}{[a + b\theta^2]^{3/2}}\,d\theta -$$

$$- \frac{1}{b^{3/2}} \left[\int_a^\infty \frac{\cos\theta}{\theta} d\theta + \int_a^\infty \frac{\sin\theta}{\theta^2} d\theta \right] -$$

$$+ \frac{3}{2} \frac{1}{b^{3/2}} a \left[\int_a^\infty \frac{\cos\theta}{\theta^5} d\theta + \int_a^\infty \frac{\sin\theta}{\theta^4} d\theta \right]$$

after putting

$$a = (r - r')^2 \qquad b = \left(\frac{V_0}{\Omega}\right)^2$$

and similarly for the other terms.

Numerical Calculations

Consider a wing of unit semispan, a given ratio of $\frac{V_0}{\Omega} = 2.5$, which may be considered as a minimum attainable practical limit in spinning, at least, within the range of validity of Prandtl's theory, and visualize a vortex leaving at the point of abscissa $r = 1$. We determine the values of the above coefficients for the points on abscissa 0 and -1 (the calculation of the coefficients B for the point on abscissa 0 is omitted, since they are obviously useless, being multiplied by the abscissa itself.)

We have:

r' =	A_1 =	B_1 =	A_2 =	B_2 =
0	0.05354	-------	- 0.0280	-------
- 1	0.03036	0.00120	- 0.00625	0.00236

The following table shows the corresponding values of $v_a' \frac{4\pi}{\Gamma}$ and $v_t' \frac{4\pi}{\Gamma}$ in comparison with those of equation (2):

r'	$v_a' \frac{4\pi}{\Gamma}$	$\bar{v}_a' \frac{4\pi}{\Gamma}$	error %	$v_t' \frac{4\pi}{\Gamma}$	$\bar{v}_t' \frac{4\pi}{\Gamma}$	error %
0	+ 0.400	+ 0.372	+ 7	- 0.972	- 0.930	+ 4.5
- 1	+ 0.172	+ 0.186	- 7.5	- 0.4990	- 0.465	+ 6.7

Since it may be assumed that the percentage of error established for the point $r' = -1$ is a maximum and besides, since the closer we approach the vortex the error must of necessity decrease also, we were constrained to deduce that the substitution of the rectilinear for the helicoidal vortex does not produce, according to the effects of this analysis, errors greater than those involved in the conventional approximation inherent to the nature of the problem itself.

II. Check on the Degree of Approximation Obtained with

a) $\cos(\tan^{-1})\, x = 1 - 0.2\, x - 0.2\, x^2$

b) $\sin(\tan^{-1})\, x = 1.05\, x - 0.35\, x^2$

x'	$\cos(\tan^{-1})\, x$			$\sin(\tan^{-1})\, x$		
	effective	for a)	error %	effective	for b)	error %
0.1	0.99553	0.988	- 0.756	0.09955	0.1015	+ 1.97
0.2	0.98058	0.972	- 0.821	0.19612	0.1960	- 0.06
0.3	0.95782	0.952	- 0.602	0.28735	0.2835	- 1.34
0.4	0.92847	0.928	- 0.050	0.37139	0.3640	- 1.99
0.5	0.89443	0.900	+ 0.675	0.44721	0.4375	- 2.15
0.6	0.85749	0.868	1.255	0.51449	0.5040	- 2.04
0.7	0.81949	0.832	1.406	0.57364	0.5633	- 1.80
0.8	0.78087	0.792	1.425	0.62469	0.6160	- 1.39
0.9	0.74329	0.748	0.634	0.66896	0.6615	- 1.11
1.0	0.70710	0.700	- 1.005	0.70710	0.7000	- 0.71

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3. Fage, A. and Johansen, F. C.: The Connection between Lift and Circulation for an Inclined Flat Plate. R. & M. 1139, British A.R.C., 1928.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

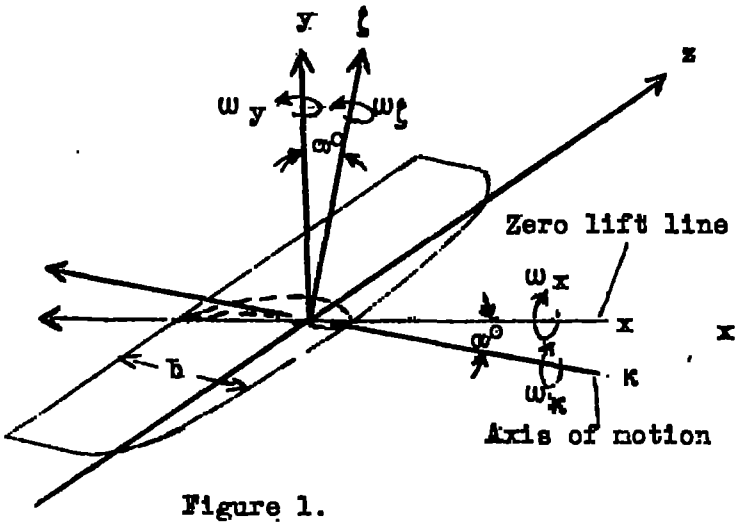


Figure 1.

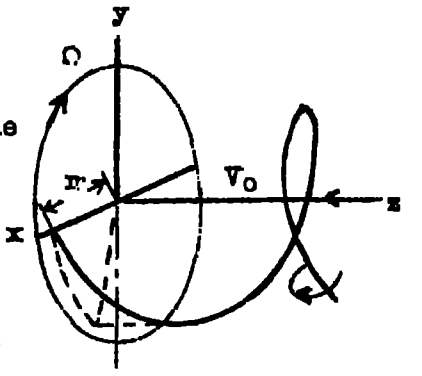


Figure 7.

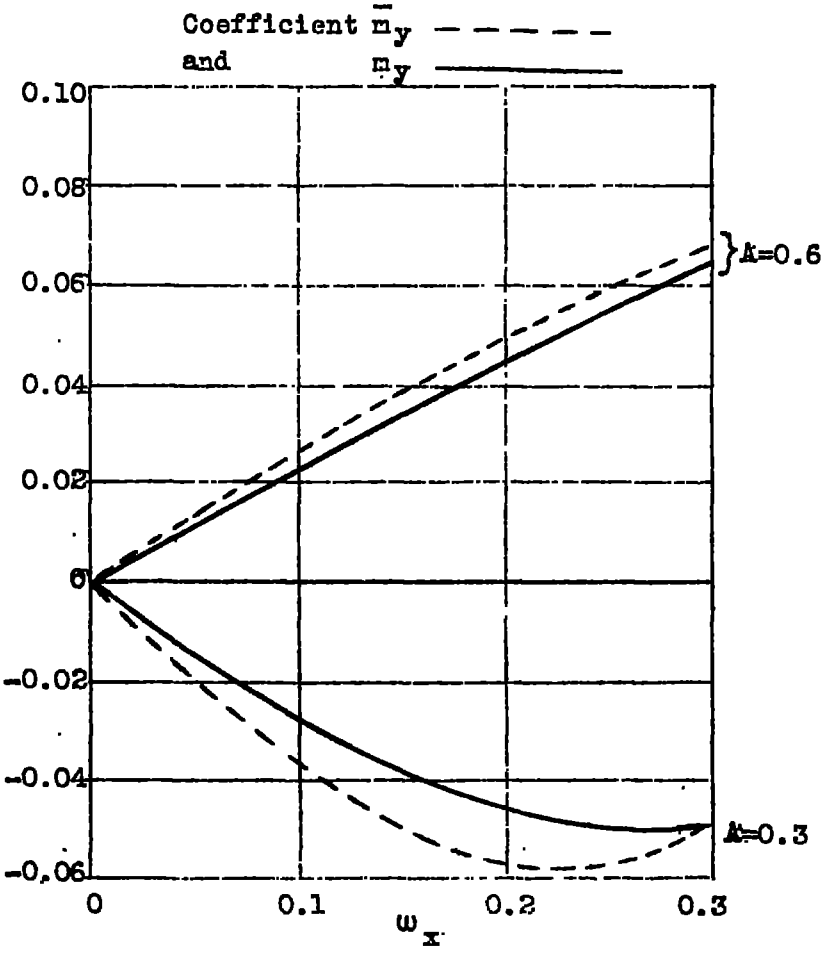
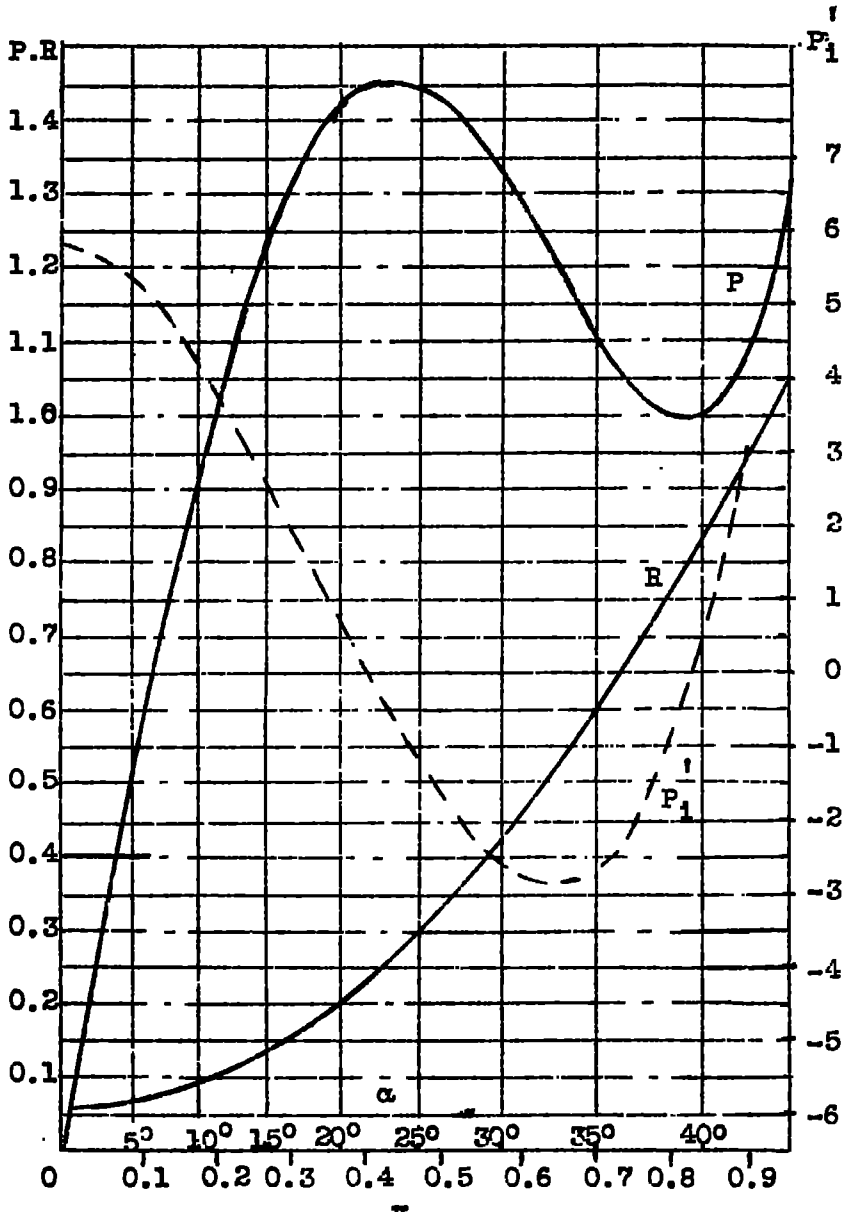


Figure 6.- Rotation about axis x.

Coefficients P,R, and P_1'



$$A = \frac{V_y}{V_x} = \tan \alpha$$

Figure 2.

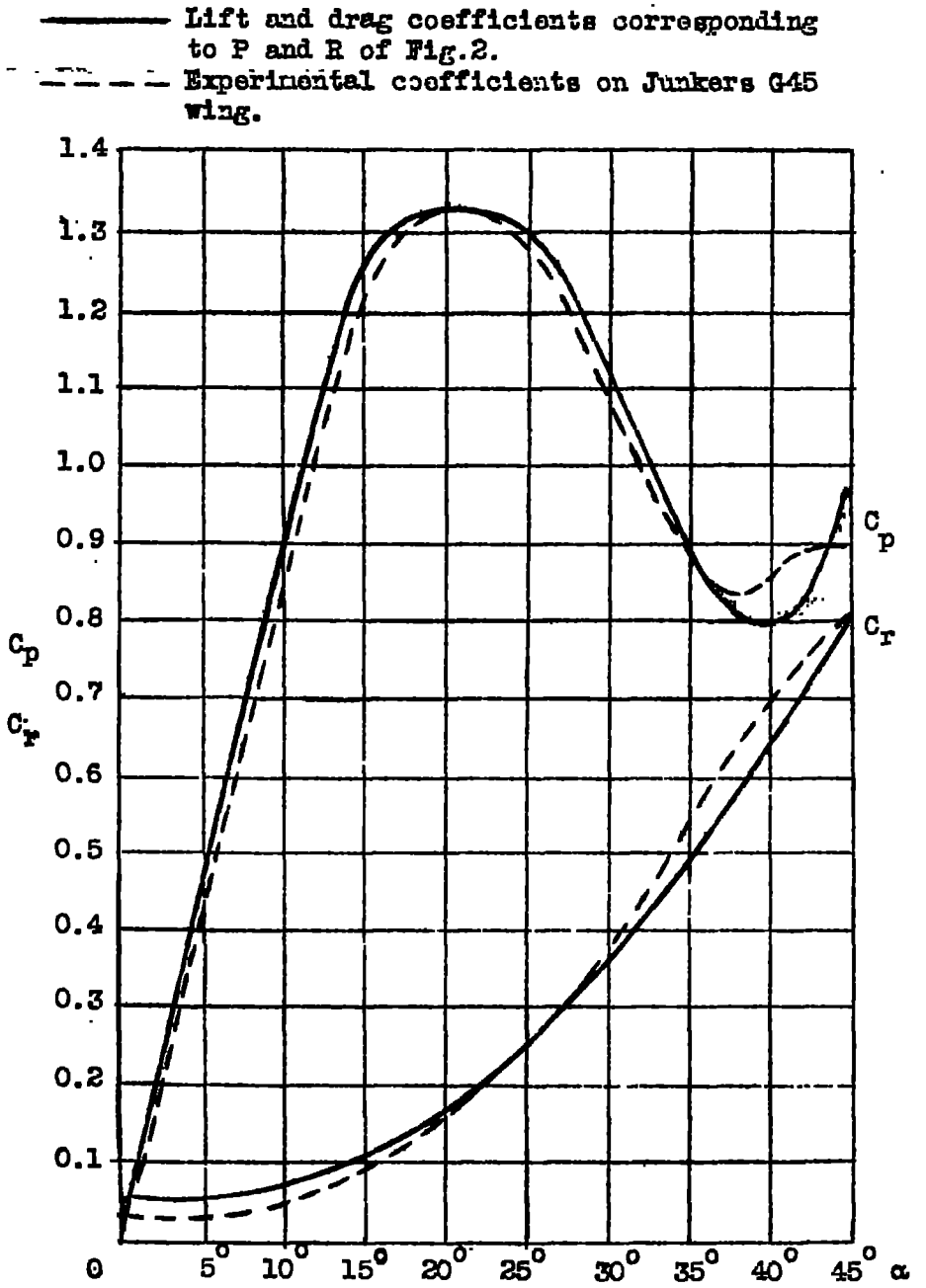

Lift and drag coefficients corresponding to P and R of Fig.2.
Experimental coefficients on Junkers G45 wing.
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
C_p
C_r
0
5°
10°
15°
20°
25°
30°
35°
40°
45°
α

Figure 3.

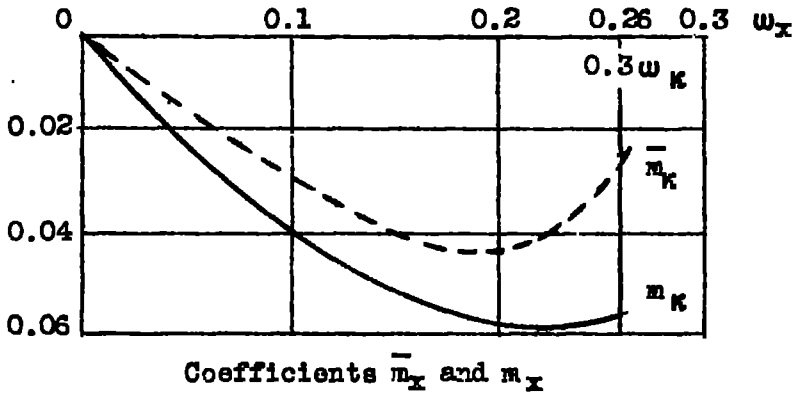


Figure 4a.- Moments due to rotation about axis of motion. $\omega_\zeta = 0$ for $A_0 = 0.6$

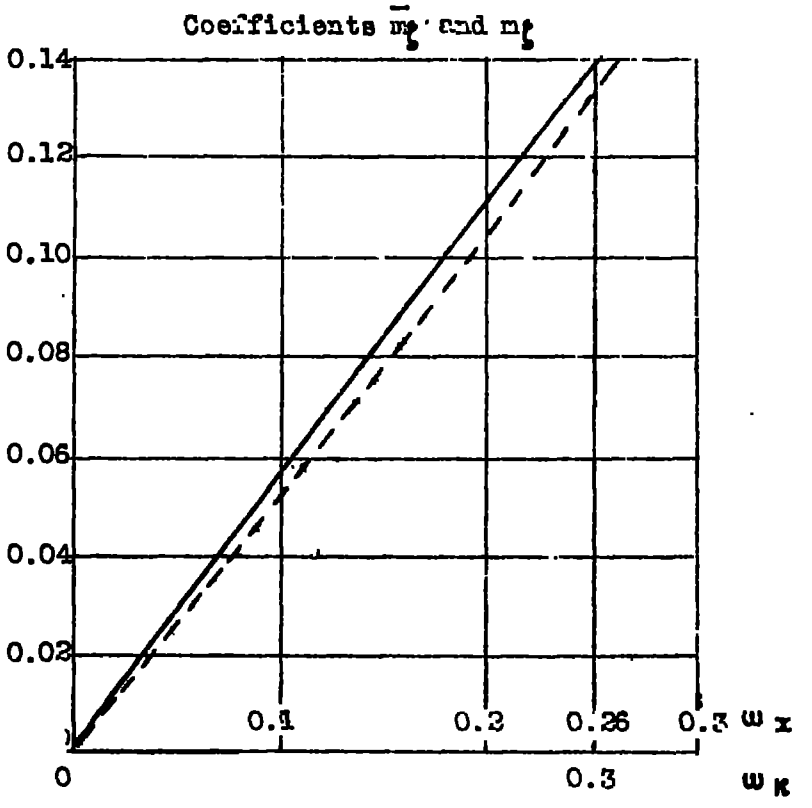


Figure 4b.- Moments due to rotations about axis of motion $\omega_\zeta = 0$ for $A_0 = 0.6$

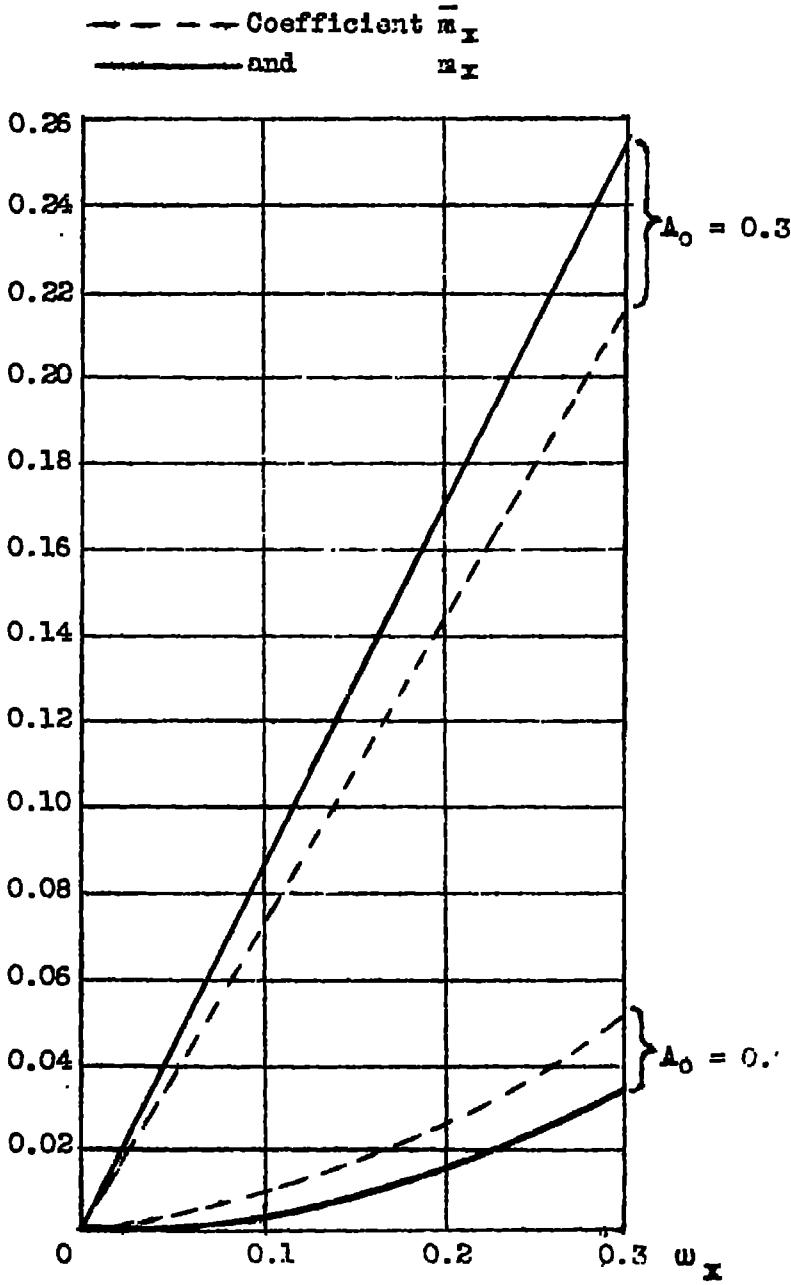


Figure 5.- Rotation about axis x

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