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SCHRÖDINGER OPERATORS WITH THE q-LADDER SYMMETRY ALGEBRAS[1]

Sergei Skorik
Department of Physics, University of Southern California, Los Angeles, CA 90089-0484, U.S.A.

Vyacheslav Spiridonov[2]
Centre de Recherches Mathématiques, Université de Montreal, C.P. 6128-A, Montréal, Québec, H3C 3J7, Canada

Abstract

A class of the one-dimensional Schrödinger operators L with the symmetry algebra $LB^{\pm} = q^{\pm 2}B^{\pm}L$, $[B^+, B^-] = \mathcal{P}_N(L)$, is described. Here $B^{\pm}$ are the 'q-ladder' operators and $\mathcal{P}_N(L)$ is a polynomial of the order N. Peculiarities of the coherent states of this algebra are briefly discussed.

1 Introduction

Exactly solvable spectral problems are of great importance. They have numerous applications in classical and quantum mechanics. In the last decades the theory of solitons once again exhibited their universal character. However, the definition of the notion of solvability (or integrability) itself is quite delicate. In particular, it involves the definition of functions which are allowed to enter the solution of the problem (in a sense these are two complementary things – it is common to define functions as solutions of some equations). Let us take for example the standard one-dimensional Schrödinger equation:

$$L\psi(x) \equiv (-d^2/dx^2 + u(x))\psi(x) = \lambda\psi(x), \tag{1}$$

endowed with some boundary conditions. The widely used tacit definition of solvability of this spectral problem consists in the requirement for $\psi(x)$ to be a finite sum of the hypergeometric functions ${}_2F_1(a, b; c; x)$ [1], or of their descendants. On the one hand, the ${}_2F_1$ indeed occupies a distinguished place among the classical special functions, but on the other hand, there are more complicated objects whose global structure has been well understood.

In the theory of nonlinear evolution equations the smooth bounded potential $u(x)$ is said to be solvable if it has a spectrum consisting of the $N+1$ permitted bands, N of the finite width and one infinitely large (these are the regions of λ for which the wave functions $\psi(x)$ are bounded). In some cases this condition leads to the Lamé equation, which is a simplest generalization of the

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[2]On leave of absence from the Institute for Nuclear Research, Russian Academy of Sciences, Moscow, Russia

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second order differential equation for hypergeometric function. Another type of functions with known global analytical properties can be defined with the help of the nonlinear second order differential equations: the Painlevé transcendents PI-PVI are the simplest examples. Different and more rich classes of functions are defined by the second order (linear or non-linear) finite-difference equations, an example is given by the basic, or q-hypergeometric function. In the latter two cases, the corresponding special functions may be taken for definition of potentials and then the condition of solvability of (1) can be thought as the requirement for wave functions $\psi(x)$ to be related to $u(x)$ by some simple formulae which do not lead to the essentially new objects. This short discussion and the systems to be described below demonstrate that the problem of classification of all exactly solvable problems is far from completion even for a simple equation (1).

The integrability of a problem is related to its symmetry properties. Unfortunately, most of the symmetries are "hidden" and, as a result, the group-theoretical treatment of special functions often emerges as a secondary problem. Rarely new function was introduced primarily from the symmetry principles. The situation however changes when the classification problem is dealt with. The so-called characterization theorems are targeted to the enumeration of specific properties which define a taken system uniquely within the given class of equations. Using them one can see what properties should be abandoned in order to get more general systems. E.g., one may ask what are the most general potentials $u(x)$ for which there exist a differential operator A of the order N such that $[L, A] = 0$. For odd N this happens to lead to the finite-gap potentials mentioned above (even N cases contain generically a functional non-uniqueness).

Recently, the characterization of all potentials leading to the ordinary ladder algebra:

$$[L, A] = \mu A, \qquad \mu > 0, \tag{2}$$

where A is the N-th order differential operator, has been given in [2]. For $N = 1$ it is easy to find that $u(x) \propto x^2$, i.e. a harmonic oscillator potential. For $N = 2$ one gets the singular oscillator potential $u(x) \propto \alpha x^2 + \beta/x^2$. The $N = 3$ case corresponds to the quite complicated situation when $u(x)$ involves the Painlevé IV transcendental function [2].

Already from (2) one expects that the spectrum of Hamiltonian L is purely discrete and equidistant. This, however, depends on the structure of zero modes of A. If all of them are physical eigenstates of L with different eigenvalues, and the conjugated operator A^+ does not break normalizability, then one has a spectrum composed from N independent arithmetic series. The wave functions are explicitly expressed through $f(x), f'(x)$ and $\int f(x)dx$, where $f(x)$ is *defined* by a system of N first order nonlinear differential equations, i.e. the problem is solved exactly in the above sense. Although it is much harder to calculate physical observables of these systems than for the potentials related to the hypergeometric function, the principally important characteristics – the spectrum – is very simple and elegant.

The aim of this note is, however, to present even more complicated potentials than just mentioned ones [3-7]. The first class of them is connected with the algebra (2) but the operator A is now a differential-difference operator [3]. The resulting potentials are more general than those of [2] since the corresponding characterization theorem does not apply. The second class is based on the q-deformed ladder relation:

$$LA = q^2 AL, \qquad q^2 \neq 0,\ 1, \tag{3}$$

which simply can not be realized with the help of differential operators of the finite order. Note that the limit $q \to 1$ does not necessarily mean that the operator A will be an integral of motion – there may be a diverging constant entering additively into L such that for $q = 1$ one gets (2) rather than $[L, A] = 0$.

2 Self-Similar Potentials

Let us briefly describe the definition of potentials leading to (3). The basic tool is the factorization, or dressing method based on the Darboux transformations. One takes a set of Hamiltonians, $L_j = -d^2/dx^2 + u_j(x)$, and represents them as products of the first-order differential operators,

$$A_j^+ = -\frac{d}{dx} + f_j(x), \qquad A_j^- = \frac{d}{dx} + f_j(x), \tag{4}$$

up to some constants λ_j:

$$L_j = A_j^+ A_j^- + \lambda_j, \tag{5}$$

i.e. $u_j(x) = f_j^2(x) - f_j'(x) + \lambda_j$. Then one imposes the following intertwining relations:

$$L_j A_j^+ = A_j^+ L_{j+1}, \qquad A_j^- L_j = L_{j+1} A_j^-, \tag{6}$$

which constrain the difference in spectral properties of L_j and L_{j+1} and are equivalent to the equations:

$$A_{j+1}^+ A_{j+1}^- + \lambda_{j+1} = A_j^- A_j^+ + \lambda_j. \tag{7}$$

Substitution of (4) into (7) yields the chain of differential equations [8]:

$$f_j'(x) + f_{j+1}'(x) + f_j^2(x) - f_{j+1}^2(x) = \lambda_{j+1} - \lambda_j \equiv \mu_j, \tag{8}$$

which is called the dressing chain.

The potentials we are interested in are defined by the following self-similarity constraints imposed upon the dressing chain [3, 6]:

$$f_{j+N}(x) = q f_j(qx + r), \qquad \mu_{j+N} = q^2 \mu_j. \tag{9}$$

The simplest example, defined by the reduction $f_j(x) = q^j f(q^j x)$, $\lambda_j = q^{2j}\lambda$, has been found in [4].

At the operator level, the relations (9) lead to the Schrödinger operators with non-trivial q-deformed symmetry algebras. Let us consider the products:

$$M_j^+ = A_j^+ A_{j+1}^+ \dots A_{j+N-1}^+, \qquad M_j^- = A_{j+N-1}^- \dots A_{j+1}^- A_j^-, \tag{10}$$

which generate the interwinings

$$L_j M_j^+ = M_j^+ L_{j+N}, \qquad M_j^- L_j = L_{j+N} M_j^-. \tag{11}$$

The structure relations complimentary to (11) look as follows:

$$M_j^+ M_j^- = \prod_{k=0}^{N-1} (L_j - \lambda_{j+k}), \qquad M_j^- M_j^+ = \prod_{k=0}^{N-1} (L_{j+N} - \lambda_{j+k}). \tag{12}$$

These identities show that if the operators L_j and L_{j+N} are related to each other through some simple transformation, e.g.

$$L_{j+N} = q^2 U L_j U^{-1} + \omega, \tag{13}$$

where U is an invertable operator, then the combinations $B_j^+ \equiv M_j^+ U$, $B_j^- \equiv U^{-1} M_j^-$, map eigenfunctions of L_j onto themselves, i.e. they are symmetry operators for L_j. The form of U is restricted by the requirement that the L_j's be of the Schrödinger form. Taking U to be the affine transformation generator, $Uf(x)U^{-1} = f(qx + r)$, fixing the indices (e.g., $j \equiv 1$) and removing them, we get the symmetry algebra [6]:

$$LB^{\pm} = q^{\pm 2} B^{\pm} L, \qquad L \equiv -d^2/dx^2 + f_1^2(x) - f_1'(x) + \lambda_1 - \omega/(1-q^2), \tag{14}$$

$$B^+ B^- = \prod_{k=1}^{N} (L + \frac{\omega}{1-q^2} - \lambda_k), \qquad B^- B^+ = \prod_{k=1}^{N} (q^2 L + \frac{\omega}{1-q^2} - \lambda_k). \tag{15}$$

For $N = 1$ this is a q-analog of the Heisenberg-Weyl algebra which for special values of the parameters serves as the spectrum generating algebra [5]. For $N = 2$ this is a q-deformation of the $su(1,1)$ algebra, and for $N > 2$ we get polynomial relations describing symmetries of the self-similar potentials. Note that the limit $q \to 1$ is not trivial. If the parameter r in (9) is not zero then we get the realizations of the algebra (2) which generalize the ones described in [2]. For $q \neq 1$ the parameter r may be set equal to zero. If the operators $B^{\pm}$ are well defined and have N normalizable zero modes, then the self-similar potentials have spectra consisting of N independent geometric series. Moreover, $u(x)$'s are reflectionless and represent initial conditions for the infinite-soliton solutions of the KdV equation.

We conclude that the $N = 1$ case describes the deformation of harmonic oscillator potential, the $N = 2$ case corresponds to the q-deformed conformal quantum mechanics [3], and the $N \geq 3$ cases correspond to the q-deformation of the Painlevé type equations.

An interesting situation takes place when the parameter q is a root of unity, i.e. $q^n = 1$. Generically these cases are related to the hyperelliptic potentials, the $N = 1$ system has been analyzed in detail in [7]. Depending on whether q is a primitive root of unity of odd or even degree, the solution may be unique or non-unique. The $q = -1$ system exists only when the initial condition $f(0) = 0$ is imposed and it provides a non-standard realization of the Heisenberg-Weyl algebra. Indeed, the equation arising from (8), (9) at $N = 1, q = -1$:

$$\frac{d}{dx}(f(x) - f(-x)) + f^2(x) - f^2(-x) = \mu, \tag{16}$$

has the general solution $f(x) = \mu x/2$. This corresponds to the operators $B^{\pm}$ satisfying $[B^-, B^+] = \mu$ with the explicit form:

$$B^- = P(d/dx + \mu x/2), \qquad B^+ = (-d/dx + \mu x/2)P, \tag{17}$$

where P is the parity operator $P\psi(x) = \psi(-x)$.

The general $q^3 = 1$ solution exists for arbitrary initial condition and is given by the equianharmonic Weierstrass function: $u_j(x) = 2\wp(x + \Omega_j), (\wp')^2 = 4(\wp^3 - 1)$, where $\Omega_{j+3} = \Omega_j$, and $q^j \Omega_j = \omega_2$ – the real semiperiod of the doubly periodic function $\wp(x)$. The analytical solutions at $q^4 = 1$ exist for special initial condition but they contain functional non-uniqueness. The particular

subcase of the $q^4 = 1$ system is defined by the (pseudo-)lemniscatic Weierstrass function satisfying the equation $(\wp')^2 = 4\wp^3 \pm \wp$. So, the group-theoretical treatment of the Schrödinger equation with these specific elliptic function potentials naturally leads to the q-oscillator algebra at roots of unity. Note that the algebra of symmetry operators in these cases does not have the spectrum generating meaning.

3 Coherent States of the q-Ladder Algebras

Coherent states are interesting objects of quantum mechanics [9]. Originally proposed for the harmonic oscillator potential, eventually they were generalized in many directions. Let us discuss briefly coherent states of the algebra (14), (15) which we define as eigenstates of the "annihilation" operator B^-:

$$B^-\psi_\alpha(x) = \alpha\psi_\alpha(x). \tag{18}$$

Looking at the definition of B^- one can realize that this is quite complicated functional equation. The simplest case ($N = 1$) has the following explicit form:

$$(d/dx + f(x))\psi_\alpha(x) = \alpha\sqrt{|q|}\psi_\alpha(qx), \tag{19}$$

where $f(x)$ is a smooth solution of the differential equation with the deviating argument:

$$\frac{d}{dx}(f(x) + qf(qx)) + f^2(x) - q^2 f^2(qx) = \mu.$$

The $\sqrt{|q|}$ factor appeared because we took U to be unitary operator so that $(B^-)^\dagger = B^+$. We also assume that $0 < q^2 \leq 1$. (The $q^2 > 1$ choice is equivalent to analytical continuation of the $q^2 < 1$ potentials to the imaginary axis. This brings pole singularities into the potential and, as a result, operators $B^\pm$ are not well defined.) Coherent states of the q-oscillators have been widely discussed (see, e.g., [10]), but the realization (19) is a principally new one since it deals with the ordinary Schrödinger equation. Unfortunately, the structure of functions $\psi_\alpha(x)$, their minimal complete subsets, and many other things are not known at present.

As it was noticed in [7] there is a particular coherent state among $\psi_\alpha(x)$ which happens to be an eigenstate of the Hamiltonian! (Such situation is characteristic for the whole algebra (14), (15), i.e. for any self-similar potential at $q^2 < 1$.) It corresponds to the zero energy state, $L\psi_{cl}(x) = 0$, the formal existence of which follows from the boundedness of the potential $\int_{-\infty}^{\infty} |u(x)|dx < \infty$. So, we have the following representation of the q-oscillator algebra:

$$B^\pm\psi_{cl}(x) = re^{\pm i\theta}\psi_{cl}(x), \qquad r \equiv \sqrt{\mu/(1-q^2)}, \tag{20}$$

i.e. $B^\pm$ are pure complex conjugated numbers. Possible existence of such "classical" states of the q-oscillator algebra has been noticed also (in the different context) in [11]. Since for $\psi_{cl}(x)$ we have two equations: (19) with $\alpha = re^{-i\theta}$ and

$$(-d/dx + f(x))\sqrt{|q|}\psi_{cl}(qx) = re^{i\theta}\psi_{cl}(x),$$

we can remove the derivative part and get pure q-difference equation:

$$r|q|^{-1/2}[e^{-i\theta}|q|\psi_{cl}(qx) + e^{i\theta}q^{-1}\psi_{cl}(q^{-1}x)] = (f(x) + q^{-1}f(q^{-1}x))\psi_{cl}(x), \tag{21}$$

which, however, again is not easy to solve.

Finally, let us consider the case $q = -1$, i.e. the coherent states associated with the realization (17). We put for convenience $\mu = 2$ and renormalize $B^{\pm} \to \sqrt{2}B^{\pm}$. The $\psi_\alpha(x)$ states form a subset of solutions of $(B^-)^2\psi_\alpha(x) = \alpha^2\psi_\alpha(x)$. Because $(B^{\pm})^2$ are purely differential operators, one can easily solve this equation. Picking out the proper linear combination of the corresponding two independent solutions, one can find:

$$\psi_\alpha(x) = 2^{-1/2}(e^{-i\pi/4}|i\alpha\rangle + e^{i\pi/4}|-i\alpha\rangle), \tag{22}$$

where $|\alpha\rangle$ are the canonical coherent states of a harmonic oscillator:

$$|\alpha\rangle = \pi^{-1/4}\exp\left(\frac{\alpha^2 - |\alpha|^2}{2} - \left(\frac{x}{\sqrt{2}} - \alpha\right)^2\right). \tag{23}$$

The states (22) are not minimal uncertainty states for the variables x and $p \equiv -id/dx$ for $\alpha \neq 0$:

$$\langle(\Delta x)^2\rangle = (1 - (\alpha - \alpha^*)^2 - (\alpha + \alpha^*)^2 e^{-4|\alpha|^2})/2,$$

$$\langle(\Delta p)^2\rangle = (1 + (\alpha + \alpha^*)^2 + (\alpha - \alpha^*)^2 e^{-4|\alpha|^2})/2,$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle > 1/4,$$

where $\Delta x = x - \langle x\rangle, \langle x\rangle = \int_{-\infty}^{\infty} x|\psi_\alpha(x)|^2 dx$, etc. However, it is easy to construct other canonical variables

$$\pi = i(B^+ - B^-)/\sqrt{2} = ixP, \quad \phi = (B^+ + B^-)/\sqrt{2} = -ipP, \quad [\phi, \pi] = i,$$

where P is the parity operator, for which $\psi_\alpha(x)$ minimize their uncertainties, $\langle(\Delta\phi)^2\rangle\langle(\Delta\pi)^2\rangle = 1/4$. A detailed consideration of the properties of these coherent states will be given elsewhere [12].

Finally, we would like to point out that the procedure of determination of potentials with fixed symmetry properties presented here may be generalized to other spectral problems. In [13] it was applied to the second order finite-difference equation. In that case many features of the continuous considerations maintain but there are also few new ones. E.g., the algebras at $q > 1$ may now have spectrum generating meaning, i.e. one can find the systems with the exponentially growing spectra. Another advantage consists in the possibility to realize q-analogs of the compact unitary algebras like $su(2)$.

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