Hoops in R^3

Daniel Asimov*

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Abstract

Hoops are defined as geometric circles of radius = 1 in 3-dimensional Euclidean space R^3. It is shown that it is possible to continuously fill an open set of R^3 with disjoint hoops. It is also shown that in any such open set, every pair of hoops must link each other, and consequently there is an upper bound to the volume that such an open set may have.

*Numerical Aerodynamic Simulation (NAS) Systems Division, MS T045-1, NASA Ames Research Center, Moffett Field, CA 94035-1000 (asimov@nas.nasa.gov). The author is an employee of Computer Sciences Corporation. This work is supported through NASA Contract NAS 2-12961.

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Daniel Asimov

Computer Sciences Corporation
Mail Stop T045-1
NASA Ames Research Center
Moffett Field, CA 94035-1000
E-mail: asimov@nas.nasa.gov

Abstract: Hoops are defined as geometric unit circles in $\mathbf{R}^3$. It is shown there exists a nonempty connected open set U in $\mathbf{R}^3$ that is foliated by leaves all of which are hoops, and that for any such foliation, every pair of leaves must be linked with each other. As a result there is an upper bound for the volume of such an open set U.

Definition: A hoop is any subset of $\mathbf{R}^3$ that is congruent (by a rigid motion) to the unit circle in the xy-plane $\{(x,y,0) \mid x^2 + y^2 = 1\}$. A hoop is determined uniquely by a choice of center, and a choice of normal line through that center. Thus the space of hoops in $\mathbf{R}^3$ may be identified with $\mathbf{R}^3 \mathbf{x} \mathbf{P}^2$.

Theorem 1: Let U be a nonempty, connected open set in $\mathbf{R}^3$. Let F be a topological foliation* of U such that every leaf of F is a hoop. Then every two leaves of F must link each other.

Corollary 1.1: There is an $M > 0$ such that, if U is any connected open set in $\mathbf{R}^3$ that is foliated by hoops, then the volume of U must be $\leq M$.

Notation: Let the least such M be denoted by V_{max}.

Theorem 2: $2\pi^2 \leq V_{max} \leq 2\pi(8\pi/3+3\sqrt{3})$, i.e., $19.739... \leq V_{max} \leq 85.286....$

Proof of Theorem 2: First we show that the volume of $2\pi^2$ is attainable. For any real number r with $0 \leq r < 1$, consider the circle C_r of radius r centered at the point (1,0,0) and lying in the xz-plane. Rotating C_r about the z-axis, we obtain a torus of revolution which shall be denoted by T_r. (For convenience we include the degenerate case $r = 0$.)

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*See Appendix for definition.

Lemma 2.1: Each T_r is foliated by hoops.

Proof of Lemma 2.1: This fact seems rather surprising at first, but was originally discovered 145 years ago by Yvon Villarceau [2,3,4]. For this reason we merely sketch the proof.

As is well-known, a defining equation for the torus Tr is given by

$$(x^2 + y^2 + z^2 + 1 - r^2)^2 = 4(x^2 + y^2).$$

Let alpha be the unique angle $0 \le \alpha < \pi/2$ such that $\sin(\alpha) = r$. Then it is not difficult to verify that the intersection of the sphere $(x - \sin(\alpha))^2 + y^2 + z^2 = 1$ and the plane $z/y = \tan(\alpha)$ is a circle of unit radius, and must lie on the torus T_r. ■

We denote this circle by $V(\alpha)$.

If $V(\alpha)$ is rotated about the z-axis through any angle φ, the resulting circle must also lie on T_r, by symmetry. Such circles are disjoint for distinct angles φ, $0 \le \varphi < 2\pi$, and their union is the torus T_r. (These circles are known as the "circles of Villarceau.")

In fact, a parametric representation of one of these circles is given by

$$x = \cos(\varphi)(\cos(\theta) + \sin(\alpha)) - \sin(\varphi)(\sin(\theta)\cos(\alpha)),$$
$$y = \sin(\varphi)(\cos(\theta) + \sin(\alpha)) + \cos(\varphi)(\sin(\theta)\cos(\alpha)),$$
$$z = \sin(\theta)\sin(\alpha),$$

where $0 \le \theta \le 2\pi$, and $r = \sin(\alpha)$ as above.

Notation: Let us denote the union of the tori T_r, for $0 \le r < 1$, by B.

Terminology: We use the term **bialy** to describe any set in $\mathbf{R}^3$ that is congruent to B (by a rigid motion).

It is clear from the parametric representation above that these circles form a real analytic foliation, by hoops, of the bialy B. (This amounts to nothing more than giving the Villarceau foliation to each of the nested tori Tr simultaneously.) The volume of B is easily calculated to be $2\pi^2$.■

Let H_0 be any hoop of the foliation. By Theorem 1, all other leaves of the foliation must link H_0. Hence every point of U must lie within a distance of 2 from the hoop H_0. This locus is congruent to the result of revolving the disk

$$\{(x,0,z) \mid (x - 1)^2 + z^2 \le 4\}$$

about the z-axis. Its volume is easily calculated, using Pappus' Theorem, to be $2\pi(8\pi/3 + 3\sqrt{3})$.■

Conjecture: $V_{max} = 2\pi^2$ (= 19.739...).

Definition: Let U be any open set in $\mathbf{R}^3$ that is foliated by hoops. (In general U will be unbounded and disconnected.) Let V(r) denoted the volume of U that lies within a distance r of the origin, and let B(r) denote the volume of the ball of radius r in $\mathbf{R}^3$. If lim as $r \to \infty$ of V(r)/B(r) exists, it is called the hoop packing fraction of the foliated open set U, and denoted by HPF(U). The supremum, over all open sets in $\mathbf{R}^3$ that are foliated by hoops, of HPF(U) where it is defined, is called the packing fraction for hoops in $\mathbf{R}^3$.

Theorem 3: The packing fraction for hoops in $\mathbf{R}^3$ is $\geq \pi^2/(8\sqrt{3})$ (= .712...).

Remark: We do not know whether a denser packing of hoops in $\mathbf{R}^3$ is possible.

Proof of Theorem 3: Slice up $\mathbf{R}^3$ into parallel adjoining slabs of thickness = 2. Within any slab, we may pack bialys hexagonally. This pattern may be repeated in each slab. The packing fraction may be determined by considering as fundamental domain the prism whose cross-section is an equilateral triangle of side = 4 and whose height= 2. This prism has its three vertical edges aligned with the axes of three bialys; its two triangular faces lie in the two bounding planes of a slab, respectively. The volume of the triangular prism is $2*4^2\sqrt{3}/4 = 8\sqrt{3}$. There is 3/6 = 1/2 of a bialy contained within, whose volume is $2\pi^2/2 = \pi^2$, and so the fraction for this fundamental domain is $\pi^2/(8\sqrt{3})$. As $r \to \infty$, the "edge effects" become negligible, and so this is also the hoop packing fraction in the limit for this open set.

Remark: Conway and Croft [1] have shown using the Axiom of Choice that with no continuity restrictions, all of $\mathbf{R}^3$ may be partitioned into disjoint hoops.

Proof of Theorem 1:

Lemma 1.1: Either every two leaves of F link each other, or else no two leaves of F link each other.

Proof of Lemma 1.1: Let L_0 and M_0 be any two distinct leaves of F, and suppose that they are linked. Let L_1 be any third leaf of F. Since U is open and connected, it is also arcwise connected. Thus there exists an arc A: $[0,1] \to U$ connecting L_0 with L_1. We let L(t) denote the leaf contining A(t) (so we must have $L(0) = L_0$ and $L(1) = L_1$). Since $\{L(t)\}$ forms an isotopy of L_0 in the complement of M_0, all leaves L(t) must link with M_0 since $L(0) = L_0$ does. We conclude similarly that L_1 is linked with any fourth leaf M_1.■

Assumption: Now we shall assume, contrary to what is to be proved, that in the foliation F, any pair of leaves is unlinked.

Let H_0 be an arbitrary leaf of U. Pick a point p of H_0, and let D be a small planar disk of radius ε, centered at p and perpendicular to H_0 at p. Let the circle C be defined as the boundary of D.

Consider the leaves of F which intersect C. For any point **x** of C, we denote by H(**x**) the leaf of F which passes through **x**.

Now consider the space HP(p) of all hoops that pass through a single point p of $\mathbf{R}^3$. Such a hoop is determined by a) the plane in which it lies, and b) the hoop's diameter which lies in that plane and contains the point p. The plane may be considered to be an element **x** of projective space $\mathbf{P}^2$, and the diameter is determined by a unit tangent vector at **x**. Thus the space HP(p) may be identified with the space $T_1(\mathbf{P}^2)$ of unit tangent vectors to $\mathbf{P}^2$, which is a 3-dimensional manifold. For each point **x** of C, the hoop H(**x**) determines an element in the space HP(**x**), and thus in $T_1(\mathbf{P}^2)$. As a result we have a continuous mapping

$$g: C \to T_1(\mathbf{P}^2)$$

which associates to each point x of C the hoop H(x) considered as a point of the space $T_1(\mathbf{P}^2)$.

Lemma 1.2: The continuous mapping g: $C \to T_1(\mathbf{P}^2)$ may be approximated by a smooth mapping g2: $C \to T_1(\mathbf{P}^2)$ whose image contains no hoop parallel to H_0.

Proof of Lemma 1.2: We may C^0-approximate g by a smooth mapping g_1: $C \to T_1(\mathbf{P}^2)$. Now consider all the hoops in $T_1(\mathbf{P}^2)$ which are parallel to the hoop H_0. These correspond to all the unit tangent vectors at a single point of $\mathbf{P}^2$, or in other words, a circle C_0.

We now wish to further approximate g_1 by another smooth mapping g_2 which avoids C_0. We may do this easily by standard transversality theory, because of the fact that $\dim(C) + \dim(C_0) = 1 + 1 < 3 = \dim(T_1(\mathbf{P}^2))$.■

Note that the hoops $g_2(x)$ for x in C must be unlinked with H_0, for they are C^0-close to the leaves H(x) of the foliation F, which by Assumption are unlinked with all other leaves and in particular with H_0.

Notation: The plane of the hoop H_0 will be denoted by P_0.

Choice of coordinates: Without loss of generality, we may choose coordinates such that P_0 is the xy-plane, and H_0 is the unit circle in the xy-plane.

Consider the individual hoops $g_2(x)$ for each x in C. By choice of g_2, each hoop $g_2(x)$ is not parallel to P_0. Consequently, each hoop $g_2(x)$ has a unique diameter diam(x) that is parallel to P_0.

Lemma 1.3: The endpoints of these diameters diam(x), for all x in C, form two closed curves.

Proof of Lemma 1.3: We first note that, since the curve C is chosen to lie close to the hoop H_0, we may assume that the planes of the hoops $g_2(x)$, for x in C, lie less than 90° from the plane P_0. Hence the plane of each hoop $g_2(x)$ may be assigned a unit normal vector in a continuous fashion (just choose the normal closest to the positive z direction $\mathbf{k} = (0, 0, 1)$). Let us denote this continuous choice of unit normal by $\mathbf{n}(x)$.

By our choice of g_2 above, $\mathbf{n}(x)$ may never be parallel to $\mathbf{k}$. The cross product $\mathbf{v}(x) = \mathbf{k} \times \mathbf{n}(x)$ is therefore never $\mathbf{0}$ for x in C. The vector $\mathbf{v}(x)$, since it is perpendicular to both $\mathbf{k}$ and $\mathbf{n}(x)$ must be parallel to diam(x) for each x in C. Consequently, the formula

$$L(x) = \text{center}(x) + \mathbf{v}(x)/|\mathbf{v}(x)|$$

where center(x) denotes the midpoint of diam(x), selects one of the endpoints of diam(x) in a continuous fashion. Similarly,

$$R(x) = \text{center}(x) - \mathbf{v}(x)/|\mathbf{v}(x)|$$

selects the other endpoint continuously. As x traverses C, L(x) and R(x) must therefore each describe a closed curve.■

Lemma 1.4: The curves C and L are freely homotopic in the complement of H_0.

Proof of Lemma 1.4: Let N denote the point (0,0,0,1) of the 3-sphere $\mathbf{S}^3$. We shall make use of the stereographic projection

$$sp: \mathbf{S}^3 - \{N\} \to \mathbf{R}^3$$

given by $sp(x, y, z, w) = (x/(1-w), y/(1-w), z/(1-w))$.

Some facts about stereographic projection:

1.The stereographic projection sp is a homeomorphism from $\mathbf{S}^3 - \{N\}$ to $\mathbf{R}^3$.

2.Stereographic projection preserves angles.

3.Any circle in $\mathbf{S}^3 - \{N\}$ is carried to a circle in $\mathbf{R}^3$.

4.Any sphere in $\mathbf{S}^3 - \{N\}$ is carried to a sphere in $\mathbf{R}^3$.

Since H_0 is the unit circle in the xy-plane, the mapping sp will take the great circle $\{(x, y, 0, 0) \mid x^2 + y^2 = 1\}$ onto H_0. Let H denote any hoop in $\mathbf{R}^3$ that is disjoint from H_0 and does not link H_0.

Notation: We denote the preimages of H_0 and H under the mapping sp by K_0 and K, respectively.

Lemma 1.4.1: There is a unique smallest great spherical cap in $\mathbf{S}^3$ whose boundary is K.

Proof of Lemma 1.4.1: First note that K will not be a great circle in $\mathbf{S}^3$. For, any two great circles of $\mathbf{S}^3$ either intersect or are linked with each other. But K cannot intersect or link K_0, since H does not intersect or linkH_0.

There is only one great $\mathbf{S}^2$ of $\mathbf{S}^3$ which contains K. For suppose that S_2 is any great 2-sphere of $\mathbf{S}^3$ containing K. Then S_2 must be the unit sphere of the unique 3-dimensional subspace of $\mathbf{R}^4$ containing K and the origin.

Hence there are exactly two great spherical caps in $\mathbf{S}^3$ whose boundary is K; their union is S_2. Since K is not great, just one of these caps is smaller than a hemisphere.■

Notation: For any non-great circle K in $\mathbf{S}^3$, we denote by cap4(K) the unique least great spherical cap whose boundary is K.

Lemma 1.4.2: The spherical cap cap4(K) is disjoint from K_0.

Proof of Lemma 1.4.2: The spherical cap cap4(K) is part of some great 2-sphere S in $\mathbf{S}^3$. Now such a great sphere lies in a unique 3-dimensional subspace W_3 of the $\mathbf{R}^4$ in which the $\mathbf{S}^3$ sits. At the same time, the great circle K_0 lies in a unique 2-dimensional subspace W_2 of the $\mathbf{R}^4$. By linear algebra, $W_3 \cap W_2$ in $\mathbf{R}^4$ may be either 1- or 2-dimensional.

i) If $W_3 \cap W_2$ is 1-dimensional, then $S \cap K_0$ will be just 2 diametrically opposite points. If one of these lies in cap4(K) then the other cannot. This would imply that K links K_0, contrary to hypothesis.

ii) If $W_3 \cap W_2$ is 2-dimensional, then $S \cap K_0$ will be a great circle, which must be all of K_0. If any point of K_0 lay in cap4(K), then K_0 would have to intersect K, contrary to hypothesis.■

Now, to each hoop H in $\mathbf{R}^3$ which is disjoint from and not linked with H_0, we assign a spherical cap cap(H) as follows: Let K be the preimage of H under the

mapping sp; let cap4(K) be the unique smallest spherical cap whose boundary is K; and finally we let cap(H) be sp(cap4(K)).

Note that, by construction, for all hoops H which are disjoint from and do not link H_0, cap(H) is disjoint from H_0. In addition, cap(H) is a continuous function of the hoop H.

Now we apply this construction to our family of hoops $\{ g_2(x) \mid x \in C \}$. Since everything in sight is continuous, the spherical cap $cap(g_2(x))$ depends continuously on $x \in C$.

Notation: The pole of cap(x), denoted pole(x), will refer to the unique point of cap(x) that is farthest from its boundary, H(x).

Now we define a free homotopy from C to L as follows: Let x be any point of C. For $0 \le t \le 1/2$, let F(x,t) be the constant-speed geodesic on cap(x) from x to pole(x). For $1/2 \le t \le 1$, let F(x,t) be the constant-speed geodesic on cap(x) from pole(x) to L(x). Thus Lemma 1.4 is proved.■

Now since C links H_0 and L is freely homotopic to C in the complement of H_0, L must link H_0 as well. Hence some point L(z) of L must intersect the disk D_0 that lies in the plane P_0 and is bounded by H_0. (Here z denotes some point of the curve C.) Let the hoop $g_2(z)$ be denoted by H. By construction, the "right" endpoint R(z) of diam(z) must also lie in the plane P_0 and in the leaf H.

Now, the distance between L(z) and R(z) is exactly 2, since they are diametrically opposite points on the hoop H. Since L(z) lies inside the unit disk D_0, it follows that R(z) must lie outside the disk D_0 in the plane P_0. This implies that the hoop H links the hoop H_0, contrary to Assumption. The conclusion now follows from Lemma 1.1.■

Appendix: Foliations

Definition: Let U denote an open set in Euclidean n-space $\mathbf{R}^n$, and let $\boldsymbol{F} = \{L_\alpha\}$ be a collection of connected subsets L_α of U. Then $\boldsymbol{F}$ is said to be a **k-dimensional foliation** of U if the following hold:
1) Each $x \in U$ lies in exactly one L_α
2) For every $x \in U$, there exists an open set V with $x \in V \subset U$, and a homeomorphism $h: V \approx \mathbf{R}^k \times \mathbf{R}^{n-k}$, such that for any $y \in \mathbf{R}^{n-k}$, $h^{-1}(\mathbf{R}^k \times \{y\})$ is a connected component of the set $L_\alpha \cap V$, where L_α is the unique subset containing the point x.■

Intuitively, this definition says that the decomposition of U into the pieces L_α looks locally just like parallel k-planes in $\mathbf{R}^n$.

If $\boldsymbol{F} = \{L_\alpha\}$ is a k-dimensional foliation of U, then each subset L_α is known as a **leaf** of the foliation, and is in fact a k-dimensional submanifold of U. The definition of a foliation given above is sometimes more specifically referred to as a C^0 (or **topological**) foliation. More generally, if each homeomorphism h in the above definition may be chosen to be a C^r diffeomorphism, then $\boldsymbol{F}$ is said to be a C^r foliation.

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Figures: (next 2 pages)

Figure 1: Circles of Villarceau are shown for torus T_r for $r = .8$
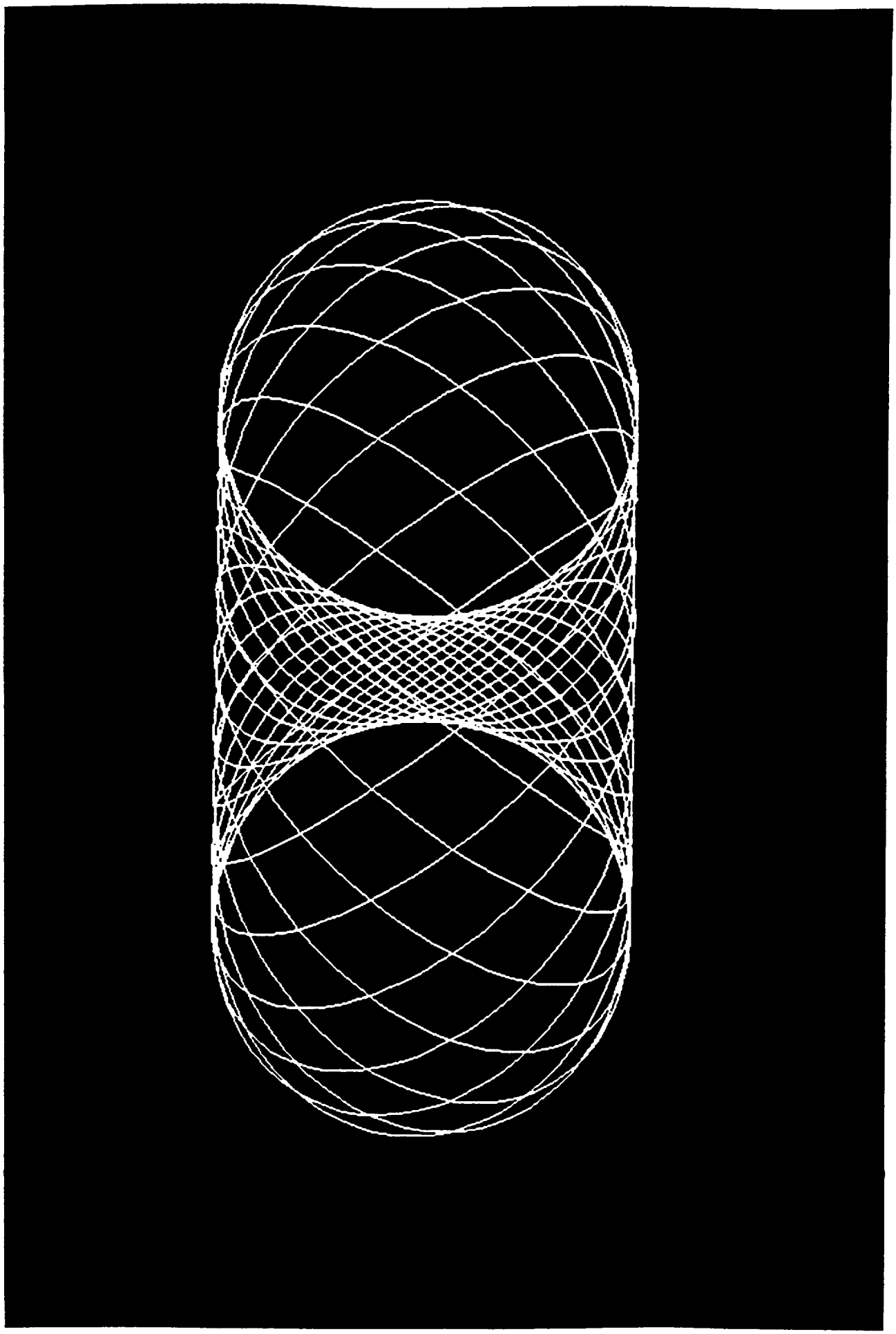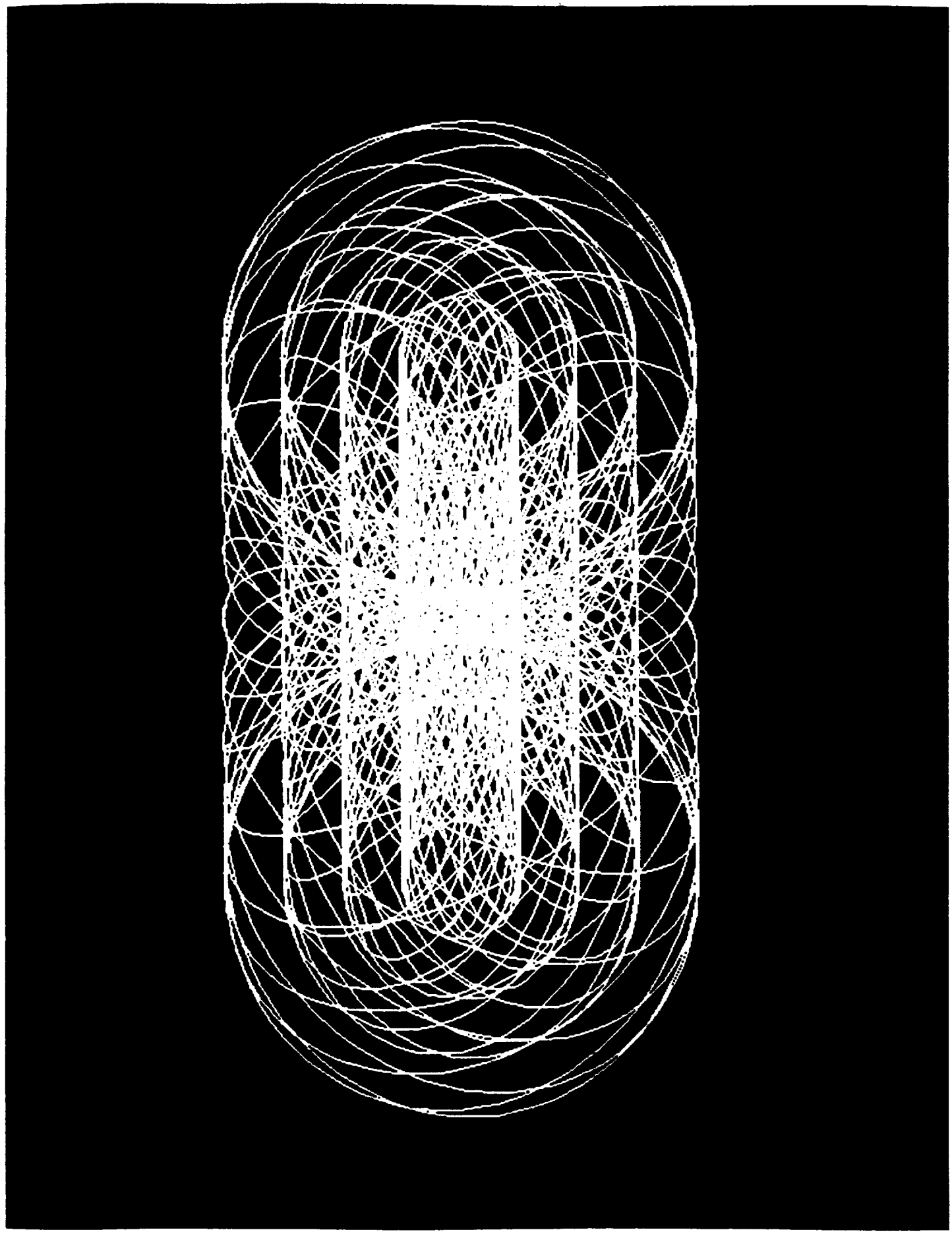
Figure 2: Nested tori T_r are shown for $r = .2, .4, .6, .8$ as part of foliation of bialy.