NASA-CR-203103

IN-47-CR
OCIT
008814

Progress Report for NASA Grant NAG5-3140

to the University of Chicago

For the Period July - December 1996

Radiative Transfer in the Lower Atmosphere Based on TOMS/ADEOS Measurements

Principal Investigator:

John E. Frederick

Department of the Geophysical Sciences

The University of Chicago

5734 South Ellis Avenue

Chicago, IL 60637

Telephone: (773) 702-3237

Facsimile: (773) 702-9505

E-mail: frederic@rainbow.uchicago.edu

December 1996

I. Objective

The long-term objective of this research is to develop and apply methods to compute the solar ultraviolet (UV) spectral irradiance at the earth's surface and especially throughout the vertical extent of the troposphere using information to be provided by TOMS/ADEOS. The broad philosophy is to view the TOMS measurements as a probe of the radiative transfer properties of the earth's atmosphere, with emphasis on the optical characteristics of the troposphere.

II. Activity During the Reporting Period

A. Validation of the TOMS Ground-Level UV Irradiance Data Product

During the summer of 1996 plans were made to conduct a workshop on validation of the ground-level UV irradiance data product to be produced by scientists from Goddard Space Flight Center using column ozone and reflectivities from TOMS. The workshop was held on October 8-9. The participants who are in a position to supply ground-based data for use in the TOMS validation effort will be providing additional written information for use in updating the UV Validation Plan which was included as an Appendix to the last Progress Report dated July 1996. The information to be provided includes the following: (1) a description of the instrumentation, (2) the frequency with which data are obtained and the GMT of the data, (3) the time lag between data collection and the availability of data for use with TOMS, (4) the latitude and longitude of the measurements, (5) the elevation of the observing sites, (6) a brief summary of calibration and quality control, and (7) a list of any additional data collected at the site. The purpose of the activity in UV validation is to identify a set of ground-based sites from which reliable data can be obtained on a timely basis, to make the logistical arrangements needed to acquire this data, and to arrange for this information to be brought to bear on the UV data product to be generated at Goddard Space Flight Center.

Based on information presented at the workshop, the number of operational sites for measuring solar UV irradiance appears to be more than adequate for use in the validation activity. The data sets identified by the workshop attendees are summarized below:

1. Japan - 4 Brewer instruments obtaining spectral data.
2. Europe - 14 sites distributed between latitude 40.5 and 69.7 degrees North. Instrumentation includes both spectral, multi-filter, and broadband devices.
3. South America - 8 sites including one spectroradiometer and seven multi-filter instruments.
4. Antarctica - 4 sites including four spectroradiometers operated by the United States and one Brewer instrument operated by Japan.
5. Canada - 10 Brewer instruments.
6. Russia - 1 broadband instrument operating in Moscow.
7. United States - A list of UV measurement efforts in the U.S.A. appears below.
   - Department of Agriculture: 30-40 broadband and multi-filter instruments.
   - Environmental Protection Agency: 7 Brewer sites with the potential to grow to 21 sites in the near future.
   - National Oceanic & Atmospheric Administration: 4 broadband instruments associated with the SURFRAD program and 10 broadband instruments associated with the ISIS program.

The above list represents a modest fraction of the 250 to 400 UV measurement sites located throughout the world. However, the availability and especially the quality of the data produced from many of the unlisted sites is uncertain. By focusing the TOMS validation effort on a subset of the sites listed above, we can have confidence in having access to data of high quality.

Plans are also being made to accommodate the different classes of instrumentation which will enter the validation effort. In principle the radiative transfer calculation done at Goddard can produce the full wavelength-dependent solar spectral irradiance to any desired

spectral resolution. The ground-based measurements, of course, do not have this capability. A viable approach to the validation effort is as follows. For each site where ground-based data are collected, the collocated TOMS data (ozone, reflectivity) can be used to compute the solar spectral irradiance. Then this high resolution calculation can be degraded to match the spectral characteristics of each observing instrument. The information needed to carry out this procedure can be supplied by the scientists in charge of the ground-based observing sites.

B. Studies in Radiative Transfer Utilizing TOMS Data

The major scientific effort supported under this grant addresses the question: How uniquely can one infer the solar UV radiation field throughout the troposphere from measurements of the radiance backscattered into space? The goal here is to estimate the tropospheric radiation field, including the influence of clouds. The relevance of the results is to the calculation of photodissociation rates as functions of altitude in the troposphere for use in chemical studies (although chemistry is not part of the proposed work).

Most previous efforts in radiative transfer, and all studies in the ultraviolet, have assumed a one-dimensional, horizontally homogeneous atmosphere. For calculations only of the UV spectral irradiance at the ground there appears to be little advantage in attempting to treat the horizontally inhomogeneous, provided one treats the result as an average over some horizontal area. The only major unknown involves whether or not the solar disk lies in the clear or cloudy portion of a partly cloudy sky. However, if the concern is with the radiation field throughout the depth of the troposphere, including horizontal variations, the situation becomes more complex. For example, the radiation field in a locally clear region of the troposphere will be influenced by the presence of a cloud located some distance away. The precise definition of the term "some distance" depends on the mean free path of a photon at the wavelength considered. This will be investigated as part of the research.

Since the last progress report the primary activity has centered on development of a radiative transfer code with the capability to calculate the radiance which emerges to space (as measured by TOMS) and the radiation field throughout the depth of an atmosphere which is partly cloudy (that is, horizontally inhomogeneous). This effort is part of the Ph.D. research of Mr. Zheng Qu at the University of Chicago. Mr. Qu is supported as a Research Assistant using funds from this grant. The Appendix is an excerpt from Mr. Qu's research prospectus, required for admission to Ph.D. candidacy. This material describes the mathematical formulation of the new model.

When the radiative transfer code is operational, we will be able to compute the emergent radiance measured by TOMS for any fractional cloudiness in the field of view. At present there are restrictions on the allowed cloud geometries, but this constraint will be weakened in future work. In addition, we will be able to investigate situations where, for example, the TOMS filed of view is clear but where the edge of a cloudy region is nearby. In this case multiple scattering would allow the cloud to influence the emergent radiance measured by TOMS even though the field of view is clear. The importance of such scenarios will be among the topics studied.

Finally, we will investigate the solar UV radiation field throughout the troposphere. The central issue here involves assessing the degree to which radiances measured by TOMS can constrain this radiation field in a horizontally inhomogeneous atmosphere.

III. Plans for the Coming Months

Attendees of the UV Validation Workshop should be providing written information on their data sets. This will be incorporated into a revised TOMS UV Validation Plan to be completed in January 1997 (provided the attendees reply according to the timetable set at the workshop). In addition, development of the new radiative transfer model for a horizontally inhomogeneous atmosphere will continue.

APPENDIX

Mathematical Formulation of a Radiative Transfer Model in an Atmosphere with Variations in the Horizontal Direction

The following pages are taken from the research prospectus of Mr. Zheng Qu, a graduate student who is supported as a Research Assistant by this grant. The mathematical formulation will be the basis of a new radiative transfer model to be used in conjunction with radiances measured by TOMS on ADEOS.

3.2 A 2-D Model for Horizontolly Inhomogeneous Atmosphere

In this section, we will discuss a simplified situation that one half the sky is clear while the other is cloud covered (Fig 4). Still we divide the atmosphere into N+1 layers, while half the n th layer is occupied by cloud. For layers far away from the cloudy layer, say, layer number m>n+1 or m<n-1, radiation calculations will be the same as those in a multiplayer homogeneous model. We now divide the n th layer into two parts, part I is solely clear and part II occupied by the cloud, as shown in Fig. This partition also applies to (n -1)th and (n +1)th layer. We set X-axis starting at partitioning line in n th layer. Within each part of the layers where partition applies, we take the media as homogeneously distributed throught the entire part.

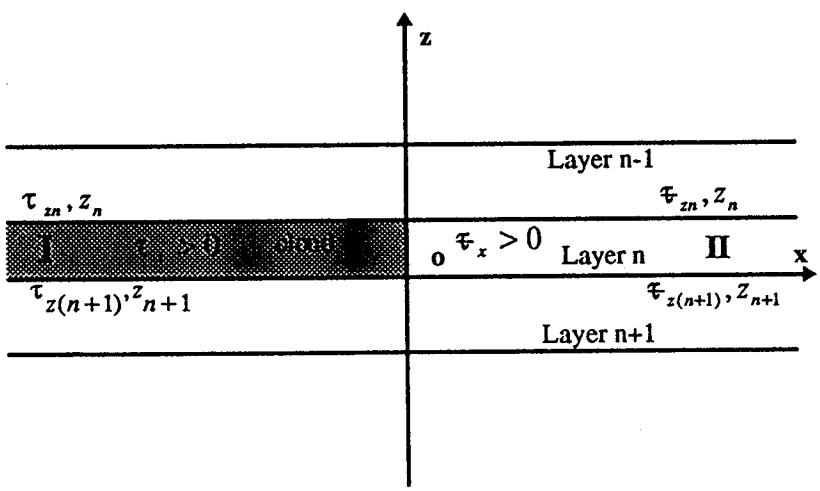

Fig. 4

3.2.1 The Radiative Transfer Equation in Two Dimensions

The original RT equation for scattered intensity is in the form

$$\frac{dI(\hat{s},x,y,z)}{n\sigma ds} = -I(\hat{s},x,y,z) + \frac{\varpi_T}{4\pi} P_T(\hat{s}_0,\hat{s}) F_0 e^{-\tau_z/\mu_0} + \frac{1}{4\pi}\int_{4\pi} \varpi_T P_T(\hat{s}';\hat{s}) I(\hat{s}';x,y,z) d\omega' \tag{3.2.1.1}$$

where

$$\frac{d}{ds} = d\hat{s} \bullet \nabla \ ;$$

$d\omega'$ is derivitive of solid angel, $d\omega' = \sin\theta' \, d\theta' \, d\phi'$;

$\hat{s}$ is the unit vector of the direction of scattered intensity, the direction is represented by (θ ,ϕ), or (μ ,ϕ), where $\mu = \cos\theta$. Fig 2 shows geometry of (θ ,ϕ);

$\hat{s}_0$ is the unit vector of the direction of incident sunlight. The direction is represented by ($-\mu_0,\phi_0$);

τ_z is vertical optical depth (defined below) at the point of interest.

We will discuss the equation in two dimensions x and z and solve the equation seperately in two parts of the layer and couple them with lateral boundary conditions. For part I, we define vertical optical depth τ_z and horizontal optical depth τ_x to replace variables (x, z) in the equation. The defination of τ_z is the same as previously defined τ ,

$$\tau_z = \int_z^{\infty} n\sigma dz \tag{3.2.1.2}$$

while τ_x is defined as

$$\tau_x = \int_x^{0} n\sigma dx \tag{3.2.1.3}$$

Rewriting the RT equation in two variables τ_x and τ_z gives

$$\sqrt{1-\mu^2}\cos\phi\frac{\partial I(\mu,\phi;\tau_x,\tau_z)}{\partial\tau_x}+\mu\frac{\partial I(\mu,\phi;\tau_x,\tau_z)}{\partial\tau_z}=I(\mu,\phi;\tau_x,\tau_z)$$
$$-\frac{\varpi_T}{4\pi}\int_0^{2\pi}d\phi'\int_{-1}^{1}d\mu' P_T(\mu',\phi';\mu,\phi)I(\mu',\phi';\tau_x,\tau_z)-\frac{\varpi_T}{4\pi}P_T(-\mu_0,\phi_0;\mu,\phi)F_0e^{-\tau_z/\mu_0} \tag{3.2.4}$$

To obtain the solution for this inhomogeneous equation, we first seek a particular solution $I^*(\mu,\phi;\tau_z)$ satisfying equation

$$\mu\frac{dI^*(\mu,\phi;\tau_z)}{d\tau_z}=I^*(\mu,\phi;\tau_z)-\frac{\varpi_T}{4\pi}P_T(-\mu_0,\phi_0;\mu,\phi)F_0e^{-\tau_z/\mu_0}$$
$$-\frac{\varpi_T}{4\pi}\int_0^{2\pi}d\phi'\int_{-1}^{1}d\mu' P_T(\mu',\phi';\mu,\phi)I^*(\mu',\phi';\tau_z) \tag{3.2.1.5}$$

as well as the vertical boundary conditions at $\tau_x \to \infty$, where we assume the horizontally homogeneous multilayer model applies. This particular solution therefore can be obtained

by any method we discussed in Section 3.1. Second, we try to find a solution $\tilde{I}(\mu,\phi;\tau_x,\tau_z)$ satisfying the homogeneous equation

$$\sqrt{1-\mu^2}\cos\phi\frac{\partial\tilde{I}(\mu,\phi;\tau_x,\tau_z)}{\partial\tau_x}+\mu\frac{\partial\tilde{I}(\mu,\phi;\tau_x,\tau_z)}{\partial\tau_z}=\tilde{I}(\mu,\phi;\tau_x,\tau_z)$$

$$-\frac{\varpi_T}{4\pi}\int_0^{2\pi}d\phi'\int_{-1}^{1}d\mu' P_T(\mu',\phi';\mu,\phi)\tilde{I}(\mu',\phi';\tau_x,\tau_z) \qquad (3.2.1.6)$$

It is easy to verify that solution

$$I(\mu,\phi;\tau_x,\tau_z)=\tilde{I}(\mu,\phi;\tau_x,\tau_z)+I^*(\mu,\phi;\tau_z) \qquad (3.2.1.7)$$

satisfy the Eq(3.2.1.4).

3.2.2 Seeking the Homogeneous Solution $\tilde{I}(\mu,\phi;\tau_x,\tau_z)$

As we did in discrete-ordinates method we discussed previously in Section 3.1.2.1, we expand the solution in Fourier series with regard to ϕ . It is reasonable to assume the solution to be symmetric with regard to x-z plane, so the Fourier series have cosine terms only,

$$\tilde{I}(\mu,\phi;\tau_x,\tau_z)=\sum_{m=0}^{\infty}I^m(\mu;\tau_x,\tau_z)\cos m\phi \qquad (3.2.2.1)$$

Still, the phase function can be expanded in associated legendre polynomials,

$$P_T(\mu',\phi';\mu,\phi)=\sum_{m=0}^{N}\sum_{l=m}^{N}\varpi_l^m P_l^m(\mu)P_l^m(\mu')\cos m(\phi'-\phi) \qquad (3.2.2.2)$$

Upon inserting Eqs. (3.2.2.1) and (3.2.2.2) into Eq (3.2.1.6) and noting that

$$\int_0^{2\pi}\cos m\phi\cos n\phi\, d\phi=\begin{cases}0, & m\neq n\\ \pi, & m=n\end{cases} \qquad (3.2.2.3)$$

$$\int_0^{2\pi}\cos\phi\cos m\phi\cos n\phi d\phi=\begin{cases}0, & n\neq m+1\\ \pi/2, & n=m+1\end{cases} \qquad (3.2.2.4)$$

Eq(3.2.1.6) splits up into (N+1) coupled equations:

$$(\sqrt{1-\mu^2}/2)\frac{\partial I^{m-1}(\mu,\tau_x,\tau_z)}{\partial\tau_x}+\mu\frac{\partial I^m(\mu,\tau_x,\tau_z)}{\partial\tau_z}=I^m(\mu,\tau_x,\tau_z)$$

$$-(1+\delta_{0,m})\frac{\varpi_T}{4}\sum_{l=m}^{N}\varpi_l^m P_l^m(\mu)\int_{-1}^{1}d\mu' P_l^m(\mu')I^m(\mu',\tau_x,\tau_z)d\mu' \qquad (3.2.2.5)$$

$(m=0,\ldots,N)$

Note

$$I^{-1}(\mu,\tau_x,\tau_z)=I^1(\mu,\tau_x,\tau_z) \qquad (3.2.2.6)$$

If we take the first two terms in Fourier expansion (3.2.2.1), *i.e.*, let N=1, the Eqs (3.2.2.5) reduce to two eqquations. Noting

$$\varpi_0^0=\varpi_0$$
$$\varpi_1^0=\varpi_1$$
$$\varpi_1^1=\varpi_1$$

$$P_0^0(\mu)=P_0(\mu)=1$$
$$P_1^0(\mu)=P_1(\mu)=\mu$$
$$P_1^1(\mu)=(1-\mu^2)^{1/2}$$

the two equations are given by

$$A(\mu)\frac{\partial}{\partial\tau_x}I^1(\mu,\tau_x,\tau_z)+\mu\frac{\partial}{\partial\tau_z}I^0(\mu,\tau_x,\tau_z)=I^0(\mu,\tau_x,\tau_z)-\int_{-1}^{1}B(\mu',\mu)I^0(\mu',\tau_x,\tau_z)d\mu'$$

(3.2.2.7)

$$A(\mu)\frac{\partial}{\partial\tau_x}I^0(\mu,\tau_x,\tau_z)+\mu\frac{\partial}{\partial\tau_z}I^1(\mu,\tau_x,\tau_z)=I^1(\mu,\tau_x,\tau_z)-\int_{-1}^{1}C(\mu',\mu)I^1(\mu',\tau_x,\tau_z)d\mu'$$

(3.2.2.8)

where

$$A(\mu)=(1-\mu^2)^{1/2}/2$$
$$B(\mu',\mu)=\frac{\varpi_T}{2}(\varpi_0+\varpi_1\mu'\mu)$$
$$C(\mu',\mu)=\frac{\varpi_T\varpi_1}{4}(1-\mu'^2)^{1/2}(1-\mu^2)^{1/2}$$

Applying Gauss's formula (3.1.2.1.7-8) in the Eqs (3.2.2.7) and (3.2.2.8) for the integration terms, we find that Eqs (3.2.2.7) and (3.2.2.8) are replaced by $2N'$ coupled first-order ordinary differential eqations,

$$A(\mu_i)\frac{\partial}{\partial\tau_x}I^1(\mu_i,\tau_x,\tau_z)+\mu\frac{\partial}{\partial\tau_z}I^0(\mu_i,\tau_x,\tau_z)$$

$$=I^0(\mu_i,\tau_x,\tau_z)-\sum_{j=-N'}^{N'}a_jB(\mu_j,\mu_i)I^0(\mu_j,\tau_x,\tau_z) \qquad (3.2.2.9)$$

$$A(\mu_i)\frac{\partial}{\partial\tau_x}I^0(\mu_i,\tau_x,\tau_z)+\mu\frac{\partial}{\partial\tau_z}I^1(\mu_i,\tau_x,\tau_z)$$

$$=I^1(\mu_i,\tau_x,\tau_z)-\sum_{j=-N'}^{N'}a_jC(\mu_j,\mu_i)I^1(\mu_j,\tau_x,\tau_z) \qquad (3.2.2.10)$$

(i=-N',...,N')

where μ_i and a_i are Gaussian points and weights respectively. For simplicity of solving the Eqs (3.2.2.9), (3.2.2.10) we take two stream approximation, *i.e.*, j=-1 and 1, and N'=1. Note that

$$\mu_1=-\mu_{-1}=1/\sqrt{3}$$

$$a_1=a_{-1}=1$$

After denoting

$$I_+^0=I^0(\mu_1,\tau_x,\tau_z)$$

$$I_-^0=I^0(\mu_{-1},\tau_x,\tau_z)$$

$$I_+^1=I^1(\mu_1,\tau_x,\tau_z)$$

$$I_-^1=I^1(\mu_{-1},\tau_x,\tau_z)$$

we have four equations in the form

$$A(\mu_1)\frac{\partial}{\partial\tau_x}I_+^1+\mu_1\frac{\partial}{\partial\tau_z}I_+^0=I_+^0-[B(\mu_1,\mu_1)I_+^0+B(-\mu_1,\mu_1)I_-^0] \qquad (3.2.2.11)$$

$$A(-\mu_1)\frac{\partial}{\partial\tau_x}I_-^1-\mu_1\frac{\partial}{\partial\tau_z}I_-^0=I_-^0-[B(\mu_1,-\mu_1)I_+^0+B(-\mu_1,-\mu_1)I_-^0] \qquad (3.2.2.12)$$

$$A(\mu_1)\frac{\partial}{\partial\tau_x}I_+^0+\mu_1\frac{\partial}{\partial\tau_z}I_+^1=I_+^1-[C(\mu_1,\mu_1)I_+^1+C(-\mu_1,\mu_1)I_-^1] \qquad (3.2.2.13)$$

$$A(-\mu_1)\frac{\partial}{\partial\tau_x}I_-^0-\mu_1\frac{\partial}{\partial\tau_z}I_-^1=I_-^1-[C(\mu_1,-\mu_1)I_+^1+C(-\mu_1,-\mu_1)I_-^1] \qquad (3.2.2.14)$$

In part I, $\tau_x>0$, apply Laplace transform with regard to τ_x

$$Z_{\pm}^{0,1} = Z_{\pm}^{0,1}(w,\tau_z) = \int_0^{\infty} I_{\pm}^{0,1}(\tau_x,\tau_z)e^{-w\tau_x}d\tau_x \qquad (3.2.2.15)$$

$$I_{\pm}^{0,1} = I_{\pm}^{0,1}(\tau_x,\tau_z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} Z_{\pm}^{0,1}(w,\tau_z)e^{w\tau_x}dw \qquad (3.2.2.16)$$

(integration is along imaginary axis)

to Eqs (3.2.2.11-14), where

$i^2 = -1$.

Eqs (3.2.2.11-14) becomes

$$A(\mu_1)wZ_+^1 - A(\mu_1)I_+^1(\tau_x = 0,\tau_z) + \mu_1\frac{d}{d\tau_z}Z_+^0 = Z_+^0 - [B(\mu_1,\mu_1)Z_+^0 + B(-\mu_1,\mu_1)Z_-^0]$$

$$A(-\mu_1)wZ_-^1 - A(-\mu_1)I_-^1(\tau_x = 0,\tau_z) - \mu_1\frac{d}{d\tau_z}Z_-^0 = Z_-^0 - [B(\mu_1,-\mu_1)Z_+^0 + B(-\mu_1,-\mu_1)Z_-^0]$$

$$A(\mu_1)wZ_+^0 - A(\mu_1)I_+^0(\tau_x = 0,\tau_z) + \mu_1\frac{d}{d\tau_z}Z_+^1 = Z_+^1 - [C(\mu_1,\mu_1)Z_+^1 + C(-\mu_1,\mu_1)Z_-^1]$$

$$A(-\mu_1)wZ_-^0 - A(-\mu_1)I_-^0(\tau_x = 0,\tau_z) - \mu_1\frac{d}{d\tau_z}Z_-^1 = Z_-^1 - [C(\mu_1,-\mu_1)Z_+^1 + C(-\mu_1,-\mu_1)Z_-^1]$$

(3.2.2.17-20)

Rearranging terms yields

$$\frac{d}{d\tau_z}\begin{pmatrix}\mathbf{Z}_+\\ \mathbf{Z}_-\end{pmatrix} = \frac{1}{\mu_1}\begin{pmatrix}\mathbf{a} & \mathbf{b}\\ -\mathbf{b} & -\mathbf{a}\end{pmatrix}\begin{pmatrix}\mathbf{Z}_+\\ \mathbf{Z}_-\end{pmatrix} + \frac{A}{\mu_1}\begin{pmatrix}\mathbf{J}_+\\ \mathbf{J}_-\end{pmatrix} \qquad (3.2.2.21)$$

where

$$\mathbf{Z}_+ = \begin{pmatrix}Z_+^0\\ Z_+^1\end{pmatrix}, \ \mathbf{Z}_- = \begin{pmatrix}Z_-^0\\ Z_-^1\end{pmatrix}, \ \mathbf{J}_+ = \begin{pmatrix}I_+^1(\tau_x = 0,\tau_z)\\ I_+^0(\tau_x = 0,\tau_z)\end{pmatrix}, \ \mathbf{J}_- = -\begin{pmatrix}I_-^1(\tau_x = 0,\tau_z)\\ I_-^0(\tau_x = 0,\tau_z)\end{pmatrix}$$

$$\mathbf{a} = \begin{pmatrix}1-\mathrm{B}_1 & -\mathrm{Aw}\\ -\mathrm{Aw} & 1-\mathrm{C}\end{pmatrix}, \ \mathbf{b} = \begin{pmatrix}-\mathrm{B}_2 & 0\\ 0 & -\mathrm{C}\end{pmatrix}, \text{ and}$$

$$A = A(\mu_1) = A(-\mu_1) = (1-\mu_1^2)^{1/2}/2$$

$$B_1 = B(\mu_1,\mu_1) = B(-\mu_1,-\mu_1) = \varpi_T(\varpi_0 + \varpi_1\mu_1^2)/2$$

$$B_2 = B(\mu_1,-\mu_1) = B(-\mu_1,\mu_1) = \varpi_T(\varpi_0 - \varpi_1\mu_1^2)/2$$

$$C = C(\mu_1,\mu_1) = C(\mu_1,-\mu_1) = C(-\mu_1,\mu_1) = C(-\mu_1,-\mu_1) = \frac{\varpi_T\varpi_1}{6}$$

Seeking solutions of the form

$$\mathbf{Z}_{\pm} = \mathbf{G}_{\pm}(w)e^{p(w)\tau_z} \qquad (3.2.2.22)$$

to the homogeneous Eqs

$$\frac{d}{d\tau_z}\begin{pmatrix} Z_+ \\ Z_- \end{pmatrix} = \frac{1}{\mu_1}\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}\begin{pmatrix} Z_+ \\ Z_- \end{pmatrix}, \tag{3.2.2.23}$$

we find

$$\frac{1}{\mu_1}\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}\begin{pmatrix} G_+ \\ G_- \end{pmatrix} = p\begin{pmatrix} G_+ \\ G_- \end{pmatrix}, \tag{3.2.2.24}$$

which is a standard eigenvalue problem of order 4x4 determining the eigenvalues of p and the eigenvectors $G_\pm$.

Because of the scpecial structure of the matrix in Eq. (3.2.24), the eigenvalues occur in pairs ($\pm p$), and the order of the algebraic eigenvalue problem [Eq. (3.2.24)] may be reduced to 2x2 as follows. Breaking Eq. (2.2.24) into

$$aG_+ + bG_- = \mu_1 p G_+ \tag{3.2.2.25}$$

$$bG_+ + aG_- = -\mu_1 p G_- \tag{3.2.2.26}$$

and then adding and subtracting these two eqations, we find

$$(a-b)(G_+ - G_-) = \mu_1 p(G_+ + G_-) \tag{3.2.2.27}$$

$$(a+b)(G_+ + G_-) = \mu_1 p(G_+ - G_-) \tag{3.2.2.28}$$

Combining Eqs(3.2.2.27) and (3.2.2.28), we obtain

$$(a-b)(a+b)(G_+ + G_-) = \mu_1^2 p^2 (G_+ + G_-) \tag{3.2.2.29}$$

Solving this eigenvalue problem of order 2x2 yields eigenvalues p and eigenvectors $(G_+ + G_-)$, we then use Eq. (3.2.2.28) to determine $(G_+ - G_-)$.

The general solutions to Eqs (3.2.2.21) is hence in the form

$$\begin{pmatrix} Z_+ \\ Z_- \end{pmatrix} = \sum_{j=1}^{4} D_j(w)\begin{pmatrix} G_{+j}(w) \\ G_{-j}(w) \end{pmatrix} e^{p_j(w)\tau_z} + \begin{pmatrix} Z_+^* \\ Z_-^* \end{pmatrix}$$

where

$D_j(w)$ are coefficients to be determined by boundary conditions, and $\begin{pmatrix} Z_+^* \\ Z_-^* \end{pmatrix}$ is a particular solution to Eqs (3.2.2.21). $\begin{pmatrix} Z_+^* \\ Z_-^* \end{pmatrix}$ are functions of w and depend on lateral boundary conditions. Now we proceed to determine the particular solution $\begin{pmatrix} Z_+^* \\ Z_-^* \end{pmatrix}$ from boundary conditions.

3.2.3 Boundary Conditions

We have lateral boundary conditions

$$I(\mu,\phi;\tau_x = 0, z) = \mathcal{I}(\mu,\phi;\mathfrak{T}_x = 0, z) \tag{3.2.3.1}$$

$$\frac{\partial}{\partial\tau_x} I(\mu,\phi;\tau_x, z)\Big|_{\tau_x=0} = -\frac{\partial}{\partial\mathfrak{T}_x}\mathcal{I}(\mu,\phi;\mathfrak{T}_x, z)\Big|_{\mathfrak{T}_x=0} \tag{3.2.3.2}$$

where $\mathcal{I}(\mu,\phi;\mathfrak{T}_x, z)$ is the scattered intensity in part II of the layer and $\mathfrak{T}_x = -\tau_x$ (hence in part II $\mathfrak{T}_x>0$ and Laplace transform with redard to $\mathfrak{T}_x$ applies). It is easy to verify that in part II, corresponding equations become,

RT equation about $\mathcal{I}(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z)$:

$$-\sqrt{1-\mu^2}\cos\phi\,\frac{\partial\mathcal{I}(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z)}{\partial\mathfrak{T}_x} + \mu\frac{\partial\mathcal{I}(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z)}{\partial\mathfrak{T}_z} = \mathcal{I}(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z)$$
$$-\frac{\varpi_T}{4\pi}\int_0^{2\pi}d\phi'\int_{-1}^{1}d\mu' P_T(\mu',\phi';\mu,\phi)\mathcal{I}(\mu',\phi';\mathfrak{T}_x,\mathfrak{T}_z) - \frac{\varpi_T}{4\pi}P_T(\mu_0,\phi_0;\mu,\phi)F_0 e^{-\mathfrak{T}_z/\mu_0} \tag{3.2.3.3}$$

and

$$\mathcal{I}(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z) = \tilde{\mathcal{I}}(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z) + \mathcal{I}^*(\mu,\phi;\mathfrak{T}_z) \tag{3.2.3.4}$$

where

$$\tilde{\mathcal{I}}(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z) = \mathcal{I}^0(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z) + \mathcal{I}^1(\mu,\phi;\mathfrak{T}_x,\mathfrak{T}_z)\cos m\phi \;; \tag{3.2.3.5}$$

$\mathcal{I}^*(\mu,\phi;\mathfrak{T}_z)$ is a particular solution to equation

$$\mu \frac{d\mathcal{I}^*(\mu,\phi;\tau_z)}{d\tau_z} = \mathcal{I}^*(\mu,\phi;\tau_z) - \frac{\varpi_T}{4\pi} P_T(\mu_0,\phi_0;\mu,\phi) F_0 e^{\tau_z/\mu_0}$$

$$-\frac{\varpi_T}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu' P_T(\mu',\phi';\mu,\phi)\mathcal{I}^*(\mu',\phi';\tau_z) \qquad (3.2.3.6)$$

After denoting

$$\mathcal{I}_+^0 = \mathcal{I}^0(\mu_1,\tau_x,\tau_z)$$
$$\mathcal{I}_-^0 = \mathcal{I}^0(\mu_{-1},\tau_x,\tau_z)$$
$$\mathcal{I}_+^1 = \mathcal{I}^1(\mu_1,\tau_x,\tau_z)$$
$$\mathcal{I}_-^1 = \mathcal{I}^1(\mu_{-1},\tau_x,\tau_z)$$

we obtain

$$-A(\mu_1)\frac{\partial}{\partial\tau_x}\mathcal{I}_+^0 + \mu_1\frac{\partial}{\partial\tau_z}\mathcal{I}_+^1 = \mathcal{I}_+^1 - [\mathcal{C}(\mu_1,\mu_1)\mathcal{I}_+^1 + \mathcal{C}(-\mu_1,\mu_1)\mathcal{I}_-^1] \qquad (3.2.3.7)$$

$$-A(-\mu_1)\frac{\partial}{\partial\tau_x}\mathcal{I}_-^0 - \mu_1\frac{\partial}{\partial\tau_z}\mathcal{I}_-^1 = \mathcal{I}_-^1 - [\mathcal{C}(\mu_1,-\mu_1)\mathcal{I}_+^1 + \mathcal{C}(-\mu_1,-\mu_1)\mathcal{I}_-^1] \qquad (3.2.3.8)$$

$$-A(\mu_1)\frac{\partial}{\partial\tau_x}\mathcal{I}_+^1 + \mu_1\frac{\partial}{\partial\tau_z}\mathcal{I}_+^0 = \mathcal{I}_+^0 - [\mathcal{B}(\mu_1,\mu_1)\mathcal{I}_+^0 + \mathcal{B}(-\mu_1,\mu_1)\mathcal{I}_-^0] \qquad (3.2.3.9)$$

$$-A(-\mu_1)\frac{\partial}{\partial\tau_x}\mathcal{I}_-^1 - \mu_1\frac{\partial}{\partial\tau_z}\mathcal{I}_-^0 = \mathcal{I}_-^0 - [\mathcal{B}(\mu_1,-\mu_1)\mathcal{I}_+^0 + \mathcal{B}(-\mu_1,-\mu_1)\mathcal{I}_-^0] \qquad (3.2.3.10)$$

where

$$A = A(\mu_1) = A(-\mu_1) = (1-\mu_1^2)^{1/2}/2$$

$$\mathcal{B}_1 = \mathcal{B}(\mu_1,\mu_1) = \mathcal{B}(-\mu_1,-\mu_1) = \varpi_T(\varpi_0 + \varpi_1\mu_1^2)/2$$
$$\mathcal{B}_2 = \mathcal{B}(\mu_1,-\mu_1) = \mathcal{B}(-\mu_1,\mu_1) = \varpi_T(\varpi_0 - \varpi_1\mu_1^2)/2$$

$$\mathcal{C} = \mathcal{C}(\mu_1,\mu_1) = \mathcal{C}(\mu_1,-\mu_1) = \mathcal{C}(-\mu_1,\mu_1) = \mathcal{C}(-\mu_1,-\mu_1) = \frac{\varpi_T\varpi_1}{6}$$

After Laplace transform with regard to τ_x

$$\mathcal{Z}_\pm^{0,1} = \mathcal{Z}_\pm^{0,1}(w,\tau_z) = \int_0^\infty \mathcal{I}_\pm^{0,1}(\tau_x,\tau_z)e^{-w\tau_x}d\tau_x \qquad (3.2.3.11)$$

$$\mathcal{I}_\pm^{0,1} = \mathcal{I}_\pm^{0,1}(\tau_x,\tau_z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\mathcal{Z}_\pm^{0,1}(iw,\tau_z)e^{iw\tau_x}dw \qquad (3.2.3.12)$$

(integration is along the imaginary axis)

Eqs (3.2.3.7-10) are transformed into

$$\frac{d}{d\mathcal{T}_z}\begin{pmatrix}\mathbf{Z}_+\\ \mathbf{Z}_-\end{pmatrix}=\frac{1}{\mu_1}\begin{pmatrix}\mathbf{c} & \mathbf{d}\\ -\mathbf{d} & -\mathbf{c}\end{pmatrix}\begin{pmatrix}\mathbf{Z}_+\\ \mathbf{Z}_-\end{pmatrix}-\frac{A}{\mu_1}\begin{pmatrix}\mathbf{K}_+\\ \mathbf{K}_-\end{pmatrix} \tag{3.2.3.13}$$

where

$$\mathbf{Z}_+=\begin{pmatrix}\mathcal{Z}_+^0\\ \mathcal{Z}_+^1\end{pmatrix},\ \mathbf{Z}_-=\begin{pmatrix}\mathcal{Z}_-^0\\ \mathcal{Z}_-^1\end{pmatrix},\ \mathbf{K}_+=\begin{pmatrix}\mathcal{I}_+^1(\mathcal{T}_x=0,\mathcal{T}_z)\\ \mathcal{I}_+^0(\mathcal{T}_x=0,\mathcal{T}_z)\end{pmatrix},\ \mathbf{K}_-=-\begin{pmatrix}\mathcal{I}_-^1(\mathcal{T}_x=0,\mathcal{T}_z)\\ \mathcal{I}_-^0(\mathcal{T}_x=0,\mathcal{T}_z)\end{pmatrix}$$

$$\mathbf{c}=\begin{pmatrix}1-\mathcal{B}_1 & Aw\\ Aw & 1-\mathcal{E}\end{pmatrix},\ \mathbf{d}=\begin{pmatrix}-\mathcal{B}_2 & 0\\ 0 & -\mathcal{E}\end{pmatrix}$$

As we did in part I, the general solutions to Eqs (3.2.3.13) are in the form

$$\begin{pmatrix}\mathbf{Z}_+\\ \mathbf{Z}_-\end{pmatrix}=\sum_{j=1}^{4}E_j(w)\begin{pmatrix}\mathbf{H}_{+j}(w)\\ \mathbf{H}_{-j}(w)\end{pmatrix}e^{q_j(w)\mathcal{T}_z}+\begin{pmatrix}\mathbf{Z}_+^*\\ \mathbf{Z}_-^*\end{pmatrix} \tag{3.2.3.14}$$

where

$q_j(w)$ and $\mathbf{H}_{\pm j}(w)$ are eigenvalues and eigenvectors of equations

$$\frac{1}{\mu_1}\begin{pmatrix}\mathbf{c} & \mathbf{d}\\ -\mathbf{d} & -\mathbf{c}\end{pmatrix}\begin{pmatrix}\mathbf{H}_+\\ \mathbf{H}_-\end{pmatrix}=q\begin{pmatrix}\mathbf{H}_+\\ \mathbf{H}_-\end{pmatrix}, \tag{3.2.3.15}$$

and $\begin{pmatrix}\mathbf{Z}_+^*\\ \mathbf{Z}_-^*\end{pmatrix}$ is a particular solution to Eqs (3.2.3.13).

Considering lateral boundary conditions (3.2.3.2) and Eqs (3.2.1.7),(3.2.2.1),(3.2.3.4-5), we denote

$$\mathbf{L}_+=\mathbf{L}_+(z)=\begin{pmatrix}\frac{\partial}{\partial\mathcal{T}_x}\mathcal{I}_+^1(\mathcal{T}_x,z)\big|_{\mathcal{T}_x=0}\\ \frac{\partial}{\partial\mathcal{T}_x}\mathcal{I}_+^0(\mathcal{T}_x,z)\big|_{\mathcal{T}_x=0}\end{pmatrix}=-\begin{pmatrix}\frac{\partial}{\partial\tau_x}I_+^1(\tau_x,z)\big|_{\tau_x=0}\\ \frac{\partial}{\partial\tau_x}I_+^0(\tau_x,z)\big|_{\tau_x=0}\end{pmatrix} \tag{3.2.3.16}$$

$$\mathbf{L}_-=\mathbf{L}_-(z)=\begin{pmatrix}\frac{\partial}{\partial\mathcal{T}_x}\mathcal{I}_-^1(\mathcal{T}_x,z)\big|_{\mathcal{T}_x=0}\\ \frac{\partial}{\partial\mathcal{T}_x}\mathcal{I}_-^0(\mathcal{T}_x,z)\big|_{\mathcal{T}_x=0}\end{pmatrix}=-\begin{pmatrix}\frac{\partial}{\partial\tau_x}I_-^1(\tau_x,z)\big|_{\tau_x=0}\\ \frac{\partial}{\partial\tau_x}I_-^0(\tau_x,z)\big|_{\tau_x=0}\end{pmatrix} \tag{3.2.3.17}$$

Rewriting the lateral boundary conditions (3.2.3.1) in the form

$$\tilde{I}(\mu,\phi;\tau_x=0,z)+I^*(\mu,\phi;z)=\tilde{\mathcal{I}}(\mu,\phi;\mathcal{T}_x=0,z)+\mathcal{I}^*(\mu,\phi;z) \tag{3.2.3.18}$$

we then denote

$$\mathbf{\mathcal{I}}^*_+ = \begin{pmatrix} \mathcal{I}^{*0}_+(z) \\ \mathcal{I}^{*1}_+(z) \end{pmatrix}, \ \mathbf{I}^*_+ = \begin{pmatrix} I^{*0}_+(z) \\ I^{*1}_+(z) \end{pmatrix}, \ \mathbf{\mathcal{I}}^*_- = \begin{pmatrix} \mathcal{I}^{*0}_-(z) \\ \mathcal{I}^{*1}_-(z) \end{pmatrix}, \ \mathbf{I}^*_- = \begin{pmatrix} I^{*0}_-(z) \\ I^{*1}_-(z) \end{pmatrix}$$

$$\hat{\mathbf{I}}_+ = \begin{pmatrix} I^0_+(\tau_x, z)\big|_{\tau_x=0} \\ I^1_+(\tau_x, z)\big|_{\tau_x=0} \end{pmatrix} = \begin{pmatrix} \mathcal{I}^0_+(\mathcal{\tau}_x, z)\big|_{\mathcal{\tau}_x=0} \\ \mathcal{I}^1_+(\mathcal{\tau}_x, z)\big|_{\mathcal{\tau}_x=0} \end{pmatrix} + \mathbf{\mathcal{I}}^*_+ - \mathbf{I}^*_+; \tag{3.2.3.19}$$

$$\hat{\mathbf{I}}_- = \begin{pmatrix} I^0_-(\tau_x, z)\big|_{\tau_x=0} \\ I^1_-(\tau_x, z)\big|_{\tau_x=0} \end{pmatrix} = \begin{pmatrix} \mathcal{I}^0_-(\mathcal{\tau}_x, z)\big|_{\mathcal{\tau}_x=0} \\ \mathcal{I}^1_-(\mathcal{\tau}_x, z)\big|_{\mathcal{\tau}_x=0} \end{pmatrix} + \mathbf{\mathcal{I}}^*_- - \mathbf{I}^*_- \tag{3.2.3.20}$$

where

$$I^*(\pm\mu_1, \phi; z) = I^{*0}_\pm(z) + I^{*1}_\pm(z)\cos\phi, \tag{3.2.3.21}$$

$$\mathcal{I}^*(\pm\mu_1, \phi; z) = \mathcal{I}^{*0}_\pm(z) + \mathcal{I}^{*1}_\pm(z)\cos\phi \tag{3.2.3.22}$$

Considering Eqs (3.2.2.11-14) and (3.2.3.7-10) at $\mathcal{\tau}_x = 0$, $\tau_x = 0$ and combining with Eqs (3.2.3.16-17) and (3.2.3.19-20) yields

$$\frac{d}{dz}\begin{pmatrix} \hat{\mathbf{I}}_+ \\ \hat{\mathbf{I}}_- \end{pmatrix} = \frac{\beta}{\mu_1}\begin{pmatrix} \mathbf{e} & \mathbf{f} \\ -\mathbf{f} & -\mathbf{e} \end{pmatrix}\begin{pmatrix} \hat{\mathbf{I}}_+ \\ \hat{\mathbf{I}}_- \end{pmatrix} - \frac{\beta A}{\mu_1}\begin{pmatrix} \mathbf{L}_+ \\ \mathbf{L}_- \end{pmatrix} \tag{3.2.3.23}$$

$$\frac{d}{dz}\left(\begin{pmatrix} \hat{\mathbf{I}}_+ \\ \hat{\mathbf{I}}_- \end{pmatrix} - \begin{pmatrix} \mathbf{I}^*_+ - \mathbf{\mathcal{I}}^*_+ \\ \mathbf{I}^*_- - \mathbf{\mathcal{I}}^*_- \end{pmatrix}\right) = \frac{\mathcal{\beta}}{\mu_1}\begin{pmatrix} \mathbf{g} & \mathbf{h} \\ -\mathbf{h} & -\mathbf{g} \end{pmatrix}\left(\begin{pmatrix} \hat{\mathbf{I}}_+ \\ \hat{\mathbf{I}}_- \end{pmatrix} - \begin{pmatrix} \mathbf{I}^*_+ - \mathbf{\mathcal{I}}^*_+ \\ \mathbf{I}^*_- - \mathbf{\mathcal{I}}^*_- \end{pmatrix}\right) - \frac{\mathcal{\beta}A}{\mu_1}\begin{pmatrix} \mathbf{L}_+ \\ \mathbf{L}_- \end{pmatrix} \tag{3.2.3.24}$$

where

$\beta = \sum_{k=s,a}\sum_{j=m,c,a} n_{jk}\sigma_{jk}$ is the extinction coefficient for part I,

$\mathcal{\beta} = \sum_{k=s,a}\sum_{j=m,c,a} \mathcal{n}_{jk}\mathcal{\sigma}_{jk}$ is the extinction coefficient for part II,

$$\mathbf{e} = \begin{pmatrix} B_1 - 1 & 0 \\ 0 & C - 1 \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} B_2 & 0 \\ 0 & C \end{pmatrix},$$

$$\mathbf{g} = \begin{pmatrix} \mathcal{B}_1 - 1 & 0 \\ 0 & \mathcal{C} - 1 \end{pmatrix}, \ \mathbf{h} = \begin{pmatrix} \mathcal{B}_2 & 0 \\ 0 & \mathcal{C} \end{pmatrix}.$$

Discretizing Eqs (3.2.1.5) and (3.2.3.6) yields

$$\frac{d}{dz}\begin{pmatrix} \mathbf{I}^*_+ \\ \mathbf{I}^*_- \end{pmatrix} = \frac{\beta}{\mu_1}\begin{pmatrix} \mathbf{e} & \mathbf{f} \\ -\mathbf{f} & -\mathbf{e} \end{pmatrix}\begin{pmatrix} \mathbf{I}^*_+ \\ \mathbf{I}^*_- \end{pmatrix} + \frac{\beta A}{\mu_1}\begin{pmatrix} \Omega_+ \\ \Omega_- \end{pmatrix} F_0 e^{[-\beta(z-z_n)-\tau_n]/\mu_0} \tag{3.2.3.25}$$

$$\frac{d}{dz}\begin{pmatrix} \mathbf{\mathcal{I}}^*_+ \\ \mathbf{\mathcal{I}}^*_- \end{pmatrix} = \frac{\mathcal{\beta}}{\mu_1}\begin{pmatrix} \mathbf{g} & \mathbf{h} \\ -\mathbf{h} & -\mathbf{g} \end{pmatrix}\begin{pmatrix} \mathbf{\mathcal{I}}^*_+ \\ \mathbf{\mathcal{I}}^*_- \end{pmatrix} - \frac{\mathcal{\beta}A}{\mu_1}\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} F_0 e^{[-\mathcal{\beta}(z-z_n)-\mathcal{\tau}_n]/\mu_0} \tag{3.2.3.26}$$

where

$$\Omega_+ = \begin{pmatrix} B(-\mu_0,\mu_1)/(2\pi) \\ C(\mu_0,\mu_1)/\pi \end{pmatrix}, \quad \Omega_- = \begin{pmatrix} -B(-\mu_0,-\mu_1)/(2\pi) \\ -C(\mu_0,\mu_1)/\pi \end{pmatrix},$$

$$\Psi_+ = \begin{pmatrix} \mathcal{B}(-\mu_0,\mu_1)/(2\pi) \\ \mathcal{C}(\mu_0,\mu_1)/\pi \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} -\mathcal{B}(-\mu_0,-\mu_1)/(2\pi) \\ -\mathcal{C}(\mu_0,\mu_1)/\pi \end{pmatrix}.$$

It is easily veryfied that the particular solutions $\mathbf{I}^*_\pm$, and $\mathbf{I}^*_\pm$ are in the form

$$\begin{pmatrix} \mathbf{I}^*_+ \\ \mathbf{I}^*_- \end{pmatrix} = \sum_{j=1}^{4} \eta_j \begin{pmatrix} \Lambda_{+j} \\ \Lambda_{-j} \end{pmatrix} e^{\rho_j \beta z} + \begin{pmatrix} \tilde{\Lambda}_+ \\ \tilde{\Lambda}_- \end{pmatrix} e^{-\beta z/\mu_0} \tag{3.2.3.27}$$

$$\begin{pmatrix} \mathbf{I}^*_+ \\ \mathbf{I}^*_- \end{pmatrix} = \sum_{j=1}^{4} \varsigma_j \begin{pmatrix} \Gamma_{+j} \\ \Gamma_{-j} \end{pmatrix} e^{\kappa_j \beta z} + \begin{pmatrix} \tilde{\Gamma}_+ \\ \tilde{\Gamma}_- \end{pmatrix} e^{-\beta z/\mu_0} \tag{3.2.3.28}$$

where

η_j, ς_j are constants to be determined by vertical boundary conditions at

$\tau_x \to \infty, \tau_x \to \infty$:

$$^n\mathbf{I}^*_\pm(z = z_n) = {}^{n-1}\mathbf{I}^*_\pm(z = z_n) \tag{3.2.3.29}$$

$$^n\mathbf{I}^*_\pm(z = z_n) = {}^{n-1}\mathbf{I}^*_\pm(z = z_n) \tag{3.2.3.30}$$

$$^n\mathbf{I}^*_\pm(z = z_{n+1}) = {}^{n+1}\mathbf{I}^*_\pm(z = z_{n+1}) \tag{3.2.3.31}$$

$$^n\mathbf{I}^*_\pm(z = z_{n+1}) = {}^{n+1}\mathbf{I}^*_\pm(z = z_{n+1}) \tag{3.2.3.32}$$

where super/subscription "n", "$n+1$", "$n-1$" denote layer number.

$\tilde{\Lambda}_\pm, \tilde{\Gamma}_\pm$ are constants to be determined by equations

$$-\frac{1}{\mu_0}\begin{pmatrix} \tilde{\Lambda}_+ \\ \tilde{\Lambda}_- \end{pmatrix} = \frac{1}{\mu_1}\begin{pmatrix} \mathbf{e} & \mathbf{f} \\ -\mathbf{f} & -\mathbf{e} \end{pmatrix}\begin{pmatrix} \tilde{\Lambda}_+ \\ \tilde{\Lambda}_- \end{pmatrix} + \frac{A}{\mu_1}\begin{pmatrix} \Omega_+ \\ \Omega_- \end{pmatrix} F_0 e^{[\beta z_n - \tau_{sn}]/\mu_0} \tag{3.2.3.33}$$

$$-\frac{1}{\mu_0}\begin{pmatrix} \tilde{\Gamma}_+ \\ \tilde{\Gamma}_- \end{pmatrix} = \frac{1}{\mu_1}\begin{pmatrix} \mathbf{g} & \mathbf{h} \\ -\mathbf{h} & -\mathbf{g} \end{pmatrix}\begin{pmatrix} \tilde{\Gamma}_+ \\ \tilde{\Gamma}_- \end{pmatrix} - \frac{A}{\mu_1}\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} F_0 e^{[\beta z_n - \tau_{sn}]/\mu_0} \tag{3.2.3.34}$$

ρ_j and $\begin{pmatrix} \Lambda_{+j} \\ \Lambda_{-j} \end{pmatrix}$ are eigenvalues and eigenvectors of equations

$$\rho\begin{pmatrix} \Lambda_+ \\ \Lambda_- \end{pmatrix} = \frac{1}{\mu_1}\begin{pmatrix} \mathbf{e} & \mathbf{f} \\ -\mathbf{f} & -\mathbf{e} \end{pmatrix}\begin{pmatrix} \Lambda_{+j} \\ \Lambda_{-j} \end{pmatrix}, \tag{3.2.3.35}$$

κ_j and $\begin{pmatrix}\Gamma_{+j}\\ \Gamma_{-j}\end{pmatrix}$ are eigen values and eigenvectors of equations

$$\kappa\begin{pmatrix}\Gamma_+\\ \Gamma_-\end{pmatrix}=\frac{1}{\mu_1}\begin{pmatrix}\mathbf{g} & \mathbf{h}\\ -\mathbf{h} & -\mathbf{g}\end{pmatrix}\begin{pmatrix}\Gamma_+\\ \Gamma_-\end{pmatrix} \tag{3.2.3.36}$$

Combine Eqs (3.2.3.23-28), we obtain equations about $\widehat{\mathbf{I}}_\pm$:

$$\begin{aligned}(\beta-\beta)\frac{d}{dz}\begin{pmatrix}\widehat{\mathbf{I}}_+\\ \widehat{\mathbf{I}}_-\end{pmatrix}=&\frac{\beta\beta}{\mu_1}\begin{pmatrix}\mathbf{g}-\mathbf{e} & \mathbf{h}-\mathbf{f}\\ -(\mathbf{h}-\mathbf{f}) & -(\mathbf{g}-\mathbf{e})\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{I}}_+\\ \widehat{\mathbf{I}}_-\end{pmatrix}\\ &-\beta\left(\frac{\beta}{\mu_1}\begin{pmatrix}\mathbf{e} & \mathbf{f}\\ -\mathbf{f} & -\mathbf{e}\end{pmatrix}+\frac{\beta}{\mu_1}\begin{pmatrix}\mathbf{g} & \mathbf{h}\\ -\mathbf{h} & -\mathbf{g}\end{pmatrix}\right)\left(\sum_{j=1}^{4}\eta_j\begin{pmatrix}\Lambda_{+j}\\ \Lambda_{-j}\end{pmatrix}e^{\rho_j\beta z}+\begin{pmatrix}\tilde{\Lambda}_+\\ \tilde{\Lambda}_-\end{pmatrix}e^{-\beta z/\mu_0}\right)\\ &-\frac{\beta^2 A}{\mu_1}\begin{pmatrix}\Omega_+\\ \Omega_-\end{pmatrix}F_0 e^{[-\beta(z-z_n)-\tau_{zn}]/\mu_0}-\frac{\beta\beta A}{\mu_1}\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}F_0 e^{[-\beta(z-z_n)-\tau_{zn}]/\mu_0}\end{aligned}$$

(3.2.3.37)

The solutions to Eqs (3.2.3.37) are in the form

$$\begin{pmatrix}\widehat{\mathbf{I}}_+\\ \widehat{\mathbf{I}}_-\end{pmatrix}=\sum_{j=1}^{4}N_j\begin{pmatrix}\mathbf{T}_{+j}\\ \mathbf{T}_{-j}\end{pmatrix}e^{r_j z}+\sum_{j=1}^{4}\begin{pmatrix}\mathbf{a}^1_{+j}\\ \mathbf{a}^1_{-j}\end{pmatrix}e^{\rho_j\beta z}+\begin{pmatrix}\mathbf{a}^2_+\\ \mathbf{a}^2_-\end{pmatrix}e^{-\beta z/\mu_0}+\begin{pmatrix}\mathbf{a}^3_+\\ \mathbf{a}^3_-\end{pmatrix}e^{-\beta z/\mu_0} \tag{3.2.3.38}$$

where

r_j and $\begin{pmatrix}\mathbf{T}_{+j}\\ \mathbf{T}_{-j}\end{pmatrix}$ are eigenvalues and eigenvectors of equations

$$(\beta-\beta)r\begin{pmatrix}\mathbf{T}_+\\ \mathbf{T}_-\end{pmatrix}=\frac{\beta\beta}{\mu_1}\begin{pmatrix}\mathbf{g}-\mathbf{e} & \mathbf{h}-\mathbf{f}\\ -(\mathbf{h}-\mathbf{f}) & -(\mathbf{g}-\mathbf{e})\end{pmatrix}\begin{pmatrix}\mathbf{T}_+\\ \mathbf{T}_-\end{pmatrix}; \tag{3.2.3.39}$$

N_j are constants to be determined by vertical boundary conditions:

$$^nI(\mu,\phi;\tau_x=0,\tau_z=\tau_{zn})={}^{n-1}I(\mu,\phi;\tau_x=0,\tau_z=\tau_{zn}), \tag{3.2.3.40}$$

$$^nI(\mu,\phi;\tau_x=0,\tau_z=\tau_{z(n+1)})={}^{n+1}I(\mu,\phi;\tau_x=0,\tau_z=\tau_{z(n+1)}) \tag{3.2.3.41}$$

or

$$^n\widehat{\mathbf{I}}_\pm(z=z_n)={}^{n-1}\widehat{\mathbf{I}}_\pm(z=z_n), \tag{3.2.3.42}$$

$$^n\widehat{\mathbf{I}}_\pm(z=z_{n+1})={}^{n+1}\widehat{\mathbf{I}}_\pm(z=z_{n+1}), \tag{3.2.3.43}$$

where sub/superscript "n" denotes n th layer.

$\mathbf{a}^1_{\pm j}$ are constants determined by equations

$$(\beta-\bar{\beta})\beta\rho_j\begin{pmatrix}\mathbf{a}^1_{+j}\\ \mathbf{a}^1_{-j}\end{pmatrix}=\frac{\bar{\beta}\beta}{\mu_1}\begin{pmatrix}\mathbf{g}-\mathbf{e} & \mathbf{h}-\mathbf{f}\\ -(\mathbf{h}-\mathbf{f}) & -(\mathbf{g}-\mathbf{e})\end{pmatrix}\begin{pmatrix}\mathbf{a}^1_{+j}\\ \mathbf{a}^1_{-j}\end{pmatrix}$$

$$-\beta\eta_j\left(\frac{\beta}{\mu_1}\begin{pmatrix}\mathbf{e} & \mathbf{f}\\ -\mathbf{f} & -\mathbf{e}\end{pmatrix}+\frac{\bar{\beta}}{\mu_1}\begin{pmatrix}\mathbf{g} & \mathbf{h}\\ -\mathbf{h} & -\mathbf{g}\end{pmatrix}\right)\begin{pmatrix}\Lambda_{+j}\\ \Lambda_{-j}\end{pmatrix} \qquad (3.2.3.44)$$

$\mathbf{a}^2_\pm$ are constants determined by equations

$$-(\beta-\bar{\beta})\frac{\beta}{\mu_0}\begin{pmatrix}\mathbf{a}^2_+\\ \mathbf{a}^2_-\end{pmatrix}=\frac{\bar{\beta}\beta}{\mu_1}\begin{pmatrix}\mathbf{g}-\mathbf{e} & \mathbf{h}-\mathbf{f}\\ -(\mathbf{h}-\mathbf{f}) & -(\mathbf{g}-\mathbf{e})\end{pmatrix}\begin{pmatrix}\mathbf{a}^2_+\\ \mathbf{a}^2_-\end{pmatrix}$$

$$-\beta\left(\frac{\beta}{\mu_1}\begin{pmatrix}\mathbf{e} & \mathbf{f}\\ -\mathbf{f} & -\mathbf{e}\end{pmatrix}+\frac{\bar{\beta}}{\mu_1}\begin{pmatrix}\mathbf{g} & \mathbf{h}\\ -\mathbf{h} & -\mathbf{g}\end{pmatrix}\right)\begin{pmatrix}\tilde{\Lambda}_+\\ \tilde{\Lambda}_-\end{pmatrix}-\frac{\beta^2 A}{\mu_1}\begin{pmatrix}\Omega_+\\ \Omega_-\end{pmatrix}F_0 e^{[-\beta z_n-\tau_{zn}]/\mu_0}$$

(3.2.3.44)

$\mathbf{a}^3_\pm$ are constants determined by equations

$$-(\beta-\bar{\beta})\frac{\bar{\beta}}{\mu_0}\begin{pmatrix}\mathbf{a}^3_+\\ \mathbf{a}^3_-\end{pmatrix}=\frac{\bar{\beta}\beta}{\mu_1}\begin{pmatrix}\mathbf{g}-\mathbf{e} & \mathbf{h}-\mathbf{f}\\ -(\mathbf{h}-\mathbf{f}) & -(\mathbf{g}-\mathbf{e})\end{pmatrix}\begin{pmatrix}\mathbf{a}^3_+\\ \mathbf{a}^3_-\end{pmatrix}-\frac{\bar{\beta}\beta A}{\mu_1}\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}F_0 e^{[-\beta z_n-\tau_{zn}]/\mu_0} \qquad (3.2.3.45)$$

Once $\hat{\mathbf{I}}_\pm$ are determined, which can be rewritten in the form

$$\begin{pmatrix}\hat{\mathbf{I}}_+\\ \hat{\mathbf{I}}_-\end{pmatrix}=\sum_{j=1}^{10}\begin{pmatrix}\mathbf{b}_{+j}\\ \mathbf{b}_{-j}\end{pmatrix}e^{\upsilon_j\tau_z}, \qquad (3.2.3.46)$$

we then have $\mathbf{J}_\pm,\mathbf{K}_\pm$ in the form

$$\mathbf{J}_\pm=\sum_{j=1}^{10}\mathbf{b}_{\pm j}e^{\upsilon_j\tau_z}, \qquad (3.2.3.47)$$

$$\mathbf{K}_\pm=\sum_{j=1}^{18}\mathbf{c}_{\pm j}e^{\vartheta_j\tau_z}. \qquad (3.2.3.48)$$

Now Solving Eqs (3.2.2.21) and (3.2.3.13) to get particular solutions

$$\mathbf{Z}^*_\pm=\mathbf{J}_\pm=\sum_{j=1}^{10}\mathbf{d}_{\pm j}(w)e^{\upsilon_j\tau_z}$$

$$\mathbf{Z}^*_\pm=\sum_{j=1}^{18}\mathbf{m}_{\pm j}(w)e^{s_j\tau_z}$$

Hence,

$$\mathbf{I}_{\pm} = \mathbf{I}_{\pm}(\tau_x,\tau_z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathbf{Z}_{\pm}(w,\tau_z)e^{w\tau_z}dw$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[\sum_{j=1}^{4} D_j(w)\mathbf{G}_{\pm j}(w)e^{p_j(w)\tau_z} + \sum_{j=1}^{10}\mathbf{d}_{\pm j}(w)e^{v_j\tau_z}]e^{w\tau_z}dw \qquad (3.2.3.49)$$

$$\mathbf{\mathcal{I}}_{\pm} = \mathbf{\mathcal{I}}_{\pm}(\tau_x,\mathcal{T}_z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathbf{\mathcal{Z}}_{\pm}(w,\mathcal{T}_z)e^{w\tau_z}dw$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[\sum_{j=1}^{4} E_j(w)\mathbf{H}_{\pm j}(w)e^{q_j(w)\mathcal{T}_z} + \sum_{j=1}^{18}\mathbf{m}_{\pm j}(w)e^{s_j\tau_z}]e^{w\mathcal{T}_z}dw \qquad (3.2.3.50)$$

Coefficients $D_j(w)$, $E_j(w)$ are determined by vertical boundary conditions:

$${}^{n}I(\mu,\phi;\tau_x,\tau_z=\tau_{zn}) = {}^{n-1}I(\mu,\phi;\tau_x,\tau_z=\tau_{zn}) \qquad (3.2.3.51)$$

$${}^{n}\mathcal{I}(\mu,\phi;\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{zn}) = {}^{n-1}\mathcal{I}(\mu,\phi;\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{zn}) \qquad (3.2.3.52)$$

$${}^{n}I(\mu,\phi;\tau_x,\tau_z=\tau_{z(n+1)}) = {}^{n+1}I(\mu,\phi;\tau_x,\tau_z=\tau_{z(n+1)}) \qquad (3.2.3.53)$$

$${}^{n}\mathcal{I}(\mu,\phi;\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{z(n+1)}) = {}^{n+1}\mathcal{I}(\mu,\phi;\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{z(n+1)}) \qquad (3.2.3.54)$$

or

$${}^{n}\mathbf{I}_{\pm}(\tau_x,\tau_z=\tau_{zn}) = {}^{n-1}\mathbf{I}_{\pm}(\tau_x,\tau_z=\tau_{zn}) \qquad (3.2.3.55)$$

$${}^{n}\mathbf{\mathcal{I}}_{\pm}(\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{zn}) = {}^{n-1}\mathbf{\mathcal{I}}_{\pm}(\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{zn}) \qquad (3.2.3.56)$$

$${}^{n}\mathbf{I}_{\pm}(\tau_x,\tau_z=\tau_{z(n+1)}) = {}^{n+1}\mathbf{I}_{\pm}(\tau_x,\tau_z=\tau_{z(n+1)}) \qquad (3.2.3.57)$$

$${}^{n}\mathbf{\mathcal{I}}_{\pm}(\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{z(n+1)}) = {}^{n+1}\mathbf{\mathcal{I}}_{\pm}(\mathcal{T}_x,\mathcal{T}_z=\mathcal{T}_{z(n+1)}) \qquad (3.2.3.58)$$

Finally, we have intensities

$$I(\pm\mu_1,\phi,\tau_x,\tau_z) = I_{\pm}^{0}(\tau_x,\tau_{zz}) + I_{\pm}^{1}(\tau_x,\tau_z)\cos\phi \qquad (3.2.3.59)$$

$$\mathcal{I}(\pm\mu_1,\phi,\mathcal{T}_x,\mathcal{T}_z) = \mathcal{I}_{\pm}^{0}(\mathcal{T}_x,\mathcal{T}_{zz}) + \mathcal{I}_{\pm}^{1}(\mathcal{T}_x,\mathcal{T}_z)\cos\phi \qquad (3.2.3.60)$$