NASA/TM-97-206265

Time Evolution of Modeled Reynolds Stresses in Planar Homogeneous Flows

T. Jongen
Swiss Federal Institute of Technology, Lausanne, Switzerland

T. B. Gatski
Langley Research Center, Hampton, Virginia

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

December 1997

Available from the following:

NASA Center for AeroSpace Information (CASI)
800 Elkridge Landing Road
Linthicum Heights, MD 21090-2934
(301) 621-0390

National Technical Information Service (NTIS)
5285 Port Royal Road
Springfield, VA 22161-2171
(703) 487-4650

Time Evolution of Modeled Reynolds Stresses in Planar Homogeneous Flows

T. Jongen
Fluid Mechanics Laboratory
Swiss Federal Institute of Technology, 1015 Lausanne, Switzerland

T. B. Gatski
Aerodynamic and Acoustic Methods Branch
NASA Langley Research Center, Hampton, VA 23681, USA

Abstract

The analytic expression of the time evolution of the Reynolds stress anisotropy tensor in all planar homogeneous flows is obtained by exact integration of the modeled differential Reynolds stress equations. The procedure is based on results of tensor representation theory, is applicable for general pressure-strain correlation tensors, and can account for any additional turbulence anisotropy effects included in the closure. An explicit solution of the resulting system of scalar ordinary differential equations is obtained for the case of a linear pressure-strain correlation tensor. The properties of this solution are discussed, and the dynamic behavior of the Reynolds stresses is studied, including limit cycles and sensitivity to initial anisotropies.

I. INTRODUCTION

The dynamical behavior of the Reynolds stresses in homogeneous flows is modeled by a tensor evolution equation. Previous studies have focused on the fixed points associated with the equilibrium states of the Reynolds stresses for several homogeneous flows[1–3] in order to assess the stability of higher order models and the ability of these models to reach the correct solution points. In recent studies, such fixed points have been obtained analytically for all planar homogeneous flows in both inertial and noninertial frames as asymptotic states of the evolution of the Reynolds stresses.[4] In the present paper, the time evolution of the Reynolds stress anisotropy tensor is obtained analytically for all planar homogeneous flows. The resulting explicit expression for the Reynolds stress anisotropy tensor is quite compact and can be expressed as ratios of sums of exponentials in time.

Such an analytical solution is obtained through a recasting of the tensor equation for the Reynolds stress anisotropy into an equivalent set of three scalar ordinary differential equations in three scalar invariants by using representation theory. This procedure can be applied to the Reynolds stress model equations in which the pressure strain correlation tensor is modeled in a general way, including quadratic or higher order terms, and additional anisotropy effects can be incorporated. The solution of the resulting set of ordinary differential equations is obtained for the case of a linear pressure-strain correlation tensor, with no

additional anisotropy effects included. The present explicit nonequilibrium stress solution predicts stress anisotropies that are quite close to the ones given by the modeled Reynolds stress anisotropy evolution equation over all times. The differences can be attributed to the assumption of a slow variation of the relative strain parameter that had to be made in obtaining the explicit expression of the stress anisotropies. All the dynamic features of the Reynolds stress evolution are captured by the explicit time solution, including limit cycles.

II. EVOLUTION OF REYNOLDS STRESS ANISOTROPY

Consider incompressible, homogeneous turbulent flow, where the velocity u_i and the kinematic pressure p are decomposed into the ensemble mean and fluctuating parts:

$$u_i = \bar{u}_i + u_i', \quad p = \bar{p} + p'. \tag{1}$$

In homogeneous conditions, the velocity gradients $\partial \bar{u}_i / \partial x_j$ are independent of position. These gradients are also assumed to be independent of the time. The Reynolds stress tensor $\tau_{ij} \equiv \overline{u_i' u_j'}$ is a solution of the time evolution equation

$$\dot{\tau}_{ij} = -\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k} + \Phi_{ij} - \varepsilon_{ij} - 2\Omega_m(\epsilon_{mkj}\tau_{ik} + \epsilon_{mki}\tau_{jk}), \tag{2}$$

which is valid in an arbitrary noninertial reference frame that can undergo a rotation with angular velocity Ω_m relative to an inertial frame. In (2), ϵ_{ijk} is the permutation tensor and

$$\Phi_{ij} = \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}, \qquad \varepsilon_{ij} = 2\nu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right), \tag{3}$$

are the pressure-strain correlation and the dissipation rate tensors (where ν is the kinematic viscosity), respectively.

With the turbulent kinetic energy $K \equiv \frac{1}{2}\overline{u_i' u_i'}$, the scalar turbulent dissipation rate $\varepsilon \equiv \frac{1}{2}\varepsilon_{ii}$, and the Reynolds stress anisotropy tensor

$$b_{ij} = \frac{\tau_{ij}}{2K} - \frac{1}{3}\delta_{ij}, \tag{4}$$

the term Φ_{ij} is modeled in the commonly used second-order closure models in the general form[2] as

$$\begin{aligned}\Phi_{ij} = &-\varepsilon\left(C_1^0 + C_1^1\tfrac{\mathcal{P}}{\varepsilon}\right)b_{ij} + C_2 K S_{ij} + C_3 K\left(b_{ik}S_{kj} + S_{ik}b_{kj} - \tfrac{2}{3}b_{mn}S_{nm}\delta_{ij}\right)\\ &- C_4 K(b_{ik}W_{kj} - W_{ik}b_{kj}) + C_4 K\Omega_m(b_{ik}\epsilon_{mkj} - \epsilon_{mik}b_{kj})\\ &+ C_5\varepsilon\left(b_{ik}b_{kj} - \tfrac{1}{3}b_{mn}b_{nm}\delta_{ij}\right),\end{aligned} \tag{5}$$

Above, the strain rate S_{ij} and rotation rate W_{ij} tensors are defined as

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \quad W_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right), \tag{6}$$

and $\mathcal{P} \equiv -\tau_{ij}S_{ij} = -2Kb_{ij}S_{ij}$ is the turbulence production. The coefficients C_1^0, C_1^1, and $C_2 - C_5$ can, in general, be functions of the invariants formed on b_{ij}, S_{ij}, and W_{ij}. Equation (5) can be shown to be the most general form for Φ_{ij}. For example, the pressure-strain model of Speziale, Sarkar and Gatski[2] (SSG) gives the following coefficients:

$$C_1^0 = 3.4,\ C_1^1 = 1.8,\ C_2 = 0.36,\ C_3 = 1.25,\ C_4 = 0.4,\ C_5 = 4.2. \tag{7}$$

The substitution of (5) into (2) yields the following general evolution equation for the Reynolds stress anisotropy tensor b_{ij}, written in matrix form and in nondimensionalized variables

$$\begin{aligned}\frac{d}{dt^*}\mathbf{b} = & -\frac{1}{g\eta}\mathbf{b} - a_3\left(\mathbf{b}\mathbf{S}^* + \mathbf{S}^*\mathbf{b} - \frac{2}{3}\{\mathbf{b}\mathbf{S}^*\}\mathbf{I}\right) + a_2(\mathbf{b}\mathbf{W}^* - \mathbf{W}^*\mathbf{b}) \\ & + \frac{1}{\eta}a_4\left(\mathbf{b}^2 - \frac{1}{3}\{\mathbf{b}^2\}\mathbf{I}\right) - a_1\mathbf{S}^* - \mathbf{L}^*,\end{aligned} \tag{8}$$

where

$$\mathbf{S}^* = \mathbf{S}/\sqrt{\{\mathbf{S}^2\}},\ \mathbf{W}^* = \tilde{\mathbf{W}}/\sqrt{\{\mathbf{S}^2\}},\ \mathbf{L}^* = \mathbf{L}/\sqrt{\{\mathbf{S}^2\}}, \tag{9}$$

$$\eta = \tau\sqrt{\{\mathbf{S}^2\}}, \qquad \zeta = \tau\sqrt{-\{\tilde{\mathbf{W}}^2\}}, t^* = t\sqrt{\{\mathbf{S}^2\}}. \tag{10}$$

The tensor $\tilde{W}_{ij}$ accounts for noninertial effects

$$\tilde{W}_{ij} = W_{ij} - c_w\Omega_m e_{mij}, \tag{11}$$

where $c_w = (C_4 - 4)/(C_4 - 2)$. The following definitions are used for the coefficients:

$$\frac{1}{g} = \left(\frac{1}{2}C_1^1 + 1\right)\frac{\mathcal{P}}{\varepsilon} + \frac{1}{2}C_1^0 - 1, \tag{12}$$

$$a_1 = \left(\frac{4}{3} - C_2\right)/2, \qquad a_2 = (2 - C_4)/2, \qquad a_3 = (2 - C_3)/2, \qquad a_4 = C_5/2. \tag{13}$$

The tensor $\mathbf{L}$ can generally contain the additional turbulence anisotropic effects, and the scalar coefficients a_i may generally be functions of the invariants η and ζ. In the current context, $\mathbf{L}$ is taken simply as $\mathbf{d}$ to represent the effects of the dissipation rate anisotropy, with the dissipation rate anisotropy tensor defined as

$$d_{ij} = \frac{\varepsilon_{ij}}{2\varepsilon} - \frac{1}{3}\delta_{ij}. \tag{14}$$

Equation (8) is equivalent to (2), but must be supplemented with an equation for the turbulent kinetic energy K

$$\dot{K} = \mathcal{P} - \varepsilon, \tag{15}$$

and for closure, an equation for the turbulent dissipation rate

$$\dot{\varepsilon} = C_{\varepsilon 1}\frac{\varepsilon}{K}\mathcal{P} - C_{\varepsilon 2}\frac{\varepsilon^2}{K}, \tag{16}$$

where $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are closure constants.

III. SOLUTION OF REYNOLDS STRESS EQUATION

A. Equivalent Scalar Representation

The tensor relation (8) governing the evolution of the stress anisotropy cannot be manipulated further because it involves matrix products and their transpose. Even with linearization ($a_4 = 0$), the terms that factor $\mathbf{b}$ cannot all be grouped to allow for the integration of the system of ordinary differential equations. The following technique, however, transforms the tensor relation into an equivalent system of scalar ordinary differential equations, which in turn can be solved.

With the evolution of the anisotropy tensor $\mathbf{b}$ governed by equation (8), the tensor $\mathbf{b}$ can be assumed to be dependent only on the tensors $\mathbf{S}^*$ and $\mathbf{W}^*$, as well as on scalar quantities such as t^*, η, and ζ. It can be shown[5] in this case that for two-dimensional flows the exact *representation* for the tensor $\mathbf{b}$ is given by

$$\mathbf{b} = \{\mathbf{bS}^*\}\mathbf{S}^* + \frac{\{\mathbf{bW}^*\mathbf{S}^*\}}{\{\mathbf{W}^{*2}\}}(\mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*) + 6\{\mathbf{bS}^{*2}\}\left(\mathbf{S}^{*2} - \frac{1}{3}\mathbf{I}\right). \tag{17}$$

Equation (17) thus shows that if the three scalar invariants $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$ can be determined independently of (17), then a knowledge of these scalar functions is equivalent to knowing $\mathbf{b}$. In addition, the representation (17) can be used to construct (see Appendix A) the nonlinear term in (8),

$$\begin{aligned}\mathbf{b}^2 - \frac{1}{3}\{\mathbf{b}^2\}\mathbf{I} = {} & 2\{\mathbf{bS}^*\}\{\mathbf{bS}^{*2}\}\mathbf{S}^* + 2\frac{\{\mathbf{bW}^*\mathbf{S}^*\}\{\mathbf{bS}^{*2}\}}{\{\mathbf{W}^{*2}\}}(\mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*) \\ & + \left(\{\mathbf{bS}^*\}^2 - 2\frac{\{\mathbf{bW}^*\mathbf{S}^*\}^2}{\{\mathbf{W}^{*2}\}} - 6\{\mathbf{bS}^{*2}\}^2\right)\left(\mathbf{S}^{*2} - \tfrac{1}{3}\mathbf{I}\right),\end{aligned} \tag{18}$$

which clearly shows the same tensor function representation as in (17), as well as a dependency on the same three scalar invariants. Independent of the representations shown in (17) and (18), equations for the three scalar invariants $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$ can be formed (see Appendix B) from the Reynolds stress anisotropy evolution equation given in (8). For simplicity, the following variables are introduced

$$B_1 = \{\mathbf{bS}^*\}, \qquad B_2 = \{\mathbf{bW}^*\mathbf{S}^*\}, \qquad B_3 = \{\mathbf{bS}^{*2}\}, \tag{19}$$

and the representation in (17) is rewritten as

$$\mathbf{b} = B_1\mathbf{S}^* - \frac{B_2}{\mathcal{R}^2}(\mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*) + 6B_3\left(\mathbf{S}^{*2} - \frac{1}{3}\mathbf{I}\right), \tag{20}$$

where

$$\mathcal{R}^2 = \frac{\zeta^2}{\eta^2} = -\frac{\{\mathbf{W}^{*2}\}}{\{\mathbf{S}^{*2}\}}.$$

The evolution equation (8) is, therefore, equivalent to the system of ordinary differential scalar equations in the scalar invariants $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$ that is obtained as shown in Appendix B (equation (B5)),

$$
\begin{aligned}
\dot{B}_1 &= (2\alpha B_1 - \frac{\beta}{\eta})B_1 + 2a_2 B_2 - 2a_3 B_3 + \frac{2a_4}{\eta} B_1 B_3 - a_1 - L_1, \\
\dot{B}_2 &= -a_2 \mathcal{R}^2 B_1 + (2\alpha B_1 - \frac{\beta}{\eta})B_2 + \frac{2a_4}{\eta} B_2 B_3 - L_2, \\
\dot{B}_3 &= -\tfrac{1}{3} a_3 B_1 + (2\alpha B_1 - \frac{\beta}{\eta})B_3 + \frac{a_4}{6\eta} B_1^2 + \frac{a_4}{3\eta\mathcal{R}^2} B_2^2 - \frac{a_4}{\eta} B_3^2 - L_3,
\end{aligned}
\tag{21}
$$

where the tensor $\mathbf{L}$ appears through the invariants

$$
L_1 = \{\mathbf{L}^*\mathbf{S}^*\}, \; L_2 = \{\mathbf{L}^*\mathbf{W}^*\mathbf{S}^*\}, \; L_3 = \{\mathbf{L}^*\mathbf{S}^{*2}\}. \tag{22}
$$

Equation (21) is a system of three algebraic ordinary differential equations in the three unknowns B_1, B_2, and B_3, which is quadratic even if $a_4 \neq 0$ because g depends on B_1,

$$
\frac{1}{g} = -2\alpha\eta B_1 + \beta, \tag{23}
$$

with $\alpha = C_1^1/2+1$ and $\beta = C_1^0/2-1$. Note that the degenerate case of $\eta = 0$ is not considered because either the absence of mean velocity gradients or the absence of a turbulence field would be implied. Of course, no such restrictions apply to ζ, so the case of $\zeta = 0$ is not precluded.

The dynamic system (21) is subjected to the initial conditions

$$
B_{1,0} = \{\mathbf{b}_0\mathbf{S}^*\}, \qquad B_{2,0} = \{\mathbf{b}_0\mathbf{W}^*\mathbf{S}^*\}, \qquad B_{3,0} = \{\mathbf{b}_0\mathbf{S}^{*2}\}, \tag{24}
$$

where $\mathbf{b}_0$ is a given initial anisotropy.

B. Solution of Reynolds Stress Equation

The system of scalar ordinary differential equations (21) with the representation in (20), which is equivalent to the original tensor evolution equation for the Reynolds stress anisotropy (8), has a significant advantage in that it is much more tractable and better suited for analysis than the original tensor equation. Any expression for the extra anisotropy tensor $\mathbf{L}$ can be provided which involves the stress anisotropy tensor to any degree of complexity. It then suffices to study the resulting dynamical system (21) to have a complete description of the evolution of the Reynolds stress anisotropy tensor (8). In the case of pressure-strain rate models that are only linear in the Reynolds stress anisotropy so that $a_4 = 0$ (compare (5) with $C_5 = 0$) and for which no additional anisotropies are included $\mathbf{L} = 0$, an explicit solution of the system of ordinary differential equations (21) can be obtained. The solution procedure for the resulting differential system is not straightforward, and the major steps of its derivation are given in Appendix C. The final expression for the Reynolds stress anisotropy tensor, which is the solution of the modeled evolution equation (8) with $a_4 = 0$ and $\mathbf{L} = 0$, is rather compact and involves ratios of characteristic functions Ψ_i:

$$
\begin{aligned}
B_1(t^*) &= \frac{1}{2\alpha}\left[\frac{\beta}{\eta(t^*)} - \frac{\Psi_3(t^*)}{\Psi_2(t^*)}\right], \\
B_2(t^*) &= \frac{a_2\mathcal{R}^2}{2\alpha}\left[1 - \frac{\beta}{\eta(t^*)}\frac{\Psi_1(t^*)}{\Psi_2(t^*)} - \frac{1}{\Psi_2(t^*)}\right] + \frac{1}{\Psi_2(t^*)}B_{2,0}, \\
B_3(t^*) &= \frac{a_3}{6\alpha}\left[1 - \frac{\beta}{\eta(t^*)}\frac{\Psi_1(t^*)}{\Psi_2(t^*)} - \frac{1}{\Psi_2(t^*)}\right] + \frac{1}{\Psi_2(t^*)}B_{3,0}.
\end{aligned} \tag{25}
$$

The characteristic functions Ψ_i are the fundamental solutions of a quadratic nonlinear system of two ordinary differential equations (see Appendix C) and are related by $\Psi_2 = \dot{\Psi}_1$ and $\Psi_3 = \dot{\Psi}_2$,

$$
\begin{aligned}
\Psi_1(t^*) &= K\left[\textstyle\sum_{r=1}^{3}\mu_r\frac{1}{\lambda_r}e^{\lambda_r t^*} + \nu(H + H^0)\right], \\
\Psi_2(t^*) &= K\textstyle\sum_{r=1}^{3}\mu_r e^{\lambda_r t^*}, \\
\Psi_3(t^*) &= K\textstyle\sum_{r=1}^{3}\mu_r\lambda_r e^{\lambda_r t^*},
\end{aligned} \tag{26}
$$

where

$$H = \frac{2}{3}a_3^2 - 2a_2^2\mathcal{R}^2, \tag{27}$$

$$H^0 = 4\alpha(a_2 B_{2,0} - a_3 B_{3,0}), \tag{28}$$

$$\mu_r = [\lambda_r^2 - 2\alpha\lambda_r B_{1,0} - (H + H^0)](\lambda_p - \lambda_q), \tag{29}$$

$$\nu = (\lambda_3 - \lambda_2)/\lambda_1 + (\lambda_1 - \lambda_3)/\lambda_2 + (\lambda_2 - \lambda_1)/\lambda_3, \tag{30}$$

and $K = [(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)]^{-1}$. In (29), the indices p and q are such that $\epsilon_{rpq} = -1$. Finally, the parameters λ_r are the eigenvalues that are obtained as roots of the following third-order characteristic polynomial equation:

$$\lambda^3 - \frac{\beta}{\eta}\lambda^2 - (H + 2\alpha a_1)\,\lambda + \frac{\beta}{\eta}H = 0. \tag{31}$$

In (25), the relative strain parameter depends on the time $\eta = \eta(t^*)$, and its evolution is governed by an additional equation. However, in the derivation of the explicit solution of the system of ordinary differential equations (21), the relative strain parameter η was assumed not to vary in time, $\dot{\eta} \sim 0$. (See Appendix C.)

IV. ANALYSIS OF DYNAMICAL BEHAVIOR

A. Transient Behavior

Equations (20) and (25) completely determine the solution of the modeled evolution equation for the Reynolds stress anisotropy tensor for all planar homogeneous turbulent

flows. Any initial stress anisotropy can be taken into account. The scalars B_i of the expansion in (20) involve ratios of the characteristic functions Ψ_i, which are expressed as the sum of three exponential functions. Because the arguments of the exponentials are the same for all characteristic functions and are given as the roots of the characteristic polynomial (31), the dynamical behavior of the stress anisotropies will essentially be determined by the location of these roots in the complex plane. If Δ_1 and Δ_2 are defined as

$$\Delta_1 = -\frac{1}{3}\left(H + 2\alpha a_1 + \frac{1}{3}\frac{\beta^2}{\eta^2}\right), \quad \Delta_2 = -\frac{1}{3}\frac{\beta}{\eta}\left(H - \alpha a_1 - \frac{1}{9}\frac{\beta^2}{\eta^2}\right), \tag{32}$$

then the discriminant of the third-degree polynomial equation (31) is given by

$$\Delta = \Delta_1^3 + \Delta_2^2. \tag{33}$$

The discriminant depends on the parameters H and η only and can be rewritten as

$$\Delta = -\frac{1}{27}\left[H\frac{\beta^4}{\eta^4} + (\alpha^2 a_1^2 + 10\alpha a_1 H - 2H^2)\frac{\beta^2}{\eta^2} + (H + 2\alpha a_1)^3\right]. \tag{34}$$

Because H is a function of $\mathcal{R}$ (see equation (27)), the value of the discriminant will be essentially determined by $\mathcal{R}$ and η. Figure 1 shows the evolution of the discriminant Δ as a function of η, for different values of the parameter $\mathcal{R}$. Three cases must be distinguished:

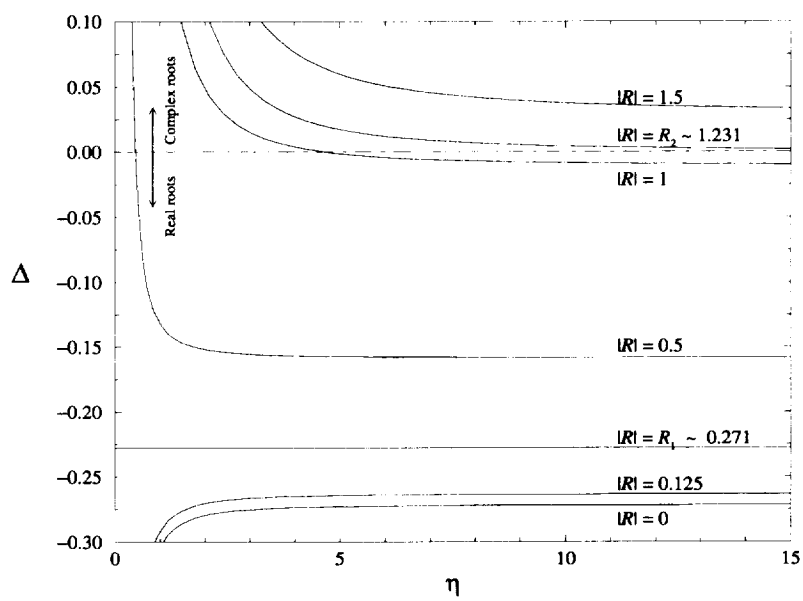

FIG. 1. Evolution of discriminant of cubic root equation as function of η for different values of parameter $\mathcal{R}$, as labeled.

(a) $\Delta < 0$

The three roots of the characteristic polynomial are real, and the characteristic functions Ψ_i are combinations of real exponentials. Because a property of the roots of a third-degree polynomial is that $\lambda_1 + \lambda_2 + \lambda_3 = \beta/\eta$ is always positive, at least one root is positive.

(b) $\Delta > 0$

Two roots are complex conjugates, for example, $\lambda_2 = \bar{\lambda}_3 = d + i\omega$ and $\lambda_1 = \lambda$. The characteristic functions can then be expressed as

$$\begin{aligned}
\Psi_1(t^*) &= K'\left[e^{\lambda t^*} + e^{dt^*}(f_{11}\cos\omega t^* + f_{12}\sin\omega t^*) + f_{13}\right], \\
\Psi_2(t^*) &= K'\left[\lambda e^{\lambda t^*} + e^{dt^*}(f_{21}\cos\omega t^* + f_{22}\sin\omega t^*)\right], \\
\Psi_3(t^*) &= K'\left[\lambda^2 e^{\lambda t^*} + e^{dt^*}(f_{31}\cos\omega t^* + f_{32}\sin\omega t^*)\right],
\end{aligned} \tag{35}$$

where K' and f_{ij} are constants. Because a property of the roots of a third-degree polynomial is that $\lambda(d^2 + \omega^2) = -H\beta/\eta$ and $\lambda + 2d = \beta/\eta$ always, λ will be positive when $H < 0$. When $H > 0$, $\lambda < 0$ and, thus, $2d = \beta/\eta - \lambda > 0$. Therefore, in this case also, one root will have a positive real part.

(c) $\Delta = 0$

The roots are all real, and two of them are equal, for example, $\lambda_2 = \lambda_3 = \lambda$. In this case, $\mu_1 = 0$, and the third root λ_1 has no effect on the characteristic functions. Because $2\lambda + \lambda_1 = \beta/\eta > 0$, at least one root will be positive.

In summary, at least one root will always have a positive real part; therefore, the characteristic functions Ψ_i will always grow exponentially. In the case of $\Delta > 0$, the characteristic functions grow with superimposed (damped) oscillations of period $T = 2\pi/\omega$. Three distinct cases can be identified on Figure 1. For mean flow fields such that $H > 0$, that is,

$$|\mathcal{R}| < \mathcal{R}_1,$$

with

$$\mathcal{R}_1 = \frac{a_3}{\sqrt{3}a_2}$$

(e.g., $|\mathcal{R}| < 0.271$ for the SSG pressure-strain correlation coefficients), the discriminant Δ is always negative, and the three roots will always be real. For values of H such that $-2\alpha a_1 < H < 0$, that is,

$$\mathcal{R}_1 < |\mathcal{R}| < \mathcal{R}_2,$$

(e.g., $0.271 < |\mathcal{R}| < 1.231$ for the SSG), where

$$\mathcal{R}_2 = \frac{1}{a_2}\sqrt{\frac{1}{3}a_3^2 + \alpha a_1},$$

the roots will have a different nature depending on the magnitude of η, and the evolution of the stress anisotropy will contain an oscillatory component for sufficiently small values of η. Finally, for values $H < -2\alpha a_1$, that is,

$$|\mathcal{R}| > \mathcal{R}_2,$$

the discriminant Δ is always positive, and the evolution of the stress anisotropy components will contain damped oscillations.

The case of a vanishing root is of particular interest. Because $\lambda_1\lambda_2\lambda_3 = -H\beta/\eta$ is always verified, one of the roots is zero (e.g., λ_2) if either $H = 0$ or $1/\eta = 0$. In the case $H = 0$ (and, thus, $\mathcal{R} = \mathcal{R}_1$), $\Delta < 0$ always, and the nonzero roots are real and given by $\lambda_{1,3} = \beta/(2\eta) \pm \sqrt{[\beta/(2\eta)]^2 + 2\alpha a_1}$. In the case $1/\eta = 0$, $\Delta = -(H + 2\alpha a_1)^3/27$. When $H > -2\alpha a_1$ (i.e., $\mathcal{R} < \mathcal{R}_2$), the roots are all real and are given by the same relations as for the case $H = 0$. When $H < -2\alpha a_1$ (i.e., $\mathcal{R} > \mathcal{R}_2$), $\Delta > 0$, and the roots are purely imaginary. The characteristic functions Ψ_2 and Ψ_3 have a purely oscillatory behavior; while Ψ_1 is increasing as $\int \Psi_2$. The period of the oscillation is $T = 2\pi/\omega$, with $\omega^2 = -(H + 2\alpha a_1)$.

B. Asymptotic States

It has been shown in the previous section that the characteristic functions Ψ_i are always increasing, except when $1/\eta = 0$. For $1/\eta \neq 0$, when the effect of the initial conditions has vanished, the exponential that corresponds to the root with the largest real part becomes dominant, and the ratios of the characteristic functions converge to the values

$$\begin{aligned} \left[\frac{\Psi_3(t^*)}{\Psi_2(t^*)}\right]_\infty &= \lim_{t^*\to\infty} \frac{\Psi_3(t^*)}{\Psi_2(t^*)} = \lambda_\infty, \\ \left[\frac{\Psi_1(t^*)}{\Psi_2(t^*)}\right]_\infty &= \lim_{t^*\to\infty} \frac{\Psi_1(t^*)}{\Psi_2(t^*)} = \frac{1}{\lambda_\infty}, \end{aligned} \tag{36}$$

where

$$\lambda_\infty = \max_{r=1,2,3} Re(\lambda_r).$$

The asymptotic values of the coefficients B_i are given by

$$\begin{aligned} B_1^\infty &= \frac{1}{2\alpha}\left(\frac{\beta}{\eta_\infty} - \lambda_\infty\right), \\ B_2^\infty &= \frac{a_2\mathcal{R}^2}{2\alpha}\left(1 - \frac{1}{\lambda_\infty}\frac{\beta}{\eta_\infty}\right), \\ B_3^\infty &= \frac{a_3}{6\alpha}\left(1 - \frac{1}{\lambda_\infty}\frac{\beta}{\eta_\infty}\right), \end{aligned} \tag{37}$$

where η_∞ is the equilibrium value achieved by the relative strain parameter. It can be shown that for any planar homogeneous flow described by the Reynolds stress model equation (2), a unique relationship holds between the equilibrium value for the production-to-dissipation ratio $(\frac{\mathcal{P}}{\varepsilon})_\infty$, the equilibrium relative strain parameter η_∞, and the rotation rate $\mathcal{R}$:

$$\begin{cases} \dfrac{1}{\eta_\infty^2} = 2g_\infty a_1 \left(\dfrac{\mathcal{P}}{\varepsilon}\right)_\infty^{-1} + g_\infty^2 H, & \text{for } -\mathcal{R}_{\lim} < \mathcal{R} < \mathcal{R}_{\lim}, \\ \dfrac{1}{\eta_\infty^2} = 0. & \text{otherwise} \end{cases} \tag{38}$$

where $g_\infty = (\alpha(\frac{\mathcal{P}}{\varepsilon})_\infty + \beta)^{-1}$ and

$$\mathcal{R}_{\text{lim}} = \frac{1}{a_2}\sqrt{\frac{1}{3}a_3^2 + \alpha a_1 + \beta a_1 \left(\frac{\mathcal{P}}{\varepsilon}\right)_\infty^{-1}}. \tag{39}$$

The equilibrium value of the production-to-dissipation ratio $(\frac{\mathcal{P}}{\varepsilon})_\infty$ is determined by the K and ϵ evolution equations. For the standard approach, where equations (15) and (16) are used, this value is given by

$$\left(\frac{\mathcal{P}}{\varepsilon}\right)_\infty = \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1}. \tag{40}$$

For values of the parameter $\mathcal{R}$ outside the range $[-\mathcal{R}_{\text{lim}}, \mathcal{R}_{\text{lim}}]$, the asymptotic value of the relative strain parameter is $1/\eta_\infty = 0$, and the representation coefficients given by (37) are

$$B_1^\infty = -\lambda_\infty/(2\alpha), \qquad B_2^\infty = a_2\mathcal{R}^2/(2\alpha), \qquad B_3^\infty = a_3/(6\alpha). \tag{41}$$

However, as shown before, for values of $1/\eta = 0$, the solution reaches a limit cycle for the anisotropy, and no asymptotic state exists. The solution (41) is, therefore, spurious because the real behavior of the anisotropy is purely oscillatory in time.

In a previous study,[6] an expression equivalent to (37) was obtained from a direct analysis of the asymptotic state of (8). Written in the present formalism, the asymptotic value for the representation coefficient B_1^∞ was obtained from the roots of a cubic polynomial in B_1^∞,

$$4\alpha^2(B_1^\infty)^3 - 4\alpha\frac{\beta}{\eta}(B_1^\infty)^2 + \left(\frac{\beta^2}{\eta^2} - H - 2\alpha a_1\right)B_1^\infty + \frac{\beta}{\eta}a_1 = 0, \tag{42}$$

which led to the problem of choosing one of the three roots so that the value of B_1^∞ was retained. This question could not be rigorously answered, and the selection of the proper root was done on the basis of continuity arguments.[6] With the present dynamic approach of the Reynolds stress equation, the proper choice for the roots in (42) is obvious and is based on the limit of a dynamical process. Clearly, the correct root is the one that controls the asymptotic behavior of the system (i.e., λ_∞). In terms of B_1^∞, B_1^∞ must be taken as the root in (42) that has *the lowest real part.*

In general, planar homogeneous flows can be described by the expression[7]

$$\frac{\partial \bar{u}_i}{\partial x_j} = \frac{1}{2}\left[(D+\omega)\delta_{i1}\delta_{j2} + (D-\omega)\delta_{i2}\delta_{j1}\right], \tag{43}$$

which yields

$$\eta^2 = \frac{1}{2}(D\tau)^2, \qquad \zeta^2 = \frac{1}{2}\left[(\omega - 2c_\omega\Omega)\tau\right]^2, \tag{44}$$

where $D/2$ is the strain rate and $\omega/2$ is the rotation rate of the flow. As shown in Table I, a wide class of homogeneous flows, both with and without system rotation, can be described in terms of $\mathcal{R}$,

TABLE I. Characterization of Common Homogeneous Turbulent Flows.

Flow	$\lvert\omega/D\rvert$	$\lvert\Omega/D\rvert$	$\lvert\mathcal{R}\rvert$	H
Plane shear	1	0	1	-1.19
Plane strain	0	0	0	0.09
Hyperbolic[a]	< 1	0	< 1	> -1.19
Elliptic[a]	> 1	0	> 1	< -1.19
Rotating plane shear	1	0.25, 0.50	0.125, 1.25[b]	0.07, -1.91

[a]See Leuchter and Benoit[7] for a description of this class of flows.

[b]These values are dependent on the pressure-strain rate model used (SSG model in this case).

$$\mathcal{R}^2 = \left(\frac{\omega}{D} - 2c_\omega \frac{\Omega}{D}\right)^2 . \tag{45}$$

As discussed above, the value $\Delta = 0$ divides the plane $(\mathcal{R}, \eta)$ into two regions in which the Reynolds stress components have distinctly different behaviors in time. These regions are illustrated in Figure 2; the solid lines are determined by the locus of points $(\mathcal{R}, \eta)$ such that $\Delta = 0$. For realizability ($\eta > 0$), only the positive root of $1/\eta$ is plotted. Figure 2

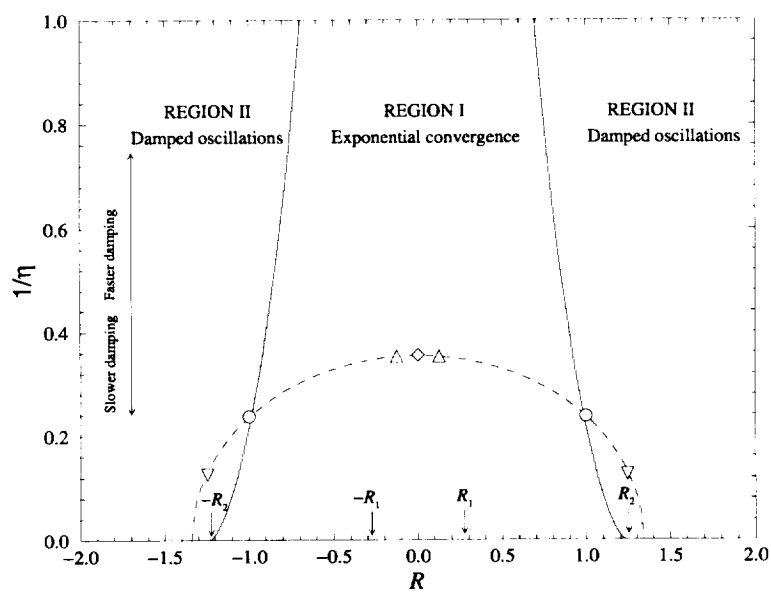

FIG. 2. Map of time-evolution types for Reynolds stress tensor in $(\mathcal{R}, 1/\eta)$ plane. Boundary $\Delta = 0$, ——; locus of equilibrium points $1/\eta_\infty$ as function of $\mathcal{R}$, – – –; plane shear, ○; plane strain, ◇; rotating plane shear $\left|\frac{\Omega}{D}\right| = 0.25$ and 0.50, △ and ▽, respectively.

also shows the locus of asymptotic solutions $1/\eta_\infty$ as a function of $\mathcal{R}$, as defined by (38). (Note the dashed line in Fig. 2.) The symbols correspond to several planar homogeneous flows. (See Table I.) When the standard equation (16) for the dissipation rate ε is used, the evolution equation for the relative strain parameter η

$$\frac{d}{dt^*}\eta = 2\eta B_1(C_{\varepsilon 1} - 1) + (C_{\varepsilon 2} - 1) \tag{46}$$

is solved in conjunction with the evolution of the coefficients B_i. Because for a given planar homogeneous flow the value of the parameter $\mathcal{R}$ is fixed, the system will evolve along vertical lines in Figure 2. For values of $(\mathcal{R}, \eta)$ situated in region I of Figure 2, the roots of the characteristic polynomial are real, and the Reynolds stress components converge to the asymptotic solution as ratios of real exponentials. In terms of dynamical systems, the asymptotic solution is a sink. For example, planar strain flows and rotating shear flows with $\Omega/D = 0.25$ will always have an evolution that is characterized by growing exponentials, for any initial condition on the anisotropy $\mathbf{b}_0$ or on the relative strain parameter η_0. Points in region II have a time evolution with a damped oscillatory character, and the asymptotic state is a spiral sink. The rate of damping of the oscillations is proportional to $1/\eta$, with no damping at all when $1/\eta = 0$. For example, a shear flow with high rotation ($\Omega/D = 0.5$) is such that $\mathcal{R} > \mathcal{R}_2$, and the stress components will evolve to their asymptotic value with damped oscillations. Note that for the homogeneous shear case ($\mathcal{R}_1 < \mathcal{R} < \mathcal{R}_2$), the two types of evolution can be experienced depending on the value of η. As already mentioned, for values $\mathcal{R} > \mathcal{R}_{\text{lim}}$, the asymptotic solution for the relative strain parameter is $1/\eta_\infty = 0$, and the solution is purely oscillatory, i.e., a limit cycle is reached.

V. ILLUSTRATIONS

First, consider a sheared flow ($\mathcal{R} = 1$) in which the turbulent field is subjected to the following initial conditions:

$$\eta_0 = 3.38, \quad b_{11,0} = b_{12,0} = b_{22,0} = 0.$$

Figure 3 shows the evolution in time of the stress anisotropies predicted by the differential equation (8) and by the present explicit time solution. Clearly, the anisotropies given by the present explicit solution are almost indistinguishable from those given by the differential equation; the difference is attributed to the assumption that $d\eta/dt^* \sim 0$, which is used in deriving the explicit solution. (See Appendix C.) From the standpoint of a dynamical system, it is more interesting to consider the evolution of the system variables η and b_{ij} in the phase plane, as shown in Figure 4.

In the case of an initial anisotropy, for instance,

$$\eta_0 = 3.38, \quad b_{11,0} = -0.1, \; b_{12,0} = 0.2, \; b_{22,0} = 0.2,$$

the present explicit nonequilibrium solution leads to stress evolutions that are almost indistinguishable from those obtained with the differential Reynolds stress equation, as shown in Figure 5. Moreover, the explicit nonequilibrium solution is remarkably close to the differential stress equation over a wide range of initial values η_0 for the relative strain parameter, as illustrated in Figure 6, which shows the initial value of η varying from 1 to 100, with isotropic initial conditions ($b_{ij,0} = 0$).

For values of the parameter $\mathcal{R}$ outside the range $[-\mathcal{R}_{\text{lim}}, \mathcal{R}_{\text{lim}}]$, the asymptotic value for $1/\eta$ is shown to be 0 (see equation (38)), and the solution reaches a limit cycle for the

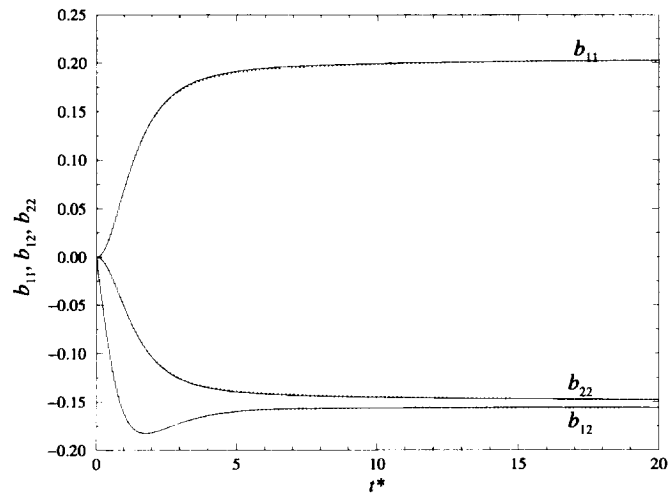

FIG. 3. Time evolution of stress anisotropies for homogeneous shear case. Initial conditions are $\eta_0 = 3.38; b_{11,0} = b_{12,0} = b_{22,0} = 0$. Present nonequilibrium solution, ——; differential Reynolds stress equation, · · · ·.

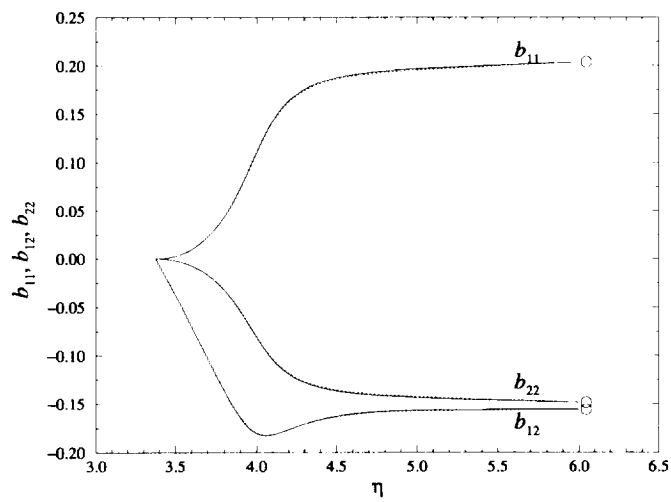

FIG. 4. Phase plane evolution of stress anisotropies for homogeneous shear case. Initial conditions are $\eta_0 = 3.38; b_{11,0} = b_{12,0} = b_{22,0} = 0$. Present nonequilibrium solution, ——; differential Reynolds stress equation, · · · ·; asymptotic solution, ∘.

anisotropy. Figure 7 shows the time evolution of the stress anisotropy components in the case of a rotation-dominated flow for which $\omega/D = 2$ (and, thus, $\mathcal{R} = 2$), which is well outside the range $[-\mathcal{R}_{\text{lim}}, \mathcal{R}_{\text{lim}}]$. The discriminant Δ is, therefore, always positive. (See Figure 2.) The initial stress field is taken to be isotropic, and the initial value of the relative strain parameter is arbitrarily set to a high value ($\eta_0 = 100$) in order to show clearly the characteristic oscillation of the dynamic system. From its initial bounded value, the relative strain parameter η grows unboundedly (so that $1/\eta \to 0$) with superimposed oscillations, as illustrated in Figure 8.

In Figures 7 and 8, clearly the present nonequilibrium explicit solution is extremely accurate in capturing the initial phase of the evolution of the anisotropy. The period of the oscillations is also captured well by the present nonequilibrium solution. When $1/\eta = 0$, the

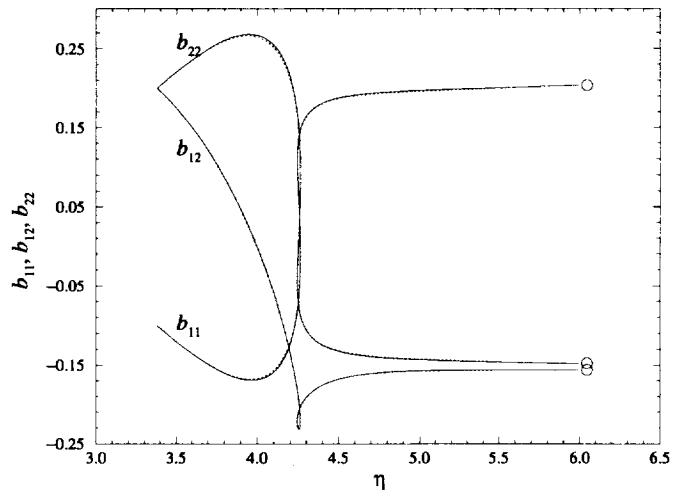

FIG. 5. Phase plane evolution of stress anisotropies for homogeneous shear case. Initial conditions are $\eta_0 = 3.38; b_{11,0} = -0.1$; $b_{12,0} = 0.2$; $b_{22,0} = 0.2$. Present nonequilibrium solution, ——; differential Reynolds stress equation, · · · ·; asymptotic solution, ∘.

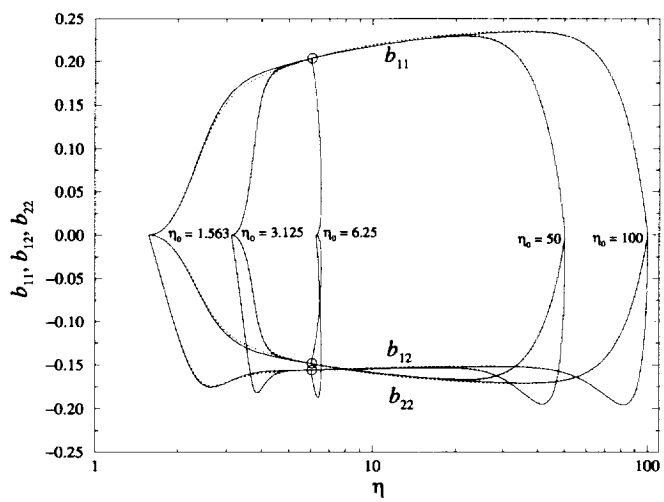

FIG. 6. Phase plane evolution of stress anisotropies for homogeneous shear case. Initial conditions are $b_{11,0} = b_{12,0} = b_{22,0} = 0$, and different values for η_0 are used, as labeled. Present nonequilibrium solution, ——; differential Reynolds stress equation, · · · ·; asymptotic solution, ∘.

solution is purely oscillatory, as discussed before. Because η starts from a bounded value, the oscillations start with a damping component, and their amplitude first decreases in time. Although the frequency of the oscillations is captured well, the amplitude clearly is not correctly represented by the present nonequilibrium solution for larger times. This finding is attributed to the hypothesis $d\eta/dt^* \sim 0$ that is used in the solution procedure. As the initial condition η_0 takes lower values, the initial damping of the oscillations is stronger, so that as the limit cycle is approached all oscillations may be nearly killed for the differential stress evolution; whereas the oscillations for the present explicit solution have not been damped fast enough, as illustrated in Figure 9, where the initial η value is set to a low value ($\eta_0 = 2$). As η grows in time, the oscillations of the nonequilibrium solution are not damped at the correct rate, and the remaining long-term amplitude of the oscillations is not correct.

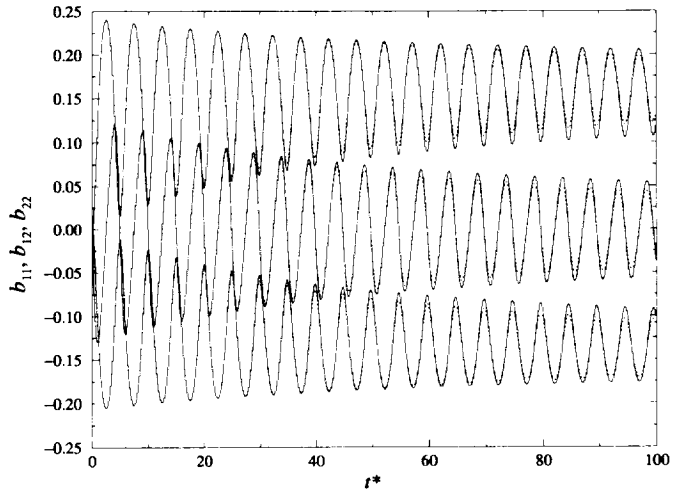

FIG. 7. Time evolution of stress anisotropies for rotation-dominated flow ($\omega/D = 2$). The initial conditions are $\eta_0 = 100$; $b_{11,0} = b_{12,0} = b_{22,0} = 0$. Present nonequilibrium solution, ——; differential Reynolds stress equation, · · · ·.

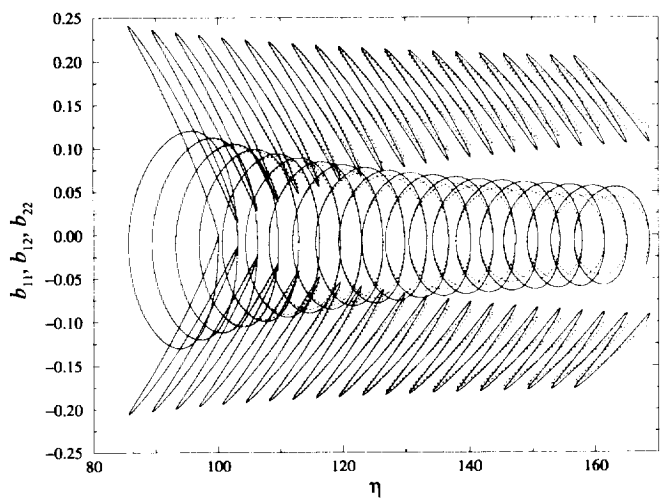

FIG. 8. Phase plane evolution of stress anisotropies for rotation-dominated flow ($\omega/D = 2$). Initial conditions are $\eta_0 = 100$; $b_{11,0} = b_{12,0} = b_{22,0} = 0$. Present nonequilibrium solution, ——; differential Reynolds stress equation, · · · ·.

VI. CONCLUSIONS

A general procedure has been developed that allows for the investigation of the time evolution of the Reynolds stress anisotropy components in all planar homogeneous turbulent flows. The procedure takes the evolution equation for the Reynolds stress anisotropy tensor and replaces it with an equivalent system of scalar ordinary differential equations. This equivalent system can then be used for assessing the dynamical behavior of a variety of turbulence closure models. This includes pressure-strain rate models which are quadratic (or higher) in the anisotropy tensor and in which other anisotropic effects, such as dissipation rate anisotropy, can be taken into account. For the case of linear pressure-strain rate models, the system of ordinary differential equations can be analytically integrated when the relative

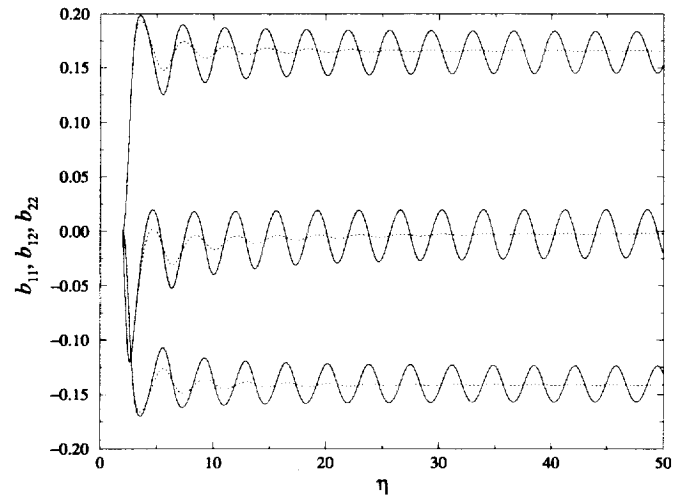

FIG. 9. Phase plane evolution of stress anisotropies for rotation-dominated flow ($\omega/D = 2$). Initial conditions are $\eta_0 = 2$; $b_{11,0} = b_{12,0} = b_{22,0} = 0$. Present nonequilibrium solution, ——; differential Reynolds stress equation, - - - -.

strain parameter is assumed to vary slowly, and an explicit expression can be found for the time evolution of the anisotropy of the Reynolds stress tensor in all planar homogeneous flow. The present nonequilibrium solution is extremely effective at capturing the initial behavior of the modeled Reynolds stress evolution, as well as the equilibrium states. In most cases, the present explicit nonequilibrium solution predicts stress anisotropies that are quite close to those given by the modeled differential Reynolds stress anisotropy evolution equation for all times; the small differences are attributed to the assumption of slow variation of the relative strain parameter used in obtaining the explicit expression of the time evolution of the modeled stress anisotropies. It has also been shown that the present nonequilibrium solution is able to predict all the dynamic features of the Reynolds stress evolution, including the oscillatory nature of the stress anisotropy for elliptic flows.

ACKNOWLEDGMENTS

The first author acknowledges the financial support from the Commission Suisse pour l'Encouragement de la Recherche Scientifique, and the support of Sulzer Brothers S.A. under contract number 3062.1.

APPENDIX A: REPRESENTATION OF $\mathbf{b}^2 - \frac{1}{3}\{\mathbf{b}^2\}\mathbf{I}$

Consider a symmetric, traceless tensor $\mathbf{b}$ for which the elements in any rectangular coordinate system are functions of the elements of two independent traceless tensors $\mathbf{S}^*$ (symmetric) and $\mathbf{W}^*$ (antisymmetric) in the same coordinate system, which is written as

$$b_{ij} = b_{ij}(S^*_{kl}, W^*_{kl}).$$

The forms of these functional relationships also must be independent of the particular coordinate system in which they are expressed; that is, the relation between $\mathbf{b}$, $\mathbf{S}^*$, and $\mathbf{W}^*$ is isotropic.[8]

For two-dimensional mean flows, $\mathbf{S}^*$ has one vanishing eigenvalue, and in the principal coordinate system of $\mathbf{S}^*$, the vorticity vector is aligned with the eigenvector of $\mathbf{S}^*$ that corresponds to the vanishing eigenvalue. If the tensor $\mathbf{b}^*$ is also assumed to have one eigenvector aligned with the eigenvector of $\mathbf{S}^*$ that corresponds to the vanishing eigenvalue, then the tensor $\mathbf{b}$ can be represented in terms of the tensors $\mathbf{S}^*$ and $\mathbf{W}^*$ and the scalar invariants $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$, as[5]

$$\mathbf{b} = \{\mathbf{bS}^*\}\mathbf{S}^* + \frac{\{\mathbf{bW}^*\mathbf{S}^*\}}{\{\mathbf{W}^{*2}\}}(\mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*) + 6\{\mathbf{bS}^{*2}\}(\mathbf{S}^{*2} - \frac{1}{3}\mathbf{I}). \tag{A1}$$

The quadratic term $\mathbf{b}^2 - \frac{1}{3}\{\mathbf{b}^2\}\mathbf{I}$ can also be represented in terms of the tensors $\mathbf{S}^*$ and $\mathbf{W}^*$ and the scalar invariants $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$.

If in expression (A1) the symmetric, traceless tensor $\mathbf{b}$ is replaced by $\mathbf{b}^2 - \frac{1}{3}\{\mathbf{b}^2\}\mathbf{I}$, then the following equation is obtained:

$$\begin{aligned}\mathbf{b}^2 - \frac{1}{3}\{\mathbf{b}^2\}\mathbf{I} = \{\mathbf{b}^2\mathbf{S}^*\}\mathbf{S}^* + \frac{\{\mathbf{b}^2\mathbf{W}^*\mathbf{S}^*\}}{\{\mathbf{W}^{*2}\}}(\mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*) \\ + 6(\{\mathbf{b}^2\mathbf{S}^{*2}\} - \frac{1}{3}\{\mathbf{b}^2\})(\mathbf{S}^{*2} - \frac{1}{3}\mathbf{I}).\end{aligned} \tag{A2}$$

Now, the scalar invariants $\{\mathbf{b}^2\mathbf{S}^*\}$, $\{\mathbf{b}^2\mathbf{W}^*\mathbf{S}^*\}$, and $\{\mathbf{b}^2\mathbf{S}^{*2}\}$ in (A2) must be expressed in terms of the scalar invariants $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$.

For conciseness, relation (A1) can be rewritten as

$$\mathbf{b} = \sum_{i=1}^{3} a_i \mathbf{T}_i, \tag{A3}$$

where the scalar coefficients a_i are

$$a_1 = \{\mathbf{bS}^*\}, \qquad a_2 = \{\mathbf{bW}^*\mathbf{S}^*\}/\{\mathbf{W}^{*2}\}, \qquad a_3 = 6\{\mathbf{bS}^{*2}\},$$

and the tensors $\mathbf{T}_i$ are given by

$$\mathbf{T}_1 = \mathbf{S}^*, \qquad \mathbf{T}_2 = \mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*, \qquad \mathbf{T}_3 = \mathbf{S}^{*2} - \frac{1}{3}\mathbf{I}.$$

Therefore,

$$\{\mathbf{b}^2\mathbf{S}^*\} = \{\mathbf{b}^2\mathbf{T}_1\}, \qquad \{\mathbf{b}^2\mathbf{W}^*\mathbf{S}^*\} = -\frac{1}{2}\{\mathbf{b}^2\mathbf{T}_2\}, \qquad \{\mathbf{b}^2\mathbf{S}^{*2}\} - \frac{1}{3}\{\mathbf{b}^2\} = \{\mathbf{b}^2\mathbf{T}_3\},$$

and if (A3) is inserted into the above expressions, then

$$\{\mathbf{b}^2\mathbf{T}_i\} = \sum_{j=1}^{3}\sum_{k=1}^{3} a_j a_k \{\mathbf{T}_j\mathbf{T}_k\mathbf{T}_i\}, \quad (i = 1,2,3). \tag{A4}$$

Finally, the 27 invariants $\{\mathbf{T}_j\mathbf{T}_k\mathbf{T}_i\}$, $(i,j,k = 1,2,3)$ must be evaluated. As a result of symmetry properties $((j,k,i) = (i,j,k) = (k,i,j))$, only 11 invariants must be computed: $(i,j,k) = (1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,2,2)$, $(1,2,3)$, $(1,3,2)$, $(1,3,3)$, $(2,2,2)$, $(2,2,3)$, $(2,3,3)$, and $(3,3,3)$. With the generalized version of the Cayley-Hamilton theorem,[9] the only resulting nonzero invariants are

$$\{\mathbf{T}_1^2\mathbf{T}_3\} = \frac{1}{6}, \qquad \{\mathbf{T}_2^2\mathbf{T}_3\} = -\frac{1}{3}\{\mathbf{W}^{*2}\}, \qquad \{\mathbf{T}_3^3\} = -\frac{1}{36},$$

together with the invariants that result from the cyclic permutations of the indices. Therefore, the relations

$$\begin{aligned}
\{\mathbf{b}^2\mathbf{S}^*\} &= \frac{1}{3}a_1a_3 = 2\{\mathbf{bS}^*\}\{\mathbf{bS}^{*2}\}, \\
\{\mathbf{b}^2\mathbf{W}^*\mathbf{S}^*\} &= \frac{1}{3}a_2a_3\{\mathbf{W}^{*2}\} = 2\{\mathbf{bW}^*\mathbf{S}^*\}\{\mathbf{bS}^{*2}\}, \\
\{\mathbf{b}^2\mathbf{S}^{*2}\} - \frac{1}{3}\{\mathbf{b}^2\} &= \frac{1}{6}a_1^2 - \frac{1}{3}a_2^2\{\mathbf{W}^{*2}\} - \frac{1}{36}a_3^2 = \frac{1}{6}\{\mathbf{bS}^*\}^2 - \frac{1}{3}\frac{\{\mathbf{bW}^*\mathbf{S}^*\}^2}{\{\mathbf{W}^{*2}\}} - \{\mathbf{bS}^{*2}\}^2,
\end{aligned} \tag{A5}$$

lead to the desired expression of the quadratic term in (18).

APPENDIX B: DERIVATION OF $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, AND $\{\mathbf{bS}^{*2}\}$ EQUATIONS

Starting from the tensor evolution equation for the Reynolds stress anisotropy (8), a system of three scalar ordinary differential equations in the three scalar unknowns $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$ can be derived.

By multiplying relation (8) by $\mathbf{S}^*$, taking the trace of the equation, and using the results of Appendix A to express $\{\mathbf{b}^2\mathbf{S}^*\}$ in terms of $\{\mathbf{bS}^*\}$ and $\{\mathbf{bS}^{*2}\}$, the following equation is obtained:

$$\{\frac{d\mathbf{b}}{dt^*}\mathbf{S}^*\} = -\frac{1}{g\eta}\{\mathbf{bS}^*\} - 2a_3\{\mathbf{bS}^{*2}\} + 2a_2\{\mathbf{bW}^*\mathbf{S}^*\} + \frac{2a_4}{\eta}\{\mathbf{bS}^*\}\{\mathbf{bS}^{*2}\} - a_1 - \{\mathbf{L}^*\mathbf{S}^*\}. \tag{B1}$$

Similarly, multiplying equation (8) by either $\mathbf{W}^*\mathbf{S}^*$ or $\mathbf{S}^{*2}$ and taking the trace of the equation leads to the following equations, respectively:

$$\{\frac{d\mathbf{b}}{dt^*}\mathbf{W}^*\mathbf{S}^*\} = -\frac{1}{g\eta}\{\mathbf{bW}^*\mathbf{S}^*\} - a_2\frac{\zeta^2}{\eta^2}\{\mathbf{bS}^*\} + \frac{2a_4}{\eta}\{\mathbf{bW}^*\mathbf{S}^*\}\{\mathbf{bS}^{*2}\} - \{\mathbf{L}^*\mathbf{W}^*\mathbf{S}^*\}, \tag{B2}$$

and

$$\{\frac{d\mathbf{b}}{dt^*}\mathbf{S}^{2*}\} = -\frac{1}{g\eta}\{\mathbf{bS}^{*2}\} - \frac{1}{3}a_3\{\mathbf{bS}^*\} + \frac{a_4}{6}\{\mathbf{bS}^*\}^2 + \frac{a_4}{3\eta}\frac{\eta^2}{\zeta^2}\{\mathbf{bW}^*\mathbf{S}^*\}^2 - \frac{a_4}{\eta}\{\mathbf{bS}^{*2}\}^2 - \{\mathbf{L}^*\mathbf{S}^{2*}\}. \tag{B3}$$

In obtaining these two equations, the following relations are used:

$$\{\mathbf{bS}^{*3}\} = \frac{1}{2}\{\mathbf{bS}^*\}, \qquad 2\{\mathbf{bW}^{*2}\mathbf{S}^*\} + \{\mathbf{bW}^*\mathbf{S}^*\mathbf{W}^*\} = -\frac{1}{2}\frac{\zeta^2}{\eta^2}\{\mathbf{bS}^*\},$$

which are consequences of the Cayley-Hamilton theorem.[9] Because the velocity gradients have been assumed independent of time, the following can easily be verified:

$$\begin{aligned} \{\frac{d\mathbf{b}}{dt^*}\mathbf{S}^*\} &= \frac{d}{dt^*}\{\mathbf{bS}^*\}, \\ \{\frac{d\mathbf{b}}{dt^*}\mathbf{W}^*\mathbf{S}^*\} &= \frac{d}{dt^*}\{\mathbf{bW}^*\mathbf{S}^*\}, \\ \{\frac{d\mathbf{b}}{dt^*}\mathbf{S}^{2*}\} &= \frac{d}{dt^*}\{\mathbf{bS}^{*2}\}. \end{aligned} \tag{B4}$$

Equations (B1), (B2), and (B3) lead, therefore, to the desired system of scalar ordinary differential equations for the invariants $\{\mathbf{bS}^*\}$, $\{\mathbf{bW}^*\mathbf{S}^*\}$, and $\{\mathbf{bS}^{*2}\}$,

$$
\begin{aligned}
\frac{d}{dt^*}\{\mathbf{bS}^*\} &= -\frac{1}{g\eta}\{\mathbf{bS}^*\} - 2a_3\{\mathbf{bS}^{*2}\} + 2a_2\{\mathbf{bW}^*\mathbf{S}^*\} \\
&\quad -a_1 - \{\mathbf{L}^*\mathbf{S}^*\} + \frac{2a_4}{\eta}\{\mathbf{bS}^*\}\{\mathbf{bS}^{*2}\}, \\
\frac{d}{dt^*}\{\mathbf{bW}^*\mathbf{S}^*\} &= -\frac{1}{g\eta}\{\mathbf{bW}^*\mathbf{S}^*\} - a_2\frac{\zeta^2}{\eta^2}\{\mathbf{bS}^*\} - \{\mathbf{L}^*\mathbf{W}^*\mathbf{S}^*\} \\
&\quad + \frac{2a_4}{\eta}\{\mathbf{bW}^*\mathbf{S}^*\}\{\mathbf{bS}^{*2}\}, \\
\frac{d}{dt^*}\{\mathbf{bS}^{*2}\} &= -\frac{1}{g\eta}\{\mathbf{bS}^{*2}\} - \frac{1}{3}a_3\{\mathbf{bS}^*\} - \{\mathbf{L}^*\mathbf{S}^{*2}\} + \frac{a_4}{6\eta}\{\mathbf{bS}^*\}^2 \\
&\quad + \frac{a_4}{3\eta}\frac{\eta^2}{\zeta^2}\{\mathbf{bW}^*\mathbf{S}^*\}^2 - \frac{a_4}{\eta}\{\mathbf{bS}^{*2}\}^2.
\end{aligned}
\tag{B5}
$$

APPENDIX C: SOLUTION OF ANISOTROPY EVOLUTION EQUATION

The change of variables

$$B_1 = \psi, \tag{C1}$$

$$B_2 = \frac{a_3}{a_2}\zeta - \frac{a_2\mathcal{R}^2}{H}\phi, \tag{C2}$$

$$B_3 = \zeta - \frac{a_3}{3H}\phi, \tag{C3}$$

transforms system (21) into the quadratic system of ordinary differential equations

$$\dot{\zeta} = \left(2\alpha\psi - \frac{\beta}{\eta}\right)\zeta, \tag{C4}$$

$$\dot{\phi} = \left(2\alpha\psi - \frac{\beta}{\eta}\right)\phi + H\psi, \tag{C5}$$

$$\dot{\psi} = \left(2\alpha\psi - \frac{\beta}{\eta}\right)\psi + \phi - a_1, \tag{C6}$$

where

$$H = \frac{2}{3}a_3^2 - 2a_2^2\mathcal{R}^2.$$

System (C4) (C6) is subjected to the following initial conditions:

$$\begin{aligned} \psi_0 = \psi(0) &= B_{1,0}, \\ \phi_0 = \phi(0) &= 2a_2B_{2,0} - 2a_3B_{3,0}, \\ \zeta_0 = \zeta(0) &= \left(\tfrac{2}{3}a_2a_3B_{2,0} - 2a_2^2\mathcal{R}^2B_{3,0}\right)/H. \end{aligned} \tag{C7}$$

In this system, the evolution of the two variables ψ and ϕ is independent of the evolution of the variable ζ. Therefore, the quadratic system of ordinary differential equations

$$\dot{\phi} = -\frac{\beta}{\eta}\phi + H\psi + 2\alpha\psi\phi, \tag{C8}$$

$$\dot{\psi} = -\frac{\beta}{\eta}\psi + \phi + 2\alpha\psi^2 - a_1, \tag{C9}$$

can be solved, and the evolution of ζ is deduced by integrating (C4),

$$\zeta(t^*) = \zeta_0 \exp\int_0^{t^*}\left[2\alpha\psi(s) - \frac{\beta}{\eta(s)}\right]ds. \tag{C10}$$

By integrating (C8), the evolution of ϕ can be given as a function of ψ,

$$\phi(t^*) = He^{\int_0^{t^*}[2\alpha\psi(s)-\beta/\eta(s)]ds}\int_0^{t^*}\psi(r)e^{-\int_0^r[2\alpha\psi(s)-\beta/\eta(s)]ds}dr + \phi_0 e^{\int_0^{t^*}[2\alpha\psi(s)-\beta/\eta(s)]ds}. \tag{C11}$$

By introducing the transformation

$$\psi(t^*) = -\frac{1}{2\alpha}\left(\frac{\dot{\omega}(t^*)}{\omega(t^*)} - \frac{\beta}{\eta(t^*)}\right), \tag{C12}$$

equation (C11) can now be rewritten as

$$\phi(t^*) = -\frac{H}{2\alpha} + \frac{1}{2\alpha\omega(t^*)}\left[\omega(0)(H + 2\alpha\phi_0) + \beta H \int_0^{t^*} \frac{\omega(s)}{\eta(s)} ds\right], \tag{C13}$$

and equation (C9) can be written in terms of ω as the following integro-differential equation:

$$\ddot{\omega} = \beta\frac{\dot{\omega}}{\eta} + (H + 2\alpha a_1 - \beta\frac{\dot{\eta}}{\eta^2})\omega - \beta H \int_0^{t^*} \frac{\omega(s)}{\eta(s)} ds - \omega(0)(H + 2\alpha\phi_0), \tag{C14}$$

where $\omega(0)$ can take any nonzero value and $\dot{\omega}(0) = -\omega(0)(2\alpha\psi_0 - \beta/\eta_0) = -\omega(0)(2\alpha B_{1,0} - \beta/\eta_0)$, with $\eta_0 = \eta(0)$ as the initial value of the relative strain parameter. Equation (C14) is integrated with the following transformation

$$\begin{aligned}
\Psi_1 &= \int \omega/\eta, \\
\Psi_2 &= \dot{\Psi}_1 = \omega/\eta, \\
\Psi_3 &= \dot{\Psi}_2 = \dot{\omega}/\eta + \omega(\dot{1/\eta}), \\
\Psi_4 &= \dot{\Psi}_3 = \ddot{\omega}/\eta + 2\dot{\omega}(\dot{1/\eta}) + \omega(\ddot{1/\eta}).
\end{aligned}$$

The functions Ψ_i are the solution of

$$\begin{aligned}
\dot{\Psi}_1 &= \Psi_2, \\
\dot{\Psi}_2 &= \Psi_3, \\
\dot{\Psi}_3 &= -\frac{\beta}{\eta}H\Psi_1 + \left(H + 2\alpha a_1 - \frac{\ddot{\eta}}{\eta}\right)\Psi_2 + \frac{1}{\eta}(\beta - 2\dot{\eta})\Psi_3 - \frac{\eta_0}{\eta}\Psi_2(0)(H + H^0),
\end{aligned} \tag{C15}$$

where the initial conditions are given by $\Psi_1(0) = 0$, $\Psi_2(0) \neq 0$, and $\Psi_3(0) = -\Psi_2(0)(2\alpha B_{1,0} - \beta/\eta_0 - \dot{\eta}_0/\eta_0)$; and $H^0 = 2\alpha\phi_0 = 4\alpha(a_2 B_{2,0} - a_3 B_{3,0})$.

In the general case for which the relative strain parameter varies in time, system (C15) is a linear system of ordinary differential equations with variable coefficients. In the case of slow variations of η, the approximations

$$\frac{\dot{\eta}}{\eta} \sim 0, \qquad \frac{\ddot{\eta}}{\eta} \sim 0 \tag{C16}$$

are valid, and the solution of the system of ordinary differential equations (C15) yields

$$\begin{aligned}
\Psi_1(t^*) &= K[\sum_{r=1}^{3} \mu_r \frac{1}{\lambda_r} e^{\lambda_r t^*} + \nu(H + H^0)], \\
\Psi_2(t^*) &= K \sum_{r=1}^{3} \mu_r e^{\lambda_r t^*}, \\
\Psi_3(t^*) &= K \sum_{r=1}^{3} \mu_r \lambda_r e^{\lambda_r t^*},
\end{aligned} \tag{C17}$$

where $K = \Psi_2(0)[(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)]^{-1}$ and

$$\mu_r = \left[\lambda_r^2 - 2\alpha B_{1,0}\lambda_r - (H + H^0)\right](\lambda_p - \lambda_q), \tag{C18}$$

$$\nu = \frac{\lambda_3 - \lambda_2}{\lambda_1} + \frac{\lambda_1 - \lambda_3}{\lambda_2} + \frac{\lambda_2 - \lambda_1}{\lambda_3}. \tag{C19}$$

In (C18), the indices p and q are chosen such that $\epsilon_{rpq} = -1$. The λ_r are eigenvalues that are obtained as roots of the following third-order characteristic polynomial:

$$\lambda^3 - \frac{\beta}{\eta}\lambda^2 - (H + 2\alpha a_1)\lambda + \frac{\beta}{\eta}H = 0. \tag{C20}$$

Finally, in terms of the original variables $B_i(t^*)$ the explicit solution is

$$\begin{aligned}
B_1(t^*) &= \frac{1}{2\alpha}\left[\frac{\beta}{\eta(t^*)} - \frac{\Psi_3(t^*)}{\Psi_2(t^*)}\right], \\
B_2(t^*) &= \frac{a_2\mathcal{R}^2}{2\alpha}\left[1 - \frac{\beta}{\eta(t^*)}\frac{\Psi_1(t^*)}{\Psi_2(t^*)} - \frac{\Psi_2(0)}{\Psi_2(t^*)}\right] + \frac{\Psi_2(0)}{\Psi_2(t^*)}B_{2,0}, \\
B_3(t^*) &= \frac{a_3}{6\alpha}\left[1 - \frac{\beta}{\eta(t^*)}\frac{\Psi_1(t^*)}{\Psi_2(t^*)} - \frac{\Psi_2(0)}{\Psi_2(t^*)}\right] + \frac{\Psi_2(0)}{\Psi_2(t^*)}B_{3,0}.
\end{aligned} \tag{C21}$$

In (C21), the initial condition $\Psi_2(0) \neq 0$ is arbitrary, and its value can, therefore, be taken as $\Psi_2(0) = 1$.

REFERENCES

[1] Speziale, C. G., and Mac Giolla Mhuiris, N. (1989). On the prediction of equilibrium states in homogeneous turbulence. *J. Fluid Mech.*, **209**, 591.

[2] Speziale, C. G., Sarkar, S., and Gatski, T. B. (1991). Modelling the pressure-strain correlation of turbulence: an invariant dynamical systems approach. *J. Fluid Mech.*, **227**, 245.

[3] Speziale, C. G., Abid, R., and Blaisdell, G. A. (1996). On the consistency of Reynolds stress turbulence closures with hydrodynamic stability theory. *Phys. Fluids*, **8**(3), 781.

[4] Girimaji, S. S. (1997). Dynamical system analysis of Reynolds stress closure equations. Eleventh Symposium on Turbulent Shear Flows, Institut National Polytechnique de Grenoble - Université Joseph Fourier.

[5] Jongen, T., and Gatski, T. B. (1997). General Explicit Algebraic stress relations and best approximation for three-dimensional flows. *Int. J. Engr. Sci.*, To Appear.

[6] Girimaji, S. S. (1996). Fully explicit and self-consistent algebraic Reynolds stress model. *Theor. Comput. Fluid Dyn.*, **8**, 387.

[7] Leuchter, O., and Benoit, J. P. (1991). Study of coupled effects of plane strain and rotation on homogeneous turbulence. Eighth Symposium on Turbulent Shear Flows, Technical University of Munich, 16-3-1.

[8] Rivlin, R. S., and Ericksen, J. L. (1955). Stress-deformation relations for isotropic materials. *Arch. Rat. Mech. Anal.*, **4**, 323.

[9] Spencer, A. J. M., and Rivlin, R. S. (1959). The theory of matrix polynomials and its application to the mechanics of isotropic continua. *Arch. Rational Mech. Anal.*, **2**, 309.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
December 1997

3. REPORT TYPE AND DATES COVERED
Technical Memorandum

4. TITLE AND SUBTITLE
Time Evolution of Modeled Reynolds Stresses in Planar Homogeneous Flows

5. FUNDING NUMBERS
WU 522-31-81-03

6. AUTHOR(S)
T. Jongen and T. B. Gatski

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, VA 23681-0001

8. PERFORMING ORGANIZATION REPORT NUMBER
L-17672

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
NASA/TM-97-206265

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Unclassified/Unlimited
Subject Category 34
Distribution: Nonstandard
Availability: NASA CASI (301) 621-0390

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
The analytic expression of the time evolution of the Reynolds stress anisotropy tensor in all planar homogeneous flows is obtained by exact integration of the modeled differential Reynolds stress equations. The procedure is based on results of tensor representation theory, is applicable for general pressure-strain correlation tensors, and can account for any additional turbulence anisotropy effects included in the closure. An explicit solution of the resulting system of scalar ordinary differential equations is obtained for the case of a linear pressure-strain correlation tensor. The properties of this solution are discussed, and the dynamic behavior of the Reynolds stresses is studied, including limit cycles and sensitivity to initial anisotropies.

14. SUBJECT TERMS
Turbulence Modeling, Nonequilibrium Flows, Homogeneous Turbulence

15. NUMBER OF PAGES
29

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT
Unclassified

18. SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

19. SECURITY CLASSIFICATION OF ABSTRACT
Unclassified

20. LIMITATION OF ABSTRACT