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The CE/SE method—A CFD Framework for the Challenges of the New Millennium

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WORKSHOP ON THE CE/SE METHOD

Abstract

The space-time conservation element and solution element (CE/SE) method, which was originated and is continuously being developed at NASA Glenn Research Center, is a high-resolution, genuinely multidimensional and unstructured-mesh compatible numerical method for solving conservation laws. Since its inception in 1991, the CE/SE method has been used to obtain highly accurate numerical solutions for 1D, 2D and 3D flow problems involving shocks, contact discontinuities, acoustic waves, vortices, shock/acoustic waves/vortices interactions, shock/boundary layers interactions and chemical reactions. Without the aid of preconditioning or other special techniques, it has been applied to both steady and unsteady flows with speeds ranging from Mach number = 0.00288 to 10. In addition, the method has unique features that allow for (i) the use of very simple non-reflecting boundary conditions, and (ii) a unified wall boundary treatment for viscous and inviscid flows.

The CE/SE method was developed with the conviction that, with a solid foundation in physics, a robust, coherent and accurate numerical framework can be built without involving overly complex mathematics. As a result, the method was constructed using a set of design principles that facilitate simplicity, robustness and accuracy. The most important among them are: (i) enforcing both local and global flux conservation in space and time, with flux evaluation at an interface being an integral part of the solution procedure and requiring no interpolation or extrapolation; (ii) unifying space and time and treating them as a single entity; and (iii) requiring that a numerical scheme be built from a nondissipative core scheme such that the numerical dissipation can be effectively controlled and, as a result, will not overwhelm the physical dissipation. Part I of the workshop will be devoted to a discussion of these principles along with a description of how the 1D, 2D and 3D CE/SE schemes are constructed. In Part II, various applications of the CE/SE method, particularly those involving chemical reactions and acoustics, will be presented. The workshop will be concluded with a sketch of the future research directions.

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A History of CFD Development

1. From 60's to 70's – many new methods were proposed and tested. Among them, the projection method was most successful. To date, most of commercial codes are based on the projection method and its branches, e.g., the SIMPLE family.

2. From early 80's to early 90's – CFD methods for aerodynamics and shock capturing were under vigorous development. In particular, significant resources were devoted to developing upwind schemes. The cost-to-value ratio in developing upwind schemes, however, has been disappointing. In general, upwind schemes are not robust and require special treatments to overcome numerical difficulties when applied to flow problems with practical geometries. Extension to flows with complex physics requires modifications in flux function formulation, which in general involve ad-hoc assumptions and approximations. 3D solutions have been scant. By and large, the upwind schemes have not been adopted by the commercial CFD community. Further development of upwind methods has not been a focal point of CFD research activities since early 90's.

3. From early 90's to now – The CFD issues have been shifted to (a) unstructured mesh method, (b) high-performance parallel computation, (c) high-accuracy unsteady flow simulation, e.g., computational aeroacoustics (CAA) and large eddy simulation (LES) and (d) extension to inter-disciplinary applications, e.g., combustion, plasma dynamics, flow with multiple phases, crystal growth, etc. It is clear that a successful CFD method must be able to address these issues.

4. The CE/SE method is designed to meet the above needs and thus can provide the CFD community a robust and efficient numerical framework for high fidelity flow simulation in practical applications.

Two Basic Beliefs

The present development is guided by the following two basic beliefs that set the present method apart from the established methods.

- In order to capture physics more efficiently and realistically, the modeling focus must be placed on the original integral form of the *space-time* physical conservation laws, rather than the differential form. The latter form follows from the integral form under the additional assumption that the physical solution is smooth, an assumption that is difficult to realize numerically in a region of rapid change, such as a shock.

- With proper modeling of the integral and differential forms themselves, the resulting numerical solution should automatically be consistent with the properties derived from the integral and differential forms, e.g., the jump conditions across a shock and the properties of characteristics. As a result, a numerical method can be greatly simplified if the derived properties are *not* explicitly used in numerical modeling.

Physics Considerations

- Flux-conservation in *space-time* be enforced locally (i.e., down to a single cell) and globally (i.e., over the entire solution domain). This is a critical requirement for accurate flow simulations, particularly if they involve long marching times and/or regions of rapid change.

- Space and time be unified and treated as a single entity.

- A numerical scheme be built from a nondissipative core scheme such that the numerical dissipation can be effectively controlled and, as a result, will not overwhelm the physical dissipation.

- A multidimensional scheme be genuinely multidimensional, i.e., it is constructed without using the dimensional-splitting approach, such that multidimensional effects and source terms (which are not aligned with special directions) can be modeled more realistically.

Mathematics Considerations

- For easy implementation and efficient computing, the method should use the simplest logical structures and approximation techniques whenever possible.

- Flux at an interface separating two conservation cells be evaluated internally *without using any ad hoc flux model or any characteristics-based technique.*

- To be compatible with unstructured meshes, the spatial meshes be built from triangles in the 2D case, and tetrahedrons in the 3D case.

- The 2D and 3D schemes should share with their 1D versions the same design principles so that they are straightforward extensions of the 1D versions and share with the 1D versions virtually identical fundamental characteristics.

- Avoid the use of special techniques that may limit application of the method and even cause undesirable side effects.

Sod's Shock Tube Problems with Non-Reflecting Boundary Conditions

Extrapolation Boundary Conditions: (i)

$$(u_m)_j^n = (u_m)_{j-1/2}^{n-1/2} \quad \text{and} \quad (u_{mx})_j^n = (u_{mx})_{j-1/2}^{n-1/2}, \qquad m = 1,2,3;\ n = 1,2,3,\ldots$$

if (j,n) is a mesh point on the right spatial boundary; and (ii)

$$(u_m)_j^n = (u_m)_{j+1/2}^{n-1/2} \quad \text{and} \quad (u_{mx})_j^n = (u_{mx})_{j+1/2}^{n-1/2}, \qquad m = 1,2,3;\ n = 1,2,3,\ldots$$

if (j,n) is a mesh point on the left spatial boundary.

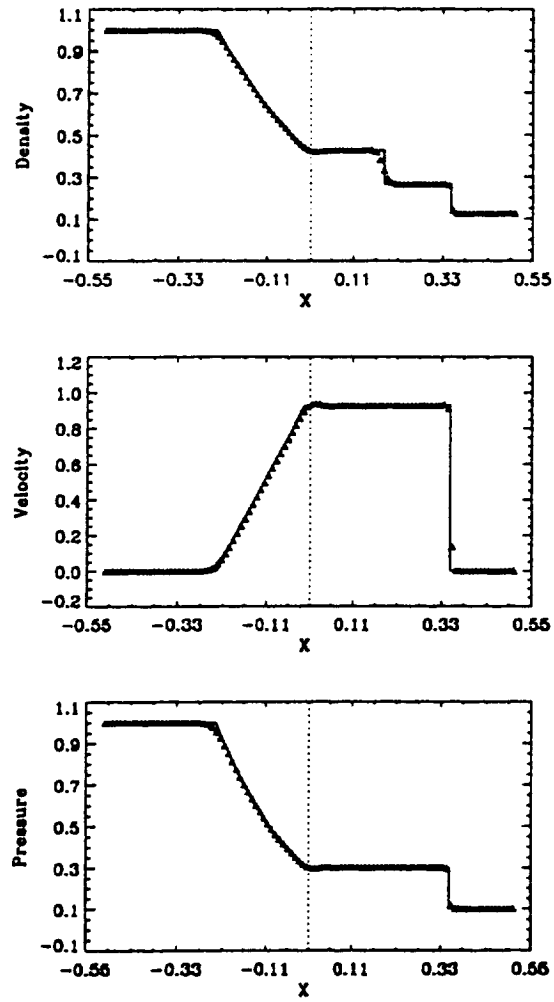
Steady-State Boundary Conditions:

$$(u_m)_j^n = (u_m)_j^0 \quad \text{and} \quad (u_{mx})_j^n = (u_{mx})_j^0, \qquad n = 1,2,3,\ldots$$

where (j,n) is any mesh point on the right or left boundary.

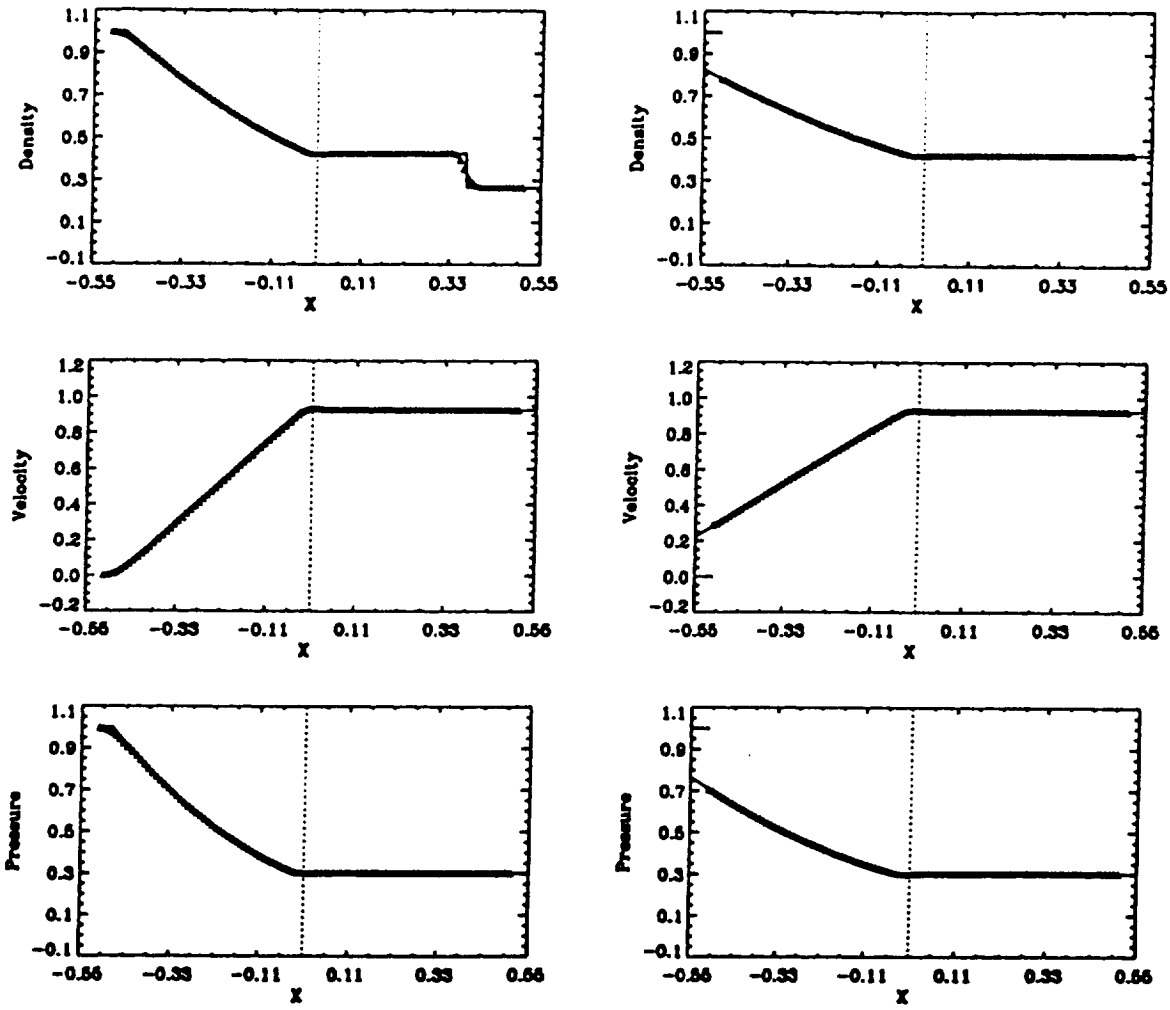
```
c                    Appendix A. A Sample Program
c
      implicit real*8(a-h,o-z)
      parameter (nxd=1000)
      dimension  q(3,nxd), qn(3,nxd), qx(3,nxd), qt(3,nxd),
     *           s(3,nxd), vxl(3), vxr(3), xx(nxd)
c
c     nx must be an odd integer.
      nx = 101
      it = 100
      dt = 0.4d-2
      dx = 0.1d-1
      ga = 1.4d0
      rhol = 1.d0
      ul = 0.d0
      pl = 1.d0
      rhor = 0.125d0
      ur = 0.d0
      pr = 0.1d0
      ia = 1
c
      nx1 = nx + 1
      nx2 = nx1/2
      hdt = dt/2.d0
      tt = hdt*dfloat(it)
      qdt = dt/4.d0
      hdx = dx/2.d0
      qdx = dx/4.d0
      dtx = dt/dx
      a1 = ga - 1.d0
      a2 = 3.d0 - ga
      a3 = a2/2.d0
      a4 = 1.5d0*a1
      u2l = rhol*ul
      u3l = pl/a1 + 0.5d0*rhol*ul**2
      u2r = rhor*ur
      u3r = pr/a1 + 0.5d0*rhor*ur**2
      do 5 j = 1,nx2
      q(1,j) = rhol
      q(2,j) = u2l
      q(3,j) = u3l
      q(1,nx2+j) = rhor
      q(2,nx2+j) = u2r
      q(3,nx2+j) = u3r
      do 5 i = 1,3
      qx(i,j) = 0.d0
      qx(i,nx2+j) = 0.d0
5     continue
c
      open (unit=8,file='for008')
      write (8,10) tt,it,ia,nx
      write (8,20) dt,dx,ga
      write (8,30) rhol,ul,pl
      write (8,40) rhor,ur,pr
c
      do 400 i = 1,it
      m = nx + i - (i/2)*2
      do 100 j = 1,m
      w2 = q(2,j)/q(1,j)
      w3 = q(3,j)/q(1,j)
      f21 = -a3*w2**2
      f22 = a2*w2
      f31 = a1*w2**3 - ga*w2*w3
      f32 = ga*w3 - a4*w2**2
      f33 = ga*w2
      qt(1,j) = -qx(2,j)
```

```
      qt(2,j) = -(f21*qx(1,j) + f22*qx(2,j) + a1*qx(3,j))
      qt(3,j) = -(f31*qx(1,j) + f32*qx(2,j) + f33*qx(3,j))
      s(1,j) = qdx*qx(1,j) + dtx*(q(2,j) + qdt*qt(2,j))
      s(2,j) = qdx*qx(2,j) + dtx*(f21*(q(1,j) + qdt*qt(1,j)) +
     *  f22*(q(2,j) + qdt*qt(2,j)) + a1*(q(3,j) + qdt*qt(3,j)))
      s(3,j) = qdx*qx(3,j) + dtx*(f31*(q(1,j) + qdt*qt(1,j)) +
     *  f32*(q(2,j) + qdt*qt(2,j)) + f33*(q(3,j) + qdt*qt(3,j)))
100   continue
      if (i.ne.(i/2)*2) goto 150
      do 120 k = 1,3
      qx(k,nx1) = qx(k,nx)
      qn(k,1) = q(k,1)
      qn(k,nx1) = q(k,nx)
120   continue
150   j1 = 1 - i + (i/2)*2
      mm = m - 1
      do 200 j = 1,mm
      do 200 k = 1,3
      qn(k,j+j1) = 0.5d0*(q(k,j) + q(k,j+1) + s(k,j) - s(k,j+1))
      vxl(k) = (qn(k,j+j1) - q(k,j) - hdt*qt(k,j))/hdx
      vxr(k) = (q(k,j+1) + hdt*qt(k,j+1) - qn(k,j+j1))/hdx
      qx(k,j+j1) = (vxl(k)*(dabs(vxr(k)))**ia + vxr(k)*(dabs(vxl(k)))
     *      **ia)/((dabs(vxl(k)))**ia + (dabs(vxr(k)))**ia + 1.d-60)
200   continue
      m = nx1 - i + (i/2)*2
      do 300 j = 1,m
      do 300 k = 1,3
      q(k,j) = qn(k,j)
300   continue
400   continue
c
      m = nx1 -it + (it/2)*2
      mm = m - 1
      xx(1) = -0.5d0*dx*dfloat(mm)
      do 500 j = 1,mm
      xx(j+1) = xx(j) + dx
500   continue
      do 600 j = 1,m
      x = q(2,j)/q(1,j)
      y = a1*(q(3,j) - 0.5d0*x**2*q(1,j))
      z = x/dsqrt(ga*y/q(1,j))
      write (8,50) xx(j),q(1,j),x,y,z
600   continue
c
      close (unit=8)
10    format(' t = ',g14.7,' it = ',i4,' ia = ',i4,' nx = ',i4)
20    format(' dt = ',g14.7,' dx = ',g14.7,' gamma = ',g14.7)
30    format(' rhol = ',g14.7,' ul = ',g14.7,' pl = ',g14.7)
40    format(' rhor = ',g14.7,' ur = ',g14.7,' pr = ',g14.7)
50    format(' x =',f8.4,' rho =',f8.4,' u =',f8.4,' p =',f8.4,
     *       ' M =',f8.4)
      stop
      end
```



(a) Profiles at t=0.2

Figure 1. — The CE/SE solution of the extended Sod's problem using the boundary conditions Eqs. (2.68) and (2.69) ($\Delta t = 0.004$, $\Delta x = 0.01$, CFL ≈ 0.88, $\epsilon = 0.5$, $\alpha = 1$)



(b) Profiles at t=0.4

(c) Profiles at t=0.6

Figure 1 (Continued).

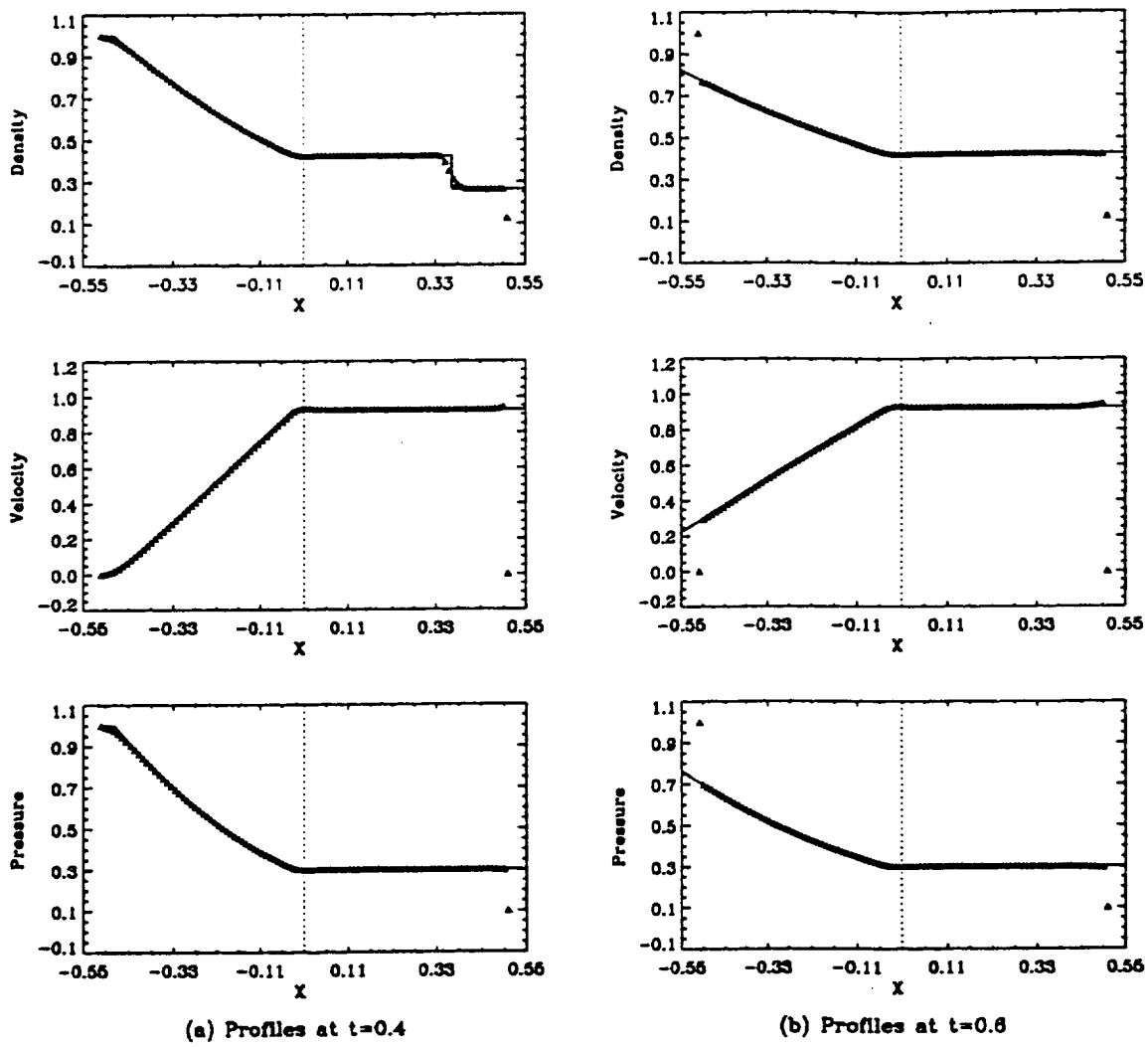


Figure 2. — The CE/SE solution of the extended Sod's problem using the boundary condition Eq. (2.70) ($\Delta t = 0.004$, $\Delta x = 0.01$, CFL ≈ 0.88, $\epsilon = 0.5$, $\alpha = 1$)

The a Scheme

Consider the PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1.1}$$

where a is a constant. Let $x_1 = x$ and $x_2 = t$ be the coordinates of a 2D Euclidean space E_2. Because Eq. (1.1) $\Leftrightarrow$

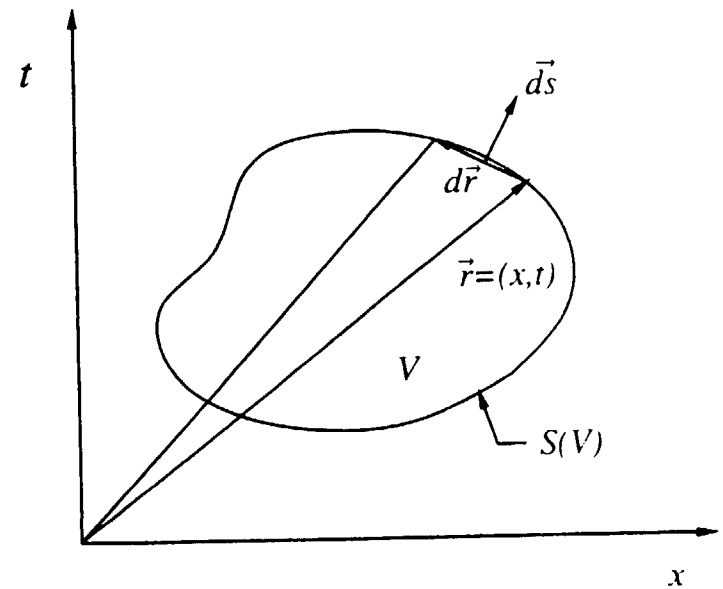
$$\frac{\partial (au)}{\partial x} + \frac{\partial u}{\partial t} = 0 \tag{1.2}$$

Gauss' divergence theorem $\Rightarrow$

$$\oint_{S(V)} \vec{h} \cdot d\vec{s} = 0 \tag{1.3}$$

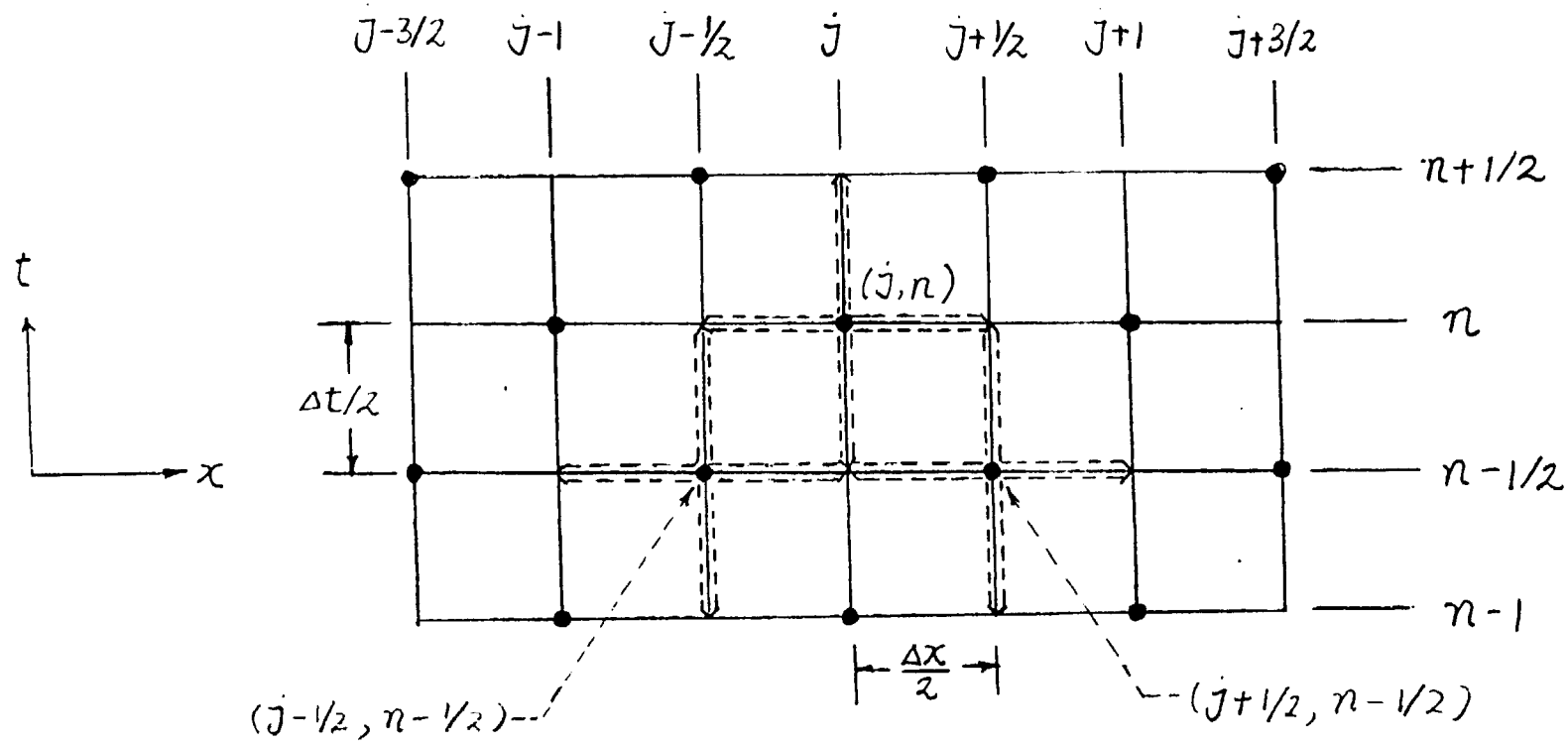
where

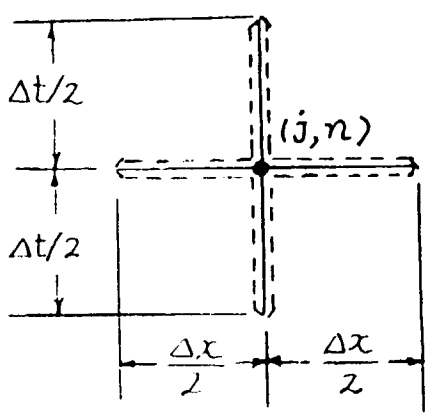
$$\vec{h} = (au,\, u), \quad \text{and} \quad d\vec{s} = d\sigma\, \vec{n} \tag{1.4}$$



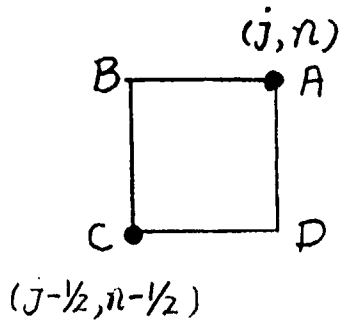
(a) Here (i) $S(V)$ is the boundary of a space-time region V in E_2; and (ii) $d\sigma$ and $\vec{n}$ are the area and the outward unit normal of a surface element on $S(V)$.

(b) $\vec{h} \cdot d\vec{s}$ is the *space-time* flux of $\vec{h}$ leaving the region V through the surface element $d\vec{s}$. Thus Eq. (1.3) simply states that the total flux leaving the space-time region V through its boundary is equal to zero.

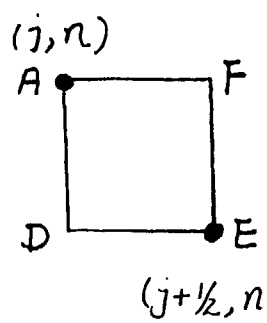




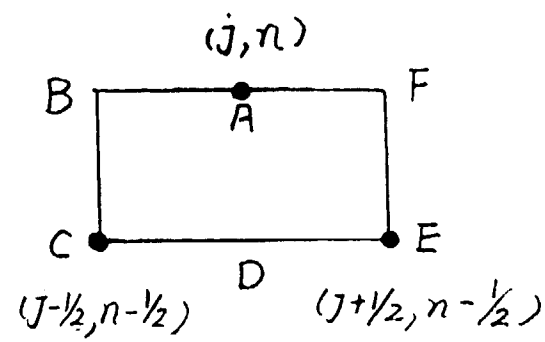
SE(j,n)



$CE_-(j,n)$



$CE_+(j,n)$



CE(j,n)

For any $(x,t) \in \mathrm{SE}(j,n)$, let

$$u(x,t) \approx u^*(x,t\,;j,n) \quad \text{and} \quad \vec{h}(x,t) \approx \vec{h}^*(x,t\,;j,n) \tag{1.5}$$

with

$$u^*(x,t\,;j,n) \stackrel{\text{def}}{=} u_j^n + (u_x)_j^n (x - x_j) + (u_t)_j^n (t - t^n) \tag{1.6}$$

$$\vec{h}^*(x,t\,;j,n) \stackrel{\text{def}}{=} \big(au^*(x,t\,;j,n),\, u^*(x,t\,;j,n)\big) \tag{1.7}$$

Here u_j^n, $(u_x)_j^n$, and $(u_t)_j^n$ are constants in $\mathrm{SE}(j,n)$.

Let $u = u^*(x,t\,;j,n)$ satisfy the convection equation. Then

$$(u_t)_j^n = -a\,(u_x)_j^n \tag{1.8}$$

Eqs. (1.6) and (1.8) $\Rightarrow$

$$u^*(x,t\,;j,n) = u_j^n + (u_x)_j^n \left[(x - x_j) - a\,(t - t^n)\right], \quad (x,t) \in \mathrm{SE}(j,n) \tag{1.9}$$

i.e., there are two independent marching variables u_j^n and $(u_x)_j^n$ for each (j,n).

Let $\overline{AB} \in \mathrm{SE}(j,n)$ be a part of $S(V)$. Then

$$\vec{h}^*(x,t\,;j,n) \stackrel{\text{def}}{=} \big(au^*(x,t\,;j,n),\, u^*(x,t\,;j,n)\big) \tag{1.7}$$

and

$$u^*(x,t\,;j,n) = u_j^n + (u_x)_j^n \left[(x - x_j) - a\,(t - t^n)\right], \quad (x,t) \in \mathrm{SE}(j,n) \tag{1.9}$$

$\Rightarrow$

$$\int_{\overline{AB}} \vec{h}^* \cdot d\vec{s} = \left[\vec{h}^*(x_c,t_c\,;j,n) \cdot \vec{n}\right] \times \ell \tag{1.10}$$

where

(i) (x_c, t_c) are the coordinates of the midpoint of $\overline{AB}$.

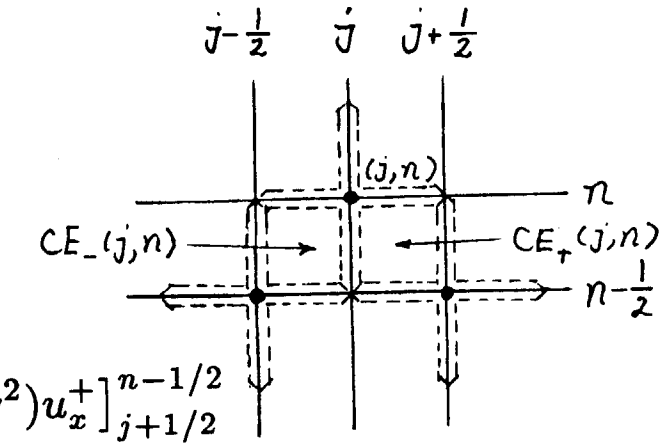
(ii) $\vec{n}$ is the outward unit normal of V at $\overline{AB}$.

(iii) ℓ is the length of $\overline{AB}$.

For any mesh point (j, n), let

$$(1.11) \qquad (u_x^+)_j^n \stackrel{\text{def}}{=} \frac{\Delta x}{4}(u_x)_j^n \quad \text{and} \quad \nu \stackrel{\text{def}}{=} \frac{a\Delta t}{\Delta x}$$

Hereafter the superscript symbol "+" is used to denote a *normalized* parameter. Using Eq. (1.11) and the assumption



$$(1.12) \qquad \oint_{S(CE_\pm(j,n))} \vec{h}^* \cdot d\vec{s} = 0$$

one has

$$(1.13) \qquad \left[(1-\nu)u + (1-\nu^2)u_x^+\right]_j^n = \left[(1-\nu)u - (1-\nu^2)u_x^+\right]_{j+1/2}^{n-1/2}$$

and

$$(1.14) \qquad \left[(1+\nu)u - (1-\nu^2)u_x^+\right]_j^n = \left[(1+\nu)u + (1-\nu^2)u_x^+\right]_{j-1/2}^{n-1/2}$$

for all (j, n). To simplify notation, in the above and hereafter we adopt a convention that can be explained using the expression on the left side of Eq. (1.13) as an example, i.e.,

$$\left[(1-\nu)u + (1-\nu^2)u_x^+\right]_j^n = (1-\nu)u_j^n + (1-\nu^2)(u_x^+)_j^n$$

Let

$$(s_+)_{j+1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[u - (1+\nu)u_x^+\right]_{j+1/2}^{n-1/2} \tag{1.15}$$

$$(s_-)_{j-1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[u + (1-\nu)u_x^+\right]_{j-1/2}^{n-1/2} \tag{1.16}$$

$$(u_x^{a+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}\left[(s_+)_{j+1/2}^{n-1/2} - (s_-)_{j-1/2}^{n-1/2}\right] \tag{1.17}$$

By adding Eqs. (1.13) and (1.14) together, and using the above definitions, one has

$$u_j^n = \frac{1}{2}\left[(1-\nu)(s_+)_{j+1/2}^{n-1/2} + (1+\nu)(s_-)_{j-1/2}^{n-1/2}\right] \tag{1.18}$$

Let $1-\nu^2 \neq 0$, i.e., $1-\nu \neq 0$ and $1+\nu \neq 0$. Then Eqs. (1.13) and (1.14) can be divided by $(1-\nu)$ and $(1+\nu)$, respectively. By subtracting the resulting equations from each other, one has

$$(u_x^+)_j^n = (u_x^{a+})_j^n \tag{1.19}$$

The a scheme is formed by Eqs. (1.18) and (1.19).

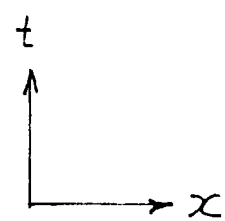
Note that the superscript symbol "a" that appears in Eqs. (1.17) and (1.19) is introduced to indicate that Eq. (1.19) is valid for the a scheme.

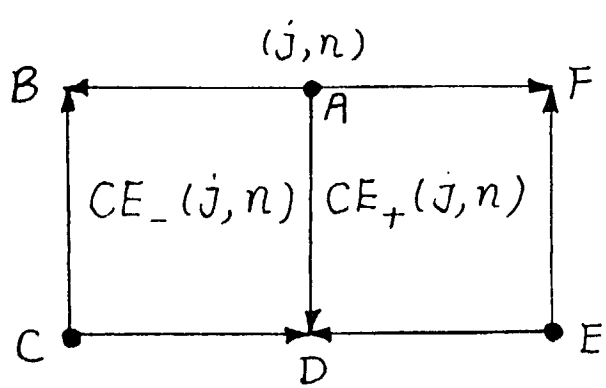
$$u_j^n = \frac{1}{2}\left[(1-\nu)(s_+)_{j+1/2}^{n-1/2} + (1+\nu)(s_-)_{j-1/2}^{n-1/2}\right] \tag{1.18}$$

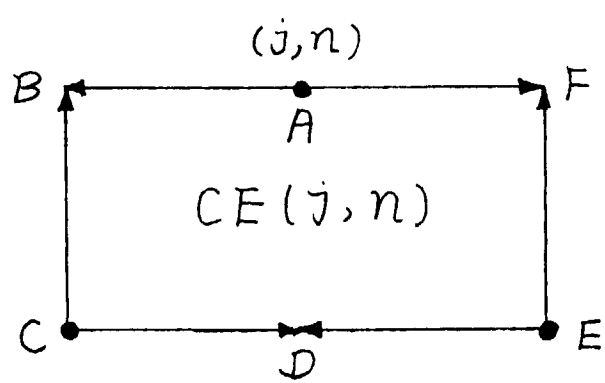
$$(u_x^+)_j^n = (u_x^{a+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}\left[(s_+)_{j+1/2}^{n-1/2} - (s_-)_{j-1/2}^{n-1/2}\right] \tag{1.19}$$

$$u^*(x,t\,;j,n) = u_j^n + (u_x)_j^n\left[(x-x_j) - a\,(t-t^n)\right], \quad (x,t) \in \mathrm{SE}(j,n) \tag{1.9}$$

$$\vec{h}^*(x,t\,;j,n) \stackrel{\text{def}}{=} \left(au^*(x,t\,;j,n),\, u^*(x,t\,;j,n)\right) \tag{1.7}$$

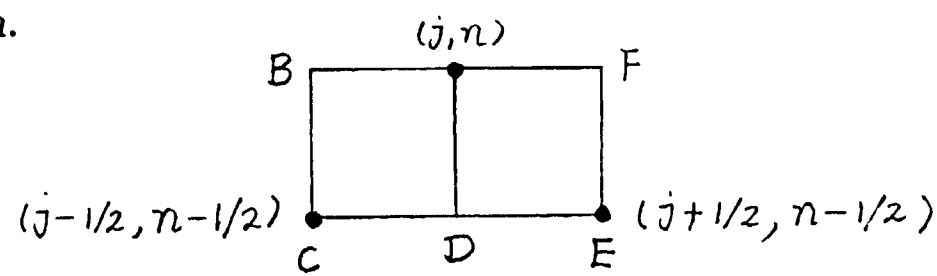






Special Features of the a Scheme

- It has the simplest stencil.
- Its two amplification factors are identical to those of the Leapfrog scheme. Therefore it is nondissipative within its stability domain: $0 \leq |\nu| \leq 1$.
- It is a two-way marching scheme, i.e., the forward marching scheme can be inverted to become the backward marching scheme and vice versa.
- For each mesh point (j, n), it is associated with two discrete equations and two independent unknowns u_j^n and $(u_x)_j^n$. As a result, the a scheme is consistent with a system of two PDEs involving two dependent variables with one of the PDEs being Eq. (1.1).
- In its construction, the surface integration over any interface separating two neighboring CEs is evaluated using the information from a single SE (no extrapolation or interpolation is used!). As a result, the local conservation relation Eq. (1.12) leads to a global flux conservation relation, i.e., *the total flux of $\vec{h}^*$ leaving the boundary of any space-time region that is the union of any combination of CEs will also vanish.*



The a-μ Scheme

The a-μ scheme is a solver of

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \mu\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.20}$$

where a and $\mu \geq 0$ are constants. Note that the integral form of the above PDE is

$$\oint_{S(V)} \vec{h} \cdot d\vec{s} = 0 \tag{1.3}$$

with

$$\vec{h} = (au - \mu\partial u/\partial x,\ u) \tag{1.21}$$

The construction procedure of the a-μ scheme is identical to that of the a scheme except that Eq. (1.7) is replaced by

$$\vec{h}^*(x,t\,;j,n) \stackrel{\text{def}}{=} \big(au^*(x,t\,;j,n) - \mu\partial u^*(x,t\,;j,n)/\partial x,\ u^*(x,t\,;j,n)\big) \tag{1.22}$$

The a-ϵ Scheme

For the a-ϵ scheme, the less stringent conservation condition

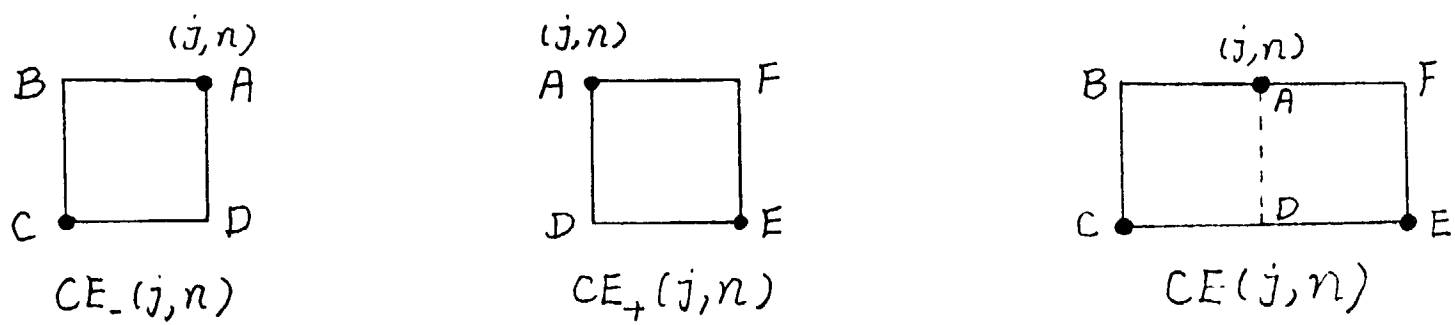
$$\oint_{S(CE(j,n))} \vec{h}^* \cdot d\vec{s} = 0 \tag{1.23}$$

is imposed at each (j,n). Again the local conservation condition Eq. (1.23) leads to a global conservation condition, i.e., *the total flux of $\vec{h}^*$ leaving the boundary of any space-time region that is the union of any combination of $CE(j,n)$, $(j,n) \in \Omega$, will also vanish.*

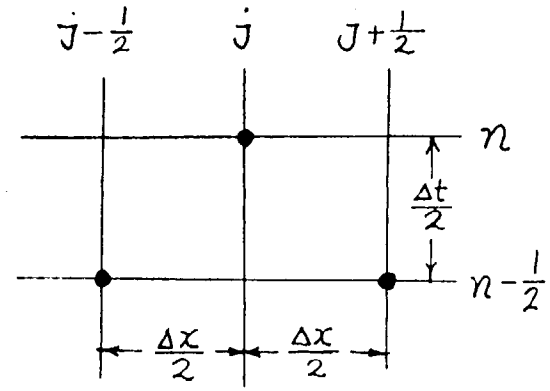
Because Eq. (1.23) $\Leftrightarrow$

$$u_j^n = \frac{1}{2}\left[(1-\nu)(s_+)_{j+1/2}^{n-1/2} + (1+\nu)(s_-)_{j-1/2}^{n-1/2}\right] \tag{1.18}$$

the a-ϵ scheme shares with the a scheme the same first (principal) constituent equation.



For any (j, n), let



$$u'^{n}_{j\pm 1/2} \stackrel{\text{def}}{=} u^{n-1/2}_{j\pm 1/2} + (\Delta t/2)(u_t)^{n-1/2}_{j\pm 1/2} \tag{1.24}$$

Using $(u_t)^n_j = -a(u_x)^n_j$ and $\nu = a\Delta t/\Delta x$, Eq. (1.24) $\Rightarrow$

$$u'^{n}_{j\pm 1/2} = \left[u - 2\nu\, u_x^+\right]^{n-1/2}_{j\pm 1/2} \tag{1.25}$$

Because $u'^{n}_{j\pm 1/2}$ is a first-order Taylor's approximation of u at $(j \pm 1/2, n)$. Thus

$$(u_x^{c+})^n_j \stackrel{\text{def}}{=} \frac{u'^{n}_{j+1/2} - u'^{n}_{j-1/2}}{4} = \frac{\Delta x}{4}\left(\frac{u'^{n}_{j+1/2} - u'^{n}_{j-1/2}}{\Delta x}\right) \tag{1.26}$$

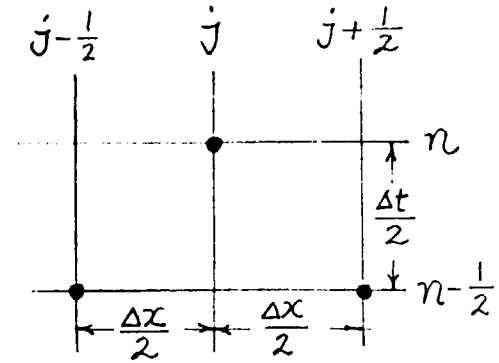
is a central-difference approximation of $\partial u/\partial x$ at (j, n), normalized by the factor $\Delta x/4$. The a-ϵ scheme is formed by Eq. (1.18) and

$$(u_x^+)^n_j = (u_x^{a+})^n_j + 2\epsilon(u_x^{c+} - u_x^{a+})^n_j \tag{1.27}$$

where ϵ is a real number.

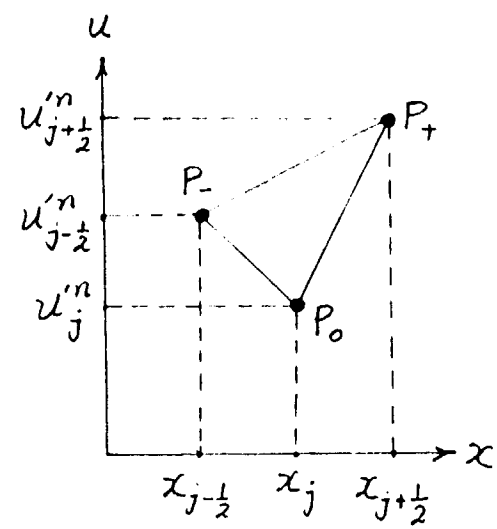
(a) The superscript symbol "c" is introduced to indicate the central-difference nature of $(u_x^{c+})^n_j$.
(b) The a-ϵ scheme is stable if $0 \le \epsilon \le 1$ and $0 \le |\nu| \le 1$;
(c) The numerical dissipation associated with the a-ϵ scheme generally increases with ϵ.
(d) The type of Numerical dissipation introduced by the second term in Eq. (1.24) is effective in damping out numerical instabilities that arise from the smooth region of a solution. But it is less effective in suppressing numerical wiggles that often occur near a discontinuity.

The a-ϵ-α-β Scheme



Let

$$(u_{x+}^{c+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}(u_{j+1/2}'^n - u_j^n) = \frac{\Delta x}{4}\left(\frac{u_{j+1/2}'^n - u_j^n}{\Delta x/2}\right) \tag{1.28}$$



$$(u_{x-}^{c+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}(u_j^n - u_{j-1/2}'^n) = \frac{\Delta x}{4}\left(\frac{u_j^n - u_{j-1/2}'^n}{\Delta x/2}\right) \tag{1.29}$$

Then

$$(u_x^{c+})_j^n = \frac{1}{2}\left[(u_{x+}^{c+})_j^n + (u_{x-}^{c+})_j^n\right] \tag{1.30}$$

Let the function W_o be defined by (i) $W_o(0,0,\alpha) = 0$ and (ii)

$$W_o(x_-, x_+; \alpha) = \frac{|x_+|^\alpha x_- + |x_-|^\alpha x_+}{|x_+|^\alpha + |x_-|^\alpha}, \qquad (|x_+| + |x_-| > 0) \tag{1.31}$$

where x_+, x_- and $\alpha \geq 0$ are real variables. Note that $W_o(x_-, x_+; \alpha)$, a nonlinear weighted average of x_- and x_+, becomes their simple average if $\alpha = 0$ or $|x_-| = |x_+|$.

The a-ϵ-α-β scheme is formed by Eq. (1.18) and

$$(u_x^+)_j^n = (u_x^{a+})_j^n + 2\epsilon(u_x^{c+} - u_x^{a+})_j^n + \beta(u_x^{w+} - u_x^{c+})_j^n \tag{1.32}$$

Here

$$(u_x^{w+})_j^n \stackrel{\text{def}}{=} W_o\left((u_{x+}^{c+})_j^n, (u_{x-}^{c+})_j^n; \alpha\right) \tag{1.33}$$

Note that:

(a) $(u^{w+})_j^n = (u_x^{c+})_j^n$ if $\alpha = 0$ or $(u_{x+}^{c+})_j^n = (u_{x-}^{c+})_j^n$.

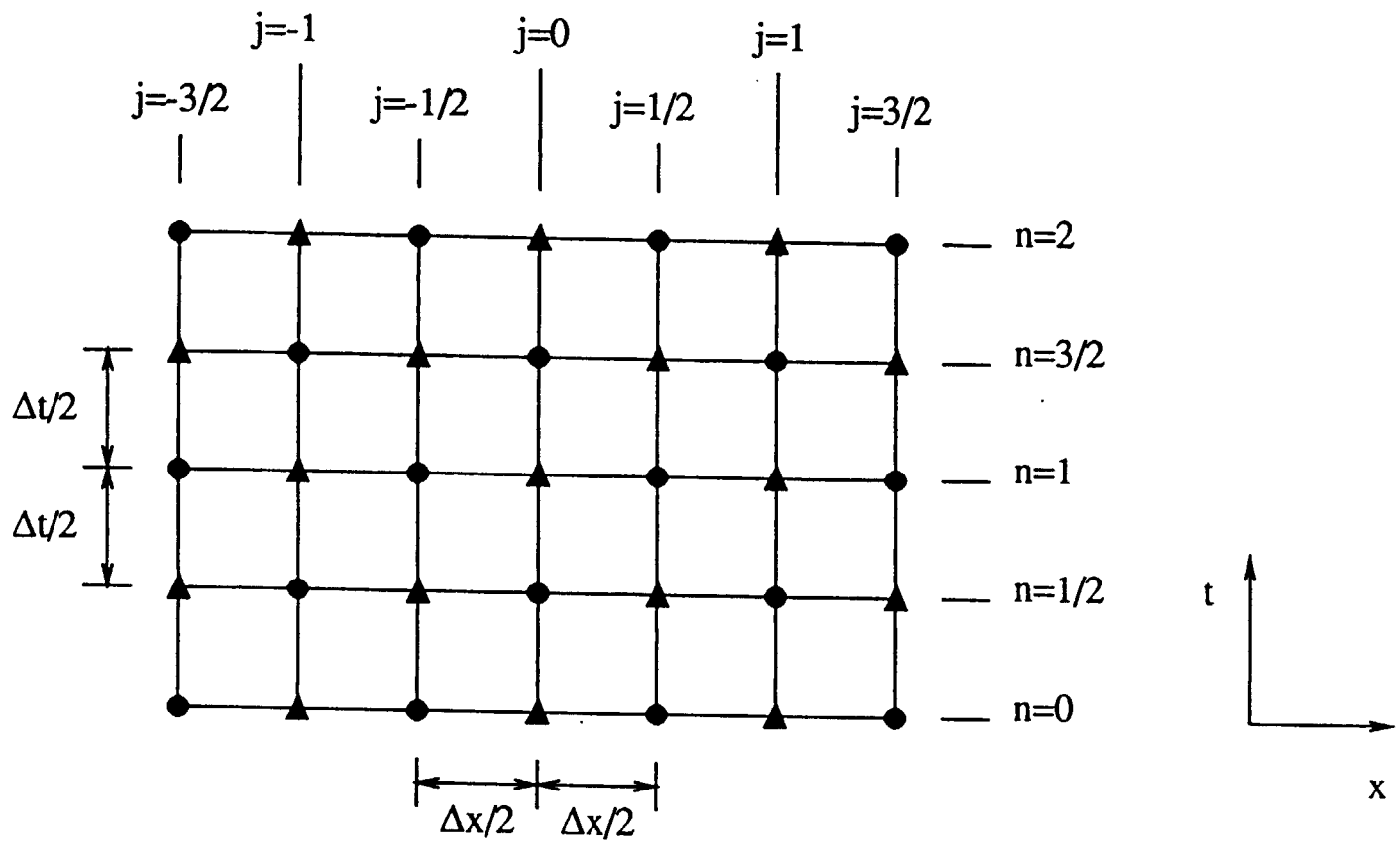
(b) The a-ϵ-α-β scheme generally is stable if $0 \le \epsilon \le 1$, $\beta \ge 0$ and $\alpha \ge 0$.

(c) We have

$$(u_x^+)_j^n = \begin{cases} (u_x^{a+})_j^n & \text{if } \epsilon = \beta = 0 \\ (u_x^{c+})_j^n & \text{if } \epsilon = 1/2 \text{ and } \beta = 0 \\ (u^{w+})_j^n & \text{if } \epsilon = 1/2 \text{ and } \beta = 1 \end{cases} \tag{1.34}$$

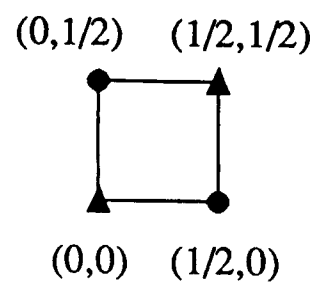
(a) In the smooth region of the solution, $(u_{x+}^{c+})_j^n$ and $(u_{x-}^{c+})_j^n$ are more or less equal and, as a result, $(u^{w+} - u_x^{c+})_j^n$ becomes negligible.

(b) Generally, with the choice of $\alpha = 1$ or $\alpha = 2$, the numerical dissipation introduced is sufficient to suppress numerical wiggles.



(a). — The dual space-time mesh

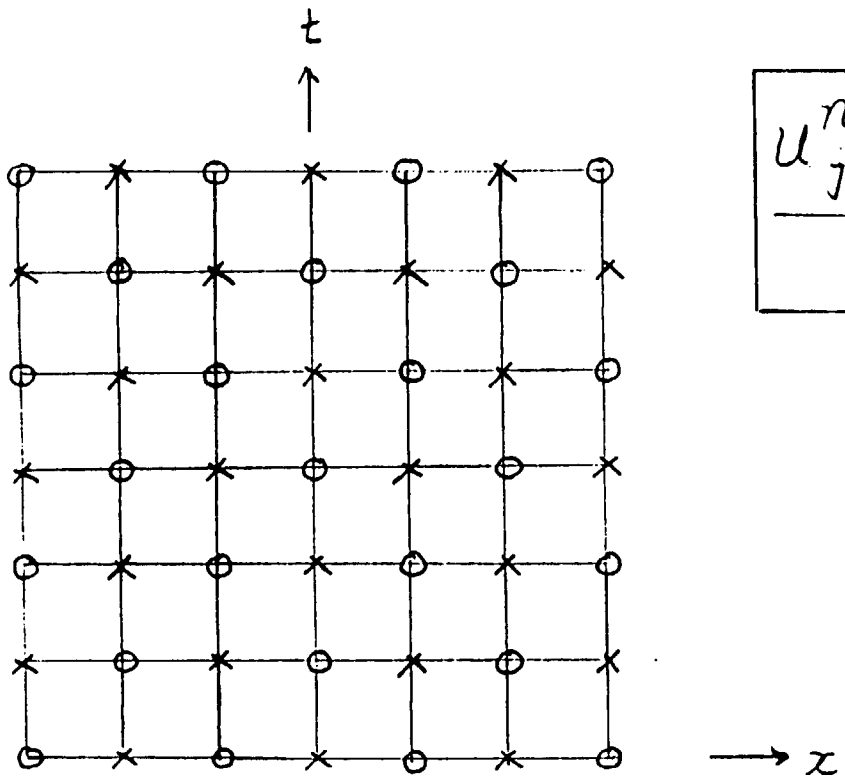
Figure 22. — Concept of dual space-time meshes



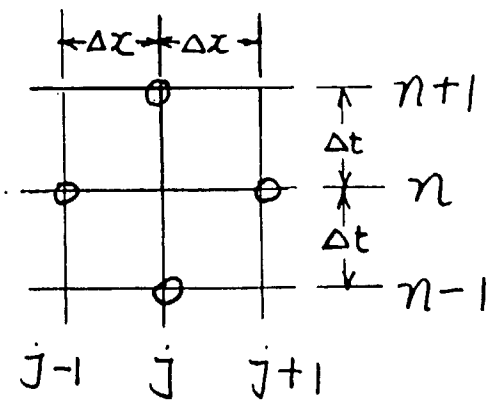
(b). — A rectangular space-time region shared by $CE_-(1/2,1/2)$ and $CE_+(0,1/2)$

Figure 22. (concluded)

The Leap-frog scheme



$$\boxed{\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + a\,\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x} = 0}$$



The Euler a Scheme

Consider a dimensionless form of the 1-D unsteady Euler equations of a perfect gas. Let ρ, v, p, and γ be the mass density, velocity, static pressure, and constant specific heat ratio, respectively. Let

$$u_1 = \rho, \qquad u_2 = \rho v, \qquad u_3 = p/(\gamma - 1) + (1/2)\rho v^2, \tag{2.1}$$

$$f_1 = u_2, \tag{2.2}$$

$$f_2 = (\gamma - 1)u_3 + (1/2)(3 - \gamma)(u_2)^2/u_1, \tag{2.3}$$

$$f_3 = \gamma u_2 u_3/u_1 - (1/2)(\gamma - 1)(u_2)^3/(u_1)^2. \tag{2.4}$$

Then the Euler equations can be expressed as

$$\frac{\partial u_m}{\partial t} + \frac{\partial f_m}{\partial x} = 0, \qquad m = 1, 2, 3. \tag{2.5}$$

The integral form of Eq. (2.5) in space-time E_2 is

$$\oint_{S(V)} \vec{h}_m \cdot d\vec{s} = 0, \qquad m = 1, 2, 3, \tag{2.6}$$

where $\vec{h}_m = (f_m, u_m)$, $m = 1, 2, 3$, are the space-time mass, momentum, and energy current density vectors, respectively.

Let (i)

$$f_{m,k} \stackrel{\text{def}}{=} \partial f_m / \partial u_k, \qquad m, k = 1, 2, 3 \tag{2.7}$$

and (ii) $(u_m)_j^n$ be the numerical version of u_m at any (j, n).

Because f_m and $f_{m,k}$ are functions of u_m, for any (j, n), we can and will define $(f_m)_j^n$ and $(f_{m,k})_j^n$ to be the values of f_m and $f_{m,k}$, respectively, when u_m, $m = 1, 2, 3$, respectively, assume the values of $(u_m)_j^n$, $m = 1, 2, 3$. Furthermore, because f_m, $m = 1, 2, 3$, are homogeneous functions of degree 1 in the variables u_m, $m = 1, 2, 3$, we have (see p. 11 of *Methods of Mathematical Physics* Vol. II, by R. Courant and D. Hilbert, interscience, 1962)

$$(f_m)_j^n = \sum_{k=1}^{3} (f_{m,k})_j^n (u_k)_j^n \tag{2.8}$$

Note that Eq. (2.8) is not essential in the development of the 1D CE/SE Euler solvers. However, in some instances, it is used to recast some equations into more convenient forms.

For any $(x,t) \in \mathrm{SE}(j,n)$, $u_m(x,t)$, $f_m(x,t)$ and $\vec{h}_m(x,t)$, respectively, are approximated by

$$u_m^*(x,t\,;j,n) \stackrel{\text{def}}{=} (u_m)_j^n + (u_{mx})_j^n(x-x_j) + (u_{mt})_j^n(t-t^n) \tag{2.9}$$

$$f_m^*(x,t\,;j,n) = (f_m)_j^n + (f_{mx})_j^n(x-x_j) + (f_{mt})_j^n(t-t^n) \tag{2.10}$$

$$\vec{h}_m^*(x,t\,;j,n) = (f_m^*(x,t\,;j,n),\ u_m^*(x,t\,;j,n)) \tag{2.11}$$

Here we assume that

$$(f_{mx})_j^n \stackrel{\text{def}}{=} \sum_{k=1}^{3}(f_{m,k})_j^n(u_{kx})_j^n, \quad \text{and} \quad (f_{mt})_j^n \stackrel{\text{def}}{=} \sum_{k=1}^{3}(f_{m,k})_j^n(u_{kt})_j^n \tag{2.12}$$

$$(u_{mt})_j^n = -(f_{mx})_j^n \tag{2.13}$$

Note that (i) the two equations in Eq. (2.12) are the numerical analogues of the analytical relations

$$\frac{\partial f_m}{\partial x} = \sum_{k=1}^{3}\frac{\partial f_m}{\partial u_k}\frac{\partial u_k}{\partial x}, \quad \text{and} \quad \frac{\partial f_m}{\partial t} = \sum_{k=1}^{3}\frac{\partial f_m}{\partial u_k}\frac{\partial u_k}{\partial t} \tag{2.14}$$

respectively, and (ii) assuming Eqs. (2.9) and (2.10), Eq. (2.13) is equivalent to

$$\frac{\partial u_m^*(x,t\,;j,n)}{\partial t} + \frac{\partial f_m^*(x,t\,;j,n)}{\partial x} = 0 \tag{2.15}$$

Note that, by their definitions,

(i) $(f_m)_j^n$ and $(f_{m,k})_j^n$, $m = 1, 2, 3$, are functions of $(u_m)_j^n$, $m = 1, 2, 3$,

(ii) $(f_{mx})_j^n$, $m = 1, 2, 3$, are functions of $(u_m)_j^n$ and $(u_{mx})_j^n$, $m = 1, 2, 3$, and

(iii) $(f_{mt})_j^n$, $m = 1, 2, 3$, are functions of $(u_m)_j^n$ and $(u_{mt})_j^n$, $m = 1, 2, 3$.

Furthermore, item (ii) and Eq. (2.13) imply that $(u_{mt})_j^n$, $m = 1, 2, 3$, are also functions of $(u_m)_j^n$ and $(u_{mx})_j^n$, $m = 1, 2, 3$. As a result, *the coefficients on the right sides of Eqs. (2.9) and (2.10) are functions of* $(u_m)_j^n$ *and* $(u_{mx})_j^n$, $m = 1, 2, 3$.

For all (j, n), we assume that

$$\oint_{S(CE_{\pm}(j,n))} \vec{h}_m^* \cdot d\vec{s} = 0, \qquad m = 1, 2, 3 \tag{2.16}$$

Using Eqs. (2.9)–(2.11), Eq. (2.16) implies that

$$\begin{aligned} &(u_m)_j^n - (u_m)_{j\pm 1/2}^{n-1/2} \pm \frac{\Delta x}{4}\left[(u_{mx})_{j\pm 1/2}^{n-1/2} + (u_{mx})_j^n\right] \\ &\pm \frac{\Delta t}{\Delta x}\left[(f_m)_{j\pm 1/2}^{n-1/2} - (f_m)_j^n\right] \pm \frac{(\Delta t)^2}{4\Delta x}\left[(f_{mt})_{j\pm 1/2}^{n-1/2} + (f_{mt})_j^n\right] = 0. \end{aligned} \tag{2.17}$$

For each mesh point (j, n), let (i)

$$(2.18) \qquad (u^+_{mx})^n_j \stackrel{\text{def}}{=} \frac{\Delta x}{4}(u_{mx})^n_j, \qquad m = 1, 2, 3$$

(ii) $\vec{u}^n_j$ and $(\vec{u}^+_x)^n_j$, respectively, be the 3×1 column matrices formed by $(u_m)^n_j$ and $(u^+_{mx})^n_j$, $m = 1, 2, 3$,

and

(iii) $(F^+)^n_j$ be the 3×3 matrix formed by $(\Delta t/\Delta x)(f_{m,k})^n_j$, $m, k = 1, 2, 3$.

Then with the aid of Eqs. (2.8), (2.12) and (2.13), one can rewrite Eq. (2.17) as a pair of matrix equations, i.e., for any mesh point (j, n),

$$(2.19) \qquad \left[(I - F^+)\vec{u} + \left(I - (F^+)^2\right)\vec{u}^+_x\right]^n_j = \left[(I - F^+)\vec{u} - \left(I - (F^+)^2\right)\vec{u}^+_x\right]^{n-1/2}_{j+1/2}$$

and

$$(2.20) \qquad \left[(I + F^+)\vec{u} - \left(I - (F^+)^2\right)\vec{u}^+_x\right]^n_j = \left[(I + F^+)\vec{u} + \left(I - (F^+)^2\right)\vec{u}^+_x\right]^{n-1/2}_{j-1/2}$$

where I is the 3×3 identity matrix.

Note that the flux conservation conditions related to the a scheme, i.e.,

$$[(1-\nu)u+(1-\nu^2)u_x^+]_j^n = [(1-\nu)u-(1-\nu^2)u_x^+]_{j+1/2}^{n-1/2} \tag{1.13}$$

and

$$[(1+\nu)u-(1-\nu^2)u_x^+]_j^n = [(1+\nu)u+(1-\nu^2)u_x^+]_{j-1/2}^{n-1/2} \tag{1.14}$$

share the same algebraic structure with their Euler counterparts, i.e.,

$$[(I-F^+)\vec{u}+(I-(F^+)^2)\,\vec{u}_x^+]_j^n = [(I-F^+)\vec{u}-(I-(F^+)^2)\,\vec{u}_x^+]_{j+1/2}^{n-1/2} \tag{2.19}$$

and

$$[(I+F^+)\vec{u}-(I-(F^+)^2)\,\vec{u}_x^+]_j^n = [(I+F^+)\vec{u}+(I-(F^+)^2)\,\vec{u}_x^+]_{j-1/2}^{n-1/2} \tag{2.20}$$

In fact, *the former pair will become the latter pair if the symbols* 1, ν, u *and* u_x^+ *are replaced by* I, F^+, $\vec{u}$ *and* $\vec{u}_x^+$, *respectively*. As a result, Eqs. (2.19) and (2.20) will be solved by a procedure similar to that used earlier to extract Eqs. (1.18) and (1.19) from Eqs. (1.13) and (1.14).

However, because

(i) matrix multiplication is not commutative,

and

(ii) the matrix $(F^+)_j^n$ is a function of $(u_m)_j^n$, $m=1,2,3$, while ν is a simple constant,

as will be shown shortly, the algebraic structure of the solution to Eqs. (2.19) and (2.20) is more complex than that of Eqs. (1.13) and (1.14).

The addition of Eqs. (2.19) and (2.20) $\Rightarrow$

$$\vec{u}_j^n = \frac{1}{2}\left\{\left[(I - F^+)\vec{s}_+\right]_{j+1/2}^{n-1/2} + \left[(I + F^+)\vec{s}_-\right]_{j-1/2}^{n-1/2}\right\} \tag{2.21}$$

where

$$(\vec{s}_+)_{j+1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[\vec{u} - (I + F^+)\vec{u}_x^+\right]_{j+1/2}^{n-1/2} \tag{2.22}$$

and

$$(\vec{s}_-)_{j-1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[\vec{u} + (I - F^+)\vec{u}_x^+\right]_{j-1/2}^{n-1/2} \tag{2.23}$$

Note that Eqs. (2.21)–(2.23) are the Euler counterparts of

$$u_j^n = \frac{1}{2}\left[(1-\nu)(s_+)_{j+1/2}^{n-1/2} + (1+\nu)(s_-)_{j-1/2}^{n-1/2}\right] \tag{1.18}$$

$$(s_+)_{j+1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[u - (1+\nu)u_x^+\right]_{j+1/2}^{n-1/2} \tag{1.15}$$

and

$$(s_-)_{j-1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[u + (1-\nu)u_x^+\right]_{j-1/2}^{n-1/2} \tag{1.16}$$

respectively.

To obtain the second part of the solution to

$$[(I - F^+)\vec{u} + (I - (F^+)^2)\,\vec{u}_x^+]_j^n = [(I - F^+)\vec{u} - (I - (F^+)^2)\,\vec{u}_x^+]_{j+1/2}^{n-1/2} \tag{2.19}$$

and

$$[(I + F^+)\vec{u} - (I - (F^+)^2)\,\vec{u}_x^+]_j^n = [(I + F^+)\vec{u} + (I - (F^+)^2)\,\vec{u}_x^+]_{j-1/2}^{n-1/2} \tag{2.20}$$

existence of the inverses of both $[I \pm (F^+)]_j^n$ must be assumed. Let Eqs. (2.19) and (2.20) be multiplied from the left by

$$[(I - F^+)_j^n]^{-1} \quad \text{and} \quad [(I + F^+)_j^n]^{-1}$$

respectively. Let the resulting expressions be subtracted from each other. Then, because

$$[(I - F^+)(I + F^+)]_j^n = [(I + F^+)(I - F^+)]_j^n = [I - (F^+)^2]_j^n \tag{2.24}$$

one obtains

$$(\vec{u}_x^+)_j^n = (\vec{u}_x^{a+})_j^n \tag{2.25}$$

where

$$(\vec{u}_x^{a+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}(\vec{S}_+ - \vec{S}_-)_j^n \tag{2.26}$$

$$(\vec{S}_+)_j^n \stackrel{\text{def}}{=} [(I - F^+)_j^n]^{-1}\,[(I - F^+)\vec{u} - (I - (F^+)^2)\,\vec{u}_x^+]_{j+1/2}^{n-1/2} \tag{2.27}$$

$$(\vec{S}_-)_j^n \stackrel{\text{def}}{=} [(I + F^+)_j^n]^{-1}\,[(I + F^+)\vec{u} + (I - (F^+)^2)\,\vec{u}_x^+]_{j-1/2}^{n-1/2} \tag{2.28}$$

Note that

$$(\vec{u}_x^{+})_j^n = (\vec{u}_x^{a+})_j^n \tag{2.25}$$

$$(\vec{u}_x^{a+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}(\vec{S}_+ - \vec{S}_-)_j^n \tag{2.26}$$

$$(\vec{S}_+)_j^n \stackrel{\text{def}}{=} \left[(I - F^+)_j^n\right]^{-1} \left[(I - F^+)\vec{u} - \left(I - (F^+)^2\right) \vec{u}_x^{+}\right]_{j+1/2}^{n-1/2} \tag{2.27}$$

and

$$(\vec{S}_-)_j^n \stackrel{\text{def}}{=} \left[(I + F^+)_j^n\right]^{-1} \left[(I + F^+)\vec{u} + \left(I - (F^+)^2\right) \vec{u}_x^{+}\right]_{j-1/2}^{n-1/2} \tag{2.28}$$

respectively, are the Euler counterparts of

$$(u_x^{+})_j^n = (u_x^{a+})_j^n \tag{1.19}$$

$$(u_x^{a+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}\left[(s_+)_{j+1/2}^{n-1/2} - (s_-)_{j-1/2}^{n-1/2}\right] \tag{1.17}$$

$$(s_+)_{j+1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[u - (1+\nu)u_x^{+}\right]_{j+1/2}^{n-1/2} \tag{1.15}$$

and

$$(s_-)_{j-1/2}^{n-1/2} \stackrel{\text{def}}{=} \left[u + (1-\nu)u_x^{+}\right]_{j-1/2}^{n-1/2} \tag{1.16}$$

Let the marching variables at the $(n-1/2)$th time level be given. Then $\vec{u}_j^n$ can be evaluated using

$$\vec{u}_j^n = \frac{1}{2}\left\{\left[(I-F^+)\vec{s}_+\right]_{j+1/2}^{n-1/2} + \left[(I+F^+)\vec{s}_-\right]_{j-1/2}^{n-1/2}\right\} \tag{2.21}$$

Because $[I \pm F^+]_j^n$ is a function of $\vec{u}_j^n$, it follows that

$$(\vec{S}_+)_j^n \stackrel{\text{def}}{=} \left[(I-F^+)_j^n\right]^{-1}\left[(I-F^+)\vec{u} - \left(I-(F^+)^2\right)\vec{u}_x^+\right]_{j+1/2}^{n-1/2} \tag{2.27}$$

$$(\vec{S}_-)_j^n \stackrel{\text{def}}{=} \left[(I+F^+)_j^n\right]^{-1}\left[(I+F^+)\vec{u} + \left(I-(F^+)^2\right)\vec{u}_x^+\right]_{j-1/2}^{n-1/2} \tag{2.28}$$

and

$$(\vec{u}_x^{a+})_j^n \stackrel{\text{def}}{=} \frac{1}{2}(\vec{S}_+ - \vec{S}_-)_j^n \tag{2.26}$$

can also be evaluated. Thus Eq. (2.21) and

$$(\vec{u}_x^+)_j^n = (\vec{u}_x^{a+})_j^n \tag{2.25}$$

form an explicit marching scheme—the Euler a scheme.

An Existence Theorem

Let

(i) v and c be the fluid speed and sonic speed, respectively (v and c are known functions of u_m, $m = 1, 2, 3$ [1]).

(ii) v_j^n and c_j^n, respectively, denote the values of v and c when u_m, $m = 1, 2, 3$, respectively, assume the values of $(u_m)_j^n$, $m = 1, 2, 3$.

and (iii)

$$(\nu_1)_j^n \stackrel{\text{def}}{=} \frac{\Delta t}{\Delta x}(v_j^n - c_j^n), \quad (\nu_2)_j^n \stackrel{\text{def}}{=} \frac{\Delta t}{\Delta x} v_j^n, \quad (\nu_3)_j^n \stackrel{\text{def}}{=} \frac{\Delta t}{\Delta x}(v_j^n + c_j^n) \tag{2.29}$$

Then it is shown in [2] that

$$\left[(I - F^+)_j^n\right]^{-1} \quad \text{and} \quad \left[(I + F^+)_j^n\right]^{-1}$$

exist if and only if

$$[(\nu_\ell)_j^n]^2 \neq 1, \qquad \ell = 1, 2, 3 \tag{2.30}$$

Obviously that Eq. (2.30) must be true if, for all (j, n), the local Courant number

$$\nu_j^n \stackrel{\text{def}}{=} \max\{|(\nu_1)_j^n|, |(\nu_2)_j^n|, |(\nu_3)_j^n|\} < 1 \tag{2.31}$$

The Euler a-ϵ Scheme

For the Euler a-ϵ scheme, the less stringent conservation conditions

$$\oint_{S(CE(j,n))} \vec{h}_m^* \cdot d\vec{s} = 0, \qquad m = 1,2,3 \tag{2.32}$$

is imposed at each (j,n). It can be shown that Eq. (2.32) $\Leftrightarrow$ Eq. (2.21).

For any (j,n), let

$$\vec{u}_{j\pm1/2}^{\prime\, n} \stackrel{\text{def}}{=} \vec{u}_{j\pm1/2}^{n-1/2} + (\Delta t/2)(\vec{u}_t)_{j\pm1/2}^{n-1/2} = (\vec{u} - 2F^+\vec{u}_x^+)_{j\pm1/2}^{n-1/2} \tag{2.33}$$

where $(\vec{u}_t)_{j\pm1/2}^{n-1/2}$ is the column matrix formed by $(u_{mt})_{j\pm1/2}^{n-1/2}$, $m = 1,2,3$. Also let

$$(\vec{u}_x^{c+})_j^n \stackrel{\text{def}}{=} \frac{\vec{u}_{j+1/2}^{\prime\, n} - \vec{u}_{j-1/2}^{\prime\, n}}{4} = \frac{\Delta x}{4}\left(\frac{\vec{u}_{j+1/2}^{\prime\, n} - \vec{u}_{j-1/2}^{\prime\, n}}{\Delta x}\right) \tag{2.34}$$

Then the Euler a-ϵ scheme is formed by Eq. (2.21) and

$$(\vec{u}_x^+)_j^n = (\vec{u}_x^{a+})_j^n + 2\epsilon(\vec{u}_x^{c+} - \vec{u}_x^{a+})_j^n \tag{2.35}$$

where ϵ is a real number.

Note that it has been shown numerically that (i) the Euler a-ϵ scheme generally is stable if

$$0 \le \epsilon \le 1, \qquad \text{and} \qquad \nu_j^n < 1 \quad \text{for all} \quad (j,n) \tag{2.36}$$

and (ii) the numerical dissipation associated with the scheme increases as the value of ϵ increases.

The Euler a-ϵ-α-β Scheme

For any (j,n) and any $m = 1,2,3$, let

$$(u^{c+}_{mx+})^n_j \stackrel{\text{def}}{=} \frac{1}{2}((u'_m)^n_{j+1/2} - (u_m)^n_j) = \frac{\Delta x}{4}\left(\frac{(u'_m)^n_{j+1/2} - (u_m)^n_j}{\Delta x/2}\right) \tag{2.37}$$

and

$$(u^{c+}_{mx-})^n_j \stackrel{\text{def}}{=} \frac{1}{2}((u_m)^n_j - (u'_m)^n_{j-1/2}) = \frac{\Delta x}{4}\left(\frac{(u_m)^n_j - (u'_m)^n_{j-1/2}}{\Delta x/2}\right) \tag{2.38}$$

with $(u'_m)^n_{j\pm 1/2}$ and $(u_m)^n_j$ being the mth components of $\vec{u}'^{\,n}_{j\pm 1/2}$ and $\vec{u}^n_j$, respectively. Let $(u^{c+}_{mx})^n_j$ be the mth component of $(\vec{u}^{c+}_x)^n_j$. Then it can be shown that

$$(u^{c+}_{mx})^n_j = \frac{1}{2}\left[(u^{c+}_{mx+})^n_j + (u^{c+}_{mx-})^n_j\right] \tag{2.39}$$

Let $(\vec{u}^{w+}_x)^n_j$ be the 3×1 column matrix formed by

$$W_o\left((u^{c+}_{mx+})^n_j, (u^{c+}_{mx-})^n_j; \alpha\right), \qquad m = 1,2,3$$

The Euler a-ϵ-α-β scheme is formed by Eq. (2.21) and

$$(\vec{u}^{+}_x)^n_j = (\vec{u}^{a+}_x)^n_j + 2\epsilon(\vec{u}^{c+}_x - \vec{u}^{a+}_x)^n_j + \beta(\vec{u}^{w+}_x - \vec{u}^{c+}_x)^n_j \tag{2.40}$$

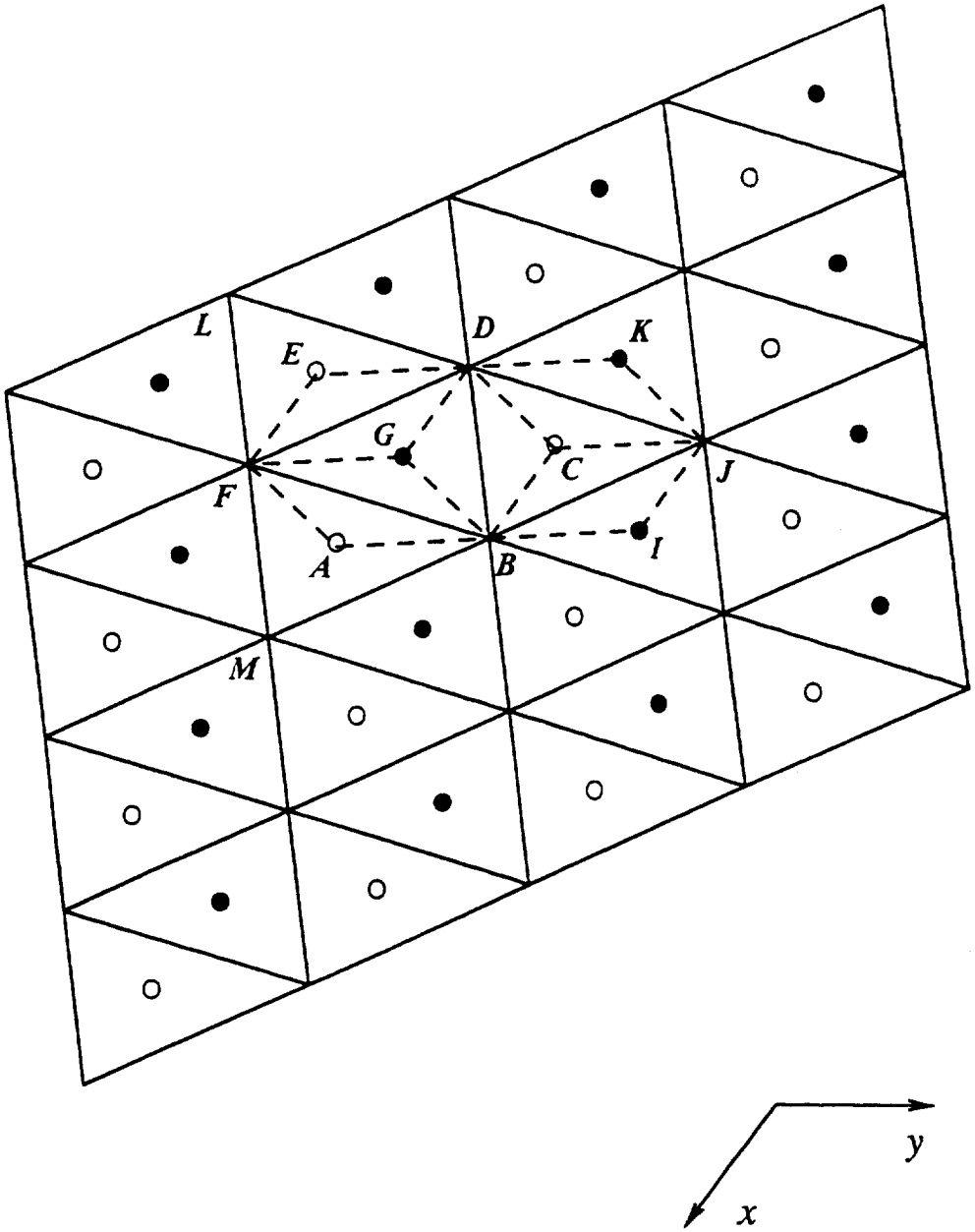


Figure 5. — A spatial domain formed from congruent triangles, showing the spatial projections of the mesh points

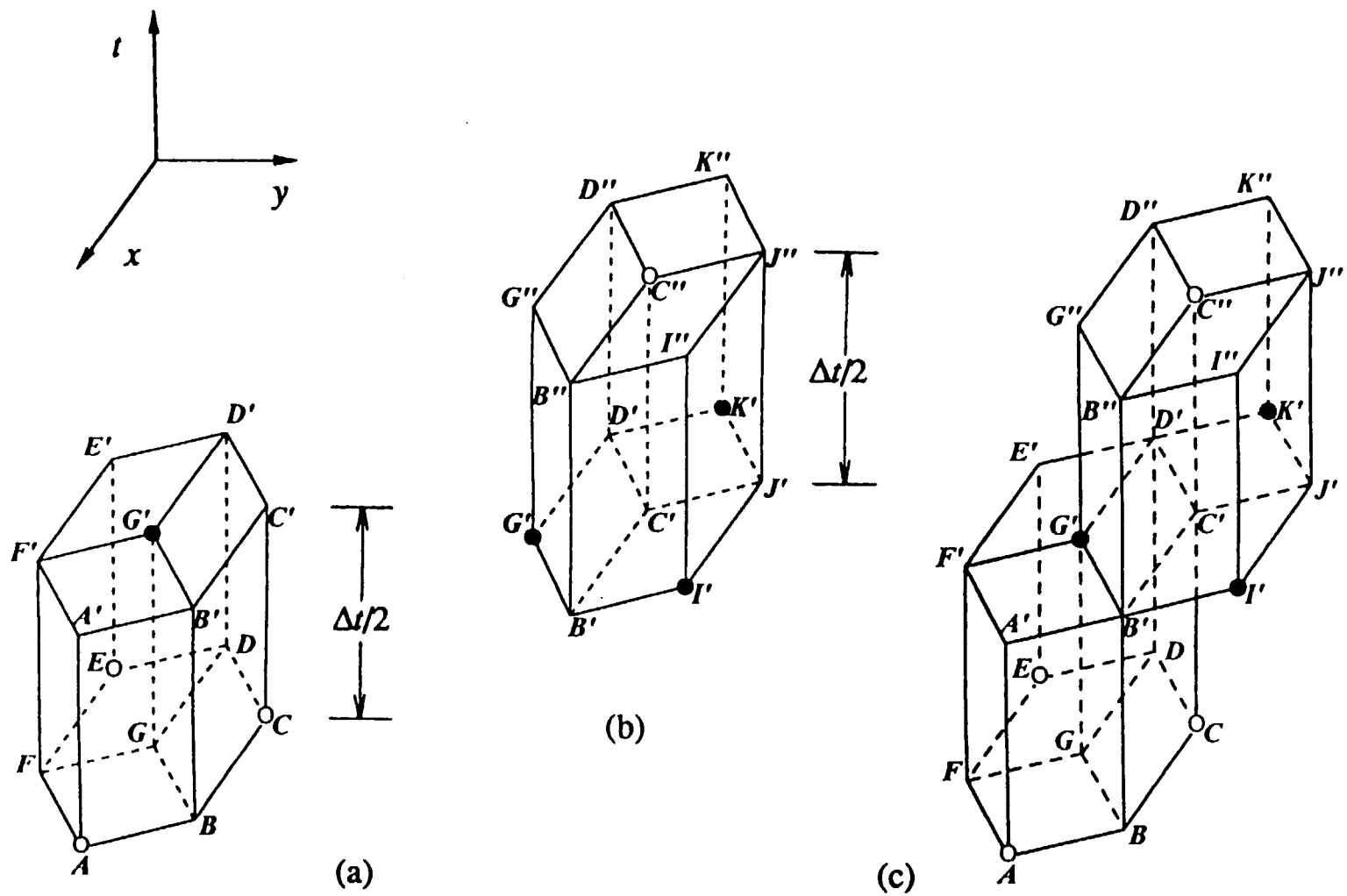


Figure 6. — (a) The CEs associated with G'. (b) the CEs associated with C''
(c) The relative positions of the CEs of successive time steps

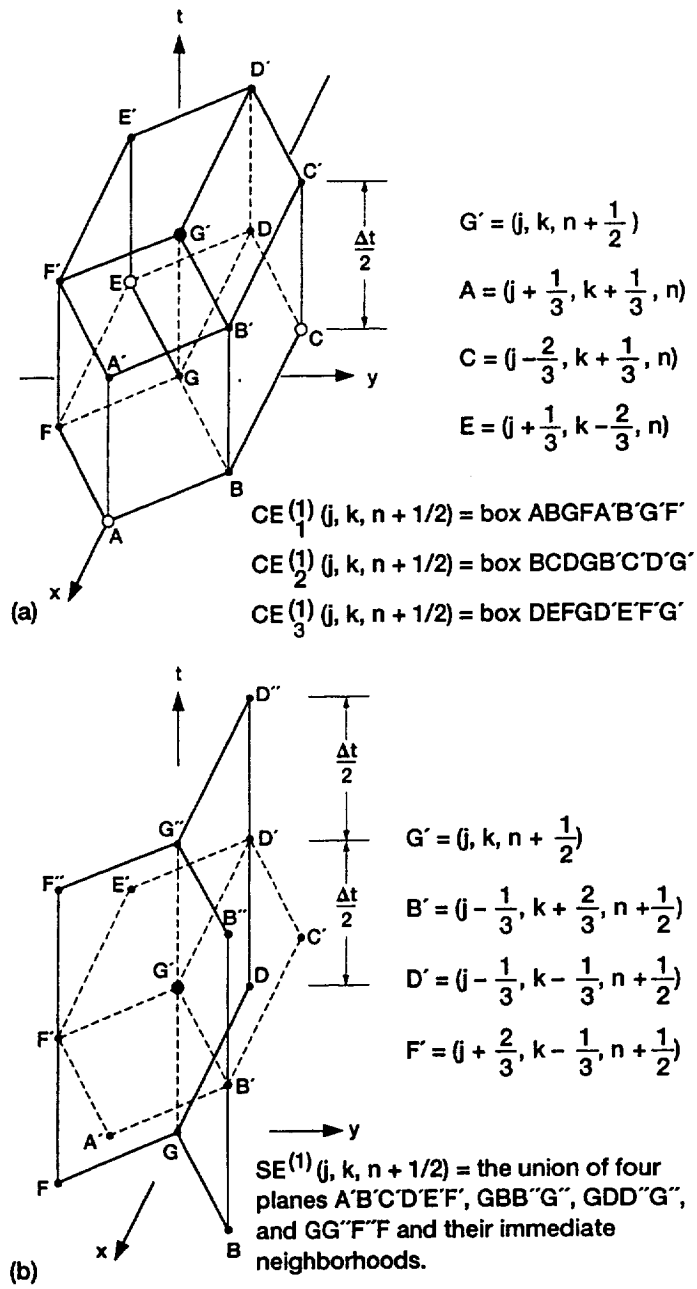
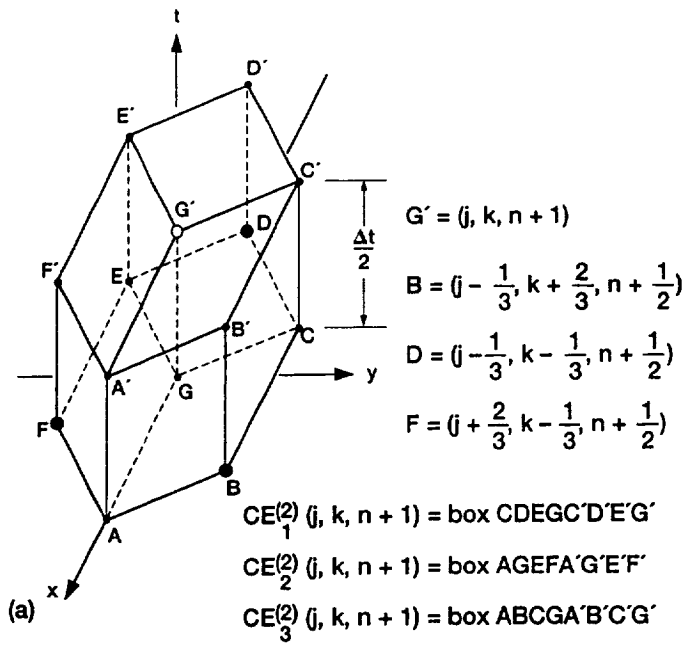


Figure 11.—(a) Conservation elements $CE^{(1)}_{\ell}(j, k, n + 1/2)$, $\ell = 1, 2, 3$, and $j, k, n = 0, \pm 1, \pm 2, \cdots$. (b) Solution elements $SE^{(1)}(j, k, n + 1/2)$, $j, k, n = 0, \pm 1, \pm 2, \cdots$).



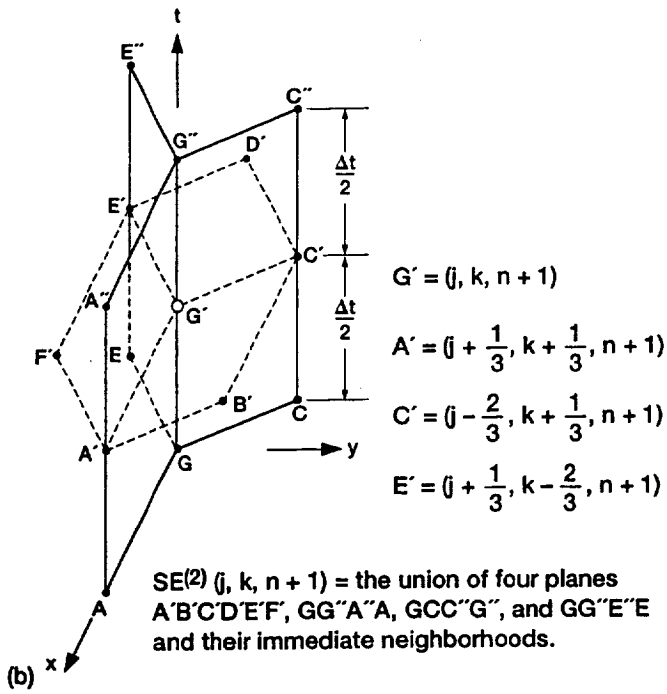


Figure 12.—(a) Conservation elements $CE_\ell^{(2)}$ (j, k, n + 1), ℓ = 1, 2, 3, j, k, = 1/3, 1/3 ±1, 1/3 ±2, ···, and n = 0, ±1, ±2, ···. (b) Solution elements $SE^{(2)}$ (j, k, n + 1), j, k = 1/3, 1/3 ±1, 1/3 ±2, ···, and n = 0, ±1, ±2, ···).

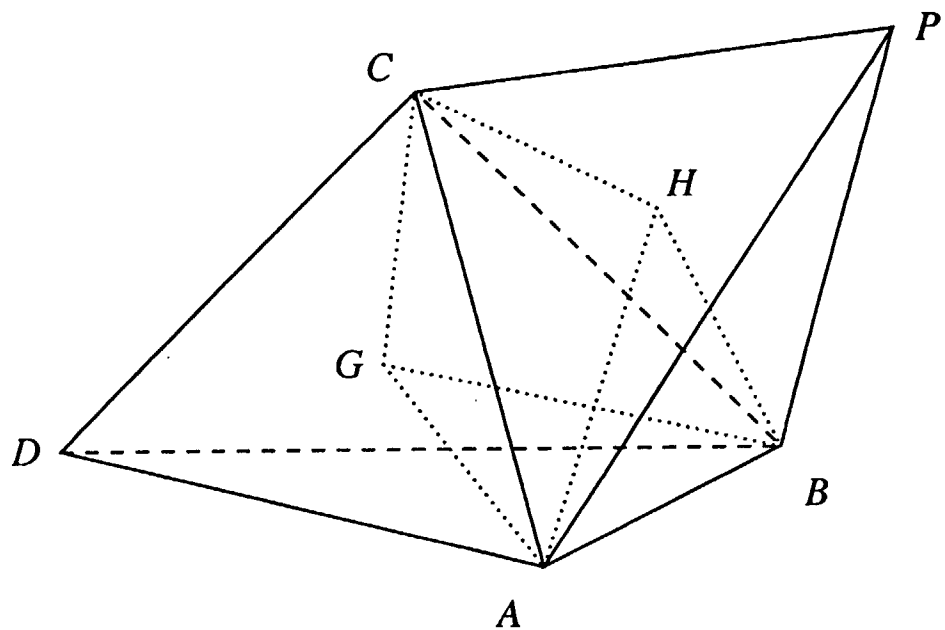


Figure 21. — Spatial projection of part of a 3D space-time mesh, showing the construction of a CE

Future Research Directions

- LES and DNS
- Incompressible flows
- Computational electromagnetics
- Multiphase flows

Conclusions

- Solid foundation in physics
- Simple and coherent numerical modeling
- Accuracy
- Robustness

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