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TURBULENT FRICTION IN THE BOUNDARY LAYER OF A FLAT PLATE IN A TWO-DIMENSIONAL COMPRESSIBLE FLOW AT HIGH SPEEDS

By F. Frankl and V. Voishel

Central Aero-Hydrodynamical Institute

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TURBULENT FRICTION IN THE BOUNDARY LAYER OF A FLAT PLATE IN A TWO-DIMENSIONAL COMPRESSIBLE FLOW AT HIGH SPEEDS*

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SUMMARY

In the present report an investigation is made on a flat plate in a two-dimensional compressible flow of the effect of compressibility and heating on the turbulent frictional drag coefficient in the boundary layer of an airfoil or wing radiator. The analysis is based on the Prandtl-Kármán theory of the turbulent boundary layer and the Stodola-Crocco theorem on the linear relation between the total energy of the flow and its velocity. (See references 1 and 2.) Formulas are obtained for the velocity distribution and the frictional drag law in a turbulent boundary layer with the compressibility effect and heat transfer taken into account. It is found that with increase of compressibility and temperature at full retardation of the flow (the temperature when the velocity of the flow at a given point is reduced to zero in case of an adiabatic process in the gas) at a constant R_x, the frictional drag coefficient c_f decreases, both of these factors acting in the same sense.

INTRODUCTION

In the present paper an attempt is made to generalize the theory of the Prandtl-Kármán turbulent boundary layer to the case of flow of a compressible gas at high velocities in the presence of a temperature gradient from the wall to the gas. Since the theory itself ("mixing path" theory) is partly empirical, the generalization suggested, which probably contains an arbitrary element, requires experimental confirmation. Moreover some error is

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introduced by certain assumptions that are made to simplify the problem. The principal of these assumptions are the following:

(1) In the fundamental differential equation for the velocity distribution in the boundary layer:

$$\tau = \kappa^2 \rho \left(\frac{\partial u}{\partial y}\right)^4 : \left(\frac{\partial^2 u}{\partial y^2}\right)^2$$

the frictional shear stress τ, having a dependence on y that is not initially known, is to a first approximation equated to its value at the wall although with increasing distance from the wall it actually tends to zero. To increase the accuracy it is possible to proceed as follows: Having found the velocity distribution in the boundary layer by the method given, τ is determined from the dynamic equilibrium taking account of the forces of inertia. From the preceding equation the velocity distribution is determined to a second approximation and so on, the process, however, being very complicated.

(2) In the laminar boundary layer the Prandtl number $Pr = \frac{c_p \mu}{\lambda}$ is equated to unity although for diatomic gases it is actually equal to 0.8. The error arising from this assumption is small, however.

(3) The nondimensional thickness of the laminar sublayer and also the jump in the velocity gradient in passing from laminar to turbulent, which, according to the Kármán theory, are absolute constants, were likewise regarded as constant although they may vary in the present case. The variation cannot, however, be theoretically determined.

(4) The condition of the transition from the pure laminar to the turbulent layer is arbitrary and is introduced only to identify the problem in question. Actually this transition depends on the turbulence of the basic flow. On account of this indeterminateness a certain additive constant enters in the formula for Re as a function of c_f of the order of 10^5 (according to tests on liquids).

The theory proposed gives somewhat more accurate results than the Kármán theory in which the resistance law obtained for small velocities is maintained the same, substituting only for μ and ρ their values at the wall but not taking account of the effect of the compressibility on the velocity profile in the boundary layer. The final answer can only be given, of course, by experiment. Qualitatively both theories arrive at the same result - namely, a decrease in c_f with increasing compressibility or heating of the wall at constant Re.

I. FUNDAMENTAL DIFFERENTIAL EQUATION

INTEGRATION - BOUNDARY CONDITIONS

Assume a thin smooth rectangular plate situated in a plane parallel stream of a compressible gas flowing with large constant velocity past a thin smooth rectangular plate set at zero angle of attack. At the same time the plate is warmed by heat supplied from some heat source and by the friction so that a difference in temperature arises between the plate and the flow and heat is transferred from the plate to the flow.

The investigation is directly concerned with the problem of the effect of large velocities and of heat transfer on the frictional drag in the turbulent boundary layer about a plate on the supposition that this layer is very thin and has a quite definite boundary.

Assume that the x-axis in the flow direction is along the plate, and the y-axis in the perpendicular direction from the boundary layer toward the free flow. The variable velocity within the boundary layer is denoted by u. Moreover, with the usual notation:

ρ density

μ viscosity

τ frictional shear stress

T absolute temperature

c_p specific heat at constant pressure

c_v specific heat at constant volume

$c_p/c_v = k$

The corresponding variables in the free flow will be denoted by

$\bar{\rho}$, $\bar{\mu}$, ..., $\bar{T}$; at the wall by ρ_*, μ_*, ..., T_*.

The pressure p, as usual in the theory of the boundary layer, is assumed as constant.

The nondimensional velocities and the magnitudes characterizing the state of the gas are functions of nondimensional coordinates and therefore depend on certain parameters which are nondimensional combinations of scale ratios, end conditions of the plate, and physical constants of the gas; the shear stress τ and the unit heat flow q and also the magnitudes p, ρ_*, T_* (pressure p assumed constant) giving the state of the gas at the plate serving as end conditions. The density and temperature ρ_* and T_* are taken as scale ratios for ρ and T. The scale of velocity is expressed by the ratio

$$v_* = \sqrt{\frac{\tau}{\rho_*}} \tag{1}$$

and the length scale as the ratio

$$\frac{\mu_*}{\rho_* v_*} = \frac{\mu_*}{\sqrt{\rho_* \tau}} \tag{2}$$

where μ_* is the value of the viscosity coefficient μ at the plate.

With the introduction of these scales the parameters of the solution can only appear as nondimensional combinations of physical constants and end conditions.

In the absence of compressibility and heat transfer such combinations do not exist, as a result of which there is obtained, as is known, a universal velocity distribution not depending on any parameters. For this reason these scales are hereinafter regarded as "universal."

Then

$$\eta = \frac{y}{\left(\frac{\mu_*}{\sqrt{\rho_* T}}\right)} \tag{3}$$

$$\varphi = \frac{u}{v_*} \tag{4}$$

$$\sigma = \frac{\rho}{\rho_*} = \frac{T_*}{T} \tag{5}$$

the square of the Bairstow number:

$$\frac{U^2}{(k - 1)\, Jc_p T_*} = B^2_{a*} \tag{6}$$

where U is the velocity in the free stream,

$$k = \frac{c_p}{c_v} = 1.4 \tag{7}$$

Moreover

$$\alpha = \frac{k - 1}{2} B^2_{a*} \tag{8}$$

$$h = \sqrt{\frac{2}{c_{f*}}} \tag{9}$$

where

$$c_{f*} = \frac{\tau}{\rho_* \frac{U^2}{2}}$$

and

$$1 - \frac{\overline{T}}{T_*} - \alpha = \omega \tag{10}$$

is the nondimensional temperature of the flow brought to rest adiabatically.

It is assumed that in the compressible gas the fundamental differential equation of the turbulence theory of Kármán remains valid; namely (reference 3),

$$\kappa^2 \rho \left(\frac{\partial u}{\partial y}\right)^4 : \left(\frac{\partial^2 u}{\partial y^2}\right)^2 = \tau \tag{11}$$

where κ is an absolute constant equal to 0.4 and τ the frictional shear at a distance y from the plate.

When the coordinate x (the distance from the leading edge of the plate) is sufficiently large, however, the inertia forces may be neglected. Because of this and from the constancy of the pressure p it follows that τ changes slowly with y. In a first approximation its value at the wall is assumed for τ.

Expressed in the universal scales, equation (11) assumes the form

$$\kappa^2 \sigma = \frac{\left(\frac{\partial^2 \varphi}{\partial \eta^2}\right)^2}{\left(\frac{\partial \varphi}{\partial \eta}\right)^4} \tag{12}$$

In order to obtain the differential equation for φ it is only necessary to express σ in terms of φ. This can be done with the aid of the known fact of the linear dependence of the total energy $Jc_pT + u^2/2$ on the velocity u. (See references 1 and 2.)

In nondimensional representation:

$$\frac{1}{\sigma} = 1 - \left(\alpha \frac{\varphi^2}{h^2} + \omega \frac{\varphi}{h}\right) \tag{13}$$

or

$$\frac{1}{\sigma}=\left(1+\frac{\omega^2}{4\alpha}\right)\left[1-\left(\frac{\sqrt{\alpha}\,\frac{\varphi}{h}+\frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1+\frac{\omega^2}{4\alpha}}}\right)^2\right] \tag{13'}$$

Substituting equation (13') in equation (12), extracting the square root of both sides of the equation, and using the minus sign in front of the radical, since the expression on the left is negative, leaves

$$\frac{d\left(\ln\frac{d\varphi}{d\eta}\right)}{d\varphi}=-\frac{\frac{\varkappa}{\sqrt{1+\frac{\omega^2}{4\alpha}}}}{\sqrt{1-\left(\frac{\sqrt{\alpha}\,\frac{\varphi}{h}+\frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1+\frac{\omega^2}{4\alpha}}}\right)^2}}. \tag{14}$$

? appears to replace $\kappa = .4$

Setting

$$\frac{\sqrt{\alpha}\,\frac{\varphi}{h}+\frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1+\frac{\omega^2}{4\alpha}}}=v \tag{15}$$

($v < 1$) and

$$\frac{\varkappa h}{\sqrt{\alpha}}=\lambda, \tag{16}$$

affords

$$d\left(\ln\frac{d\varphi}{d\eta}\right)=-\frac{\lambda\,dv}{\sqrt{1-v^2}}. \tag{17}$$

whence, after integration

$$\ln\frac{d\varphi}{d\eta}=\lambda\cos^{-1}v+\ln C$$

or

$$\frac{d\varphi}{d\eta}=Ce^{\lambda\cos^{-1}v}. \tag{18}$$

At the meeting of the laminar sublayer and the turbulent layer

$$\left(\frac{d\varphi}{d\eta}\right)_{\eta=s}=f \quad [1] \tag{19}$$

where

$$s=11.5\,. \quad [1] \tag{20}$$

The determination of the boundary conditions proceeds from the assumption of linear distribution of u in the laminar sublayer

$$u=\frac{\tau}{\mu_*}y \quad \text{(reference 4)} \tag{21}$$

or

$$\varphi=\eta \tag{21'}$$

and hence

$$\varphi(s)=s. \tag{22}$$

Further, setting

$$\frac{\sqrt{\alpha}\,\frac{s}{h}+\frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1+\frac{\omega^2}{4\alpha}}}=v_1. \tag{23}$$

results in

$$\frac{d\varphi}{d\eta}=fe^{-\lambda(\cos^{-1}v_1-\cos^{-1}v)} \tag{24}$$

A second integration with the given end conditions gives

$$\eta=s+\frac{b}{\lambda^2+1}\left\{(\lambda\sqrt{1-v^2}+v)e^{\lambda(\cos^{-1}v_1-\cos^{-1}v)}-\right.$$
$$\left.-(\lambda\sqrt{1-v_1^2}+v_1)\right\}, \tag{25}$$

where

$$b=\frac{h\sqrt{1+\frac{\omega^2}{4\alpha}}}{f\sqrt{\alpha}}.$$

[1]The magnitudes s and f to a first approximation are assumed constant, their values being taken from Kármán.

Or, if the original magnitudes are used again:

$$\eta = s + \frac{1}{f\left(\varkappa^2 + \frac{\alpha}{h^2}\right)}\left\{\left[\varkappa\sqrt{1-\left(\alpha\frac{\varphi^2}{h^2}+\omega\frac{\varphi}{h}\right)}+\frac{\alpha}{h^2}\varphi+\frac{\omega}{2h}\right]\times\right.$$

$$\times e^{\frac{\varkappa h}{\sqrt{\alpha}}\left[\cos^{-1}\left(\frac{\sqrt{\alpha}\frac{s}{h}+\frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1+\frac{\omega^2}{4\alpha}}}\right)-\cos^{-1}\left(\frac{\sqrt{\alpha}\frac{\varphi}{h}+\frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1+\frac{\omega^2}{4\alpha}}}\right)\right]}-$$

$$\left.-\left[\varkappa\sqrt{1-\left(\alpha\frac{s^2}{h^2}+\omega\frac{s}{h}\right)}+\frac{\alpha}{h^2}s+\frac{\omega}{2h}\right]\right\}. \tag{27}$$

Expanding the expression on the right in $\sqrt{\alpha}$ and $\frac{\omega}{2\sqrt{\alpha}}$ and retaining terms through the second order approximates to

$$u \cong w_1 + \frac{1}{f}\left\{\left[\frac{\alpha}{6z^2}w^3+\left(\frac{\omega}{4z}-\frac{\alpha}{2z^2}\right)w^2-\left(\frac{\omega}{2z}-\frac{\alpha}{z^2}\right)w+\right.\right.$$

$$\left.\left.+\left(1+\frac{\omega}{2z}-\frac{\alpha}{6z^2}w_1^3-\frac{\omega}{4z}w_1^2\right)\right]e^{w-w_1}-1\right\}, \tag{28}$$

where

$$\begin{cases}\varkappa\eta = u & \varkappa h = z\\ \varkappa\varphi = w\\ \varkappa s = w_1\end{cases} \tag{29}$$

As a result of the substitution of numerical values, equation (28) becomes

$$u \cong 1{,}15 + \left[\frac{\alpha}{z^2}\left(\frac{w^3}{6}-\frac{w^2}{2}+w-16.222\right)+\right.$$

$$\left.+\frac{\omega}{2z}\left(\frac{w^2}{2}-w-9.58\right)+1\right]\cdot 0.034662\,e^{\omega} \tag{30}$$

As $\alpha \to 0$ and $\omega \to 0$ in equation (28), there is obtained in the limit

$$\eta = s + \frac{1}{\varkappa f}\left(e^{\varkappa(\varphi - s)}-1\right).$$

hence

$$\varphi = s + \frac{1}{\varkappa}\ln\left[(\eta - s)\varkappa f + 1\right] \tag{31}$$

or, let

$$\frac{1}{\varkappa f}-s=\alpha; \quad s+\frac{\ln \varkappa f}{\varkappa}=b, \tag{32}$$

then equation (31) may be written in the form

$$\varphi=\frac{1}{\varkappa}\ln(\eta+a)+b, \tag{31'}$$

thus the well-known logarithmic law is obtained. Further, for

$$y=\delta, \tag{33}$$

where δ is the boundary layer thickness, or in nondimensional form,

$$\bar{\eta}=R_\delta\cdot\sqrt{\frac{c_{f*}}{2}}=R_\delta\cdot\frac{1}{h}; \tag{34}$$

$$\bar{\varphi}=\sqrt{\frac{2}{c_{f*}}}=h. \tag{35}$$

From equation (28) is found the relation

$$R_\delta\sqrt{\frac{c_{f*}}{2}}=s+\frac{1}{\varkappa f}\left\{\left[1-\frac{\alpha}{2}-\frac{\omega}{2}+\varkappa\left(\frac{\alpha}{6}+\frac{\omega}{4}\right)\sqrt{\frac{2}{c_{f*}}}+\right.\right.$$
$$\left.\left.+\frac{1}{\varkappa}\left(\alpha+\frac{\omega}{2}-\frac{\omega}{4}w_1^2\right)\sqrt{\frac{c_{f*}}{2}}-\frac{\alpha}{6\varkappa^2}\frac{c_{f*}}{2}w_1^3\right]e^{\varkappa\left(\sqrt{\frac{2}{c_{f*}}}-s\right)}-1\right\}. \tag{36}$$

II. DETERMINATION OF THE DRAG AT EACH POINT OF THE PLATE

The integral relation of Kármán reads:

$$\frac{d}{dx}\int_0^\delta \rho u(U-u)\,dy=\tau. \tag{1}$$

Let

$$\frac{xU\rho_*}{\mu_*} = R_{x*} \text{ and } \frac{xU\bar{\rho}}{\bar{\mu}} = R_x. \tag{2}$$

On introduction of the universal scales formula (1) assumes the form

$$dRx = h^2 \left\{ d \left[\frac{1}{h} \int_0^{\bar{\eta}} \varphi (h - \varphi)\, d\eta \right] \right\}. \tag{3}$$

The integration of equation (3) is carried out for the laminar sublayer and the turbulent layer (fig. 4) (on the assumption that the turbulent layer begins at the point where the thickness of the laminar sublayer is equal to that of the purely laminar layer). In the laminar sublayer

$$\varphi = \eta, \tag{4}$$

that is,

$$R_{x*} = \int_0^s h^2 \left\{ d \left[\frac{1}{h} \int_0^h \frac{\eta (h - \eta)\, d\eta}{1 - \left(\frac{\alpha \eta^2}{h^2} + \frac{\omega \eta}{h} \right)} \right] \right\} +$$

$$+ \int_s^h h^2 \left\{ d \left[\frac{1}{h} \int_0^s \frac{\eta (h - \eta)\, d\eta}{1 - \left(\frac{\alpha \eta^2}{h^2} + \frac{\omega \eta}{h} \right)} \right] + d \left[\frac{1}{h} \int_s^h \frac{\varphi (h - \varphi)\, d\varphi \frac{d\eta}{d\varphi}}{1 - \left(\frac{\alpha \varphi^2}{h^2} + \frac{\omega \varphi}{h} \right)} \right] \right\}, \tag{5}$$

where the first integral corresponds to the purely laminar layer, the second to the laminar sublayer of the turbulent layer, and the third to the turbulent layer. The first two integrals are small, as will be shown.

Something is missing here

It resulted in:

$$\frac{d\eta}{d\varphi} = \frac{1}{f} \cdot e^{\frac{\varkappa h}{\sqrt{\alpha}} \left[\cos^{-1} \left(\frac{\sqrt{\alpha} \frac{s}{h} + \frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1 + \frac{\omega^2}{4\alpha}}} \right) - \cos^{-1} \left(\frac{\sqrt{\alpha} \frac{\varphi}{h} + \frac{\omega}{2\sqrt{\alpha}}}{\sqrt{1 + \frac{\omega^2}{4\alpha}}} \right) \right]} \tag{6}$$

Developing the right-hand side of equation (6) in a Taylor series in $\sqrt{\alpha}$ and $\frac{\omega}{2\sqrt{\alpha}}$ to terms of the second order approximates to

$$\frac{d\eta}{d\varphi} \cong \frac{1}{f} e^{\varkappa(\varphi - s)\left[1 + \frac{\alpha}{6h^2}(\varphi^2 + s\varphi + s^2) + \frac{\omega}{4h}(\varphi + s)\right]} \tag{7}$$

$$\cong \frac{1}{f} e^{\varkappa(\varphi - s)} \left\{ 1 + \left[\frac{\alpha}{6h^2}(\varphi^2 + s\varphi + s^2) + \frac{\omega}{4h}(\varphi + s) \right] \varkappa(\varphi - s) \right\}$$

or, from the original notation

$$\frac{d\eta}{d\varphi} = \frac{du}{dw} = \frac{1}{f} e^{w - w_1} \left\{ 1 + \left[\frac{\alpha}{6z^2}(w^2 + w_1 w + w_1^2) + \frac{\omega}{4z}(w + w_1) \right] (w - w_1) \right\}$$

or

$$\frac{d\eta}{d\varphi} = \frac{e^{w - w_1}}{f} \left\{ 1 + \frac{\alpha}{6z^2} w^3 + \frac{\omega}{4z} w^2 - \left(\frac{\alpha}{6z^2} w_1^3 + \frac{\omega}{4z} w_1^2 \right) \right\}. \tag{7'}$$

Substituting this expression in the last integral of equation (5) and carrying out the integration gives

$$\begin{aligned} R_{x*} = \frac{1}{\varkappa^4} \Bigg\{ \frac{e^{z - w_1}}{f} \Bigg[& \frac{1}{2}\left(\frac{\alpha}{3} + \frac{\omega}{2}\right) z^3 + \left(1 - \frac{2}{3}\alpha - \omega\right) z^2 - \\ & - \left\{ 4 - \frac{4}{3}\alpha + \frac{\omega}{2}\left(\frac{w_1^2}{2} - 5\right) \right\} z + \left\{ 6 - \alpha\left(\frac{w_1^3}{6} - \frac{2}{3}\right) + \omega\left(w_1^2 - 3\right) \right\} + \\ & + (w_1^3 - 12)\frac{\alpha}{z} \Bigg] - \frac{2e^{-w_1}}{f}\left[\alpha\left(\frac{w_1^3}{6} - 2\right) + \omega\frac{w_1^2}{2} \right] \cdot E_i(z) - \frac{3\alpha k_1}{z} + \\ & + (\alpha k_2 - \omega k_3)(2 \ln z - 1) + (k_4 - \omega k_5) z + \left(\frac{1}{3} + \frac{1}{6}\omega + \frac{1}{10}\alpha \right) \frac{w_1^4}{4} - L \Bigg\}, \end{aligned} \tag{8}$$

where L is the value of the last two integrals in equation (5) for $z = w_1$ (the lower limit) and

$$E_i(z) = \int_{-\infty}^{z} \frac{e^z dz}{z} \tag{9}$$

is the well-known integral exponential function for which tables are available (for example, Jahnke and Emde). For z equal to 16, 17, and 18 the values of $E_i(z)$ were computed with the aid of the expansion of the function $f(z) = \frac{e^z}{z}$ in a McLaurin series which was then integrated term by term:

$$\int \frac{e^z dz}{z} = \ln z + z + \frac{z^2}{2 \cdot 2!} + \frac{z^3}{3 \cdot 3!} + \cdots \tag{10}$$

By taking 15 terms in equation (10), an accuracy to the fifth decimal place is assured. Terms of the form $z^n/n\,n!$ were computed with the aid of 10-place logarithm tables (which assure the required accuracy). The following results were obtained:

$$\begin{cases} z = 15 & E_i(15) = 234\,956 \\ z = 16 & E_i(16) = 595\,560 \\ z = 17 & E_i(17) = 1\,516\,637 \\ z = 18 & E_i(18) = 3\,877\,898 \end{cases}$$

By substitution of numerical values, formula (8) becomes

$$\begin{aligned} R_{x*} = 1.354 e^z \Bigg[& \frac{1}{2}\left(\frac{\alpha}{3} + \frac{\omega}{2}\right) z^3 + \left(1 - \frac{2}{3}\alpha - \omega\right) z^2 - \\ & - \left\{4 - \frac{4}{3}\alpha + 2.79\,\omega\right\} z + \left\{6 - 15.556\,\alpha + 18.160\,\omega\right\} + \frac{85.336\alpha}{z} \Bigg] - \\ & - 2.708\,(14.223\alpha + 10.58\omega)\, E_i(z) + 39.0625 \Bigg[- \frac{1181.901\alpha}{z} + \\ & + (7.748\alpha - 19.403\omega)(2 \ln z - 1) - (15.693 + 2.169\omega) z + \\ & + \left(\frac{1}{12} + \frac{1}{24}\omega + \frac{1}{40}\alpha\right) \cdot 447.746 + 41.981 + 231.866\alpha + 42.498\omega \Bigg] \end{aligned} \tag{8'}$$

If the small terms not having e^z and $E_i(z)$ as factors are disregarded, it affords the approximation

$$R_{x*} \cong 1.354\, e^z (z^2 - 4z + 6) + 1.354 \left\{ \left(\frac{1}{6}\alpha + \frac{1}{4}\omega \right) z^3 - \right.$$

$$- \left(\frac{2}{3}\alpha + \omega \right) z^2 + \left(\frac{4}{3}\alpha - 2.79\,\omega \right) z - (15.556\alpha - 18.160\omega) +$$

$$\left. + \frac{85.336\alpha}{z} \right\} - 2.708\,(14.223\alpha + 10.58\omega)\, E_i(z). \tag{11}$$

From the values of $R_{x*} = \frac{xU\rho^*}{\mu^*}$ computed by the preceding formula the values of $R_x = \frac{xU\bar{\rho}}{\bar{\mu}}$ referred to the magnitudes in the free flow were computed from the relation

$$R_x = R_{x*} \cdot \frac{\bar{\rho}}{\rho_*} \cdot \frac{\mu_*}{\bar{\mu}}$$

or, since

$$\frac{\bar{\rho}}{\rho_*} = \frac{T_*}{\bar{T}}$$

and

$$\frac{\mu_*}{\bar{\mu}} = \sqrt{\frac{T_*}{\bar{T}}},$$

hence

$$R_x = R_{x*} \left(\frac{T_*}{\bar{T}} \right)^{3/2}.$$

With the aid of the relation

$$1 - \frac{\bar{T}}{T_*} - \alpha = \omega$$

there is obtained

$$R_x = \frac{R_{x*}}{[1 - (\alpha + \omega)]^{3/2}}$$

and

$$\log R_x = \lg R_{x*} - \frac{3}{2} \log [1 - (\alpha + \omega)] \tag{12}$$

$$c_f = c_{f*} [1 - (\omega + \alpha)] \tag{13}$$

The values of $\log R_x$ and c_f were computed for: (1) $\alpha = 0$, $\omega = 0$; (2) $\omega = 0$, $\alpha = 0.05$, and $\alpha = 0.1$; (3) $\alpha = 0$, $\omega = 0.05$, and $\omega = 0.1$; (4) $\alpha = 0.05$, $\omega = 0.05$, and $\omega = 0.1$; and (5) $\alpha = 0.1$, $\omega = 0.05$, and $\omega = 0.1$ in the interval from $z = 7$ to $z = 18$. For these values of α and ω the velocity distributions were found for $z = 7$, 12, and 18. These results are given in tables 1 and 2, and shown graphically in figures 1 to 6 (notation according to equation (29)). From inspection of figures 3 to 6 it is seen that, with increasing α and ω for the same R_x, the coefficient c_f decreases.

Translation by S. Reiss,
National Advisory Committee
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TABLE I.- VELOCITY DISTRIBUTION

w	u	u for $\omega=0$, $\alpha=0{,}05$			u for $\omega=0$; $\alpha=0{,}1$			u for $\alpha=0$, $\omega=0{,}05$			u for $\alpha=0$; $\omega=0{,}1$			u for $\alpha=0{,}05$; $\omega=0{,}05$			u for $\alpha=0{,}5$; $\omega=0{,}1$			u for $\alpha=0{,}1$; $\omega=0{,}05$			u for $\alpha=0{,}1$; $\omega=0{,}1$		
	$\omega=0$ $\alpha=0$	$z=7$	$z=12$	$z=18$	$z=7$	$z=12$	$z=18$	$z=7$	$z=12$	$z=18$	$z=7$	$z=12$	$z=18$	$z=7$	$z=12$	$z=18$	$z=7$	$z=12$	$z=18$	$z=7$	$z=12$	$z=18$	$z=7$	$z=12$	$z=18$
5	6,29	6,27	6,28	6,29	6,26	6,28	6,29	6,25	6,27	6,28	6,21	6,25	6,26	11,37	11,40	11,42	11,33	11,38	11,40	11,36	11,40	11,42	11,32	11,38	11,40
6	15,13	15,24	15,17	15,15	15,36	15,20	15,16	15,25	15,20	15,18	15,37	15,27	15,22	29,34	29,22	29,18	29,46	29,29	29,27	29,46	29,25	29,19	29,58	29,32	29,23
7	39,16	40,07	39,47	39,30	40,98	39,78	39,43	40,24	39,79	39,58	41,31	40,41	40,00	79,16	78,11	77,73	80,23	78,73	78,15	80,07	78,42	77,86	81,14	79,04	78,28
8	104,45	109,2	106,1	105,2	114,0	107,7	105,9	109,8	107,5	106,5	115,1	110,7	108,6	217,8	212,6	210,7	223,2	215,7	212,7	222,6	214,1	211,3	228,0	217,3	213,4
9	282,0	303,2	289,2	285,2	324,3	296,4	288,5	304,0	294,9	290,6	326,0	307,7	299,2	606,2	583,1	574,8	628,1	595,8	583,3	627,2	590,2	578,0	649,1	603,0	586,6
10	764,6	850,7	794,0	777,6	936,7	823,2	790,7	847,6	813,1	796,9	930,6	861,5	829,2	1697	1607	1573	1780	1654	1606	1783	1635	1586	1866	1684	1619
11	2076	2407	2189	2127	2738	2301	2177	2372	2249	2192	2668	2422	2307	4778	4437	4318	5074	4610	4433	5109	4549	4368	5405	4722	4483
12	5642	6862	6057	5827	8081	6472	6011	6658	6235	6037	7674	6828	6433	13519	12291	11863	14535	12884	12259	14738	12706	12048	15754	13299	12443
13	15336	19693	16819	15995	24049	18302	16653	18727	17314	16868	22119	19292	18400	38419	34132	32863	41811	36110	34394	42775	35615	33520	46167	37593	35052
14	41686	56876	46854	43982	72065	52023	46279	52765	48147	45992	63845	54612	50302	109640	95000	89973	120720	101465	94283	124829	100169	92270	135909	106634	96580
15	113312	165197	130966	121164	217093	148619	129005	148891	134048	127147	184471	154818	140982	314087	265013	248310	349667	285783	262145	365983	282666	256151	401563	303436	269986
16	308011	482284	367304	334377	656525	426596	360712	420682	373741	351811	533352	439470	395640	902965	741044	686187	1015635	806773	730016	1077206	800337	712522	1189876	866065	756351

TABLE II.- VALUES OF LOG R_x AND c_f REFERRED TO THEIR FREE STREAM VALUES

z	$\alpha=0$, $\omega=0$		$\alpha=0$, $\omega=0{,}05$		$\alpha=0$, $\omega=0{,}1$		$\alpha=0{,}05$; $\omega=0$		$\alpha=0{,}1$, $\omega=0$		$\alpha=0{,}05$, $\omega=0{,}05$		$\alpha=0{,}05$, $\omega=0{,}1$		$\alpha=0{,}1$, $\omega=0{,}05$		$\alpha=0{,}1$, $\omega=0{,}1$	
	$\log R_x$	c_f	$\log R_x$	c_f	$\log R_x$	c_f	$\log R_x$	c_f	$\log R_x$	c_f	$\log R_x$	c_f	$\log R_x$	c_f	$\log R_x$	c_f	$\log R_x$	c_f
7	4,60304	0,00653	4,661	0,00620	4,720	0,00588	4,656	0,00620	4,711	0,00588	4,715	0,00588	4,773	0,00555	4,771	0,00555	4,832	0,00522
8	5,18576	0,00500	5,250	0,00475	5,315	0,00450	5,253	0,00475	5,302	0,00450	5,309	0,00450	5,372	0,00425	5,368	0,00425	5,433	0,00400
9	5,74784	0,00396	5,818	0,00376	5,888	0,00356	5,810	0,00376	5,872	0,00356	5,880	0,00356	5,950	0,00337	5,942	0,00337	6,012	0,00317
10	6,29411	0,00320	6,370	0,00304	6,445	0,00288	6,360	0,00304	6,425	0,00288	6,435	0,00288	6,510	0,00272	6,500	0,00272	6,574	0,00256
11	6,82794	0,00264	6,910	0,00251	6,989	0,00238	6,898	0,00251	6,965	0,00238	6,976	0,00238	7,054	0,00224	7,044	0,00224	7,123	0,00211
12	7,35175	0,00222	7,439	0,00211	7,522	0,00200	7,424	0,00211	7,496	0,00200	7,510	0,00200	7,592	0,00189	7,580	0,00189	7,661	0,00178
13	7,86735	0,00189	7,960	0,00180	8,048	0,00170	7,943	0,00180	8,019	0,00170	8,033	0,00170	8,119	0,00161	8,106	0,00161	8,191	0,00151
14	8,37609	0,00163	8,473	0,00155	8,565	0,00147	8,456	0,00155	8,534	0,00147	8,551	0,00147	8,638	0,00139	8,623	0,00139	8,712	0,00130
15	8,87903	0,00142	8,981	0,00135	9,076	0,00128	8,962	0,00135	9,043	0,00128	9,060	0,00128	9,152	0,00121	9,136	0,00121	9,226	0,00114
16	9,37699	0,00125	9,484	0,00119	9,584	0,00112	9,464	0,00119	9,546	0,00112	9,565	0,00112	9,660	0,00106	9,643	0,00106	9,736	0,00100
17	9,87065	0,00111	9,983	0,00105	10,085	0,00100	9,960	0,00105	10,046	0,00100	10,065	0,00100	10,163	0,00094	10,148	0,00094	10,242	0,00089
18	10,36053	0,00099	10,477	0,00094	10,584	0,00089	10,454	0,00094	10,542	0,00089	10,564	0,00089	10,663	0,00084	10,644	0,00084	10,742	0,00079

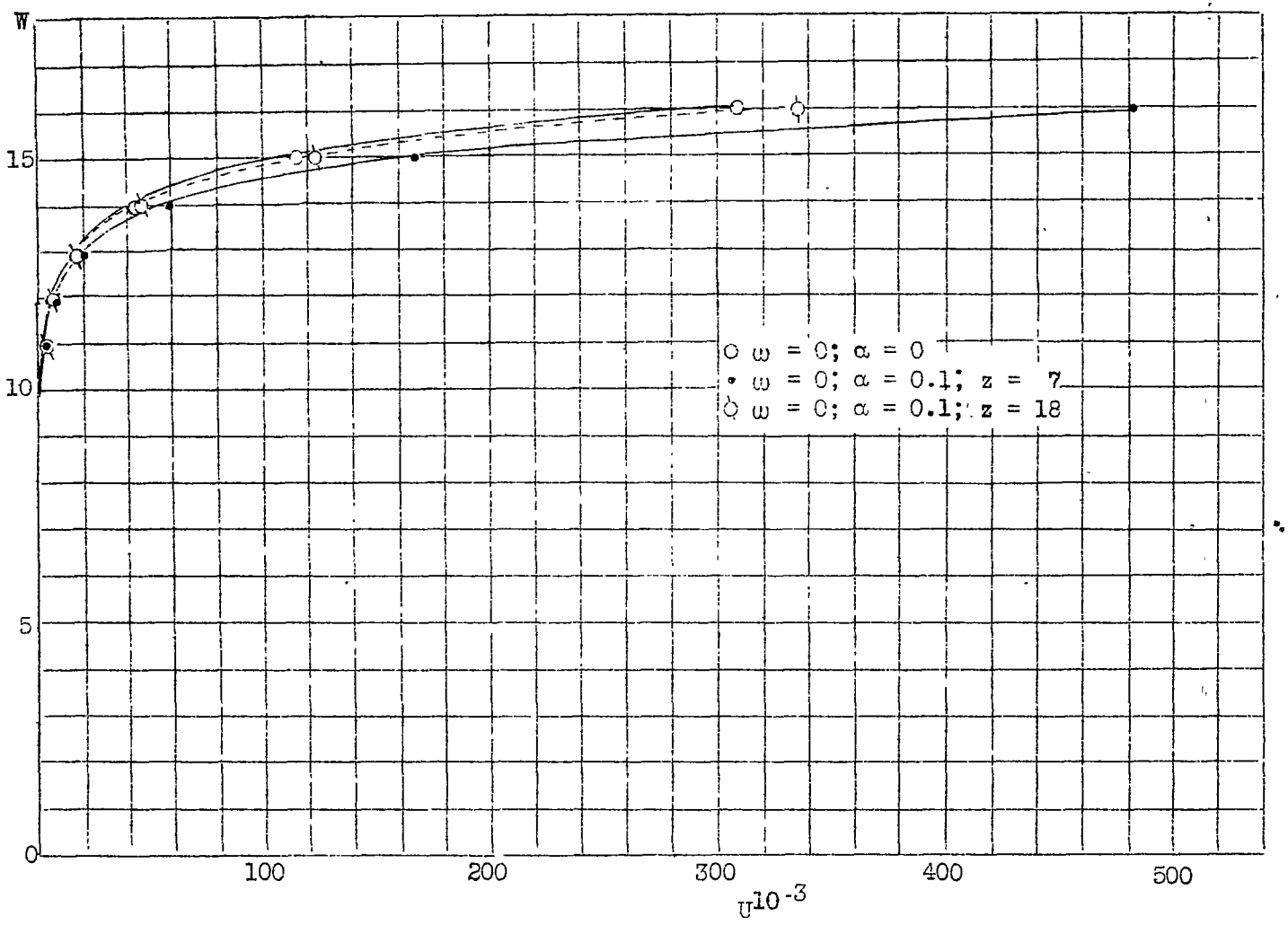

Figure 1.- Velocity distribution in the absence of heat transfer.

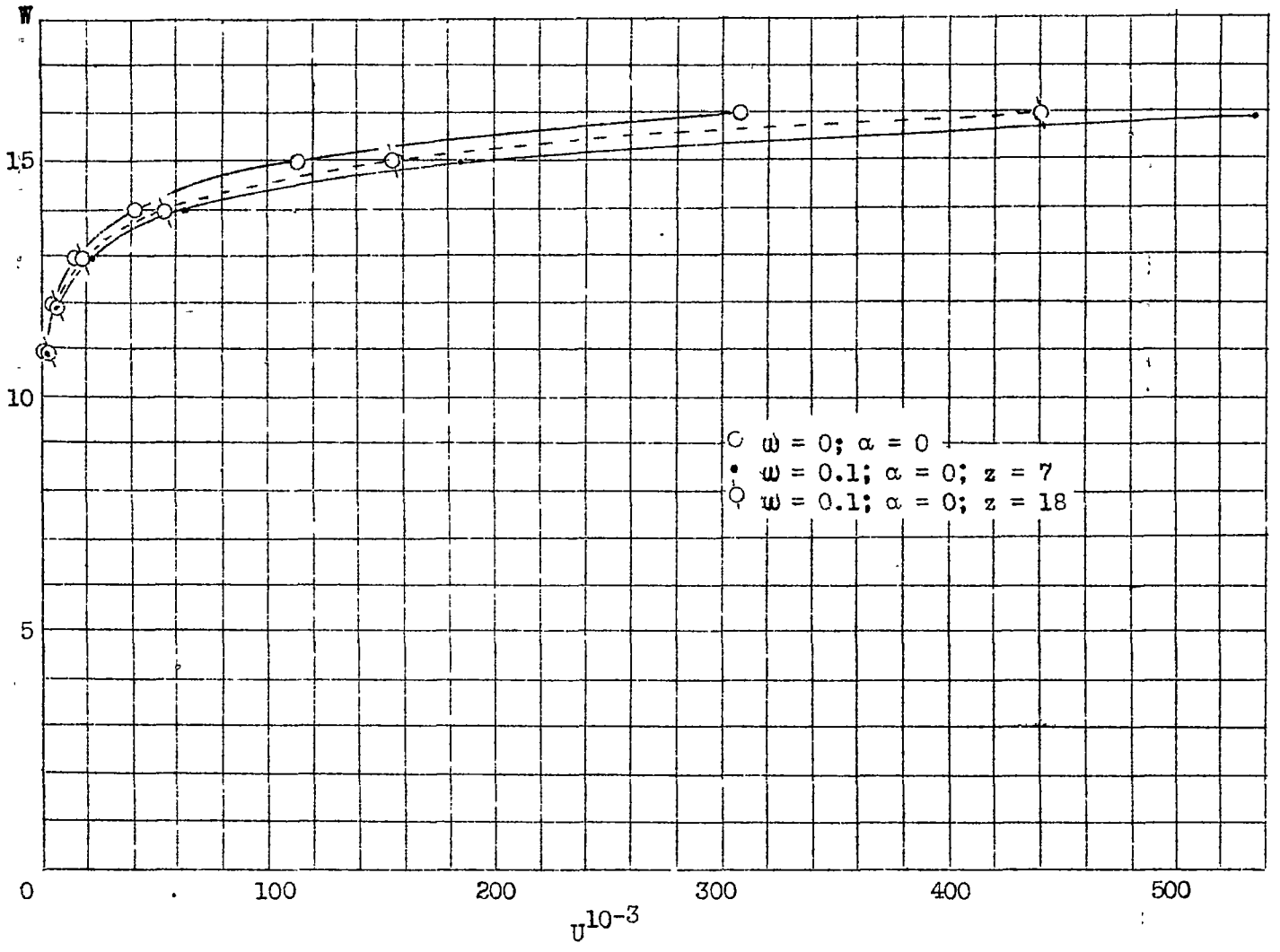

Figure 2.- Velocity distribution in the presence of heat transfer, for small velocities.

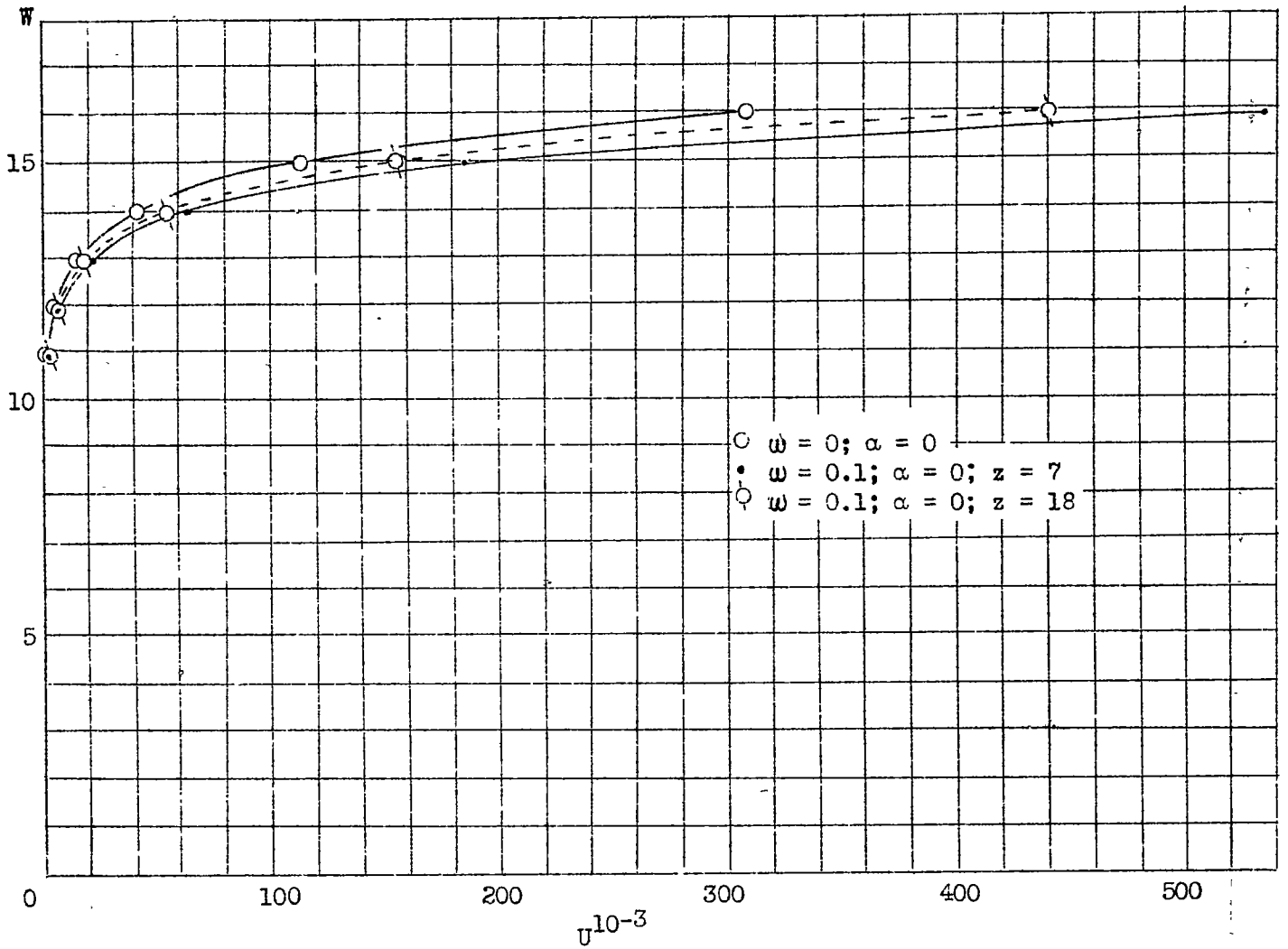

Figure 2.- Velocity distribution in the presence of heat transfer, for small velocities.

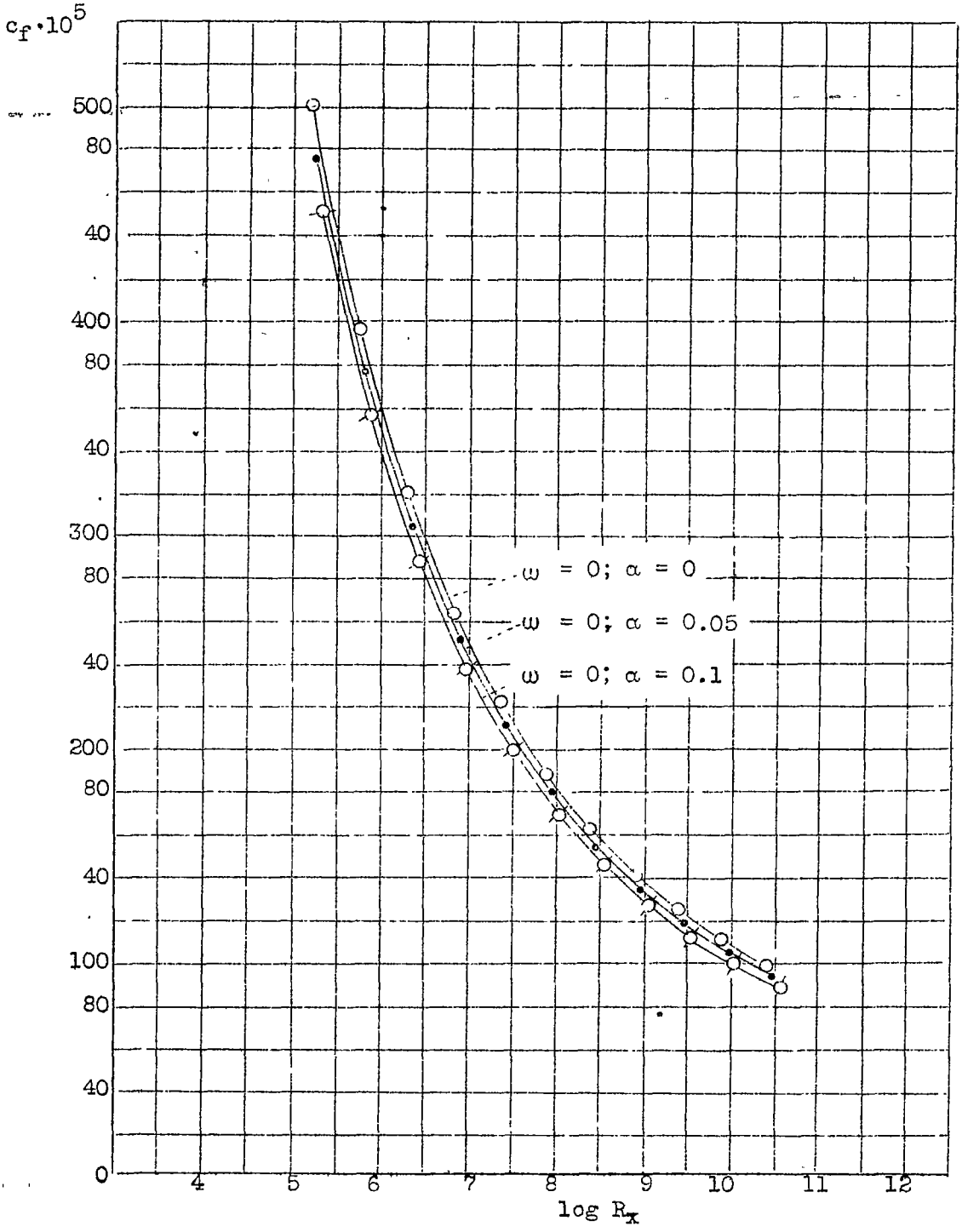

Figure 4a.- Dependence of the drag coefficient c_f, referred to the density $\bar{\rho}$ in the free stream, on log R_x for $\omega = 0$ (absence of heat transfer but with compressibility effect).

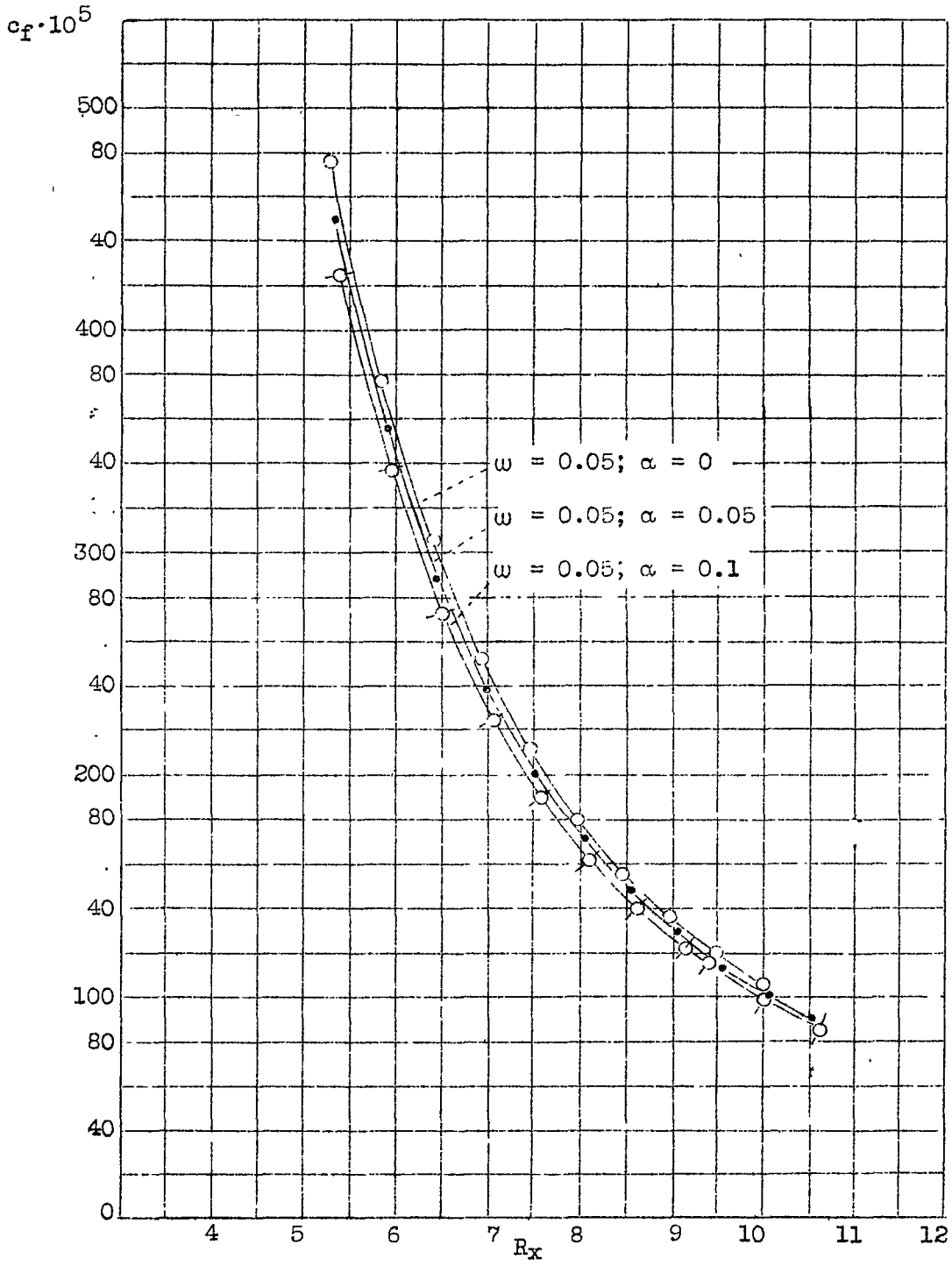

Figure 5.- Dependence of c_f on log R_x for $\omega = 0.05$ (at small heat transfer).
(Reference to $\bar{\rho}$ in the free stream)

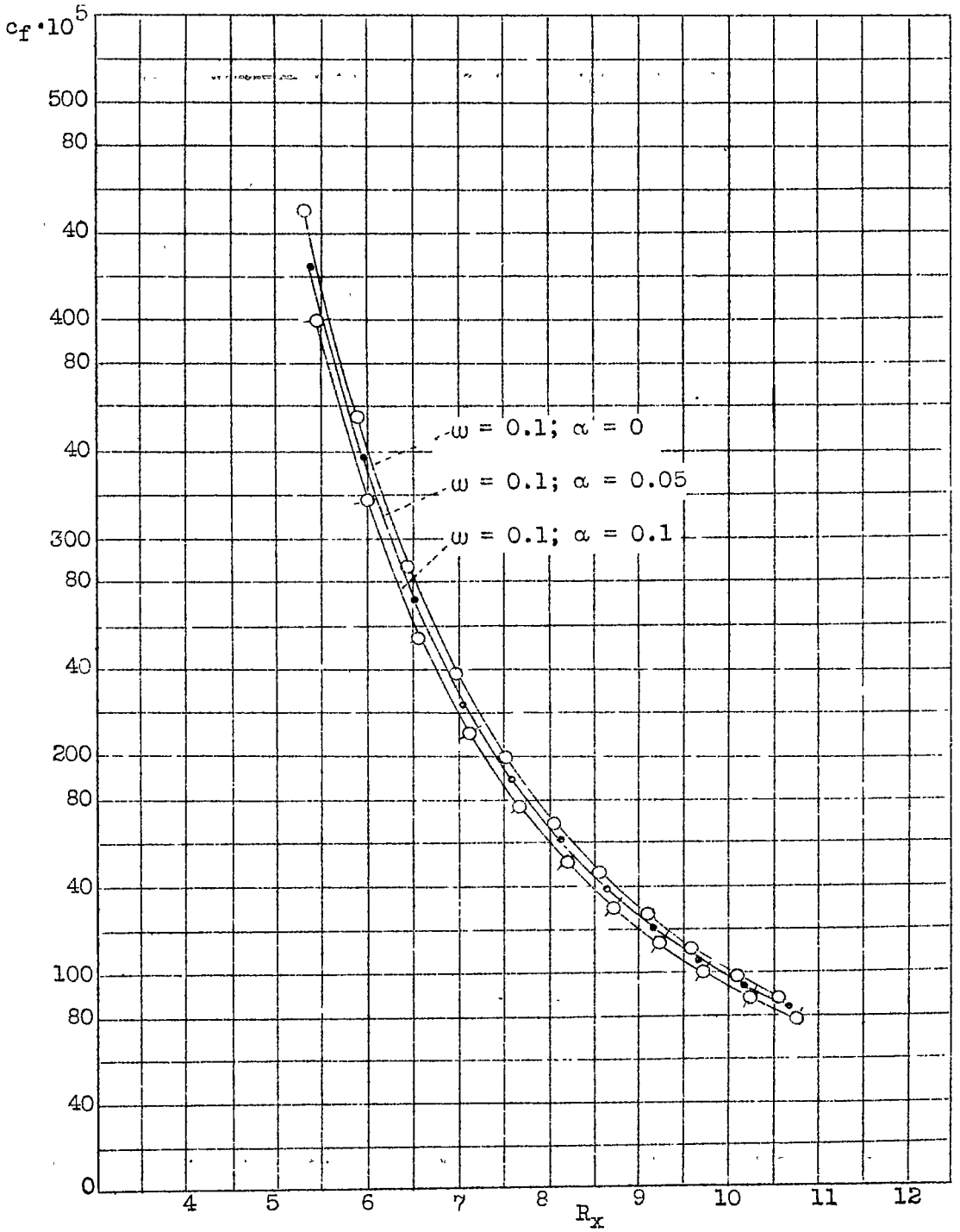

Figure 6.- Dependence of c_f on log R_x for $\omega = 0.1$ (for large heat transfer).

(Reference to $\bar{\rho}$ in the free stream)