State and Output Feedback Certainty Equivalence M-MRAC for Systems with Unmatched Uncertainties

Vahram Stepanyan and Kalmanje Krishnakumar

ABSTRACT

The paper presents a certainty equivalence indirect adaptive control design method for parametric strict feedback nonlinear systems of any relative degree with unmatched uncertainties in state and output feedback settings. The approach is based on the parameter identification (estimation) model, which is completely separated from the control design and is capable of producing parameter estimates as fast as the computing power allows. It is shown that the system’s input and output tracking errors can be systematically decreased by the proper choice of the design parameters.

Key Words: adaptive control, unmatched uncertainties, certainty equivalence, guaranteed transient bounds, disturbedance rejection

I. Introduction

Control design for systems with unmatched uncertainties is a challenging task, and the majority of design methods is based on the backstepping technique outlined in [10]. There, it has been shown that certainty equivalence principle leads to over parametrization, which can be avoided by the departure from the certainty equivalence. In this case, the adaptive laws enter into the control law, which can result in the high magnitude control signals in the case of large adaptive rates (fast adaptation). An alternative certainty equivalence control design method is presented in [2], which avoids over parametrization for linear systems, but not for nonlinear systems with the relative degree greater than two.

Although the backstepping method provides a systematic way of handling the unmatched uncertainties, it suffers from the problem of ”explosion of terms” because of repeated differentiations of virtual control functions. To reduce the computational load, the standard backstepping was combined with a sliding mode control approach in [1, 6]. The idea was to use an auxiliary second order subsystem to which a second order sliding mode control is applied instead of last two steps in the standard backstepping design. In [20], Multiple surface sliding control was introduced to simplify the backstepping design procedure. Some drawbacks of this approach were addressed by the Dynamic surface control [19], where a first order filter was applied to proceed with the backstepping recursions without differentiation.

Alternative approaches include approximation of the virtual control derivatives using sliding mode filters [11, 18], neural networks [14, 15], fuzzy systems [7], first order linear filters [21], and second order linear filters [5], where the singular perturbation method and Tikhonov’s theorem are used to prove the closed-loop stability and to obtain the performance bounds.

In this paper, we present a certainty equivalence indirect adaptive control approach without over parametrization for parametric strict feedback nonlinear systems of any relative degree in state and output feedback settings, which is the main contribution of the paper. The approach is based on the identification scheme, which is completely separated from the control design. To enable a fast adaptation without generating high frequency oscillations in the adaptive signals, it employs an error feedback term, like in...
the modified reference model MRAC (M-MRAC) architecture introduced in [16]. It is shown that the state prediction error converges to zero independent of the control design. Moreover, it is shown that transient of the state prediction error and the combined parameter estimation error can be regulated by the proper choice of the error feedback gain and the adaptation rate. The control design follows the command filtered backstepping procedure [5]. It is shown that the input tracking error (difference between ideal control and command filtered certainty equivalence control signal) and output tracking error can be regulated by the proper choice of design parameters. State feedback part of the paper was originally reported in [17]. Here, we have completed the proofs, expanded the class of systems to include the external disturbances, and extended the results to the output feedback systems.

The rest of the paper is organized as follows. In Section II, we give the problem statement in the state feedback framework and the assumptions. In Section III, we introduce the identification model and give its properties. The control design is presented in Section IV, and the controller’s performance is analyzed in Section V. In Section VI we extend the design for the systems with external disturbances. The output feedback design is presented in Section VII. Simulation examples are presented in Section VIII, and some concluding remarks are given in Section IX.

II. Problem Statement

Consider an uncertain single input single output (SISO) system in the parametric strict feedback form [10] (p. 99)

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) + \theta_1^T \varphi_1(x_1) \\
\dot{x}_2(t) &= x_3(t) + \theta_2^T \varphi_2(x_1, x_2) \\
&\vdots \\
\dot{x}_n(t) &= u(t) + \theta_n^T \varphi_n(x)
\end{align*}
\]

with \(x(0) = x_0\), where \(x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n\) and \(u \in \mathbb{R}\) are the state and the input of the system, \(\theta_i \in \mathbb{R}^{p_i}\) are vectors of unknown constant parameters, and \(\varphi_i : \mathbb{R}^1 \rightarrow \mathbb{R}^{p_i}\), \(i = 1, \ldots, n\) are sufficiently smooth known vector-functions. The system (1) is written in the vector form

\[
\dot{x}(t) = Ax(t) + bu(t) + \eta(t),
\]

where we denote

\[
A = \begin{bmatrix}
0_{(n-1) \times 1} & I_{(n-1) \times (n-1)} & 0_{1 \times (n-1)}
\end{bmatrix},
\]

and \(\eta(t) = [\theta_1^T \varphi_1(x_1) \, \theta_2^T \varphi_2(x_1, x_2) \cdots \theta_n^T \varphi_n(x)]^T\). Here, \(I_{(n-1) \times (n-1)}\) denotes the \(n-1\) dimensional identity matrix.

The objective is to design a control signal \(u(t)\) such that the closed-loop signals are bounded, and the system’s output \(y(t) = x_1(t)\) tracks the output \(y_r(t) = x_{r,1}\) of the reference model

\[
x_r(t) = A_r x_r(t) + b_r r(t), \quad x_r(0) = x_0,
\]

where \(A_r = A - b k^T\), \(b_r = k_r b\), and \(r(t)\) is a piece-wise continuous and bounded external command. Here, the feedback gain \(k\) and feedforward gain \(k_r\) are chosen to make \(A_r\) Hurwitz and meet the performance specifications.

III. Identification Model

In order to generate the necessary parameter estimates, we introduce the following identification model

\[
\begin{align*}
\dot{\hat{x}}(t) &= A \hat{x}(t) + b u(t) + \hat{\eta}(t) + c \tilde{x}(t) \\
\dot{\hat{x}}(0) &= \hat{x}_0,
\end{align*}
\]

where \(\hat{x}(t)\) is the state prediction, \(\hat{x}(t) = x(t) - \hat{x}(t)\) is the state prediction error, \(c > 0\) is a design parameter, \(\hat{\eta}(t) = [\hat{\theta}_1^T(t) \varphi_1(x_1) \, \hat{\theta}_2^T(t) \varphi_2(x_1, x_2) \cdots \hat{\theta}_n^T(t) \varphi_n(x)]^T\), and \(\hat{\theta}_i(t)\) is the estimate of the unknown parameter \(\theta_i\) for each \(i = 1, \ldots, n\). These estimates are generated according to adaptive laws

\[
\dot{\hat{\theta}}_i(t) = \gamma \hat{x}_i(t) \varphi_i(x_1, \ldots, x_i), \quad i = 1, \ldots, n,
\]

where \(\gamma > 0\) is the adaptation rate. The state prediction error dynamics do not explicitly depend on the control signal

\[
\dot{\hat{x}}(t) = (A - c I_{n \times n}) \hat{x}(t) + \hat{\eta}(t),
\]

where we define \(\hat{\eta}(t) = \eta(t) - \hat{\eta}(t)\) with \(\hat{\theta}_i(t) = \theta_i - \hat{\theta}_i(t), \quad i = 1, \ldots, n\) being the parameter estimation error.

Lemma III.1 The error signals \(\hat{x}(t)\) and \(\hat{\theta}_i(t), \quad i = 1, \ldots, n\) are globally uniformly bounded, and \(\hat{x}(t) \to 0\) as \(t \to \infty\).

Proof. Consider a candidate Lyapunov function

\[
V(t) = \frac{1}{2} \sum_{i=1}^{n} [\hat{x}_i^2(t) + \frac{1}{\gamma} \hat{\theta}_i^T(t) \hat{\theta}_i(t)],
\]

\(© 2014 John Wiley and Sons Asia Pte Ltd and Chinese Automatic Control Society Prepared using asjcauth.cls\)
the derivative of which is computed along the trajectories of the prediction error dynamics (6) and the adaptive laws (5)

\[ \dot{V}(t) = x^T(t)(A - cI_{n \times n})\hat{x}(t) + \sum_{i=1}^{n} \hat{\theta}_i(t) \left[ \hat{x}_i(t)\varphi_i(x_1, \ldots, x_i) + \frac{\hat{\theta}_i(t)}{\gamma} \right]. \]

Substituting the adaptive laws and completing the squares results in

\[ \dot{V}(t) = -(c - 1) \sum_{i=1}^{n} \hat{x}_i^2(t) - \frac{1}{2} (\hat{x}_i(t)) = (8) \]

\[ -\frac{1}{2} \sum_{i=1}^{n-1} [\hat{x}_i(t) - \hat{x}_{i+1}(t)]^2 - \frac{1}{2} (c - 1) \sum_{i=1}^{n} \hat{x}_i^2(t). \]

With \( c > 1 \), the LaSalle-Yoshizawa theorem ([10], p.24) guarantees that \( \hat{x}(t), \hat{\theta}_i(t), i = 1, \ldots, n \) are globally uniformly bounded, and \( \hat{x}(t) \to 0 \) as \( t \to \infty \). In particular, there exists \( \beta_1 > 0 \) such that \( \sum_{i=1}^{n} ||\hat{\theta}_i(t)||^2 \leq \beta_1^2 \).

**Lemma III.2** If \( \hat{x}_0 = x_0 \), then the state prediction error \( \hat{x}(t) \) satisfies the bound

\[ ||\hat{x}(t)||_{\infty} \leq \beta_1 \gamma^{-1/2}. \]

**Proof.** The proof follows from the fact that \( ||\hat{x}(t)||^2 \leq 2V(t) \leq 2V(0) \leq \gamma^{-1} \beta_1^2 \) for all \( t \geq 0 \).

The next lemma gives the bound on the state prediction error when \( \hat{x}_0 \neq x_0 \).

**Lemma III.3** If \( \hat{x}_0 \neq x_0 \), then \( \hat{x}(t) \) satisfies the bound

\[ ||\hat{x}(t)|| \leq \beta_2 e^{-(c-1)t} + \frac{\beta_1}{\sqrt{\gamma}}, \]

where \( \beta_2 = \sqrt{2V(0) - \frac{\beta_1^2}{\gamma}} \).

**Proof.** The inequality (8) can be written as

\[ \dot{V}(t) \leq -2(c-1)V(t) + \frac{c-1}{\gamma} \beta_1^2, \]

which implies that

\[ V(t) \leq \left[ V(0) - \frac{\beta_1^2}{2\gamma}\right] e^{-2(c-1)t} + \frac{\beta_1^2}{2\gamma}, \]

Recalling that \( ||\hat{x}(t)||^2 \leq 2V(t) \), we obtain

\[ ||\hat{x}(t)|| \leq \sqrt{\left[ V(0) - \frac{\beta_1^2}{2\gamma}\right] e^{-2(c-1)t} + \frac{\beta_1^2}{2\gamma}}, \]

The bound (10) follows from the fact that \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for any \( a \geq 0, b \geq 0 \).

Since the effect of the initialization of the state prediction error decays exponentially with the rate \( c - 1 \), and \( c \) will be assigned to large values in order to damp the high frequency oscillations in adaptive estimates for large adaptation rates, we set \( \hat{x}_0 = x_0 \) in the following derivations.

The next lemma gives the properties of the error signal \( \eta(t) \) and a tighter bound on \( \hat{x}(t) \), when \( x(t) \) and \( u(t) \) are bounded (which will be provided with the control design in the next section).

**Lemma III.4** Let the estimates \( \hat{x}(t) \) and \( \hat{\eta}(t) \) be generated by the identification model (4) and (5). In addition, let \( x(t) \) and \( u(t) \) be bounded. Then \( \eta(t) \to 0 \) as \( t \to \infty \), and \( \hat{\eta}(t) \) and \( \hat{x}(t) \) satisfy the following bounds

\[ ||\hat{\eta}(t)|| \leq \beta_3 e^{-\nu_1 t} + \frac{\beta_4}{\sqrt{\nu_2}}, \]

\[ ||\hat{x}(t)|| \leq \beta_5 e^{-\nu_2 t} + \frac{\beta_6}{\sqrt{\nu_3}}, \]

where the constants \( \beta_i > 0, i = 3, \ldots, 6 \) and \( \nu_1 > \nu_2 > 0 \) are defined in the proof.

**Proof.** From III.1 it follows that \( \hat{x}(t), \hat{\theta}_i(t), \hat{\theta}_i(t) \in \mathcal{L}_\infty (i = 1, \ldots, n) \). Therefore \( \hat{x}(t) \) is uniformly continuous. Since \( \varphi_i(x_1, \ldots, x_i) \) is smooth and \( x(t), u(t) \in \mathcal{L}_\infty \), it follows that \( \varphi_i(x_1, \ldots, x_i), \hat{x}(t), \varphi_i(x_1, \ldots, x_i), \hat{\eta}(t) \in \mathcal{L}_\infty (i = 1, \ldots, n) \). On the other hand, we conclude fro the adaptive laws that \( \hat{\theta}_i(t) \in \mathcal{L}_\infty (i = 1, \ldots, n) \). Therefore \( \hat{\eta}(t) \in \mathcal{L}_\infty \). It follows that \( \hat{\eta}(t) \in \mathcal{L}_\infty \), hence \( \hat{\eta}(t) \) is uniformly continuous. Then, (6) implies that \( \hat{x}(t) \) is uniformly continuous. Since its integral \( \hat{x}(t) \) has a finite limit, it follows from Barbilat’s lemma ([13], p.19) that \( \hat{x}(t) \to 0 \) as \( t \to \infty \), which in the view of (6) results in \( \hat{\eta}(t) \to 0 \) as \( t \to \infty \).

Next we derive upper bounds for the error signals \( \hat{\eta}(t) \) and \( \hat{x}(t) \). To this end we notice that \( \hat{\eta}(t) \) satisfies the dynamics

\[ \dot{\hat{\eta}}_i(t) + c\hat{\rho}_i(t) + \gamma \rho_i(t)\hat{\eta}_i(t) = -\gamma \rho_i(t)\hat{x}_{i+1}(t) - \gamma \rho_i(t)\hat{x}_i(t) + ch_i(t) + \dot{h}_i(t) \]

for all \( i = 1, \ldots, n \), where for the notational convenience we introduce a variable \( \hat{x}_{n+1} = 0 \). The other variables are defined as \( \rho_i(t) = \varphi_i^T(t)\varphi_i(t), h_i(t) = \hat{\theta}_i^T(t)\varphi_i(t) \). All variables involved in (16) are bounded. In particular, there exist positive constants \( \delta_1, \delta_2, \delta_3 \) such that \( ||\rho_i(t)||_{\mathcal{L}_\infty} \leq \delta_1, ||\hat{\rho}_i(t)||_{\mathcal{L}_\infty} \leq \delta_2 \).
and \( \| h_i(t) \|_{\mathcal{L}_\infty} \leq \delta_3 \). Then, it can be concluded from equation (16) that \( c \) determines the damping and \( \gamma \) determines the frequency of the signal \( \bar{h}_i(t) \). It follows from the results of [16], that choosing \( \rho \geq 2\sqrt{\beta_1}\gamma \) damps the oscillations in \( \bar{h}_i(t) \) and guarantees the bound

\[
\| \bar{h}_i(t) \| \leq \beta_3 e^{-\rho_1 t} + \delta_4 \| \bar{x}(t) \| + \frac{\delta_5}{\sqrt{\gamma}} \| h(t) \|, \quad (17)
\]

where \( \nu_1 \) is proportional to \( \sqrt{\gamma} \), and the positive constants \( \beta_3, \delta_4 \) and \( \delta_5 \) are independent of \( \gamma \) (see details in [16]). Substituting (9) we arrive to (14) with \( \beta_4 = \delta_4 \beta_1 + \delta_3 \delta_5 \).

Since \( A_r \) is Hurwitz, there exist positive constants \( \delta_0 \) and \( \nu_2 \) such that the state transition matrix satisfies the inequality \( \| e^{A_r t} \| \leq \delta_0 e^{-\nu_2 t} \). It follows that \( \| e^{(A-\bar{c} \alpha_n I)t} \| \leq \delta_0 e^{-\nu_2 t} + \frac{\delta_0}{\nu_2 + \nu} \). Then we obtain from (6) by the direct integration that

\[
\| \bar{x}(t) \| \leq \frac{\beta_3 \delta_0}{\nu_2 + c - \nu_1} \left[ e^{-\nu_1 t} - e^{-(\nu_2 + c)t} \right] + \frac{\delta_4}{\nu_2 + c} \left[ 1 - e^{-(\nu_2 + c)t} \right] \leq \beta_5 e^{-\nu_2 t} + \frac{\delta_0}{c \sqrt{\nu}}, \quad (18)
\]

since \( c \) and \( \nu_1 \) are proportional to \( \sqrt{\gamma} \), which is much greater than \( \nu_2 \) (adaptation is faster than the reference model).

### IV. Control Design

In this section we design four controllers. First one is a conventional backstepping control, which is designed assuming that \( \theta \) is known and omitting the actual differentiations of the stabilizing functions. It is called the ideal control, and the system’s corresponding response is the ideal output. They are used for comparison with the actual control and output of the uncertain system. The second one is a certainty equivalence controller designed for the identification model, which is obtained from the backstepping controller by replacing the uncertainties with their adaptive estimates without actual differentiation of the stabilizing functions. The third one is a command filtered backstepping controller introduced in [5]. The output tracking results of the second and third designs along with the corresponding control signals are used to prove the main result, which is presented in the next section. The forth controller is the actual one.

#### 4.1. Known System

Let \( \theta \) be known. Following the standard backstepping procedure [10], we define new variables

\[
z_i^{0}(t) = x_i^{0}(t) - \alpha_{i-1}^{0}(t), \quad i = 1, \ldots, n \quad (19)
\]

(superscript 0 indicates that the variables corresponds to the ideal control signal) and stabilizing functions

\[
\alpha_i^{0}(t) = 0, \quad \dot{\alpha}_1(t) = -\eta_1(t) \quad \alpha_i^{0}(t) = -\eta_i(t) + \dot{\alpha}_{i-1}^{0}(t), \quad i = 2, \ldots, n. \quad (20)
\]

The system (2) can be written in new variables as

\[
\dot{z}^{0}(t) = A z^{0}(t) + b(u^{0}(t) - \alpha_n^{0}(t)). \quad (21)
\]

Obviously, the control signal

\[
u^{0}(t) = -k^T z^{0}(t) + k_r r(t) + \alpha_n^{0}(t) \quad (22)
\]

reduces the system into the reference model (3), hence the error \( e^{0}(t) = z^{0}(t) - x_r(t) \) satisfies dynamics

\[
\dot{e}^{0}(t) = A_e e^{0}(t). \quad (23)
\]

**Lemma IV.1** The controller defined by (19), (20) and (22) guarantees the control objective for the system (2).

**Proof.** Since \( A_r \) is Hurwitz, it follows that \( e^{0}(t) \in \mathcal{L}_\infty \) and \( e^{0}(t) \) exponentially converges to zero. Hence, \( y^{0}(t) = y_r(t) + e_{[n]}^{0}(t) \) exponentially converges to \( y_r(t) \). In addition, from \( r(t) \in \mathcal{L}_\infty \) it follows that \( x_r(t) \in \mathcal{L}_\infty \), which along with \( e^{0}(t) \in \mathcal{L}_\infty \) implies that \( z^{0}(t) \in \mathcal{L}_\infty \). Boundedness of \( \alpha_n^{0}(t), \quad i = 1, \ldots, n \) and \( x^{0}(t) \) is obtained recursively starting with \( x^{0}(t) = z_{[n]}^{0}(t) \in \mathcal{L}_\infty \). Then, \( u^{0}(t) \in \mathcal{L}_\infty \) follows.

#### 4.2. Certainty Equivalence Control

Next, we design a controller for the identification model (4), by replacing the unknown parameter \( \theta \) with its estimate \( \hat{\theta}(t) \) in the stabilizing functions

\[
\dot{\alpha}_0(t) = 0, \quad \dot{\alpha}_1(t) = -\eta_1(t) \quad (24)
\]

\[
\dot{\alpha}_i(t) = -\eta_i(t) + \dot{\alpha}_{i-1}(t), \quad i = 2, \ldots, n,
\]

and introducing new variables as

\[
\dot{z}_i(t) = \dot{x}_i(t) - \dot{\alpha}_{i-1}(t), \quad i = 1, \ldots, n. \quad (25)
\]

The identification model in new variables takes the form

\[
\dot{z}(t) = A \dot{z}(t) + b[u(t) - \dot{\alpha}_n(t)] + c\ddot{x}(t). \quad (26)
\]

Defining the control signal as

\[
\dot{u}(t) = -k^T \dot{z}(t) + k_r r(t) + \dot{\alpha}_n(t). \quad (27)
\]
we obtain
\[ \dot{z}(t) = A_r \dot{z}(t) + b_r r(t) + c \dot{x}(t), \]
which is in the form of modified reference model introduced in [16]. In this case the error signal \( \dot{e}(t) = \hat{z}(t) - x_r(t) \) evolves according to the dynamics
\[ \dot{\hat{e}}(t) = A_r \hat{e}(t) + c \dot{x}(t). \]

**Lemma IV.2** The controller defined by (24), (25) and (27) guarantees the control objective.

**Proof.** According to Lemma III.1, \( \tilde{x}(t) \in \mathcal{L}_\infty \) and \( \tilde{x}(t) \to 0 \) as \( t \to \infty \). Since \( A_r \) is Hurwitz and \( r(t) \in \mathcal{L}_\infty \), it follows that \( \dot{z}(t) \in \mathcal{L}_\infty \), \( \hat{e}(t) \in \mathcal{L}_\infty \) and \( \dot{\hat{e}}(t) \to 0 \). Therefore, application of the controller defined by (24), (25) and (27) to both the system and the identification model results in
\[ y(t) = y_r(t) + \dot{\hat{e}}_1(t) + \dot{x}_1(t) \to y_r(t) \] (30)

In addition, from \( y_r(t) \in \mathcal{L}_\infty \) it follows that \( x_1(t) \in \mathcal{L}_\infty \), hence \( \dot{\hat{e}}_1(t) \in \mathcal{L}_\infty \), since \( \varphi_1(x_1) \) is continuous and \( \theta_1(t) \in \mathcal{L}_\infty \) according to Lemma III.1. This implies that \( \dot{x}(t) \in \mathcal{L}_\infty \) and \( x_2(t) \in \mathcal{L}_\infty \). Continuing this recursion we conclude that \( \dot{x}(t) \in \mathcal{L}_\infty \), \( \tilde{x}(t) \in \mathcal{L}_\infty \), \( \varphi_1(x_1, \ldots, x_i) \in \mathcal{L}_\infty \), \( \dot{e}(t) \in \mathcal{L}_\infty \), \( i = 1, \ldots, n-1 \), and \( \dot{\hat{e}}_1(t) \in \mathcal{L}_\infty \).

We notice that using the state \( x \) in the identification model (4) simplifies the stability analysis in the identification stage. However, the control design becomes problematic, because \( \dot{\hat{e}}_1(t) \) contains unknown parameter \( \theta \) through the state derivative \( \dot{x}(t) \). One way to overcome this issue is to replace \( \varphi_1(x_1, \ldots, x_i) \) with \( \varphi_1(\tilde{x}_1, \ldots, \tilde{x}_i) \) in the identification model (4), which brings additional terms \( \lambda_i \| \theta \| \| \tilde{x}(t) \| \tilde{x}(t) \) into the \( V(t) \) expression (8), where \( \lambda_i \) is the Lipschitz constant of \( \varphi_1(x_1, \ldots, x_i) \). In this case the same stability properties are guaranteed with the choice of \( c > 1 + \sum_{i=1}^{n} \lambda_i \| \theta \| \), and \( \dot{\hat{e}}_1(t) \) becomes implementable. However, the repetitive differentiations of \( \theta(t) \) introduces the multiple powers of the adaptation rate \( \gamma(t) \) into the control law, which suggests to keep the adaptation slow from the perspective of the control constraints. This is in conflict with the improvement of the performance by speeding up the adaptation. From this perspective, we use the command filtered backstepping approach from [5]. Although the method was introduced to simplify the process of determining the command derivatives in the backstepping procedure, it also allows to completely separate the identifier design from the controller design. Therefore, the identification process can be made as fast as the computational power allows. In the meantime, the high frequency oscillations associated with the fast adaptation are avoided with the proposed identification model by the proper choice of design parameters.

### 4.3. Command Filtering

Following [5], we introduce the command filtered approach for the design (19), (20) and (22), which will be used for the analysis purposes, and for the design (24), (25) and (27), which will actually be implemented. In the case of known system, the new variables are introduced as (superscript \( f \) indicates the command filtered version)
\[ z_i^f(t) = x_i^f(t) - \sigma_{i-1,1}^f(t), \quad i = 1, \ldots, n, \] (31)
where the command filter is designed as
\[ \dot{\sigma}_{i,1}^0(t) = \omega \sigma_{i,2}^0(t) \]
\[ \dot{\sigma}_{i,2}^0(t) = -2\zeta \sigma_{i,2}^0(t) - \omega [\sigma_{i,1}^0(t) - \alpha_i^f(t)] \]
\[ \alpha_i^f(t) = 0, \quad \alpha_i^f(t) = -\eta_i^f(t) \]
\[ \alpha_i^f(t) = -\eta_i^f(t) + \omega \sigma_{i-1,2}^0(t), \quad i = 2, \ldots, n, \] (33)
where the superscript \( f \) indicates that the corresponding quantities are computed when the command filtered control is in the loop. In (32), \( \omega \) and \( \zeta \) are respectively the command filter’s frequency and damping ratio. The system (2) is written in \( z^f \)-variable as
\[ \dot{z}^f(t) = A z^f(t) + b[u(t) - \alpha_n^f(t)] - \alpha_i^f(t) + \sigma^0(t), \]
where
\[ \alpha_i^f(t) = \begin{bmatrix} \alpha_1^f(t) \\ \vdots \\ \alpha_{n-1}^f(t) \\ 0 \end{bmatrix}, \quad \sigma^0(t) = \begin{bmatrix} \sigma_{1,1}^0(t) \\ \vdots \\ \sigma_{n-1,1}^0(t) \\ 0 \end{bmatrix}. \]

The compensated state is introduced as \( v^0(t) = z^f(t) - \xi^0(t) \), where \( \xi^0(t) \) is defined dynamically as
\[ \dot{\xi}_i^0(t) = \sigma_{i,1}^0(t) - \alpha_i^f(t) + \xi_{i+1}^0(t) \]
with \( \xi_i^0(0) = 0 \) for \( i = 1, \ldots, n-1 \), and \( \xi_n^0(0) = 0 \). In compensated state the system (2) takes the form
\[ \dot{v}^0(t) = A v^0(t) + b[u(t) - \alpha_n^f(t)], \] (36)
The control signal $u(t)$ in this procedure is defined as
\[ u^f(t) = -k^T v^0(t) + k_r r(t) + \alpha_t(t). \] (37)

Whereas the compensated error signal $e^0(t) = v^0(t) - x_r(t)$ satisfies the dynamics
\[ \dot{e}^0(t) = A_r e^0(t) \] (38)

and obviously is exponentially stable, the uncompensated error $e^f(t) = z^f(t) - x_r(t)$ has dynamics
\[ e^f(t) = A_r e^f(t) + bk^T \xi^0(t) - \tilde{\alpha}^f(t) + \sigma^0(t). \] (39)

**Lemma IV.3** The command filtered controller defined by (31), (32), (33), (35) and (37) guarantees the following relationships
\[
\begin{align*}
\epsilon^f(t) - \epsilon^0(t) &= \mathcal{O}(\epsilon), \quad \xi^0(t) = \mathcal{O}(\epsilon) \\
\sigma^0_{i,1}(t) - \alpha^0(t) &= \mathcal{O}(\epsilon), \quad i = 1, \ldots, n - 1 \\
\omega \sigma^0_{i,2}(t) - \tilde{\alpha}^f(t) &= \mathcal{O}(\epsilon), \quad i = 1, \ldots, n - 1,
\end{align*}
\]

where $\epsilon = 1/\omega$ (the proper choice of $\zeta$ and $\omega$ is discussed in [5]), and the notation $\mathcal{O}(\epsilon)$ is adopted from [8] (p. 383).

**Proof.** Although the error systems (38) and (39) are not in the standard backstepping form, the proof still follows the steps from [5].

### 4.4. Actual Control Design

Here, we design the command filtered version of the certainty equivalente control, which is the actual controller applied to the uncertain system (2). The uncompensated state is now introduced as
\[ \hat{z}^f_i(t) = \hat{x}^f_i(t) - \sigma_{i-1,1}(t), \quad i = 1, \ldots, n. \] (41)

where $\sigma_{i-1,1}(t)$ is the filter’s state given by
\[
\begin{align*}
\dot{\sigma}_{i,1}(t) &= \omega \sigma_{i,2}(t) \\
\dot{\sigma}_{i,2}(t) &= -2\zeta \omega \sigma_{i,2}(t) - \omega [\sigma_{i,1}(t) - \tilde{\alpha}^f(t)] \\
&\text{with the initial conditions } \sigma_{i,1}(0) = \hat{\sigma}_{i,1}(0) \text{ and } \sigma_{i,2}(0) = 0,
\end{align*}
\]

with the initializations $\sigma_{i,1}(0) = \tilde{\sigma}_{i,1}(0)$ and $\sigma_{i,2}(0) = 0$, and the stabilizing functions have the form
\[
\begin{align*}
\hat{\sigma}_0(t) &= 0, \quad \tilde{\alpha}^f(t) = -\hat{\eta}_1(t) \\
\dot{\hat{\sigma}}_1(t) &= -\hat{\eta}_1(t) + \sigma_{i-1,2}(t), \quad i = 2, \ldots, n.
\end{align*}
\]

Here, we introduce a short hand notation $\hat{\eta}_i(t) = \dot{\theta}_i(t) \varphi(t)$ and $\varphi(t) = \varphi(x^f_i(t), \ldots, x^f_1(t))$. The identification model in $z$-variables takes the form
\[
\dot{\hat{z}}^f(t) = A \hat{z}^f(t) + b[u(t) - \tilde{\alpha}_n(t)] + c \hat{\alpha}^f(t) - \tilde{\alpha}^f(t) + \sigma(t), \] (44)

where $\tilde{\alpha}^f(t) = z^f(t) - \hat{z}^f(t)$. The compensated state and its dynamics for the identification model are similarly introduced
\[ \dot{\xi}_{i,1}(t) = \sigma_{i,1}(t) - \alpha^f(t) + \xi_{i+1}(t) \] (45)

with $\xi_{i}(0) = 0$ for $i = 1, \ldots, n - 1$, $\xi_n(t) = 0$, and $v(t) = \hat{z}^f(t) - \xi(t)$.

\[
\dot{\hat{v}}(t) = A \hat{v}(t) + b[u(t) - \tilde{\alpha}^f(t)] + c \hat{\alpha}^f(t). \] (46)

The control signal to be implemented has the form
\[ u(t) = \hat{u}^f(t) = -k^T v(t) + k_r r(t) + \alpha^f(t). \] (47)

The resulting compensated error signal $e^c(t) = v(t) - x_r(t)$ and uncompensated error signal $e^f(t) = z^f(t) - x_r(t)$ satisfy dynamics
\[
\begin{align*}
\dot{e}^c(t) &= A_r e^c(t) + c \tilde{\alpha}^f(t) \\
\dot{e}^f(t) &= A_r e^f(t) + c \tilde{\alpha}^f(t) + \sigma(t) + c \hat{\alpha}^f(t).
\end{align*}
\]

**Lemma IV.4** The command filtered controller defined by (41), (42), (43), (45) and (47) guarantees the following relationships
\[
\begin{align*}
\dot{e}^f(t) - \hat{e}(t) &= \mathcal{O}(\epsilon), \quad \xi(t) = \mathcal{O}(\epsilon) \\
\sigma_{i,1}(t) - \tilde{\alpha}_{i}(t) &= \mathcal{O}(\epsilon), \quad i = 1, \ldots, n - 1 \\
\omega \sigma_{i,2}(t) - \tilde{\alpha}_{i}(t) &= \mathcal{O}(\epsilon), \quad i = 1, \ldots, n - 1.
\end{align*}
\]

**Proof.** Since the exponential convergence of $\tilde{\alpha}^f(t)$ is not guaranteed, Tikhonov’s theorem ([8], Theorem 11.2) cannot be directly applied to the system comprised of (44), (45) and (49). However, since $\tilde{\alpha}^f(t)$ does not depend on $\epsilon$, a simple state transformation $s(t) = \tilde{\alpha}^f(t) - \mu(t)$, where $\mu(t)$ is dynamically defined as $\mu(t) = A_r \mu(t) + c \tilde{\alpha}^f(t)$, results in the system $s(t) = A_r s(t) + bk^T \xi(t) - \tilde{\alpha}^f(t) + \sigma(t)$, for which the hypothesis of the Tikhonov’s theorem can be verified following the steps from [5], and the last three relationships in (49) can be concluded along with $s(t) - \hat{s}(t) = \mathcal{O}(\epsilon)$, where $\hat{s}(t) = \hat{e}(t) - \mu(t)$ and satisfies the exponentially stable dynamics $\hat{s}(t) = A_r \hat{s}(t)$. It follows that $\tilde{\alpha}^f(t) - \hat{e}(t) = s(t) - \hat{s}(t) = \mathcal{O}(\epsilon)$, which completes the proof.

### V. Performance Analysis

The following two lemmas are needed to prove our main result.
Lemma V.1 Let the command filtered controller for system (2) be defined by (31), (32), (33), (35) and (37). Then all closed-loop signals are bounded and

\[ x^f(t) - x^0(t) = \mathcal{O}(\varepsilon) . \]  

(50)

In addition, if \( \omega \) is sufficiently large, then

\[ u^f(t) - u^0(t) = \mathcal{O}(\varepsilon) . \]  

(51)

**Proof.** Since \( e^0(t) \in \mathcal{L}_\infty \) it follows from (40) that \( e^f(t) \in \mathcal{L}_\infty \), implying that \( x^f(t) \in \mathcal{L}_\infty \). It follows form Lemma IV.1 and Lemma IV.3 that \( \sigma_{i,1}^0(t) \in \mathcal{L}_\infty \) and \( \sigma_{i,2}^0(t) \in \mathcal{L}_\infty \) for \( i = 1, \ldots, n - 1 \). Then, (31) implies that \( x^f(t) \in \mathcal{L}_\infty \). Therefore \( \eta^f(t) \in \mathcal{L}_\infty \), hence \( \alpha^f_i(t) \in \mathcal{L}_\infty \) for all \( i = 1, \ldots, n \). Since \( e_i(t) \) is exponentially stable, it follows that \( v^0(t) \in \mathcal{L}_\infty \), and hence \( u^f(t) \in \mathcal{L}_\infty \).

It is straightforward to compute the difference

\[ x^f_i(t) - x^0_i(t) = e^0_i(t) + \sigma_{i,1}^0(t) - \alpha^0_i(t) . \]  

Since \( e^0_i(t) = \mathcal{O}(\varepsilon) \), we have \( \sigma_{i,1}^0(t) = \mathcal{O}(\varepsilon) \), and \( \alpha^0_i(t) = \mathcal{O}(\varepsilon) \), it follows that \( x^f_i(t) - x^0_i(t) = \mathcal{O}(\varepsilon) \) for all \( i = 1, \ldots, n \). Next, we compute the difference

\[ u^f(t) - u^0(t) = - k^T [v^0(t) - \hat{z}(t)] + \alpha^0_i(t) . \]  

Since \( \hat{z}(t) \in \mathcal{L}_\infty \), it follows that \( \alpha^f_i(t) \in \mathcal{L}_\infty \). On the other hand \( \alpha^f_i(t) = \eta_i(t) \), \( \eta_i(t) = \eta_i(t) + \omega \sigma_{n-1,2}(t) - \alpha_{n-1}(t) \). Since \( \varphi_n(x) \) is smooth, we have \( \varphi_n(x^f(t)) = \varphi_n(x^0(t)) = \mathcal{O}(\varepsilon) \). Then it follows that \( u^f(t) - u^0(t) \in \mathcal{L}_\infty \) if \( \omega > \max(\|k\|, \|\theta_n\|) \).

Lemma V.2 Let the command filtered controller for system (2) and identification model (4) be defined by (31), (32), (33), (35) and (37). Then all signals are bounded and

\[ \hat{x}^f(t) - \hat{x}(t) = \mathcal{O}(\varepsilon) . \]  

(52)

In addition, if \( \omega \) is sufficiently large, then

\[ \hat{u}^f(t) - \hat{u}(t) = \mathcal{O}(\varepsilon) . \]  

(53)

**Proof.** Since \( \hat{e}(t) \in \mathcal{L}_\infty \) it follows from (49) that \( \hat{e}^f(t) \in \mathcal{L}_\infty \), implying that \( \hat{z}(t) \in \mathcal{L}_\infty \). It follows form Lemma IV.2 and Lemma IV.4 that \( \sigma_{i,1}(t) \in \mathcal{L}_\infty \) and \( \sigma_{i,2}(t) \in \mathcal{L}_\infty \) for \( i = 1, \ldots, n - 1 \), therefore (41) implies that \( \hat{x}^f(t) \in \mathcal{L}_\infty \). Since \( \hat{x}^f(t) \in \mathcal{L}_\infty \), it follows that \( \hat{z}^f(t) \in \mathcal{L}_\infty \). Therefore \( \hat{\eta}^f(t) \in \mathcal{L}_\infty \), hence \( \hat{\alpha}^f_i(t) \in \mathcal{L}_\infty \) for all \( i = 1, \ldots, n \). Since \( e_i(t) \) is exponentially stable, it follows that \( \hat{v}(t) \in \mathcal{L}_\infty \), and hence \( \hat{u}^f(t) \in \mathcal{L}_\infty \).

It is straightforward to compute the difference

\[ \hat{x}^f_i(t) - \hat{x}_i(t) = \hat{e}^f_i(t) - \hat{e}_i(t) + \hat{\sigma}_{i,1}(t) - \hat{\alpha}_i(t) . \]  

(54)

Since \( \hat{e}^f_i(t) = \mathcal{O}(\varepsilon) \) and \( \hat{\sigma}_{i,1}(t) = \mathcal{O}(\varepsilon) \), it follows that \( \hat{x}^f_i(t) - \hat{x}_i(t) = \mathcal{O}(\varepsilon) \) for all \( i = 1, \ldots, n \). Next, we compute the difference

\[ \hat{u}^f(t) - \hat{u}(t) = - k^T [\hat{v}(t) - \hat{z}(t)] + \hat{\alpha}_n(t) . \]  

Since \( \hat{e}_n(t) = \mathcal{O}(\varepsilon) \), it follows that \( \hat{v}(t) - \hat{z}(t) = \mathcal{O}(\varepsilon) \). Then it follows from Lemma V.1 that \( \mathcal{O}(\varepsilon) \) holds if \( \omega > \max(\|k\|, \|\theta_n\|) \).

**Theorem V.1** Let the system's controller be defined according to command filtered scheme given by (41), (42), (43), (45) and (47). Then the input and output tracking errors satisfy the following upper bounds

\[ |\hat{u}(t)| \leq \beta_T e^{-\nu t} + \frac{\beta_{\varphi}}{\sqrt{\gamma}} + \mathcal{O}(\varepsilon) \]  

(54)

\[ |e(t)| \leq \beta_T e^{-\nu t} + \frac{\beta_{\varphi}}{\sqrt{\gamma}} + \mathcal{O}(\varepsilon) , \]  

(55)

where \( \beta_T, \beta_{\varphi}, \beta_{\theta}, \beta_n, \beta_{\beta_0}, \beta_{\gamma_0} \) and \( \nu \) are positive constants defined in the proof.

**Proof.** It is easy to see that

\[ \hat{u}(t) = u^0(t) - u^f(t) + u^f(t) - u(t) = \mathcal{O}(\varepsilon) - k^T \hat{v}(t) + \hat{\alpha}_n(t) , \]  

(56)

where \( \hat{v}(t) = v^0(t) - v(t) \) and \( \hat{\alpha}_n(t) = \alpha^f_n(t) - \alpha^0_n(t) \). Obviously, \( \hat{v}(t) \) satisfies the dynamics

\[ \hat{v}(t) = A_v \hat{v}(t) - c \hat{x}^f(t) \]  

(57)

with the initial conditions \( \hat{v}(0) = \hat{x}^f(0) - \hat{y}(0) \), where we denote \( \hat{y}_i(t) = \eta_i(t) - \hat{y}_i(t) = \hat{\theta}_i(t) \varphi_i(t) \) for each \( i = 1, \ldots, n \). Similar to (18), for some positive constants \( \beta_{\varphi_1}, \beta_{\varphi_2} \), one can obtain from (57) that

\[ |\hat{v}(t)| \leq \beta_{\varphi_1} e^{-\nu t} + \frac{\beta_{\varphi_2}}{\sqrt{\gamma}} , \]  

(58)

Next, we observe that the signals \( \hat{\sigma}_{i,1}(t) = \sigma^0_{i,1}(t) - \sigma_{i,1}(t) \) and \( \hat{\sigma}_{i,2}(t) = \sigma^0_{i,2}(t) - \sigma_{i,2}(t) \) satisfy the operator equations

\[ \hat{\sigma}_{i,1}(s) = G_1(s) \hat{\alpha}_i(s), \quad G_1(s) = \frac{\omega^2}{s^2 + 2(\omega s + \omega^2)} \]  

(59)

\[ \hat{\sigma}_{i,2}(s) = G_2(s) \hat{\alpha}_i(s), \quad G_2(s) = \frac{\omega s}{s^2 + 2(\omega s + \omega^2)} , \]
where \( \hat{\alpha}_1(t) = \alpha_1(t) - \hat{\alpha}_1(t) \). Since \( \|G_1(s)\|_{\mathcal{H}_\infty} = 1 \) for \( \zeta \geq \sqrt{2}/2 \) and \( \|G_2(s)\|_{\mathcal{H}_\infty} = (2\zeta)^{-1} \), it follows from [8] (p. 201) that
\[
\|\hat{\sigma}_{i,1}(t)\|_{\mathcal{L}_\infty} \leq \|\hat{\alpha}_1(t)\|_{\mathcal{L}_\infty} \quad (60)
\]
\[
\|\hat{\sigma}_{i,2}(t)\|_{\mathcal{L}_\infty} \leq (2\zeta)^{-1}\|\hat{\alpha}_1(t)\|_{\mathcal{L}_\infty} .
\]

Now we can recursively compute the bound of \( \hat{\alpha}_1(t) \) using the definitions (33) and (43). For \( i = 1 \), we have \( \hat{\alpha}_1(t) = \hat{\eta}_1(t) \), therefore
\[
|\hat{\alpha}_1(t)| \leq \beta_3 e^{-\nu_3 t} + \frac{\beta_4}{\sqrt{\gamma}} .
\]

For \( i = 2, \ldots, n \), we have \( \hat{\alpha}_i(t) = -\hat{\eta}_i(t) + \omega \hat{\sigma}_{i,2}(t) \), hence
\[
|\hat{\alpha}_n(t)| \leq q_n \beta_1 e^{-\nu_1 t} + q_n \beta_4 \sqrt{\gamma} , \quad (62)
\]
where \( q_n = 1 + \frac{\nu_1}{\beta_1} + \cdots + \left( \frac{\nu_1}{\beta_1} \right)^n \).

Combining the relationships (56), (58) and (62) we obtain (54), where \( \beta_7 = \|k\|_{\mathcal{L}_{11}} + q_n \beta_3 \), \( \beta_8 = \|k\|_{\mathcal{L}_{12}} + q_n \beta_4 \), since \( \nu_1 > \nu_2 \) for large values of \( \gamma \) (fast adaptation).

To prove (55), we notice that \( e(t) = y_i(t) - y_p(t) = \hat{e}_i(t) + \hat{e}_1(t) - x_{1,1}(t) = \hat{x}_i(t) + \hat{e}_1(t) \). Since \( \hat{e}_1(t) = \hat{e}_1(t) + \mathcal{O}(e) \), it follows that \( e(t) = \hat{x}_i(t) + \hat{e}_1(t) + \mathcal{O}(e) \). Using (18) one can obtain from (29) that
\[
|\hat{e}(t)| \leq \beta_{13} e^{-\nu_2 t} + \frac{\beta_{14}}{\sqrt{\gamma}} ,
\]
for some positive constants \( \beta_{13} \) and \( \beta_{14} \), and the relationship (55) follows. The proof is complete.

It follows from Theorem V.1 that the bounds on the input and output tracking errors can be systematically decreased by choosing large values for \( \omega \) and \( \gamma \).

**Remark V.1** The proposed approach can be extended to systems
\[
\dot{x}(t) = \psi(x(t)) + Ax(t) + bu(t) + \eta(t) , \quad (64)
\]
where \( \psi(x(t)) \) is a vector of known smooth functions. In this case, the identification model includes the known terms
\[
\hat{x}(t) = \psi(x(t)) + A\hat{x}(t) + bu(t) + \hat{\eta}(t) + c\hat{x}(t) , \quad (65)
\]
which does not alter the prediction error dynamics (6).
The stabilizing functions and the control signal are modified to include the components of \( \psi(x(t)) \)
\[
\hat{x}(t) = \psi(x(t)) - \psi_1(x(t)) n_1(t) \quad (66)
\]
\[
\hat{u}(t) = -k^T z^0(t) + k_r r(t) - \psi_n(x(t)) + u_n(t) , \quad (69)
\]
The corresponding changes are made in the command filtered and certainty equivalence designs.

Also, the presented approach can be extended to multi-input-multi-output systems with unmatched uncertainties, and to the systems with time variant parameters.

**VI. Disturbance Rejection**

In the presence of external disturbances the system (1) can be written in the form
\[
\dot{x}(t) = Ax(t) + bu(t) + \eta(t) + d(t) , \quad (67)
\]
where \( d(t) = [d_1(t) \ldots d_n(t)]^T \) is piecewise continuous and bounded. From the perspective of the backstepping control, it is required that \( d_{j-1}^n(t) \) be bounded for \( j = 1, \ldots, n \). To streamline the derivations we also assume that \( d_n(t) \) has a bounded derivative. These derivations can be easily generalized to the case when \( d_n(t) \) is only bounded (see [16] for details).

The identification model takes the form
\[
\hat{x}(t) = A\hat{x}(t) + bu(t) + \hat{\eta}(t) + \hat{d}(t) + c\hat{x}(t) , \quad (68)
\]
where \( \hat{d}(t) \) is the estimate of the disturbance signal. The adaptive estimates are generated according to laws
\[
\dot{\hat{\theta}}_i(t) = \gamma \Pr\left( \hat{\theta}_i(t), \hat{x}_i(t); \varphi_i(x_1, \ldots, x_i) \right)
\]
\[
\hat{d}_i(t) = \gamma \Pr\left( \hat{d}_i(t), \hat{x}_i(t) \right) \quad i = 1, \ldots, n , \quad (69)
\]
where \( \Pr(\cdot, \cdot) \) denotes the projection operator [12], which is introduced to prevent the parameter drift. The state prediction error dynamics are written as
\[
\dot{\hat{x}}(t) = (A - c\hat{n}_n)\hat{x}(t) + \hat{\kappa}(t) , \quad (70)
\]
where we denote \( \hat{\kappa}(t) = \kappa(t) - \hat{\kappa}(t) = \eta(t) + d(t) - \hat{\eta}(t) - \hat{d}(t) = \hat{\eta}(t) + \hat{d}(t) \).

No asymptotic convergence can be guaranteed in the presence of external disturbances, but the transient bounds similar to the disturbance free case can be obtained.

**Lemma VI.1** The error signals \( \hat{x}(t), \hat{d}(t) \) and \( \hat{\theta}_i(t), i = 1, \ldots, n \) are uniformly ultimately bounded, and the following bounds hold
\[
\|\hat{\kappa}(t)\| \leq \beta_{15} e^{-\nu_1 t} + \frac{\beta_{16}}{\sqrt{\gamma}}
\]
\[
\|\hat{x}(t)\| \leq \beta_{17} e^{-\nu_2 t} + \frac{\beta_{18}}{\sqrt{\gamma}} , \quad (71)
\]

© 2014 John Wiley and Sons Asia Pte Ltd and Chinese Automatic Control Society

Prepared using asjauth.cls
where \( \nu_1, \nu_2 \) are the same as in Section III, and \( \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18} \) are positive constants defined in the proof.

**Proof.** Consider a candidate Lyapunov function
\[
V_1(t) = \frac{1}{2} \sum_{i=1}^{n} \left[ \ddot{x}^2_i(t) + \frac{1}{\gamma} \dot{\theta}_i^T(t) \dot{\theta}_i(t) + \frac{1}{\gamma} \ddot{\theta}_i^2(t) \right],
\]
(72)
the derivative of which computed along the trajectories of the prediction error dynamics (6) and the adaptive laws (5) has the form
\[
\dot{V}_1(t) = \sum_{i=1}^{n} \dot{\theta}_i^T(t) \left[ \ddot{x}_i(t) \varphi_i(x_1, \ldots, x_i) + \dot{\theta}_i(t) \right] + \ddot{x}^T(t) (A - cI_{n \times n}) \ddot{x}(t) + \frac{d}{\gamma} \frac{d^T(t) d(t)}{\gamma}.
\]
Substituting the adaptive laws and using the properties of the projection operator (see [12] for details) we obtain the following upper bound
\[
\dot{V}_1(t) \leq \ddot{x}^T(t) (A - cI_{n \times n}) \ddot{x}(t) + \frac{d}{\gamma} \frac{d^T(t) d(t)}{\gamma}.
\]
(73)
Therefore, the trajectories of the error system stay inside the Lyapunov level set
\[
L = \{(\dot{x}, \dot{\theta}, \ddot{d}) : V(\dot{x}, \dot{\theta}, \ddot{d}) = V_* \}.
\]
(77)
From the definition of \( V_1(t) \) we have
\[
\|e(t)\|^2 \leq 2V_1(t) \leq 2V_*.
\]
(78)
Hence, the following conservative bound can be derived
\[
\|\ddot{x}(t)\| \leq 2 \frac{d^T d}{\gamma (c-1)} + 2d^2 + \frac{\beta_{16}}{\sqrt{\gamma}}.
\]
(79)

We derive \( \bar{\kappa}(t) \)-dynamics following the steps from [16]. It can be shown that selecting initial estimates inside the region defined by the projection operator results in the equation
\[
\ddot{x}(t) = \dot{\theta}_i^T(t) \varphi_i(t) + \dot{\theta}_i^T(t) \varphi_i(t) + \dot{d}(t)
\]
(80)
for all \( i = 1, \ldots, n \), on some initial interval \( (0, t_1) \).
Here, we denote \( \bar{\rho}_i(t) = 1 + \varphi_i(t) \varphi_i(t), \bar{h}_i(t) = \dot{\theta}_i^T(t) \varphi_i(t) + \dot{d}(t) \).

Next, we compute the bounds of the signals \( \ddot{x}(t) \) and \( \kappa(t) = \bar{\rho}_i(t) + \ddot{d}(t) \). It follows from inequality (74) that \( \dot{V}_1(t) \leq 0 \) whenever
\[
V_1(t) > V_* = \frac{2d^*d_1}{\gamma (c-1)} + \frac{2d^2 + 4\theta^2}{\gamma}.
\]
(76)
VII. Output Feedback

In this section we show that the output feedback control problem for conventional parametric strict feedback systems can be cast as a disturbance rejection problem considered in the previous section. Let only the output of system (1) be available for feedback, and assume that the equations \( f_j, j = 1, \ldots, n \) depend only on the output \( y \). Then the system (1) can be written in the observer form

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + k_0 y(t) + b u(t) + \Phi^T(y) \theta \\
y(t) &= x_1(t),
\end{align*}
\]

where the vector \( k_0 \) is selected such that the matrix \( A_0 = A - k_0 b_0^T \) is Hurwitz, \( b_0 = [1 \ 0 \ \ldots \ 0]^T \), \( \Phi(y) = \text{diag}[\varphi_1(y), \ldots, \varphi_n(y)] \) is a block diagonal matrix, and \( \theta = [\theta_1, \ldots, \theta_n]^T \) is the combined parameter vector.

Following the steps from [10] (p.329), we transform the system (85) by means of the filters

\[
\dot{\xi}(t) = A_0 \xi(t) + \Phi(y),
\]

(86)

It is straightforward to show that

\[
x(t) = s(t) + \Xi^T(t) \theta + \delta(t),
\]

where \( \delta(t) \) is a solution of the exponentially stable system

\[
\dot{\delta}(t) = A_0 \delta(t).
\]

Using the representation 87 we transform the output dynamics as follows

\[
\begin{align*}
\dot{y}(t) &= x_2(t) + \theta_1^T \varphi_1(y) \\
&= s_2(t) + \Xi^T_{(2)}(t) \theta + \delta_2(t) + \varphi_1(y)^T \theta_1 \\
&= s_2(t) + h^T(t) \theta + \delta_2(t),
\end{align*}
\]

(89)

where \( \Xi_{(2)}(t) \) denotes the second column of matrix \( \Xi(t) \), \( h(t) = [\Xi_{(2,1)}(t) + \varphi_1^T \Xi_{(2,2)}(t), \ldots, \Xi_{(2,n)}(t)]^T \), and \( \Xi_{(2,j)}(t), j = 1, \ldots, n \) is the partition of \( \Xi_{(2)}(t) \) into \( p_j \) dimensional sub-vectors. Combining (89) with the \( s \)-dynamics we obtain

\[
\begin{align*}
\dot{y}(t) &= s_2(t) + h^T(t) \theta + \delta_2(t) \\
\dot{s}_1(t) &= s_{j+1}(t) + k_0[y(t) - s_1(t)], \quad j = 2, \ldots, n - 1 \\
\dot{s}_n(t) &= u(t) + k_0n[y(t) - s_1(t)],
\end{align*}
\]

(90)

We notice that \( \delta_2(t) \) is bounded and has a bounded derivative, hence the approach from Section VI can be applied considering \( \delta_2(t) \) as an external disturbance. In this case the prediction model is given by a scalar equation

\[
\dot{\hat{y}}(t) = s_2(t) + \hat{\kappa}(t) - c \tilde{y}(t),
\]

(91)

where \( \hat{\kappa}(t) = h^T(t) \hat{\theta}(t) + \hat{\delta}_2(t), \) and \( \hat{\theta}(t) \) and \( \hat{\delta}_2(t) \) are corresponding adaptive estimates given by the adaptive laws (69).

VIII. Simulation Examples

8.1. Example 1

Consider a third order uncertain system presented in [3] and [4].

\[
\begin{align*}
\dot{z}_1(t) &= 2 \sin(z_1(t)) + 1.5 z_2(t) + \omega_1(z, t) \\
\dot{z}_2(t) &= 0.8 z_1(t) x_2(t) + z_3(t) + \omega_2(z, t) \\
\dot{z}_3(t) &= -1.5 z_3^2(t) + 2 u(t) + \omega_3(z, t),
\end{align*}
\]

(92)

where the functions

\[
\begin{align*}
\omega_1(z, t) &= 0.2 \sin(t) + 0.1 z_1 + 0.12 \\
\omega_2(z, t) &= 0.3 \sin(2t) + 0.2 z_1 + 0.2 z_2 - 0.4 \\
\omega_3(z, t) &= 0.2 \sin(2t) + 0.2 z_1 + 0.3 z_2 + 0.2 z_3 + 0.3,
\end{align*}
\]

(93)

are treated as unknown disturbances, assuming the rest of the system is known. First, we transform the system (92) using substitutions \( x_1 = z_1, \ x_2 = 1.5 z_2, \ x_3 = 1.5 z_3 \). The resulting system is in the form of (67) with

\[
b = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \quad d(t) = \begin{bmatrix} 0.2 \sin(t) + 0.12 \\ 0.45 \sin(2t) - 0.6 \\ 0.3 \sin(2t) + 0.45 \end{bmatrix},
\]

\[
\eta = \begin{bmatrix} \theta_{1,1} \sin(x_1) + \theta_{1,2} x_1 \\ \theta_{2,1} x_1 x_2 + \theta_{2,2} x_1 + \theta_{2,3} x_2 \\ \theta_{3,1} x^3_1 + \theta_{3,2} x_1 + \theta_{3,3} x_2 \theta_{3,4} x_3 \end{bmatrix},
\]

where the values of unknown parameters are straightforward to compute. Unlike [3] and [4], we do not assume any known bounds and treat the nonlinearities as unknown terms. The external command is set to \( r(t) = 2 \sin(0.15 t) + 4 \cos(0.1 t) - 4 \) as in [3] and [4]. The reference model is selected with \( k^T = [8 \ 10.4 \ 5.2] \) and \( k_r = 8 \). The identification model is designed with \( c = 2 \sqrt{7} \). For the command filtering we set \( \omega = 500 \) and \( \zeta = 0.8 \). Figure 1 displays state prediction performance for \( \gamma = 400 \) on the interval \([0, 60] \) sec. The adaptive estimates of the components of \( \hat{\kappa}(t) = \eta(t) + d(t) \) are presented in Figure 2. The systems response to the external command along with the command filtered...
It can be observed that good tracking is achieved in all signals. To show the $L_\infty$ bound improvement of the error signals with the increase of adaptation rate we run simulations with $\gamma = 400, 800, 1600, 3200, 6400$ and compute the maximum of the magnitude of all error signals on the interval $[0, 160]$ sec after the effect of the parameter initialization dies out. The results are presented in Table 1. It can be learned that a four-fold increase in the adaptation rate results in about 50\% reduction in all error signals.

### Table 1. Errors magnitudes for different values of $\gamma$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>400</th>
<th>800</th>
<th>1600</th>
<th>3200</th>
<th>6400</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(t)$</td>
<td>0.2462</td>
<td>0.1799</td>
<td>0.1286</td>
<td>0.0913</td>
<td>0.0648</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>8.3972</td>
<td>6.1552</td>
<td>4.5944</td>
<td>3.5168</td>
<td>3.2853</td>
</tr>
<tr>
<td>$\hat{\kappa}_1(t)$</td>
<td>0.1746</td>
<td>0.1146</td>
<td>0.0760</td>
<td>0.0513</td>
<td>0.0351</td>
</tr>
<tr>
<td>$\hat{\kappa}_2(t)$</td>
<td>0.3158</td>
<td>0.2459</td>
<td>0.1741</td>
<td>0.0976</td>
<td>0.0488</td>
</tr>
<tr>
<td>$\hat{\kappa}_3(t)$</td>
<td>0.3678</td>
<td>0.3596</td>
<td>0.3127</td>
<td>0.2950</td>
<td>0.2371</td>
</tr>
<tr>
<td>$\hat{x}_1(t)$</td>
<td>0.0022</td>
<td>0.0010</td>
<td>0.0005</td>
<td>0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\hat{x}_2(t)$</td>
<td>0.0043</td>
<td>0.0024</td>
<td>0.0012</td>
<td>0.0005</td>
<td>0.0002</td>
</tr>
<tr>
<td>$\hat{x}_3(t)$</td>
<td>0.0037</td>
<td>0.0027</td>
<td>0.0023</td>
<td>0.0016</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

#### 8.2. Example 2

For the output feedback case we consider a third order uncertain system

$$
\dot{x}_1(t) = x_2(t) + \theta x_1^3(t) \sin(x_1(t)) \tag{94}
$$

$$
\dot{x}_2(t) = x_3(t)
$$

$$
\dot{x}_3(t) = u(t)
$$

which was presented in state feedback framework in [19] with $\theta = 1$. The goal is to design a controller such that the system’s output $y(t) = x_1(t)$ tracks the output $y_r(t) = x_{r,1}(t)$ of the reference model with $k^T = [8 \ 10.4 \ 5.2]$ and $k_\gamma = 8$, where $v(t) = \sin(t)$ for this simulation example. To implement the filters, we choose $k_0 = [6 \ 11 \ 6]^T$. The identification model is implemented with $c = 2\sqrt{\gamma}$, and the command filters are selected with $\omega = 500$ and $\zeta = 0.8$. Figure 4 displays state prediction performance and the adaptive estimate of $\kappa(t) = \theta x_1^3(t) \sin(x_1(t)) + \delta_3(t)$ for $\gamma = 2000$. The system’s sinusoidal response and the
command filtered certainty equivalent control signal vs the ideal response and ideal control are presented in Figure 5. It can be observed that good tracking is achieved in all signals. The $L_\infty$ bounds of the error signals are summarized in Table 1. Once again the increase in the adaptation rate results in corresponding decrease in all error signals as predicted.

![Graph](image1)

**Fig. 4.** Adaptive estimates $\hat{y}(t)$ and $\hat{\kappa}(t)$ vs true values in output feedback for $\gamma = 2000$.

![Graph](image2)

**Fig. 5.** Actual output and control signals vs ideal ones in output feedback for $\gamma = 2000$.

**IX. Concluding remarks**

We have presented an indirect state and output feedback adaptive control method for nonlinear systems with unmatched uncertainties and external disturbances that follows the certainty equivalence principle. The approach uses a fast identification model, which is independent of the control design and achieves desired transient and steady state properties by the proper choice of the design parameters. The controller is in the form of the command filtered backstepping control. The resulting tracking errors can be decreased as desired by speeding up the adaptation and command filtering processes, subject to available computational power.

**REFERENCES**


7. **S. I. Han and J. M. Lee,** “Improved Prescribed Performance Constraint Control for a Strict **Table 2.** Errors magnitudes in output feedback.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>4000</th>
<th>8000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{y}(t)$</td>
<td>0.0380</td>
<td>0.0262</td>
<td>0.0186</td>
<td>0.0133</td>
<td>0.0095</td>
</tr>
<tr>
<td>$\hat{u}(t)$</td>
<td>0.5872</td>
<td>0.4148</td>
<td>0.2919</td>
<td>0.2056</td>
<td>0.1452</td>
</tr>
<tr>
<td>$\hat{\kappa}(t)$</td>
<td>0.0420</td>
<td>0.0288</td>
<td>0.0202</td>
<td>0.0142</td>
<td>0.0101</td>
</tr>
<tr>
<td>$\hat{\gamma}(t)$</td>
<td>0.0013</td>
<td>0.0006</td>
<td>0.0003</td>
<td>0.0002</td>
<td>0.0001</td>
</tr>
</tbody>
</table>


