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Steady Secondary Momentum and Enthalpy Streaming in the Pulse Tube Refrigerator

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ABSTRACT

Our study investigates the steady secondary momentum and enthalpy streaming that occurs in the pulse tube refrigerator. The linearized mass, momentum and energy conservation laws that are described by N. Rott[1] are applied to a pulse tube, with the phase and amplitude of the axial velocity boundary conditions treated as independent parameters. Heat transfer between the gas and the tube wall is included. Heat transfer is shown to affect enthalpy flow by modifying the dynamic temperature amplitude and the temperature phase angle of the gas. We also calculate the steady mass flow circulation due to Reynolds stresses in the pulse tube. The length scale of the circulation is shown to be of the order of the tube length. Mass flow circulation is a loss mechanism because it results in a direct convection of enthalpy between the cold and hot ends.

INTRODUCTION

The one-dimensional enthalpy flow model for the pulse tube refrigerator (PTR) has been extensively developed[2]. It is applicable in the limit of negligible heat transfer between the gas and the tube wall. For a system with transport in the axial z-direction parallel to the tube length, the steady enthalpy flow is calculated from the time-averaged quantity $\langle H \rangle = \langle \dot{m}\, C_p T_d \rangle$ where the quantities in brackets are time averages. The time-averaged enthalpy flow is $\langle H \rangle$, the heat capacity at constant pressure is C_p, the mass flow is $\dot{m}$, and the dynamic temperature is T_d. Though the one-dimensional enthalpy flow model has been very successful, it does not include the effects of momentum and thermal diffusion.

Including momentum and thermal diffusion into the analysis will provide a greater understanding of the flow physics within a pulse tube. With thermal diffusion, one can examine how heat transfer between the gas and the tube wall affects enthalpy flow. One can then determine whether it helps or hinders. With momentum diffusion, one can understand how the Reynolds stresses in the viscous layer produce the forces that drive steady momentum streaming[3]. Momentum streaming is important for the PTR because it is a loss mechanism, i.e., it results in steady mass flow circulation between the cold and warm ends.

The present work addresses how thermal and momentum diffusion affect enthalpy flow and momentum streaming in the PTR. A two-dimensional axisymetric model – based on the mass, momentum and energy conservation laws – is the foundation of this analysis. The model is used to understand how heat transfer between the gas and the tube wall changes the temperature amplitude and phase angle of the gas. The changes are then assessed in terms of their impact on enthalpy flow. Steady momentum streaming is then addressed. The resulting analysis gives a general understanding of the transport mechanisms within the pulse tube.

SCALING PARAMETERS

The model is based upon the set of linearized conservation equations examined by Rott[1] and used by Swift[4] and by Merkli and Thomann[5] to describe thermoacoustic effects. The equations are recast in dimensionless form and applied to a cylindrical tube geometry with boundary conditions representative of the PTR. The basic state problem is then solved for the case of negligible axial temperature gradient. A complete description of the governing equations and the solution method would leave little room for the main purpose of this paper – that of showing calculated results. In the Appendix we give a brief description of the equations. For full details, the reader is referred to elsewhere[6]. In this section we define the system and outline the problem.

The axisymetric system under study is shown in Figure 1. All variables are dimensionless and are scaled from 0 to 1. The system is an ideal gas inside a tube that has a finite wall thickness. The problem is divided into two domains. The gas domain, which considers only the gas inside the tube, extends from $r = 0$ to $r = 1$ and $z = 0$ to $z = 1$. The tube wall domain, which considers only the tube wall thickness, extends from $y = 0$ to $y = 1$ and $z = 0$ to $z = 1$. A planar geometry is used for the tube wall domain. This is valid in the limit of a very thin-walled tube relative to the inner tube radius. The two problems are linked together through the boundary conditions between the gas and the tube wall interface; temperature and heat flux must be continuous across the interface.

For the gas domain, the velocity field and the thermodynamic variables of temperature, T, and density, ρ, are functions of spatial coordinates r and z, and of time, t. The pressure, p, depends only on z and t . This approximation is valid when the ratio of the tube length to tube radius is large. For the case of the PTR, it is usually greater than 10. For the tube wall domain, the temperature is a function of y and t.

The normalized velocity field for the gas domain is composed of axial velocity, u, and radial velocity, ν. The boundary conditions for u are at $r = 0$ and $z = 0$, $u = U_o e^{i2\pi t}$ where $U_o = 1$; and at $r = 0$ and $z = 1$, $u = U_L e^{i2\pi(t + \phi_U)}$ where U_L is the normalized velocity amplitude. The imaginary number $\sqrt{-1}$ is given by i. The boundary conditions for temperature are at the centerline ($r = 0$) the radial heat flux is zero; at the inner surface of the tube wall ($r = 1$, $y = 0$), the temperature and heat flux are continuous between the gas and the wall; and at the outer surface of the tube ($y = 1$) the radial heat flux is zero (adiabatic).

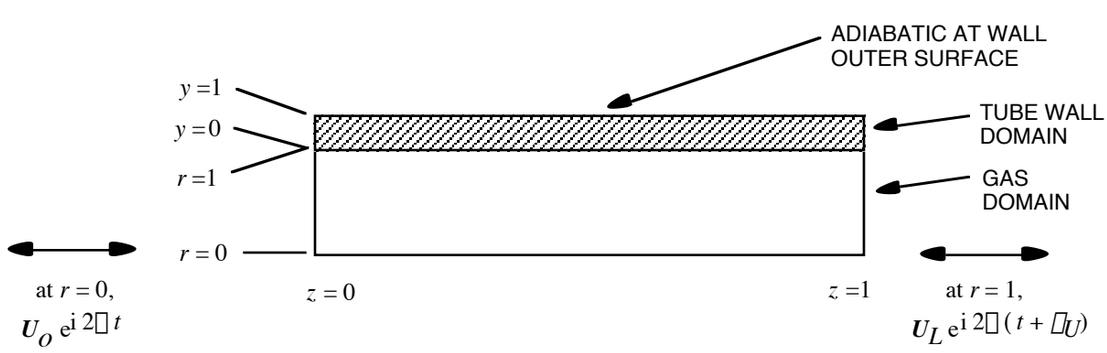

Figure 1. Axisymetric system studied. The gas domain extends from $r = 0$ to $r = 1$ and $z = 0$ to $z = 1$. The tube wall domain extends from $y = 0$ to $y = 1$ and $z = 0$ to $z = 1$.

Table 1. Definitions of scaling parameters

	Name	Definition	Physical Meaning
δ	expansion parameter	$U_o^* \omega^* / L^*$	ratio of displacement length to tube length
Va	Valensi Number	$r_w^{*2} / (\omega^* \nu^*)$	ratio of tube inner radius to viscous diffusion length
Pr	Prandtl Number	ν^* / α_g^*	ratio of viscous to thermal diffusion length scales
M	Mach Number	$U_o^* / \sqrt{\gamma R T_o^*}$	ratio of velocity at r=0 and z=0 to speed of sound
Fo	Fourier Number	$\alpha_t^* \omega^* / l_t^{*2}$	ratio of thermal diffusion length to tube wall thickness
U_L	velocity ratio	U_l^* / U_o^*	ratio of velocity at r=0 and z=1 to velocity at r=0 and z=0
ϕ_U	velocity phase angle		phase angle at r=0 and z=1 relative to velocity at r=0 and z=0

The quasi-steady, periodic solution for temperature, pressure, density and velocity field is analytic in r, z, t, and has non-dimensional scaling parameters δ, Va, Pr, M, Fo, U_L and ϕ_U. Table 1 defines the scaling parameters in terms of *dimensional* (starred) quantities, and describes their physical meaning. U_o^* is the velocity at $z = 0$, U_l^* is the velocity at $z = 1$, T_o^* is the reference temperature, l_t^* is the tube thickness, ω^* is the period of oscillation, r_w^* is the tube inner radius, L^* is the tube length, ν^* is the kinematic viscosity, α_t^* is the tube wall thermal diffusivity, α_g^* is the gas thermal diffusivity, γ is the heat capacity ratio and R is the ideal gas constant.

RESULTS

The Effect of Heat Transfer on the Periodic Gas Temperature

Heat transfer affects enthalpy flow by changing the dynamic temperature amplitude and phase angle of the gas. The quasi-steady periodic solution to the basic state problem for the gas temperature has the form

$$T_1 = \left\{ f_1[r,z] + f_2[r]\, T_w e^{i2\pi\phi_T} \right\} e^{i2\pi t} \tag{1}$$

where T_1 is the oscillating temperature of the gas, T_w is the temperature at the interface between the gas and the tube wall and ϕ_T is the phase angle between the temperature at the interface and U_o. The temperatures and phase angle are dimensionless and range from 0 to 1. Note that only the real part of Eq. (1) has physical meaning. The first term of Eq. (1) represents isothermal wall conditions. The second term is due to heat transfer coupling between the gas and the tube wall. Heat transfer adds a lagging component that is strongest in the thermal diffusion layer. Figures 2a and 2b plot T_w and ϕ_T as functions of Fo for the given parameters δ, M, U_L , ϕ_U and VaPr (product of Va and Pr). The parameter values are representative of an OPT. For Fo << 1, $T_w \to 0$ which means that isothermal wall conditions are approached. For Fo $\to \infty$ adiabatic conditions are approached. Figure 2a shows that near isothermal conditions are reached when Fo $\leq$ 10.

A typical operating point for an OPT is at VaPr $\approx$ 10 and Fo $\approx$ 10. In this case, Fig. 2a shows that T_w is $\boldsymbol{O}(10^{-2})$. The symbol $\boldsymbol{O}$ is used to mean "of order". This is about one order less than the

dynamic temperature amplitude, which is $\boldsymbol{O}(10^{-1})$ relative to the reference temperature. This means that for OPTs, the temperature at the interface between the gas and tube wall is close to

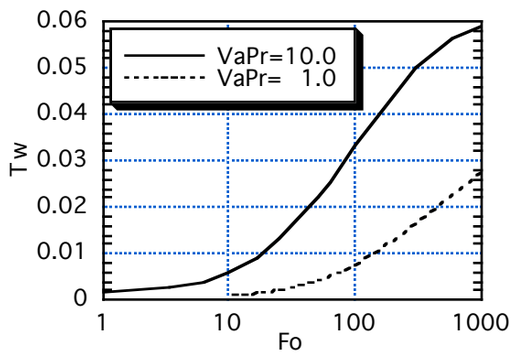

Fig. 2a

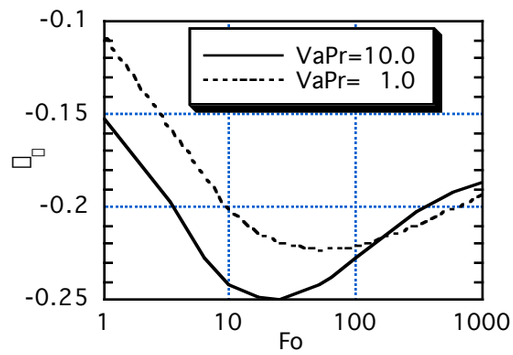

Fig. 2b

Figure 2. The effect of heat transfer on the dynamic temperature amplitude T_w , and the phase angle ϕ_T, at the tube wall inner surface (at $r = 1$). Fo is the Fourier Number for the tube wall (defined in Table 1). Other parameters; $U_L = 0.3$, $\phi_U = -0.25$, $\phi = 0.667$, M = 5.1 x 10^{-4}.

being isothermal, i.e., heat transfer is present. This is important because for one-dimensional models, it is assumed that the conditions at the interface are adiabatic. Figure 2b plots Fo vs. ϕ_T. The plot shows that ϕ_T is less than zero which means that the temperature at the interface lags the reference velocity at $z = 0$.

The BPT which operates at $U_L = 0$ and VaPr $\gg 1$ ideally operates at the isothermal limit $T_w \to 0$. In this case, heat transfer between the gas and tube wall produce the proper phase shift between velocity and temperature for enthalpy to flow. Calculations show that ϕ_T remains relatively constant at $\phi_T \approx -0.25$ even for large Fo. However, since proper phasing takes place only in the diffusion layer, maximum efficiency is obtained when thermal diffusion fills the entire tube, i.e., VaPr $\sim\boldsymbol{O}(1)$. Stack arrangements are then necessary to increase the total enthalpy flow, such as used in acoustic refrigerators.

Figure 3 shows an example of how heat transfer affects radial temperature distribution and phase angle relative to an isothermal wall. The parameter values are the same as those in Fig. 2 with VaPr = 10 and Fo = 100. Three times are shown: $t = 0.2$, $t = 0.35$ and $t = 0.5$. The plot shows how the oscillating wall temperature modifies the temperature primarily in the thermal diffusion layer ($r > 0.5$). At the core ($r < 0.5$) the temperatures are similar, both in amplitude and phase. At $r = 0$, the temperature amplitude and phase are close to one-dimensional predictions. For one-dimensional models, the temperature is not a function of r. Figure 3 clearly shows that when diffusion is considered, the temperature profiles are very different from idealized one-dimensional models.

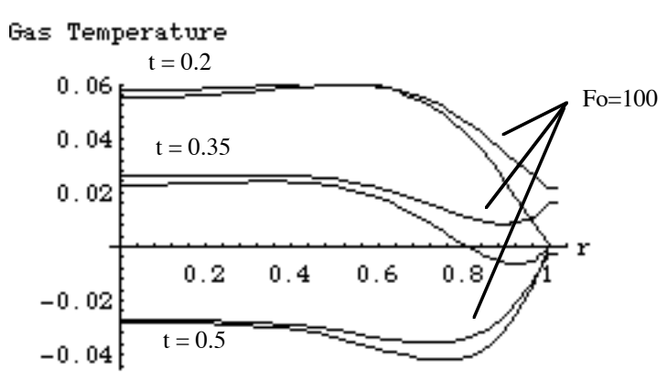

Figure. 3. The effect of heat transfer on the oscillating gas temperature as a function of radial distance for Fo = 100 and for an isothermal tube wall; $U_L = 0.3$, $\Phi_U = -0.25$, $\Phi = 0.667$, M = 5.1 x 10^{-4}, VaPr = 10.

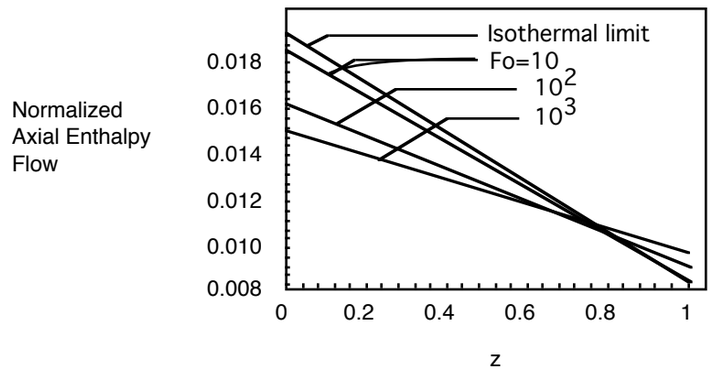

Figure 4. Normalize enthalpy flow vs. z for an OPT configuration: $\Phi = 0.55$, VaPr = 10, $U_L = 0.3$, $\Phi_U = -0.25$ and M = 2.0 x 10^{-4}; The reference enthalpy flow is 58.7W.

The Effect of Heat Transfer on Enthalpy Flow

Enthalpy flow is calculated from the quantity

$$\langle H \rangle = 2\pi \int_0^z \int_0^1 \left(\overline{u_O T_1 r} \right) dr \, dz \qquad (2)$$

where the overbar means time-averaged. The reference boundary condition on $\langle H \rangle$ at $z = 0$ is $\langle H \rangle = 0$. The heat capacity and density are not explicit in Eq. (2) because the variables are dimensionless, i.e., the dimensionless heat capacity and dimensionless density are both 1.

Figure 4 plots the normalized enthalpy flow vs. z for an OPT configured system. The strict interpretation of Figure 4 is that it shows the enthalpy flow at a time immediately after a PTR is started from isothermal conditions. Figure 4 shows that the effect of heat transfer is to steepen the enthalpy flow gradient; enthalpy flow decreases as z increases. This means that enthalpy is continuously rejected to the wall. This reduces the amount of enthalpy that can be rejected out of the system at the hot heat exchanger, thereby reducing refrigeration. Isothermal wall conditions are effectively attained for Fo $\leq$ 10. OPTs typically operate at Fo $\gg$ 10. Figure 4 shows that when there is heat transfer, enthalpy flow near the hot heat exchanger ($z = 1$) decreases when compared to the more adiabatic case of Fo >>> 1.

Enthalpy Flux Fields

Figures 5a and 5b show an OPT enthalpy flux field for both an isothermal tube wall and for Fo = 1000, respectively. The scaling parameters are the same as in Fig. 4 for VaPr = 10. For the

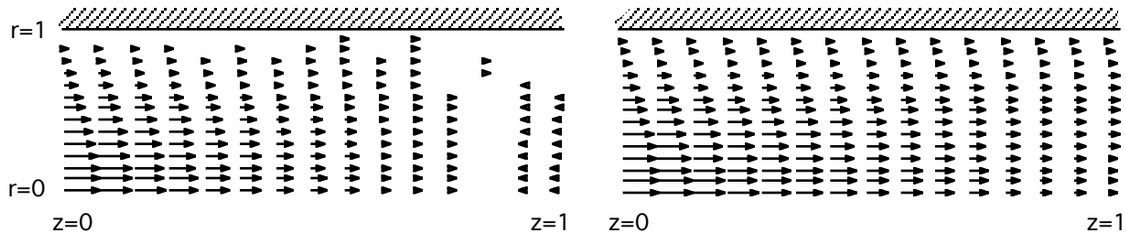

Fig. 6a. Isothermal tube wall Fig. 6b. Fo = 1000

Figure 6. Enthalpy flux field for an OPT: $\Phi = 0.55$, VaPr = 1.0, U_L=1, $\Phi_U = -0.25$, M = 5.1 x 10^{-4}. For Fo = 1000, $\Phi_T = -0.143$, $T_w = 0.0086$.

isothermal case the flux field is unidirectional. For the case in which Fo = 1000, the flux field reverses itself in the thermal diffusion layer near $z = 0$. Near $z = 1$, however, the flux field has a larger positive flow over the isothermal case, resulting in greater enthalpy flow near the warm end as previously indicated in Fig. 4.

Figure 6 shows an OPT configuration in which VaPr = 1 and $U_L = 1$. These parameters illustrate a case in which the velocity amplitude at the tube ends are equal, and diffusion fills the entire tube. For the isothermal case in Fig. 6a there is a reversal of enthalpy flux in the vicinity of $z = 1$. This means that the highest temperature will be located at a position where $z < 1$. For the case of Fo = 1000, the entire enthalpy field flows in one direction.

An interesting case is a comparison between isothermal and near adiabatic conditions for a BPT. Figure 7 shows this comparison. Isothermal conditions in Fig. 7a are seen to produce unidirectional enthalpy flow. However, for adiabatic conditions in which there is little heat transfer, there is significant enthalpy streaming in the diffusion layer that is in a direction opposite to the core flow. For the BPT then, adiabatic conditions can be detrimental.

Steady Secondary Mass Flux

The steady secondary mass flux due to the Reynolds stresses is a quadratic quantity and therefore nonvanishing when time-averaged. Steady mass flow is strongly dependent on the velocity amplitude, velocity gradient, and on the velocity phase angle between the tube ends. It is less dependent on heat transfer with the tube wall. Figure 8 plots the steady z-component of mass flux as a function of r at $z = 0$ for a BPT configuration at different Va. The oscillating end of the tube is located at $z = 0$ and the closed end is located at $z = 1$. The Reynolds stresses, which are strongest in the momentum diffusion layer, are the driving forces[6]. By mass conservation for a closed system the time-averaged mass flow at any z location must be zero. This requires that the gas in the core move in a direction counterflow to the gas in the diffusion layer.

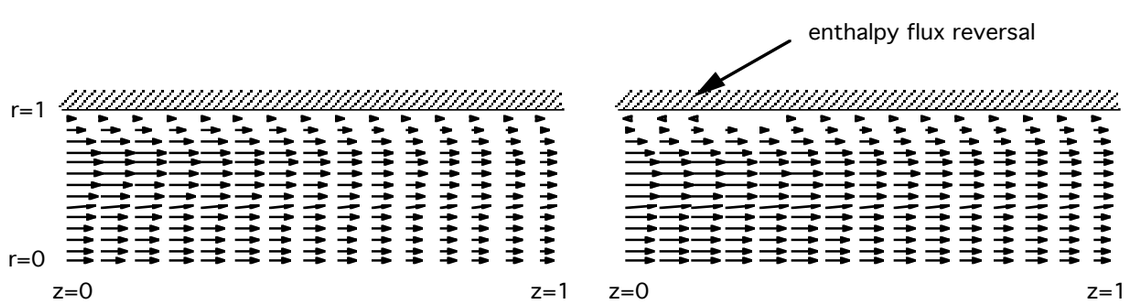

Fig. 5a. Isothermal tube wall Fig. 5b. Fo = 1000

Figure. 5 Enthalpy flux field for an OPT. The enthalpy flux in the thermal diffusion layer is seen to reverse for Fo = 1000. System parameters: $\Phi = 0.55$, VaPr = 10, $U_L = 0.3$, $\Phi_U = -0.25$, M = 2.0 x 10^{-4}. For Fo = 1000, $\Phi_T = -0.187$, $T_w = 0.049$.

Figure 8 shows the axial component of velocity as a function of radial distance for a BPT configured system for Va = 10, 20, 40 and 80. For Va < 35, the gas in the diffusion layer flows

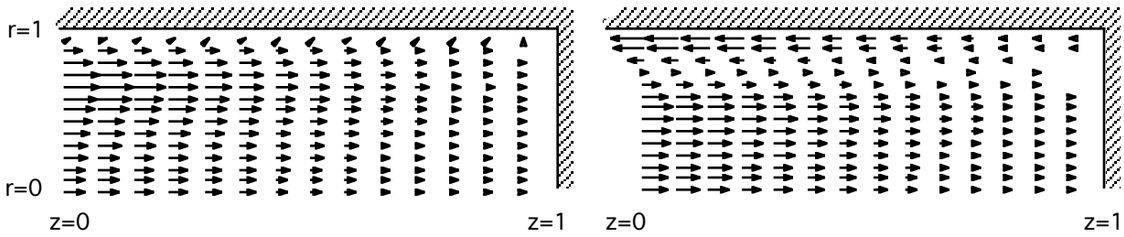

Fig. 7a. Isothermal tube wall Fig. 7b. Fo = 1000

Figure. 7 Enthalpy flux field for an BPT configuration: $\square = 0.55$, VaPr = 10, $U_L = 0$, M = 5.1 x 10^{-4}. For Fo = 1000, $\square_T = -0.262$ and $T_w = 0.021$.

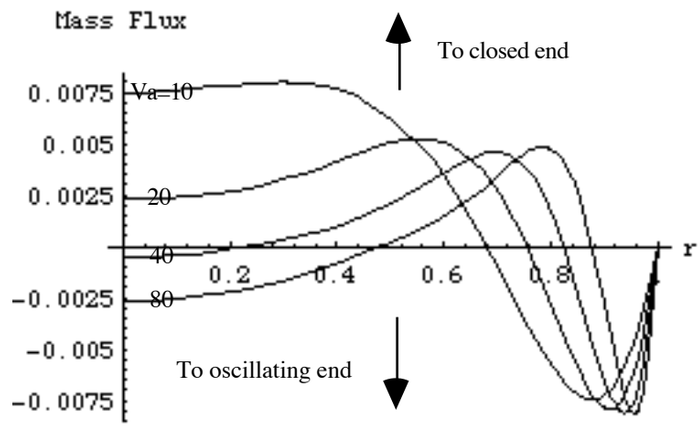

Figure 8. Normalized mass flux distribution for a BPT: $\square = 0.10$, M = 7.7 x 10^{-4}.

towards the oscillating end while the gas in the core flows towards the closed end. As Va is increased, a double "boundary layer" is produced. In the first layer very near the wall the gas continues to flow towards the oscillating end. The gas in the core, however, has switched direction and now it too flows towards the oscillating end. Between the core and the first viscous layer is a second viscous layer – a transition layer – which flows towards the closed end. This phenomena has also been predicted for incompressible oscillating flow systems with slightly varying tube diameters[7].

Figure 9 shows the steady velocity field for an flow system configured as an OPT for Va = 80, $U_L = 0.3$ and $\square_U = -0.25$. This is an example of the complexity of the steady flow patterns within a pulse tube. The radial velocity component is very apparent and is a result of the Reynolds stresses producing a body force on the fluid. We have experimentally confirmed the direction of the core flow for both an OPT and BPT configured system using smoke-wire flow visualization[6] for values of Va > 40.

The major consequences of mass streaming for the PTR are the losses associated with a direct enthalpy convection between the cold and warm ends. For axial temperature gradients of $\boldsymbol{O}(1)$, convected enthalpy flow is approximately one order of magnitude smaller than the enthalpy flow used to do refrigeration as calculated from a one-dimensional model.

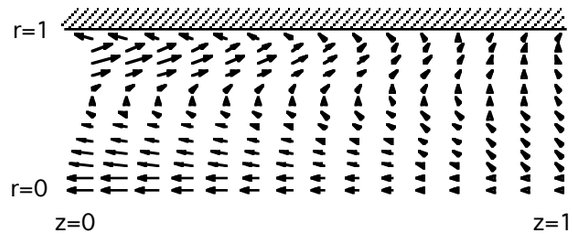

Figure 9. Normalized velocity field for an OPT with isothermal tube wall conditions: $\square = 0.15$, Va = 80, Pr = 0.7, $U_L = 0.3$ and $\square_U = -0.25$, M = 5.7 x 10^{-4}.

SUMMARY

Thermal and momentum diffusion give rise to steady secondary flows that can have important implications for the pulse tube refrigerator. We have developed a two-dimensional mathematical model which can be used to understand steady enthalpy and momentum flows. Included is the effect of heat transfer between the gas and tube wall. Some insightful results show that:

- Heat transfer between the gas and the tube wall produces axial enthalpy flow gradients. This is a possible loss mechanism, since the enthalpy flow available to do refrigeration is rejected to the tube wall and not to the hot heat exchanger.
- For real PT systems heat transfer is *not* negligible;
- Convection of enthalpy due to mass streaming is a loss mechanism. It is approximately one order of magnitude smaller than the enthalpy flow used to do refrigeration;
- The dimensionless parameters of ϵ, Va, Pr,, M, U_L, ϵ_U and Fo allow for an extended range of scaling that can include modeling of heat exchangers and regenerators.

The model is being further developed to optimize PTR performance through an entropy analysis that includes axial temperature gradients and viscous dissipation for geometries in which Va << 1. Steady mass streaming of enthalpy and the corresponding losses are also being examined through velocity amplitude and velocity phase angle dependency, and system geometry.

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APPENDIX

Summary of Mathematical Formulation

A series expansion solution in the small parameter ϵ allows linearization of the non-dimensional conservation equations. The expansion parameter is equal to the inverse of the Strouhal Number which is a frequency parameter. The resulting set of coupled linear partial differential equations are derived in the limit of lubrication theory. They are presented here as the non-dimensional equivalent to the equations first examined comprehensively by Rott[1]. Periodic solutions in the limit of negligible axial temperature gradients are sought for the basic state problem. Axial velocity boundary conditions are parametric in amplitude and phase angle. The resulting basic state solution is extended to the time-averaged $\boldsymbol{O}(\epsilon)$ problem which describes the steady secondary motion of interest. The steady momentum flow is found to be driven by the Reynolds stresses in the viscous layer near the tube wall. The steady enthalpy flow is generated by the conditions on axial velocity boundary conditions and is influenced by thermal diffusion whose temperature amplitude and temperature phase angle are intimately connected with the tube wall Fourier Number, Fo.

Basic State Problem

The axisymetric system under examination is shown in Figure 1 of the main paper. The normalized velocity field is composed of scalars axial velocity, u, and radial velocity, Φ. The velocity field and the thermodynamic variables of temperature, T, and density, ρ are functions of spatial coordinates r and z, and of time, t. The thermodynamic pressure, p, depends only on z and t, in accordance with lubrication theory.

The conservation equations for mass, momentum, energy and the equation of state for an ideal gas are scaled to non-dimensional form, from which the expansion parameter ε is identified. The series expansion solution is of the form $\Phi(r, z, t) = [\Phi_o + \varepsilon\Phi_1(r, z,) + \varepsilon^2\Phi_2(r, z,) + \boldsymbol{O}(\varepsilon^3)]\ e^{i2\pi t}$ where t is the dimensionless time bounded between 0 and 1, and Φ represents the dependent variables $u(r, z, t)$, $\Phi(r, z, t)$, $T(r, z, t)$, $\rho(r, z, t)$, and $p(z, t)$. The symbol $\boldsymbol{O}$ means "of order". The terms for the dimensionless quantities p_o, ρ_o and T_o are equal to one while u_o and Φ_o are the basic state velocities to be solved. We use the notation $\Phi_{,\Phi}$ to mean the partial derivative of Φ with respect to Φ. The linearized equations for the basic state problem are:

$$u_{o,t} = \frac{\gamma p_{1,z}}{\gamma \mathrm{M}^2 \mathrm{Str}} + \frac{1}{\mathrm{Va}} \frac{\left(r u_{o,r}\right)_{,r}}{r} \tag{A1}$$

$$T_{1,t} = \frac{\gamma(\gamma-1)}{\gamma} p_{1,t} + \frac{1}{\mathrm{Pr}\mathrm{Va}} \frac{\left(r T_{1,r}\right)_{,r}}{r} \tag{A2}$$

$$0 = \rho_{1,t} + \frac{\left(\rho_o r\right)_{,r}}{r} + u_{o,z} \tag{A3}$$

$$p_1 = \rho_1 + T_1 \ . \tag{A4}$$

The non-dimensional parameters written in terms of dimensional (starred) quantities are

$$\varepsilon = \frac{U_o^* \tau^*}{L^*}; \qquad \mathrm{Va} = \frac{r_w^{*\,2}}{\nu_o^* \tau^*}; \qquad \mathrm{Pr} = \frac{\nu_o^*}{\alpha_g^*}; \qquad \mathrm{M} = \frac{U_o^*}{\sqrt{\gamma R T_o^*}}$$

where ε is the expansion parameter and equal to the inverse of the Strouhal Number , Str; Va is the Valensi Number, Pr is the Prandtl Number, M is the Mach Number, U_o^* is the reference velocity, and τ^* is the period; L^* is the tube length and r_w^* is the inner radius of the tube; ν_o^* is the kinematic viscosity, α_g^* is the gas thermal diffusivity and T_o^* is the reference temperature; γ is the heat capacity ratio, and R is the ideal gas constant.

The velocity boundary conditions are normalized with respect to U_o^* which is the velocity at $r = z = 0$. They are

$$\begin{aligned} &r = 0,\ u_{o,r} = 0,\ \Phi_o = 0 \\ &r = 1,\ u_o = \Phi_o = 0 \\ &r = 0 \text{ and } z = 0\ ,\ u_o = U_o = 1 \\ &r = 0 \text{ and } z = 1\ ,\ u_o = U_L\, e^{i2\pi\phi_U} \end{aligned} \tag{A5}$$

where U_L is the dynamic velocity amplitude ratio at $z = 1$ (see Table 1). The conditions on velocity at the tube ends allow the velocity amplitude U_L and the velocity phase angle, ϕ_U, to be independent parameters. The temperature boundary conditions are

$$r = 0,\ T_{1,r} = 0 \tag{A6}$$
$$r = 1,\ T_1 = T_w e^{i2\pi\phi_T}$$
where T_w is the temperature amplitude, ϕ_T is the phase angle between T_1 and the reference velocity U_o .

Periodic diffusion heat transfer applies over the tube wall domain while temperature and heat flux are continuous between the gas and tube wall. For the limit in which the tube wall thickness, l_t^*, is much less than the tube inner radius, r_w^*, planar geometry can be used for the tube wall. In this case, the unsteady thermal diffusion equation for the tube wall domain is

$$\theta_{1,t} = \mathrm{Fo}\,\theta_{1,yy} \tag{A7}$$

where θ_1 is the dynamic temperature amplitude for the tube wall. The Fourier Number, Fo, is defined as ratio of the square of the thermal penetration length to the tube wall thickness:

$$\mathrm{Fo} = \frac{\alpha_t^* \tau^*}{l_t^{*2}} \tag{A8}$$

where α_t^* is the thermal diffusivity of the tube wall and is equal to the thermal conductivity divided by the product of density and heat capacity. The boundary conditions are that temperature be continuous at the inner surface of the tube wall (interface between the gas and the tube) and that adiabatic conditions apply at the outer surface:

$$y = 0,\ \theta_1 = T_1(\text{at } r = 1) = T_w e^{i2\pi\phi_T} \tag{A9}$$
$$y = 1,\ \theta_{1,y} = 0\ .$$

The additional unknown of ϕ_T is found from the condition that heat flux be continuous across the interface between the gas and the tube:

$$\text{at } r = 1,\ y = 0,\ z = 0;\ \{k_t \theta_{1,y}\}_{\text{at y=0}} = \{k_g T_{1,r}\}_{\text{at r=1, z=0}} \tag{A10}$$

where k_t and k_g are the thermoconductivities of the tube wall and gas, respectively. A numerical search for the value of ϕ_T and T_w that satisfies Eq. (A10) is required. Equations (A1) to (A4), (A7) and (A10) along with the boundary conditions comprise the basic state problem for the unknowns u_o, θ_o, θ_1, T_1, p_1, θ_1, T_w and ϕ_T. The equation set is solved yielding a complicated solution contorted with analytic functions in r, z, t, and with dimensionless scaling parameters τ, Va, Pr, M, Fo, U_L and ϕ_U. In general, the effect of decreasing heat transfer (increasing Fo, Fo > 1) from the isothermal wall case (Fo << 1) is to add a lagging component to θ_o, θ_1, T_1, p_1, while u_o remains unaffected.

O(τ) Steady Secondary Flow

The steady secondary momentum streaming problem is obtained from the ***O***(τ) time-averaged momentum equation:

$$\frac{\left(\overline{\theta_o u_o} r\right)_{,r}}{r} + \left(\overline{u_o^2}\right)_{,z} = -\frac{\overline{p}_{2,z}}{\tau \mathrm{M}^2\,\mathrm{Str}} + \frac{1}{\mathrm{Va}}\frac{\left(r\overline{u}_{1,r}\right)_{,r}}{r} \tag{A11}$$

where the overbars represent time-averaged quantities. The boundary conditions are

$$\text{at } r = 0,\ \overline{u}_{1,r} = 0 \tag{A12}$$

at $r = 1$, $\overline{u_1} = 0$

The terms on the left-hand-side of Eq. (A11) are the steady Reynolds stresses. Mass flow is constrained within the tube so that the *axial* net mass flow averaged over one cycle at any axial location must be zero:

$$0 = \int_0^1 \overline{u_1}\, r\, dr + \int_0^1 \overline{\rho_1 u_o}\, r\, dr\,. \tag{A13}$$

Equations (A11) and (A13) along with the boundary conditions Eq. (A12) are used to find u_1 and the pressure field p_2. The quantity u_1 by itself is not of physical significance; it represents only the streaming mass flux due to the Reynolds stresses. The observed steady mass flux scalars – a combination of the Reynolds stresses and the counterflow pressure gradient – are defined as $(\overline{\rho_s u_s})$ and $(\overline{\rho_s \rho_s})$ and are related through the $\boldsymbol{O}(\varepsilon)$ steady mass conservation equation:

$$\left(\overline{\rho_s u_s}\right) \equiv \overline{u_1} + \left(\overline{\rho_1 u_o}\right) \tag{A14}$$

$$\left(\overline{\rho_s \rho_s}\right) \equiv \overline{\rho_1} + \left(\overline{\rho_1 \rho_o}\right) \tag{A15}$$

$$0 = \overline{u_1}_{,z} + \left(\overline{\rho_1 u_o}\right)_{,z} + \frac{\left(\overline{\rho_1} r\right)_{,r}}{r} + \frac{\left(\overline{\rho_1 \rho_o} r\right)_{,r}}{r}\,. \tag{A16}$$

The steady $\boldsymbol{O}(\varepsilon)$ axial enthalpy flux is found by integrating the z-component of the $\boldsymbol{O}(\varepsilon)$ energy equation

$$\overline{h_2}_{,z}(r,z) = \left(\overline{u_o T_1}\right)_{,z} \tag{A17}$$

with the boundary condition at $z = 0$, $\overline{h_2} = 0$.

Equations (A14) to (A16) are solved for the steady mass flux field while Eq. (A17) is solved for the axial enthalpy flux. The complete solution to this problem is given in detail elsewhere[6].