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GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

On a Consistent System of Astrodynamic Constants

By

Helmut G. L. Krause

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ABSTRACT

An internally consistent system of astrodynamic constants is derived based upon theoretical coupling relationships and the most recent available experimental data. A previously existing discrepancy in the value of the gravitational parameter of the earth (as derived by different methods) has been eliminated. Likewise, several inconsistencies in the previously available system of lunar constants have been removed.

A new method of determining the ratio of the masses of the Earth and Moon has been derived and the results are in agreement with other determinations.

An error study of each constant is presented; both relative and absolute probable errors are listed.

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PROPULSION AND VEHICLE ENGINEERING DIVISION

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An internally consistent system of astrodynamic constants is derived based upon theoretical coupling relationships and the most recent available experimental data. A previously existing discrepancy in the value of the gravitational parameter of the earth (as derived by different methods) has been eliminated. Likewise, several inconsistencies in the previously available system of lunar constants have been removed.

A new method of determining the ratio of the masses of the Earth and Moon has been derived and the results are in agreement with other determinations.

An error study of each constant is presented; both relative and absolute probable errors are listed.

The results of this study can be summarized in the following list of astrodynamic constants:

1. GENERAL CONSTANTS

1. Newton's gravitation constant:

$$G = 6.670\,(1 \pm 0.0007) \times 10^{-8} = (6.670 \pm 0.005) \times 10^{-8}\ \mathrm{cm^3\,g^{-1}\,s^{-2}}(\mathrm{dyn\ \ cm^2\,g^{-2}})$$

2. Velocity of light:

$$c = 299\,792.5\,(1 \pm 3.3 \times 10^{-7}) = 299\,792.5 \pm 0.1\ \ \mathrm{km/s}$$

3. Solar parallax (Sun's equatorial horizontal parallax):

$$\pi_\odot = 8.''79414\,(1 \pm 5.8 \times 10^{-6}) = 8.''79414 \pm 0.''00005$$

4. Astronomical unit (mean Earth-Sun distance $= R_e / \pi_\odot \sin 1''$):

$$\mathrm{a.\,u.} = 149\,598\,700\,(1 \pm 2.7 \times 10^{-6}) = 149\,598\,700 \pm 400\ \ \mathrm{km}$$

5. Light year (distance which light travels in a year $= P_\oplus\ c$):

$$\mathrm{l.\,y.} = 31\,556\,925.9747\ \ \mathrm{c} = (9.460\,530 \pm 0.000\,003) \times 10^{12}\ \mathrm{km} = 63\,239.39 \pm 0.15\ \mathrm{a.\,u.}$$

6. Parsec (distance in which 1 a. u. appears as $1'' = 1$ a. u. $/\sin 1''$):

$$\mathrm{pc} = 206\,264.806\,247\ \mathrm{a.\,u.} = (3.085\,695 \pm 0.000\,008) \times 10^{13}\,\mathrm{km} = 3.261\,651 \pm 0.000\,008\ \ \mathrm{l.\,y.}$$

7. Light time for 1 a. u. :

$$\tau = \mathrm{a.u.}/c = 499.008\,(1 \pm 3.6 \times 10^{-5}) = 499.008 \pm 0.018\ \ \mathrm{s}$$

8. Constant of aberration:

$$K = 20.''4956\,(1 \pm 3.5 \times 10^{-5}) = 20.''4956 \pm 0.''0007$$

9. Obliquity of ecliptic

$$\epsilon = 23^\circ 27' 8.''26 - 46.''844\ \mathrm{T} - 0.''0060\ \mathrm{T}^2 + 0.''00183\ \ \mathrm{T}^3$$

$$\cos\epsilon = 0.917\,3917\ ;\ \sin\epsilon = 0.397\,985\,5\ (1900.0)$$

10. Newcomb's constant of precession (per tropical century):

$$P = \frac{p_0}{\cos\epsilon} = 5493.''62 - 0.''00364\ \mathrm{T} = (N + 2.''96)^*$$

11. Luni-solar precession in longitude:

$$p_0 = p_\odot + p_\mathrm{l} = 5039.''804 + 0.''4930\ \mathrm{T} - 0.''000\,04\ \mathrm{T}^2 = (N + 2.''72)$$

---

* N refers to Newcomb's precessional data

12. Geodetic precession in longitude:

$$p_g = \frac{3}{2} (v_\oplus / c)^2 \; n_\oplus = 1.''9188 \pm 0.''0002 = (N + 1.''92)$$

13. Observed luni-solar precession in longitude:

$$p_1 = p_0 - p_g = 5037.''885 + 0.''4930 \; T - 0.''00004 \; T^2 = (N + 0.''80)$$

14. Planetary precession in right ascension:

$$\lambda = - m + n \cot \epsilon = 12.''473 - 1.''8870 \; T - 0.''00014 \; T^2 = (N + 0.''00)$$

15. General precession in longitude:

$$p = p_1 - \lambda \cos \epsilon = m \cos \epsilon + n \sin \epsilon = 5026.''441 + 2.''2229 \; T + 0.''00026 \; T^2 = (N + 0.''80)$$

16. General precession in right ascension:

$$m = p_1 \cos \epsilon - \lambda = 4609.''236 + 2.''7945 \; T + 0.''00012 \; T^2 = (N + 0.''73)$$

$$= 307.^s 2824 + 0.^s 18630 + 0.^s 000008 \; T^2 = (N + 0.^s 0487)$$

17. General precession in declination:

$$n = p_1 \sin \epsilon = 2005.''005 - 0.''8533 \; T - 0.''00037 \; T^2 = (N + 0.''32)$$

$$= 133.^s 6670 - 0.^s 05689 \; T - 0.^s 000025 \; T^2 = (N + 0.^s 0213)$$

18. Mean siderial time rate (for 1950.0):

$$\dot{\Theta}_m = 7.292\,115\,854\,79 \times 10^{-5} \text{rad/s} = 1.002\,737\,909\,294 \;\; d_*/d \; (s_*/s)$$

$$= 15.041\,068\,639\,41 \;\; ''/s \; (°/h) = 360.985\,647\,346\,0 \;\; °/d$$

2. EARTH CONSTANTS

19. Semi-major axis of the Earth's orbit:

$$a_\oplus = 149\,598\,700 \; (1 \pm 2.7 \times 10^{-6}) = 149\,598\,700 \pm 400 \text{ km}$$

20. Siderial mean orbital motion (for 1950.0):

$$n_\oplus = 0.985\,609\,108\,0 \;\; °/d = 0.041\,067\,046\,15 \;\; ''/s \;\; (°/h)$$

$$= 1.990\,986\,581\,7 \times 10^{-7} \text{ rad/s}$$

21. Mean orbital velocity:

$$v_\oplus = a_\oplus n_\oplus = 29\,784.90\,(1 \pm 2.7 \times 10^{-6}) = 29\,784.90 \pm 0.08 \quad \text{m/s}$$

22. Mass ratio of the Sun to the Earth-Moon system:

$$\nu = \frac{M_\odot}{M_\oplus + M_\odot} = 328\,898.6\,(1 \pm 1.6 \times 10^{-5}) = 328\,898.6 \pm 5.2$$

23. Mass ratio of the Sun to the Earth:

$$\nu\,(1 + \kappa) = \frac{M_\odot}{M_\oplus} = 332\,947.6\,(1 \pm 2.0 \times 10^{-5}) = 332\,947.6 \pm 6.7$$

24. Gravitational parameter of the Earth:

$$\mu_\oplus = G M_\oplus = 398\,606.4\,(1 \pm 1.23 \times 10^{-5}) = 398\,606.4 \pm 4.9 \ \text{km}^3/\text{s}^2$$

25. Mass:

$$M_\oplus = 5.9761 \times 10^{27}\,(1 \pm 7.2 \times 10^{-4}) = (5.9761 \pm 0.0043) \times 10^{27}\ \text{g}$$

26. Equatorial radius:

$$R_e = 6\,378\,170\,(1 \pm 3.14 \times 10^{-6}) = 6\,378\,170 \pm 20 \ \text{m}$$

27. Polar radius:

$$R_p = 6\,356\,788\,(1 \pm 3.70 \times 10^{-6}) = 6\,356\,788 \pm 24 \ \text{m}$$

28. Flattening (oblateness, ellipticity):

$$f = \frac{R_e - R_p}{R_e} = 0.003\,352\,33\,(1 \pm 1.7 \times 10^{-4}) = 0.003\,352\,33 \pm 0.000\,000\,56 = 1{:}\,(298.30 \pm 0.05)$$

$$1 - f = (1 - e^2)^{1/2} = (1 + \epsilon^2)^{-1/2} = R_p / R_e = 0.996\,647\,67\,(1 \pm 5.6 \times 10^{-7})$$

29. First eccentricity of the meridian ellipse:

$$e = 0.081\,813\,33 + 0.000\,006\,80$$

$$e^2 = f\,(2 - f) = 0.006\,693\,42 + 0.000\,001\,11$$

30. Second eccentricity of the meridian ellipse:

$$\epsilon = 0.082\,088\,52 \pm 0.000\,006\,87$$

$$\epsilon^2 = \frac{e^2}{1 - e^2} = 0.006\,738\,53 \pm 0.000\,001\,13$$

31. Mean radius:

$$\bar{R} = (2\,R_e + R_p)\,/\,3 = 6\,371\,043\,(1 \pm 3.3 \times 10^{-6}) = 6\,371\,043 \pm 21 \ \text{m}$$

32. Radius for geodetic latitude $\varphi = \sin^{-1}\sqrt{1/3} = 35°15'51.''8$

$$R_1 = 6\,371\,083\,(1 \pm 3.3 \times 10^{-6}) = 6\,371\,083 \pm 21 \text{ m}$$

33. Radius for geocentric latitude $\phi = \sin^{-1}\sqrt{1/3} = 35°15'51.''8$

$$R_2 = 6\,371\,019\,(1 \pm 3.3 \times 10^{-6}) = 6\,371\,019 \pm 21 \text{ m}$$

34. Radius for sphere of same surface area:

$$R_S = 6\,371\,041\,(1 \pm 3.3 \times 10^{-6}) = 6\,371\,041 \pm 21 \text{ m}$$

35. Radius for sphere of same volume:

$$R_V = 6\,371\,035\,(1 \pm 3.3 \times 10^{-6}) = 6\,371\,035 \pm 21 \text{ m}$$

36. Surface area:

$$S_\oplus = 5.100\,711 \times 10^{14}\,(1 \pm 6.6 \times 10^{-6}) = (5.100\,711 \pm 0.000\,034) \times 10^{14} \text{ m}^2$$

37. Volume:

$$V_\oplus = 1.083\,225 \times 10^{21}(1 \pm 1.0 \times 10^{-5}) = (1.083\,225 \pm 0.000\,011) \times 10^{21}\text{m}^3$$

38. Mean density:

$$\bar{\rho}_\oplus = 5.5170\,(1 \pm 7.3 \times 10^{-4}) = 5.5170 \pm 0.0040 \text{ g/cm}^3$$

39. Angular velocity of the Earth's rotation:

$$\Omega_m = 7.292\,115\,146\,46 \times 10^{-5}\text{rad/s} = 1.002\,737\,811\,891 \text{ rot/d}$$

$$= 15.041\,067\,178\,37 \text{ ''/s } (°/\text{h}) = 360.985\,612\,280\,8 \text{ °/d}$$

40. Rotational velocity at the equator:

$$\Omega_m R_e = 465.1035\,(1 \pm 3.2 \times 10^{-6}) = 465.1035 \pm 0.0015 \text{ m/s}$$

41. Centrifugal acceleration at the equator:

$$\Omega_m^2 R_e = 0.033\,915\,88\,(1 \pm 3.2 \times 10^{-6}) = 0.033\,915\,88 \pm 0.000\,000\,11 \text{ m/s}^2$$

42. Centrifugal acceleration factor:

$$\varpi_\oplus = \frac{\Omega^2 R_e^3}{G M_\oplus} = 3461.369 \times 10^{-6}(1 \pm 2.2 \times 10^{-5}) = (3461.369 \pm 0.076) \times 10^{-6}$$

43. Oblateness coefficients of the Earth's potential:

$$J = 1623.48 \times 10^{-6}(1 \pm 1.8 \times 10^{-4}) = (1623.48 \pm 0.29) \times 10^{-6}$$

$$K = \frac{6}{7}\ D = 8.85 \times 10^{-6}$$

$$J_2 = \frac{2}{3}\ J = 1082.32 \times 10^{-6}(1 \pm 1.8 \times 10^{-4}) = (1082.32 \pm 0.19) \times 10^{-6}$$

$$J_4 = -\ \frac{4}{15}\ K = -\ \frac{8}{35}\ D = -\ 2.36 \times 10^{-6}$$

44. Coefficients of the Earth's gravity formula:

$$\beta = 5302.92 \times 10^{-6}; \quad \gamma = -\ 5.85 \times 10^{-6}$$

45. Mass of the Earth's atmosphere:

$$M_{atm} = \frac{p_0}{g} \cdot S_\oplus = (10\,332.275\ \mathrm{kg/m^2}) \cdot S_\oplus$$

$$= 5.270\,195 \times 10^{18}(1 \pm 6.6 \times 10^{-6}) = (5.270\,195 \pm 0.000\,035) \times 10^{18}\ \mathrm{kg}$$

46. Relative mass of the Earth's atmosphere:

$$A = M_{atm}/M_\oplus = 0.881\,88 \times 10^{-6}(1 \pm 7.3 \times 10^{-4}) = (0.881\,88 \pm 0.00064) \times 10^{-6}$$

47. Gravity acceleration correction factor:

$$\chi = g_e/(GM_\oplus/R_e^2) = 0.998\,165\,66\ (1 \pm 4.0 \times 10^{-7}) = 0.998\,165\,66 \pm 0.000\,000\,40$$

$$1 - \chi = A + \tilde{\omega} - J - \frac{1}{2}\ K = 1834.34 \times 10^{-6}\ (1 \pm 2.2 \times 10^{-4}) = (1834.34 \pm 0.40) \times 10^{-6}$$

48. Gravity acceleration at the Earth's equator:

$$g_e = 9.780\,362\ (1 \pm 3.3 \times 10^{-6}) = 9.780\,362 \pm 0.000\,032\ \ \mathrm{m/s^2}$$

49. Dynamic oblateness:

$$H = \frac{C - A}{C} = 3272.09 \times 10^{-6}\ (1 \pm 1.6 \times 10^{-4}) = (3272.09 \pm 0.54) \times 10^{-6} = 1/305.615 \pm 0.05$$

50. Moment of inertia parameter:

$$q = \frac{3}{2}\ \frac{C}{M_\oplus R_e^2} = \frac{J}{H} = 0.496\,16\ (1 \pm 3.4 \times 10^{-4}) = 0.496\,16 \pm 0.000\,17$$

51. Dimensionless moments of inertia:

$$\frac{A}{M_\oplus R_e^2} = J_2\left(\frac{1}{H} - 1\right) = 0.329\,69 \pm 0.000\,11$$

$$\frac{C}{M_\oplus R_e^2} = J_2/H = 0.330\,77 \pm 0.000\,11$$

52. Unit for the Earth's moments of inertia:

$$M_{\oplus} R_e^2 = 2.4311_4 \times 10^{38} (1 \pm 7.3 \times 10^{-4}) = (2.4311_4 \pm 0.0018) \times 10^{38} \text{ kg m}^2$$

53. Earth's moments of inertia:

$$A = 0.801\,52 \times 10^{38} (1 \pm 1.07 \times 10^{-3}) = (0.801\,52 \pm 0.000\,86) \times 10^{38} \text{ kg m}^2$$

$$C = 0.804\,15 \times 10^{38} (1 \pm 1.07 \times 10^{-3}) = (0.804\,15 \pm 0.00086) \times 10^{38} \text{ kg m}^2$$

$$C - A = 2.6313 \times 10^{35} (1 \pm 9.1 \times 10^{-4}) = (2.6313 \pm 0.0024) \times 10^{35} \text{ kg m}^2$$

54. Angular momentum:

$$C\,\Omega = 5.8640 \times 10^{33} (1 \pm 1.07 \times 10^{-3}) = (5.8640 \pm 0.0063) \times 10^{33} \text{ kg m}^2/\text{s}$$

55. Rotational energy:

$$\frac{1}{2} C\Omega^2 = 2.1380 \times 10^{29} (1 \pm 1.07 \times 10^{-3}) = 2.1380 \pm 0.0023) \times 10^{29} \text{ kg m}^2/\text{s}^2 \text{(joule)}$$

56. Circular velocity at the Earth's equator:

$$v_{\text{cir}} = \sqrt{\mu_{\oplus}/R_e} = 7905.404\ (1 \pm 7.7 \times 10^{-6}) = 7905.404 \pm 0.061 \text{ m/s}$$

3. LUNAR CONSTANTS

57. Mean observed distance from the Earth:

$$\bar{r}_{☾} = 384\,402.0\ (1 \pm 2.6 \times 10^{-6}) = 384\,402.0 \pm 1.0 \text{ km}$$

58. Relative mean lunar distance:

$$\bar{r}_{☾}/R_e = 1/\sin \pi_{☾} = 60.268\,38\ (1 \pm 5.8 \times 10^{-6}) = 60.268\,38 \pm 0.000\,35$$

59. Constant part of the sine of the perturbed lunar parallax:

$$\pi_{☾}' = \frac{\sin \pi_{☾}}{\sin 1''} = 3422.''438\ (1 \pm 5.8 \times 10^{-6}) = 3422.''438 \pm 0.''020$$

60. Mean perturbed equatorial horizontal parallax:

$$\pi_{☾} = 3422.''595\ (1 \pm 5.8 \times 10^{-6}) = 3422.''595 \pm 0.''020$$

61. Semi-major axis of the Moon's orbit:

$$a_{☾} = 1.000\,907\,681\ \bar{r}_{☾} = 384\,750.9\ (1 \pm 2.6 \times 10^{-6}) = 384\,750.9 \pm 1.0 \text{ km}$$

62. Siderial mean orbital motion (for 1950.0):

$$n_{☾} = 13.176\,358\,259\,8\ ^\circ/\text{d} = 0.549\,014\,912\,685\ ''/\text{s}\ (^\circ/\text{h})$$

$$= 2.661\,699\,407\,99 \times 10^{-6} \text{ rad/s}$$

63. Mean orbital velocity:

$$v_{☾} = a_{☾}\ n_{☾} = 1024.091\ (1 \pm 2.9 \times 10^{-6}) = 1024.091 \pm 0.003\ \mathrm{m/s}$$

64. Lunar inequality in the Moon's ecliptic longitude:

$$L = 6.''4439\ (1 \pm 3.0 \times 10^{-4}) = 6.''4439 \pm 0.''0019$$

65. Parallactic inequality in the Moon's ecliptic longitude:

$$P_{*} = 124.''986\ (1 \pm 3.3 \times 10^{-5}) = 124.''986 \pm 0.''004$$

66. Mass ratio of the Earth to the Moon:

$$1/\kappa = M_{\oplus}/M_{☾} = 81.250\ (1 \pm 3.0 \times 10^{-4}) = 81.250 \pm 0.024$$

67. Mass ratio of the Earth-Moon system to the Earth:

$$1 + \kappa = (M_{\oplus} + M_{☾})/M_{\oplus} = 1.012\,307\,7\ (1 \pm 3.7 \times 10^{-6}) = 1.012\,307\,7 \pm 0.000\,0037$$

68. Gravitational parameter of the Moon:

$$\mu_{☾} = G M_{☾} = 4905.92\ (1 \pm 3.1 \times 10^{-4}) = 4905.92 \pm 1.52\ \mathrm{km^3/s^2}$$

69. Mass:

$$M_{☾} = 7.3552 \times 10^{25}\ (1 \pm 1.02 \times 10^{-3}) = (7.3552 \pm 0.0075) \times 10^{25}\ \mathrm{g}$$

70. Moon's semi-diameter at mean distance:

$$s_{☾} = 932.''72\ (1 \pm 1.0 \times 10^{-4}) = 932.''72 \pm 0.''09$$

71. Relative radius of the visible disk of the Moon:

$$k = R_{☾}/R_{e} = 0.2725\,289\ (1 \pm 1.0 \times 10^{-4}) = 0.2725\,289 \pm 0.0000\,273$$

72. Radius of the visible disk of the Moon:

$$R_{☾} = \frac{b + c}{2} = 1738\ 236\ (1 \pm 1.0 \times 10^{-4}) = 1738\ 236 \pm 174\ \mathrm{m}$$

73. Longest semi-axis directed to the Earth:

$$a = R_{☾}/0.9995\,918 = 1738\,946\ (1 \pm 1.07 \times 10^{-4}) = 1738\,946 \pm 186\ \mathrm{m}$$

74. Medium semi-axis in orbital direction:

$$b = 0.9998\,116\ a = 1738\,618\ (1 \pm 1.20 \times 10^{-4}) = 1\,738\,618 \pm 209\ \mathrm{m}$$

75. Shortest semi-axis (rotational or polar radius):

$$c = 0.9993\,720\ a = 1\,737\,854\ (1 \pm 1.08 \times 10^{-4}) = 1\,737\,854 \pm 188\ \mathrm{m}$$

76. Volume:

$$V_{☾} = \frac{4}{3}\ \pi\ a\,b\,c\ = 2.20086 \times 10^{25}\,(1 \pm 3.35 \times 10^{-4}) = (2.20086 \pm 0.00074) \times 10^{25}\,\mathrm{cm}^3$$

77. Mean density:

$$\bar{\rho}_{☾} = 3.3420\ (1 \pm 1.5 \times 10^{-3}) = 3.3420 \pm 0.0050\ \ \mathrm{g/cm}^3$$

78. Surface density:

$$\rho_0 = 3.290\ \ \mathrm{g/cm}^3$$

79. Central density:

$$\rho_c = 3.420\ \ \mathrm{g/cm}^3$$

80. Inhomogenity factor of the Moon:

$$\lambda = 0.1991\ (1 \pm 5.0 \times 10^{-4}) = 0.1991 \pm 0.0001$$

81. Dimensionless moment of inertia parameters:

$$f = \frac{C - B}{C - A} = 0.70\ (1 \pm 2.86 \times 10^{-2}) = 0.70 \pm 0.02$$

$$\alpha = \frac{C - B}{C} = 0.0004395\ (1 \pm 3.03 \times 10^{-2}) = 0.0004395 \pm 0.0000133$$

$$\beta = \frac{C - A}{C} = 0.0006279\ (1 \pm 1.6 \times 10^{-4}) = 0.0006279 \pm 0.0000010$$

$$\gamma = \frac{B - A}{C} = 0.0001884\ (1 \pm 6.85 \times 10^{-2}) = 0.0001884 \pm 0.0000129$$

$$g = \frac{3}{2}\ \frac{C}{M_{☾}\ a^2} = 0.5972\ (1 \pm 5.0 \times 10^{-4}) = 0.5972 \pm 0.0003$$

$$J = \frac{3}{2}\ \frac{C - 1/2\ (A + B)}{M_{☾}\ a^2} = 0.0003187\ (1 \pm 1.38 \times 10^{-2}) = 0.0003187 \pm 0.0000044$$

$$K = \frac{3}{2}\ \frac{B - A}{M_{☾}\ a^2} = 0.0001125\ (1 \pm 6.84 \times 10^{-2}) = 0.0001125 \pm 0.0000077$$

$$L = \frac{3}{2}\ \frac{C - A}{M_{☾}\ a^2} = 0.0003750\ (1 \pm 2.1 \times 10^{-3}) = 0.0003750 \pm 0.0000008$$

82. Dimensionless moment of inertia differences:

$$\frac{C - A}{M_{☾}\ a^2} = 0.0002500\ (1 \pm 2.1 \times 10^{-3}) = 0.0002500 \pm 0.0000005$$

$$\frac{C - B}{M_{☾}\ a^2} = 0.0001750\ (1 \pm 3.09 \times 10^{-2}) = 0.0001750 \pm 0.0000054$$

82. Dimensionless moment of inertia differences (Continued):

$$\frac{B-A}{M_{☾}\,a^2} = 0.000\,0750\,(1 \pm 6.84 \times 10^{-2}) = 0.000\,0750 \pm 0.000\,0051$$

83. Dimensionless moments of inertia:

$$\frac{A}{M_{☾}\,a^2} = 0.397\,87_7\,(1 \pm 5.0 \times 10^{-4}) = 0.397\,87_7 \pm 0.00020$$

$$\frac{B}{M_{☾}\,a^2} = 0.397\,95_2\,(1 \pm 5.0 \times 10^{-4}) = 0.397\,95_2 \pm 0.000\,20$$

$$\frac{C}{M_{☾}\,a^2} = 0.398\,12_7\,(1 \pm 5.0 \times 10^{-4}) = 0.398\,12_7 \pm 0.000\,20$$

84. Axial ratios of the Moon:

$$\frac{b}{a} = \sqrt{1 - \frac{2\gamma}{1+\gamma}} = 0.9998\,116\,(1 \pm 1.3 \times 10^{-5}) = 0.9998\,116 \pm 0.0000\,129$$

$$\frac{c}{a} = \sqrt{1 - \frac{2\beta}{1+\gamma}} = 0.9993\,720\,(1 \pm 1.0 \times 10^{-6}) = 0.9993\,720 \pm 0.0000\,010$$

$$\sigma = R_{☾}/a = \frac{1}{2}\left(\frac{b}{a} + \frac{c}{a}\right) = 0.9995\,918\,(1 \pm 7.0 \times 10^{-6}) = 0.9995\,918 \pm 0.0000\,070$$

85. Oblateness coefficients of the potential function of the Moon:

$$J_2 = \frac{C - 1/2\,(A+B)}{M_{☾}\,a^2} = \frac{2}{3}\,J = 0.000\,2125 \pm 0.000\,0029$$

$$J_2^{(2)} = \frac{B-A}{4\,M_{☾}\,a^2} = \frac{1}{6}\,K = 0.000\,0188 \pm 0.000\,0013$$

86. Unit for the Moon's moments of inertia:

$$M_{☾}\,a^2 = 2.224\,1_6 \times 10^{35}(1 \pm 1.2 \times 10^{-3}) = (2.224\,1_6 \pm 0.0027) \times 10^{35}\ \mathrm{kg\ m^2}$$

87. Moment of inertia differences:

$$C - \frac{1}{2}\,(A+B) = 0.000\,473 \times 10^{35}(1 \pm 1.50 \times 10^{-2}) = (0.000\,473 \pm 0.000\,007) \times 10^{35}\mathrm{kg\ m^2}$$

$$B - A = 0.000\,167 \times 10^{35}(1 \pm 6.96 \times 10^{-2}) = (0.000\,167 \pm 0.000\,012) \times 10^{35}\ \mathrm{kg\ m^2}$$

88. Moon's moments of inertia:

$$A = 0.884\,94 \times 10^{35}(1 \pm 1.73 \times 10^{-3}) = (0.884\,94 \pm 0.001\,53) \times 10^{35}\mathrm{kg\ m^2}$$

$$B = 0.885\,11 \times 10^{35}(1 \pm 1.73 \times 10^{-3}) = (0.885\,11 \pm 0.001\,53) \times 10^{35}\mathrm{kg\ m^2}$$

$$C = 0.885\,50 \times 10^{35}(1 \pm 1.73 \times 10^{-3}) = (0.885\,50 \pm 0.001\,53) \times 10^{35}\mathrm{kg\ m^2}$$

1. THE ASTRONOMICAL UNIT AND THE SOLAR PARALLAX

The astronomical unit (A.U.), or the Earth's mean distance from the Sun, is connected with the solar parallax ($\pi_\odot$) by the following relation (with $R_e = 6378.170 \pm 0.020$ km the equatorial radius of the Earth):

$$1 \text{ A.U.} = \frac{R_e}{\sin \pi_\odot} = \frac{R_e}{\pi_\odot'' \sin 1''} = \frac{206264.''806247}{\pi_\odot''} R_e = \frac{1\,315\,592\,000 \pm 4000}{\pi_\odot''} \quad (1)$$

Modern determinations of the solar parallax usually are included between the two values $\pi_\odot = 8.''790 \pm 0.''001$ (H. Spencer Jones, 1941) and $\pi_\odot = 8.''79835 \pm 0.''00039$ (E. Rabe, 1949). The mean value of both determinations, $\pi_\odot = 8.''794 \pm 0.''002$, has been accepted by C. W. Allen (Ref. 1, p. 131) in his book on "Astrophysical Quantities" (1955). Exactly in the middle of these two values are also the recently obtained data of radar echoes from Venus, which have a considerably higher accuracy than previous determinations. Furthermore, agreement of the different radio observatories is also very good, as shown in the following table:

Radio Observatory	Author (Year)	Ref.	Radar Frequency (MC/sec)	Astronomical Unit	Solar Parallax (R_e = 6378.170 km)
Millstone (Lincoln Lab., M.I.T.)	Pettengill, Price et. al. (1961)	2	440	149 597 850 ± 400	$8.''79419_1$
Goldstone (J.P.L.)	Victor, Stevens and Muhlemann (1961)	3	2388	149 598 845 ± 250	$8.''79413_2$
Jodrell Bank (U. of Manchester)	Thomson et. al. (1961)		408	149 601 000 ± 5000	$8.''79400_5$
Moorestown (R.C.A.)	Maron et. al. (1961)		----	149 596 000	$8.''79429_9$
U. S. S. R.	Kotelnikov (1961)		700	149 599 500 ± 800	$8.''79409_4$
					$8.''79414_4$

According to Newcomb and de Sitter the semimajor axis of the Earth's orbit around the Sun is given by $a_\oplus = 1.000\,000\,236 + 0.000\,000\,004$ A.U. (or approximately 35 km more than 1 A.U.). For practical purposes both distances will be assumed equal to

$$a_\oplus = 149\,598\,700 \pm 400 = 149\,598\,700 \; (1 \pm 2.7 \times 10^{-6}) \text{ km} \quad (2)$$

The corresponding solar parallax will be

$$\pi_\odot = \frac{R_e}{a_\oplus \sin 1''} = 206\,264.''806\,247 \; \frac{6378.170 \pm 0.020}{149\,598\,700 \pm 400} = 8.''79414 \pm 0.''00005 \quad (3)$$

Taking as best value for the light velocity the value determined by Froome (Ref. 4) in 1958:

$$c = 299\,792.5 \pm 0.1 = 299\,792.5 \; (1 \pm 3.3 \times 10^{-7}) \text{ km/sec} \quad (4)$$

the light-time for unit distance (1 A.U.) is therefore

$$\tau = \frac{a_\oplus}{c} = 499.007_5 \pm 0.018 = 499.007_5 \ (1 \pm 3.6 \times 10^{-5}) \text{ sec} \tag{5}$$

2. DEFINITION OF TIME UNITS, MEAN ORBITAL MOTIONS, AND ROTATIONAL ANGULAR VELOCITIES OF EARTH AND MOON

There are three different times which are in use, namely, the Greenwich mean solar time or universal time (U.T.), the Greenwich mean siderial time (G.M.S.T.) and the ephemeris time (E.T.) or Newtonian time. Due to the variable rotation of the Earth, the mean solar time and the mean siderial time do not have a constant rate. The observations are therefore functions of a variable time, while the gravitational theories for the Sun and the planets use a uniform time. The ephemeris time, having a constant rate is defined by the orbital motion of the Earth as given by Newcomb's Tables of the Sun. It is therefore necessary to apply corrections to our practical determinations of time. In addition to the fluctuations and the tidal slowing down of the Earth's rotation, the Moon also shows a real diminution in the angular mean motion which is not given by Brown's lunar theory.

The correction to Newcomb's tabulated tropical mean longitude of the Sun (Ref. 5)

$$L_\odot = 279^\circ\ 41'\ 48.''04 + 129\,602\,768.''13\ T_E + 1.''089\ T_E^2 \tag{6}$$

is, according to H. Spencer Jones (Ref. 6),

$$\Delta L_\odot = +\ 1.''00 + 2.''97\ T + 1.''23\ T^2 + 0.0748\ B \tag{7}$$

when the observation times are in U.T. The time T is in Julian centuries of 36525^d counted from 1900 Jan. 0, 12^h U.T. (Greenwich mean noon) and B is the irregular fluctuation in the Moon's mean longitude in arc seconds (time of observation again expressed in U.T.). The Sun's tropical mean longitude, $L_\odot$, increases at the rate of $1''$ in $86400/(0.985\,647\,3354 \times 3600) = 24.349\,48$ sec, so that the correction to universal time, required to obtain ephemeris time is, according to H. Clemence (Ref. 7),

$$\Delta t \equiv t_E - t_U = 24.349\,48\ \Delta L_\odot = +\ 24.^s\,349 + 72.^s\,318\ T + 29.^s\,950\ T^2 + 1.82134\ B \tag{8}$$

H. Spencer Jones gives for the irregular fluctuation (Ref. 6)

$$B = (L_{☾,\,\text{obs.}} - L_{☾,\,\text{tabular}}) + 10.''71 \sin(140.^\circ 0\ T + 240.^\circ 7) - 4.''65 - 12.''96\ T - 5.''22\ T^2 \tag{9}$$

The periodic term is Brown's empirical term in his lunar theory. Therefore the correction to the Moon's mean longitude, as given by Brown's Tables of the Motion of the Moon (Ref. 8), is

$$\Delta L_☾ \equiv L_{☾,\,\text{obs.}} - L_{☾,\,\text{tabular}} = +\ 4.''65 + 12.''96\ T + 5.''22\ T^2 + B - 10.''71 \sin(140.^\circ 0\ T + 240.^\circ 7) \tag{10}$$

in order to obtain the actual mean longitude determined by observations in U.T. In the time

interval $\Delta t = t_E - t_U$ the Moon's mean longitude increases by ($n = 13.176\,396\,5268 \times 3600/86400 = 0.549\,016\,522''/\text{sec}$)

$$\Delta L_t = 0.549\ \Delta t = 13.368\ \Delta L_\odot = +\ 13.''37 + 39.''70\ T + 16.''44\ T^2 + B \tag{11}$$

Therefore the correction to Brown's Tables is

$$\Delta L_\text{☾} - \Delta L_t = -\ 8.''72 - 26.''74\ T - 11.''22\ T^2 - 10.''71 \sin(140.^\circ 0\ T + 240.^\circ 7) \tag{12}$$

when the observations are in ephemeris time. Brown's theory is now reduced to a gravitational theory with the same measure of time as defined by Newcomb's Tables of the Sun. Clemence's corrected value for the Moon's mean longitude (Ref. 7)

$$L_\text{☾} = 270^\circ\, 26'\, 2.''99 + 1\,732\,564\,379.''31\ T_E - 4.''08\ T_E^2 + 0.''0068\ T_E^3 \tag{13}$$

is used in the American Ephemeris and Nautical Almanac.

By means of equation (6) the tropical year, from mean equinox to mean equinox, thus has the length

$$P_\text{trop} = \frac{2\pi}{L_\odot^\bullet} = \frac{1\,296\,000'' \times 36\,525^{dE}}{129\,602\,768.''13 + 2.''178\ T} = 365.^{dE}242\,198\,78 - 0.^{dE}000\,006\,138\ T$$

$$= 365^{dE}05^{hE}\,48^{mE}\,45.^{sE}\,9747 - 0.^{sE}\,5303\ T = 31\,556\,925.^{sE}\,9747 - 0.^{sE}\,5303\ T \tag{14}$$

In 1957 the ephemeris second has been adopted as the fundamental invariable unit of time, and it is the fraction 1/31 556 925.9747 of the tropical year for 1900 Jan. 0, 12^h E.T. (Ref. 9)

The basis for all civil time-keeping is the universal time which is non-uniform. In practical life, however, the difference between mean solar time and ephemeris time can be neglected because there is $1^d \approx (1 \pm 10^{-8})^{dE}$. To define universal time Newcomb introduced a fictitious mean Sun which moves with the same constant siderial rate, in the equator, as the mean siderial motion is for the true Sun, affected by aberration (20.''50) in the ecliptic. According to Newcomb, the right ascension of the fictitious mean Sun is (neglecting nutation in right ascension)

$$R_E = 279^\circ\, 41'\, 27.''54 + 129\,602\,768.''13\ T_E + 1.''394\ T_E^2$$

$$= 18^h\, 38^m\, 45.^s 836 + 8\,640\,184.^s 542\ T_E + 0.^s 0929\ T_E^2 \tag{15}$$

Defining a point on the equator whose right ascension, measured from the mean equinox of date, is

$$R_U = 18^h\, 38^m\, 45.^s 836 + 8\,640\,184.^s 542\ T_U + 0.^s 0929\ T_U^2\,, \tag{16}$$

and where R_E differs from R_U by $0.002738\ \Delta t$ (see equation 8), the Greenwich hour angle

$\tau_{Gr(R_U)}$ of the point whose right ascension is R_U [equal to universal time (U.T.) $\pm 12^h$], increased by the right ascension R_U, is the Greenwich hour angle, $\tau_{Gr(\Upsilon)}$, of the mean vernal equinox of date which is Greenwich mean siderial time, Θ_{Gr}. That is,

$$\text{U.T.} \pm 12^h + R_U = \tau_{Gr(R_U)} + R_U = \tau_{Gr(\Upsilon)} \equiv \Theta_{Gr} \tag{17}$$

Adding the East longitude to both sides gives the local mean time on the left side of the equation and the local mean siderial time on the right side, because

$$\Theta = \Theta_{Gr} + \lambda_{\text{east.}}$$

The time rate of the right ascension, given by differientation of equation (16), is

$$\begin{aligned}
\dot{R}_U &= 8\,640\,184.542 + 0.1858\ T_U\ [\text{sec/Jul. century}] \\
&= 129\,602\,768.13 + 2.788\ T_U\ [\,''/\text{Jul.century}] \\
&= 3548.330\,4074 + 0.000\,076\,33\ T_U\ [\,''/\text{d}] \\
&= 0.985\,647\,3354 + 2.1203 \times 10^{-8}\ T_U\ [\,°/\text{d}],
\end{aligned} \tag{18}$$

Adding to this the time rate of the hour angle, $\dot{\tau} = 360\ °/\text{d} = 1296\,000\ ''/\text{d}$, the time rate of the mean siderial time is then

$$\begin{aligned}
\dot{\Theta}_m &= 1\,299\,548.330\,4074 + 0.000\,076\,33\ T\ [\,''/\text{d}] \\
&= 360.985\,647\,3354 + 2.1203 \times 10^{-8}\ T\ [\,°/\text{d}] \\
&= 15.041\,068\,638\,97 + 8.835 \times 10^{-10}\ T\ [\,''/\text{s or } °/\text{h}] \\
&= 1.002\,737\,909\,265 + 0.5890 \times 10^{-10}\ T\ [\text{d}_*/\text{d or s}_*/\text{s}] \\
&= 7.292\,115\,854\,58 \times 10^{-5} + 4.283 \times 10^{-15}\ T\ [\text{rad/s}]
\end{aligned} \tag{19}$$

This motion is the result of the spin of the Earth and the motion of the vernal equinox (precession). Because the latter motion takes place in the ecliptic the equatorial component or the general precession in right ascension, m, must be used here. The mean angular velocity of the Earth's rotation is, therefore,

$$\Omega_m = \dot{\Theta}_m - m \tag{20}$$

It is very probable that Newcomb's value for the general precession in longitude, p, must be increased by $\Delta p = +\ 0.''80$ per tropical century (see Part 9), thus

$$\begin{aligned}
p &= 5026.441 + 2.2229\ T + 0.00026\ T^2\ [\,''/\text{trop. century}] \\
&= 0.137\,619\,4 + 0.000\,060\,861\ T + 0.712 \times 10^{-8}\ T^2\ [\,''/\text{d}] \\
&= 0.000\,038\,227\,61 + 0.000\,000\,016\,906\ T + 1.98 \times 10^{-12}\ T^2\ [\,°/\text{d}]
\end{aligned} \tag{21}$$

Neglecting the correction in planetary precession ($\Delta\lambda = 0$) the correction for the general precession in right ascension would be $\Delta m = \Delta p \cdot \cos\epsilon = 0.''80 \times 0.917 = +0.''73$. Thus,

$$m = 4609.236 + 2.7945\ T + 0.00012\ T^2\ [''/\text{trop. century}]$$

$$= 0.1261967 + 0.000076511\ T + 0.33 \times 10^{-8}\ T^2\ [''/\text{d}]$$

$$= 0.00003505464 + 0.000000021253\ T + 0.92 \times 10^{-12} T^2\ [°/\text{d}] \qquad (22)$$

The angular velocity of the Earth's rotation is, therefore,

$$\Omega_m = 1\,299\,548.2042107 - 0.00000018\ T\ [''/\text{d}]$$

$$= 360.985\,612\,2808 - 0.0050 \times 10^{-8}\ T\ [°/\text{d}]$$

$$= 15.041\,067\,178\,37 - 0.021 \times 10^{-10}\ T\ [''/\text{s or } °/\text{h}]$$

$$= 1.002\,737\,811\,891 - 0.0014 \times 10^{-10} T\ [\text{rot/d}]$$

$$= 7.292\,115\,146\,46 \times 10^{-5} - 0.010 \times 10^{-15}\ T\ [\text{rad/s}] \qquad (23)$$

Using equations (19) and (23) the following periods are obtained:

1. Mean solar day (culmination period of the mean Sun)

$$1^d = 1.^d{*}\ 002\,737\,909\,265 + 0.^d{*}\ 589 \times 10^{-10}\ T$$

$$= 86\,636.^s{*}\ 555\,360\,50 + 0.^s{*}050\,889\,6 \times 10^{-4}\ T$$

$$= 24^h{*}\,03^m{*}\,56.^s{*}\ 555\,360\,50 + 0.^s{*}\ 050\,889\,6 \times 10^{-4}\ T$$

$$= 1.^{rot}\ 002\,737\,811\,891 - 0.^{rot}\ 001\,4 \times 10^{-10}\ T$$

$$= (1 + 10^{-8})^{dE} \qquad (24)$$

2. Mean siderial day or mean equinoctial day (culmination period of the vernal equinox)

$$1^{d*} = 1^d - 0.^d002\,730\,433\,586 - 0.^d\ 587 \times 10^{-10}\ T$$

$$= 0.^d\ 997\,269\,566\,414 - 0.^d\ 587 \times 10^{-10} T$$

$$= 86164.^s\ 090\,538\,17 - 0.^s\ 050\,716\,8 \times 10^{-4} T$$

$$= 23^h56^m\,04.^s\,090\,538\,17 - 0.^s\ 050\,716\,8 \times 10^{-4}\ T$$

$$= \frac{\Omega}{\Theta} = 1 - \frac{m}{\Theta} = (1 - 0.000\,000\,097\,108)^{rot} - 0.^{rot}\ 589 \times 10^{-10}\ T$$

$$= 0.^{rot}\ 999\,999\,902\,892 - 0.^{rot}\ 589 \times 10^{-10}\ T \qquad (25)$$

3. Period of the Earth's rotation (culmination period of an equatorial star without proper motion)

$$I^{rot} = \frac{\dot{\Theta}}{\Omega} = 1 + \frac{m}{\Omega} = 1^{d}.000\,000\,097\,108 + 0^{d}.589 \times 10^{-10}\ T$$

$$= 86400^{s}.008\,390\,13 + 0^{s}.050\,889\,6 \times 10^{-4}\ T$$

$$= 1^{d} - 0^{d}.002\,730\,336\,743 + 0^{d}.001\,4 \times 10^{-10}\ T$$

$$= 1^{d} - 235^{s}.901\,094\,60 + 0^{s}.000\,120\,96 \times 10^{-4}\ T$$

$$= 0^{d}.997\,269\,663\,257 + 0^{d}.001\,4 \times 10^{-10}\ T$$

$$= 86\,164^{s}.098\,905\,40 + 0^{s}.001\,209\,6 \times 10^{-5}\ T \qquad (26)$$

Because

$$1^{d} - 1^{d}* = 236^{s}.555\,360\,50 + 0^{s}.050\,889\,6 \times 10^{-4}\ T$$

$$= 235^{s}.909\,461\,83 + 0^{s}.050\,7168 \times 10^{-4}\ T$$

or

$$1^{h} - 1^{h}* = 9^{s}.856\,473 = 9^{s}.829\,561$$

the change of mean siderial time against mean solar time is $9^{s}.856\,47$ in a mean solar hour and $9^{s}.829\,56$ in a mean siderial hour.

In order to apply Kepler's third law, the siderial mean angular motions of the Earth about the Sun and the Moon about the Earth will be needed. Differentiation of equation (6) gives the tropical mean motion of the Earth:

$$n_{\oplus}\,(\text{trop.}) = \dot{L}_{\odot} = 129\,602\,768.13 + 2.178\ T\ [\,''/\text{Jul. century}\,]$$

$$= 3548.330\,407\,4 + 0.000\,059\,63\ T\ [\,''/\text{d}\,]$$

$$= 0.985\,647\,335\,4 + 1.6564 \times 10^{-8}\ T\ [\,°/\text{d}\,] \qquad (27)$$

Subtracting from this the general precession in longitude (equation 21) yields the siderial mean motion of the Earth:

$$n_{\oplus}\,(\text{sid.}) = \dot{L}_{\odot} - p = 3548.192\,788\,0 - 0.000\,001\,23\ T\ [\,''/\text{d}\,]$$

$$= 0.985\,609\,1078 - 0.0342 \times 10^{-8}\ T\ [\,°/\text{d}\,]$$

$$= 0.041\,067\,046\,16 - 0.1424 \times 10^{-10} T\ [\,''/\text{s}\,]$$

$$= 1.990\,986\,582\,0 \times 10^{-7} - 0.690\,374\,68 \times 10^{-16}\ T\ [\text{rad/s}\,] \qquad (28)$$

Differentiating equation (13) provides the tropical mean motion of the Moon:

$$n_{☾}\ (\text{trop.}) = \dot{L}_{☾} = 1\,732\,564\,379.31 - 8.16\,T + 0.0204\,T^2\ [\,''/\text{Jul. century}]$$

$$= 47\,435.027\,496\,5 - 2.234 \times 10^{-4}\ T + 0.559 \times 10^{-6} T^2\ [\,''/\text{d}]$$

$$= 13.176\,396\,5268 - 6.206 \times 10^{-8}\ T + 1.55 \times 10^{-10} T^2\ [\,^\circ/\text{d}]$$

$$= 0.549\,016\,521\,950 - 2.586 \times 10^{-9}\ T + 6.46 \times 10^{-12} T^2\ [\,''/\text{s}] \tag{29}$$

and subtraction of the general precession in longitude, p, gives the siderial mean motion of the Moon:

$$n_{☾}\ (\text{sid.}) = \dot{L}_{☾} - p = 47\,434.889\,8771 - 2.843 \times 10^{-4}\,T + 0.552 \times 10^{-6} T^2\ [\,''/\text{d}]$$

$$= 13.176\,358\,2992 - 7.897 \times 10^{-8}\ T + 1.53 \times 10^{-10} T^2\ [\,^\circ/\text{d}]$$

$$= 0.549\,014\,929\,133 - 3.290 \times 10^{-8}\ T + 6.38 \times 10^{-12} T^2 [\,''/\text{s}]$$

$$= 2.661\,699\,487\,73 \times 10^{-6} - 1.595 \times 10^{-13} T + 3.09 \times 10^{-17} T^2 [\,\text{rad/s}] \tag{30}$$

Because the distance of the Earth to the Sun is now known more accurately than before, it is possible to give the mean orbital velocity of the Earth about the Sun with high accuracy, namely

$$v_{\oplus} = a_{\oplus}\ n_{\oplus} = 29784.90 \pm 0.08\ [\text{m/s}] \tag{31}$$

There are two constants connected with this velocity. Taking $e = 0.01675$ for the orbital eccentricity of the Earth the value for the constant of aberration will be

$$K = \frac{v_{\oplus}/c}{\sqrt{1-e^2}\,\sin 1''} = \frac{r\,n_{\oplus}}{\sqrt{1-e^2}\,\sin 1''} = 20.''4956 \pm 0.''0007, \tag{32}$$

and using the formula of de Sitter (Ref. 10) the geodetic precession, due to the special theory of relativity, is

$$p_g = \frac{3}{2}\,(v_{\oplus}/c)^2\ n_{\oplus} = \frac{3}{2}\,(K \sin 1''\sqrt{1-e^2})^2 n_{\oplus} = 1.''9188 \pm 0.''0002 \tag{33}$$

3. THE LUNAR DISTANCE AND THE LUNAR PARALLAX

The mean observed distance, $\overline{r}_{☾}$, of the Moon from the Earth is connected with the mean perturbed lunar parallax, $\pi_{☾}$, and the constant $\pi'_{☾}$ of the sine of the perturbed lunar parallax by the following relation:

$$\frac{\overline{r}_{☾}}{R_e} = \frac{1}{\sin \pi_{☾}} = \frac{1}{\pi'_{☾}\ \sin 1''} \tag{34}$$

Dividing both sides of the series development

$$\pi_{☾} = \sin \pi_{☾} + \frac{1}{6}\ \sin^3 \pi_{☾} + \cdots$$

by sin 1″ yields (because $\pi_{☾}' = \sin \pi_{☾} / \sin 1''$)

$$\pi_{☾}'' = \pi_{☾}' + \frac{1}{6} (\pi_{☾}')^3 (\sin 1'')^2 + \cdots = \pi_{☾}' \left[1 + \frac{1}{6} \left(\frac{R_e}{\bar{r}_{☾}} \right)^2 + \cdots \right]$$

$$= \pi_{☾}' \times 1.000045885 = \pi_{☾}' + 0.''157 \tag{35}$$

Newer determinations are

$\pi_{☾}' = 3422.''54$	$\pi_{☾} = 3422.70$	(E.W. Brown, Ref. 8)
3422.526 ± 0.009	3422.683 ± 0.009	(W. deSitter, Ref. 10)
3422.419 ± 0.024	3422.576 ± 0.024	(H. Jeffreys, Ref. 11, p. 193)
3422.493	3422.650	(Herrick, Baker, Ref. 12)

Recent determinations of the mean lunar distance, $\bar{r}_{☾}$, by means of radar echoes to the Moon are in very close agreement (see Reference 13). This value is given by

$$\bar{r}_{☾} = 384\,402.0 + 1.0 = 384\,402.0\,(1 \pm 2.6 \times 10^{-6})\ \text{km} \tag{36}$$

thus

$$\frac{\bar{r}_{☾}}{R_e} = \frac{384\,402.0\,(1 \pm 2.6 \times 10^{-6})}{6378.170\,(1 \pm 3.2 \times 10^{-6})} = 60.26838\,(1 \pm 5.8 \times 10^{-6}) = 60.26838 \pm 0.00035 \tag{37}$$

and

$$\pi_{☾}' = \frac{\sin \pi_{☾}}{\sin 1''} = \frac{206\,264.''806\,247}{(\bar{r}_{☾} / R_e)} = 3422.''438 \pm 0.''020 \tag{38}$$

$$\pi_{☾} = 3422.''595 + 0.''020 \tag{39}$$

To obtain the semi-major axis, $a_{☾}$, it is necessary to add to the mean lunar distance the constant part of the solar perturbations according to Brown's lunar theory. There is now

$$a_{☾} = 1.000\,907\,681\ \bar{r}_{☾} = 384\,750.9 \pm 1.0\ \text{km} \tag{40}$$

and the mean orbital velocity of the Moon about the Earth is

$$v_{☾} = a_{☾}\, n_{☾} = 1024.091 \pm 0.003\ \text{m/s} \tag{41}$$

4. MASS RATIOS OF THE SUN AND THE EARTH-MOON SYSTEM

Taking the already given values for $a_{\oplus}$, $n_{\oplus}$ (sid.), and $a_{☾}$, $n_{☾}$ (sid.) then Kepler's third law gives

$$n_{\oplus}^2\, a_{\oplus}^3 = G\,(M_{\odot} + M_{\oplus} + M_{☾}) = G\,M_{\oplus}\,(\nu + 1)(1 + \kappa) \tag{42}$$

$$n_{☾}^2\, a_{☾}^3 = G\,(M_{\oplus} + M_{☾}) = G\,M_{\oplus}\,(1 + \kappa) \tag{43}$$

with

$$\nu = \frac{M_\odot}{M_\oplus + M_☾} \gg 1 \qquad ; \qquad \kappa = \frac{M_☾}{M_\oplus} \ll 1 \tag{44}$$

Dividing the two equations yields

$$\nu \equiv \frac{M_\odot}{M_\oplus + M_☾} = \left(\frac{n_\oplus}{n_☾}\right)^2 \left(\frac{a_\oplus}{a_☾}\right)^3 - 1 = \left(\frac{v_\oplus}{v_☾}\right)^2 \left(\frac{a_\oplus}{a_☾}\right) - 1$$

$$= 328\,898.6 \ (1 \pm 1.6 \times 10^{-5}) = 328\,898.6 \pm 5.2 \tag{45}$$

This value is approximately in the middle between the value 329 390 obtained by S. Newcomb (Ref. 14), adopted by Am. Ephemeris, and the value 328 446 ± 43 determined by E. Rabe (Ref. 15).

The second equation gives

$$\mu_\oplus (1 + \kappa) = n_☾^2 \ a_☾^3 = v_☾^2 \ a_☾ = 403\,512.3 \quad \pm 3.2 \ \ \mathrm{km^3/s^2} \tag{46}$$

which connects the gravitational parameter $\mu_\oplus = G M_\oplus$ for the Earth with κ, the ratio of the Moon's mass to the Earth's mass.

5. THE CONSTANT OF LUNAR INEQUALITY AND THE PARALLACTIC INEQUALITY IN MOON'S ECLIPTIC LONGITUDE

The Parallactic Inequality in the Moon's ecliptic longitude is given by E. W. Brown's lunar theory as follows

$$P_* = (49853.''2 \pm 1.''2) \ \frac{1-\kappa}{1+\kappa} \ \frac{\pi_\odot}{\pi_☾'} = (49853.''2 \pm 1.''2) \ \frac{1/\kappa - 1}{1/\kappa + 1} \ \frac{\bar{r}_☾}{a_\oplus} \tag{47}$$

or with the newest data for the lunar distance and the astronomical unit

$$P_* = (128.''1005 \pm 0.''0037) \ \frac{1/\kappa - 1}{1/\kappa + 1} \tag{48}$$

Newer determinations are:

$P_* = 124.''86 \pm 0.''15$	(J. Bauschinger, Ref. 16)
$= 125.154$	(E. W. Brown, Ref. 8)
$= 124.93$	(H. Battermann, Ref. 17)
$= 124.969 \pm 0.042$	(D. Brouwer and O. B. Watts, Ref. 18)

On the other hand, the constant of Lunar Inequality is defined by W. deSitter (Ref. 10) as

$$L = \frac{\kappa}{1+\kappa} \ \frac{\pi_\odot}{\sin \pi_☾} = \frac{206\,264.''806\,247}{1/\kappa + 1} \ \frac{\pi_\odot}{\pi_☾'} = \frac{530.''0089 \pm 0.''0028}{1/\kappa + 1} \tag{49}$$

Newcomb used the lunar inequality in the Sun's longitude which is, according to deSitter, $L_s = 1.00450\,L$. The ratio of P_* and L, depending only on κ, is

$$P_*/L = \frac{49\,853.''2 \pm 1.''2}{206\,264.''\,806\,247}\,(1/\kappa - 1) = (0.241\,695 \pm 0.000\,006)\,(1/\kappa - 1) \tag{50}$$

The mass ratio is therefore

$$1/\kappa = (4.13744_4 \pm 0.00010)\,\frac{P_*}{L} + 1 = \frac{530.''0089 \pm 0.''0028}{L} - 1 = \frac{128.''1005 + P_*}{128.''1005 - P_*} \tag{51}$$

Observed values for the constant of Lunar Inequality are:

$L = 6.''456 \pm 0.''012$	(Newcomb, Ref. 14)	From observations of	Sun
6.414 ± 0.009	(D. Gill, Ref. 19)	"	Victoria
6.4305 ± 0.0031	(A.R. Hinks, Ref. 20)	"	Eros (opp. of 1901)
6.4390 ± 0.0015	(H. Spencer Jones, Ref.21)	"	Eros (opp. of 1931)
6.450 ± 0.010	(Morgan and Scott, Ref. 22)	"	Sun
6.4378 ± 0.0015	(H. Jeffreys, Ref. 23)	"	Eros (opp. of 1931)
6.4356 ± 0.0028	(E. Rabe, Ref. 15)	"	Eros (opp. of 1931)
6.4428 ± 0.0014) 6.4430 ± 0.0017)	(E. Delano, Ref. 24)	"	Eros (opp. of 1931)

The latest reevaluation of all Eros observations during the opposition of 1930/31 by E. Delano (Ref. 24) gave

$L = 6.''4428 \pm 0.''0014$ (from right ascensions of Eros)
$L = 6.4430 \pm 0.0017$ (from declinations of Eros)

Delano used the old value $\pi_\odot = 8.''790$ for the solar parallax and obtained therefore $1/\kappa = 81.222 \pm 0.027$ and 81.219 ± 0.030, respectively. With the newest values for $\pi_\odot$ and $\pi_☾'$ there is now

$$1/\kappa = 81.263_7 \quad \text{and} \quad 1/\kappa = 81.261_2$$

respectively.

6. ANOTHER METHOD FOR THE DETERMINATION OF THE RATIO OF THE MASSES OF EARTH AND MOON

The mass of the Earth is given by

$$M_\oplus = \frac{4}{3}\,\pi\,R_e^3\,(1 - f)\,\rho_\oplus \tag{52}$$

where f is the flattening (oblateness) and $\rho_\oplus$ is the mean density of the Earth. On the other hand the mass of the triaxial figure of the Moon is given by

$$M_☾ = \frac{4}{3}\,\pi\,a\,b\,c\,\rho_☾ = \frac{4}{3}\,\pi\,a^3\left(\frac{b}{a}\right)\left(\frac{c}{a}\right)\rho_☾ \tag{53}$$

Because the longest axis is always directed to the Earth (neglecting the small librations) and can never be seen, the lunar radius of the visible disk is

$$R_{☾} = \frac{b+c}{2} \quad ; \quad \frac{R_{☾}}{a} = \frac{1}{2}\left(\frac{b}{a} + \frac{c}{a}\right) = \sigma \tag{54}$$

That yields for the mass of the moon

$$M_{☾} = \frac{4}{3}\,\pi\, R_{☾}^3 \;\; \frac{(b/a)(c/a)}{\sigma^3}\, \rho_{☾} \tag{55}$$

The mass ratio is therefore

$$1/\kappa \equiv M_{\oplus}/M_{☾} = \left(\frac{R_{\oplus}}{R_{☾}}\right)^3 \frac{(1-f)\,\sigma^3}{(b/a)(c/a)} \left(\frac{\rho_{\oplus}}{\rho_{☾}}\right) \frac{(1-f)\,\sigma^3}{(b/a)(c/a)} \;\; \frac{\rho_{\oplus}/\rho_{☾}}{k^3} \tag{56}$$

where

$$k = \frac{R_{☾}}{R_{\oplus}} = \frac{\sin s_{☾}}{\sin \pi_{☾}} = \frac{s_{☾}'}{\pi_{☾}'} = \frac{s_{☾} - 0.''003}{\pi_{☾} - 0.''157} \tag{57}$$

is given by the lunar parallax, $\pi_{☾}$, and the apparent semi-diameter of the Moon, $s_{☾}$.

A reevaluation of Sir Harold Jeffrey's best data on the Moon's figure by the author gave (see section 12)

$$\frac{b}{a} = 0.9998\,116 \quad ; \quad \frac{c}{a} = 0.9993\,720 \quad ; \quad \sigma = 0.9995\,918$$

From the secular perturbations of artificial Earth satellites there follows as best value for the Earth oblateness

$$1/f = 298.30 \quad ; \quad 1 - f = 0.9966\,4767$$

so that

$$1/\kappa = 0.9962\,409 \;\; \frac{\rho_{\oplus}/\rho_{☾}}{k^3} \tag{58}$$

Taking for the mean densities the well-known and frequently used values

$$\rho_{\oplus} = 5.517 \pm 0.004 \;\; g/cm^3 \text{ (Heyl)} \; ; \; \rho_{☾} = 3.342 \pm 0.005 \;\; g/cm^3 \text{ (Jeffreys)}$$

the density ratio, independent from the assumed value of the gravitational constant, G, becomes $\rho_{\oplus}/\rho_{☾} = 1.6508$. Therefore

$$1/\kappa = 1.6446/k^3 \tag{59}$$

With $\pi_{☾}' = 3422.''438$ and various values for $s_{☾}$ the following table is obtained

$s_{☾}$	k	$1/\kappa$
932.''58 (Newcomb, Ref. 25)	0.2724891	81.286
932.63 (American Ephem., Ref. 26)	0.2725037	81.272
932.80 + 0.07 (Hirose & Manabe, Ref. 27)	0.2725534	81.228

The American Ephemeris is using $k = 0.2724953$, based on Brown's lunar parallax. The values for $1/\kappa$ obtained in the previous paragraph are between the two latter values in this table. The arithmetic mean of these two latter values will be taken as the presently best value, namely

$$1/\kappa \equiv \frac{M_{\oplus}}{M_{☾}} = 81.250\,(1 \pm 3 \times 10^{-4}) = 81.250 \pm 0.024 \tag{60}$$

and therefore

$$k \equiv \frac{R_{☾}}{R_e} = 0.2725289 \pm 0.0000273 \tag{61}$$

The last equation of paragraph 3 now gives

$$\mu_{\oplus} \equiv GM_{\oplus} = \frac{403512.3}{1.0123077} = 398606.4 \pm 4.9 \ \mathrm{km}^3/\mathrm{sec}^2 \tag{62}$$

for the Earth, while, for the Moon,

$$\mu_{☾} \equiv GM_{☾} = \kappa\mu_{\oplus} = 4905.92 \pm 1.52 \ \mathrm{km}^3/\mathrm{sec}^2 \tag{63}$$

7. GEODYNAMIC (TERRESTRIAL) RELATIONS

The surface of the Earth (geoid) can be approximated as the surface of an spheroid assumed as an equipotential surface. The equation for the Earth's radius, as function of the latitude, is then given by

$$R = R_e\left[1 - f\sin^2\varphi + \left(\frac{5}{8}f^2 - \kappa\right)\sin^2 2\varphi\right] = R_e\left[1 - f\sin^2\phi - \left(\frac{3}{8}f^2 + \kappa\right)\sin^2 2\phi\right] \tag{64}$$

where φ is the geodetic (geographic) latitude and ϕ the geocentric latitude. They are related by

$$\tan\phi = (1-f)^2 \tan\varphi = (1-e^2)\tan\varphi \tag{65}$$

where $e = \sqrt{f(2-f)}$ is the eccentricity of the meridian ellipse of the Earth. The equation of the Earth-ellipsoid is obtained by setting $\kappa = 0$. The maximum depression, $-\kappa R_e$, of the spheroid from the ellipsoid is reached at the latitude 45°. It will never be more than 5.17 m. For the For the spheroid as equipotential surface there is

$$U = \frac{GM}{R}\left[1 - \frac{2}{3}J\left(\frac{R}{R_e}\right)^2 P_2(\sin\phi) + \frac{4}{15}K\left(\frac{R}{R_e}\right)^4 P_4(\sin\phi) + \cdots\right] + \frac{1}{2}\Omega^2 R^2\cos^2\phi$$

$$= \frac{GM}{R}\left[1 - \sum_{n=2}^{\infty} J_n\left(\frac{R}{R_e}\right)^n P_n(\sin\phi)\right] + \frac{1}{2}\Omega^2 R^2\cos^2\phi = \text{const.} \tag{66}$$

and the acceleration of gravity at this surface is

$$g = |\operatorname{grad} U| = g_e [1 + \beta \sin^2 \varphi + \gamma \sin^2 2\varphi] = g_e [1 + \beta \sin^2 \phi + (\gamma + \beta f) \sin^2 2\phi] \tag{67}$$

where Ω is the angular velocity of the Earth's rotation, β and γ are constant gravity coefficients and J_n or J and K are constant oblateness coefficients. These coefficients depend only on f, κ, and a parameter containing Ω^2 (centrifugal force parameter). This latter parameter is a little different in the various second-order theories which have been developed. Taking

$$\Omega = 7.292\,115\,146 \times 10^{-5} \text{ rad/sec}$$

$$1/f = 298.30 \quad ; \quad 1 - f = 0.996\,647\,670$$

$$G = 6.670 \times 10^{-8} \text{cm}^3/(\text{g}\cdot\text{sec}^2) \quad ; \quad \rho_m = 5.517 \text{ g/cm}^3 \quad ; \quad G\rho_m = 3.679\,839 \times 10^{-7} \tag{68}$$

then the following parameters could be used

$$\varpi = \frac{\Omega^2 R_e^3}{GM} = \frac{\Omega^2}{4/3\,\pi\, G\rho_m (1-f)} = 0.003461\,369 \text{ [Herrick, (Ref.12]} \tag{69}$$

$$\varpi' = \frac{\Omega^2 R_v^3}{GM} = \frac{\Omega^2}{4/3\,\pi G \rho_m} = \varpi(1-f) = 0.003449\,766 \text{ [Jeffreys, Ref.11]} \tag{70}$$

$$\varpi_1 = \frac{\Omega^2 R_1^3}{GM} = \varpi\left(1 - f + 2f^2 - \frac{8}{3}\kappa\right) = \varpi'(1 + 2f^2 - \frac{8}{3}\kappa) = 0.003449\,843 \text{ [deSitter, Ref.10]} \tag{71}$$

$$m = \frac{\Omega^2 R_e}{g_e} = \frac{\varpi}{1 - A - \frac{3}{2}\varpi + f + f^2 - \frac{3}{7} f\varpi + \frac{16}{7}\kappa} = \frac{\varpi}{1 - A - \varpi + J + 1/2\,K}$$

$$= 0.003467\,730 \text{ [Darwin, Ref.28; Helmert, Ref.29]} \tag{72}$$

where $R_v = R_e (1 - f)^{1/3}$ is the radius for a sphere of same volume as the Earth and $R_I = R_e (1 - 1/3\, f + 5/9\, f^2 - 8/9\kappa)$ is the mean radius for which $P_2 (\sin \varphi) = 0$ or $\varphi = \sin^{-1}\sqrt{1/3} = 35°15'51.''8$. $A = 0.88 \times 10^{-6}$ is the mass of the Earth's atmosphere (expressed in mass of the Earth) which does not contribute to the surface gravity of the Earth. Different assumptions have been made for κ. Bullard (Ref.30) found $10^6 \kappa = 0.68$. This value was accepted later by Herrick, Baker, and Hilton (Ref.12). On the other hand, deSitter found values of only $10^6 \kappa = 0.47$ to 0.52 and used the round mean value $10^6 \kappa = 0.50$. The theoretical limits are according to deSitter (Ref. 31)

$$0 \leqslant \kappa \leqslant \frac{5}{16} f\varpi - \frac{1}{4} f^2 = (3.62 - 2.81) \times 10^{-6} = 0.81 \times 10^{-6} \tag{73}$$

The different formula systems now yield

$$J = f - \frac{1}{2}\varpi - \frac{1}{2}f^2 + \frac{9}{14}f\varpi + \frac{4}{7}\kappa$$

$$= f - \frac{1}{2}\varpi' - \frac{1}{2}f^2 + \frac{1}{7}f\varpi' + \frac{4}{7}\kappa$$

$$= f - \frac{1}{2}\varpi_1 - \frac{1}{2}f^2 + \frac{1}{7}f\varpi_1 + \frac{4}{7}\kappa$$

$$= f - \frac{1}{2}m - \frac{1}{2}f^2 + \frac{1}{7}fm + \frac{3}{4}m^2 + \frac{4}{7}\kappa \qquad (74)$$

$$K = \frac{6}{7}D = 3f^2 - \frac{15}{7}f\varpi + \frac{24}{7}\kappa$$

$$= 3f^2 - \frac{15}{7}f\varpi' + \frac{24}{7}\kappa$$

$$= 3f^2 - \frac{15}{7}f\varpi_1 + \frac{24}{7}\kappa$$

$$= 3f^2 - \frac{15}{7}fm + \frac{24}{7}\kappa \qquad (75)$$

$$\beta = \frac{5}{2}\varpi - f - \frac{26}{7}f\varpi + \frac{15}{4}\varpi^2 + \frac{8}{7}\kappa$$

$$= \frac{5}{2}\varpi' - f - \frac{17}{14}f\varpi' + \frac{15}{4}\varpi'^2 + \frac{8}{7}\kappa$$

$$= \frac{5}{2}\varpi_1 - f - \frac{17}{14}f\varpi_1 + \frac{15}{4}\varpi_1^2 + \frac{8}{7}\kappa$$

$$= \frac{5}{2}m - f - \frac{17}{14}fm + \frac{8}{7}\kappa \qquad (76)$$

$$\gamma = \frac{1}{8}f^2 - \frac{5}{8}f\varpi - 3\kappa$$

$$= \frac{1}{8}f^2 - \frac{5}{8}f\varpi' - 3\kappa$$

$$= \frac{1}{8}f^2 - \frac{5}{8}f\varpi_1 - 3\kappa$$

$$= \frac{1}{8}f^2 - \frac{5}{8}fm - 3\kappa \qquad (77)$$

$$\chi \equiv g_e/(GM/R_e^2) = 1 - A - \varpi + J + \frac{1}{2} K$$

$$= 1 - A - \frac{3}{2}\varpi + f + f^2 - \frac{3}{7} f\varpi + \frac{16}{7}\kappa$$

$$= 1 - A - \frac{3}{2}\varpi' + f + f^2 - \frac{27}{14} f\varpi' + \frac{16}{7}\kappa$$

$$= 1 - A - \frac{3}{2}\varpi_1 + f + f^2 - \frac{27}{14} f\varpi_1 + \frac{16}{7}\kappa$$

$$= 1 - A - \frac{3}{2} m + f + f^2 - \frac{27}{14} fm + \frac{9}{4} m^2 + \frac{16}{7}\kappa \tag{78}$$

Using these equations and the above-given constants for the Earth then the following table is obtained with $J_2 \equiv (C - A)/(M R_e^2) = \frac{2}{3} J$ and $J_4 = -\frac{4}{15} K = -\frac{8}{35} D$:

Coefficient	$\kappa = 0$	$\kappa = 0.50 \times 10^{-6}$	$\kappa = 0.68 \times 10^{-6}$
$10^6 J$ $10^6 K$	1623.48 8.85	1623.77 10.56	1623.87 11.18
$10^6 J_2$ $10^6 J_4$	1082.32 − 2.36	1082.51 − 2.82	1082.58 − 2.98
$10^6 \beta$ $10^6 \gamma$	5302.92 − 5.85	5303.49 − 7.35	5303.70 − 7.89
χ	0.9981 6566	0.9981 6680	0.9981 6721

The numerical values for $|J_4|$ are a little higher than the values derived from the observed secular perturbations of artificial satellites. Thus the data for $\kappa = 0$ will be used here.

The gravitational parameter of the Earth is now given by

$$\mu_\oplus \equiv GM_\oplus = \frac{4}{3}\pi G\rho_\oplus R_e^3 (1 - f) = \frac{g_e R_e^2}{\chi} = 398\,606.4 \pm 4.9 \ \text{km}^3/\text{sec}^2 \tag{79}$$

which corresponds to $1/\kappa = 81.250$. Taking, furthermore, $\frac{4}{3}\pi = 4.188\,790\,204$; $1 - f = 0.996\,647\,670$; $\chi = 0.998\,165\,66$ that yields

$$g_e R_e^2 = \chi\mu_\oplus = 397\,875.2 \pm 4.9 \ \text{km}^2/\text{sec}^2 = 3.978\,752 \times 10^{14} \ \text{m}^3/\text{sec}^2 \tag{80}$$

and

$$\frac{g_e}{R_e} = \frac{4}{3}\pi (1 - f)\chi G \rho_\oplus = 4.167\,090\,090\, G\rho_\oplus \tag{81}$$

or with $G = 6.670 \times 10^{-8}\ \mathrm{cm}^3/\mathrm{g}\cdot\mathrm{sec}^2$

$$\rho_\oplus = \frac{(g_e/R_e)\times 10^7}{2.779\,45} \quad [\mathrm{g/cm}^3] \tag{82}$$

8. EXPERIMENTAL VALUES OF THE GRAVITATIONAL ACCELERATION OF THE EARTH

The observed gravitational acceleration at the surface of the rotating Earth can be represented by the formula

$$g = g_e\,[1 + \beta \sin^2 \varphi + \gamma \sin^2 2\varphi + \delta \cos^2 \varphi \cos 2(\lambda - \lambda_0)] \tag{83}$$

where λ is the geographic longitude measured eastwards of Greenwich meridian. The first term corresponds to a sphere. The next two terms give the contribution due to the oblateness of the Earth spheroid, while the longitude term is due to the non-ellipticity of the equator when the Earth is assumed as a triaxial figure. The longitude, λ_0, gives the direction of the longest semi-axis of the equator. δ is connected with the difference $B - A$ of the equatorial moments of inertia or with the flattening, f_e, of the equator by the relation

$$\delta = \frac{9}{4}\,\frac{B - A}{M_\oplus R_e^2} = \frac{9}{4}\,\Lambda f_e(2 - f_e) \approx \frac{3}{8}\,f_e \tag{84}$$

because the inhomogenity factor of the Earth is given by

$$\Lambda = \frac{C}{M_\oplus(a^2 + b^2)} \approx \frac{C}{2M_\oplus R_e^2} \approx \frac{1}{6} \tag{85}$$

The most important determinations from gravity measurements since 1915 have been compiled in a table on the following page.

All these gravity measurements are still based on the standard gravity value of Potsdam ($\varphi = 52°\,22.'86$; $\lambda = +\,13°\,4.'06$; $h = 87$ m)

$$g = 981.2740 \text{ gal}$$

obtained by F. Kuhnen and Ph. Furtwangler (Ref. 43). It is necessary to revise the Potsdam system. For the correction of the Potsdam value, the following data are given (Refs. 44 and 45):

P. R. Heyl and G. S. Cook (Wash. D.C.)	: – 20	milligal	according to absolute gravity measurements
Bullard (Teddington, G. Brit.)	: – 15	"	
J. S. Clarke (Teddington, Gt. Brit.)	: – 13	"	
Ivanoff (Leningrad, U.S.S.R.)	: – 4	"	
P. R. Heyl	: – 15	"	according to recalculations
Bullard and Browne	: – 16	"	
Morelli (1954)	: – 16	"	
H. Jeffreys	: – 13.4	"	
Wollard	: – 14 to – 18	"	
A. Berroth	: – 12.5	"	
Morelli (1959)	: – 12.9	"	

Author	Year	Ref.	g_e (gal)	$10^6 \beta$	$10^6 \gamma$	$10^6 \delta$	λ_0	ΔR_e (m)	$1:f_1$	$1:f_2$	$1:f_m$	$1:f_e$
F. R. Helmert	1915	32	978.052 ± 3	5285 ± 5		18 ± 3	− 17° ± 4	230 ± 51			296.7 ± 0.4	28 000
A. Berroth	1916	33	978.046	5296		11.6 ± 4	− 10	150 ± 58	296.7	298.8	297.8 ± 0.7	42 000
W. Heiskanen	1924	34	978.048 ± 3	5293 ± 6		---	---	---	---	---	297.4 ± 0.5	---
"	1924	34	978.052 ± 3	5285 ± 6	− 7	27 ± 3	+ 18 ± 5	345 ± 38	294.3 ± 0.6	299.0 ± 0.6	296.7 ± 0.5	18 700
"	1928	35	978.049 ± 1	5289		---	---	---	---	---	297.06	---
"	1928	35	978.049 ± 1	5293		19 ± 3	0 ± 5	242	295.7	299.0	297.3	26 703
"	1938	36	978.0451	5302.7		---	---	---	---	---	298.25 ± 0.3	---
"	1938	36	978.0524	5297.0		27.6	− 25	352 ± 30	295.3 + 0.4	300.2 + 0.4	297.8 ± 0.4	18 314
N. F. Shuravlev	1940	37	978.0484	5303		---	---	---	---	---	298.3	---
E. Niskanen	1945	38	978.0468	5297.8	−5.9	23.0	− 3.9	293	295.7 ± 0.2	299.8 ± 0.2	297.8 ± 0.2	21 550
H. Jeffreys	1948	39	978.0513	5285.9		---	---	---	---	---	296.85 ± 0.66	---
H. Schütte	1950	40	978.0520 ± 33	5282.7 ± 6.0		---	---	---	---	---	296.3	---
U. A. Uotila	1957	41	978.0516	5291.0		10.6	− 6	---	---	---	297.2	47 600
Heiskanen & Uotila	1957	42	978.0496	5293.4		---	---	---	---	---	297.4	---
Mean Value			978.049_5	$5292._4$							297.3	

Taking for g_e the latest determination of Heiskanen and Uotila (Ref. 42) which is nearly in agreement with the average value of all determinations, and using the correction due to H. Jeffreys (Ref. 39) the following value is obtained

$$g_e = 978.0496 - 0.0134 = 978.0362 \text{ gal} = 9.780362 \text{ m/sec}^2 \tag{86}$$

9. THE DYNAMIC OBLATENESS AND THE CONSTANTS OF PRECESSION AND NUTATION

The dynamic flattening $H = (C - A)/C$ is connected with Newcomb's constant of precession, P, by the relation

$$\frac{P}{H} = 530977.''04 + \frac{94419319''}{1/\kappa + 1} = 1678932.''29 \quad \text{for } 1/\kappa = 81.250 \tag{87}$$

while the constant of nutation, N, is given by

$$\frac{N}{H} = \frac{252871''}{1/\kappa + 1} \cos\epsilon = \frac{231982''}{1/\kappa + 1} = 2820.''45 \quad \text{for } 1/\kappa = 81.250 \tag{88}$$

where $\cos\epsilon = 0.9173917$ (for 1900.0) has been used for the cosine of the obliquity, ϵ, of the ecliptic. The constants in the equations are obtained from Brown's theory of the motion of the Moon and are well known. Both equations yield

$$\frac{P}{N} = 2.288872\,(1/\kappa + 1) + 407.01140 = 595.271 \quad \text{for } 1/\kappa = 81.250$$

while observed modern values of P and N lead to

$$\frac{P}{N} = \frac{5493.''62}{9.''208} = 596.614 \quad \text{and thus} \quad 1/\kappa = \frac{P/N}{2.288872} - 178.8218 = 81.84$$

This value for $1/\kappa$ is by far too large. H. Jeffreys (Ref. 39) has shown that in the equation for the constant of nutation, N, another constant H' for the dynamic flattening must be used due to the deviation of the Earth's interior from the isostatic equilibrium ($H' < H$). Therefore H_2 can be determined only from P and $1/\kappa$. With $p_0 = p_\odot + p_\ell$, the lunisolar precession, $p_g = \frac{3}{2}\frac{v_\oplus}{c^2} n_\oplus$, the geodetic precession (a relativistic term due to W. deSitter), p, the general precession in longitude, and λ, the planetary precession in right ascension, Newcomb's precessional constant is

$$P = \frac{p_0}{\cos\epsilon} = \frac{p + p_g}{\cos\epsilon} + \lambda \tag{89}$$

Values for 1900.0 derived from observations are (for a tropical century)

$P = 5490.''66$	$p = 5025.''641$	$\lambda = 12.''473$	$p_g = 0$	(Newcomb & Andoyer, Ref. 46)
5493.156 ± 0.175	5026.000	12.493	1.915	(deSitter & Brouwer, Ref. 10)
5493.847	$5026.''650$	12.469	1.921	(Clemence, Ref. 7)

According to newer investigations, Newcomb's value of the general precession in longitude

must be corrected by $\Delta p = +\ 0.''75$ (H. R. Morgan, Ref. 47), $\Delta p = +\ 0.''71$ (J. H. Oort, Ref. 48), $\Delta p = +\ 0.''86$(Dirk Brouwer, Ref. 48), $\Delta p = +\ 0.''84$ (Poulkovo Obs., Ref. 48). The average value for the correction may be $\Delta p = +\ 0.''80$. All these investigators take $\Delta\lambda = 0$. In another paper, J. H. Oort (1943, Ref. 49) takes $\Delta\lambda = +\ 0.''02$, a correction also used by deSitter. The correction for P is therefore

$$\Delta P = \frac{\Delta p + \Delta p_g}{\cos \epsilon} + \Delta\lambda = \frac{0.''80 + 1.92}{0.917\,3917} + 0.00 = 2.''96 \tag{90}$$

The value

$$P = 5490.''66 + 2.''96 = 5493.''62 \tag{91}$$

will be accepted here. The dynamic flattening is now

$$H \equiv \frac{C - A}{C} = \frac{5493.''62}{1\,678\,932.''29} = 0.003\,272\,091 = \frac{1}{305.615 \pm 0.05} \tag{92}$$

and thus

$$q \equiv \frac{3}{2}\ \frac{C}{M_\oplus R_e^2} = \frac{J}{H} = 0.49\,616_0 \pm 0.00017 \tag{93}$$

and

$$\frac{C}{m_\oplus R_e^2} = \frac{J_2}{H} = 0.330\,77_3 \pm 0.00011 \tag{94}$$

The quantity q may be calculated in another way. Clairant's theory for the Earth in hydrostatic equilibrium has been developed to the second order by Radau (Ref. 50), Callandreau (Ref. 51), and Darwin (Ref. 28). deSitter (Ref. 31) gives

$$q \equiv \frac{3}{2}\ \frac{C}{M R_e^2} = 1 - \frac{1}{3}\ \varpi_1 - \frac{2}{5}\left(1 - \frac{2}{3}\ f\right) \frac{\sqrt{1 + \eta_1}}{1 + \lambda_1} \tag{95}$$

where

$$\eta_1 = \frac{\frac{5}{2}\varpi_1 + \frac{10}{21}\varpi_1^2 + \frac{4}{7} f^2 - \frac{6}{7} f \varpi_1}{f - \frac{5}{42}\ f^2 + \frac{4}{7}\ \kappa} - 2 \tag{96}$$

and $1 + \lambda_1$ is an average value of Radau's function, $f\ (\eta)$, depending on the internal density distribution of the Earth. The most reliable value, $1 + \lambda_1 = 1.00016$, was derived by Bullard (Ref. 30). With the above data for f and ϖ_1 the above-mentioned equations give, for $\kappa = 0$,

$$\eta_1 = 0.57440 \quad ; \quad q = 0.49815$$

$$\frac{C}{M R_e^2} = \frac{2}{3}\ q = 0.33210$$

These data are not compatible with the previously derived data (eqs. 93 and 94), showing that the hypothesis of hydrostatic equilibrium is not fulfilled for the Earth.

10. DENSITY DISTRIBUTION WITHIN THE MOON

It is very difficult to derive a consistent system of lunar constants. Most reports on this subject are based on the work of Sir Harold Jeffreys. However, not even this source is free of inconsistencies. The reason is that many lunar constants are coupled with each other by relations. Therefore a systematic investigation of these relations will be necessary.

It is assumed that the density ρ within the Moon is constant over concentric ellipsoidal shells

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = \mu^2 \tag{97}$$

where μ varies from 0 at the center to 1 at the surface, and where

$$\begin{aligned} \xi/a &= \mu \cos\phi \cos\theta \\ \eta/b &= \mu \cos\phi \sin\theta \\ \zeta/c &= \mu \sin\phi \end{aligned} \tag{98}$$

are the relative coordinates of the mass element

$$dm = \rho(\mu)\ d\xi\ d\eta\ d\zeta = \rho(\mu)\ a\,b\,c\,\mu^2 \cos\phi\, d\mu\, d\phi\, d\theta \tag{99}$$

The angle ϕ is the lunicentric latitude and θ the longitude. a is the longest semi-axis of the surface ellipsoid pointing toward the Earth, b the smallest semi-axis in the lunar equator, and c the rotational or polar semi-axis.

Using equations (98) and (99) after observing that

$$\int_0^{2\pi} \sin^2\theta\ d\theta = \int_0^{2\pi} \cos^2\theta\ d\theta = \pi\ ; \quad \int_{-\pi/2}^{\pi/2} \cos^3\phi\ d\phi = \frac{4}{3}\ ; \int_{-\pi/2}^{\pi/2} \sin^2\phi \cos\phi\ d\phi = \frac{2}{3}$$

the moments of inertia around the a, b, c axes, respectively, become

$$\begin{aligned} A &= \int_0^M (\eta^2 + \zeta^2)\ dm = \lambda M\,(b^2 + c^2) \\ B &= \int_0^M (\zeta^2 + \xi^2)\ dm = \lambda M\,(c^2 + a^2) \\ C &= \int_0^M (\xi^2 + \eta^2)\ dm = \lambda M\,(a^2 + b^2) \end{aligned} \tag{100}$$

where the integrations are taken from 0 to 2π with respect to θ, from $-\pi/2$ to $\pi/2$ with respect to ϕ and from 0 to 1 with respect to μ. In the last equation $M = 4/3\ \pi a\,b\,c\,\rho_m$ is the total mass (ρ_m is the mean density) and the inhomogenity factor, λ, is given by

$$\lambda = \frac{\int_0^1 \rho\mu^4 d\mu}{\rho_m} = \frac{\int_0^1 \rho\mu^4\, d\mu}{3\int_0^1 \rho\mu^2\, d\mu} \tag{101}$$

Because $a > b > c$, there is $A < B < C$. A constant density model gives $\lambda = 1/5 = 0.2$. For Jeffreys compressional model with constant bulk modulus (Ref. 52) Roche's density law [with $\rho_m = 3.342$ g/cm^3 (mean density), $\rho_0 = 3.290$ g/cm^3 (surface density) and $\rho_c = 3.420$ g/cm^3 (central density)] becomes

$$\rho = \rho_c - (\rho_c - \rho_0)\,\mu^2 = 3.420 - 0.130\,\mu^2 \ [\mathrm{g/cm^3}] \tag{102}$$

The inhomogenity factor is, therefore,

$$\lambda = \frac{1}{5}\,\frac{\rho_c - \frac{5}{7}(\rho_c - \rho_0)}{\rho_m} = \frac{1}{5}\,\frac{\rho_c - \frac{5}{7}(\rho_c - \rho_0)}{\rho_c - \frac{3}{5}(\rho_c - \rho_0)} = \frac{0.9955}{5} =$$

$$= 0.1991 \pm 0.0001 \tag{103}$$

11. CONSTANTS OF THE PHYSICAL LIBRATION OF THE MOON

The values of $f \equiv \dfrac{C-B}{C-A}$ and the inclination of the Moon's equator to the ecliptic can be determined from observations of the physical libration of the Moon. Due to the difficulty of observations near the irregular limb of varied illumination the values for f scatter widely, as can be seen from the following table (Refs. 11 and 13):

Author	Year	$f \equiv \dfrac{C-B}{C-A}$
F. Hayn	1907	0.75 ± 0.04
F. Hayn		0.85 ± 0.07
J. Stratton	1909	0.50 ± 0.03
I. V. Belkovich	1936	0.84 ± 0.08
I. V. Belkovich	1949	0.67 ± 0.03
K. Koziel	1949	0.71 ± 0.051
K. Koziel	1949	0.60 ± 0.055
A.A. Nefedjev	1950	0.65 ± 0.045
A.A. Yakovkin	1950	0.85 ± 0.03
T. Weimer	1954	0.60
Mean Value		0.70_2

Sir Harold Jeffreys used $f = 0.84$ in his book *The Earth* (Ref. 11). Later he recommended $f = 0.67$ (Ref. 53) and used $f = 0.639 \pm 0.014$ in his latest paper (Ref.54).

The secular motions of the perigee and node of the lunar orbit are also influenced by the Moon's oblateness coefficients (L and K). From Jeffreys equation for the perigee motion follows (Ref. 11):

$$380\,L - 1192\,K = 6.420 - 3896\,J_{\oplus} \gtrsim 0$$

and thus

$$f = 1 - \frac{K}{L} \geqslant 1 - \frac{380}{1192} = \frac{812}{1192} = 0.6812$$

The mean value of the table is consistent with this lower limit for f and therefore

$$f = 0.70 \pm 0.02 \tag{104}$$

will be adopted in this paper.

A new investigation of the libration of the Moon's axis by H. Jeffreys (Ref. 54) leads to

$$\beta \equiv \frac{C - A}{C} = 0.0006279 \pm 0.0000010 \tag{105}$$

taking into account a solar effect not evaluated by Hayn.

The three quantities λ, f, and β are sufficient to calculate all other quantities, provided the mass and the mean radius of the Moon are known.

12. RELATIONS AND NUMERICAL VALUES FOR THE DIMENSIONLESS MOMENT OF INERTIA PARAMETERS

The same symbols for moment of inertia parameters will be used as they have been introduced mainly by H. Jeffreys (Ref. 11). The numerical values are based on the above-given parameters λ, f, and β; namely

$$\lambda = \frac{A}{M(b^2 + c^2)} = \frac{B}{M(c^2 + a^2)} = \frac{C}{M(a^2 + b^2)}$$

$$= \frac{C - B}{M(b^2 - c^2)} = \frac{C - A}{M(a^2 - c^2)} = \frac{C - B}{M(b^2 - c^2)} = 0.1991 \pm 0.0001 \tag{106}$$

$$f = \frac{\alpha}{\beta} = \frac{C - B}{C - A} = \frac{2J - K}{2J + K} = \frac{J - 1/2\ K}{L} = \frac{b^2 - c^2}{a^2 - c^2} = 0.70 \pm 0.02 \tag{107}$$

$$\beta = \frac{C - A}{C} = \frac{J + 1/2\ K}{g} = \frac{L}{g} = \frac{a^2 - c^2}{a^2 + b^2} = 0.0006279 \pm 0.0000010 \tag{108}$$

The other parameters can be derived from these as follows:

$$1 - f = \frac{\gamma}{\beta} = \frac{B - A}{C - A} = \frac{K}{J + 1/2\ K} = \frac{K}{L} = \frac{a^2 - b^2}{a^2 - c^2} = \frac{3\lambda - g}{L} = \frac{3\lambda/g - 1}{\beta} =$$

$$= 0.30 \pm 0.02 \tag{109}$$

$$\alpha = \frac{C - B}{C} = f\beta = \frac{J - 1/2\ K}{g} = \frac{L - K}{g} = \frac{b^2 - c^2}{a^2 + b^2} = 0.000\,4395 \pm 0.000\,0133 \tag{110}$$

$$\gamma = \frac{B - A}{C} = \beta - \alpha = \frac{K}{g} = (1 - f)\beta = \frac{a^2 - b^2}{a^2 + b^2} = 0.000\,1884 \pm 0.000\,0129 \tag{111}$$

$$g = \frac{3}{2}\ \frac{C}{Ma^2} = \frac{3\lambda}{1 + \gamma} = \frac{3}{2}\ \lambda\ \frac{a^2 + b^2}{a^2} = 0.5972 \pm 0.0003 \tag{112}$$

$$L = \frac{3}{2}\ \frac{C - A}{Ma^2} = J + \frac{1}{2}\ K = g\beta = \frac{3}{2}\ \lambda\ \frac{a^2 - c^2}{a^2} = 0.000\,3750 \pm 0.000\,0008 \tag{113}$$

$$J = \frac{3}{2}\ \frac{C - \frac{A + B}{2}}{Ma^2} = L - \frac{1}{2}\ K = \frac{1 + f}{2}\ L = \frac{1}{2}\ \frac{1 + f}{1 - f}\ K = \frac{3}{2}\lambda\ \frac{\frac{a^2 + b^2}{2} - c^2}{a^2}$$

$$= g\left(\beta - \frac{1}{2}\ \gamma\right) = g\ \frac{\alpha + \beta}{2} = g\left(\alpha + \frac{1}{2}\ \gamma\right) = 0.000\,3187_5 \pm 0.000\,0044 \tag{114}$$

$$K = \frac{3}{2}\ \frac{B - A}{Ma^2} = (1 - f)\ L = g\gamma = \frac{3}{2}\ \lambda\ \frac{a^2 - b^2}{a^2} = 0.000\,1125 \pm 0.000\,0077 \tag{115}$$

The dimensionless moments of inertia and their differences are obtained from the above-mentioned data as:

$$\frac{C - A}{Ma^2} = \frac{2}{3}\ L = \frac{2\,J + K}{3} = \lambda\ \frac{a^2 - c^2}{a^2} = 0.000\,2500 \pm 0.000\,0005 \tag{116}$$

$$\frac{C - B}{Ma^2} = \frac{2}{3}\ fL = \frac{2\,J - K}{3} = \lambda\ \frac{b^2 - c^2}{a^2} = 0.000\,1750 \pm 0.000\,0054 \tag{117}$$

$$\frac{B - A}{Ma^2} = \frac{2}{3}\ (1 - f)\ L = \frac{2}{3}\ K = \lambda\ \frac{a^2 - b^2}{a^2} = 0.000\,0750 \pm 0.000\,0051 \tag{118}$$

$$\frac{A}{Ma^2} = \frac{2}{3}\ g\ (1 - \beta) = \frac{2}{3}\ (g - L) = \lambda\ \frac{b^2 + c^2}{a^2} = 0.3978_{77} \pm 0.0002 \tag{119}$$

$$\frac{B}{Ma^2} = \frac{2}{3}\ g\,(1 - \alpha) = \frac{2}{3}\ (g - fL) = \lambda\ \frac{c^2 + a^2}{a^2} = 0.3979_{52} \pm 0.0002 \tag{120}$$

$$\frac{C}{Ma^2} = \frac{2}{3}\, g = \frac{2\lambda}{1+\gamma} = \lambda\, \frac{a^2 + b^2}{a^2} = 0.3981_{27} \pm 0.0002 \tag{121}$$

The ratios of the semi-axes now become

$$\frac{b}{a} = \sqrt{\frac{1-\gamma}{1+\gamma}} = \sqrt{1 - \frac{2\gamma}{1+\gamma}} = 0.9998116 \pm 0.0000129 \tag{122}$$

$$\frac{c}{\bar{a}} = \sqrt{1 - \frac{2\beta}{1+\gamma}} = \sqrt{1 - \frac{2\gamma}{(1-f)(1+\gamma)}} = 0.9993720 \pm 0.0000010 \tag{123}$$

These data seem, at present, to be the most reasonable. H. Jeffreys' value

$$g \equiv \frac{3}{2}\, \frac{C}{Ma^2} = 0.5956 \pm 0.0010$$

is slightly low, and affords a higher density concentration towards the center as has been assumed by Jeffreys.

13. THE FINAL DETERMINATION OF THE DIMENSIONS, MASSES AND MOMENTS OF INERTIA FOR THE EARTH AND THE MOON

Using the obtained value of the gravitational acceleration, g_e, at the equator (eq. 86) in the relations (eq. 80) and (eq. 82) at the end of Section 7 there follows at once $R_e = \sqrt{\chi \mu_\oplus / g_e} = 6378169.835$ m and $\rho_\oplus = 5.516964$ g/cm³. The final values, adopted for the Earth, will be taken as

$$R_e + 6378170\ (1 \pm 3.2 \times 10^{-6})\ \mathrm{m} = 6378170 \pm 20\ \mathrm{m} \tag{124}$$

and

$$\rho_\oplus = 5.5170\,(1 \pm 7.3 \times 10^{-4})\ \mathrm{g/cm^3} = 5.5170 \pm 0.0040\ \mathrm{g/cm^3} \tag{125}$$

The volume of the Earth is

$$V_\oplus = \frac{4}{3}\pi R_e^3 (1-f) = 1.083225 \times 10^{27}\ (1 \pm 1.02 \times 10^{-5})\ \mathrm{cm^3}$$

$$= (1.083225 \pm 0.000011) \times 10^{27}\ \mathrm{cm^3} \tag{126}$$

while the mass is given by

$$M_\oplus = \frac{\mu_\oplus}{G} = V_\oplus\, \rho_\oplus = 5.9761 \times 10^{27}\ (1 \pm 7.2 \times 10^{-4})\ \mathrm{g} = (5.9761 \pm 0.0043) \times 10^{27}\ \mathrm{g} \tag{127}$$

and the polar radius now becomes

$$R_p = R_e\,(1-f) = 6356788\ (1 \pm 3.7 \times 10^{-6})\ \mathrm{m} = 6356788 \pm 24\ \mathrm{m} \tag{128}$$

The unit for the moments of inertia is

$$M_{\oplus} R_e^2 = 2.4311_4 \times 10^{45} (1 \pm 7.3 \times 10^{-4}) \text{g cm}^2 = (2.4311_4 \pm 0.0018) \times 10^{45} \text{g cm}^2 \tag{129}$$

and therefore

$$C - A = J_2 \times M_{\oplus} R_e^2 = 2.6313 \times 10^{42} (1 \pm 9.1 \times 10^{-4}) \text{g cm}^2 = (2.6313 \pm 0.0024) \times 10^{42} \text{g cm}^2 \tag{130}$$

$$C = \frac{2}{3} q \times M_{\oplus} R_e^2 = 8.0415 \times 10^{44} (1 \pm 1.07 \times 10^{-3}) \text{g cm}^2 = (8.0415 \pm 0.0086) \times 10^{44} \text{g cm}^2 \tag{131}$$

It is now possible to give corresponding data for the Moon. The mass is given by

$$M_{☾} = \kappa M_{\oplus} = 7.3552 \times 10^{25} (1 \pm 1.02 \times 10^{-3}) \text{ g} = (7.3552 \pm 0.0075) \times 10^{25} \text{ g} \tag{132}$$

while the mean visible radius is

$$R_{☾} = k R_e = 1\,738\,236 (1 \pm 1.0 \times 10^{-4}) = 1\,738\,236 \pm 174 \text{ m} \tag{133}$$

and therefore the semi-axes of the three-axial Moon are

$$a = \frac{R_{☾}}{\sigma} = 1\,738\,946 \pm 186 \text{ m} \tag{134}$$

$$b = a \left(\frac{b}{a}\right) = 1\,738\,618 \pm 209 \text{ m} \tag{135}$$

$$c = a \left(\frac{c}{a}\right) = 1\,737\,854 \pm 188 \text{ m} \tag{136}$$

The unit for the Moon's moments of inertia is

$$M_{☾} a^2 = 2.2241_6 \times 10^{42} (1 \pm 1.23 \times 10^{-3}) \text{ g cm}^2 = (2.2241_6 \pm 0.0027) \times 10^{42} \text{g cm}^2 \tag{137}$$

and thus the moments of inertia are

$$A = 0.8849_{42} \times 10^{42} (1 \pm 1.73 \times 10^{-3}) \text{ g cm}^2 = (0.8849_{42} \pm 0.0015_3) \times 10^{42} \text{g cm}^2 \tag{138}$$

$$B = 0.8851_{09} \times 10^{42} (1 \pm 1.73 \times 10^{-3}) \text{ g cm}^2 = (0.8851_{09} \pm 0.0015_3) \times 10^{42} \text{g cm}^2 \tag{139}$$

$$C = 0.8854_{98} \times 10^{42} (1 \pm 1.73 \times 10^{-3}) \text{ g cm}^2 = (0.8854_{98} \pm 0.0015_3) \times 10^{42} \text{g cm}^2 \tag{140}$$

$$C - \frac{A + B}{2} = (0.000\,473 \pm 0.000\,007) \times 10^{42} \text{ g cm}^2 \tag{141}$$

$$B - A = (0.000\,167 \pm 0.000\,012) \times 10^{42} \text{ g cm}^2 \tag{142}$$

The oblateness coefficients of the potential function of the Moon are

$$J_2 = \frac{C - (A + B)/2}{M_{☾} a^2} = \frac{2}{3} J = 0.000\,2125 \pm 0.000\,0029 \tag{143}$$

$$J_2^{(2)} = \frac{B-A}{4\,M_{☾}\,a^2} = \frac{1}{6}\,K = 0.000\,0188 \pm 0.000\,0013 \tag{144}$$

The derived value for the equatorial radius of the Earth (eq. 124) is in good agreement with the following values:

Author	Year	Ref.	R_e
W. M. Kaula	1961	55	6 378 163 ± 21 m
V. C. Clarke, Jr.	1962	56	6 378 165 ± 25
I. Fischer	1962	57	6 378 166
Present Report	1962		6 378 170 ± 20

I. Fisher's value for $1/\kappa = M_{\oplus}/M_{☾} = 81.268$ is also in good agreement with the value in this report. The presented system of constants is not only a consistent one, but the most serious discrepancy has been removed in determining the gravitational parameter $\mu_{\oplus}$ from terrestrial data and, on the other hand, from the lunar mean motion in combination with radar measurements of the Moon's distance.

Finally, the present data for the Moon's moments of inertia are compared with the values of other authors in the following table:

Author	Ref.	A 10^{35} kg·m^2	B 10^{35} kg·m^2	C 10^{35} kg·m^2	$C-(A+B)/2$ 10^{35} kg·m^2	$B-A$ 10^{35} kg·m^2
B. E. Kalensher	58	0.87976	0.87985	0.88032	0.00051	0.00009
Makemson, Baker, Westrom	13	0.88837	0.88856	0.88893	0.00047	0.00019
V. C. Clarke, Jr.	56	0.88746	0.88764	0.88801	0.00046	0.00018
Present Report	---	0.88494	0.88511	0.88550	0.00047	0.00017

The values of V. C. Clarke, Jr. are used for the Ranger Program.

14. THE EARTH ELLIPSOID

The equation of the rotational ellipsoid or spheroid is

$$\frac{x^2+y^2}{R_e^2} + \frac{z^2}{R_p^2} = 1 \tag{145}$$

where

$$x = R\cos\phi\cos\lambda = R_e\cos\psi\cos\lambda = \rho_n\cos\varphi\cos\lambda \tag{146}$$

$$y = R\cos\phi\sin\lambda = R_e\cos\psi\sin\lambda = \rho_n\cos\varphi\sin\lambda \tag{147}$$

$$z = R\sin\phi = R_p\sin\psi = \rho_n(1-e^2)\sin\varphi \tag{148}$$

and R_e is the equatorial Earth radius, R_p the polar Earth radius, R the local Earth radius, ρ_n the normal radius of curvature, λ the geographic longitude (positive eastward of Greenwich), ϕ the geocentric latitude, ψ the reduced latitude, φ the geodetic or geographic ($\approx$ astronomical) latitude, and e the first eccentricity of the meridian ellipse. Introducing the second eccentricity, ϵ, and the flattening (oblateness, ellipticity), f, the following relations hold

$$f = \frac{R_e - R_p}{R_e} = 1 - \sqrt{1 - e^2} = 1 - \frac{1}{\sqrt{1 + \epsilon^2}} \approx \frac{1}{2} e^2 + \frac{1}{8} e^4 + \cdots + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2\kappa - 3)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2\kappa)} e^{2\kappa} + \cdots \tag{149}$$

$$e^2 = \frac{R_e^2 - R_p^2}{R_e^2} = \frac{\epsilon^2}{1 + \epsilon^2} = f(2 - f) = 2f - f^2 \tag{150}$$

$$\epsilon^2 = \frac{R_e^2 - R_p^2}{R_e^2} = \frac{e^2}{1 - e^2} = \frac{f(2 - f)}{(1 - f)^2} = 2f + 3f^2 + \cdots + (\kappa + 1) f^\kappa + \cdots \tag{151}$$

thus

$$\frac{R_p}{R_e} = 1 - f = \sqrt{1 - e^2} = \frac{1}{\sqrt{1 + \epsilon^2}} \tag{152}$$

The different latitude angles are related by

$$\tan \phi = \sqrt{1 - e^2} \tan \psi = (1 - e^2) \tan \varphi \tag{153}$$

$$\tan \phi = (1 - f) \tan \psi = (1 - f)^2 \tan \varphi \tag{154}$$

By differentiation the relation

$$\frac{R^2}{R_p} d\phi = R_e \, d\psi = \rho \, d\varphi \tag{155}$$

follows.

Thus the line element is

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\phi^2 + r^2 \cos^2 \phi \, d\lambda^2 = R_e^2 (1 - e^2 \cos^2 \psi) \, d\psi^2 + R_e^2 \cos^2 \psi \, d\lambda^2 \tag{156}$$

The parameter ρ is the mean radius of curvature, and is correlated with the normal radius of curvature

$$\rho_n = \frac{R_e}{(1 - e^2 \sin^2 \varphi)^{1/2}} = \frac{R_e^2 / R_p}{(1 + \epsilon^2 \cos^2 \varphi)^{1/2}} = R \frac{\cos \phi}{\cos \varphi} = \tag{157}$$

$$= R_e \left[1 + \frac{1}{2} e^2 \sin^2 \varphi + \frac{3}{8} e^4 \sin^4 \varphi + \frac{5}{16} e^6 \sin^6 \varphi + \cdots \right]$$

and to the meridional radius of curvature

$$\rho_m = \frac{R_e(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}} = \frac{R_p^2/R_e}{(1-e^2\sin^2\varphi)^{3/2}} = \frac{R_e^2/R_p}{(1+\epsilon^2\cos^2\varphi)^{3/2}}$$

$$= R_e(1-e^2)\left[1+\frac{3}{2}e^2\sin^2\varphi+\frac{15}{8}e^4\sin^4\varphi+\frac{35}{16}e^6\sin^6\varphi+\cdots\right] \tag{158}$$

by the relation

$$\rho = \sqrt{\rho_m\rho_n} = \frac{R_p}{1-e^2\sin^2\varphi} = \frac{R_e^2/R_p}{1+\epsilon^2\cos^2\varphi} = R_p\left[1+e^2\sin^2\varphi+e^4\sin^4\varphi+e^6\sin^6\varphi+\cdots\right] \tag{159}$$

The radius of a parallel of latitude is $\rho_n\cos\varphi = R_e\cos\psi = R\cos\phi$. Because f or e^2 are small quantities the latitudes ϕ, ψ, and φ will not differ very much from each other. Therefore it is very useful to have rapidly converging series developments available for the differences $\phi-\varphi$ and $\psi-\varphi$. With

$$m = \frac{R_e^2-R_p^2}{R_e^2+R_p^2} = \frac{1-(1-e^2)}{1+(1-e^2)} = \frac{e^2}{2-e^2} = f+\frac{1}{2}f^2-\frac{1}{4}f^4+\cdots \tag{160}$$

there is

$$\phi = \varphi - m\sin 2\varphi + \frac{m^2}{2}\sin 4\varphi - \frac{m^3}{3}\sin 6\varphi \pm\cdots \tag{161}$$

$$\varphi = \phi + m\sin 2\phi + \frac{m^2}{2}\sin 4\phi + \frac{m^3}{3}\sin 6\phi+\cdots \tag{162}$$

and with

$$n = \frac{R_e-R_p}{R_e+R_p} = \frac{1-(1-f)}{1+(1-f)} = \frac{f}{2-f} = \frac{1}{2}f+\frac{1}{4}f^2+\cdots \tag{163}$$

there is

$$\psi = \varphi - n\sin 2\varphi + \frac{n^2}{2}\sin 4\varphi - \frac{n^3}{3}\sin 6\varphi\pm\cdots \tag{164}$$

$$\varphi = \psi + n\sin 2\psi + \frac{n^2}{2}\sin 4\psi + \frac{n^3}{3}\sin 6\psi+\cdots \tag{165}$$

An accurate formula for the difference $\varphi-\phi$ is given by

$$\tan(\varphi-\phi) = \frac{e^2\tan\varphi}{1+(1-e^2)\tan^2\varphi} = \frac{e^2\sin\varphi\cos\varphi}{1-e^2\sin^2\varphi} \tag{166}$$

The local Earth radius (radius vector) can be accurately calculated from the relations

$$R = \sqrt{x^2+y^2+z^2} = \frac{R_p}{\sqrt{1-e^2\cos^2\phi}} = R_e\sqrt{1-e^2\sin^2\psi} = R_e\sqrt{\frac{1-e^2(2-e^2)\sin\varphi}{1-e^2\sin^2\varphi}}$$

$$= R_e\sqrt{\frac{1-4m(1+m)^{-2}\sin^2\varphi}{1-4n(1+n)^{-2}\sin^2\varphi}} = \frac{1+n}{1+m}R_e\sqrt{\frac{1+m^2+2m\cos 2\varphi}{1+n^2+2n\cos 2\varphi}} \tag{167}$$

By taking the logarithm of the last relation there follows

$$\ln \frac{R}{R_e} = \ln \frac{1+n}{1+m} + \frac{1}{2} \ln (1 + m^2 + 2m \cos 2\varphi) - \frac{1}{2} \ln (1 + n^2 + 2n \cos 2\varphi)$$

or, using a known series development,

$$\log \frac{R}{R_e} = \log \frac{1+n}{1+m} + M \left[(m-n) \cos 2\varphi - \frac{m^2 - n^2}{2} \cos 4\varphi + \frac{m^3 - n^3}{3} \cos 6\varphi \mp \cdots \right] \quad (168)$$

where M is the module ($M = 0.43\,429\,44819$). This series is due to Encke.

Conventional power series for R/R_e can be obtained as follows:

$$\frac{R}{R_e} = (1 + \epsilon^2 \sin^2 \phi)^{-1/2} = 1 - \frac{1}{2} \epsilon^2 \sin^2 \phi + \frac{3}{8} \epsilon^4 \sin^4 \phi - \frac{5}{16} \epsilon^6 \sin^6 \phi \pm \cdots \quad (169)$$

The above-mentioned power series for the local Earth radius can also be written

$$\frac{R}{R_e} = 1 - \left(\frac{1}{2} \epsilon^2 - \frac{3}{8} \epsilon^4 + \frac{5}{16} \epsilon^6 \right) \sin^2 \phi - \left(\frac{3}{32} \epsilon^4 - \frac{5}{64} \epsilon^6 \right) \sin^2 2\phi + \frac{5}{64} \epsilon^6 \sin^2 \phi \sin^2 2\phi + \cdots \quad (170)$$

or

$$\frac{R}{R_e} = 1 - f \sin^2 \phi - \left(\frac{3}{8} f^2 + \frac{1}{2} f^3 \right) \sin^2 2\phi + \frac{5}{8} f^3 \sin^2 \phi \sin^2 2\phi + \cdots \quad (171)$$

using eq. (151).

In order to obtain power series for the local Earth radius as a function of the geodetic latitude, φ, it is useful to set

$$k = \frac{4m}{(1+m)^2} = e^2 (2 - e^2) \; ; \; k' = \frac{4n}{(1+n)^2} = f (2 - f) = e^2 \; ; \; k/k' = 2 - e^2 \quad (172)$$

the non-dimensional local Earth radius is now

$$\frac{R}{R_e} = 1 - \frac{1}{2} (k - k') \sin^2 \varphi - \frac{1}{8} (k^2 + 2kk' - 3k'^2) \sin^4 \varphi - \frac{1}{16} (k^3 + k^2 k' + 3kk'^2 - 5k'^3) \cdot \sin^6 \varphi - \cdots \quad (173)$$

or

$$\frac{R}{R_e} = 1 - \frac{1}{2} e^2 [(2 - e^2) - 1] \sin^2 \varphi - \frac{1}{8} e^4 [(2 - e^2)^2 + 2(2 - e^2) - 3] \sin^4 \varphi$$

$$- \frac{1}{16} e^6 [(2 - e^2)^3 + (2 - e^2)^2 + 3(2 - e^2) - 5] \sin^6 \varphi - \cdots \quad (174)$$

Because

$$e^2 = 2f - f^2 \; ; \quad 2 - e^2 = 2 - 2f + f^2$$

there is, also,

$$\frac{R}{R_e} = 1 - \left(f - \frac{5}{2}f^2 + 2f^3\right)\sin^2\varphi - \left(\frac{5}{2}f^2 - \frac{17}{2}f^3\right)\sin^4\varphi - \frac{13}{2}f^3\sin^6\varphi - \cdots$$

$$= 1 - f\sin^2\varphi + \left(\frac{5}{8}f^2 - \frac{1}{2}f^3\right)\sin^2 2\varphi + \frac{13}{8}f^3\sin^2\varphi\,\sin^2 2\varphi + \cdots$$

$$= 1 - \frac{1}{2}f + \frac{5}{16}f^2 + \frac{5}{32}f^3 + \left(\frac{1}{2}f - \frac{13}{64}f^3\right)\cos 2\varphi - \left(\frac{5}{16}f^2 + \frac{5}{32}f^3\right)\cos 4\varphi + \frac{13}{64}f^3\cos 6\varphi + \cdots \tag{175}$$

It is also important to know the arc corresponding to 1° in longitude, namely

$$\nu = \frac{\pi}{180}\rho_n\cos\varphi = \frac{\pi}{180}\frac{R_e\cos\varphi}{(1 - e^2\sin^2\varphi)^{1/2}}$$

$$= \frac{\pi}{180}R_e\cos\varphi\left(1 + \frac{1}{2}e^2\sin^2\varphi + \frac{3}{8}e^4\sin^4\varphi + \cdots\right)$$

$$= \frac{\pi}{180}R_e\left(1 + \frac{1}{8}e^2 + \frac{3}{64}e^4\right)\cos\varphi - \left(\frac{e^2}{8} + \frac{9}{128}e^4\right)\cos 3\varphi + \frac{3}{128}e^4\cos 5\varphi \mp \cdots \tag{176}$$

and the arc corresponding to 1° in latitude, namely

$$\mu = \frac{\pi}{180}\rho_m = \frac{\pi}{180}\frac{R_e(1 - e^2)}{(1 - e^2\sin^2\varphi)^{3/2}} = \frac{\pi}{180}R_e(1 - e^2)\left[1 + \frac{3}{2}e^2\sin^2\varphi + \frac{15}{8}e^4\sin^4\varphi + \cdots\right]$$

$$= \frac{\pi}{180}\frac{R_p^2}{R_e}\left[1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 - \left(\frac{3}{4}e^2 + \frac{15}{16}e^4\right)\cos 2\varphi + \frac{15}{64}e^4\cos 4\varphi \mp \cdots\right] \tag{177}$$

The length of a meridian quadrant is given by

$$Q_m = \frac{\pi}{2}\overline{\rho}_m = \int_0^{\pi/2}\frac{R_e(1 - e^2)}{(1 - e^2\sin^2\varphi)^{3/2}}\,d\varphi = R_e(1 - e^2)\int_0^1\frac{dz}{(1 - e^2z^2)\sqrt{(1 - z^2)(1 - e^2z^2)}}$$

$$= R_e(1 - e^2)\prod\left(\frac{\pi}{2}, -e^2, e\right) = R_e(1 - e^2)\int_0^{\pi/2}\left[1 + \frac{3}{2}e^2\sin^2\varphi + \frac{15}{8}e^4\sin^4\varphi + \frac{35}{16}e^6\sin^6\varphi + \ldots\right]d\varphi$$

$$= \frac{\pi}{2}R_e(1 - e^2)\left[1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \frac{175}{256}e^6 + \cdots\right]$$

$$= \frac{\pi}{2}R_e\left(1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \cdots\right)$$

$$= \frac{\pi}{2}R_e\left(1 - \frac{1}{2}f + \frac{1}{16}f^2 + \frac{1}{32}f^3 + \cdots\right) \tag{178}$$

The volume of the Earth spheroid is simply given by

$$V = \frac{4}{3}\pi R_e^2 R_p = \frac{4}{3}\pi R_e^3 (1-f) \tag{179}$$

while the surface area follows from

$$S = 4\pi R_e R_p \int_0^1 \sqrt{1+\epsilon^2 z^2}\, dz = 2\pi R_e R_p \left[\sqrt{1+\epsilon^2} + \frac{1}{\epsilon}\sinh^{-1}\epsilon\right]$$

$$= 2\pi R_e^2 \left[1 + \frac{\sinh^{-1}\epsilon}{\epsilon\sqrt{1+\epsilon^2}}\right] = 2\pi R_e^2 \left[1 + \frac{\ln(\epsilon+\sqrt{1+\epsilon^2})}{\epsilon\sqrt{1+\epsilon^2}}\right]$$

$$= 2\pi R_e^2 \left[1 + \left(1 - \frac{1}{6}\epsilon^2 + \frac{3}{40}\epsilon^4 - \frac{5}{112}\epsilon^6 \pm \cdots\right)\left(1 - \frac{1}{2}\epsilon^2 + \frac{3}{8}\epsilon^4 - \frac{5}{16}\epsilon^6 \pm \cdots\right)\right]$$

$$= 4\pi R_e^2 \left(1 - \frac{1}{3}\epsilon^2 + \frac{4}{15}\epsilon^4 - \frac{8}{35}\epsilon^6 - \cdots\right) = 4\pi R_e^2 \left(1 - \frac{2}{3}f + \frac{1}{15}f^2 + \frac{4}{105}f^3 + \cdots\right) \tag{180}$$

These formulas will now be used to derive the Earth's dimensions and other Earth parameters. Taking $R_e = 6378170 \pm 20$ m and $1/f = 298.30 \pm 0.05$ yields

$$e^2 = 0.006\,693\,422 \quad , \quad e = 0.081\,813\,334$$

$$\epsilon^2 = 0.006\,738\,525 \quad , \quad \epsilon = 0.082\,088\,522$$

$$m = 0.003\,357\,949 = 692.''627$$

$$n = 0.001\,678\,979 = 346.''314$$

The mean radius is

$$\overline{R} = \frac{2R_e + R_p}{3} = 6\,371\,043 \pm 21 \text{ m} \tag{181}$$

The radius for the geodetic latitude $\varphi = \sin^{-1}\sqrt{1/3} = 35°\,15'\,51.''8$ ($\phi = 35°\,4'\,59.''5$) is

$$R_1 = R_e \sqrt{1 - \frac{e^2(1-e^2)}{3-e^2}} = R_e \left[1 - \frac{1}{3}f + \frac{5}{9}f^2 + \frac{1}{3}f^2 + \cdots\right] = 6.371\,083 \pm 21 \text{ m} \tag{182}$$

The radius for the geocentric latitude $\phi = \sin^{-1}\sqrt{1/3} = 35°\,15'\,51.''8$ ($\varphi = 35°\,26'\,45.''5$) is

$$R_2 = R_e \sqrt{1 - \frac{e^2}{3-2e^2}} = R_e \left[1 - \frac{1}{3}f - \frac{1}{3}f^2 - \frac{7}{27}f^3 - \cdots\right] = 6\,371\,019 \pm 21 \text{ m} \tag{183}$$

The radius for a sphere of equal area is

$$R_S = (S/4\pi)^{1/2} = R_e \left(1 - \frac{1}{3}f^2 - \frac{1}{45}f^3 + \frac{11}{945}f + \cdots\right) = 6\,371\,041 \pm 21 \text{ m} \tag{184}$$

while the radius for a sphere of equal volume is

$$R_V = (3V/4\pi)^{1/3} = R_e (1-f)^{1/3} = R_e \left(1 - \frac{1}{3}f - \frac{1}{9}f^2 - \frac{5}{81}f^3 - \cdots\right) = 6\,371\,035 \pm 21 \text{ m} \tag{185}$$

The surface area is

$$S = 4\pi R_e^2 \left(1 - \frac{2}{3} f + \frac{1}{15} f^2 + \frac{4}{105} f^3 + \cdots\right) = (5.100\,711 \pm 0.000\,034) \times 10^{14}\,\mathrm{m}^2 \quad (186)$$

while the volume is given by

$$V = \frac{4}{3}\pi R_e^3 (1 - f) = (1.083\,225 \pm 0.000\,011) \times 10^{21}\,\mathrm{m}^3 \quad (187)$$

The radius of curvature at the pole is

$$\rho_p = R_e^2 / R_p = R_e / (1 - f) = 6\,399\,624 \pm 24\ \mathrm{m} \quad (188)$$

while the meridional radius of curvature at the equator is given by

$$\rho_{m_e} = R_p^2 / R_e = R_e (1 - f)^2 = R_e (1 - e^2) = 6\,335\,478 \pm 27\ \mathrm{m} \quad (189)$$

The length of an equatorial quadrant is

$$Q_e = \frac{\pi}{2} R_e = 10\,018\,806 \pm 31\ \mathrm{m} \quad (190)$$

while the length of a meridional quadrant is

$$Q_m = \frac{\pi}{2} R_e \left(1 - \frac{1}{2} f + \frac{1}{16} f^2 + \frac{1}{32} f^3 + \cdots\right) = 10\,002\,020 \pm 34\ \mathrm{m} \quad (191)$$

Therefore the arc corresponding to 1° in longitude is at the equator

$$\nu_e = Q_e / 90 = \frac{\pi}{180} R_e = 111\,320.07 \pm 0.35\ \mathrm{m} \quad (192)$$

while the arc corresponding to 1° in latitude is at the equator

$$\mu_e = \frac{\pi}{180} \rho_{m_e} = \frac{\pi}{180} R_e (1 - f)^2 = 110\,574.95 \pm 0.47\ \mathrm{m} \quad (193)$$

at the pole

$$\mu_p = \frac{\pi}{180} \rho_p = \frac{\pi}{180} R_e / (1 - f) = 111\,694.51 \pm 0.41\ \mathrm{m} \quad (194)$$

and in the average

$$\bar{\mu} = Q_m / 90 = \frac{\pi}{180} R_e \left(1 - \frac{1}{2} f + \frac{1}{16} f^2 + \frac{1}{32} f^3 + \cdots\right) = 111\,133.56 \pm 0.38\ \mathrm{m} \quad (195)$$

Finally, a few series developments are given for the Earth radius, for the various definitions of latitude, and for the radii of curvature:

$$\phi = \varphi - 692.''627 \sin 2\varphi + 1.''163 \sin 4\varphi \mp \cdots$$

$$\varphi = \phi + 692.''627 \sin 2\phi + 1.''163 \sin 4\phi + \cdots \tag{196}$$

and

$$\psi = \varphi - 346.''314 \sin 2\varphi + 0.''291 \sin 4\varphi \mp \cdots$$

$$\varphi = \psi + 346.''314 \sin 2\psi + 0.''291 \sin 4\psi + \cdots \tag{197}$$

furthermore

$$\log \frac{R}{R_e} = 9.9992\,7266 + 0.0007\,2917 \cos 2\varphi - 0.0000\,0184 \cos 4\varphi + \cdots \tag{198}$$

and

$$\begin{aligned}\frac{R}{R_e} &= 1 - 0.00336\,9263 \sin^2\phi + 0.00001\,7028 \sin^4\phi - 0.00000\,0096 \sin^6\phi \pm \cdots \\ &= 1 - 0.00335\,2330 \sin^2\phi - 0.00000\,4233 \sin^2 2\phi + 0.00000\,0024 \sin^2\phi \sin^2 2\phi + \cdots \\ &= 0.99832\,1724 + 0.00167\,6162 \cos 2\phi + 0.00000\,2111 \cos 4\phi + 0.00000\,0003 \cos 6\phi + \cdots\end{aligned} \tag{199}$$

or

$$\begin{aligned}R &= 6378\,170.0 - 21489.7 \sin^2\phi + 108.6 \sin^4\phi - 0.6 \sin^6\phi \pm \cdots \\ &= 6378\,170.0 - 21381.7 \sin^2\phi - 27.0 \sin^2 2\phi + 0.2 \sin^2\phi \sin^2 2\phi + \cdots \\ &= 6367\,465.7 + 10690.8 \cos 2\varphi + 13.5 \cos 4\varphi + 0.0_2 \cos 6\varphi + \cdots\end{aligned} \tag{200}$$

and

$$\begin{aligned}\frac{R}{R_e} &= 1 - 0.00332\,4310 \sin^2\varphi - 0.00002\,7777 \sin^4\varphi - 0.00000\,0241 \sin^6\varphi - \cdots \\ &= 1 - 0.00335\,2330 \sin^2\varphi - 0.00000\,7004 \sin^2 2\phi + 0.00000\,0060 \sin^2\phi \sin^2 2\phi + \cdots \\ &= 0.99832\,0349 + 0.00167\,6156 \cos 2\varphi + 0.00000\,3487 \cos 4\varphi + 0.00000\,0008 \cos 6\varphi + \cdots\end{aligned} \tag{201}$$

or

$$\begin{aligned}R &= 6378\,170.0 - 21203.0 \sin^2\varphi - 177.2 \sin^4\varphi - 1.5 \sin^6\varphi - \cdots \\ &= 6378\,170.0 - 21381.7 \sin^2\varphi - 44.7 \sin^2 2\varphi + 0.4 \sin^2\varphi \sin^2 2\varphi + \cdots \\ &= 6367\,456.9 + 10690.8 \cos 2\varphi + 22.2 \cos 4\varphi + 0.0_5 \cos 6\varphi + \cdots\end{aligned} \tag{202}$$

and

$$\frac{\mu}{\mu_e} = \frac{\rho_m}{\rho_{m_e}} = 1 + 0.01004\,0132 \sin^2 \varphi + 0.00008\,4004 \sin^4 \varphi$$

$$= 1.00505\,1568 - 0.00506\,2068 \cos 2\varphi + 0.00001\,0500 \cos 4\varphi \mp \cdots \tag{203}$$

or

$$\mu = 110\,574.95 + 1110.19 \sin^2 \varphi + 9.29 \sin^4 \varphi + \cdots$$

$$= 111\,133.53 - 559.74 \cos 2\varphi + 1.16 \cos 4\varphi \mp \cdots \tag{204}$$

$$\frac{\nu}{\nu_e} = \frac{\rho_n}{R_e} \cos \varphi = 1.00083\,8785 \cos \varphi - 0.00083\,9841 \cos 3\varphi + 0.00000\,1059 \cos 5\varphi + \cdots \tag{205}$$

or

$$\nu = 111\,413.44 \cos \varphi - 93.49 \cos 3\varphi + 0.12 \cos 5\varphi \tag{206}$$

and

$$\frac{\rho_n}{R_e} = 1 + 0.00334\,6711 \sin^2 \varphi + 0.00001\,6801 \sin^4 \varphi + 0.00000\,0094 \sin^6 \varphi + \cdots$$

$$= 1 + 0.00336\,3605 \sin^2 \varphi - 0.00000\,4224 \sin^2 2\varphi - 0.00000\,0023 \sin^2 \varphi \sin^2 2\varphi$$

$$= 1.00167\,9685 - 0.00168\,1800 \cos 2\varphi + 0.00000\,2118 \cos 4\varphi - 0.00000\,0003 \cos 6\varphi \tag{207}$$

or

$$\rho_n = 6378\,170.0 + 21345.9 \sin^2 \varphi + 107.2 \sin^4 \varphi + 0.6 \sin^6 \varphi + \cdots$$

$$= 6378\,170.0 + 21453.6 \sin^2 \varphi - 26.9 \sin^2 2\varphi - 0.1_5 \sin^2 \varphi \sin^2 2\varphi + \cdots$$

$$= 6388\,883.\dot{3} - 10726.8 \cos 2\varphi + 13.5 \cos 4\varphi - \cdots \tag{208}$$

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APPROVAL

ON A CONSISTENT SYSTEM OF ASTRODYNAMIC CONSTANTS

By

Helmut G.L. Krause

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