25 p.

N 62 14462

American Rocket Society

LUNAR MISSIONS

MEETING

JULY 17-19, 1962

CLEVELAND AUDITORIUM

CLEVELAND, OHIO

A STATISTICAL OPTIMIZING NAVIGATION PROCEDURE FOR SPACE FLIGHT

by
Richard H. Battin
Massachusetts Institute of Technology
Cambridge, Mass.

2461-62

ACKNOWLEDGMENT

This report was prepared under the auspices of DSR Project 55-191, sponsored by the National Aeronautical Space Administration, under contract NAS-9-153.

The publication of this report does not constitute approval by NASA of the findings or the conclusions contained herein. It is published only for the exchange and stimulation of ideas.

ABSTRACT

14462

In a typical self-contained space navigation system celestial observation data are gathered and processed to produce estimated velocity corrections. The results of this paper provide a basis for determining the best celestial measurements and the proper times to implement velocity corrections.

Fundamental to the navigation system is a procedure for processing celestial measurement data which permits incorporation of each individual measurement as it is made to provide an improved estimate of position and velocity. In order to "optimize" the navigation, a statistical evaluation of a number of alternative courses of action is made. The various alternatives, which form the basis of a decision process, concern the following:

1. Which star and planet combination provide the "best" available observation ?
2. Does the best observation give a sufficient reduction in the predicted target error to warrant making the measurement?
3. Is the uncertainty in the indicated velocity correction a small enough percentage of the correction itself to justify an engine restart and propellant expenditure?

Numerical results are presented which illustrate the effectiveness of

1. INTRODUCTION

During the past two years, the problems of guiding a space vehicle during the midcourse phase of its mission have been extensively explored at the MIT Instrumentation Laboratory. Following the specific demonstration of the technical feasibility of an unmanned photographic reconnaissance flight to the planet Mars reported by Laning, Frey, and Trageser [1], the detailed navigational aspects of such a venture were developed [2] by Dr. J. H. Laning, Jr., and the present author. Later, a variable time of arrival navigation theory was devised [3] and contrasted with the earlier fixed time of arrival scheme. More recently, the question of optimum utilization of navigation data has been given considerable study. It is the solution of this problem which forms the subject of the present paper.

The general method of navigation is based on perturbation theory so that only deviations in position and velocity from a reference path are utilized. Data is gathered by an optical angle measuring device and processed by a spacecraft digital computer. Periodically, small changes in the spacecraft velocity are implemented by a propulsion system as directed by the computer.

Basically, three problems are considered in this paper: (1) to identify the best sources of data available to the space vehicle navigator; (2) to define the optimum linear operations for processing the data in a manner consistent with the mission objectives; and (3) to minimize both the amount of navigational data and the number of corrective maneuvers required without unduly compromising mission accuracy.

The formulation of an optimum linear estimator as a recursion operation in which the current best estimate is combined with newly acquired information to produce a still better estimate was presented by Kalman [4]. The author is indebted to Dr. Stanley F. Schmidt for directing his attention to Kalman's excellent work. In fact, the original application of Kalman's theory to space navigation was made by Schmidt [5] and his associates.

BATTIN-1

The research described in the following sections of this paper was performed without any detailed knowledge of Schmidt's activities. As a result of this independent approach, several new and interesting ideas have developed: (1) an extremely simple derivation of the optimum linear operator has been achieved using only the basic technique of least squares estimation; (2) the mathematical problem of determining the optimum plane in which to make a star-planet angular measurement has been solved; (3) a procedure for incorporating cross-correlation effects of random measurement errors in determining the optimum linear operation has been developed. The author is indebted to Mr. Gerald L. Smith for correcting a basic mistake in the original treatment of cross-correlation errors.

Throughout the paper, we shall deal exclusively with discrete information; observations or velocity corrections are made at specific points in time which are termed "decision points." The interval between decision points is not necessarily uniform and may be selected somewhat arbitrarily; e.g., the interval length required for accurate numerical integration of the trajectory equations was used in preparing the computational data presented in Section 6.

Finally, a few remarks relevant to notational conventions are appropriate. We shall deal generally with both three- and six-dimensional vectors. A column vector of any dimension is represented by a lower case underscored letter. Matrices are denoted by capital letters and can be either square or rectangular arrays. The transpose of a vector or a matrix will be denoted by a superscript T. Thus, the scalar product of two vectors $\underline{a}$ and $\underline{b}$ will be written as $\underline{a}^T\underline{b}$. In like manner a quadratic form associated with a square matrix A will be written as $\underline{x}^T A \underline{x}$. The expected value of a random vector $\underline{x}$ will be indicated by an overscore; thus, $\overline{\underline{x}}$ denotes the average value of $\underline{x}$.

The author wishes to acknowledge the extensive services of Peter Phillion who prepared the numerical data reported in Section 6.

2. OUTLINE OF THE NAVIGATION AND GUIDANCE PROCEDURE

2.1 A Deterministic Method

The basic process involved in determining spacecraft position by means of a celestial fix consists fundamentally of a sequence of measurements of the angles between selected pairs of celestial objects. Three independent and precise angular measurements made at a known instant of time suffice to determine uniquely the position of the vehicle. Practical constraints, however, preclude simultaneous measurements without severely complicating the instrumentation. On the other hand, if the vehicle dynamics are governed by known laws and if deviations from a pre-determined reference trajectory are kept sufficiently small to permit a linearization of the navigation problem, then the question of simultaneous measurements loses it significance.

Under the assumptions of a linearized theory, a single observation serves to fix the position of the spacecraft in one coordinate. For example, if A_n is the angle measured at time t_n and is defined by the lines-of-sight from the vehicle to a star and to a nearby celestial body, the position of the vehicle is established along a line normal to the direction toward the near body and in the plane of the measurement. It is shown in Appendix A that the deviation in position $\delta\underline{r}_n$ of the spacecraft from the reference position is related to the deviation in angular measurement δA_n by

$$\delta A_n = \underline{h}_n^T \, \delta\underline{r}_n \qquad (2.1)$$

if the observation is made at a known instant of time t_n. The vector $\underline{h}_n$ depends upon the geometrical configuration of the relevant celestial objects at time t_n as well as the type of measurement made.

Because of the inherent dynamic coupling of position and velocity, the result at a later time t_{n+1} of a measurement made at time t_n does not lend itself to simple geometric interpretation. In order to provide a geometrical description, it is convenient to introduce the concept of a six dimensional space in which the coordinates represent the components of both position and velocity deviations of the vehicle from the reference path as functions of time.

BATTIN-2

Points in this space are defined by the six dimensional deviation vector

$$\delta \underline{x}_n = \begin{Vmatrix} \delta \underline{r}_n \\ \delta \underline{v}_n \end{Vmatrix} \tag{2.2}$$

where $\delta \underline{v}_n$ is the deviation in the vector velocity of the vehicle from the reference value. The vector $\delta \underline{x}_n$ defines the "state" of the vehicle dynamics at time t_n. Transition from one state to another is provided by the matrix operation

$$\Phi_{n+1,n} = \Phi(t_{n+1}, t_n)$$

which is frequently referred to as the "transition matrix". Indeed, the relationship between $\delta \underline{x}_{n+1}$ and $\delta \underline{x}_n$ is simply

$$\delta \underline{x}_{n+1} = \Phi_{n+1,n}\, \delta \underline{x}_n \tag{2.3}$$

as shown in Section 3.4.

By means of the rectangular matrix K defined by

$$K = \begin{Vmatrix} I \\ O \end{Vmatrix} \tag{2.4}$$

Eq. (2.1) may be written in terms of $\delta \underline{x}_n$ as

$$\delta A_n = \underline{h}_n^T K^T \delta \underline{x}_n \tag{2.5}$$

The submatrices I and O are, respectively, the three dimensional identity and zero matrices. Now, by combining Eqs. (2.3) and (2.5)

$$\delta A_n = \underline{h}_n^T K^T \Phi_{n+1,n}^{-1}\, \delta \underline{x}_{n+1} \tag{2.6}$$

it is clear that the effect at time t_{n+1} of an observation at time t_n is to determine the component of the six dimensional deviation vector in the direction defined by the vector $\Phi^{T-1}_{n+1,n} K\underline{h}_n$. Six observations made at different times would provide a set of six equations of the form of Eq. (2.6). If no two of the component directions were parallel, then the deviation vector could be obtained by inverting the six dimensional coefficient matrix.

2.2 Statistical Parameters of the Navigation Problem

Because of the presence of instrument inaccuracies additional observations may be used to reduce the errors associated with the simple deterministic process just described. By applying least square techniques to the observed data, a more accurate estimate of position and velocity is frequently possible than could be obtained from the minimum number of measurements. For this purpose, it is necessary to know certain statistical information with respect to the instrument inaccuracies. In a linear least squares estimation procedure all statistical calculations are based on first and second order averages and no additional statistical data is needed.

At this point of the discussion it is necessary to distinguish measured values, estimated values and true values of various quantities; e.g., $\delta \tilde{A}_n$ will be the measured value of the deviation in the angle A_n from its reference value at time t_n, δA_n the true value of the deviation, and $\delta \hat{A}_n$ the estimated value. If we write

$$\delta \tilde{A}_n = \delta A_n + \alpha_n \tag{2.7}$$

then α_n will be the error in the measurement. In the subsequent analysis α_n will be regarded as a random variable with an average value $\bar{\alpha}_n$ and a variance

$$\sigma_n^2 = \overline{\alpha_n^2} - \bar{\alpha}_n^2 \tag{2.8}$$

The possibility of cross-correlation of measurement errors will not be excluded; i.e., in general, the average $\overline{\alpha_n \alpha_m}$ may be different from $\bar{\alpha}_n \bar{\alpha}_m$.

In Section 4 an estimation procedure is developed for determining an optimal linear estimate of $\delta \underline{x}_n$, denoted by $\delta \hat{\underline{x}}_n$. As each measurement is made, the estimate $\delta \hat{\underline{x}}_n$ is updated by a simple recursive formula and, thereby, the problem associated with inverting sixth order matrices is avoided. An integral part of the estimation technique is the correlation matrix of the errors in the estimate. If we write

$$\delta \hat{\underline{x}}_n = \delta \underline{x}_n + \underline{e}_n \tag{2.9}$$

then

$$\underline{e}_n = \begin{Vmatrix} \underline{\epsilon}_n \\ \underline{\delta}_n \end{Vmatrix} \tag{2.10}$$

is the six dimensional error vector and may be partitioned as shown using $\underline{\epsilon}_n$ and $\underline{\delta}_n$ to denote, respectively, the position and velocity errors. The correlation matrix is thus defined by

$$E_n = \overline{\underline{e}_n \underline{e}_n^T} = \begin{Vmatrix} \overline{\underline{\epsilon}_n \underline{\epsilon}_n^T} & \overline{\underline{\epsilon}_n \underline{\delta}_n^T} \\ \overline{\underline{\delta}_n \underline{\epsilon}_n^T} & \overline{\underline{\delta}_n \underline{\delta}_n^T} \end{Vmatrix} = \begin{Vmatrix} E_n^{(1)} & E_n^{(2)} \\ E_n^{(3)} & E_n^{(4)} \end{Vmatrix} \tag{2.11}$$

BATTIN-3

For later use in a statistical analysis of the guidance problem, the correlation matrix of the actual deviation vector will be needed. This matrix is defined by

$$X_n = \overline{\delta \underline{x}_n \, \delta \underline{x}_n^T} \tag{2.12}$$

and may be calculated recursively using

$$X_n = \Phi_{n,n-1} X_{n-1} \Phi_{n,n-1}^T \tag{2.13}$$

Initially, i.e., at injection

$$\delta \hat{\underline{x}}_o = \delta \underline{x}_o + \underline{e}_o = 0 \tag{2.14}$$

so that

$$X_o = E_o \tag{2.15}$$

provides an initial value for the X_n matrix.

It is important to distinguish between a new estimate $\delta \hat{\underline{x}}_n$, obtained by incorporating an observation at time t_n, and an estimate simply extrapolated from a previous estimate. For the latter case, the notation $\delta \hat{\underline{x}}'_n$ is used where

$$\delta \hat{\underline{x}}'_n = \Phi_{n,n-1} \delta \hat{\underline{x}}_{n-1} \tag{2.16}$$

In like manner, we define an extrapolated error vector $\underline{e}'_n$. The extrapolated correlation matrix is readily shown to be

$$E'_n = \Phi_{n,n-1} E_{n-1} \Phi_{n,n-1}^T \tag{2.17}$$

Note that an estimate of the deviation in the angle to be measured at time t_n may be obtained from the extrapolated estimate of $\delta \hat{\underline{x}}_{n-1}$. We have

$$\delta \hat{A}'_n = \underline{h}_n^T K^T \delta \hat{\underline{x}}'_n \tag{2.18}$$

and it is this quantity, compared with the measured deviation $\delta \tilde{A}_n$, which is used in arriving at a revised estimate of $\delta \underline{x}_n$.

When cross-correlation of measurement errors is considered, it is convenient to use an augmented deviation vector having seven dimensions and defined as

$$\delta \underline{x}_n = \begin{Vmatrix} \delta \underline{r}_n \\ \delta \underline{v}_n \\ \alpha_n \end{Vmatrix} \tag{2.19}$$

Since, in this case, the error in a measurement at time t_n may be predicted on the basis of previous observations, we may define

$$\hat{\alpha}_n = \alpha_n + \beta_n \tag{2.20}$$

as the best estimate of the error to be expected in the measurement of A_n. The term β_n is then the error in the estimation of the measurement error. The error vector $\underline{e}_n$ will, of course, be seven dimensional and expressible as

$$\underline{e}_n = \begin{Vmatrix} \underline{\epsilon}_n \\ \underline{\delta}_n \\ \beta_n \end{Vmatrix} \tag{2.21}$$

Correspondingly, the correlation matrix becomes

$$E_n = \begin{Vmatrix} \overline{\underline{\epsilon}_n \underline{\epsilon}_n^T} & \overline{\underline{\epsilon}_n \underline{\delta}_n^T} & \overline{\underline{\epsilon}_n \beta_n} \\ \overline{\underline{\delta}_n \underline{\epsilon}_n^T} & \overline{\underline{\delta}_n \underline{\delta}_n^T} & \overline{\underline{\delta}_n \beta_n} \\ \overline{\beta_n \underline{\epsilon}_n^T} & \overline{\beta_n \underline{\delta}_n^T} & \overline{\beta_n^2} \end{Vmatrix} \tag{2.22}$$

It will be convenient in our later work to define the correlation vector $\underline{\phi}_n$ as the last column of the matrix E_n.

For purposes of illustration consider the following model for correlated measurement errors. Let the error at time t_{n+1}, be composed of two parts.

$$\alpha_{n+1} = \alpha'_{n+1} + \zeta_{n+1}$$

$$\alpha'_{n+1} = \alpha_n \exp[-\lambda(t_{n+1} - t_n)] \tag{2.23}$$

BATTIN-4

where α_n and ζ_{n+1} are independent random numbers, λ is a positive constant, and $\bar{\zeta}_{n+1}$ is zero. It follows that

$$\hat{\alpha}'_{n+1} = \hat{\alpha}_n \exp\left[-\lambda(t_{n+1} - t_n)\right] \qquad (2.24)$$

and

$$\beta'_{n+1} = \beta_n \exp\left[-\lambda(t_{n+1} - t_n)\right] \qquad (2.25)$$

Hence, the extrapolated error vector $\underline{e}'_{n+1}$ is calculated from

$$\underline{e}'_{n+1} = P_{n+1,n}\, e_n \qquad (2.26)$$

where $P_{n+1,n}$ is the augmented transition matrix

$$P_{n+1,n} = \begin{Vmatrix} \Phi_{n+1,n} & 0 \\ 0 & \exp\left[-\lambda(t_{n+1} - t_n)\right] \end{Vmatrix} \qquad (2.27)$$

The augmented extrapolated correlation matrix is then computed from

$$E'_{n+1} = P_{n+1,n}\, E_n\, P^T_{n+1,n} \qquad (2.28)$$

2.3 Summary of the Navigation and Guidance Equations

In the navigation and guidance theory presented here, the problem of launch guidance from Earth is ignored. It is assumed that the main propulsion stages are completed at time t_L and that the correlation matrix $E_o = E(t_L)$ is specified initially from a statistical knowledge of injection guidance errors. The initial estimate of position and velocity deviation $\delta\hat{\underline{x}}_o = \delta\hat{\underline{x}}(t_L)$ is zero, since, in the absence of any observation, the best unbiased estimate is that the spacecraft is on course.

The time interval from launch to arrival time t_A at the target point is considered to be subdivided into a number of smaller intervals by the sequence of times t_1, t_2, . . . called "decision points". At each decision point one of three possible courses of action is followed: (1) a single observation is made; (2) a velocity correction is implemented; or (3) no action is taken. A revised estimate of the deviation vector $\delta\underline{x}(t)$ is made at each such point -- the form of the revision depending, of course, on the nature of the decision. Specifically, as shown in Section 4, for uncorrelated measurement errors the revised estimate at the decision time t_n is one of the following:

$$\delta\hat{\underline{x}}_n = \begin{cases} \delta\hat{x}'_n + a_n^{-1} E'_n K \underline{h}_n (\delta\tilde{A}_n - \delta\hat{A}'_n) & \text{(measurement)} \\ (I + JB_n)\, \delta\hat{\underline{x}}'_n & \text{(correction)} \\ \delta\hat{\underline{x}}'_n & \text{(no action)} \end{cases} \qquad (2.29)$$

The scalar coefficient a_n is computed from

$$a_n = \underline{h}_n^T K^T E'_n K \underline{h}_n + \overline{\alpha_n^2} \qquad (2.30)$$

The rectangular matrix J has six rows and three columns

$$J = \begin{Vmatrix} 0 \\ I \end{Vmatrix} \qquad (2.31)$$

and is just the reverse of the K matrix. The matrix B_n is also rectangular having three rows and six columns and is partitioned as shown

$$B_n = \begin{Vmatrix} C_n^* & -I \end{Vmatrix} \qquad (2.32)$$

where C_n^* is one of the fundamental navigation matrices described in Section 3.2.

At each decision point it is also necessary to update the correlation matrix E_n. Thus

$$E_n = \begin{cases} E'_n - a_n^{-1} (E'_n K\underline{h}_n)(E'_n K\underline{h}_n)^T & \text{(measurement)} \\ E'_n + J\, \overline{\eta_n \eta_n^T}\, J^T & \text{(correction)} \\ E'_n & \text{(no action)} \end{cases} \qquad (2.33)$$

BATTIN-5

The vector $\underline{\eta}_n$ is the difference between the commanded velocity correction and the actual velocity change implemented at time t_n.

The above collection of formulae provides the means of maintaining an up to date estimate of the deviation vector $\delta\hat{\underline{x}}_n$ but, in themselves, do not provide any clue as to what decision should be made at each point. Suggestions for reasonable decision rules are discussed in Section 6.2 and in Appendix B.

When measurement errors are correlated, the only significant change arises in the method of processing a measurement to obtain a revised estimate in the augmented deviation vector and the associated correlation matrix. Thus

$$\delta\hat{\underline{x}}_n = \delta\hat{\underline{x}}'_n + a_n^{-1} (E'_n K \underline{h}_n + \underline{\phi}'_n) [\delta \tilde{A}_n - (\delta \hat{A}'_n + \hat{\alpha}'_n)] \tag{2.34}$$

$$E_n = E'_n - a_n^{-1} (E'_n K \underline{h}_n + \underline{\phi}'_n)(E'_n K \underline{h}_n + \underline{\phi}'_n)^T \tag{2.35}$$

where

$$a_n = \underline{h}_n^T K^T E'_n K \underline{h}_n + 2\underline{h}_n^T K^T \underline{\phi}'_n + (\overline{\beta'^2_n} + \overline{\zeta^2_n}) \tag{2.36}$$

The remaining equations are unaltered; however, certain obvious changes are required in the definition of the matrices J, K, and B_n in order that they be dimensionally compatible with the seven dimensional deviation vector.

3. FUNDAMENTAL NAVIGATION MATRICES

Basic to the solution of the navigation problem is a certain collection of matrices. The objective here is to introduce these matrices, indicate their role in the navigation theory, and show how they may be obtained as solutions of differential equations.

3.1 General Solution of the Linearized Trajectory Equations

Let $\underline{r}_s(t)$ and $\underline{v}_s(t)$ denote the position and velocity vectors of the spacecraft in an inertial coordinate system, and let $\underline{g}(\underline{r}_s, t)$ denote the gravitational acceleration at position $\underline{r}_s$ and time t. Then

$$\frac{d\underline{r}_s}{dt} = \underline{v}_s \quad , \quad \frac{d\underline{v}_s}{dt} = \underline{g}(\underline{r}_s, t) \tag{3.1}$$

are the basic equations of motion of the spaceship except for those brief periods during which propulsion is applied.

Let the vectors $\underline{r}_o(t)$ and $\underline{v}_o(t)$ represent the position and velocity at time t associated with the prescribed reference trajectory, and define

$$\delta\underline{r}(t) = \underline{r}_s(t) - \underline{r}_o(t) \qquad \delta\underline{v}(t) = \underline{v}_s(t) - \underline{v}_o(t) \tag{3.2}$$

Then, the deviations $\delta\underline{r}$ and $\delta\underline{v}$ may be approximately related by means of the linearized differential equations:

$$\frac{d(\delta\underline{r})}{dt} = \delta\underline{v} \qquad \frac{d(\delta\underline{v})}{dt} = G(\underline{r}_o, t)\ \delta\underline{r} \tag{3.3}$$

where $G(\underline{r}_o, t)$ is a matrix whose elements are the partial derivatives of the components of $\underline{g}(\underline{r}_o, t)$ with respect to the components of $\underline{r}_o$.

A particularly useful fundamental set of solutions of Eqs. (3.3) may be developed in the following way. Let t_L and t_A be, respectively, the time of launch and the time of arrival at the target. Then, define the matrices R(t), $R^*(t)$, V(t), $V^*(t)$ as the solutions of the matrix differential equations

$$\frac{dR}{dt} = V \qquad \frac{dR^*}{dt} = V^*$$
$$\frac{dV}{dt} = GR \qquad \frac{dV^*}{dt} = GR^* \tag{3.4}$$

BATTIN-6

which satisfy the initial conditions

$$R(t_L) = O \quad , \quad R^*(t_A) = O$$
$$V(t_L) = I \quad , \quad V^*(t_A) = I \qquad (3.5)$$

Here O and I denote, respectively, the zero and identity matrix. If we now write

$$\delta \underline{r}(t) = R(t)\,\underline{c} + R^*(t)\,\underline{c}^* \qquad (3.6)$$

$$\delta \underline{v}(t) = V(t)\,\underline{c} + V^*(t)\,\underline{c}^* \qquad (3.7)$$

where $\underline{c}$ and $\underline{c}^*$ are arbitrary constant vectors, it follows that these expressions satisfy the perturbation differential equations (3.3), and contain precisely the required number of unspecified constants to meet any valid set of initial or boundary conditions.

The elements of the R and V matrices represent deviations in position and velocity from the corresponding reference quantities as the result of certain specific deviations in the launch velocity from its reference value. For example, the first columns of these matrices are the vector deviations at time t due to a unit change in the first component of the velocity at time t_L. Corresponding interpretations may be ascribed to the other columns as well. A similar discussion will provide a physical meaning for the elements of R^* and V^*. For this purpose, however. it is convenient to imagine the roles of launch and target points as reversed.

3.2 The Vector Velocity Correction

Associated with the position $\underline{r}_s$ and the time t is the vector velocity required by the spacecraft to travel in free fall from $\underline{r}_s(t)$ to the target point $\underline{r}_o(t_A)$ in the time $t_A - t$. An expression for this velocity vector is readily obtained from Eqs. (3.6) and (3.7). The condition that the vehicle pass through the target point is met by the requirement

$$\delta \underline{r}(t_A) = 0 = R(t_A)\,\underline{c} + R^*(t_A)\,\underline{c}^*$$

Since $R^*(t_A) = O$, it follows that $\underline{c} = 0$. Eliminating $\underline{c}^*$ between Eqs. (3.6) and (3.7) gives for the required velocity deviation* at time t

$$\delta \underline{v}^+(t) = V^*(t)\,R^*(t)^{-1}\,\delta \underline{r}(t) \qquad (3.8)$$

*The superscripts- and + are used to distinguish the velocity just prior to correction from the velocity immediately following the correction.

Hence, the required velocity correction $\Delta \underline{v}^*$ is given by

$$\Delta \underline{v}^*(t) = C^*(t)\,\delta \underline{r}(t) - \delta \underline{v}^-(t) \qquad (3.9)$$

where the C^* matrix is defined by

$$C^*(t) = V^*(t)\,R^*(t)^{-1} \qquad (3.10)$$

The elements of the C^* matrix are deviations in vehicle velocity from the reference values, as required to place the vehicle on a trajectory to the target point, which arise from certain specific deviations in the vehicle position. The interpretation applied to the columns is made in the manner described earlier in connection with the R and V matrices.

If the spacecraft has been in a free-fall status since launch, then, by employing arguments similar to those used in establishing Eq. (3.8), it can be shown that

$$\delta \underline{v}^-(t) = C(t)\,\delta \underline{r}(t) \qquad (3.11)$$

where

$$C(t) = V(t)\,R(t)^{-1} \qquad (3.12)$$

In this case Eq. (3.9) takes the form

$$\Delta \underline{v}^*(t) = [C^*(t) - C(t)]\,\delta \underline{r}(t) \qquad (3.13)$$

Since $\delta \underline{r}(t)$ is different from zero solely as a result of an injection velocity error $\delta \underline{v}(t_L)$, it follows, from the definition of the R matrix, that

$$\Delta \underline{v}^*(t) = -\Lambda(t)\,\delta \underline{v}(t_L) \qquad (3.14)$$

Thus, the Λ matrix, defined by

$$\Lambda(t) = V(t) - C^*(t)\,R(t) \qquad (3.15)$$

relates a deviation in launch velocity to the velocity impulse required at time t. A starred form of the Λ matrix

$$\Lambda^*(t) = V^*(t) - C(t)\,R^*(t) \qquad (3.16)$$

will occur in the subsequent discussions.

3.3 Differential Equation Solutions

The matrices C, C^*, Λ, Λ^* may also be generated directly as solutions of differential equations. However, for C and C^*, a difficulty arises in prescribing appropriate initial conditions. From the initial values of the R and R^* matrices, it follows that $C(t_L)$ and $C^*(t_A)$ are both infinite. The singularities may be avoided by working directly with the differential equation for the inverse matrices C^{-1} and C^{*-1}.

BATTIN-7

By differentiating the identity

$$C(t)^{-1}\, V(t) = R(t) \tag{3.17}$$

and using Eq. (3.4), the following equation for C^{-1} results

$$\frac{dC^{-1}}{dt} + C^{-1}\, G\, C^{-1} = I \tag{3.18}$$

Similarly, we obtain

$$\frac{dC^{*-1}}{dt} + C^{*-1}\, G\, C^{*-1} = I \tag{3.19}$$

Equations (3.18) and (3.19) may be used to demonstrate an interesting property possessed by C and C^*. It is easy to show that the G matrix is symmetrical. It follows at once that the matrices C and C^* will be symmetrical for all values of t in the interval (t_L, t_A) if they are symmetrical for any particular time. But from Eq. (3.17) and a similar one involving starred matrices, we have

$$C(t_L)^{-1} = 0 \quad , \quad C^*(t_A)^{-1} = 0 \tag{3.20}$$

so that C and C^* are, indeed, symmetrical for t equal to t_L and t_A respectively. Hence C(t) and $C^*(t)$ are symmetrical for all t in the interval from launch to the target point.

In an entirely analogous manner, differential equations may be developed for Λ and Λ^*. By differentiating Eqs. (3.15) and (3.16) and using Eq. (3.4), one readily obtains the equations

$$\frac{d\Lambda}{dt} + C^*\Lambda = 0 \tag{3.21}$$

and

$$\frac{d\Lambda^*}{dt} + C\,\Lambda^* = 0 \tag{3.22}$$

with the initial conditions

$$\Lambda(t_L) = I \quad , \quad \Lambda^*(t_A) = I \tag{3.23}$$

3.4 The State Transition Matrix

Let $\delta \underline{r}_n = \delta \underline{r}(t_n)$ and $\delta \underline{v}_n = \delta \underline{v}(t_n)$ be the deviations in position and velocity at time t_n, and let R_n, V_n, . . . be the corresponding values of the fundamental matrices. The $\underline{c}$ and $\underline{c}^*$ must be obtained as solutions of

$$\delta \underline{r}_n = R_n\, \underline{c} + R_n^*\, \underline{c}^* \tag{3.24}$$

$$\delta \underline{v}_n = V_n\, \underline{c} + V_n^*\, \underline{c}^* \tag{3.25}$$

Multiplying Eq. (3.24) by R_n^{-1}, we obtain for $\underline{c}$

$$\underline{c} = R_n^{-1}\,(\delta \underline{r}_n - R_n^*\, \underline{c}^*) \tag{3.26}$$

Then, by substituting this expression into Eq. (3.25) and using Eqs. (3.12) and (3.16), there results

$$\underline{c}^* = -\Lambda_n^{*-1}\,(C_n\, \delta \underline{r}_n - \delta \underline{v}_n) \tag{3.27}$$

Finally, from Eq. (3.26) we have

$$\underline{c} = -\Lambda_n^{-1}\,(C_n^*\, \delta \underline{r}_n - \delta \underline{v}_n) \tag{3.28}$$

after some simplification. Thus, with $\underline{c}$ and $\underline{c}^*$ determined, the position and velocity deviations at any other time t are given by Eqs. (3.6) and (3.7).

In terms of the six dimensional deviation vector defined by Eq. (2.2), the result may be written in the form

$$\delta \underline{x}(t) = \begin{Vmatrix} R(t) & R^*(t) \\ V(t) & V^*(t) \end{Vmatrix} \begin{Vmatrix} \underline{c} \\ \underline{c}^* \end{Vmatrix} \tag{3.29}$$

Consider now a specific value of $t = t_{n+1}$. Then substituting from Eqs. (3.27) and (3.28) into Eq. (3.29), a relationship between $\delta \underline{x}_{n+1}$ and $\delta \underline{x}_n$ is displayed

$$\delta \underline{x}_{n+1} = \Phi_{n+1,n}\, \delta \underline{x}_n \tag{3.30}$$

where $\Phi_{n+1,n}$, the six-dimensional state transition matrix, is computed from

$$\Phi_{n+1,n} = \begin{Vmatrix} R_{n+1} & R_{n+1}^* \\ V_{n+1} & V_{n+1}^* \end{Vmatrix} \begin{Vmatrix} (C_n^{*-1}\Lambda_n)^{-1} & 0 \\ 0 & (C_n^{-1}\Lambda_n^*)^{-1} \end{Vmatrix} \begin{Vmatrix} -I & C_n^{*-1} \\ -I & C_n^{-1} \end{Vmatrix} \tag{3.31}$$

It is not difficult to show than an alternate calculation of the transition matrix may be made directly as the solution of the sixth order matrix differential equation

$$\frac{d\,\Phi(t, t_n)}{dt} = F(t)\,\Phi(t, t_n) \tag{3.32}$$

subject to the initial condition $\Phi(t_n, t_n)$ equal to the six dimensional identity matrix. The matrix F(t) is

$$F(t) = \begin{Vmatrix} 0 & I \\ G(t) & 0 \end{Vmatrix} \tag{3.33}$$

Finally, it has been shown(5) that the inverse of the matrix $\Phi_{n+1,n}$ is directly obtained as

$$\Phi_{n+1,n}^{-1} = \Phi_{n,n+1} = \begin{Vmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{Vmatrix}^{-1} = \begin{Vmatrix} \Phi_4^T & -\Phi_2^T \\ -\Phi_3^T & \Phi_1^T \end{Vmatrix} \tag{3.34}$$

BATTIN-8

4. DERIVATION OF THE OPTIMUM LINEAR ESTIMATE

4.1 Uncorrelated Measurement Errors

As noted in the Introduction, the optimum linear estimate of the deviation vector may be expressed as a recursion formula. Therefore, assume $\delta \hat{\underline{x}}_{n-1}$ and E_{n-1} are known and that a single measurement of the type described in Appendix A is made at time t_n. The observed deviation in the measured quantity A_n is $\delta \tilde{A}_n$, and the best estimate for δA_n, as obtained from the extrapolated estimate of $\delta \hat{\underline{x}}_{n-1}$, is given by Eq. (2.18). Then a linear estimate for the deviation vector $\delta \underline{x}_n$ at time t_n is expressible as a linear combination of the extrapolated estimate of $\delta \underline{x}_{n-1}$ and the difference between the observed and estimated deviations in the measured quantity A_n. Thus, for uncorrelated measurement errors,

$$\delta \hat{\underline{x}}_n = \delta \hat{\underline{x}}_n' + \underline{w}_n (\delta \tilde{A}_n - \delta \hat{A}_n') \tag{4.1}$$

where the vector $\underline{w}_n$ is a weighting factor which will be chosen so as to minimize the mean-squared error in the estimate.

For this purpose use Eqs. (2.9), (2.7) and (2.5) to write

$$\begin{aligned} \underline{e}_n(\underline{w}_n) &= \delta \hat{\underline{x}}_n - \delta \underline{x}_n \\ &= \delta \hat{\underline{x}}_n' + \underline{w}_n (\delta A_n + \alpha_n - \delta \hat{A}_n') - \delta \underline{x}_n \\ &= (I - \underline{w}_n \underline{h}_n^T K^T)(\delta \hat{\underline{x}}_n' - \delta \underline{x}_n) + \underline{w}_n \alpha_n \\ &= (I - \underline{w}_n \underline{h}_n^T K^T)\, \underline{e}_n' + \underline{w}_n \alpha_n \end{aligned} \tag{4.2}$$

where I is the six-dimensional identity matrix. Then the correlation matrix E_n defined by Eq. (2.11) may be expressed as a function of the weighting vector $\underline{w}_n$ as

$$E_n(\underline{w}_n) = (I - \underline{w}_n \underline{h}_n^T K^T)\, E_n' \,(I - K \underline{h}_n \underline{w}_n^T) + \underline{w}_n \underline{w}_n^T \overline{\alpha_n^2} \tag{4.3}$$

The mean-squared errors in the estimate of position and velocity deviations $\overline{\epsilon_n^2}$ and $\overline{\delta_n^2}$ are simply the respective traces of the submatrices $E_n^{(1)}$ and $E_n^{(4)}$. If the six-dimensional weighting vector $\underline{w}_n$ is partitioned into two three-dimensional vectors

$$\underline{w}_n = \begin{Vmatrix} \underline{w}_n^{(1)} \\ \underline{w}_n^{(2)} \end{Vmatrix} \tag{4.4}$$

then from Eq. (4.3) it is easy to show that $E_n^{(1)}$ is a function only of $\underline{w}_n^{(1)}$ and $E_n^{(4)}$ is a function only of $\underline{w}_n^{(2)}$. Therefore, for the purposes of the following discussion, it is legitimate formally to treat the mean-squared error in the estimate $\overline{e_n^2(\underline{w}_n)}$ as the trace of the six-dimensional correlation matrix $E_n(\underline{w}_n)$. The subvectors of the optimum weighting vector $\underline{w}_n$ will then each be optimum for the respective estimates of position and velocity deviations.

In order to determine the optimum weighting vector, one may apply the usual technique of the variational calculus. Let $\underline{w}_n$ take on a variation $\delta \underline{w}_n$ and obtain from Eq. (4.3)

$$\delta \overline{e_n^2(\underline{w}_n)} = 2\, \mathrm{tr} \left[- \delta \underline{w}_n \underline{h}_n^T K^T E_n' \,(I - K \underline{h}_n \underline{w}_n^T) + \delta \underline{w}_n \underline{w}_n^T \overline{\alpha_n^2} \right] \tag{4.5}$$

If $\delta \overline{e^2(\underline{w}_n)}$ is to vanish for all variations $\delta \underline{w}_n$, then it must follow that

$$a_n \underline{w}_n = E_n' K \underline{h}_n \tag{4.6}$$

where the positive scalar quantity a_n is defined by Eq. (2.30).

It can be readily shown that the $\underline{w}_n$ determined from Eq. (4.6) actually does minimize $e_n^2(\underline{w}_n)$. Suppose that the optimum $\underline{w}_n$ is replaced by another weighting factor $\underline{w}_n - \underline{y}_n$. Then from Eqs. (4.3) and (2.17)

$$\overline{e_n^2(\underline{w}_n - \underline{y}_n)} = \mathrm{tr} \left[E_n' - 2 (\underline{w}_n - \underline{y}_n) \underline{h}_n^T K^T E_n' + a_n (\underline{w}_n - \underline{y}_n)(\underline{w}_n^T - \underline{y}_n^T) \right] \tag{4.7}$$

and using Eq. (4.6)

$$\overline{e_n^2(\underline{w}_n - \underline{y}_n)} = \mathrm{tr} \left[E_n' - a_n (\underline{w}_n - \underline{y}_n)(\underline{w}_n^T + \underline{y}_n^T) \right] \tag{4.8}$$

so that

$$\overline{e_n^2(\underline{w}_n - \underline{y}_n)} = \overline{e_n^2(\underline{w}_n)} + a_n\, \mathrm{tr}\, (\underline{y}_n \underline{y}_n^T) \tag{4.9}$$

Thus, the mean-squared error is not decreased by perturbing $\underline{w}_n$ if Eq. (4.6) holds.

Having obtained the optimum weighting vector, the expression for the correlation matrix of the estimate errors E_n given by Eq. (4.3) may be written in a more convenient form. Thus, from the definition of a_n in Eq. (2.30),

BATTIN-9

there results

$$E_n = E_n' (I - K \underline{h}_n \underline{w}_n^T) - \underline{w}_n h_n^T K^T E_n' + a_n \underline{w}_n \underline{w}_n^T \qquad (4.10)$$

Substituting from Eq. (4.6), the final expression may be written as

$$E_n = E_n' - a_n^{-1} (E_n' K \underline{h}_n)(E_n' K \underline{h}_n)^T \qquad (4.11)$$

Equations (4.1) and (4.11) then serve as recursive relations to be used in obtaining improved estimates of position and velocity deviations at each of the measurement times t_1, t_2, . . .

4.2 Correlated Measurement Errors

If the measurement errors are correlated, the derivation is only slightly altered. The linear estimate for the seven dimensional deviation vector $\delta\underline{x}_n$ at time t_n is again expressible as a linear combination of the extrapolated estimate of $\delta\underline{x}_{n-1}$ and the difference between the observed and estimated deviations in the measured quantity A_n. However, the estimated deviation in A_n must also include the estimate of the error in the observation. Thus

$$\delta\hat{\underline{x}}_n = \delta\hat{\underline{x}}_n' + \underline{w}_n [\delta\tilde{A}_n - (\delta\hat{A}_n' + \hat{\alpha}_n')] \qquad (4.12)$$

where now the weighting vector $\underline{w}_n$ is seven dimensional.

The error in the estimate may be written as

$$\begin{aligned} \underline{e}_n &= \delta\hat{\underline{x}}_n - \delta\underline{x}_n \\ &= \delta\hat{\underline{x}}_n' + \underline{w}_n (\delta A_n - \beta_n' + \zeta_n - \delta\hat{A}_n') - \delta\underline{x}_n \\ &= (I - \underline{w}_n \underline{h}_n^T K^T)(\delta\hat{\underline{x}}_n' - \delta\underline{x}_n) - \underline{w}_n (\beta_n' - \zeta_n) \qquad (4.13) \\ &= (I - \underline{w}_n \underline{h}_n^T K^T)\underline{e}_n' - \underline{w}_n (\beta_n' - \zeta_n) \end{aligned}$$

The correlation matrix, expressed as a function of the weighting vector $\underline{w}_n$, is then

$$\begin{aligned} E_n(\underline{w}_n) &= (I - w_n \underline{h}_n^T K^T) E_n' (I - K \underline{h}_n \underline{w}_n^T) \\ &\quad - (I - \underline{w}_n \underline{h}_n^T K^T) \underline{\phi}_n' \underline{w}_n^T \\ &\quad - \underline{\phi}_n'^T (I - K \underline{h}_n \underline{w}_n^T) + \underline{w}_n \underline{w}_n^T (\overline{\beta_n'^2} + \overline{\zeta_n^2}) \qquad (4.14) \end{aligned}$$

Again if we require $\overline{\delta e^2(\underline{w}_n)}$ to vanish for all variations $\delta \underline{w}_n$, it is readily shown that

$$a_n \underline{w}_n = E_n' K \underline{h}_n + \underline{\phi}_n' \qquad (4.15)$$

where a_n is defined by Eq. (2.36).

4.3 Correlation Between the Estimate and the Error

An important property of the optimum estimate, which is needed for the development of the statistical analysis procedures described in Section 5, will be derived here. The result may be stated simply as

$$\overline{\underline{e}_n \, \delta\hat{\underline{x}}_n^T} = 0 \qquad (4.16)$$

if $\delta\hat{\underline{x}}_n$ is the optimum estimate; i.e., the optimum estimate and the associated error in the estimate are uncorrelated. In the proof we consider, for simplicity, only the case of uncorrelated measurement errors, but the property is readily established in general.

From Eq. (4.5) we have

$$\underline{w}_n \overline{\alpha_n^2} - (I - \underline{w}_n \underline{h}_n^T K^T) E_n' K \underline{h}_n = 0 \qquad (4.17)$$

or alternately,

$$\underline{w}_n \overline{\alpha_n^2} - [(I - \underline{w}_n \underline{h}_n^T K^T)\overline{\underline{e}_n']\underline{e}_n'^T} K \underline{h}_n = 0 \qquad (4.18)$$

Substituting, for the bracketed quantity from Eq. (4.2) gives

$$\underline{w}_n \overline{\alpha_n^2} + \overline{(\underline{w}_n \alpha_n - \underline{e}_n) \underline{e}_n'^T} K \underline{h}_n = 0 \qquad (4.19)$$

But since $\overline{\alpha_n \underline{e}_n'^T} = 0$, we have

$$\overline{(\underline{w}_n \alpha_n) \alpha_n} - \overline{\underline{e}_n \underline{e}_n'^T} K \underline{h}_n = 0 \qquad (4.20)$$

Again substituting for $\underline{w}_n \alpha_n$ from Eq. (4.2) gives

$$\overline{[\underline{e}_n - (I - \underline{w}_n \underline{h}_n^T K^T) \underline{e}_n'] \alpha_n} - \overline{\underline{e}_n \underline{e}_n'^T} K \underline{h}_n = 0 \qquad (4.21)$$

or, simply,

$$\overline{\underline{e}_n (\alpha_n - \underline{e}_n'^T K \underline{h}_n)} = 0 \qquad (4.22)$$

Thus, $\underline{e}_n$ and the scalar quantity $\alpha_n - \underline{e}_n'^T K \underline{h}_n$ are uncorrelated. Hence,

$$\overline{\underline{e}_n [\underline{w}_n^T (\alpha_n - \underline{e}_n'^T K \underline{h}_n)]} = 0 \qquad (4.23)$$

or, from Eq. (4.2)

$$\overline{\underline{e}_n (\underline{e}_n^T - \underline{e}_n'^T)} = 0 \qquad (4.24)$$

Therefore,

$$\overline{\underline{e}_n [\delta \underline{x}_n^T + \underline{e}_n^T - (\delta \underline{x}_n^T + \underline{e}_n'^T)]} = 0 \qquad (4.25)$$

or

$$\overline{\underline{e}_n \delta \hat{\underline{x}}_n^T} = \overline{\underline{e}_n \delta \hat{\underline{x}}_n'^T} \qquad (4.26)$$

From this final relationship it is easy to show that $\underline{e}_n$ and $\delta \hat{\underline{x}}_n$ are uncorrelated. For if we substitute from Eqs. (4.2) and (2.16), it follows that

$$\begin{aligned}\overline{\underline{e}_n \delta \hat{\underline{x}}_n^T} &= [(I - \underline{w}_n \underline{h}_n^T K^T) \Phi_{n,n-1} \overline{\underline{e}_{n-1} + \underline{w}_n \alpha_n]\, \delta \hat{\underline{x}}_{n-1}^T} \Phi_{n,n-1}^T \\ &= (I - \underline{w}_n \underline{h}_n^T K^T) \Phi_{n,n-1} \overline{\underline{e}_{n-1} \delta \hat{\underline{x}}_{n-1}^T} \Phi_{n,n-1}^T \qquad (4.27)\end{aligned}$$

Then by continuing the reduction of $\overline{\underline{e}_{n-1} \delta \hat{\underline{x}}_{n-1}^T}$ we have, finally, $\overline{\underline{e}_n \delta \hat{\underline{x}}_n^T}$ related to $\overline{\underline{e}_o \delta \hat{\underline{x}}_o}$ which is zero. Thus, Eq. (4.16) is established and the proof is complete.

5. STATISTICAL ANALYSIS OF THE GUIDANCE PROCEDURE

From exact knowledge of the six-dimensional deviation vector $\delta \underline{x}_n$ at time t_n, a velocity correction may be calculated which, if implemented, will insure the vehicle's arrival at a fixed point in space at the required time. However, only the estimate $\delta \hat{\underline{x}}_n$ is available. From this, an estimate of the velocity correction vector $\Delta \hat{\underline{v}}_n$ may be determined from

$$\Delta \hat{\underline{v}}_n = B_n \delta \hat{\underline{x}}_n \qquad (5.1)$$

where B_n is defined by Eq. (2.32). (Refer to the discussion leading to Eq. (3.9).)

The need for a velocity correction arises solely from improper injection into orbit. If the first such correction is executed perfectly, then, of course, no further corrections are required. However, because of imperfect knowledge of position and velocity obtained from navigational measurements, the commanded velocity change will be in error. Furthermore, the actual velocity change experienced will differ from that commanded because of imperfect instrumentation. Therefore, subsequent corrections will be required to remove the effects produced by earlier inaccuracies.

5.1 Correlation Matrix of the Velocity Correction Vector

An estimate of the required velocity correction vector $\Delta \hat{\underline{v}}_n$, as computed from Eq. (5.1), may be determined at each decision time whether or not the correction is actually implemented. The correlation matrix of the velocity correction vector may be expressed directly in terms of the extrapolated matrices E_n' and X_n'.

From Eq. (5.1) we have

$$\Delta \hat{\underline{v}}_n = B_n (\delta \underline{x}_n' + \underline{e}_n') \qquad (5.2)$$

so that

$$\overline{\Delta \hat{\underline{v}}_n \Delta \hat{\underline{v}}_n^T} = B_n (\overline{\delta \underline{x}_n' \delta \underline{x}_n'^T} + \overline{\underline{e}_n' \delta \underline{x}_n'^T} + \overline{\delta \underline{x}_n' \underline{e}_n'^T} + E_n') B_n^T \qquad (5.3)$$

BATTIN-11

On the other hand

$$\delta \hat{\underline{x}}_n'^T = \delta \underline{x}_n'^T + \underline{e}_n'^T \tag{5.4}$$

from which

$$\overline{\underline{e}_n' \, \delta \hat{\underline{x}}_n'^T} = \overline{\underline{e}_n' \, \delta \underline{x}_n'^T} + E_n' = 0 \tag{5.5}$$

according to the theorem proved in Section 4.3.

Hence

$$\overline{\Delta \hat{\underline{v}}_n \, \Delta \hat{\underline{v}}_n^T} = B_n (X_n' - E_n') B_n^T \tag{5.6}$$

The correlation matrix X_n may be calculated using Eq. (2.13) when no velocity correction is made. If the velocity is corrected at time t_n, the following procedure is valid.

Using Eq. (2.29) we may write

$$\begin{aligned} \delta \underline{x}_n &= \delta \underline{x}_n' + J B_n \, \delta \hat{\underline{x}}_n' - J \underline{\eta}_n \\ &= (I + J B_n) \, \delta \underline{x}_n' + J B_n \, \underline{e}_n' - J \underline{\eta}_n \end{aligned} \tag{5.7}$$

Hence,

$$\begin{aligned} \overline{\delta \underline{x}_n \, \delta \underline{x}_n^T} &= (I + J B_n) \, \overline{\delta \underline{x}_n' \, \delta \underline{x}_n'^T} \, (I + J B_n)^T \\ &+ J B_n E_n' (J B_n)^T + J \overline{\underline{\eta}_n \, \underline{\eta}_n^T} J^T \\ &+ (I + J B_n) \, \overline{\delta \underline{x}_n' \, \underline{e}_n'^T} \, (J B_n)^T \\ &+ J B_n \, \overline{\underline{e}_n' \, \delta \underline{x}_n'^T} \, (I + J B_n)^T \end{aligned} \tag{5.8}$$

which may be further reduced using Eq. (5.5). In summary, then

$$X_n = \begin{cases} X_n' & \text{(no correction)} \\ (I + J B_n)(X_n' - E_n')(I + J B_n)^T + E_n' + J \overline{\underline{\eta}_n \, \underline{\eta}_n^T} J^T & \text{(correction)} \end{cases} \tag{5.9}$$

Just as the extrapolated error vector and the associated correlation matrix are altered at an observation point, so also will they change at a correction point. Thus,

$$\underline{e}_n = \underline{e}_n' + \begin{Vmatrix} 0 \\ \underline{\eta}_n \end{Vmatrix} \tag{5.10}$$

and

$$E_n = E_n' + \begin{Vmatrix} 0 & 0 \\ 0 & \overline{\underline{\eta}_n \, \underline{\eta}_n^T} \end{Vmatrix} \tag{5.11}$$

The mean-squared estimate of the velocity correction is determined as the trace of the matrix $\Delta \hat{\underline{v}}_n \, \Delta \hat{\underline{v}}_n^T$. As a basis for a decision theory, it is important to know something of the precision of the estimate. Clearly, a velocity correction having a large uncertainty should not be commanded if it is possible to improve substantially the estimate by future observations. The uncertainty $\underline{d}_n$ in the estimate $\Delta \hat{\underline{v}}_n$ is simply

$$\underline{d}_n = \Delta \hat{\underline{v}}_n - B_n \, \delta \underline{x}_n = B_n \, \underline{e}_n' \tag{5.12}$$

Hence, the mean-squared uncertainty is determined as the trace of the matrix

$$\overline{\underline{d}_n \, \underline{d}_n^T} = B_n E_n' B_n^T \tag{5.13}$$

5.2 Uncertainty in the Applied Velocity Correction

In order to complete the statistical analysis of the velocity correction, it is necessary to examine more carefully the vector uncertainty $\underline{\eta}$ in the velocity correction. The inaccuracy in establishing a commanded velocity correction $\Delta \hat{\underline{v}}$ is due to errors in both magnitude and orientation. In the following analysis the two sources of error will be assumed independently random with zero means.

Consider a coordinate system in which the estimated velocity correction vector is along one of the coordinate axes. Then if M is the transformation matrix which relates the selected axis system and the original reference system, we may write

$$\Delta \hat{\underline{v}} = \Delta \hat{v} \, M \begin{Vmatrix} 0 \\ 0 \\ 1 \end{Vmatrix} \tag{5.14}$$

Now, define a random variable κ such that

$$\Delta v = (1 + \kappa) \Delta \hat{v} \tag{5.15}$$

BATTIN-12

and let γ be the random angle between $\Delta\hat{\underline{v}}$ and $\Delta\underline{v}$. It will be assumed that both κ and γ are small quantities so that powers and products are negligible compared with unity. The actual vector velocity correction is then

$$\Delta\underline{v} = (1+\kappa)\,\Delta\hat{v}\,M \begin{Vmatrix} \gamma\cos\beta \\ \gamma\sin\beta \\ 1 \end{Vmatrix} \tag{5.16}$$

where β is a polar angle defining the rotation of $\Delta\underline{v}$ with respect to $\Delta\hat{\underline{v}}$. Hence, the uncertainty vector $\underline{\eta}$ is expressible as

$$\underline{\eta} = \Delta\hat{\underline{v}} - \Delta\underline{v} = -\Delta\hat{v}\,M\left\{(1+\kappa)\gamma \begin{Vmatrix} \cos\beta \\ \sin\beta \\ 0 \end{Vmatrix} + \kappa \begin{Vmatrix} 0 \\ 0 \\ 1 \end{Vmatrix}\right\} \tag{5.17}$$

Assume that κ, γ, β are statistically independent random variables with zero means. Further assume that β is uniformly distributed over the interval $-\pi$ to π. Then one obtains for the correlation matrix of the velocity correction uncertainty

$$\overline{\underline{\eta}\underline{\eta}^T} = \overline{\kappa^2}\;\overline{\Delta\hat{\underline{v}}\,\Delta\hat{\underline{v}}^T} + \frac{\overline{\gamma^2}}{2}\,\overline{\Delta\hat{v}^2}\,M \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix} M^T \tag{5.18}$$

$$= \overline{\kappa^2}\;\overline{\Delta\hat{\underline{v}}\,\Delta\hat{\underline{v}}^T} + \frac{\overline{\gamma^2}}{2}\left(\overline{\Delta\hat{\underline{v}}^T\,\Delta\hat{\underline{v}}}\,I - \overline{\Delta\hat{\underline{v}}\,\Delta\hat{\underline{v}}^T}\right)$$

where I is the three-dimensional identity matrix and $\overline{\kappa^2}$ and $\overline{\gamma^2}$ are the mean-squared valued of κ and γ.

5.3 Miss Distance at the Target

Turning now to the problem of guidance accuracy, the determination of the position deviation vector at the nominal time of arrival at the target is made by extrapolating the deviation vector from the point of the final velocity correction. Thus, if t_N is the time of the last correction and $\delta\underline{x}_A$ is the deviation vector at the time of arrival t_A, then

$$\delta\underline{x}_A = \Phi_{A,N}\,\delta\underline{x}_N^+ \tag{5.19}$$

But from Eq. (3.31) and the terminal conditions for the navigation matrices, we have

$$\Phi_{A,N} = \begin{Vmatrix} -R_A\Lambda_N^{-1} & 0 \\ -V_A\Lambda_N^{-1} & -\Lambda_N^{*-1} \end{Vmatrix} \begin{Vmatrix} C_N^* & -I \\ C_N & -I \end{Vmatrix} \tag{5.20}$$

Hence, the position deviation vector at the target $\delta\underline{r}_A$ may be written as

$$\delta\underline{r}_A = -R_A\,\Lambda_N^{-1}\,B_N\,\delta\underline{x}_N^+ \tag{5.21}$$

with a similar expression obtainable for the velocity deviation at time t_A.

The target position error may be written ultimately in terms of the error vector $\underline{e}_N$ according to the following self-evident steps

$$\begin{aligned} \delta\underline{r}_A &= -R_A\,\Lambda_N^{-1}\,B_N\,(\delta\underline{x}_N^- + J\,\Delta\underline{v}_N) \\ &= -R_A\,\Lambda_N^{-1}(B_N\,\delta\underline{x}_N^- - \Delta\underline{v}_N) \\ &= R_A\,\Lambda_N^{-1}(B_N\,\underline{e}_N' - \underline{\eta}_N) \\ &= R_A\,\Lambda_N^{-1}B_N\,\underline{e}_N \end{aligned} \tag{5.22}$$

The mean square position error at the target is then computed as the trace of the matrix $\overline{\delta\underline{r}_A\;\delta\underline{r}_A^T}$.

BATTIN - 13

6. APPLICATION TO TRANS-LUNAR NAVIGATION

6.1 Decision Rules

As a necessary step in the application of the navigation and guidance scheme formulated in this paper, certain rules must be adopted concerning the course of action to be taken at each of the "decision points" described in Section 2.3. The number and frequency of observations must be controlled in some manner -- ideally by a decision rule which is realistically compatible with both the mission objectives and the capabilities of the measuring device. If an observation is to be made, a decision is required regarding the type of measurement and the celestial objects to be used. Periodic velocity corrections must be applied and the number of impulses and times of occurrence must be decided.

Once the decision rules have been specified, it is necessary to test their effectiveness according to some measure of performance. A typical objective is to minimize the miss distance at the target. However, a reduction in miss distance usually implies an increase in either the required number of measurements or a greater expenditure of corrective propulsion or both. In the face of these conflicting objectives, compromises are clearly necessary and statistical simulation provides a means of arriving at an acceptable balance.

In the interest of minimizing the number of simulator runs, Monte Carlo techniques should be avoided if possible. Fortunately, it is unnecessary to generate the true spacecraft trajectory, as would be required for Monte Carlo simulation, in order to analyze the effects of a particular set of decision rules. The reader may readily verify that (Eq. (2.29), which defines the estimate $\delta \hat{\underline{x}}_n$ and depends on actual measurement data, is never involved in any of the statistical calculations.

A specific example of a set of decision rules to be applied at each decision point is as follows:

1. The estimated mean-squared velocity correction $\overline{\Delta \hat{v}_n^2}$ and the mean-squared uncertainty $\overline{d_n^2}$ associated with the estimate are computed from Eqs. (5.6) and (5.13). If the ratio

$$R_v = \sqrt{\overline{d_n^2} / \overline{\Delta \hat{v}_n^2}} \tag{6.1}$$

is less than a specified amount $R_{v(min)}$, a velocity correction is made at time t_n.

2. If the criteria is not met which would call for initiation of a velocity correction, the desirability of making an observation is examined. For this purpose, an abbreviated star catalog is postulated together with selected planets. Each star and planet measurement combination is analyzed to determine its effect on the reduction in position uncertainty at the target. The particular star-planet combination producing the greatest mean-square reduction is then defined as the best potential measurement.

Now let $\overline{\delta r_A^{2+}}$ and $\overline{\delta r_A^{2-}}$ be the respective mean-square position uncertainties at the target which would result with and without the best possible observation. Then, if the ratio

$$R_p = \sqrt{\frac{\overline{\delta r_A^{2-}} - \overline{\delta r_A^{2+}}}{\overline{\delta r_A^{2-}}}} \tag{6.2}$$

is greater than a specified value $R_{p(max)}$, the best potential measurement is made at time t_n. In other words, for a measurement to be made, a significant reduction in the potential miss distance must result. If, on the other hand, the above criterion is not met, no action is taken at the decision point t_n.

6.2 Numerical Example

In this section, the decision rules presented previously are applied to the circumlunar navigation problem. It was found that the velocity correction criterion worked quite well to establish the times of mid-course maneuvers with the exception of the final correction. The required velocity change increases quite rapidly as the target is approached and the timing of this last correction is critical. After preliminary experimentation with different values of R_v, it was decided to fix apriori the correction times for the remainder of the study of the navigation problem. Cross correlation between measurement errors was ignored and only the Earth and Moon together with the 20 brightest stars were considered for potential measurements.

The date and time of orbital injection was Julian Day 2440043.6088 with the closest point of approach some 60 miles from the lunar surface. The nominal total time of flight from injection was 126.4 hours.

The correlation matrix of injection errors E_0 was obtained from the following assumed root-mean-square injection errors,

Altitude	Track	Range
10,000 ft	15,000 ft	5000 ft
15 ft/sec	6 ft/sec	4 ft/sec

The correlation matrix below was obtained by a transformation from the altitude, track, range coordinate system to a coordinate system with the x axis along the vernal equinox, z axis along the Earth polar axis and the y axis chosen to make a right handed coordinate system. The basic units in the E_0 matrix are miles and miles per hour.

$$E_o = \begin{Vmatrix} 0.918 & 0.063 & 0.203 & 0 & 0 & 0 \\ 0.063 & 4.58 & -1.86 & 0 & 0 & 0 \\ 0.203 & -1.86 & 7.04 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.73 & 4.65 & 2.72 \\ 0 & 0 & 0 & 4.65 & 83.8 & 36.0 \\ 0 & 0 & 0 & 2.72 & 36.0 & 36.1 \end{Vmatrix}$$

At each decision point, forty potential measurements were examined and evaluated according to the decision criterion. The minimum time between observations was required to be 15 minutes. For simplicity, only star elevations above an illuminated horizon of either the Earth or Moon were considered. Certain practical constraints were imposed so that physically unrealizable measurements were screened out. For example, in order to keep the field of view requirements reasonable, the lines of sight to the star and to the horizon were required not to exceed seventy degrees. Also no measurement could be made if the line of sight to either star or planet edge were closer than fifteen degrees from the direction to the Sun. Furthermore, if the illuminated face of the Moon formed the background of the edge of the Earth from which a star elevation was to be reckoned, that particular measurement would not be made.

The optical measuring device used for the observations was assumed to be unbiased with a random error whose variance was

$$\sigma_E^2 = (0.00005)^2 + \left(\frac{1}{r_{SE}}\right)^2 \text{ radians}$$

for the Earth, and

$$\sigma_M^2 = (0.00005)^2 + \left(\frac{0.5}{r_{SM}}\right)^2 \text{ radians}$$

for the Moon where r_{SE} and r_{SM} are the distances in miles from the spacecraft to the Earth and Moon respectively. In this manner it was possible to account for the larger uncertainty in defining the horizon which would exist when the spacecraft is close to a planet. At large distances the rms error is approximately 0.05 milliradians.

The magnitude error in applying a velocity correction was assumed to be isotropic and proportional to the commanded correction. Specifically, the relation

$$\overline{\eta_n^2} = 0.0001 \, \overline{\Delta v_n^2}$$

was adopted so that the rms error would be one percent of the rms correction. The orientation error assumed was 0.01 radians.

Preliminary results of an analysis of this sample trajectory are summarized in the accompanying tables. A number of simulated guidance flights were made for which the strategy parameters R_v and R_p had various assigned values. Then, in order to evaluate the effect on the navigation data of a variation in the time of year, a set of pseudo-trajectories was generated by the simple device

BATTIN-15

of rotating the direction of the Sun as viewed from the Earth. The trajectory was considered to be unchanged by this process--the assumption being quite adequate for the purpose of this preliminary analysis. In this manner different illuminated portions of the Earth and Moon were visible to the spacecraft resulting, thereby, in different measurements.

In general, as R_p is increased, one requires each measurement to have a proportionately greater significance in the reduction of the potential target error, with the result that the required total number of measurements decreases. There may be a corresponding penalty, of course, in that the resulting uncertainties in position and velocity at the target can increase. The objective in preparing a measurement schedule is to arrive at an acceptable compromise.

The number of velocity corrections as well as the times of their occurrence is, of course, controlled by R_v. On the other hand, the number of measurements is not sensibly affected by variations in this parameter. As an example, in Table 1 navigation data for the Earth to Moon trajectory is given for two values of the velocity correction uncertainty ratio R_v. Although the final position uncertainties are of the order of two miles, the deviations from the reference path are approximately twelve miles. This large difference results from the fact that measurement data was gathered after the final velocity correction so that knowledge of the orbit improved although no attempt was made to reduce the target error. It should be noted that if one elects to eliminate the final position deviation by a velocity correction one tenth hour before the nominal arrival time, velocity corrections of 104 mph and 68 mph, respectively, are required. There will, of course, be an accompanying increase in the final velocity deviations of 51 mph and 52 mph, respectively.

In Table 2 the navigation data for the Earth to Moon trajectory is given as a function of the miss distance reduction ratio R_p for velocity corrections made at 5, 20, 52, and 61.5 hours. For the case $R_p = 0.6$, there is a noticeable decrease in the final position uncertainty compared to that for $R_p = 0.5$. This apparent anomaly arises from the fact that for the $R_p = 0.6$ case, three observations are made after the last velocity correction, while, correspondingly, only two observations are made for the $R_p = 0.5$ case. Table 3 presents similar data for the Moon to Earth trajectory.

In order to study the effect of variations in the illuminated portions of the planet's surfaces, one set of values for R_p and times for velocity corrections was selected and the Sun direction altered in sixty degree steps except for the 70° and 250° cases. These two directions were singled out because they form a line approximately perpendicular to the Earth-Moon line at launch. Table 4 gives the results for the Earth to Moon trajectory and further shows that the 70° and 180° cases produce significantly larger uncertainties. For the 120° case the total velocity correction of 114 mph is somewhat higher. However this can be improved since the times selected for velocity corrections were not optimum for all cases. Table 5 presents similar data for the Moon to Earth trajectory.

In all cases the final velocity correction just prior to arrival at perilune is significantly larger than the previous two mid-course corrections. The result is a rather large velocity deviation from the nominal value at the target point. On the return flight this deviation causes the first velocity correction to be substantial which accounts for the increase in fuel requirements required for the Moon to Earth trip. If the objective of the flight does not include passage through a preassigned perilune position, then, obviously, the total of velocity corrections can be reduced.

Table 6 summarizes Earth to Moon flight navigation data for various Moon horizon uncertainties. The number of measurements remained constant (76 and 77) for the cases investigated. Total velocity corrections, final velocity deviations and final position deviations did not increase until the uncertainty reached 5 miles. However final position and velocity uncertainties are sensitive to Moon horizon determination as would be expected.

Table 7 presents the same data for the Moon to Earth flight for various Earth horizon uncertainties. The number of measurements and total velocity correction did not vary appreciably. However all uncertainties and deviations are sensitive to Earth horizon determination.

Finally, in Tables 8 and 9, a complete history of a circumlunar mission is given corresponding to the starred cases summarized in Tables 4 and 5.

BATTIN-16

Table 1. Earth to Moon flight navigation data as a function of velocity correction uncertainty ratio.

Miss Distance Reduction Ratio = { 0.1 start to 8 hrs; 0.5 8 hrs to 62.5 hrs

Sun Line = 250°

Velocity Correction Uncertainty Ratio	Number of Measurement	Times for Velocity Corrections	Total Velocity Correction (mph)	Final Position Uncertainty (miles)	Final Velocity Uncertainty (mph)	Final Position Deviation (miles)	Final Velocity Deviation (mph)
0.2	39	7.0 hrs 18.0 hrs 61.8 hrs	107	2.5	11.1	12.5	95
0.3	40	5.5 hrs 11.5 hrs 26.0 hrs 61.4 hrs	77	1.8	4.6	12.0	39

Table 2. Earth to Moon flight navigation data as a function of miss distance reduction ratio.

(miss distance reduction ratio constant at 0.1 from 0 to 8 hrs)

Velocity Corrections at 5, 20, 52, 61.5 hrs
Sun Line = 250°

Miss Distance Reduction Ratio (from 8 hrs to 62.5 hrs)	Number of Measurements	Total Velocity Correction (mph)	Final Position Uncertainty (miles)	Final Velocity Uncertainty (mph)	Final Position Deviation (miles)	Final Velocity Deviation (mph)
0.2	115	52	0.70	1.7	3.9	16
0.3	77	56	1.10	3.7	7.1	23
0.4	55	59	1.10	3.7	8.7	26
0.5	40	78	1.20	4.0	11.0	60
0.6	32	68	0.84	3.1	17.4	66

Table 3. Moon to Earth flight navigation data as a function of miss distance reduction ratio.

Velocity Corrections at 64, 88, 120, 125 hrs
Sun Line = 250°

Miss Distance Reduction Ratio	Number of Measurements	Total Velocity Correction (mph)	Final Position Uncertainty (miles)	Final Velocity Uncertainty (mph)	Final Position Deviation (miles)	Final Velocity Deviation (mph)
0.2	97	82	1.5	2.8	10.0	22
0.3	44	89	1.6	3.1	12.6	28
0.4	28	99	1.8	3.4	13.9	33
0.5	12	197	2.5	4.8	15.2	94
0.6	10	211	4.3	8.0	28.0	123

Table 4. Earth to Moon flight navigation data for pseudo trajectories as a function of sun direction rotation.

Miss Distance Reduction Ratio = { 0.1 start to 8 hrs; 0.5 8 hrs to 62.5 hrs

Velocity Corrections at 5, 20, 52, 61.5 hrs

Sun Direction Rotation (degrees)	Number of Measurements	Total Velocity Correction (mph)	Final Position Uncertainty (miles)	Final Velocity Uncertainty (mph)	Final Position Deviation (miles)	Final Velocity Deviation (mph)
0	41	68	1.3	3	3	31
70	39	64	6.5	18	8	39
120	39	114	1.6	3	12	92
180	40	66	5.2	21	12	48
250	40	78	1.2	4	11	60
300	39	88	1.2	4	4	46

BATTIN-17

Table 5. Moon to Earth flight navigation data for pseudo trajectories as a function of sun direction rotation.

Miss Distance Reduction Ratio = 0.4

Velocity Corrections at 64, 88, 120, 125 hrs (* first corr. at 70 hrs)

Sun Direction Rotation (degrees)	Number of Measurements	Total Velocity Correction (mph)	Final Position Uncertainty (miles)	Final Velocity Uncertainty (mph)	Final Position Deviation (miles)	Final Velocity Deviation (mph)
70	16	80	3.5	6.5	24	53
120	16	227	2.9	5.5	21	66
180*	20	94	2.1	4.3	10	41
250	28	99	1.8	3.4	14	33
300	14	163	2.3	3.9	31	99

Table 6. Earth to Moon flight navigation data as a function of Moon horizon uncertainty.

Miss Distance Reduction Ratio = { 0.1 start to 8 hrs; 0.3 8 hrs to 62.5 hrs }

Sun Line = 250°

Velocity Corrections at 5, 20, 52, 61.5 hrs

Moon Horizon Uncertainty (miles)	Number of Measurements	Total Velocity Correction (mph)	Final Position Uncertainty (miles)	Final Velocity Uncertainty (mph)	Final Position Deviation (miles)	Final Velocity Deviation (mph)
0.5	77	56	1.1	3.7	7.1	23
1.0	76	54	2.0	8.7	7.8	23
2.0	76	54	2.9	10.6	7.9	23
3.0	76	55	3.6	10.3	8.1	23
5.0	76	68	5.4	16.9	8.7	27

Table 7. Moon to Earth flight navigation data as a function of Earth horizon uncertainty.

Miss Distance Reduction Ratio = 0.3

Sun Line = 250°

Velocity Corrections at 64, 88, 120, 125 hrs

Earth Horizon Uncertainty (miles)	Number of Measurements	Total Velocity Correction (mph)	Final Position Uncertainty (miles)	Final Velocity Uncertainty (mph)	Final Position Deviation (miles)	Final Velocity Deviation (mph)
1	44	89	1.6	3.1	12.6	28
2	44	88	2.6	4.8	15.3	32
3	42	89	3.8	7.1	19.1	38
5	42	91	5.8	10.7	21.6	43

Table 8. Typical navigation data for Earth to Moon flight.

Miss Distance Reduction Ratio = { 0.1 start to 8 hrs.; 0.5 8 hrs to 62.5 hrs }

Sun Line = 250°

Time (hours)	Observation		Velocity Correction (mph)	Reduction in Position Uncertainty at Target (miles)	Position Uncertainty at Target (miles)	Indicated Velocity Correction (mph)	Uncertainty in Velocity Correction (mph)	Position Uncertainty (miles)	Velocity Uncertainty (mph)	Position Deviation (miles)	Velocity Deviation (mph)
0.6	Moon	Antares		262	2528	0	11.9	4.8	10.9	4.9	11.0
0.9	Earth	Fomalhaut		2031	1504	1.3	12.9	4.5	7.9	7.4	11.3
1.2	Earth	Deneb		540	1404	11.0	9.2	5.3	7.9	10.4	12.0
1.5	Earth	Aldebaran		412	1342	12.6	9.2	6.5	7.6	13.9	12.7
1.8	Earth	Aldebaran		370	1290	14.0	9.2	8.2	7.6	17.6	13.5
2.2	Earth	Aldebaran		408	1224	15.5	9.4	10.6	7.6	23.1	14.6
2.6	Earth	Pollux		456	1136	16.9	9.5	12.0	7.2	28.9	15.5
3.0	Earth	Procyon		515	1013	18.3	9.2	13.2	6.8	35.2	16.4
3.4	Earth	Procyon		405	928	19.7	8.7	14.6	6.5	41.8	17.2
3.8	Earth	Pollux		426	825	20.9	8.3	15.2	6.0	48.7	17.9
4.5	Earth	Procyon		403	719	22.8	8.0	17.0	5.6	61.5	19.1
5.0			24.1								
5.5	Earth	Pollux		435	573	0	7.6	18.4	4.9	70.0	7.8
6.0	Earth	Procyon		273	504	4.6	6.4	18.3	4.4	69.3	7.6
6.5	Earth	Pollux		244	441	5.7	5.8	18.1	4.1	69.0	7.5
7.0	Moon	Antares		196	395	6.5	5.3	18.2	3.7	69.1	7.5
7.5	Earth	Pollux		186	348	7.2	4.9	17.9	3.4	69.5	7.5
8.5	Moon	Antares		187	294	7.9	4.7	18.3	3.0	71.4	7.6
9.5	Moon	Antares		158	248	8.7	4.1	18.3	2.7	74.3	7.7
10.0	Earth	Pollux		135	208	9.3	3.6	17.1	2.4	76.2	7.8
10.5	Moon	Antares		107	179	9.7	3.1	16.0	2.1	78.3	7.9
12.0	Moon	Antares		95	151	10.5	2.9	16.7	1.8	85.8	8.3
12.5	Earth	Pollux		78	130	10.8	2.5	15.9	1.7	88.7	8.4
13.5	Moon	Antares		68	111	11.3	2.3	15.6	1.5	94.9	8.6
15.0	Moon	Antares		56	96	12.0	2.0	16.0	1.3	105.2	8.9
16.0	Earth	Pollux		48	83	12.6	1.8	16.0	1.2	112.7	9.1
17.0	Moon	Antares		41	72	13.0	1.7	15.7	1.1	120.6	9.3
19.5	Moon	Antares		36	62	14.3	1.6	16.9	1.0	141.9	9.8
20.0			14.5								
22.0	Earth	Pollux		32	54	0	1.5	18.2	0.9	137.4	4.6
23.5	Moon	Antares		28	47	0.6	1.4	18.0	0.8	131.2	4.4
28.5	Moon	Antares		23	40	1.1	1.4	20.2	0.7	112.4	4.0
29.5	Earth	Pollux		20	35	1.4	1.3	30.2	0.7	109.0	3.9
37.0	Moon	Antares		17	30	1.9	1.6	23.3	0.6	86.2	3.6
40.5	Earth	Pollux		15	26	2.4	1.6	24.6	0.5	77.4	3.5
52			5.4								
53.5	Moon	Antares		13	22	0	3.4	28.7	0.4	54.4	5.1
57.5	Earth	Regulus		11	19	2.8	5.1	28.5	0.3	40.9	4.9
60.0	Moon	Regulus		10	17	6.8	8.7	18.2	0.7	35.5	4.8
60.5	Earth	Procyon		9	14	11.1	7.7	13.2	0.8	34.7	5.0
61.4	Moon	Regulus		8.5	11.2	21.1	10.9	11.9	1.7	33.3	6.7
61.8			33.5								
62.1	Moon	Aldebaran		10.5	3.9	0	24.6	8.4	4.4	23.6	36.2
62.4	Moon	α Crucis		3.8	1.2	59.9	22.7	1.3	2.4	14.9	51.2
62.56								1.2	3.9	11.2	60.1

BATTIN-18

Table 9. Typical navigation data for Moon to Earth flight.

Miss Distance Reduction Ratio = 0.4
Sun Line = 250°

Time (hours)	Observation		Velocity Correction (mph)	Reduction in Position Uncertainty at Target (miles)	Position Uncertainty at Target (miles)	Indicated Velocity Correction (mph)	Uncertainty in Velocity Correction (mph)	Position Uncertainty (miles)	Velocity Uncertainty (mph)	Position Deviation (miles)	Velocity Deviation (mph)
62.6	Moon	Achemar		86	154	59.8	4.9	1.1	3.8	11.1	60.2
64.0	Moon	Fomalhaut		96	120	62.7	2.5	1.9	1.1	86.1	63.2
64.4			63.4								
65.0	Moon	Altair		105	172	0	1.4	1.7	1.1	111.4	3.4
79.0	Earth	Pollux		70	157	1.0	1.3	12.7	0.8	100.6	2.0
79.5	Moon	Fomalhaut		65	143	1.3	1.1	11.6	0.7	100.1	2.0
80.0	Earth	Aldebaran		61	130	1.4	1.0	11.2	0.7	99.6	2.0
80.5	Moon	Antares		53	118	1.4	0.9	8.3	0.5	99.1	2.0
81.0	Earth	Pollux		56	104	1.6	0.7	7.7	0.5	98.6	2.0
81.5	Earth	Pollux		44	95	1.6	0.6	7.4	0.4	98.1	2.0
83.5	Earth	Pollux		38	87	1.7	0.6	7.8	0.4	96.2	2.0
86.5	Earth	Pollux		35	79	1.8	0.6	8.5	0.4	93.5	2.0
88.5			1.9								
90.5	Earth	Pollux		32	72	0	0.6	9.4	0.4	88.7	1.7
95.5	Earth	Aldebaran		29	66	0.3	0.7	10.6	0.3	80.9	1.8
96.0	Moon	Antares		27	60	0.4	0.7	9.8	0.3	80.1	1.8
97.0	Earth	Regulus		24	55	0.5	0.6	9.5	0.3	78.5	1.8
105.0	Earth	Regulus		22	51	0.8	0.9	11.3	0.3	65.1	2.0
105.5	Moon	Antares		20	46	0.9	0.8	10.8	0.3	64.2	2.0
113.0	Earth	Regulus		19	43	1.8	1.3	12.2	0.3	50.3	2.4
114.0	Moon	Antares		17	39	2.0	1.4	11.8	0.3	48.4	2.5
120.5			5.7								
121.0	Moon	Antares		16	36	0	3.7	13.2	0.8	33.9	5.5
122.5	Moon	Fomalhaut		14	33	2.4	5.2	13.2	1.2	27.7	5.6
123.8	Earth	Procyon		14	29	5.4	8.3	11.1	1.7	23.5	6.0
124.2	Earth	Pollux		19	22	8.4	9.0	9.3	1.7	22.6	6.4
124.6	Earth	Canopus		14	17	13.1	9.1	7.9	1.7	22.1	7.0
125.0	Earth	Canopus		11	14	19.9	9.8	7.0	1.9	22.3	8.2
125.3			28.2								
125.6	Earth	Canopus		11	9	0	16.0	5.5	2.5	17.4	21.4
125.9	Earth	α Centauri		8	3	22.2	18.3	2.1	1.5	13.3	21.0
126.2	Earth	Antares		2	2	66.7	14.4	1.6	1.7	12.2	24.0
126.4								1.8	3.4	13.9	33.1

APPENDIX A

NAVIGATIONAL MEASUREMENTS

The mathematical processes are considered here in some detail for determining spacecraft position by means of both celestial observation and ground based radar measurements. It is assumed throughout the analysis that approximations to spacecraft position and velocity are already known so that perturbation techniques may be employed.

Secondary effects arising from the finite speed of light, the finite distance or stars, etc. are ignored in this analysis. Such effects may be lumped together for a particular reference point on the trajectory as a modification to the stored data which represent reference values for the quantities to be measured at that point.

For simplicity in the present analysis, it will be assumed that the spacecraft clock is perfect so that all measurements are made at known instants of time. Methods of including clock errors in the computation are discussed thoroughly in reference 2.

As indicated in Section 2.1 each measurement establishes a component of spacecraft position along some direction in space. If Q is the quantity to be measured and δQ is the difference between the true and the reference values, then it will be shown that the relation between δQ and the deviation in spacecraft position $\delta \underline{r}$ is

$$\delta Q = \underline{h}^T \, \delta \underline{r} \qquad \text{(A.1)}$$

regardless of the type of measurement. Thus, the $\underline{h}$ vector alone will characterize the kind of measurement.

Sun-Planet Measurement

The first type of measurement to be considered is that of the angle from the Sun to a planet. By passing to the limit of infinite distance from one or the other of these bodies, corresponding relations for the Sun-star or planet-star type of measurement may be obtained.

BATTIN-19

Let S_o and P_o be, respectively, the reference positions of the spacecraft and a planet at the time of the measurement. Let $\underline{r}$ be the vector from the Sun to S_o and $\underline{z}$ the vector from S_o to P_o. With A denoting the angle from the Sun line to the planet line, we have

$$\cos A = -(\underline{r} \cdot \underline{z})/rz \qquad (A.2)$$

where r and z denote magnitudes of the respective vectors $\underline{r}$ and $\underline{z}$. Treating all changes as first-order differentials, it can be shown that

$$\delta A = \left(\frac{\underline{m} - (\underline{n} \cdot \underline{m})\,\underline{n}}{r \sin A} + \frac{\underline{n} - (\underline{n} \cdot \underline{m})\,\underline{m}}{z \sin A} \right) \cdot \delta \underline{r} \qquad (A.3)$$

For details the reader is referred to reference 2. Here $\underline{n}$ and $\underline{m}$ are, respectively, the unit vectors from S_o toward the Sun and toward P_o. The two individual vector coefficients of $\delta\underline{r}$ in Eq. (A.3) are vectors in the plane of the measurement and normal, respectively, to the lines-of-sight to the Sun and to the planet.

Planet Diameter Measurement

If D is the actual diameter of a planet, the apparent angular diameter A is found from

$$\sin (A/2) = D/2z \qquad (A.4)$$

Again taking differentials as before, one can show that

$$\delta A = \frac{D\underline{m} \cdot \delta \underline{r}}{z^2 \cos (A/2)} \qquad (A.5)$$

Star Occultations

The next type of measurement to be considered is that of noting the time at which a star is occulted by a planet. Let $\underline{z}$ be the vector from S_o to P_o, $\underline{r}$ the vector from the Sun to S_o and $\underline{n}$ a unit vector in the direction of the star to be occulted. With γ denoting the angle from the star line to the planet line as shown in Fig. A-1, we have, at the nominal instant of occultation,

$$\underline{n} \cdot \underline{z} = z \cos \gamma \qquad (A.6)$$

Treating changes as first order differentials we obtain

$$\underline{n} \cdot \delta \underline{z} = \cos \gamma\, \delta z - z \sin \gamma\, \delta\gamma \qquad (A.7)$$

$$= \cos \gamma\, \underline{m} \cdot \delta \underline{z} - z \sin \gamma\, \delta \gamma$$

where $\underline{m}$ is a unit vector from S_o toward P_o.

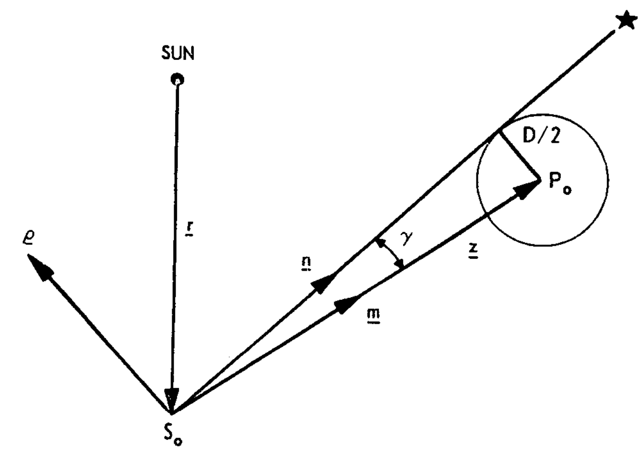

Fig. A-1. Measurement of time of a star occultation.

The angle deviation $\delta\gamma$ is computed from a first order differential of $2z \sin\gamma = D$. There results

$$\delta\gamma = -D\underline{m} \cdot \delta\underline{z}/2z^2 \cos \gamma \qquad (A.8)$$

Furthermore, if $\underline{v}_p$ and $\underline{v}_s$ are the respective velocity vectors of the planet and the spacecraft and if $\delta\tau$ is the difference between the observed and the reference occultation times, we have

$$\delta \underline{z} = \underline{v}_p\, \delta\tau - (\delta\underline{r} + \underline{v}_s\, \delta\tau) \qquad (A.9)$$

$$= -\delta\underline{r} - \underline{v}_r\, \delta\tau$$

where $\underline{v}_r$ is the velocity of the spacecraft relative to the planet. Then by combining Eqs. (A.7), (A.8) and (A.9) we have finally

$$\delta\tau = -\frac{\underline{\rho} \cdot \delta\underline{r}}{\underline{\rho} \cdot \underline{v}_r} \qquad (A.10)$$

where $\underline{\rho}$ is a unit vector perpendicular to $\underline{n}$ and lying in the plane determined by the lines-of-sight to the planet and the star.

Star Elevation Measurement

Consider next the measurement of the angle between the lines-of-sight to a star and the edge of a planet disc. From Fig. A-2 we have

$$\underline{n} \cdot \underline{z} = z \cos (A + \gamma) \qquad (A.11)$$

BATTIN-20

where A is the angle to be measured. Again taking total differentials and noting that $\delta \underline{r} = -\delta \underline{z}$, we obtain

$$\frac{1}{z}\underline{\rho}\cdot\delta\underline{r} = \delta A + \delta\gamma = \delta A + D\underline{m}\cdot\delta\underline{r}/2z^2\cos\gamma = \delta A + \tan\gamma\,\underline{m}\cdot\delta\underline{r}/z \qquad (A.12)$$

or finally

$$\delta A = \frac{\underline{\rho}\cdot\delta\underline{r}}{z\cos\gamma} \qquad (A.13)$$

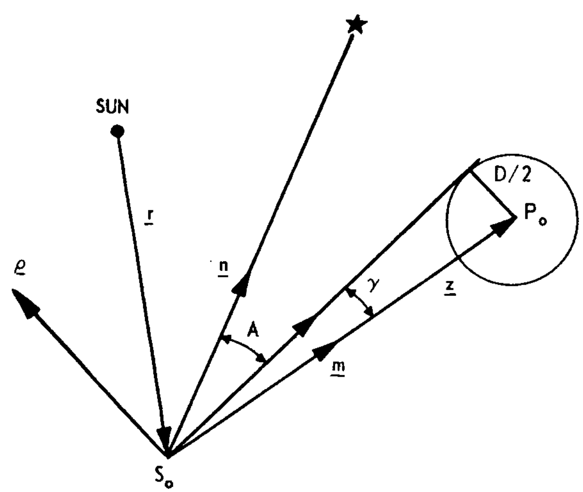

Fig. A-2. Measurement of star elevation angle.

Landmark Measurement

For the measurement of the angle between a landmark on a planet surface and a star, let $\underline{\rho}$ be a unit vector perpendicular to the line-of-sight to the landmark and in the plane of the measurement. Then if $\underline{p}$ is the vector position of the landmark relative to the center of the planet, we have

$$\delta A = \frac{\underline{\rho}\cdot\delta\underline{r}}{|\underline{z}+\underline{p}|} \qquad (A.14)$$

Radar Range, Azimuth, and Elevation Measurements

Assume the radar site to be the origin of the coordinate system although other origins could equally well be used. Let a cartesian coordinate system be chosen such that the z axis is radially out from the center of the Earth through the radar site; the x axis is positive in the direction from which radar azimuths are to be measured; the y axis completes the coordinate system. Then, we may write

$$\underline{r} = r\left\|\begin{matrix}\cos\beta\cos\theta\\ \cos\beta\sin\theta\\ \sin\beta\end{matrix}\right\| \qquad (A.15)$$

where r, θ, β are, respectively, the range, azimuth, and elevation of the vehicle from the radar site. Taking differentials separately for each of the three variables gives

$$\frac{\partial\underline{r}}{\partial r}\delta r = \left\|\begin{matrix}\cos\beta\cos\theta\\ \cos\beta\sin\theta\\ \sin\beta\end{matrix}\right\|\delta r \qquad (A.16)$$

$$\frac{\partial\underline{r}}{\partial\beta}\delta\beta = r\left\|\begin{matrix}-\sin\beta\cos\theta\\ -\sin\beta\sin\theta\\ \cos\beta\end{matrix}\right\|\delta\beta \qquad (A.17)$$

$$\frac{\partial\underline{r}}{\partial\theta}\delta\theta = r\left\|\begin{matrix}-\cos\beta\sin\theta\\ \cos\beta\cos\theta\\ 0\end{matrix}\right\|\delta\theta \qquad (A.18)$$

Then, by expressing each of these relations in the form of Eq. (A.1), we obtain

$$\delta r = \left\|\cos\beta\cos\theta \quad \cos\beta\sin\theta \quad \sin\beta\right\|\delta\underline{r} \qquad (A.19)$$

$$\delta\beta = \frac{1}{r}\left\|-\sin\beta\cos\theta \quad -\sin\beta\sin\theta \quad \cos\beta\right\|\delta\underline{r} \qquad (A.20)$$

$$\delta\theta = \frac{1}{r\cos\beta}\left\|-\sin\theta \quad \cos\theta \quad 0\right\|\delta\underline{r} \qquad (A.21)$$

The vector coefficients in Eqs. (A.19) – (A.21) are each unit vectors in the direction of increasing r, β, θ, respectively.

BATTIN - 21

APPENDIX B

OPTIMUM SELECTION OF NAVIGATION MEASUREMENTS

In the main body of this paper a method of processing measurement data in an optimum linear manner has been developed. The purpose of this appendix is to treat the associated problem of selecting those measurements which are, in some sense, most effective. For example, the requirement might be to select the measurement to be made at time t_n in order to get the maximum reduction in mean-squared positional or velocity uncertainty at time t_n. Of perhaps greater significance would be the requirement of selecting the measurement which minimizes the uncertainty in any linear combination of position and velocity deviations. Specifically, one might select the measurement which minimizes the uncertainty in the required velocity correction. As a further example, one might wish to select that measurement which, if followed immediately by a velocity correction, would result in the smallest position error at the target.

Consider first the simplest case, i.e., minimizing the mean-squared positional uncertainty at time t_n. From Eq. (2.29) the mean-squared positional uncertainty is expressible as

$$\overline{\epsilon_n^2} = \mathrm{tr}\left(E_n^{(1)'}\right) - \frac{\underline{h}_n^T E_n^{(1)'} E_n^{(1)'} \underline{h}_n}{\underline{h}_n^T E_n^{(1)'} \underline{h}_n + \overline{\alpha_n^2}} \qquad (B.1)$$

assuming the measurement errors to be uncorrelated. In the absence of any measurement error ($\alpha_n^2 = 0$) , the problem of minimizing either mean-squared error is equivalent to finding a direction for the $\underline{h}_n$ vector which maximizes the ratio of two quadratic forms. For the case of the mean-squared positional error, the geometrical interpretation is clear. Since the principal directions of $E_n^{(1)'}$ and $E_n^{(1)'} E_n^{(1)'}$ are the same, the optimal direction for $\underline{h}_n$ coincides with the major principal direction of $E_n^{(1)}$.

The problem of minimizing the mean-squared velocity uncertainty at time t_n by proper choice of the $\underline{h}_n$ vector is not as easily solved or interpreted. Again, from Eq. (2.29) the mean-squared velocity uncertainty may be written as

$$\overline{\delta_n^2} = \mathrm{tr}\,(E_n^{(4)'}) - \frac{\underline{h}_n^T E_n^{(2)'} E_n^{(3)'} \underline{h}_n}{\underline{h}_n^T E_n^{(1)'} \underline{h}_n + \overline{\alpha_n^2}} \qquad (B.2)$$

Denote by p and q the two quadratic forms

$$p = \underline{h}_n^T E_n^{(2)'} E_n^{(3)'} \underline{h}_n \quad , \quad q = \underline{h}_n^T E_n^{(1)'} \underline{h}_n \qquad (B.3)$$

From the theory of quadratic forms there exists an orthogonal transformation which will reduce q to a diagonal form. Thus

$$\underline{h}_n = Q\,\underline{d} \qquad (B.4)$$

gives

$$q = \underline{d}^T Q^T E_n^{(1)'} Q\,\underline{d} = \mu_1 d_1^2 + \mu_2 d_2^2 + \mu_3 d_3^2 \qquad (B.5)$$

where μ_1, μ_2, μ_3 are the characteristic roots of the matrix $E_n^{(1)'}$ and the columns of the Q matrix are the associated characteristic unit vectors. Since $E_n^{(1)'}$ is a positive definite matrix, the characteristic roots are positive and a further transformation

$$\underline{f} = D\,\underline{d} \qquad (B.6)$$

gives

$$q = \underline{f}^T \underline{f} = f_1^2 + f_2^2 + f_3^2 \qquad (B.7)$$

where D is a diagonal matrix whose diagonal elements are $\sqrt{\mu_1}$, $\sqrt{\mu_2}$, $\sqrt{\mu_3}$.

The same transformation from $\underline{h}_n$ to $\underline{f}$ applied to the quadratic form p produces

$$p = \underline{f}^T D^{-1} Q^T E_n^{(2)'} E_n^{(3)'} Q D^{-1} \underline{f} \qquad (B.8)$$

One final transformation applied to $\underline{f}$ will reduce Eq. (B.8) to a diagonal form thus

$$\underline{f} = S\,\underline{m} \qquad (B.9)$$

results in

$$p = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 \qquad (B.10)$$

BATTIN-22

where the columns of the S matrix are the characteristic unit vectors of the matrix $D^{-1} Q^T E_n^{(2)\prime} E_n^{(3)\prime} Q D^{-1}$ and $\lambda_1, \lambda_2, \lambda_3$, the corresponding characteristic roots. The same transformation (B.9) applied to (B.7) gives

$$q = \underline{m}^T S^T S \underline{m} = m_1^2 + m_2^2 + m_3^2 \tag{B.11}$$

since S is an orthogonal matrix.

In summary, then, the transformation

$$\underline{h}_n = Q D^{-1} S \underline{m} \tag{B.12}$$

produces for the ratio of the two quadratic forms

$$\frac{p}{q} = \frac{\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2}{m_1^2 + m_2^2 + m_3^2} \tag{B.13}$$

Furthermore, if the matrix $E_n^{(2)\prime}$ is nonsingular, the product $E_n^{(2)\prime} E_n^{(3)\prime} = E_n^{(2)\prime} E_n^{(2)\prime T}$ is positive definite and it would then follow that $\lambda_1, \lambda_2, \lambda_3$ are all real and positive.

The problem of maximizing the ratio p/q is now readily solved. Since no measurement error is assumed, one cannot hope to determine more than the direction for the optimum $\underline{h}_n$ or, equivalently, the optimum $\underline{m}$. Therefore, it may be assumed that $\underline{m}$ is a unit vector. Let

$$\lambda_k = \max(\lambda_1, \lambda_2, \lambda_3) \tag{B.14}$$

Then the optimum value of $\underline{m}$ is

$$m_j = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \tag{B.15}$$

The same technique can be used to select that direction for $\underline{h}_n$ which minimizes the uncertainty in any linear combination of position and velocity deviations. Specifically, consider the selection of that measurement which minimizes the uncertainty in the velocity correction which would be required immediately following the measurement.

The correlation matrix of the velocity correction uncertainty is

$$\overline{\underline{d}_n \underline{d}_n^T} = B_n E_n B_n^T \tag{B.16}$$

and the mean-squared uncertainty may be expressed as

$$\overline{d_n^2} = \mathrm{tr}(B_n E_n' B_n^T) - \frac{\underline{h}_n^T W \underline{h}_n}{\underline{h}_n^T E_n^{(1)\prime} \underline{h}_n + \overline{\alpha_n^2}} \tag{B.17}$$

Here W is a symmetric matrix defined by

$$W = \left\| E_n^{(1)\prime} \; E_n^{(2)\prime} \right\| B_n^T B_n \left\| \begin{matrix} E_n^{(1)\prime} \\ E_n^{(2)\prime T} \end{matrix} \right\| \tag{B.18}$$

so that if $\left\| E_n^{(1)\prime} \; E_n^{(2)\prime} \right\| B_n^T$ is nonsingular, the matrix W will be positive definite. Under any circumstances, if the identification

$$E_n^{(2)\prime} \sim \left\| E_n^{(1)\prime} \; E_n^{(2)\prime} \right\| B_n^T$$

is made, then the exact same procedure may be used to select the optimum direction for the $\underline{h}_n$ vector as was used previously to minimize the mean-squared velocity uncertainty.

In all cases of practical interest the determination of the optimum direction for the $\underline{h}_n$ vector must be made subject to certain constraints. For example, one might wish to select the "best" star to be used in measuring the angle between the line of sight to the center of a planet disc and the line of sight to the star. For such a measurement the $\underline{h}_n$ vector is required to be perpendicular to the line of sight to the planet. If $\underline{z}_n$ is the position vector of the planet from the space vehicle, then we must have

$$\underline{h}_n^T \underline{z}_n = 0 \tag{B.19}$$

Applying the transformation defined in Eq. (B.12) gives

$$\underline{m}^T S^T D^{-1} Q^T \underline{z}_n = 0 \tag{B.20}$$

Let $\underline{p}$ be a unit vector in the direction of $S^T D^{-1} Q^T \underline{z}_n$. Then the problem of selecting the optimum direction for $\underline{h}_n$ or, equivalently, for $\underline{m}$ is to maximize

$$\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2$$

subject to the conditions of constraint

$$\underline{m}^T \underline{p} = 0 \text{ and } \underline{m}^T \underline{m} = 1 \tag{B.21}$$

In terms of the Lagrange multipliers ρ and σ, this is equivalent to the problem of obtaining a free maximum for

$$\sum_{j=1}^{3} \lambda_j m_j^2 - 2\rho \sum_{j=1}^{3} p_j m_j - \sigma \left\{ \sum_{j=1}^{3} m_j^2 - 1 \right\}$$

Setting the partial derivatives with respect to each of the m_j's equal to zero, we have

$$m_j = \frac{\rho p_j}{\lambda_j - \sigma} \qquad j = 1, 2, 3 \qquad \text{(B. 22)}$$

where ρ and σ are to be determined from the requirements of Eq. (B-21).

The condition that $\underline{m}$ be orthogonal to $\underline{p}$ leads to a quadratic equation for σ,

$$\sigma^2 - \left[p_1^2 (\lambda_2 + \lambda_3) + p_2^2 (\lambda_1 + \lambda_3) + p_3^2 (\lambda_1 + \lambda_2)\right] \sigma \qquad \text{(B. 23)}$$

$$+ p_1^2 \lambda_2 \lambda_3 + p_2^2 \lambda_1 \lambda_3 + p_3^2 \lambda_1 \lambda_2 = 0$$

If the λ's are ordered $\lambda_1 < \lambda_2 < \lambda_3$, then the two roots σ_1 and σ_2 will be such that $\lambda_1 < \sigma_1 < \lambda_2 < \sigma_2 < \lambda_3$. The other Lagrange multiplier ρ is determined so that $\underline{m}$ will be a unit vector. With the optimum vector $\underline{m}$ selected, the corresponding value for $\underline{h}_n$ is found from Eq. (B. 12).

It is easy to show that σ_2 provides the desired maximum while σ_1 gives the minimum. From Eq. (B. 22) one obtains

$$\sum_{j=1}^{3} \lambda_j m_j^2 - \sigma \sum_{j=1}^{3} m_j^2 = \rho \sum_{j=1}^{3} p_j m_j \qquad \text{(B. 24)}$$

Using this and Eqs. (B. 21) it follows that

$$\sigma = \sum_{j=1}^{3} \lambda_j m_j^2 \qquad \text{(B. 25)}$$

Hence, σ_1 and σ_2 are the respective minimum and maximum of the original expression to be maximized.

BIBLIOGRAPHY

1. Laning, J. H., Jr., Frey, E. J., and Trageser, M. B., Preliminary Considerations on the Instrumentation of a Photographic Reconnaissance of Mars, Vistas in Astronautics, Vol. 2, Pergamon Press, 1959.

2. Battin, R. H., and Laning, J. H., Jr., A Navigation Theory for Round-Trip Reconnaissance Missions to Venus and Mars, Planetary and Space Science, Vol. 7, pp. 40-56, Pergamon Press, July 1961.

3. Battin, R. H., A Comparison of Fixed and Variable Time of Arrival Navigation for Interplanetary Flight, Proceedings of the Fifth AFBMD/STL Aerospace Symposium on Ballistic Missile and Space Technology, Academic Press, 1960.

4. Kalman, R. E., A New Approach to Linear Filtering and Prediction Problems, Journal of Basic Engineering, March 1960.

5. McLean, J. D., Schmidt, S. F., and McGee, L. A., Optimal Filtering and Linear Prediction Applied to a Space Navigation System for the Circumlunar Mission, NASA TN D-1208.

BATTIN-24