THE RESTRICTED THREE-BODY-PROBLEM AS A PERTURBATION OF EULER'S PROBLEM OF TWO FIXED CENTERS AND ITS APPLICATION TO LUNAR TRAJECTORIES

by

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Abstract

The restricted Three-Body-Problem considers the motion of an infinitesimal mass under the gravitational attraction of two finite masses, which revolve about their common center of gravity in coplanar circles. It is well known that Euler's problem of two fixed centers, consisting of the motion of an infinitesimal mass under the gravitational attraction of two finite masses fixed in space, can be solved by elliptic functions.

The idea presented here is to take the solution of Euler's problem as the solution of the restricted Three-Body-Problem by allowing the initial values to be functions of time now. Differential equations for the perturbed initial values are established. These equations can be given in closed form by using the fact that the transformation to the perturbed initial values of Euler's problem is canonical. Thus, an approximation can be obtained for the solution of the restricted Three-Body-Problem. The method can also be used to represent classes of neighboring trajectories for guidance purposes.

Notations

dot: differentiation with respect to time t, e.g. $\dot{x}_1 = dx_1/dt$, $\dot{x} = dx/dt$

small circle: partial differentiation with respect to time t, e.g. $\mathring{x} = \partial x/\partial t$

4-vectors: e.g., $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $x_o = \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{pmatrix}$, $\frac{\partial H}{\partial x} = \begin{pmatrix} \partial H/\partial x_1 \\ \partial H/\partial x_2 \\ \partial H/\partial x_3 \\ \partial H/\partial x_4 \end{pmatrix}$

with the matrix $J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ a canonical system $\dot{x}_1 = \partial H/\partial x_3$, $\dot{x}_2 = \partial H/\partial x_4$, $\dot{x}_3 = -\partial H/\partial x_1$, $\dot{x}_4 = -\partial H/\partial x_2$ is written as $\dot{x} = J(\partial H/\partial x)$.

Introduction

For the purpose of guidance of lunar vehicles it would be useful and convenient to possess analytical expressions that give position and velocity at a certain time as functions of position and velocity at an earlier time; in addition, the formulas should be as simple as possible and of sufficient accuracy. This general problem was one of the incentives to a study that shall be outlined in its basic ideas in this paper. Only the simplest case of the problem of motion and guidance in the real earth-moon space has been attacked so far, idealizing as much as possible. Accordingly, in the following the restricted three-body problem in a plane will be considered, treated as a perturbation problem of Euler's problem of two fixed centers, which can be solved in closed form by elliptic functions.

The procedure of attacking the problem is the following: Represent the restricted three-body problem in a rotating coordinate system in which the two finite masses are at rest. Then the Hamiltonian H of the restricted problem can be written as the sum of two terms, $H = H_1 + H_2$, one of which, say H_1, is exactly the Hamiltonian of Euler's problem. Solve Euler's problem - this can be done in principle - and consider the initial values of the solution of Euler's problem as the new variables of the restricted problem. It can be shown that this transformation of the dependent variables is canonical, and that the new variables satisfy canonical differential equations, such that the Hamiltonian of this system is the other term H_2 of the Hamiltonian $H = H_1 + H_2$ of the restricted problem. This process is applied once more to $H_2 = H_3 + H_4$, and the problem is reduced to the solution of a system of canonical differential equations with the Hamiltonian H_4. The solutions of this last system of equations are slowly varying functions for not too large intervals of time.

The object of this study is not so much the approximation of the true trajectory by another simpler trajectory - mathematically usually an initial value problem -, but the development of methods towards the solution of the above-mentioned prediction problem - mathematically closer to a boundary value problem.

Theorem on Canonical Initial Values

The following theorem will be used essentially in the subsequent developments; it will be formulated here without proof. As already mentioned, it states that in certain cases the initial values of the solution of a system of canonical differential equations are canonical variables for another canonical system.

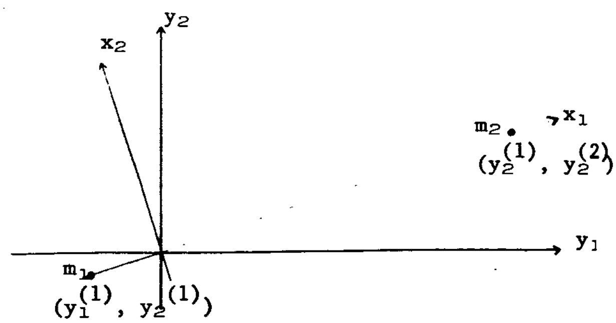
Theorem: Let $\dot{\alpha} = J(\partial H/\partial \alpha)$ be a canonical system with the Hamiltonian $H = H(t, \alpha)$ and initial values t_o, α_o.

Let $H = H_1 + H_2$ and let $\beta = B(t, \beta_o)$ be a solution of the canonical system $\dot{\beta} = J(\partial \tilde{H}_1/\partial \beta)$ with the Hamiltonian $\tilde{H}_1 = \tilde{H}_1(t, \beta) = H_1(t, \beta)$ and initial values t_o, β_o.

Define functions $\gamma = \Gamma(t, \delta) = B(t, \delta)$ and determine the functions $\delta = \Delta(t, \delta_o)$ so that the Γ are a solution of the (original) canonical system $\dot{\gamma} = J(\partial \tilde{H}/\partial \gamma)$ with the Hamiltonian $\tilde{H} = \tilde{H}(t, \gamma) = H(t, \gamma)$ and initial values t_o, $\gamma_o = \alpha_o$.

Then the functions $\Delta(t, \delta_o)$ are a solution of the canonical system $\dot{\delta} = J(\partial H_2^*/\partial \delta)$ with the Hamiltonian $H_2^* = H_2^*(t, \delta) = H_2(t, \Gamma(t, \delta))$ and initial values $\delta_o = \alpha_o$.

The Restricted Three-Body Problem in the Plane



Let y_1, y_2 be cartesian, barycentric, and space fixed coordinates in the plane of motion; let the finite masses m_1 and m_2 revolve in circles about their common center of mass with angular velocity n. The differential equations of motion of a particle with mass zero and position (y_1, y_2) in the gravitational field of m_1 and m_2 read

$$\ddot{y}_1 = -\gamma m_1 \frac{y_1 - y_1^{(1)}}{\rho_1^3} - \gamma m_2 \frac{y_1 - y_1^{(2)}}{\rho_2^3}$$

$$\ddot{y}_2 = -\gamma m_1 \frac{y_2 - y_2^{(1)}}{\rho_1^3} - \gamma m_2 \frac{y_2 - y_2^{(2)}}{\rho_2^3}$$

where

$$\rho_i = \sqrt{(y_1 - y_1^{(i)})^2 + (y_2 - y_2^{(i)})^2} \qquad i = 1,2.$$

The specific kinetic energy is

$$\tilde{T} = \frac{1}{2}(\dot{y}_1^2 + \dot{y}_2^2),$$

the specific potential

$$\tilde{V} = -\gamma m_1 \frac{1}{\rho_1} - \gamma m_2 \frac{1}{\rho_2},$$

and the Lagrangian

$$\tilde{L} = \tilde{T} - \tilde{V}.$$

The equations of motion can also be written as

$$\ddot{y}_1 = -\frac{\partial \tilde{V}}{\partial y_1}; \quad \ddot{y}_2 = -\frac{\partial \tilde{V}}{\partial y_2}.$$

Introduce new coordinates x_1, x_2 by the rotation

$$x_1 = y_1 \cos n(t - t_o) + y_2 \sin n(t - t_o)$$

$$x_2 = -y_1 \sin n(t - t_o) + y_2 \cos n(t - t_o)$$

so that the coordinates of m_1 and m_2 are constant and $x_1^{(1)}$, 0 and $x_2^{(1)}$, 0. In this new coordinate system we get:

specific kinetic energy

$$T = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2)^2 + \frac{1}{2} n^2(x_1^2 + x_2^2) + n(x_1\dot{x}_2 - \dot{x}_1 x_2)$$

$$= \frac{1}{2}[(\dot{x}_1 - nx_2)^2 + (\dot{x}_2 + nx_1)^2],$$

specific potential

$$V = V(x_1, x_2) = -\gamma m_1 \frac{1}{r_1^3} - \gamma m_2 \frac{1}{r_2^3},$$

where

$$r_i = \sqrt{(x_1 - x_1^{(i)})^2 + x_2^2},$$

the Lagrangian

$$L = T - V,$$

and the differential equations of motion

$$\ddot{x}_1 - 2n\dot{x}_2 - n^2 x_1 = -\frac{\partial V}{\partial x_1}$$

$$\ddot{x}_2 + 2n\dot{x}_1 - n^2 x_2 = -\frac{\partial V}{\partial x_2}.$$

For the following we will write the differential equations of motion in canonical form; introduce in the usual manner the generalized momenta x_3 and x_4 by

$$x_3 = \frac{\partial L}{\partial \dot{x}_1} = \dot{x}_1 - nx_2, \qquad x_4 = \frac{\partial L}{\partial \dot{x}_2} = \dot{x}_2 + nx_1$$

and the Hamiltonian

$$H = H(x) = H(x_1, x_2, x_3, x_4) = x_3\dot{x}_1 + \dot{x}_2 x_4 - L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}(x_3^2 + x_4^2) + n(x_2x_3 - x_1x_4) + V(x_1, x_2).$$

The canonical equations of motion read

$$\dot{x}_1 = \frac{\partial H}{\partial x_3}, \quad \dot{x}_2 = \frac{\partial H}{\partial x_4}, \quad \dot{x}_3 = -\frac{\partial H}{\partial x_1}, \quad \dot{x}_4 = -\frac{\partial H}{\partial x_2},$$

or in matrix form $\dot{x} = J \frac{\partial H}{\partial x}$.

Let the initial values be t_o, x_o.

Euler's Problem of Two Fixed Centers

Write $H = H_1 + H_2$,

where $H_1 = \frac{1}{2}(x_3^2 + x_4^2) + V(x_1, x_2)$

and $H_2 = n(x_2x_3 - x_1x_4)$;

H_1 is the Hamiltonian for Euler's problem of two fixed centers, as can be derived directly or from the fact that $H_1 = H(n = 0)$. Introduce new coordinates w instead of x for the (formal) solution of Euler's problem, and define a Hamiltonian $\tilde{H}_1$ by

$$\tilde{H}_1 = \tilde{H}_1(w) = H_1(x).$$

The differential equations of motion are

$$\dot{w} = J \frac{\partial \tilde{H}_1}{\partial w},$$

and denote the initial values by t_o, w_o.

Let $w = W(t, w_o)$ be the solution of this system of differential equations, i.e. of Euler's problem; it follows from the theory of differential equations that the $W_j(t, \tilde{w})$ are holomorphic functions of the variables $\tilde{w}_i$ in a sufficiently small neighborhood of the point $\tilde{w}_o = w_o$. Since $W(t_o, \tilde{w}) = \tilde{w}$, one finds, expanding W in a power series of $t - t_o$:

$$W(t, \tilde{w}) = \tilde{w} + \left(\frac{\partial W}{\partial t}\right)_{t_o} (t - t_o) + \ldots$$

and therefore

$$\left(\frac{\partial W_j(t, \tilde{w})}{\partial \tilde{w}_i}\right)_{t_o} = \delta_{ji},$$

and this holds especially for $\tilde{w} = w_o$.

Perturbation

Introduce functions Ω_j, $j = 1, \ldots, 4$, by

$$\omega = \Omega(t, \xi) = W(t, \xi)$$

and determine the functions $\xi_j = \Xi_j(t, x_o)$ in such a way that ω is the solution of the original restricted problem, i.e. that

$$\overset{\circ}{\omega} = J \frac{\partial \hat{H}}{\partial \omega}$$

with the Hamiltonian $\hat{H} = \hat{H}(\omega) = H(\omega)$ and initial values t_o, $\omega_o = x_o$.

According to the theorem formulated at the beginning one finds that the functions $\xi_j = \Xi_j(t, x_o)$ are determined by the canonical system

$$\dot{\xi} = J \frac{\partial \tilde{H}_2}{\partial \xi},$$

with the Hamiltonian $\tilde{H}_2 = \tilde{H}_2(t, \xi) = H_2(\Omega(t, \xi))$ and initial values t_o, $\xi_o = x_o$.

New Expression for $\tilde{H}_2$

Let us derive another explicit expression for the Hamiltonian $\tilde{H}_2(t, \xi)$ in order to be able to split $\tilde{H}_2$ into the sum of two terms and to apply this perturbation procedure once more. From the definition of H_2 it follows that

$$\tilde{H}_2 = n(\omega_2\omega_3 - \omega_1\omega_4), \qquad \omega_j = W(t, \xi).$$

The ω_j are the solution of Euler's problem and consequently satisfy the following set of equations:

$$\overset{\circ}{\omega}_1 = \omega_3, \qquad \overset{\circ}{\omega}_2 = \omega_4$$

$$\overset{\circ}{\omega}_3 = -\frac{\partial V(\omega_1, \omega_2)}{\partial \omega_1} = -\gamma m_1 \frac{\omega_1 - \omega_1^{(1)}}{r_1^3} - \gamma m_2 \frac{\omega_1 - \omega_1^{(1)}}{r_2^3}$$

$$\overset{\circ}{\omega}_4 = -\frac{\partial V(\omega_1, \omega_2)}{\partial \omega_2} = -\gamma m_1 \frac{\omega_2}{r_1^3} - \gamma m_2 \frac{\omega_2}{r_2^3},$$

where $\omega_1^{(i)}$, 0 is the fixed position of m_i, and

$$r_i = \sqrt{(\omega_1 - \omega_1^{(i)})^2 + \omega_2^2}.$$

It follows that

$$\begin{aligned}
\frac{\partial \tilde{H}_2}{\partial t} &= n(\overset{\circ}{\omega}_2\omega_3 + \omega_2\overset{\circ}{\omega}_3 - \overset{\circ}{\omega}_1\omega_4 - \omega_1\overset{\circ}{\omega}_4) \\
&= n(\omega_4\omega_3 + \omega_2\overset{\circ}{\omega}_3 - \omega_3\omega_4 - \omega_1\overset{\circ}{\omega}_4) \\
&= n(\omega_2\overset{\circ}{\omega}_3 - \omega_1\overset{\circ}{\omega}_4) \\
&= n\left[\gamma m_1\omega_1^{(1)} \frac{\omega_2}{r_1^3} + \gamma m_2\omega_1^{(2)} \frac{\omega_2}{r_2^3}\right] \\
&= n\gamma m_1\omega_1^{(1)}\omega_2 \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right),
\end{aligned}$$

since

$$m_1\omega_1^{(1)} + m_2\omega_1^{(2)} = 0$$

in the used barycentric coordinate system.

Integration of $\overset{\circ}{\tilde{H}}_2$ with respect to time t (the ξ are considered as constants) yields

$$\tilde{H}_2(t, \xi) = \tilde{H}_2(t_o, \xi) + n\gamma m_1\omega_1^{(1)} \int_{t_o}^{t} \omega_2\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) dt;$$

for $\tilde{H}_2(t_o, \xi)$ we find, since $\Omega(t_o, \xi) = \xi$:

$$\tilde{H}_2(t_o, \xi) = n(\xi_2\xi_3 - \xi_1\xi_4).$$

Second Perturbation

Write $\tilde{H}_2(t, \xi) = \tilde{H}_3 + \tilde{H}_4$,

where $\tilde{H}_3 = n(\xi_2\xi_3 - \xi_1\xi_4)$

and $\tilde{H}_4 = n\gamma m_1\omega_1^{(1)} \int_{t_o}^{t} \omega_2\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) dt,$

and $\omega_2 = \Omega_2(t, \xi)$, $r_i = r_i(t, \xi)$.

As in the first perturbation process, we will solve the canonical system with the Hamiltonian $\tilde{H}_3$ and introduce the initial values of the solution as new variables into the full canonical system with the Hamiltonian $\tilde{H}_2$.

Introduce new variables g_1, g_2, g_3, g_4 and solve the canonical system

$$\dot{g} = \frac{\partial H_3^*}{\partial g}$$

with the Hamiltonian

$$H_3^* = H_3^*(g) = \tilde{H}_3(g) = n(g_2 g_3 - g_1 g_4)$$

and initial values t_o, g_o. The solution $g = G(t,g_o)$ can be given explicitly as the simple rotation

$$g = \begin{pmatrix} \cos n(t-t_o) & \sin n(t-t_o) & 0 & 0 \\ -\sin n(t-t_o) & \cos n(t-t_o) & 0 & 0 \\ 0 & 0 & \cos n(t-t_o) & \sin n(t-t_o) \\ 0 & 0 & -\sin n(t-t_o) & \cos n(t-t_o) \end{pmatrix} g_o$$

or

$$g = R(t)g_o.$$

Now introduce functions $\gamma_j = \Gamma_j(t,\lambda) = G_j(t,\lambda)$ and determine the functions $\lambda_j = \Lambda_j(t,x_o)$ so that the functions Γ_j are solutions of the full canonical system with $\tilde{H}_2$, i.e. of

$$\dot{\gamma} = J\frac{\partial H_2^*}{\partial \gamma}$$

with the Hamiltonian $H_2^* = H_2^*(t,\gamma) = \tilde{H}_2(t,\gamma)$ and initial values t_o, λ_o.

Again it follows by the theorem formulated in the beginning, that the functions $\lambda = \Lambda(t,x_o)$ are determined by the canonical system

$$\dot{\lambda} = J\frac{\partial H_4^*}{\partial \lambda}$$

with the Hamiltonian $H_4^* = H_4^*(t,\lambda) = \tilde{H}_4(t,\Gamma(t,\lambda))$ and initial values t_o, $\lambda_o = x_o$. The Hamiltonian H_4^* can be written also as

$$H_4^*(t,\lambda) = n\gamma m_1 \omega_1^{(1)} \int_{t_o}^{t} \omega_2 \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) dt,$$

where ω and r_i have to be expressed as functions of t and λ (λ is a constant for the integration).

Concluding Remarks

By the foregoing developments the restricted three-body problem has been reduced principally to the computation of the functions $\lambda_j = \Lambda_j(t,x_o)$ as the solutions of a system of canonical differential equations.

One can try, for instance, to expand $\Lambda(t,x_o)$ in a power series of $t-t_o$ or in a series of suitably chosen polynomials in t. It is easily seen that

$$\left(\frac{\partial \Lambda}{\partial t}\right)_{t_o} = 0,$$

therefore the power series reads

$$\Lambda(t,x_o) = x_o + \frac{1}{2}\left(\frac{\partial^2 \Lambda(t,x_o)}{\partial t^2}\right)_{t_o} (t-t_o)^2 + \ldots,$$

i.e., the linear term vanishes. This indicates that $\lambda \approx x_o$ is a good approximation in some neighborhood of t_o.

The coordinates x of the restricted three-body problem can now be written as

$$x = x(t,x_o) = W(t,R(t)\Lambda(t,x_o))$$

and, solved for λ,

$$\lambda = \lambda(t,x_o) = R(-t)W(-t,x(t,x_o)).$$

The formulas for the coordinates x in the restricted problem show - as one expects of course from the construction -, that x is represented by the solution W of Euler's problem with varying initial values. Thus, the trajectory of the restricted problem is not approximated in the usual manner by one solution of Euler's problems with the same fixed initial values, but by a one-dimensional point set of a one parameter family of solutions of Euler's problem.

The differential equation for λ is of a very complicated form and it has not been tried yet to approximate its solution analytically. However, the last formula can be used - and it has been used extensively - to compute λ numerically and to fit it then by suitably chosen functions. The results are very promising so far, and the numerical computations show clearly that λ is a slowly varying function of time for the first half (outbound leg) of a circumlunar trajectory; λ is also a slowly varying function for small changes in the initial values, so that families of trajectories can be represented by one expression for λ.