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# Thermodynamic Extrapolation of Rocket Performance Parameters

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First and second partial derivatives of some thermodynamically defined rocket performance parameters are presented. These derivatives are useful for the extrapolation and interpolation of thermodynamic performance calculations for changes in combustion pressure, combustion enthalpy, pressure ratio, and area ratio. The accuracy of the extrapolation or interpolation is indicated for a typical propellant combination.

FOR SOME propellant systems, performance data are needed over a wide range of conditions such as chamber pressure, pressure ratio or area ratio, oxidant to fuel ratio, and, occasionally, initial enthalpy. The thermodynamic calculation of rocket performance is usually sufficiently difficult to require the use of automatic computers. Because of the cost and time involved in the computations, it is often not feasible to explore the complete range of independent variables. For this reason, the problem of extrapolating or interpolating a limited number of performance calculations occurs frequently.

The use of partial derivatives in extrapolating performance data with considerable accuracy for moderate changes in the independent variables was discussed previously by Gordon and Huff (1).<sup>3</sup> In Ref. 1, expressions were derived for first partial derivatives of the logarithm of specific impulse, characteristic velocity, and area ratio with respect to the logarithm of chamber pressure. In addition, first- and second-order corrections to impulse for a change in combustion enthalpy were also given. A similar first-order correction for the effect of combustion enthalpy (or heat of formation) on impulse was described by Gordon (2).

In this report, the list of partial derivatives is extended to include the derivatives of specific impulse  $I$ , specific impulse in vacuum  $I_{vac}$ , area per unit mass flow rate  $A/w$ , area ratio  $\epsilon$ , and characteristic velocity  $c^*$  with respect to combustion pressure  $P_c$ , combustion enthalpy  $h_c$ , pressure ratio  $P_c/P$ , and area ratio  $\epsilon$ . Examples are given to illustrate the use of the derivatives.

Analytical expressions for the derivatives of parameters with respect to oxidant to fuel ratio  $O/F$  can be derived; however, the method of derivation of these expressions is somewhat different from that used to obtain derivatives for a fixed  $O/F$ , and, therefore will not be presented in this paper. All derivatives in this paper are for a fixed  $O/F$ .

## Effect of Change in Initial Conditions on End Point of a Process

All the thermodynamic properties of a system of known composition can be specified uniquely in terms of any two thermodynamic functions, say  $\alpha$  and  $\beta$ . Thus the system can be represented by a point in a two-dimensional space with coordinates  $(\alpha, \beta)$ . At any point in  $(\alpha, \beta)$  space, not only are all the thermodynamic properties of the system known, but it is also possible to determine the rate of change of these properties along some curve in  $(\alpha, \beta)$  space. Thus if  $\psi$  is a third thermodynamic function, the derivative  $(\partial\beta/\partial\alpha)_\psi$  expresses the rate of change of  $\beta$  with respect to a change in  $\alpha$  along a curve of constant  $\psi$ . This partial derivative is the usual thermodynamic first partial derivative discussed in thermodynamic textbooks.

## Thermodynamic Processes

A process in thermodynamics means that a system originally at some point  $(\alpha_0, \beta_0)$  has been moved to a new point  $(\alpha, \beta)$ . An infinitesimal process can be completely characterized by a derivative of the form  $(\partial\beta/\partial\alpha)_\psi$ . A finite process can be specified by giving a starting point  $(\alpha_0, \beta_0)$ , a path, say a curve of constant  $\psi$ , and one of the coordinates of the end point  $\alpha$ . For a given path, the only independent variables of the process are the coordinates of the initial point  $(\alpha_0, \beta_0)$  and a coordinate, say  $\alpha$ , of the final point. The other

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coordinate of the final point,  $\beta$ , and all other thermodynamic functions are dependent variables in the process.

Let  $\varphi$  be any dependent thermodynamic variable associated with the end point of a finite process. The rates of change of  $\varphi$  with respect to the independent thermodynamic variables of the process are of two different types. For a change in  $\alpha$ , the usual type of derivative  $(\partial\varphi/\partial\alpha)_\psi$  is needed. However, for a change in one of the coordinates of the initial points, derivatives of the type  $(\partial\varphi/\partial\alpha_0)_{\beta_0}$  and  $(\partial\varphi/\partial\beta_0)_{\alpha_0}$  are needed. Expressions for the latter type of derivative are obtained in the following section.

### Derivatives Associated With a Finite Process

As indicated in the previous section, the process is defined as taking place along a curve of constant  $\psi$ . Therefore

$$\psi_0(\alpha_0, \beta_0) = \psi(\alpha, \beta) = \psi(\alpha, \varphi) \quad [1]$$

Whatever change in  $\psi_0$  results from a change in the initial point  $(\alpha_0, \beta_0)$  must be equal to the change in  $\psi$ . This may be expressed in differential form as

$$\left(\frac{\partial\psi_0}{\partial\alpha_0}\right) d\alpha_0 + \left(\frac{\partial\psi_0}{\partial\beta_0}\right) d\beta_0 = \left(\frac{\partial\psi}{\partial\alpha}\right) d\alpha + \left(\frac{\partial\psi}{\partial\varphi}\right) d\varphi \quad [2]$$

Imposing in turn the conditions of constant  $\alpha_0$  and constant  $\beta_0$  gives the following desired expressions for the partial derivatives of a function at the end point of a process with respect to the initial coordinates:

$$\left(\frac{\partial\varphi}{\partial\beta_0}\right)_{\alpha_0} = \frac{(\partial\psi_0/\partial\beta_0)_{\alpha_0}}{(\partial\psi/\partial\varphi)_\alpha} + \left(\frac{\partial\varphi}{\partial\alpha}\right)_\psi \left(\frac{\partial\alpha}{\partial\beta_0}\right)_{\alpha_0} \quad [3]$$

$$\left(\frac{\partial\varphi}{\partial\alpha_0}\right)_{\beta_0} = \frac{(\partial\psi_0/\partial\alpha_0)_{\beta_0}}{(\partial\psi/\partial\varphi)_\alpha} + \left(\frac{\partial\varphi}{\partial\alpha}\right)_\psi \left(\frac{\partial\alpha}{\partial\alpha_0}\right)_{\beta_0} \quad [4]$$

In Eqs. [3 and 4] all the derivatives except  $(\partial\alpha/\partial\beta_0)_{\alpha_0}$  and  $(\partial\alpha/\partial\alpha_0)_{\beta_0}$  are the standard thermodynamic first derivatives and can be immediately evaluated. The two exceptions can be evaluated for a specified form of the relation

$$\alpha = \alpha(\alpha_0, \beta_0) \quad [5]$$

Two forms of Eq. [5] are considered in this paper:  $\alpha = k_1$  and  $\alpha = k_2\alpha_0$ , where  $k_1$  and  $k_2$  are constants. For  $\alpha = k_1$ , Eqs. [3 and 4] reduce to

$$\left(\frac{\partial\varphi}{\partial\beta_0}\right)_{\alpha_0, \alpha} = \left(\frac{\partial\psi_0}{\partial\beta_0}\right)_{\alpha_0} / \left(\frac{\partial\psi}{\partial\varphi}\right)_\alpha \quad [6]$$

$$\left(\frac{\partial\varphi}{\partial\alpha_0}\right)_{\beta_0, \alpha} = \left(\frac{\partial\psi_0}{\partial\alpha_0}\right)_{\beta_0} / \left(\frac{\partial\psi}{\partial\varphi}\right)_\alpha \quad [7]$$

For the particular choice  $\alpha = k_2\alpha_0$ , Eqs. [3 and 4] give

$$\left(\frac{\partial\varphi}{\partial\beta_0}\right)_{\alpha_0, \alpha_0/\alpha} = \left(\frac{\partial\psi_0}{\partial\beta_0}\right)_{\alpha_0} / \left(\frac{\partial\psi}{\partial\varphi}\right)_\alpha \quad [8]$$

$$\left(\frac{\partial\varphi}{\partial\alpha_0}\right)_{\beta_0, \alpha_0/\alpha} = \left(\frac{\partial\psi_0}{\partial\alpha_0}\right)_{\beta_0} / \left(\frac{\partial\psi}{\partial\varphi}\right)_\alpha + \frac{\alpha}{\alpha_0} \left(\frac{\partial\varphi}{\partial\alpha}\right)_\psi \quad [9]$$

It may be seen that the right-hand sides of Eqs. [6 and 8] are identical.

### Application to Rocket Performance

#### First Derivatives of Thermodynamic Functions With Respect to $P_c$ , $h_c$ , and $P_c/P$ for an Isentropic Process

In this paper, Eqs. [6-9] are applied to an isentropic process. For an isentropic expansion from  $P_c$  and  $h_c$  to an exit pressure  $P$ , the following correspondences to functions in Eqs. [6-9] apply:  $s_c = \psi_0$ ,  $P_c = \alpha_0$ ,  $P = \alpha$ ,  $h_c = \beta_0$ , and  $s = \psi$ . From Eqs. [6 and 8]

$$\begin{aligned} \left(\frac{\partial\varphi}{\partial h_c}\right)_{P_c, P} &= \left(\frac{\partial\varphi}{\partial h_c}\right)_{P_c, P_c/P} \\ &= \left(\frac{\partial s_c}{\partial h_c}\right)_{P_c} / \left(\frac{\partial s}{\partial\varphi}\right)_P = \frac{1}{T_c(\partial s/\partial\varphi)_P} \end{aligned} \quad [10]$$

From Eq. [7]

$$\left(\frac{\partial\varphi}{\partial P_c}\right)_{h_c, P} = \left(\frac{\partial s_c}{\partial P_c}\right)_{h_c} / \left(\frac{\partial s}{\partial\varphi}\right)_P = \frac{-v_c}{T_c} / \left(\frac{\partial s}{\partial\varphi}\right)_P \quad [11]$$

From Eq. [9]

$$\begin{aligned} \left(\frac{\partial\varphi}{\partial P_c}\right)_{h_c, P_c/P} &= \left(\frac{\partial s_c}{\partial P_c}\right)_{h_c} / \left(\frac{\partial s}{\partial\varphi}\right)_P + \frac{P}{P_c} \left(\frac{\partial\varphi}{\partial P}\right)_s \\ &= \frac{-v_c}{T_c} / \left(\frac{\partial s}{\partial\varphi}\right)_P + \frac{P}{P_c} \left(\frac{\partial\varphi}{\partial P}\right)_s \end{aligned} \quad [12]$$

For any  $\varphi$  selected, the right-hand sides of Eqs. [10-12] are in terms of the usual first partial derivatives. Bridgman (3) presents a convenient scheme (given in many thermodynamic textbooks) for expressing all first partial derivatives in terms of three first partial derivatives, namely,  $(\partial h/\partial T)_P = c_p$ ,  $(\partial v/\partial T)_P$ , and  $(\partial v/\partial P)_T$ . An equation of state is required to evaluate these derivatives. In rocket performance calculations it is convenient to assume that the reaction products obey an ideal equation of state with a variable molecular weight

$$Pv = RT/M \quad [13]$$

It has been found useful in performance calculations to calculate molecular weight derivatives rather than volume derivatives to indicate changes in composition. From Eq. [13] there follows

$$\left(\frac{\partial v}{\partial T}\right)_P = \frac{v}{T} \left[ 1 - \left(\frac{\partial \ln M}{\partial \ln T}\right)_P \right] \quad [14]$$

$$\left(\frac{\partial v}{\partial P}\right)_T = \frac{-v}{P} \left[ 1 + \left(\frac{\partial \ln M}{\partial \ln P}\right)_T \right] \quad [15]$$

A further discussion of the molecular weight derivatives and a method for their numerical evaluation is given by Gordon et al. (4). With the aid of Bridgman's tables (3) and Eqs. [13-15], all other first partial derivatives can be expressed in terms of  $c_p$ ,  $(\partial \ln M/\partial \ln T)_P$  and  $(\partial \ln M/\partial \ln P)_T$ . As an example, assuming  $\varphi = T$ , Eq. [12] becomes

$$\begin{aligned} \left(\frac{\partial T}{\partial P_c}\right)_{h_c, P_c/P} &= \frac{-v_c}{T_c} / \left(\frac{\partial s}{\partial T}\right)_P + \frac{P}{P_c} \left(\frac{\partial T}{\partial P}\right)_s \\ &= \frac{-v_c}{T_c} \frac{T}{c_p} + \frac{P}{P_c} \frac{T}{c_p} \left(\frac{\partial v}{\partial T}\right)_P \\ &= \frac{-R}{P_c M c_p} \frac{T}{c_p} + \frac{RT}{P_c M c_p} \left[ 1 - \left(\frac{\partial \ln M}{\partial \ln T}\right)_P \right] \end{aligned} \quad [16]$$

In a manner similar to the previous example, derivatives were obtained from Eqs. [10-12] for  $\varphi = T$ ,  $h$ , and  $\rho$ . Expressions for the logarithmic form of these derivatives are given in Table 1. The logarithmic form was selected to present the results in dimensionless form. Expressions for molecular weight derivatives are also given in Table 1. These are obtained by considering molecular weight to be a function of temperature and pressure and expanding by the chain rule in differentiation. Derivatives of the form  $[\partial \ln \varphi / \partial \ln (P_c/P)]_{P_c, h_c} = [\partial \ln \varphi / \partial \ln (P_c/P)]_s = -(\partial \ln \varphi / \partial \ln P)_s$  are also given in Table 1. Henceforth, derivatives at constant  $P_c$  and  $h_c$  will be shown as derivatives at constant  $s$ . The use of these derivatives will be discussed in a later section.

A word of caution is required about the derivatives with respect to  $\ln h_c$ . If a reference level for enthalpy is chosen so that  $h_c$  is negative, the logarithmic derivative must be replaced by a derivative with respect to  $h_c$ .

Table 1 First derivatives of some thermodynamic properties and rocket performance parameters

$\lambda$	$\left(\frac{\partial \ln \lambda}{\partial \ln P_c}\right)_{P_c/P, h_c}$	$\left(\frac{\partial \ln \lambda}{\partial \ln h_c}\right)_{P_c/P, P_c}$	$\left(\frac{\partial \ln \lambda}{\partial \ln P_c}\right)_{h_c, P}$	$\left(\frac{\partial \ln \lambda}{\partial \ln P_c}\right)_{P_c, h_c} = -\left(\frac{\partial \ln \lambda}{\partial \ln P}\right)_s$
$T$	$\frac{R}{c_p M} \left[ 1 - \left(\frac{\partial \ln M}{\partial \ln T}\right)_P \right] - \frac{R}{c_p M_c}$	$\frac{h_c}{c_p T_c}$	$-\frac{R}{c_p M_c}$	$\frac{R}{c_p M} \left[ \left(\frac{\partial \ln M}{\partial \ln T}\right)_P - 1 \right]$
$h$	$\frac{R T}{h} \left( \frac{1}{M} - \frac{1}{M_c} \right)$	$\frac{h_c T}{h T_c}$	$-\frac{R T}{h M_c}$	$-\frac{R T}{h M}$
$\rho$	$\frac{R}{c_p M_c} \left[ 1 - \left(\frac{\partial \ln M}{\partial \ln T}\right)_P \right] + \frac{1}{\gamma}$	$-\frac{h_c}{c_p T_c} \left[ 1 - \left(\frac{\partial \ln M}{\partial \ln T}\right)_P \right]$	$\frac{R}{M c_p} \left[ 1 - \left(\frac{\partial \ln M}{\partial \ln T}\right)_P \right]$	$-\left(\frac{\partial \ln \rho}{\partial \ln P}\right)_s = -\frac{1}{\gamma}$
$M$	$\left(\frac{\partial \ln M}{\partial \ln T}\right)_P \left(\frac{\partial \ln P_c}{\partial \ln P_c}\right)_{P_c/P, h_c} + \left(\frac{\partial \ln M}{\partial \ln P}\right)_T$	$\left(\frac{\partial \ln M}{\partial \ln T}\right)_P \left(\frac{\partial \ln h_c}{\partial \ln h_c}\right)_{P_c/P, P_c}$	$\left(\frac{\partial \ln M}{\partial \ln T}\right)_P \left(\frac{\partial \ln P_c}{\partial \ln P_c}\right)_{h_c, P}$	$\left(\frac{\partial \ln M}{\partial \ln T}\right)_P \left(\frac{\partial \ln P_c}{\partial \ln P_c}\right)_s - \left(\frac{\partial \ln M}{\partial \ln P}\right)_T$
$I$	$\frac{R T}{2(h_c - h)} \left( \frac{1}{M_c} - \frac{1}{M} \right)$	$\frac{h_c}{2 T_c} \left( \frac{T_c - T}{h_c - h} \right)$	$\frac{R T}{2(h_c - h) M_c}$	$\frac{R T}{2(h_c - h) M}$
$A/w$	$-\left(\frac{\partial \ln I}{\partial \ln P_c}\right)_{P_c/P, h_c} - \left(\frac{\partial \ln \rho}{\partial \ln P_c}\right)_{P_c/P, h_c}$	$-\left(\frac{\partial \ln I}{\partial \ln h_c}\right)_{P_c/P, P_c} - \left(\frac{\partial \ln \rho}{\partial \ln h_c}\right)_{P_c/P, P_c}$	$-\left(\frac{\partial \ln I}{\partial \ln P_c}\right)_{h_c, P} - \left(\frac{\partial \ln \rho}{\partial \ln P_c}\right)_{h_c, P}$	$-\left(\frac{\partial \ln I}{\partial \ln P_c}\right)_s - \left(\frac{\partial \ln \rho}{\partial \ln P_c}\right)_s$
$\epsilon$	$\left(\frac{\partial \ln A/w}{\partial \ln P_c}\right)_{P_c/P, h_c} - \left[\frac{\partial \ln A/w}{\partial \ln P_c}\right]_{P_c/P, h_c}$	$\left(\frac{\partial \ln A/w}{\partial \ln h_c}\right)_{P_c/P, P_c} - \left[\frac{\partial \ln A/w}{\partial \ln h_c}\right]_{P_c/P, P_c}$	$\left(\frac{\partial \ln A/w}{\partial \ln P_c}\right)_{h_c, P} - \left[\frac{\partial \ln A/w}{\partial \ln P_c}\right]_{h_c, P}$	$\left(\frac{\partial \ln A/w}{\partial \ln P_c}\right)_s$
$c^*$	$1 + \left[\frac{\partial \ln A/w}{\partial \ln P_c}\right]_{P_c/P, h_c}$	$\left[\frac{\partial \ln A/w}{\partial \ln h_c}\right]_{P_c/P, P_c}$	$1 + \left[\frac{\partial \ln A/w}{\partial \ln P_c}\right]_{h_c, P}$	0
$I_{vac}$	$\frac{1}{I_{vac}} \left\{ I \left(\frac{\partial \ln I}{\partial \ln P_c}\right)_{P_c/P, h_c} + \frac{P A}{w} \left[ 1 + \left(\frac{\partial \ln A/w}{\partial \ln P_c}\right)_{P_c/P, h_c} \right] \right\}$	$\frac{1}{I_{vac}} \left\{ I \left(\frac{\partial \ln I}{\partial \ln h_c}\right)_{P_c/P, P_c} + \frac{P A}{w} \left[ 1 + \left(\frac{\partial \ln A/w}{\partial \ln h_c}\right)_{P_c/P, P_c} \right] \right\}$	$\frac{1}{I_{vac}} \left\{ I \left(\frac{\partial \ln I}{\partial \ln P_c}\right)_{h_c, P} + \frac{P A}{w} \left[ 1 + \left(\frac{\partial \ln A/w}{\partial \ln P_c}\right)_{h_c, P} \right] \right\}$	$\frac{1}{I_{vac}} \left\{ I \left(\frac{\partial \ln I}{\partial \ln P_c}\right)_s + \frac{P A}{w} \left[ 1 + \left(\frac{\partial \ln A/w}{\partial \ln P_c}\right)_s \right] \right\}$
$c_F$	$\left(\frac{\partial \ln I}{\partial \ln P_c}\right)_{P_c/P, h_c} - \left(\frac{\partial \ln c^*}{\partial \ln P_c}\right)_{P_c/P, h_c}$	$\left(\frac{\partial \ln I}{\partial \ln h_c}\right)_{P_c/P, P_c} - \left(\frac{\partial \ln c^*}{\partial \ln h_c}\right)_{P_c/P, P_c}$	$\left(\frac{\partial \ln I}{\partial \ln P_c}\right)_{h_c, P} - \left(\frac{\partial \ln c^*}{\partial \ln P_c}\right)_{h_c, P}$	$\left(\frac{\partial \ln I}{\partial \ln P_c}\right)_s$

Table 2 Second partial derivatives of  $I^2$ 

$$\left[ \frac{\partial(\partial I^2/\partial \ln P_c)_{h_c, P_c/P}}{\partial \ln P_c} \right]_{h_c, P_c/P} = 2I^2 \left\{ \left( \frac{\partial \ln I}{\partial \ln P_c} \right)_{h_c, P_c/P} \left( \frac{\partial \ln T}{\partial \ln P_c} \right)_{h_c, P_c/P} - \left( \frac{\partial \ln I}{\partial \ln P_c/P} \right)_s \left[ \frac{M}{M_c} \left( \frac{\partial \ln M_c}{\partial \ln P_c} \right)_{h_c} - \left( \frac{\partial \ln M}{\partial \ln P_c} \right)_{h_c, P_c/P} \right] \right\}^a$$

$$\left[ \frac{\partial(\partial I^2/\partial \ln P_c)_{h_c, P_c/P}}{\partial h_c} \right]_{P_c, P_c/P} = \left[ \frac{\partial(\partial I^2/\partial h_c)_{P_c, P_c/P}}{\partial \ln P_c} \right]_{h_c, P_c/P} = \frac{I^2}{h_c - h} \frac{T}{T_c} \left[ \left( \frac{\partial \ln T_c}{\partial \ln P_c} \right)_{h_c} - \left( \frac{\partial \ln T}{\partial \ln P_c} \right)_{h_c, P_c/P} \right]^b$$

$$\left[ \frac{\partial(\partial I^2/\partial \ln P_c)_{h_c, P_c/P}}{\partial \ln P_c/P} \right]_s = \left[ \frac{\partial(\partial I^2/\partial \ln P_c/P)_s}{\partial \ln P_c} \right]_{h_c, P_c/P} = 2I^2 \left( \frac{\partial \ln I}{\partial \ln (P_c/P)} \right)_s \left[ \left( \frac{\partial \ln T}{\partial \ln P_c} \right)_{h_c, P_c/P} - \left( \frac{\partial \ln M}{\partial \ln P_c} \right)_{h_c, P_c/P} \right]$$

$$\left[ \frac{\partial(\partial I^2/\partial h_c)_{P_c, P_c/P}}{\partial h_c} \right]_{P_c, P_c/P} = \frac{I^2}{h_c - h} \frac{T}{T_c^2} \left( \frac{1}{c_{Pc}} - \frac{1}{c_P} \right)$$

$$\left[ \frac{\partial(\partial I^2/\partial h_c)_{P_c, P_c/P}}{\partial \ln P_c/P} \right]_s = \left[ \frac{\partial(\partial I^2/\partial \ln P_c/P)_s}{\partial h_c} \right]_{P_c, P_c/P} = \frac{I^2}{h_c - h} \frac{RT}{M c_P T_c} \left[ 1 - \left( \frac{\partial \ln M}{\partial \ln T} \right)_P \right]$$

$$\left[ \frac{\partial(\partial I^2/\partial \ln P_c/P)_s}{\partial \ln P_c/P} \right]_s = 2I^2 \left( \frac{1 - \gamma}{\gamma} \right) \left( \frac{\partial \ln I}{\partial \ln P_c/P} \right)_s$$

$$^a \left( \frac{\partial \ln M_c}{\partial \ln P_c} \right)_{h_c} = \frac{2R}{M_c c_{Pc}} \left[ 1 - \left( \frac{\partial \ln M_c}{\partial \ln T_c} \right)_{P_c} \right] - \frac{R}{M_c c_{Pc}} - \frac{\gamma_c - 1}{\gamma_c}$$

$$^b \left( \frac{\partial \ln T_c}{\partial \ln P_c} \right)_{h_c} = - \frac{R}{M_c c_{Pc}} \left( \frac{\partial \ln M_c}{\partial \ln T_c} \right)_{P_c}$$

### Derivatives of Rocket Performance Parameters With Respect to $P_c$ , $h_c$ , and $P_c/P$ for Isentropic Expansion

Parameters that are usually of interest in performance calculations are defined in the following equations, where it is assumed that a consistent set of units is employed:

$$I = [2(h_c - h)]^{1/2} \quad [17]$$

$$(A/w) = 1/\rho u = 1/\rho I \quad [18]$$

$$\epsilon = (A/w)/(A/w)_i = A/A_i \quad [19]$$

$$c^* = P_c(A/w)_i \quad [20]$$

$$I_{\text{vac}} = I + (A/w)P \quad [21]$$

$$C_F = I/c^* \quad [22]$$

A specific set of units and associated dimensional constants for Eqs. [17–22] is given in Ref. 4.

Derivatives of the logarithm of  $I$  can be obtained directly from the derivatives of the logarithm of  $h$  in Table 1. Thus, for example

$$\left( \frac{\partial \ln I}{\partial \ln P_c} \right)_{P_c/P, h_c} = \frac{-h}{I^2} \left( \frac{\partial \ln h}{\partial \ln P_c} \right)_{P_c/P, h_c} \quad [23]$$

Eqs. [18–22] can be used to obtain derivatives of the remaining parameters in terms of the derivatives of the logarithm  $I$  and the thermodynamic functions. The results are included in Table 1.

### Second Derivatives of Specific Impulse With Respect to $P_c$ , $h_c$ , and $P_c/P$

An examination of Table 1 shows that the expressions for the derivatives of the logarithm of  $I$  include only the thermodynamic functions  $h$ ,  $T$ , and  $M$ . Therefore, the second derivative of the logarithm of  $I$  can be obtained by the methods previously discussed. However, for purposes of extrapolation,  $I^2$  is a better form than  $\ln I$ . (This will be discussed further in a later section.) The first and second derivatives of  $I^2$  can be expressed in terms of logarithmic derivatives as follows:

$$\frac{\partial I^2}{\partial y} = 2I^2 \frac{\partial \ln I}{\partial y} \quad [24a]$$

$$\frac{\partial^2 I^2}{\partial x \partial y} = 2I^2 \left[ \frac{\partial^2 \ln I}{\partial x \partial y} + 2 \left( \frac{\partial \ln I}{\partial y} \right) \left( \frac{\partial \ln I}{\partial x} \right) \right] \quad [24b]$$

Detailed expressions for the second partial derivatives of  $I^2$  obtained by means of [24b] are given in Table 2.

The expressions for the first derivatives of the remainder of the performance parameters in Table 1, unlike specific impulse, include first derivatives of thermodynamic functions. The second derivatives of these performance parameters are therefore not included in this paper, inasmuch as they involve second derivatives of thermodynamic functions which are generally not available.

### First Derivatives of Rocket Performance Parameters With Respect to $P_c$ , $h_c$ , and $\epsilon$

The previous discussion of thermodynamic processes showed that a process could be specified by giving the value of two thermodynamic functions at the initial point, a path connecting the final point and initial point, and one of the coordinates of the final point. Thus for a given process, any parameter of the process  $\lambda$  can be considered a function of the coordinates of the initial point and the coordinate of the final point. For the isentropic expansion of gases in a rocket nozzle, it is permissible to write

$$\lambda = \lambda(P_c, h_c, P_c/P) \quad [25]$$

and for area ratio in particular

$$\epsilon = \epsilon(P_c, h_c, P_c/P) \quad [26]$$

Eliminating  $P_c/P$  between the previous two equations gives

$$\lambda = \lambda(P_c, h_c, \epsilon) \quad [27]$$

where the functional forms of Eqs. [25 and 27] are not the same.

If the functional relation in Eq. [25] were known, then this would provide an alternate method of obtaining the first, second, and fourth columns of derivatives in Table 1, i.e.,  $(\partial \ln \lambda / \partial \ln P_c)_{P_c/P, h_c}$ ,  $(\partial \ln \lambda / \partial \ln h_c)_{P_c/P, P_c}$ , and  $[\partial \ln \lambda / \partial \ln (P_c/P)]_s$ . Similarly, if the functional relation in Eq. [27] were known, the partial derivatives  $(\partial \ln \lambda / \partial \ln P_c)_{\epsilon, h_c}$ ,  $(\partial \ln \lambda / \partial \ln h_c)_{\epsilon, P_c}$ , and  $(\partial \ln \lambda / \partial \ln \epsilon)_s$  could be obtained. However, expressions for these last three derivatives need not be

Table 3 Theoretical rocket performance assuming equilibrium composition during expansion

FUEL OXIDANT		H 2 O 2		CHEMICAL FORMULA	ENTHALPY CAL/MOL												STATE		TEMP DEG K		O/F= 7.936508, PERCENT FUEL= 11.1901, EQUIVALENCE RATIO= 1.0000																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																						
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obtained explicitly inasmuch as they can be related to the first set of three derivatives as shown in the next paragraph.

The total differentials of Eqs. [25-27] are

$$d \ln \lambda = \left( \frac{\partial \ln \lambda}{\partial \ln P_c} \right)_{P_c/P, h_c} d \ln P_c + \left( \frac{\partial \ln \lambda}{\partial \ln h_c} \right)_{P_c/P, P_c} \times d \ln h_c + \left[ \frac{\partial \ln \lambda}{\partial \ln (P_c/P)} \right]_s d \ln P_c/P \quad [28]$$

$$d \ln \epsilon = \left( \frac{\partial \ln \epsilon}{\partial \ln P_c} \right)_{P_c/P, h_c} d \ln P_c + \left( \frac{\partial \ln \epsilon}{\partial \ln h_c} \right)_{P_c/P, P_c} \times d \ln h_c + \left[ \frac{\partial \ln \epsilon}{\partial \ln (P_c/P)} \right]_s d \ln P_c/P \quad [29]$$

$$d \ln \lambda = \left( \frac{\partial \ln \lambda}{\partial \ln P_c} \right)_{\epsilon, h_c} d \ln P_c + \left( \frac{\partial \ln \lambda}{\partial \ln h_c} \right)_{\epsilon, P_c} \times d \ln h_c + \left( \frac{\partial \ln \lambda}{\partial \ln \epsilon} \right)_s d \ln \epsilon \quad [30]$$

Eliminating  $d \ln \epsilon$  in Eq. [30] with Eq. [29] and then comparing the coefficients of the differentials with those of Eq. [28] with some rearrangement, the following relations give:

$$\left( \frac{\partial \ln \lambda}{\partial \ln P_c} \right)_{\epsilon, h_c} = \left( \frac{\partial \ln \lambda}{\partial \ln P_c} \right)_{P_c/P, h_c} - \left[ \frac{\partial \ln \lambda}{\partial \ln (P_c/P)} \right]_s \times \left( \frac{\partial \ln \epsilon / \partial \ln P_c}{\partial \ln \epsilon / \partial \ln (P_c/P)} \right)_{P_c/P, h_c} \quad [31]$$

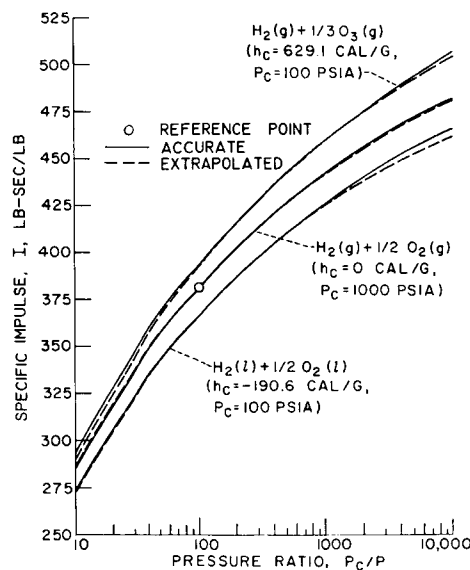


Fig. 1 Comparison of accurately calculated specific impulse data for stoichiometric  $H_2-O_2$  or  $H_2-O_3$  with data extrapolated from the reference point

$$\left(\frac{\partial \ln \lambda}{\partial \ln h_c}\right)_{\epsilon, P_c} = \left(\frac{\partial \ln \lambda}{\partial \ln h_c}\right)_{P_c/P, P_c} - \left(\frac{\partial \ln \lambda}{\partial \ln (P_c/P)}\right)_s \times \left(\frac{\partial \ln \epsilon / \partial \ln h_c}{\partial \ln \epsilon / \partial \ln (P_c/P)}\right)_s \quad [32]$$

$$\left(\frac{\partial \ln \lambda}{\partial \ln \epsilon}\right)_s = \left[\frac{\partial \ln \lambda / \partial \ln (P_c/P)}{\partial \ln \epsilon / \partial \ln (P_c/P)}\right]_s \quad [33]$$

#### Second Derivative of Impulse With Respect to Area Ratio

An approximate second derivative of impulse with respect to area ratio for constant  $P_c$  and  $h_c$  can be obtained by differentiation of [33], assuming that  $[\partial \ln \epsilon / \partial \ln (P_c/P)]_s$  is constant. This is a very good approximation for pressure ratios greater than 10. With this assumption, there follows

$$\left[\frac{\partial^2 (I^2)}{\partial (\ln \epsilon)^2}\right]_s = \frac{2I^2 \left(\frac{1-\gamma}{\gamma}\right) \left[\frac{\partial \ln I}{\partial \ln (P_c/P)}\right]_s}{\left[\frac{\partial \ln \epsilon}{\partial \ln (P_c/P)}\right]_s^2} \quad [34]$$

#### Partial Derivatives for Extrapolation

The first and second partial derivatives given in Tables 1 and 2 can be used for extrapolation purposes by assuming that the change in any one of the parameters  $\lambda$  or functions  $\varphi$  can be obtained from a truncated Taylor series. Thus, for any function  $f$ , neglecting derivatives higher than the second

Table 5 Comparison of accurately calculated characteristic velocity data for stoichiometric  $H_2-O_2$  with data extrapolated from reference point

$P_c$ , psia	$h_c = 0$ cal/g Accu- rate	$h_c = -190.6$ cal/g Extrapo- lated	$h_c = 629.1$ cal/g Accu- rate	$h_c = 629.1$ cal/g Extrapo- lated	$h_c = 629.1$ cal/g Accu- rate	$h_c = 629.1$ cal/g Extrapo- lated
1000	7208 <sup>a</sup>	7208	7075	7077	7625	7641
600	7165	7166	7035	7034	7573	7599
100	7008	7017	6887	6886	7386	7451

<sup>a</sup> Reference point for extrapolation.

$$f(x_i) = f(x_i^0) + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^0 (x_i - x_i^0) + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^0 (x_i - x_i^0)(x_j - x_j^0) \quad [35]$$

When only first derivatives are available, the last term in Eq. [35] is omitted.

The accuracy obtained with Eq. [35] depends considerably on the form of  $f$  and  $x_i$ . Those forms whose first derivatives are the most nearly constant over the range of interest can be expected to give the most accurately extrapolated values.

An indication of the desirable forms for  $f$  and  $x_i$  can be obtained by an examination of some performance data and derivatives. In Table 3, data are given for stoichiometric  $H_2(g)-O_2(g)$  for  $P_c = 1000$  psia assuming equilibrium composition during isentropic expansion. Table 3 is the direct output from an IBM 704 program.

The symbols used in Table 3 for some parameters are somewhat different from those used in the rest of this paper because the IBM printer does not contain characters such as lower case letters, Greek letters, subscripts, or superscripts. The following examples illustrate the differences:

$$(DLI/DLPC)PC/P = \left(\frac{\partial \ln I}{\partial \ln P_c}\right)_{P_c/P, h_c}$$

$$(DLCS/DHC)PC/P = \left(\frac{\partial \ln c^*}{\partial \ln h_c}\right)_{P_c/P, P_c}$$

$$(DLAR/DLPCP)S = \left[\frac{\partial \ln \epsilon}{\partial \ln (P_c/P)}\right]_s$$

It can be seen from Table 3 that the derivatives  $[\partial \ln \epsilon / \partial \ln (P_c/P)]_s$  are very nearly constant over a considerable range of pressure ratio. It is to be expected, therefore, that a good form of Eq. [35] for extrapolating area ratios to other pressure ratios is

$$\ln \epsilon = (\ln \epsilon)^0 + \left[\frac{\partial \ln \epsilon}{\partial \ln (P_c/P)}\right]_s^0 \left[\ln \left(\frac{P_c}{P}\right) - \ln \left(\frac{P_c}{P}\right)^0\right] \quad [36]$$

In contrast, the large differences in the values for  $[\partial \ln I / \partial \ln (P_c/P)]_s$  indicate that using  $\ln I$  for  $f$  and  $\ln (P_c/P)$  for  $x$  in Eq. [35] would not give particularly good results. A more

Table 4 Comparison of accurately calculated area ratio data for stoichiometric  $H_2-O_2$  with data extrapolated from reference point

$P_c/P$	$h_c = 0.0$ cal/g $P_c = 1000$ psia		$h_c = -190.6$ cal/g $P_c = 100$ psia		$h_c = 629.1$ cal/g $P_c = 100$ psia	
	Accurate	Extrapolated	Accurate	Extrapolated	Accurate	Extrapolated
10.0	2.440	2.228	2.468	2.289	2.466	2.337
40.83	7.009	6.941	7.151	7.130	7.169	7.279
68.05	10.50	10.49	10.75	10.77	10.81	11.00
100.0	14.31 <sup>a</sup>	14.31	14.69	14.70	14.81	15.01
300.0	35.05	34.75	36.28	35.70	37.02	36.45
400.0	44.39	43.84	46.04	45.04	47.22	45.98
600.0	61.92	60.83	64.44	62.49	66.66	63.80
800.0	78.38	76.75	81.77	78.83	85.21	80.49
1000.0	94.08	91.90	98.31	94.40	103.2	96.38

<sup>a</sup> Reference point for extrapolation.

nearly constant set of derivatives can be obtained in this case by multiplying the logarithmic derivatives by  $I$ , i.e.

$$\left[ \frac{\partial I}{\partial \ln (P_c/P)} \right]_s = I \left[ \frac{\partial \ln I}{\partial \ln (P_c/P)} \right]_s \quad [37]$$

A still more nearly constant set of derivatives are

$$\left[ \frac{\partial I^2}{\partial \ln (P_c/P)} \right]_s = 2I^2 \left[ \frac{\partial \ln I}{\partial \ln (P_c/P)} \right]_s \quad [38]$$

For specific impulse, therefore, a good form of Eq. [35] for extrapolating to other pressure ratios is

$$I^2 = (I^2)^0 + \left[ \frac{\partial I^2}{\partial \ln (P_c/P)} \right]_s^0 \Delta \ln (P_c/P) + \frac{1}{2} \left[ \frac{\partial^2 I^2}{\partial (\ln P_c/P)^2} \right]_s^0 (\Delta \ln P_c/P)^2 \quad [39]$$

A different situation occurs for temperature. In Table 3 the derivative  $[\partial \ln T / \partial \ln (P_c/P)]_s$  is fairly constant over part of the range of interest (from  $P_c/P = 1$  to about 100), whereas another derivative form  $[\partial T / \partial \ln (P_c/P)]_s$  is more nearly constant over the remainder of the pressure ratio interval. A further complication arises because the derivative of temperature or logarithm of temperature with respect to pressure ratio has a maximum. Extrapolation may, therefore, give very poor results no matter which derivative form is used. For most chemical systems, the derivatives  $[\partial \ln T / \partial \ln (P_c/P)]_s$  are more nearly constant as a function of pressure ratio than are the derivatives  $[\partial T / \partial \ln (P_c/P)]_s$ .

## Numerical Examples of Extrapolation

### Specific Impulse

The use of the first and second derivatives permits the extrapolation of specific impulse data with considerable accuracy. This is illustrated by two numerical examples. Both examples start with the data in Table 3 for a pressure ratio of 100.

#### Example 1

Extrapolate  $I$  from  $P_c/P = 100$  to  $P_c/P = 1000$  ( $\Delta \ln (P_c/P) = 2.30259$ ).

From Table 3,  $I = 382.0$ ,  $[\partial \ln I / \partial \ln (P_c/P)]_s = 0.08396$ , and  $\gamma = 1.1215$ . From Eq. [24],  $[\partial I^2 / \partial \ln (P_c/P)]_s = 24,504$ , whereas from the equation in Table 2,  $[\partial^2 (I^2) / \partial (\ln P_c/P)^2]_s = -2654.6$ . From Eq. [35]

$$I^2 = (382.0)^2 + 24,504 (2.30259) + \frac{1}{2} (-2654.6) (2.30259)^2$$

$$I^2 = 195,309$$

$$I = 441.9$$

This is an excellent agreement with the accurately calculated value of 442.2 given in Table 3.

#### Example 2

Extrapolate  $I$  from  $P_c/P = 100$  to 1000 ( $\Delta \ln (P_c/P) = 2.30259$ ), from  $P_c = 1000$  to 100 psia ( $\Delta \ln P_c = -2.30259$ ), from  $h_c = 0$  to  $-190.6$  cal/g (from gaseous to liquid propellant).

Using the data in Table 3 for combustion chamber conditions and for a pressure ratio of 100, the three first derivatives of  $I^2$  from Eq. [24] are

$$(\partial I^2 / \partial \ln P_c)_{P_c/P, h_c} = 2600.4$$

$$(\partial I^2 / \partial h_c)_{P_c/P, P} = 28,648$$

$$[\partial I^2 / \partial \ln (P_c/P)]_s = 24,504$$

Table 6 Comparison of interpolated performance parameters using two points and two slopes with accurately calculated data

$P_c/P$	Temperature $T$ , °K	Interpolated	Molecular weight $M$			Specific impulse $I$			Interpolated	Area ratio $\epsilon$			Vacuum specific impulse, $I_{vac}$			Thrust coefficient $C_F$		
			Accurate	Error	Interpolated	Accurate	Error	Interpolated		Accurate	Error	Interpolated	Calculated, Eq. [40]	Calculated, Eq. [22]	Calculated, Eq. [22]	Accurate	Error	Calculated, Eq. [22]
100 <sup>a</sup>	2469	0	17.423	0	382.0	382.0	0	14.31	14.31	0	0	144.1	414.1	1.705	1.705	0	0	1.705
300	2218	0	17.701	-0.001	413.8	413.8	0	35.05	35.05	0	0	440.0	440.0	1.847	1.847	0	0	1.847
400	2151	0	17.761	0	421.2	421.2	0	44.35	44.35	-0.04	0	446.1	446.0	1.880	1.880	0	0	1.880
600	2054	-1	17.834	0	431.0	430.9	+0.1	61.83	61.92	-0.09	0	454.1	454.0	1.924	1.923	+0.001	+0.001	1.924
800	1984	-1	17.878	0	437.4	437.4	0	78.31	78.38	-0.07	0	459.4	459.3	1.953	1.952	+0.001	+0.001	1.953
1000 <sup>a</sup>	1930	0	17.907	0	442.0	442.0	0	94.08	94.08	0	0	463.3	463.3	1.974	1.974	0	0	1.974

<sup>a</sup> Points used to obtain interpolated data.

Numerical values for all the quantities that appear in the expressions for the second derivatives in Table 2 are given in Table 3 with the following three exceptions:

$$\left(\frac{\partial \ln M_c}{\partial \ln P_c}\right)_{h_c} = 0.015198 \quad (\text{see footnote, Table 2})$$

$$\left(\frac{\partial \ln T_c}{\partial \ln P_c}\right)_{h_c} = 0.041197 \quad (\text{see footnote, Table 2})$$

$$\left(\frac{\partial \ln M}{\partial \ln P_c}\right)_{P_c/P, h_c} = 0.007821 \quad (\text{see Table 1})$$

The six second derivatives of  $I^2$  are (with subscripts omitted for brevity)

$$\partial^2(I^2)/\partial (\ln P_c)^2 = -182.69$$

$$\partial^2(I^2)/\partial \ln P_c \partial h_c = 1.5610$$

$$\partial^2(I^2)/\partial \ln P_c \partial \ln (P_c/P) = 162.67$$

$$\partial^2(I^2)/\partial h_c^2 = -0.0036205$$

$$\partial^2(I^2)/\partial h_c \partial \ln (P_c/P) = 5.3690$$

$$\partial^2(I^2)/\partial [\ln (P_c/P)]^2 = -2654.6$$

Using the first and second derivatives in Eq. [35] gives

$$\begin{aligned} I^2 = & (382.0)^2 + 2600.4(-2.30259) + 28,648(-0.1906) + \\ & 24,504(2.30259) + \frac{1}{2}(-182.69)(-2.30259)^2 + \\ & \frac{1}{2}(-0.0036205)(-190.6)^2 + \frac{1}{2}(-2654.6)(2.30259)^2 \\ & + 1.5610(-2.30259)(-190.6) + 162.67 \times \\ & (-2.30259)(2.30259) + 5.3690(-190.6)(2.30259) \end{aligned}$$

$$I^2 = 180,778$$

$$I = 425.2$$

This is in very good agreement with an accurately calculated value of 426.5.

Some additional numerical comparisons of accurate data with data extrapolated for various changes in  $P_c$ ,  $h_c$ , and  $P_c/P$  are given in Fig. 1 for specific impulse, in Table 4 for area ratio, and in Table 5 for characteristic velocity. The extrapolated data are all obtained from one accurately calculated reference point corresponding to  $P_c = 1000$  psia,  $h_c = 0$ , and  $P_c/P = 100$ . The derivative forms used were first and second derivatives of  $I^2$ , first derivatives of  $\ln \epsilon$ , and first derivatives of  $c^*$ . It is apparent that the extrapolated values are in excellent agreement with the accurately calculated values for a considerable range of extrapolation.

## Partial Derivatives for Interpolation

Derivatives can be used to increase the accuracy of interpolation in a specified range. This is because each derivative is approximately equivalent to having an additional point in the specified interval. For example, if only functions are known at two points, only linear interpolation is possible. However, if the first derivatives of these functions are also known at the two points, cubic interpolation is possible. With second derivatives also known, quintic interpolation is possible. Thus, for example, the following equation may be obtained:

$$I^2 = A + B[\ln(P_c/P)] + C[\ln(P_c/P)]^2 + D[\ln(P_c/P)]^3 \quad [40]$$

where the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are determined by the solution of four simultaneous equations involving the values of  $I^2$  and first derivatives of  $I^2$ , each at two pressure ratios.

To illustrate the accuracy of interpolation which can be obtained by this technique, values were interpolated for  $T$ ,  $M$ ,  $\epsilon$ , and  $I$  at pressure ratios of 300, 400, 600, and 800 using cubic

equations derived from the data of Table 3 at pressure ratios of 100 and 1000. The functional forms for  $I$  and  $\epsilon$  which gave the best results as functions of the logarithm of  $P_c/P$  are  $I^2$  and  $\ln \epsilon$ . Both the linear and logarithmic forms gave essentially the same results for  $T$  and  $M$ .

The interpolated results are compared in Table 6 with the accurately calculated values of Table 3. Included in Table 6 are values of  $C_F$  and  $I_{vac}$ . Values of  $C_F$  were calculated from  $c^*$  and the interpolated values of  $I$  by means of Eq. [22], using consistent units. Values of  $I_{vac}$  were calculated from  $c^*$  and the interpolated values of  $I$  and  $\epsilon$  by means of an alternate form of Eq. [19]:

$$I_{vac} = I + c^*\epsilon/(P_c/P) \quad [41]$$

again using consistent units. As may be seen from Table 6, the interpolated results are in excellent agreement with the accurate results.

In the case of specific impulse, a quintic equation can be obtained from data at two pressure ratios, inasmuch as both first and second derivatives are available. Specific impulse values interpolated over the entire range of pressure ratios from 10 to 10,000 from a quintic formed from data at these two points were within 0.1 lb-sec/lb of accurately calculated values.

Data are often desired at assigned area ratios. By use of derivatives obtained from Eq. [33], accurately interpolated parameters corresponding to assigned area ratios can be obtained in a manner similar to that just discussed for assigned pressure ratio interpolation.

## Nomenclature

$A, B, C, D$	= coefficients, Eq. [40]
$A/w$	= area per unit mass flow rate
$c^*$	= characteristic velocity
$c_p$	= heat capacity at constant pressure per unit mass
$C_F$	= thrust coefficient
$h$	= enthalpy per unit mass
$I$	= specific impulse, Eq. [17]
$I_{vac}$	= specific impulse in vacuum, Eq. [21]
$k_1, k_2$	= constants
$M$	= molecular weight
$O/F$	= oxidant to fuel weight ratio
$P$	= pressure
$P_c/P$	= pressure ratio
$R$	= universal gas constant, 1.98726 cal/mole-°K
$s$	= entropy per unit mass
$T$	= absolute temperature
$u$	= velocity
$v$	= specific volume
$x, y, z$	= any variable
$\alpha, \beta, \phi, \psi$	= any thermodynamic variable
$\epsilon$	= area ratio
$\lambda$	= process parameter
$\rho$	= density
$\gamma$	= isentropic exponent ( $\partial \ln P / \partial \ln \rho$ ) <sub>s</sub>

## Subscripts

$c$	= combustion chamber
$0$	= initial coordinate of a finite process
$t$	= throat

## Superscript

$0$	= reference condition
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