NASA TN D-1415

TECHNICAL NOTE

D-1415

CALCULATION OF EPHEMERIDES FROM INITIAL VALUES

Karl Stumpff

Goddard Space Flight Center
Greenbelt, Maryland

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON December 1962

CALCULATION OF EPHEMERIDES FROM INITIAL VALUES

by

Karl Stumpff

Goddard Space Flight Center

SUMMARY

A short report is presented on a method for calculating undisturbed ephemerides (coordinates of location and velocity) of a planet or a comet when its initial time values are given. Thus far this method has been available only in German language publications. The orbital elements need not be known for this method, which is applicable without formal variations for all types of orbits. In particular, the singularity of classical methods is avoided by the transfer from elliptical to hyperbolic orbits. In place of Kepler's equation, a transcendent *main equation* appears which is valid for all types of orbits and becomes rational for circular and parabolic orbits. The formulas for the calculation of location and velocity coordinates are simple and especially well suited for electronic computers. The optimal area for application (small and medium intermediate times) coincides with the requirements for orbit determination, orbit correction, and the calculation of special perturbations.

CONTENTS

CALCULATION OF EPHEMERIDES FROM INITIAL VALUES

by
Karl Stumpff*
Goddard Space Flight Center

THE PROBLEM

The undisturbed orbit of a celestial body orbiting around the sun is definitely determined if the coordinates of its location and velocity are given in relation to the sun. From these six quantities, x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ — the *local elements*† of the orbit — the six classical orbital elements, i, Ω, ω, a, e, T, can be derived. And then $x, \cdots, \dot{z}$ can be computed for any other given time t with the help of the orbital elements. This process requires the solution of the transcendental Kepler equation for every t if the orbit is an ellipse with a relatively small eccentricity. For parabolas, near-parabolic ellipses and hyperbolas, as well as for greatly eccentric hyperbolas ($e \gg 1$), other relationships occur in place of the Kepler equation, so that very different computational processes must be used for these orbits.

The possibility of arriving at a solution without using the classical orbit elements and without variations owing to individual case differences is indicated by the Taylor development of coordinates; for instance

$$x(\tau) = x_0 + \dot{x}_0 \tau + \frac{1}{2} \ddot{x}_0 \tau^2 + \frac{1}{6} \dddot{x}_0 \tau^3 + \cdots , \tag{1}$$

where $\tau = k(t - t_0)$ denotes the "intermediate time" expressed in units of $1/k = 58^{d}\!.13244$, and the higher derivatives $\ddot{x}_0$, $\dddot{x}_0$, $\cdots$, valid for t_0, can be derived from the equations of motion

$$\ddot{x} + \frac{x}{r^3} = 0, \cdots \ddot{z} + \frac{z}{r^3} = 0 \qquad \left(r^2 = x^2 + y^2 + z^2\right) . \tag{2}$$

But these formulas which are independent of orbit shape are only sufficiently convergent for small intermediate times τ, and thus are utilized only occasionally for first orbit

*NAS-NASA Research Associate; Professor Emeritus, Göttingen University.
†The local elements of the orbit are so named because their values depend upon the location of the orbit.

determination. The method described herein shows that it is also possible to develop exact closed formulas where the coordinates $x(\tau) \cdots \dot{z}(\tau)$ can be represented directly as functions of the local elements and an intermediate time of any given length, τ.

THE LOCAL INVARIABLES

Two-body motion occurs in a plane given by the two vectors $\mathbf{p_0}(x_0, y_0, z_0)$, $\dot{\mathbf{p}}_0(\dot{x}_0, \dot{y}_0, \dot{z}_0)$, which define the location and velocity of the planet at the time t_0. With the exception of the situation wherein $\mathbf{p_0}$ and $\dot{\mathbf{p}}_0$ are coincidental or oriented opposite to one another (and this case, which indicates straight-line motion either directly toward or away from the sun, shall be discounted). Since $\mathbf{p}(\tau)$, and also $\dot{\mathbf{p}}(\tau)$, can be broken down into components according to the directions $\mathbf{p_0}$ and $\dot{\mathbf{p}}_0$, we have

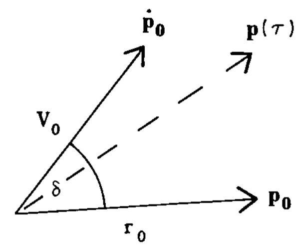

Figure 1—Geometric representation of two-body motion

$$\mathbf{p}(\tau) = \mathbf{p_0}F + \dot{\mathbf{p}}_0 G, \quad \dot{\mathbf{p}}(\tau) = \mathbf{p_0}\dot{F} + \dot{\mathbf{p}}_0 \dot{G}, \tag{3}$$

where F and G are functions of the local elements and the intermediate time τ.

Further, it is clear that the relative locations of the vector $\mathbf{p}(\tau)$ and the scalar quantities F and G for any given value of τ, depend only upon the geometric characteristics of Figure 1, defined by $\mathbf{p_0}$ and $\dot{\mathbf{p}}_0$ — in other words, by the values r_0 or V_0 of these vectors and the angle δ between them — and not upon the orientation of this figure in an area, or space, coordinate system. From this, however, it follows that F and G contain the local elements only in such form as

$$\left.\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= (\mathbf{p_0}\mathbf{p_0}) = p_{11} = r_0^2 , \\ x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0 &= (\mathbf{p_0}\dot{\mathbf{p}}_0) = p_{12} = r_0 V_0 \cos\delta , \\ \dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 &= (\dot{\mathbf{p}}_0\dot{\mathbf{p}}_0) = p_{22} = V_0^2 , \end{aligned}\right\} \tag{4}$$

which we call *local invariables,* because they are independent of the coordinate system selected, although they vary from point to point of the orbit. In actuality, if we set

$$\frac{1}{r^3} = \mu , \tag{5}$$

then, according to Equation 2,

$$\left.\begin{aligned} \ddot{x} &= -x\mu\,, \\ \dddot{x} &= -x\dot{\mu} - \dot{x}\mu\,, \\ x^{iv} &= -x\ddot{\mu} - 2\dot{x}\dot{\mu} - \ddot{x}\mu = x(\mu^2 - \ddot{\mu}) - 2\dot{x}\dot{\mu}\,, \text{ etc.}; \end{aligned}\right\} \quad (6)$$

and we find that, in the development of Equation 1, all higher derivations of x can be represented linearly through x and $\dot{x}$, with coefficients which are dependent only upon $\mu, \dot{\mu}, \ddot{\mu} \cdots$.

Again, these quantities can be expressed through the invariables (Equation 4) as can be demonstrated easily through step by step differentiation of Equation 5. We introduce the quantities

$$\left.\begin{aligned} \mu &= \frac{1}{r^3} = p_{11}^{-\frac{3}{2}}\,, \\ \sigma &= \frac{\dot{r}}{r} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{r^2} = \frac{p_{12}}{p_{11}}\,, \\ \omega &= \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{r^2} = \frac{p_{22}}{p_{11}} \end{aligned}\right\} \quad (7)$$

as the *fundamental invariables*. By differentiating them and substituting $-\mu x, \cdots -\mu z$ for $\ddot{x}, \cdots \ddot{z}$, we find the relations:

$$\left.\begin{aligned} \dot{\mu} &= -3\mu\sigma\,, \text{ or } \dot{r} = r\sigma\,; \\ \dot{\sigma} &= \omega - \mu - 2\sigma^2\,; \\ \dot{\omega} &= -2\sigma(\mu + \omega)\,; \end{aligned}\right\} \quad (8)$$

and also

$$\left.\begin{aligned} \ddot{\mu} &= 15\mu\sigma^2 - 3\mu\omega + 3\mu^2\,; \\ \dddot{\mu} &= -105\mu\sigma^3 + 45\mu\sigma\omega - 42\mu^2\sigma\,; \end{aligned}\right\} \quad (9)$$

etc.

Thus, all derivations of μ (or r), σ, ω are themselves functions of these quantities. It follows that F and G are functions of the intermediate time τ and of μ_0, σ_0, ω_0, if these are the values of the invariables for $t = t_0$ $(\tau = 0)$. Finally, from Equation 1 (when Equations 6 and 9 are introduced, we see that μ and ω always occur in connection with τ^2, and that σ always occurs in connection with τ, and that F and $(1/\tau)$G can be written as power series of $\mu_0\tau^2$, $\sigma_0\tau$, and $\omega_0\tau^2$.

However, experience has shown that it is advantageous to introduce, in addition to Equation 7, other complex invariables such as

$$\epsilon = \omega - \mu, \qquad \rho = 2\mu - \omega, \qquad \vartheta = \omega - \sigma^2 \tag{10}$$

which have a special meaning in two-body motion, and can replace one or the other of the invariables (Equation 7) if necessary. The derivatives of Equation 10 are

$$\dot{\epsilon} = -\sigma(2\omega - \mu), \qquad \dot{\rho} = -2\rho\sigma, \qquad \dot{\vartheta} = -4\vartheta\sigma . \tag{11}$$

In particular, by the elimination of σ from $\dot{r} = r\sigma$, $\dot{\rho} = -2\rho\sigma$, $\dot{\vartheta} = -4\vartheta\sigma$, we obtain

$$\frac{\dot{\rho}}{\rho} = -2\frac{\dot{r}}{r}, \frac{\dot{\vartheta}}{\vartheta} = -4\frac{\dot{r}}{r},$$

whose integration yield the integrals of energy and areas:

$$r^2\rho = \text{constant} = \frac{1}{a}, \text{ and } r^4\vartheta = \text{constant} = p = a(1 - e^2) . \tag{12}$$

It follows from this that the quantity ρ serves as a criterion for the orbit shape, since $\rho > 0$ for ellipses $(a > 0)$, $\rho = 0$ for parabolas $(a = \infty)$ and $\rho < 0$ for hyperbolas $(a < 0)$.

All geometric quantities of a conic-section orbit can be expressed in more or less simple form by means of the invariables (Equations 7 and 10); for example, we readily find that

$$\left.\begin{aligned}
e\cos v &= \frac{p}{r} - 1 = \frac{\vartheta}{\mu} - 1 ; & e\cos E &= 1 - \frac{r}{a} = \frac{\epsilon}{\mu} ; \\
e\sin v &= \dot{r}\sqrt{p} = \frac{\sigma\sqrt{\vartheta}}{\mu} ; & e\sin E &= \frac{r\sin v}{\sqrt{ap}} = \frac{\sigma\sqrt{\rho}}{\mu} ; \\
\dot{v} &= \sqrt{\vartheta} ; & \dot{E} &= \sqrt{\rho} ; \\
e^2 &= \frac{\epsilon^2 + \rho\sigma^2}{\mu^2} ; & n &= a^{-\frac{3}{2}} = \frac{\rho\sqrt{\rho}}{\mu} .
\end{aligned}\right\} \tag{13}$$

Finally, we obtain

$$\left.\begin{aligned} \dot{r} &= r\sigma , \\ \ddot{r} &= r\left(\epsilon - \sigma^2\right) , \\ \dddot{r} &= - r\sigma\left[3\left(\epsilon - \sigma^2\right) + \mu\right], \quad \mu = \frac{1}{r^3} \end{aligned}\right\} \tag{14}$$

from which, by the elimination of σ and $\epsilon - \sigma^2$, there follows

$$\dddot{r} + 3\,\frac{\dot{r}\,\ddot{r}}{r} + \frac{\dot{r}}{r^3} = 0 . \tag{15}$$

INTRODUCTION OF A NEW INDEPENDENT VARIABLE

In order to integrate Equation 15, we introduce a new variable q in place of the time τ :

$$\dot{q} = \frac{1}{r} , \quad d\tau = r\,dq , \quad \tau = \int_0^q r(q)\,dq .$$

For the initial epoch $\tau = 0$, and also $q = 0$; since $1/r$ is essentially positive as long as r remains finite, q increases proportionally in the same sense as τ, so that $q(\tau)$ represents a definitely reversible function. Therefore, if we set*

$$\left.\begin{aligned} \dot{r} &= r'\dot{q} = \frac{r'}{r} , \\ \ddot{r} &= \left(\frac{r''}{r} - \frac{r'^2}{r^2}\right)\dot{q} = \frac{r''}{r^2} - \frac{r'^2}{r^3} , \\ \dddot{r} &= \left(\frac{r'''}{r^2} - 4\,\frac{r'r''}{r^3} + 3\,\frac{r'^3}{r^4}\right)\dot{q} = \frac{r'''}{r^3} - 4\,\frac{r'r''}{r^4} + 3\,\frac{r'^3}{r^5} , \end{aligned}\right\} \tag{16}$$

then Equation 15 becomes

$$r''' - \frac{r'r''}{r} + \frac{r'}{r} = r''' + \frac{1 - r''}{r}\,r' = 0 . \tag{17}$$

*Derivations with respect to q are indicated by primes.

However,

$$\frac{d}{dq}\left(\frac{1-r''}{r}\right) = -\frac{1}{r}\left(r''' + \frac{1-r''}{r}r'\right) = 0 ;$$

thus $\alpha^2 = (1-r'')/r$ is constant and Equation 17 assumes the simple form

$$r''' + \alpha^2 r' = 0 \tag{18}$$

whose complete integral, with three constants a, b, c is, in closed form,

$$\left.\begin{aligned} r &= a + b\cos\alpha q + c\sin\alpha q & &(\text{for } \alpha^2 > 0) , \\ r &= a + bq + cq^2 & &(\text{for } \alpha^2 = 0) , \\ r &= a + b\cosh\beta q + c\sinh\beta q & &(\text{for } -\beta^2 = \alpha^2 < 0) , \end{aligned}\right\} \tag{19}$$

The integrals (Equation 19) still contain the case variations of the classic two-body theory. Our requirements for formulas applicable without difference to all orbit shapes can be satisfied by utilizing the following expedient:

The differential Equation 18 has the particular integral

$$r = \cos\alpha q .$$

By continued integration with respect to q, a series of auxiliary functions are obtained:

$$c_n = \frac{1}{q^n}\int_0^q \cdots \int_0^q \cos\alpha\xi \, (d\xi)^n . \tag{20}$$

The beginning of this series of auxiliary functions (which we shall term c-functions) is:

$$c_0 = \cos\alpha q ; \quad c_1 = \frac{\sin\alpha q}{\alpha q} ; \quad c_2 = \frac{1-\cos\alpha q}{(\alpha q)^2} ; \quad c_3 = \frac{\alpha q - \sin\alpha q}{(\alpha q)^3} . \tag{21}$$

By setting $\lambda = \alpha q$, we can also write the c-functions in the form of the constantly convergent power series

$$c_n(\lambda^2) = \frac{1}{n!} - \frac{\lambda^2}{(n+2)!} + \frac{\lambda^4}{(n+4)!} - \frac{\lambda^6}{(n+6)!} + \cdots . \tag{22}$$

For this there applies the differential relationship

$$\frac{d}{dq}\left(c_{n+1}\, q^{n+1}\right) = c_n q^n \tag{23}$$

and the recurrence formula

$$\frac{1}{n!} = c_n + \lambda^2 c_{n+2} \quad . \tag{24}$$

Now we can write $r(q)$ as a Taylor series:

$$r(q) = r_0 + r_0{}' q + \frac{1}{2!} r_0{}'' q^2 + \frac{1}{3!} r''' q^3 + \cdots ,$$

or, by substituting Equation 24 for the reciprocal factorials

$$r(q) = r_0 + r_0{}'\left(c_1 + \lambda^2 c_3\right) q + r_0{}''\left(c_2 + \lambda^2 c_4\right) q^2 + r_0{}'''\left(c_3 + \lambda^2 c_5\right) + \cdots .$$

If we again substitute $\lambda^2 = \alpha^2 q^2$ and arrange the preceding equation according to powers of q, we obtain

$$r(q) = r_0 + c_1 r_0{}' q + c_2 r_0{}''^2 + c_3\left(r_0{}''' + \alpha^2 r_0{}'\right) q^3 + c_4\left(r_0{}^{IV} + \alpha^2 r_0{}''\right) q^4 + \cdots$$

in which all terms of third and higher orders disappear because of Equation 18.

We now have the closed expression

$$r(q) = r_0 + c_1 r_0{}' q + c_2 r_0{}'' q^2 \ . \tag{25}$$

But now, because of Equations 12, 14, and 16,

$$r' = \frac{\dot{r}}{\dot{q}} = r\,\dot{r} = r^2\sigma ,$$

$$r'' = \frac{1}{\dot{q}}\left(\dot{r}^2 + \dot{r}\,\ddot{r}\right) = r^3\epsilon ,$$

$$\alpha^2 = \frac{1 - r''}{r} = \frac{1 - r^3\epsilon}{r} = \frac{1 - r^3(\mu - \rho)}{r} = r^2\rho = \frac{1}{a} ,$$

we obtain

$$\lambda^2 = \alpha^2 q^2 = \frac{q^2}{a} = \rho(rq)^2$$

as the argument of the c-functions. The closed expression of Equation 25 can now be written

$$r(q) = r_0\left[1 + c_1 \sigma_0 (r_0 q) + c_2 \epsilon_0 (r_0 q)^2\right] , \quad (26)$$

where $c_n = c_n\left[\rho_0 (r_0 q)^2\right]$.

Finally, if we set

$$\xi_0 = \mu_0 \tau^2, \quad \eta_0 = \sigma_0 \tau, \quad \zeta_0 = \epsilon_0 \tau^2, \quad \chi_0 = \rho_0 \tau^2 \quad (27)$$

and, in place of q, introduce the variable

$$z = \frac{r_0 q}{\tau} , \quad (28)$$

we obtain, instead of Equation 26:

$$r = r_0 (1 + c_1 \eta_0 z + c_2 \zeta_0 z^2) , \quad (29a)$$

where $c_n = c_n (\chi_0 z^2)$; or

$$r = r_0 (c_0 + c_1 \eta_0 z + c_2 \xi_0 z^2) , \quad (29b)$$

where $c_0 = 1 - c_2 \chi_0 z^2$ and $\chi_0 = \xi_0 - \zeta_0$.

THE MAIN EQUATION

To clarify the relationship between z and τ, consider that

$$d\tau = r\,dq = r_0 \left(1 + c_1 q \cdot r_0 \sigma_0 + c_2 q^2 \cdot r_0^2 \epsilon_0\right) dq . \quad (30)$$

Upon integrating and setting

$$\int c_n q^n dq = c_{n+1} q^{n+1}$$

according to Equation 23, Equation 30 becomes

$$\tau = r_0 q + c_2 r_0^2 \sigma_0 q^2 + c_3 r_0^3 \epsilon_0 q^3 .$$

If we divide this by τ and substitute z (Equation 28), we finally obtain

$$1 = z + c_2 \sigma_0 \tau z^2 + c_3 \epsilon_0 \tau^2 z^3$$

or

$$\boxed{1 = z + c_2 \eta_0 z^2 + c_3 \zeta_0 z^3; \qquad \text{for} \quad c_n = c_n\left(\chi_0 z^2\right) .} \tag{31a}$$

If we add $0 = z\left(c_1 + c_3 \chi_0 z^2 - 1\right)$ to the above equation, where $\chi_0 = \xi_0 - \zeta_0$, it can also be written:

$$1 = c_1 z + c_2 \eta_0 z^2 + c_3 \xi_0 z^3 , \tag{31b}$$

This transcendental equation represents the Kepler equation as developed here and the analogous equations for parabolic and hyperbolic motion. The Equations 29 and 31 are valid for all forms of orbits: in hyperbolic motion, the argument of the c-functions becomes negative, since ρ_0 and $\chi_0 = \rho_0 \tau^2$ become negative, but the functions themselves remain real. In the case of the parabola, the c_n are approaching their constant term $1/n!$; thus $\lambda^2 = \chi_0 z^2$ becomes zero; and the main equation becomes a rational third degree equation:

$$1 = z + \frac{1}{2} \eta_0 z^2 + \frac{1}{6} \zeta_0 z^3 . \tag{32}$$

For circular orbits $(e = 0)$, $\sigma_0 = \epsilon_0 = 0$ from Equation 13, and $\eta_0 = \zeta_0 = 0$; therefore Equation 31 has the trivial form $z = 1$.

Equation 31a, termed the *main equation* of the two-body problem, can also be developed directly from the Kepler equation. If we write t_0 and t for two times,

$$E - e \sin E = M = k \frac{t - T}{\sqrt{a^3}} ,$$

and

$$E_0 - e \sin E_0 = M_0 = k \frac{t_0 - T}{\sqrt{a^3}} ,$$

where T is the time of passage through the perihelion, and then set $\lambda = E - E_0$, the difference of both equations is

$$\lambda - e\left[\sin\left(E_0 + \lambda\right) - \sin E_0\right] = \frac{\tau}{\sqrt{a^3}} ,$$

or

$$\left(1 - e\cos E_0\right)\sin\lambda + e\sin E_0(1 - \cos\lambda) + (\lambda - \sin\lambda) = \frac{\tau}{\sqrt{a^3}} .$$

By substituting in the foregoing the expressions of Equation 13 for $e\cos E_0$, $e\sin E_0$, $a^{-3/2}$, and the c-functions

$$c_1\left(\lambda^2\right) = \sin\lambda/\lambda, \quad c_2\left(\lambda^2\right) = (1 - \cos\lambda)/\lambda^2, \quad \text{and } c_3\left(\lambda^2\right) = (\lambda - \sin\lambda)/\lambda^3 ,$$

we obtain

$$c_1 \frac{\rho_0}{\mu_0}\lambda + c_2\sigma_0 \frac{\sqrt{\rho_0}}{\mu_0}\lambda^2 + c_3\lambda^3 = \frac{\tau\rho_0\sqrt{\rho_0}}{\mu_0} .$$

If we then divide through by term on the right side and put

$$c_1 = 1 - \lambda^2 c_3, \quad \lambda = z\tau\sqrt{\rho_0}, \quad \sigma_0\tau = \eta_0, \quad \epsilon_0\tau^2 = \zeta_0,$$

we obtain the main equation in the form of Equation 31. The quantity $\lambda = E - E_0 = z\tau\sqrt{\rho_0}$, introduced in the argument of the c-functions, is real only for positive ρ_0 or for elliptical orbits, and is identical with $\lambda = \alpha q$ (which we used in the previous section) since $\alpha = 1/\sqrt{a} = r_0\sqrt{\rho_0}$ and $\lambda = r_0 q\sqrt{\rho_0} = z\tau\sqrt{\rho_0}$, from Equation 28.

From $z = \left(r_0/\tau\right) q$ and $q = \int_0^\tau d\tau/r$ it follows that

$$z = \frac{1}{\tau}\int_0^\tau \frac{r_0}{r}\, d\tau$$

or, if we set

$$r = r_0\Delta = r_0\left(1 + c_1\,\eta_0 z + c_2\,\xi_0\, z^2\right) , \tag{33}$$

that

$$z = \frac{1}{\tau}\int_0^\tau \frac{d\tau}{\Delta} . \tag{34}$$

Therefore z is the mean value for $1/\Delta = r_0/r$ over the time interval $(0, \tau)$ and is always a positive quantity. For circular orbits $r = r_0$, and therefore $\Delta = 1$ from Equation 33 and $z = 1$ from Equation 34, which values we could also ascertain from the main equation.

The form of the main equation valid for parabolic orbits (Equation 32) can be derived from the well known cubic equation for the tangent of the half true anomaly, which replaces the Kepler equation for $e = 1$:

$$\tan\frac{v}{2} + \frac{1}{3}\tan^3\frac{v}{2} = 2k\,\frac{t - T}{\sqrt{p^3}} ,$$

where p = the parameter of the parabola. By subtracting

$$\tan\frac{v_0}{2} + \frac{1}{3}\tan^3\frac{v_0}{2} = 2k\,\frac{t_0 - T}{\sqrt{p^3}}$$

(for $t = t_0$) from the above equation and by letting

$$\left.\begin{aligned} A &= \tan\frac{v}{2} - \tan\frac{v_0}{2} , \\ B &= 1 + \tan\frac{v}{2}\tan\frac{v_0}{2} , \end{aligned}\right\} \qquad (35)$$

we obtain the relation

$$A\left(B + \frac{1}{3}A^2\right) = \frac{2\tau}{\sqrt{p^3}} . \qquad (36)$$

Since, from Equation 13,

$$\tan v = \frac{2\tan\frac{v}{2}}{1 - \tan^2\frac{v}{2}} = \frac{\sigma\sqrt{\vartheta}}{\mu}$$

$$\tan^2\frac{v}{2} + 2\,\frac{\vartheta - \mu}{\sigma\sqrt{\vartheta}}\tan\frac{v}{2} = 1 ,$$

we deduce as a solution of this quadratic equation

$$\tan\frac{v}{2} = \frac{1}{\sigma\sqrt{\vartheta}}\left[\mu - \vartheta \pm \sqrt{\mu^2 - \vartheta\left(2\mu - \vartheta - \sigma^2\right)}\right]$$

According to Equation 10,

$$2\mu - \vartheta - \sigma^2 = (2\mu - \omega) + \omega - \sigma^2 - \vartheta = \rho \;;$$

but in the case of the parabola $\rho = 0$, either

$$\tan\frac{v}{2} = \frac{2\mu - \vartheta}{\sigma\sqrt{\vartheta}} = \frac{\sigma}{\sqrt{\vartheta}}$$

or

$$\tan\frac{v}{2} = -\frac{\sqrt{\vartheta}}{\sigma} .$$

Here the first of these solutions applies, since σ and v disappear in the perihelion.

Now, from Equation 16 $r' = dr/dq = r^2\sigma, r'' = r^3\epsilon$; and from Equation 25

$$\begin{aligned} r' &= r_0'\left(c_1 q\right)' + r_0''\left(c_2 q^2\right)' \\ &= r_0^2 \sigma_0 c_0 q^0 + r_0^3 \epsilon_0 \left(c_1 q\right) \\ &= r_0^2\left(c_0\sigma_0 + c_1\epsilon_0 r_0 q\right) \\ &= r_0^2\left(c_0\sigma_0 + c_1\epsilon_0 z\tau\right) . \end{aligned}$$

Therefore in the case of the parabola $\left(c_0 = c_1 = 1,\ \rho = \mu - \epsilon = 0\right)$, we have

$$\sigma = \frac{r'}{r^2} = \frac{1}{\Delta^2}\left(\sigma_0 + \epsilon_0 z\tau\right) = \frac{\sigma_0 + \mu_0 z\tau}{\Delta^2} .$$

Furthermore $r^4\vartheta = r_0^4\vartheta_0$ from Equation 12, thus

$$\sqrt{\vartheta} = \frac{\sqrt{\vartheta_0}}{\Delta^2} .$$

Since

$$\tan\frac{v_0}{2} = \frac{\sigma_0}{\sqrt{\vartheta_0}} ,$$

and

$$\tan\frac{v}{2} = \frac{\sigma}{\sqrt{\vartheta}} = \frac{\sigma_0 + \mu_0 z\tau}{\sqrt{\vartheta_0}} \quad ,$$

we find from Equation 35 that

$$A = \frac{\mu_0 z\tau}{\sqrt{\vartheta_0}} \; ,$$

$$B = 1 + \frac{\sigma_0}{\vartheta_0}\left(\sigma_0 + \mu_0 z\tau\right) = \frac{\mu_0}{\vartheta_0}\left(2 + \sigma_0 z\tau\right) \; .$$

Now, because $\rho = 0$, we may set

$$\sigma_0^2 + \vartheta_0 = 2\mu_0 \; .$$

Equation 36 then has the form

$$\frac{\mu_0 z\tau}{\sqrt{\vartheta_0}}\left[\frac{\mu_0}{\vartheta_0}\left(2 + \sigma_0 z\tau\right) + \frac{1}{3}\,\frac{\mu_0^2 z^2 \tau^2}{\vartheta_0}\right] = \frac{2\tau}{\sqrt{p^3}}$$

or

$$z + \frac{1}{2}\eta_0 z^2 + \frac{1}{6}\xi_0 z^3 = \frac{1}{\mu_0^2}\sqrt{\left(\frac{\vartheta_0}{p}\right)^3} \quad , \qquad (37)$$

where we let $\xi_0 = \mu_0\tau^2$, $\eta_0 = \sigma_0\tau$ according to Equation 27. This equation is identical with Equation 32; however,

$$\frac{p}{2} = r_0\cos^2\frac{v_0}{2} = \frac{r_0}{1 + \tan^2\frac{v_0}{2}} = \frac{r_0}{1 + \frac{\sigma_0^2}{\vartheta_0}} = \frac{r_0\,\vartheta_0}{\vartheta_0 + \sigma_0^2} = \frac{r_0\,\vartheta_0}{2\mu_0} \quad ,$$

so that

$$\left(\frac{\vartheta}{p}\right)^3 = \left(\frac{\mu_0}{r_0}\right)^3 = \mu_0^4 \quad ;$$

and therefore the right side of Equation 37 equals unity. Furthermore, for parabolas, $\mu_0 = \epsilon_0$; thus $\xi_0 = \zeta_0$.

F AND G AS FUNCTIONS OF τ

If, with the help of the main equation, z has been determined as a function of τ and of the local invariables of the initial epoch, the quantities F and G as well as their derivatives $\dot{F}$ and $\dot{G}$ can be expressed easily as functions of z. If in the vectorial differential equation

$$\ddot{\mathbf{p}} = -\mu\mathbf{p}$$

we set

$$\mathbf{p} = \mathbf{p}_0 F + \dot{\mathbf{p}}_0 G \,, \quad \ddot{\mathbf{p}} = \mathbf{p}_0 \ddot{F} + \dot{\mathbf{p}}_0 \ddot{G}$$

in the manner of Equation 3, there follows that the identity

$$\mathbf{p}_0 (\ddot{F} + \mu F) + \dot{\mathbf{p}}_0 (\ddot{G} + \mu G) = 0$$

is fulfilled only if

$$\ddot{F} + \mu F = 0 \,, \quad \ddot{G} + \mu G = 0 \,. \tag{38}$$

If a new variable q is introduced (with $\dot{q} = 1/r$ as above), we then have

$$\left. \begin{aligned} \dot{F} &= F'\dot{q} = \frac{F'}{r} \,, \\ \ddot{F} &= \frac{F''}{r^2} - \frac{F'\sigma}{r} \,, \end{aligned} \right\} \tag{39}$$

since $\dot{r} = r\sigma$; and we obtain

$$rF'' - r^2\sigma F' + F = 0$$

instead of Equation 38. If we differentiate the above equation again with respect to q, we obtain

$$rF''' + r'F'' - 2rr'\sigma F' - r^2\sigma' F' - r^2\sigma F'' + F' = 0$$

or, since

$$r' = r^2\sigma ,$$

$$\sigma' = \frac{\dot{\sigma}}{\dot{q}} = r\dot{\sigma} = r\left(\epsilon - 2\sigma^2\right) ,$$

$$1 = r^3\mu ,$$

$$rF''' + r^3(\mu - \epsilon)F' = r\left(F''' + r^2\rho F'\right) = 0 .$$

But $r^2\rho = \alpha^2$ is constant; thus

$$F''' + \alpha^2 F' = 0 ,$$

and

$$G''' + \alpha^2 G' = 0 ,$$

which means that the same differential equation will suffice for $F(q)$ and $G(q)$ as for $r(q)$. Therefore, since exactly the same operations can be used on these two functions as on $r(q)$ in Equation 25, we have

$$\left.\begin{aligned} F(q) &= F_0 + c_1 F_0' q + c_2 F_0'' q^2 , \\ G(q) &= G_0 + c_1 G_0' q + c_2 G_0'' q^2 . \end{aligned}\right\} \qquad (40)$$

Now from

$$x(\tau) = x_0 F(\tau) + \dot{x}_0 G(\tau) ,$$

$$\dot{x}(\tau) = x_0 \dot{F}(\tau) + \dot{x}_0 \dot{G}(\tau)$$

it follows that, for $\tau = 0$,

$$F_0 = 1, \quad \dot{F}_0 = 0 ;$$

$$G_0 = 0, \quad \dot{G}_0 = 1 .$$

Also from Equation 38,

$$\ddot{F}_0 = -\mu_0 F_0 = -\mu_0 ;$$

$$\ddot{G}_0 = -\mu_0 G_0 = 0 .$$

If we put

$$F_0' = r_0 \dot{F}_0 = 0 ,$$

$$F_0'' = r_0^2 \ddot{F}_0 + r_0 \sigma_0 F_0' = -r_0^2 \mu_0 ,$$

$$G_0' = r_0 \dot{G}_0 = r_0 ,$$

$$G_0'' = r_0^2 \ddot{G}_0 + r_0 \sigma_0 G_0' = r_0^2 \sigma_0 .$$

(obtained from Equation 39) into Equation 40, we obtain

$$F(q) = 1 - \mu_0 r_0^2 c_2 q^2 ,$$

and

$$G(q) = r_0 \left[c_1 q + \sigma_0 r_0 c_2 q^2\right] .$$

Differentiating with respect to q and regarding Equation 23, we have

$$F'(q) = -\mu_0 r_0^2 c_1 q ,$$

$$G'(q) = r_0 \left[c_0 + \sigma_0 r_0 c_1 q\right] .$$

By substituting $r_0 q = z\tau$, $\xi_0 = \mu_0 \tau^2$, and $\eta_0 = \sigma_0 \tau$ into the above equation, we have:

$$\left.\begin{aligned} F &= 1 - c_2 \xi_0 z^2 , \\ G &= z\tau\left(c_1 + c_2 \eta_0 z\right) = \tau\left(1 - c_3 \xi_0 z^3\right) \end{aligned}\right] \quad (41)$$

because of Equation 31b; and

$$\left.\begin{aligned} \dot{F} &= \frac{1}{r} F' = -c_1 \frac{\mu_0 z\tau}{\Delta} , \\ \dot{G} &= \frac{1}{r} G' = \frac{c_0 + c_1 \eta_0 z}{\Delta} = \frac{\Delta - c_2 \xi_0 z^2}{\Delta} = 1 - \frac{1 - F}{\Delta} \end{aligned}\right\} \quad (42)$$

because of Equation 29b.

SOLUTION OF THE MAIN EQUATION

With the formulas of Equations 41 and 42, the problem of calculating the ephemerides from given initial values is led back to the solution of the main equation. As was previously mentioned, this equation is rational for two cases: (1) the circular orbit, where it has the trivial form $z = 1$; and (2) the parabolic orbit, where it takes the form of a cubic equation. In all other cases the main equation is transcendental with the outer form of a cubic equation whose coefficients are only slightly dependent upon the unknown z. If the intermediate time τ is not overly large — and this either is never the case in practical applications, or it can be easily avoided — the equation can always be solved by means of a rapidly converging iteration process if a suitable approximate solution $z = z_0$ is available. For slightly eccentric orbits, but also for orbits of any given eccentricity, if the intermediate time τ is short, success is always possible with $z_0 = 1$, and for near-parabolic ellipses or hyperbolas with z_0 as the solution of the cubic equation (Equation 32).

For the iteration, the Newton approximation method is preferable; i.e., we try to find the zero of the function

$$H(z) = z + c_2 \eta_0 z^2 + c_3 \zeta_0 z^3 - 1 .$$

For $z = z_0$, $H(z_0)$ is a small quantity. Now from Equation 23,

$$\frac{d}{dz} \left(c_{n+1} z^{n+1} \right) = c_n z^n ,$$

if we set $q = z\tau / r_0$ and indicate that here τ represents a constant which is also contained in η_0 and ζ_0. Thus, by differentiation with respect to z, we obtain

$$\frac{dH}{dz} = 1 + c_1 \eta_0 z + c_2 \zeta_0 z^2 = \Delta .$$

Therefore, if $z_1 = z_0 + \delta z$ is an improved value of the unknown,

$$H(z_1) = H(z_0) + \delta z \cdot \Delta(z_0) + \cdots \approx 0$$

and

$$\delta z \approx -\frac{H(z_0)}{\Delta(z_0)} . \tag{43}$$

The versatile properties of the c-functions also make it possible to transform the Taylor series

$$0 = H(z_0) + \delta z \cdot H'(z_0) + \frac{1}{2}(\delta z)^2 \cdot H''(t_0) + \cdots$$

(in which the derivatives with respect to z are indicated by primes) into a closed expression. If we set

$$z = \frac{r_0}{\tau} q = \beta q ,$$

where β is constant, and if we consider that

$$\frac{d^3\Delta}{dq^3} + \alpha^2 \frac{d\Delta}{dq} = 0 \qquad \left(\alpha^2 = \rho_0 r_0^2\right)$$

follows from Equation 18 and from $\Delta = r/r_0$, we can also set

$$\frac{d\Delta}{dq} = \Delta' \cdot \frac{dz}{dq} = \Delta'\beta ,$$

and

$$\frac{d^3\Delta}{dq^3} = \Delta''' \beta^3 .$$

Therefore

$$\Delta''' + \left(\frac{\alpha}{\beta}\right)^2 \Delta' = 0 ,$$

where $(\alpha/\beta)^2 = \rho_0\tau^2 = \chi_0$. Furthermore since $H' = \Delta$,

$$H^{iv} + \chi_0 H'' = 0 .$$

Now if we develop $H(z)$ around the small approximative value $H(z_0) = H_0$ into a Taylor series

$$H = H_0 + H_0'\,\delta z + \frac{1}{2!} H_0''(\delta z)^2 + \frac{1}{3!} H_0'''(\delta z)^3 + \cdots ,$$

we can then apply the method used for $r(q)$ and, by introducing the c-functions

$$c_n = c_n\left[\chi_0(\delta z)^2\right]$$

and eliminating the reciprocal factorials by the recurrence formula (Equation 24), we obtain the rigid expression:

$$H = H_0 + H_0'\,\delta z + c_2 H_0''(\delta z)^2 + c_3 H_0'''(\delta z)^3 = 0 . \tag{44}$$

But, if the index 0 in all cases denotes the fact that the respective quantity for $z = z_0$ is to be taken,

$$H_0' = \Delta_0$$

$$H_0'' = \Delta_0'$$

$$= c_0\eta_0 + c_1\zeta_0 z_0$$

$$= \eta_0\left(1 - c_2\chi_0 z_0^2\right) + \zeta_0 z_0\left(1 - c_3\chi_0 z_0^2\right)$$

$$= \eta_0 + \zeta_0 z_0 - \chi_0\left(H_0 - z_0 + 1\right)$$

$$H_0''' = \zeta_0 - \chi_0\left(H_0' - 1\right)$$

$$= \zeta_0 - \chi_0\left(\Delta_0 - 1\right) .$$

By inserting these values into Equation 44, we can use the approximation of Equation 43 for the calculation of the small factors $(\delta z)^2$ and $(\delta z)^3$. This also applies for c_2 and c_3 in a still greater degree; since the argument $\chi_0(\delta z)^2$ is very small, $c_2 = 1/2$ and $c_3 = 1/6$ could always be used.

If the orbit is a parabola, z will be a solution of the cubic equation (Equation 32):

$$z + az^2 + bz^3 = 1 ,$$

where $a = \eta_0/2$ and $b = \zeta_0/6$. If we introduce $u = z - a/3b$ in place of the unknown z,

$$u^3 + \alpha u = \beta ,$$

where

$$\alpha = \frac{3b - a^2}{3b^2} ,$$

$$\beta = \frac{27 b^2 - 2a^3 + 9 ab}{27b^3} .$$

Now the only real solution of the above equation is given by the Cardanic formula

$$u = \sqrt[3]{\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \left(\frac{\alpha}{3}\right)^3}} + \sqrt[3]{\frac{\beta}{2} - \sqrt{\left(\frac{\beta}{2}\right)^2 + \left(\frac{\alpha}{3}\right)^3}} ,$$

because $3b - a^2 = 1/4\left(2\xi_0 - \eta_0^2\right) = \tau^2/4\left(2\epsilon_0 - \sigma_0^2\right)$ and since the discriminant $(\beta/2)^2 + (\alpha/3)^3$ is always positive, for the parabola $(\rho = 0)$, $\omega = 2\mu = 2\epsilon$; therefore $2\epsilon - \sigma^2 = \omega - \sigma^2 = \vartheta = p/r^4$. This quantity is always positive since the orbital parameter is positive (except for the case of the straight-line orbit, where $p = 0$). Therefore $\alpha > 0$.

If we set

$$x = \frac{1}{\sqrt{1 + \frac{4}{27}\frac{\alpha^3}{\beta^2}}} ,$$

then

$$u = \sqrt[3]{\frac{\beta}{2x}} \left(\sqrt[3]{1 + x} - \sqrt[3]{1 - x}\right)$$

$$= \sqrt[3]{\frac{\beta}{2x}} \frac{2x}{\sqrt[3]{(1+x)^2} + \sqrt[3]{1-x^2} \sqrt[3]{(1-x)^2}} . \qquad (45)$$

On the other hand,

$$\frac{4}{27}\frac{\alpha^3}{\beta^2} = \frac{4\beta}{27}\left(\frac{\alpha}{\beta}\right)^3 = \frac{1 - x^2}{x^2} ,$$

$$\beta = 2\,\frac{1 - x^2}{x^2}\left(\frac{3\beta}{2\alpha}\right)^3 ,$$

and therefore

$$\frac{\beta}{2x} = \left(1 - x^2\right)\left(\frac{3\beta}{2\alpha x}\right)^3 ,$$

which is inserted into Equation 45 to yield

$$u = \frac{\beta}{\alpha}\,\frac{3m}{1 + m + m^2}$$

with

$$m = \sqrt[3]{\frac{1 - x}{1 + x}} .$$

This solution process can also be used for near-parabolic orbits (it is immaterial whether they are elliptical or hyperbolic). If $A = 2c_2\eta_0$, $B = 6c_3\zeta_0$, $f = 2B - A^2$, $g = 3B(A + B) - A^3$, then the main equation is solved by using the equations:

$$x = \frac{1}{\sqrt{1 + \frac{f^3}{g^2}}} ,$$

$$m = \sqrt[3]{\frac{1 - x}{1 + x}} ;$$

$$u = \frac{2g}{Bf}\,\frac{m}{1 + m + m^2} ;$$

$$z = u + \frac{A}{B} .$$

Here the integration will start with $A = \eta_0$, $B = \zeta_0$, since $2c_2$ and $6c_3$ only vary slightly from 1; and c_2 and c_3 are determined anew with the argument $\chi_0 z^2$ in second approximation.

CALCULATION OF AN EPHEMERIS FROM INITIAL VALUES

The ephemerides of the coordinates of location and velocity of a celestial body moving in an undisturbed conic section orbit around the sun can be calculated easily from given initial values (x_0, y_0, z_0) and $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ by the following method which is suitable for electronic computers.

Given Quantities

The given quantities are: $x_0, y_0, z_0; \dot{x}_0, \dot{y}_0, \dot{z}_0; \tau = k(t - t_0)$, where $k = 0.0172021$. If an ephemeris is to be calculated for equally spaced times $t_n = t_0 + n\omega$, where ω = table interval in days, $n = 0, 1, 2, \cdots$, then the intermediate times are $\tau = nk\omega$ (unit $1/k = 58^d{\cdot}13244$).

Invariables

The invariables are:

$$r_0^2 = x_0^2 + y_0^2 + z_0^2 ;$$

$$r_0^2 \sigma_0 = x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0 ;$$

$$r_0^2 \omega_0 = \dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 .$$

From these, r_0, σ_0, ω_0 are calculated, then

$$\mu_0 = \frac{1}{r_0^3}, \quad \epsilon_0 = \omega_0 - \mu_0, \quad \rho_0 = \mu_0 - \epsilon_0,$$

$$\xi_0 = \mu_0\tau^2, \quad \eta_0 = \sigma_0\tau, \quad \zeta_0 = \epsilon_0\tau^2, \quad \chi_0 = \rho_0\tau^2 .$$

Solution of the Main Equation

The main equation for the transfer from t_0 to t_1 with $\tau = k(t_1 - t_0)$ is

$$H = z + c_2\,\eta_0 z^2 + c_3\,\zeta_0 z^3 - 1 = 0.$$

Beginning with $z = z_0$ (see the comments on page 24), we calculate:

$$\lambda^2 = \chi_0 z_0^2 \;;$$

$$c_2 = \frac{1}{2!} - \frac{\lambda^2}{4!} + \frac{\lambda^4}{6!} - \cdots \;;$$

$$c_3 = \frac{1}{3!} - \frac{\lambda^2}{5!} + \frac{\lambda^4}{7!} - \cdots \;;$$

$$c_1 = 1 - \lambda^2 c_3 \;;$$

$$\Delta_0 = 1 + c_1 \eta_0 z_0 + c_2 \zeta_0 z_0^2 \;;$$

$$H_0 = z_0 + c_2 \eta_0 z_0^2 + c_3 \zeta_0 z_0^3 - 1 \;;$$

$$\delta z = -\frac{H_0}{\Delta_0}, \quad z_1 = z_0 + \delta z, \quad \lambda^2 = \chi_0 z_1^2, \quad \cdots$$

This computation is repeated until the values of z_0, z_1, $z_2, \cdots$ no longer change.

Coordinates for $t = t_1$

Let z, λ^2, c_1, c_2, c_3, Δ be the values from the last iteration for which $H = 0$ is fulfilled. We then form:

$$1 - F = c_2 \xi_0 z^2 \;;$$

$$G = \tau\left(1 - c_3 \xi_0 z^3\right) \;;$$

$$\dot{F} = -c_1 \frac{\xi_0 z}{\Delta\tau} \;;$$

$$1 - \dot{G} = \frac{1 - F}{\Delta} \;;$$

$$x_1 - x_0 = -(1-F)\,x_0 + G\,\dot{x}_0 \;;$$

$$\dot{x}_1 - \dot{x}_0 = \dot{F}\,x_0 - (1-\dot{G})\,\dot{x}_0 \;.$$

The appropriate formulas apply for y_1, z_1; $\dot{y}_1$, $\dot{z}_1$.

Controls

Once we have calculated the invariables $\mu_1, \sigma_1, \omega_1, \cdots$ with the new coordinates $(x_1, y_1, z_1; \dot{x}_1, \dot{y}_1, \dot{z}_1)$, then the integral relations

$$\rho_1 = \frac{\rho_0}{\Delta_0^2},$$

$$\omega_1 - \sigma_1^2 = \frac{\omega_0 - \sigma_0^2}{\Delta_0^4}$$

must be fulfilled.

COMMENTS

In computing an equidistant ephemeris, we have two possibilities: (1) after the first step has yielded the coordinates x_1, y_1, z_1; $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$, we use these elements as initial values and t_1 as the initial time in the second step. The computation then yields $x_2, \cdots \dot{z}_2$; the procedure is continued, always with the same intermediate time τ. (2) We use the same initial values $(x_0, \cdots \dot{z}_0)$ and their derived invariables for a larger selection of steps with increasing intermediate times.

Each method has advantages and disadvantages. In the first, since τ is always small, the iteration for the solution of the main equation can usually be begun with $z_0 = 1$ and it converges rapidly. A disadvantage, however, is that the rounding-off errors in the series

$$x_1 = x_0 + (x_1 - x_0),\ x_2 = x_0 + (x_1 - x_0) + (x_2 - x_1),\ \cdots$$

accumulate rapidly. Also, every step requires the computation of the velocities (which are usually not used in practice), and the invariables. The second method does not have these disadvantages: τ becomes larger for each step; yet the main equation can be solved easily, even if $z = 1$ does not suffice as the initial hypothesis. In this case we can always begin with the z which appeared as a solution of the main equation in the last step. Thus if $z_1, z_2, z_3, \cdots z_n$ are the solutions of the main equations for the transfer from t_0 to $t_1, t_2, t_3 \cdots t_n$, the determination of z_n is begun with $z = z_{n-1}$, and that of z_1 with $z_0 = 1$.

Therefore the second method is generally preferred since it is time-saving and more exact. Nevertheless, from time to time, depending upon the size of the table interval selected — perhaps after every fifth or tenth or fifteenth step — we will transfer to new initial elements so that τ and ξ, η, ζ, χ do not become too large. As long as $\chi = \rho\tau^2$ is of moderate magnitude, the power series for the c-functions converge fairly rapidly. For programming in electronic computers, the following expression is recommended:

$$c_n(\lambda^2) = \frac{1}{n!}\left\{1 - \frac{\lambda^2}{(n+1)(n+2)}\left[1 - \frac{\lambda^2}{(n+3)(n+4)}\left(1 - \frac{\lambda^2}{(n+5)(n+6)}\cdots ,\right)\right]\right\}$$

which allows the automatic computation from within, where the number of terms in the brackets depends upon the magnitude of λ^2. For large values of λ^2 this method converges too slowly; and the trigonometric form, which is not well suited for electronic computation, can be used:

$$c_1 = \frac{\sin\lambda}{\lambda}, \quad c_2 = \frac{1 - \cos\lambda}{\lambda^2} = \frac{1}{2}\left(\frac{\sin\frac{\lambda}{2}}{\frac{\lambda}{2}}\right)^2, \quad c_3 = \frac{\lambda - \sin\lambda}{\lambda^3} .$$

This unsuitability is the principal reason for not allowing the intermediate time to increase too rapidly.

The major areas for application of the methods described above are: (1) the improvement of an orbit which originally was determined using the Laplacian method; and (2) the computation of special perturbations. In the former application, the first orbit determination always yields the local elements for a given time t_0. The improvement of these elements by the use of actual observations, which can be spread out over a long period of time, necessitates the calculation of the coordinates at these times for correlation with the observations in question.

To apply the method to the special perturbations of the right-angle coordinates of a planet or of a comet, ephemerides of the undisturbed orbit of the celestial body are required, which can be readily calculated by means of the above method. As far as the coordinates of the perturbing planets are concerned, these values were taken from the yearbooks for the times of the calculation, which would be unsuitable for electronic computation. In this case it would be better to use the above method, by computing (beginning with location and velocity coordinates of the perturbing planet at the initial epoch) an undisturbed ephemeris (osculating orbit) of the planet which is sufficiently exact for determining the perturbations. The latter method has been utilized in Germany with good results.

NASA-Langley, 1962 G-217